Mod.3] Fourier Series & Fourier Transform

* **Trigonometric Series :**

$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ where all a's and b's are constants is called as Trigonometric series.

The exapansion of any periodic function $f(x)$ (satisfying Dirichlet's conditions) in the form of the above series is called as Fourier series.

* **Dirichlet's conditions :-**

A funⁿ $f(x)$ defined in the interval $c_1 \leq x \leq c_2$ can be expressed as Fourier series if in the interval -

i] $f(x)$ and its integrals are finite & single valued.

ii] $f(x)$ has discontinuities finite in number.

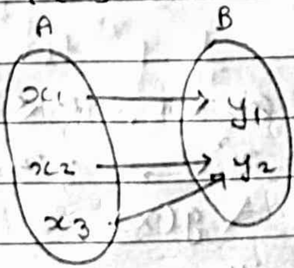
iii] $f(x)$ has finite number of maxima & minima.

These conditions are known as Dirichlet's conditions.

* **Single-valued function :-**

eg. $f(x) = x^2$

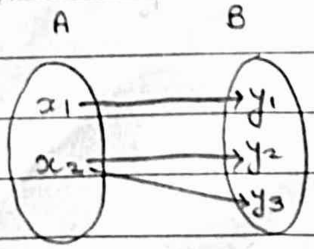
$f(x) : A \rightarrow B$



* **Multiple-valued function :-**

eg. $f(x) = \sqrt{x}$

$f(x) : A \rightarrow B$



1) i) <u>Even funⁿ</u> :- If $f(-x) = f(x)$

$\Rightarrow f(x)$ is an even funⁿ.

eg. $f(x) = \cos x$, x^2, $|x|$

ii) <u>Odd funⁿ</u> :- If $f(-x) = -f(x)$

$\Rightarrow f(x)$ is an odd funⁿ.

eg. $f(x) = \sin x$, x^3

iii) <u>Neither even nor odd funⁿ</u> :-

eg. $\log x$, e^x, 10^x

eg. $f(x) = x+1$ & $f(-x) = -x+1$

$= -(x-1)$

Q. Check whether funⁿ is even or odd.

$$g(x) = \begin{cases} \cos x, & -2 \leq x \leq 0 \\ x^2, & 0 < x \leq 2 \end{cases}$$

$\Rightarrow$

For $g(-x) = \begin{cases} \cos(-x), & -2 \leq -x \leq 0 \\ (-x)^2, & 0 < -x \leq 2 \end{cases}$

$= \begin{cases} \cos x, & -2 \leq -x \leq 0 \\ x^2, & 0 < -x \leq 2 \end{cases}$

$= \begin{cases} \cos x, & 0 \leq x \leq 2 \\ x^2, & -2 \leq x \leq 0 \end{cases}$

here, $g(-x) \neq g(x)$

$\therefore$ This is neither even nor odd.

$g(x) = f_1(x) \cdot f_2(x)$

$f_1(x)$	$f_2(x)$	$g(x)$
E	O	O
O	E	O
E	E	E
O	O	E

(imp) 2] If n is an integer, then i] $\sin n\pi = 0$ &

ii] $\cos n\pi = (-1)^n$

iii] $\cos 2n\pi = 1$

iv] $\sin 2n\pi = 0$

$(\cos n\pi \cos \pi \mp \sin n\pi \sin \pi)$ v] $\cos(n \pm 1)\pi = -\cos n\pi$

$(\sin n\pi \cos \pi \pm \cos n\pi \sin \pi)$ vi] $\sin(n \pm 1)\pi = 0$

vii] $\sin(2n\pi + x) = \sin x$

viii] $\cos(2n\pi + x) = \cos x$

3] $\int_{-a}^{a} f(x)dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd} \\ 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even} \end{cases}$

4] $\int_c^{c+2\pi} \cos nx\, dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0, \quad (n \neq 0)$

$\int_c^{c+2\pi} \sin nx\, dx = \left[\frac{-\cos nx}{n}\right]_c^{c+2\pi} = 0, \quad (n \neq 0)$

5] a] $\int_{-\pi}^{\pi} \cos nx\, dx = 2\int_0^{\pi} \cos nx\, dx$

b] $\int_{-\pi}^{\pi} \sin nx\, dx = 0$

c] $\int_{-\pi}^{\pi} \underset{(O)}{\sin mx} \underset{(E)}{\cos nx}\, dx = 0, \quad \forall\, m,n \ (m \neq n)$

d] $\int_{-\pi}^{\pi} \underset{(E)}{\cos mx} \underset{(E)}{\cos nx}\, dx = 2\int_0^{\pi} \cos mx \cos nx\, dx$

e] $\int_{-\pi}^{\pi} \underset{(O)}{\sin mx} \underset{(O)}{\sin nx}\, dx = 2\int_0^{\pi} \sin mx \sin nx\, dx$

6] $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

* Fourier series in $(0, 2\pi)$ & $(-\pi, \pi)$:-

$$\boxed{\begin{array}{l}\text{Fourier series in } (c, c+2\pi) \text{ is -} \\ f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\end{array}}$$

here, a_0, a_n, b_n are called fourier coefficients of $f(x)$

$$a_0 = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Case 1 : If $c = 0$, the interval becomes $(0, 2\pi)$ and F.S of $f(x)$ is given by $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

where, $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

Q. Expand $f(x) = x^2$ as F.S in $(0, 2\pi)$.

$\Rightarrow$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ ... (I)

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{8\pi^3}{3} - 0\right] = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\, dx = \frac{1}{\pi}\left[x^2 \cdot \frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left[\frac{-\sin nx}{n^3}\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(0 - 4\pi\left[\frac{-1}{n^2}\right] + 0\right) - \left(0 - 0 + 0\right)\right]$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\, dx = \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left[\frac{\cos nx}{n^3}\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left((2\pi)^2\left(\frac{-1}{n}\right) - 0 + 2\left(\frac{1}{n^3}\right)\right) - \left(0 - 0 + 2\left(\frac{1}{n^3}\right)\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \cancel{\frac{2}{n^3}} - \cancel{\frac{2}{n^3}}\right]$$

$$= -\frac{4\pi}{n}.$$

Putting in (I),
$$x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx + \left(\frac{-4\pi}{n}\sin nx\right)\right)$$

$$x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n}.$$

Q. Expand $f(x) = e^{-x}$ in F.S in $(0, 2\pi)$

$\Rightarrow$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{2\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} = \frac{-1}{2\pi}\left[e^{-2\pi} - e^{0}\right]$$

$$= \frac{1}{2\pi}\left[1 - e^{-2\pi}\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx = \frac{1}{\pi}\left[e^{-x}\left(\cancel{\frac{\sin nx}{n}}\right) - e^{-x}(-1)\left(\cancel{\frac{-\cos nx}{n^2}}\right) + \right.$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left[-1\cos nx + n\sin nx\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{1+n^2}(-1+0) - \frac{1}{1+n^2}(-1+0)\right]$$

$$= \frac{1}{\pi}\left[\frac{1 - e^{-2\pi}}{1+n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left[-1\sin nx - n\cos nx\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{1+n^2}(0-n) - \frac{1}{1+n^2}(0-n)\right]$$

$$= \frac{n}{\pi}\left[\frac{1 - e^{-2\pi}}{1+n^2}\right].$$

$$\therefore \text{F.S} \Rightarrow \quad e^{-x} = \frac{1}{2\pi}\left[1 - e^{-2\pi}\right] + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\left(\frac{1-e^{-2\pi}}{1+n^2}\right)\cos nx + \frac{n}{\pi}\left(\frac{1-e^{-2\pi}}{1+n^2}\right)\sin nx\right].$$

H.W

Q.1] Expand $f(x) = x \cdot \sin x$ in F.S in $(0, 2\pi)$.

Hint: $a_n = \frac{2}{n^2-1}$, if $n \neq 1$, $b_n = 0$, if $n \neq 1$

Calculate a_1 & b_1 by putting $n=1$.

Q.2] Obtain b_5 and a_0 for $f(x) = \begin{cases} \sin x, & 0 \leq x < \pi \\ 0, & \pi < x \leq 2\pi \end{cases}$

Hint: $a_0 = \frac{1}{\pi}$, $b_5 = 0$

Q.3] Find F.S for $f(x) = \frac{3x^2 - 6x\pi + 2\pi^2}{12}$ in $(0, 2\pi)$

Q.4] Find a_3, b_n and a_0 for $f(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ 2\pi - x, & \pi \leq x \leq 2\pi \end{cases}$

Q.5] Obtain F.S for $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$

Also derive the value of b_5, a_3.

Q.3] $\Rightarrow$ $a_0 = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{3x^2 - 6x\pi + 2\pi^2}{12} \right) dx$

$= \frac{1}{24\pi} \left[\frac{3x^3}{3} - \frac{6x^2\pi}{2} + 2\pi^2 x \right]_0^{2\pi}$

$= \frac{1}{24\pi} \left[(8\pi^3 - 3(4\pi^3) + 4\pi^3) - (0 + 0 + 0) \right]$

$= 0$

P.T.O

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(3x^2 - 6x\pi + 2\pi^2)}{12}\cos nx\,dx$$

$$= \frac{1}{12\pi}\left[(3x^2 - 6x\pi + 2\pi^2)\left(\frac{\sin nx}{n}\right) - (6x - 6\pi)\left(\frac{-\cos nx}{n^2}\right) + (6)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[\left(\frac{0 + 6x - 6\pi + 0}{n^2}\right) - \left(\frac{6x - 6\pi}{n^2}\right)\right]$$

$$= \frac{1}{12\pi}\left[\frac{12\pi - \cancel{6}\pi - 0 + \cancel{6}\pi}{n^2}\right]$$

$$= \frac{1}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(3x^2 - 6x\pi + 2\pi^2)}{12}\sin nx\,dx$$

$$= \frac{1}{12\pi}\left[(3x^2 - 6x\pi + 2\pi^2)\left(-\frac{\cos nx}{n}\right) - (6x - 6\pi)\left(\frac{-\sin nx}{n^2}\right) + (6)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(3(4\pi^2) - 6\pi(2\pi) + 2\pi^2)\left[\frac{-1}{n}\right] + 6\left[\frac{1}{n^3}\right] - 2\pi^2\left(\frac{-1}{n}\right) - 6\left(\frac{1}{n^3}\right)\right]$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi^2}{n} + \frac{6}{n^3} + \frac{2\pi^2}{n} - \frac{6}{n^3}\right]$$

$$= 0$$

$$\therefore f(x) = 0 + \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos(nx) + 0\sin(nx)\right).$$

* Parseval's Identity :-

If $f(x)$ converges uniformly in $(c, c+2l)$ then,

$$\boxed{\frac{1}{2l}\int_c^{c+2l}[f(x)]^2dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)}$$

This is known as Parseval's Identity for $f(x)$ in $(c, c+2l)$

case i : If $l=\pi$, the interval is $(c, c+2\pi)$ and
$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

For $(0,2\pi)$ $\quad \frac{1}{2x}\int_0^{2\pi}[f(x)]^2dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$

Q. I Find the Fourier Series for $f(x) = \begin{cases} x, & 0<x<\pi \\ 2\pi - x, & \pi<x<2\pi \end{cases}$

in $f(x) \to (0,2\pi)$. Hence deduce that, $\frac{\pi^2}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$

$\Rightarrow$

let $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ ... (I)

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi}f(x)dx$$

$$a_0 = \frac{1}{2\pi}\left[\int_0^{\pi}x\,dx + \int_{\pi}^{2\pi}(2\pi - x)dx\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}[\pi^2 - 0] + \frac{1}{2}\left[(4\pi(2\pi) - 4\pi^2) - [(4\pi^2) - \pi^2]\right]\right]$$

$$= \frac{1}{4\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - 3\pi^2\right] = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cdot\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} +$$

$$\frac{1}{\pi}\left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] + \left[\frac{-1}{n^2} + \frac{(-1)^n}{n^2}\right]\right\}$$

$$= \frac{-2}{n^2\pi}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{-4}{n^2\pi}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cdot\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} +$$

$$\frac{1}{\pi}\left[(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[\left(-\frac{\pi(-1)^n}{n}\right) - (0)\right] + \left[-\frac{\pi}{n}(-1)^n\right]\right\}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n}\right]$$

$$= 0$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{-2}{\pi n^2}(1-(-1)^n)\cos nx + 0\right)$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(1-(-1)^n)\cos nx}{n^2}\right].$$

now, $f(x) = \frac{\pi}{2} - \frac{2}{\pi}\left[\frac{2\cos x}{1^2} + \frac{2}{3^2}\cos 3x + \frac{2}{5^2}\cos 5x + \dots\right]$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{3^2\pi}\cos 3x - \frac{4}{5^2\pi}\cos 5x$$

To prove that, $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$

we need to use Parsenal's Identity,

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2) \dots \text{(II)}$$

now, $\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = \frac{1}{2\pi}\left[\int_0^{\pi} x^2 dx + \int_{\pi}^{2\pi}(2\pi-x)^2 dx\right]$

$$= \frac{1}{2\pi}\left[\left[\frac{x^3}{3}\right]_0^{\pi} + \left.\frac{(2\pi-x)^3}{3(-1)}\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{\pi^3}{3} - 0\right) + \left(\frac{-1}{3}\right)\left[(0)^3 - (\pi^3)\right]\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= \frac{\pi^2}{3}.$$

now, $\frac{\pi^2}{3} = \left(\frac{\pi}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\left[\frac{-2}{\pi n^2}(1-(-1)^n)\right]^2 + 0^2\right)$

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{1}{2}\left(\frac{4}{\pi^2}\right)\sum_{n=1}^{\infty}\frac{(1-(-1)^n)^2}{n^4}.$$

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{2}{\pi^2}\left[\frac{2^2}{1^4} + \frac{2^2}{3^4} + \frac{2^2}{5^4} + \frac{2^2}{7^4} + \dots\right]$$

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{2}{\pi^2}\left[\frac{4}{1^4} + \frac{4}{3^4} + \frac{4}{5^4} + \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{8}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

Hence Proved.

Q.3] $f(x) = \frac{3x^2 - 6\pi x + 2\pi^2}{12}$, for $x \in (0, 2\pi)$, Deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(from pg.no. 50 continue)

$\Rightarrow$ Fourier Series : $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$

i.e. $\frac{3x^2 - 6\pi x + 2\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$

For deduction here, substitution can be done.
Put $x = 0$ in above series,

$$\therefore \frac{3(0)^2 - 6\pi(0) + 2\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n(0)$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Hence Proved.

Q.3] Obtain F.S for $f(x) = \frac{1}{2}(\pi - x)$ in $(0, 2\pi)$ with period 2π & hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

$\Rightarrow$ here, $f(x) = \frac{1}{2}(\pi - x)$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\,dx$$

$$= \frac{1}{4\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(2\pi^2 - \frac{4\pi^2}{2}\right) - (0-0)\right]$$

$$= \frac{1}{4\pi}[0]$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\cos nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{(+1)}{n^2} - \left(-\frac{(+1)}{n^2}\right)\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{\pi(1)}{n} - 0\right) - \left(-\frac{\pi(1)}{n} - 0\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n}\right]$$

$$b_n = \frac{1}{n}$$

$$\therefore F.S = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx.$$

$$\frac{1}{2}(\pi-x) = \frac{1}{1}\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots$$

For deduction, put $x = \frac{\pi}{2}$ in above equ.

$$\Rightarrow \frac{1}{2}\left(\frac{\pi}{2}\right) = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7}$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Q.4] Obtain FS of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $(0, 2\pi)$.

Also, deduce that, $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty$

$\Rightarrow$

$$f(x) = \frac{1}{4}(\pi - x)^2$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 dx$$

$$= \frac{1}{8\pi} \left.\frac{(\pi - x)^3}{3(-1)}\right|_0^{2\pi}$$

$$= \frac{-1}{24\pi}\left[(-\pi)^3 - (\pi)^3\right]$$

$$= \frac{-1}{24\pi} \times -2\pi^3$$

$$= \frac{\pi^2}{12}$$

$$\cancel{a_\pi = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \cos nx\, dx}$$

$$\cancel{= \frac{1}{4\pi}\left[\left(\frac{(\pi - x)^2}{(-1)}\right)\left[\frac{\sin nx}{n}\right] - (+2)(\pi-x)(-1)\left[\frac{-\cos nx}{n^2}\right]\right]_0^{2\pi}}$$

$$\cancel{= \frac{1}{4\pi}\left[\frac{-1}{n^2} - \left(\frac{+2}{n^2}\right)\right]}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi-x)^2 \cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left[\frac{\sin nx}{n}\right] - (2(\pi-x)(-1))\left[\frac{-\cos nx}{n^2}\right] + (-2)\left[\frac{-\sin nx}{n^3}\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[+2(-\pi)\left(\frac{-1}{n^2}\right) - \left[2(\pi)\left(\frac{-1}{n^2}\right)\right]\right]$$

$$= \frac{1}{4\pi}\left[\frac{4\pi}{n^2}\right]$$

$$= \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi-x)^2 \sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left[\frac{-\cos nx}{n}\right] - (2(\pi-x)(-1))\left[\frac{-\sin nx}{n^2}\right] + (-2)\left[\frac{\cos nx}{n^3}\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left[(-\pi)^2\left(\frac{-1}{n}\right) + 0 + (-2)\left(\frac{1}{n^3}\right)\right] - \left[\pi^2\left(\frac{-1}{n}\right) + 0 + (-2)\left(\frac{1}{n^3}\right)\right]\right]$$

$$= \frac{1}{4\pi}[0]$$

$$= 0$$

$\therefore$ F.S $\Rightarrow$ $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$

$$\therefore \frac{1}{4}(\pi-x)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx \quad \ldots \text{(I)}$$

For deduction, Put $x=0$ in equ. (I)

$$\Rightarrow \frac{1}{4}\pi^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}(1)$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{\pi^2}{12} + \left(\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots\right)$$

$$\Rightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \quad \text{(II)}$$

now, put $x = \pi$ in equ. (I)

$$\therefore \frac{1}{4}(\pi-\pi)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi$$

$$\therefore -\frac{\pi^2}{12} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \dots$$

$$\therefore \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots \quad \text{(III)}$$

now, adding equ. (II) & (III),

$$\therefore \frac{\pi^2}{6} + \frac{\pi^2}{12} = 2 + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \dots$$

$$\therefore \frac{3\pi^2}{12} = 2\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right]$$

$$\therefore \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

Hence Proved.

H.W

1] Obtain F.S $f(x) = \cos px$ where p is not an integer and $x \in (0, 2\pi)$ and deduce that,

1] $\pi \operatorname{cosec} \pi x = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{p+n} + \frac{1}{p-n}\right]$

2] $\pi \cot 2\pi p = \frac{1}{2p} + p \sum_{n=1}^{\infty} \frac{1}{p^2 - n^2}$

Q.4] Find the F.S $f(x) = \left(\frac{\pi - x}{2}\right)^2$ and Continue

Prove that, $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$

$\Rightarrow$ for $f(x) = \left(\frac{\pi - x}{2}\right)^2$

F.S $\Rightarrow$ $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx.$

For proving the following, we have to use parseval's identity –

$$\frac{1}{2\pi}\int_0^{2\pi} (f(x))^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

L.H.S $\Rightarrow$ $\frac{1}{32\pi}\int_0^{2\pi}\left(\frac{\pi - x}{\cancel{2}}\right)^4 dx$

$= \frac{1}{32\pi}\left[\frac{1}{5}\left(\frac{\pi - x}{\cancel{2}}\right)^5 \times \frac{1}{(-1)}\right]_0^{2\pi}$

$= \frac{1}{32\pi}\left[\frac{1}{5}\left(\frac{-\pi}{\cancel{2}}\right)^5 \times \frac{1}{(-1)} - \left[\frac{1}{5}\left(\frac{\pi}{\cancel{2}}\right)^5 \times \frac{1}{(-1)}\right]\right]$

$= \frac{1}{32\pi}\left[\frac{\pi^5}{5\cancel{32}} + \frac{\pi^5}{5\cancel{32}}\right]$

$= \frac{\pi^4}{80}$

R.H.S $\Rightarrow$ $a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \left(\frac{\pi^2}{12}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\left(\frac{1}{n^2}\right)^2 + 0^2\right)$

$\therefore \frac{\pi^4}{80} = \frac{\pi^4}{144} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$

$\therefore \frac{\pi^4}{80} - \frac{\pi^4}{144} + \frac{1}{2}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots\right]$

$\therefore \frac{\cancel{64}\pi^4 \times \cancel{2}^1}{\cancel{80}_{10} \times \cancel{144}_{\cancel{72}_9}} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$

$\therefore \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$

Fourier series in $(-\pi, \pi)$:-

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

→ If $f(x)$ is odd funⁿ,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{O} \underbrace{\cos nx}_{E}}_{O} dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{O} \underbrace{\sin nx}_{O}}_{E} dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx dx$$

i.e. If $f(x)$ is odd then, $a_0 = 0$ & $a_n = 0$.

→ If $f(x)$ is even funⁿ,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{0}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{E} \underbrace{\cos nx}_{E}}_{E} dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{E} \underbrace{\sin nx}_{O}}_{O} dx = 0$$

i.e. If $f(x)$ is even then, $b_n = 0$.

* Step for fourier series in $(-\pi, \pi)$

Step 1 - Check whether funⁿ is even or odd.

Step 2 - a] If $f(x)$ is neither even nor odd -
then, Calculate all 3 coefficients in $(-\pi, \pi)$.

b] If $f(x)$ is odd -
Show that funⁿ is odd & state that $a_0 = 0$, $a_n = 0$.
Calculate b_n & substitute in F.S.

c] If $f(x)$ is even -
Show that funⁿ is even & state that $b_n = 0$
Calculate a_0, a_n & substitute in F.S.

Q. Find F.S for $f(x) = \begin{cases} x+\pi, & 0 \leq x \leq \pi \\ -x-\pi, & -\pi \leq x \leq 0 \end{cases}$

$\Rightarrow$

Consider $f(-x) = \begin{cases} -x+\pi, & 0 \leq -x \leq \pi \\ -(-x)-\pi, & -\pi \leq -x \leq 0 \end{cases}$

$$= \begin{cases} -x+\pi, & 0 \geq x \geq -\pi \\ x-\pi, & \pi \geq x \geq 0 \end{cases}$$

$$= \begin{cases} -x+\pi, & -\pi \leq x \leq 0 \\ x-\pi, & 0 \leq x \leq \pi \end{cases}$$

$\neq f(x)$ also $\neq f(-x)$

$\therefore$ $f(x)$ is neither even nor odd.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} -(x+\pi)\,dx + \int_{0}^{\pi} (x+\pi)\,dx \right]$$

$$= \frac{1}{2\pi}\left[-\left[\frac{x^2}{2}+\pi x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}+\pi x\right]_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\left[0+0-\left(\frac{\pi^2}{2}-\pi^2\right)\right] + \left[\left(\frac{\pi^2}{2}+\pi^2\right)-(0+0)\right]\right]$$

$$= \frac{1}{2\pi}\left[\frac{-\pi^2}{2}+\frac{3\pi^2}{2}\right]$$

$$= \frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -(x+\pi)\cos nx\,dx + \int_{0}^{\pi}(x+\pi)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[-(x+\pi)\left(\frac{\sin nx}{n}\right)-(-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \right.$$

$$\left.\left[(x+\pi)\left(\frac{\sin nx}{n}\right)-(1)\left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\left(0-\frac{1}{n^2}\right)-\left(0-\frac{(-1)^n}{n^2}\right)\right) + \left(\cancel{\left(2\pi\left(\frac{-(-1)^n}{n}\right)\right)} - \cancel{\left(\pi\left(\frac{-1}{n}\right)+0\right)}\right)\right]$$

$$= \cancel{\frac{1}{\pi}\left[\frac{-1}{n^2}+\frac{(-1)^n}{n^2}+\frac{-2\pi(-1)^n}{n}+\frac{\pi}{n}\right]}$$

$$+\left(\left(0+\frac{(-1)^n}{n^2}\right)-\left(0+\frac{1}{n^2}\right)\right)$$

$$= \frac{1}{\pi}\left[\frac{-1}{n^2}+\frac{(-1)^n}{n^2}+\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n-1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -(x+\pi)\sin nx\,dx + \int_{0}^{\pi}(x+\pi)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -(x+\pi)\sin nx\,dx + \int_{0}^{\pi}(x+\pi)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-(x+\pi)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[(x+\pi)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{n}(1)+0\right) - (0-0)\right] + \frac{1}{\pi}\left[\left(\frac{-2\pi}{n}(-1)^n + 0\right) - \left(\frac{-\pi}{n}(1)+0\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{2\pi}{n}(-1)^n + \frac{\pi}{n}\right]$$

$$= \frac{2}{n}\left[1-(-1)^n\right]$$

$$\therefore F\cdot S \Rightarrow \quad f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi n^2}\left((-1)^n - 1\right)\cos nx + \frac{2}{n}\left[1-(-1)^n\right]\sin nx\right].$$

Q.3] Obtain F.S for $f(x) = \begin{cases} x + \pi/2, & -\pi < x < 0 \\ \pi/2 - x, & 0 < x < \pi \end{cases}$

Hence deduce that, i] $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$ ii] $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$

* Parseval Identity for $f(x)$ in $(-\pi, \pi)$:-

If $f(x)$ converges uniformly in open interval $-\pi$ to π, then F.S for $f(x)$ is given by -

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)}$$

And parseval's identity in $(-\pi, \pi)$ is given by -

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right]}$$

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$$\Rightarrow \quad f(-x) = \begin{cases} -x + \pi/2 & , -\pi < -x < 0 \\ \pi/2 - (-x) & , 0 < -x < \pi \end{cases}$$

$$f(-x) = \begin{cases} \pi/2 - x & , \pi > x > 0 \\ \pi/2 + x & , 0 > x > -\pi \end{cases}$$

$$f(-x) = f(x)$$

$\therefore$ It is an even fun^n.

$\therefore$ $b_n = 0$

now, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)dx$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[(\pi^2 - \pi^2) - (0 - 0)\right]$$

$$= 0$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi/2 - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi/2 - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 - \frac{(-1)^n}{n^2}\right) - \left(0 - \frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi n^2}\left[1-(-1)^n\right]$$

$\therefore$ Fourier series $\Rightarrow$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[1-(-1)^n\right] \cos nx$$

now, at $x=0$, fun is discontinuous.

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]$$

$$= \frac{\pi}{2}$$

For deduction, put $x=0$ in F.S

$$\therefore \quad f(0) = \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[1-(-1)^n\right] \cos n(0)$$

$$\therefore \quad \frac{\pi}{2} = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^2}$$

$$\therefore \quad \frac{\pi^2}{4} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \dots$$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Hence Proved.

now, To deduce, $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$

we need to use Parseval's Identity.

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$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}[a_n^2+b_n^2] \quad \text{... Parsevals Identity}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}[1-(-1)^n]\right)^2$$

$$= \frac{1}{\pi}\int_{0}^{\pi}[f(x)]^2\,dx = \frac{1}{2}\times\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]^2}{n^4}$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2}-x\right)^2dx = \frac{2}{\pi^2}\left[\frac{4}{1^4}+\frac{4}{3^4}+\frac{4}{5^4}+\dots\right]$$

$$\frac{1}{\pi}\left[\frac{1}{3}\frac{\left(\frac{\pi}{2}-x\right)^3}{(-1)}\right]_0^{\pi} = \frac{8}{\pi^2}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots\right]$$

$$\frac{1}{\pi}\times\frac{-1}{3}\left[\left(\frac{-\pi}{2}\right)^3-\left(\frac{\pi}{2}\right)^3\right] = \frac{8}{\pi^2}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots\right]$$

$$\frac{-1}{3\cancel{\pi}}\left[\frac{-2\pi^{\cancel{3}\,2}}{8}\right] = \frac{8}{\pi^2}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots\right]$$

$$\therefore \quad \frac{\pi^4}{96} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots$$

hence Proved.

H.W

Q. Find Fourier series of $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$,

hence deduce that, $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Parseval's Identity

* Fourier series in $(c, c+2l)$:

Form of FS in $(c, c+2l)$ i.e. in $(0, 2l)$ or $(-l, l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

For interval in $(0,2l)$:	For interval in $(-l,l)$:
$a_0 = \frac{1}{2l}\int_0^{2l} f(x)dx$	$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx$
$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$	$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$
$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$	$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

* Parseval's identity in $f(x)$ in interval $(c, c+2l)$:

$$\boxed{\frac{1}{2l}\int_c^{c+2l}[f(x)]^2dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}[a_n^2+b_n^2]}$$

* Parseval's identity in $(0$ to $2l)$.

$$\frac{1}{2l}\int_0^{2l}[f(x)]^2dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}[a_n^2+b_n^2]$$

* Parseval's identity in $(-l$ to $l)$.

$$\frac{1}{2l}\int_{-l}^{l}(f(x))^2dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}[a_n^2+b_n^2].$$

Q.1 Find FS for $f(x) = \begin{cases}\pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2\end{cases}$ with period 2.

Hence deduce that, $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}$. ...

$\Rightarrow$ here, $l=1$

$\therefore$ F.S $\Rightarrow$ $f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)]$.

$$a_0 = \frac{1}{2(1)} \int_0^2 f(x)dx$$

$$= \frac{1}{2}\left[\int_0^1 \pi x\, dx + \int_1^2 \pi(2-x)dx\right]$$

$$= \frac{1}{2}\left[\frac{\pi x^2}{2}\Big|_0^1 + \pi\left(2x - \frac{x^2}{2}\right)\Big|_1^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) + \pi\left(\left(4 - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \pi\left(2 - \frac{3}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2}.$$

$$a_n = \frac{1}{[1]} \int_0^2 f(x) \cos\left(\frac{n\pi x}{1}\right) dx$$

$$= \int_0^1 \pi x \cos n\pi x\, dx + \int_1^2 \pi(2-x) \cos n\pi x\, dx$$

$$= \left[\pi x \left(\frac{\sin n\pi x}{n\pi}\right) - (\pi)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right)\right]_0^1 + \left[\pi(2-x)\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{(n\pi)^2}\right)\right]_1^2$$

$$= \left[\left(\frac{1}{n}\sin n\pi + \frac{1}{n^2\pi}\cos n\pi\right) - \left(0 + \frac{1}{n^2\pi}\cos 0\right)\right] +$$

$$\left[\left(0 \cancel{+} \frac{1}{n^2\pi}\cos 2n\pi\right) - \left(\frac{1}{n}\sin n\pi \cancel{+} \frac{1}{n^2\pi}\cos n\pi\right)\right]$$

$$= \left[\left(0 + \frac{1}{n^2\pi}(-1)^n\right) - \left(\frac{1}{n^2\pi}\right) \cancel{+} \left(\frac{1}{n^2\pi}\right) + \left(0 + \frac{(-1)^n}{n^2\pi}\right)\right]$$

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$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{[1]}\int_0^2 f(x)\sin\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \left[\pi x\left(\frac{-\cos n\pi x}{n\pi}\right) - (\pi)\left(\frac{-\sin n\pi x}{(n\pi)^2}\right)\right]_0^1 +$$

$$\left[\pi(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-\pi)\left(\frac{-\sin n\pi x}{(n\pi)^2}\right)\right]_1^2$$

$$= \left[\left(-\frac{1}{n}\cos n\pi + \frac{1}{n^2\pi}\sin n\pi\right) - \left(0 + \frac{1}{n^2\pi}\sin 0\right)\right] +$$

$$\left[\left(0 - \frac{1}{n^2\pi}\sin 2n\pi\right) - \left(\frac{-1}{n}\cos n\pi - \frac{1}{n^2\pi}\sin n\pi\right)\right]$$

$$= \left[-\frac{(-1)^n}{n} + 0 - 0 + 0 + \frac{(-1)^n}{n} + 0\right]$$

$$= 0.$$

$$\therefore F.S. \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2[(-1)^n - 1]}{n^2\pi}\cos n\pi x\right].$$

now,

$$\pi x = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos n\pi x$$

fro deduction, put $x = 0$

$$\therefore 0 = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$-\frac{\pi}{2} = \frac{2}{\pi}\left[\frac{-2}{(1)^2} + 0 - \frac{2}{(3)^2} + 0 - \frac{2}{(5)^2} + \cdots\right]$$

$$\frac{\pi^2}{4} = -2\left[\frac{1}{(1)^2} + \frac{1}{(3)^2} + \frac{1}{(5)^2} + \frac{1}{(7)^2} + \cdots\right]$$

$$\Rightarrow \pi/8 = 1/1^2 + 1/3^2 + 1/5^2 + \ldots$$ Hence Proved.

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Q.3 Obtain F.S for $f(x) = 2x - x^2$ in $(0,3)$ and hence deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \infty = \frac{\pi^2}{12}$.

$\Rightarrow$ here, $l = \frac{3}{2}$.

So, let $f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi x}{3}\right) + b_n \sin\left(\frac{2n\pi x}{3}\right)\right]$

$$a_0 = \frac{1}{2(3/2)}\int_0^3 (2x - x^2)\,dx$$

$$= \frac{1}{3}\int_0^3 \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{1}{3}\left[\left(9 - \frac{27}{3}\right) - (0 - 0)\right]$$

$$= \frac{1}{3}\left[\frac{-\cancel{105}}{4}\right]$$

$$= \frac{-\cancel{15}}{\cancel{4}},\ 0$$

$$a_n = \frac{1}{3/2}\int_0^3 (2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[\frac{(2x - x^2)\sin\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}} - (2 - 2x)\left[\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right] + (-2)\left[\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{(2n\pi/3)^3}\right]\right]_0^3$$

$$= \frac{2}{3}\left\{\left[\frac{(-3)\sin 2n\pi}{2n\pi/3} + \frac{(-4)\cos 2n\pi}{(2n\pi/3)^2} + \frac{(2)\sin 2n\pi}{(2n\pi/3)^3}\right] - \left[0 + \frac{(2)\cos 0}{(2n\pi/3)^2} + 0\right]\right\}$$

$$= \frac{2}{3}\left[\frac{-\cancel{4}\times 9\,(1)}{\cancel{4}n^2\pi^2} - \frac{\cancel{4}\times 9\,(1)}{2\cancel{4}n^2\pi^2}\right]$$

$$= \frac{\cancel{2}}{3}\left[\frac{-\cancel{2}\times 9}{\cancel{4}n^2\pi^2}\right] \qquad \frac{2}{3}\left[\frac{-9}{n^2\pi^2} - \frac{9}{2n^2\pi^2}\right]$$

$$= \frac{-9}{3n^2\pi^2} \qquad = \frac{\cancel{2}}{\cancel{3}}\left[-\frac{\cancel{3}}{\cancel{2}}\left[\frac{9}{n^2\pi^2}\right]\right] = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{3/2}\int_0^3 (2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{2n\pi/3}\right) - (2-2x)\left(\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{(2n\pi/3)^2}\right) + (-2)\left(\frac{\cos(2n\pi x/3)}{(2n\pi/3)^3}\right)\right]_0^3$$

$$= \frac{2}{3}\left\{\left[\frac{3\cdot\cos 2n\pi}{2n\pi/3} + \frac{(-4)\sin 2n\pi}{(2n\pi/3)^2} + \frac{(-2)\cos 2n\pi}{(2n\pi/3)^3}\right] - \left[0 + \frac{(2)\sin 0}{(2n\pi/3)^2} + \frac{(-2)\cos 0}{(2n\pi/3)^3}\right]\right\}$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi} + 0 - \cancel{\frac{2}{(2n\pi/3)^3}} - 0 + 0 + \cancel{\frac{2}{(2n\pi/3)^3}}\right]$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi}\right]$$

$$= \frac{3}{n\pi}.$$

$$\therefore F.S \Rightarrow f(x) = \sum_{n=1}^{\infty}\left[\frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]$$

Put $x=0$

$$\therefore \quad 0 = \sum_{n=1}^{\infty}\frac{-9}{\pi^2}\cdot\frac{1}{n^2}(1)$$

$$\therefore \frac{-\pi^2}{9} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots$$

put $x = 3/2$

$$\left[2\left(\frac{3}{2}\right) - \left(\frac{3}{2}\right)^2\right] = \sum_{n=1}^{\infty}\left[\frac{-9}{n^2\pi^2}\cos(n\pi) + \frac{3}{n\pi}\sin(n\pi)\right]$$

$$\left[3 - \frac{9}{4}\right] = \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{\cancel{3}}{4} \times \frac{-\pi^2}{\cancel{9}3} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Q.3] Obtain F.S for $f(x) = |x|$, $-2 < x < 2$

Hence deduce that, $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

$\Rightarrow$

Here, $l = 2$

$$\therefore F.S \Rightarrow |x| = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{2}\right) + b_n\sin\left(\frac{n\pi x}{2}\right)\right]$$

here,

$|-x| = x = |x|$

$\Rightarrow f(x) = |x|$ is an even fun^n.

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{l}\int_0^{l} f(x)\,dx$$

$$= \frac{1}{2}\int_0^{2} |x|\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2$$

$$= 1.$$

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$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{2l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{2}\int_{0}^{2} |x|\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[\frac{x^2}{2}\right]_0^2 \qquad \left[\frac{x\sin\left(\frac{n\pi x}{2}\right)}{n\pi/2} - (1)\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^2}\right)\right]_0^2$$

$$= \frac{1}{2}(4-0) \qquad = \left[\frac{2\sin n\pi}{n\pi/2} + \frac{\cos n\pi}{(n\pi/2)^2}\right] - \left[0 + \frac{\cos 0}{(n\pi/2)^2}\right]$$

$$= 2. \qquad = 0 + \frac{4(-1)^n}{n^2\pi^2} - \frac{4(1)}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right].$$

$\therefore$ Since $f(x)$ is even funⁿ

$\therefore b_n = 0$

$$F.S \Rightarrow \quad |x| = 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]\cos\left(\frac{n\pi x}{2}\right)$$

To deduce, put $x = 0$

$$0 = 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[(-1)^n - 1\right](1)$$

$$-1 \times \frac{\pi^2}{4} = \frac{-2}{1^2} + 0 - \frac{2}{3^2} + 0 - \frac{2}{5^2} + \cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

* Fourier Transform :-

In real life problems not all funⁿ/events are periodic in nature. Fourier Transform deals with such non periodic funⁿ & hence it is very useful in practical applications.

If a funⁿ $f(x)$ is defined on $(-\infty,\infty)$, is piecewise continuous in each finite interval & is absolutely integrable in $(-\infty,\infty)$ then integral $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$ is called Fourier Transform of $f(x)$ & it is denoted by $F(s)$.

i.e. $$\boxed{F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx}$$

eg. $f(x) = x^2, \quad 0 < x < 2$

then, $$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{2} x^2 \cdot e^{isx}\,dx$$

* Fourier sine transform & cosine transform :-

If $f(x)$ is defined in $(0,\infty)$, we define its -

1) F.S.T by
$$\boxed{F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx}$$

2) F.C.T by
$$\boxed{F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx}$$

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Q.1] Find the Fourier Transform of $f(x) = \begin{cases} 1 + x/a, & -a < x < 0 \\ 1 - x/a, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow$

By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{0} \left(1 + \frac{x}{a}\right) e^{isx} dx + \int_{0}^{a} \left(1 - \frac{x}{a}\right) e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\left(1 + \frac{x}{a}\right) \frac{e^{isx}}{is} - \left(\frac{1}{a}\right) \left[\frac{e^{isx}}{(is)^2}\right] \right)_{-a}^{0} + \left(\left(1 - \frac{x}{a}\right) \frac{e^{isx}}{is} - \left(\frac{-1}{a}\right) \left[\frac{e^{isx}}{(is)^2}\right] \right)_{0}^{a} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\left(\frac{1}{is} - \frac{1}{a(is)^2} \right) - \left(0 - \frac{e^{-isa}}{a(is)^2} \right) \right] + \left[\left(0 + \frac{e^{isa}}{a(is)^2} \right) - \left(\frac{1}{(is)} + \frac{1}{a(is)^2} \right) \right] \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\cancel{\frac{1}{is}} - \frac{1}{a(is)^2} + \frac{e^{-isa}}{a(is)^2} + \frac{e^{isa}}{a(is)^2} - \cancel{\frac{1}{is}} - \frac{1}{a(is)^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{as^2} - \frac{e^{isa}}{as^2} - \frac{e^{-isa}}{as^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{as^2} - \frac{1}{as^2} \left[\cos as + \cancel{i\sin as} + \cos as - \cancel{i\sin as} \right] \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{as^2} - \frac{2\cos as}{as^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{as^2} \left[1 - \cos as \right].$$

Q.2] Find Fourier cosine transform of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

$*\left[\int_0^\infty \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2}\right]$ (imp)

$\Rightarrow$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a \cos x \cos sx \, dx + \int_a^\infty (0) \cos sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a \frac{2 \cos x \cos sx}{2} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^a (\cos(s+1)x + \cos(s-1)x) dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{(s+1)} + \frac{\sin(s-1)x}{(s-1)} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{(s+1)} + \frac{\sin(s-1)a}{(s-1)} - 0 - 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{(s+1)} + \frac{\sin(s-1)a}{(s-1)} \right]$$

Q.3) Find Fourier sine transform of $\frac{1}{x}$.

$\Rightarrow$

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty \frac{\sin sx}{x} dx \right]$$

Put $x = \frac{\theta}{s} \Rightarrow \frac{1}{x} = \frac{s}{\theta}$

x	0	∞
θ	0	∞

$$dx = \frac{1}{s} d\theta$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty \frac{\cancel{s} \sin\theta}{\theta} \times \frac{d\theta}{\cancel{s}} \right]$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2}$$

$$= \sqrt{\frac{\pi}{2}}$$

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* $I = \int f(x,a)\,dx$

$\Rightarrow \dfrac{dI}{da} = \int \dfrac{\partial (f(x,a))}{\partial a}\,dx$

$\Rightarrow I = \int \text{Answer}\; da.$ [DUIS]

*** Q.4] Find the Fourier transform of $f(x) = e^{-x^2}$.

$\Rightarrow$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{isx}\,dx$$

even fun^n odd fun^n

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} [\cos sx + i\sin sx]\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x^2} \cos sx\,dx + 0$$

Let $I(s) = \int_0^\infty e^{-x^2} \cos sx\,dx$

$$\frac{dI(s)}{ds} = \int_0^\infty x \cdot e^{-x^2} \cdot -\sin sx\,dx$$

$$= \frac{1}{2} \int_0^\infty \{e^{-x^2}(-2x)\} \sin sx\,dx$$

here, $\int e^{-x^2}(-2x)\,dx = e^{-x^2}$

$$= \frac{1}{2}\left[\left(\sin sx\,(e^{-x^2})\right)_0^\infty - \int_0^\infty s \cdot \cos sx \cdot e^{-x^2}\,dx \right]$$

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