Mod 2] Inverse Laplace Transform

We know that, $y = f(t) \Rightarrow t = f^{-1}(y)$

similarly, $L[f(t)] = \phi(s)$

$\Rightarrow L^{-1}(\phi(s)) = f(t)$

If $L[f(t)] = \phi s = \int_0^\infty e^{-st} f(t)dt$,

then $f(t)$ is called the inverse Laplace Transform of $\phi(s)$ and it is denoted by $L^{-1}[\phi(s)] = f(t)$

eg. 1] $L^{-1}\left[\frac{4}{s^2+16}\right] = \sin 4t$

2] $L^{-1}\left[\frac{1}{s^2-4}\right] = ?$

$\because L(\sinh 2t) = \frac{2}{s^2-4} \Rightarrow L^{-1}\left(\frac{2}{s^2-4}\right) = \sinh 2t$

$\Rightarrow L^{-1}\left(\frac{1}{s^2-4}\right) = \frac{1}{2}\sinh 2t$

* Inverse Laplace Transform of Standard fun^n.

1] $L^{-1}\left[\frac{1}{s}\right] = 1 \quad \because \left[L(1) = \frac{1}{s}\right]$

2] $L^{-1}\left[\frac{1}{s-a}\right] = e^{at} \quad \because \left[L[e^{at}] = \frac{1}{s-a}\right]$

3] $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{1}{\Gamma(n+1)} t^n \quad \because \left[L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}\right]$

or $\boxed{L^{-1}\left[\frac{1}{s^n}\right] = \frac{1}{\Gamma n} t^{n-1}}$ ***

eg. $L^{-1}\left(\frac{1}{s^6}\right) = \frac{t^5}{\Gamma 6} = \frac{t^5}{5!}$

$L^{-1}\left[\frac{1}{s^{5/2}}\right] = \frac{t^{3/2}}{\Gamma 5/2} = \frac{t^{3/2}}{3/2 \cdot 1/2 \cdot \sqrt{\pi}} = \frac{4t^{3/2}}{3\sqrt{\pi}}$

4] $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$ $\quad \because \left[L(\sin at) = \frac{a}{s^2+a^2}\right]$

5] $L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$ $\quad \because \left[L(\sinh at) = \frac{a}{s^2-a^2}\right]$

6] $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$ $\quad \because \left[L(\cos at) = \frac{s}{s^2+a^2}\right]$

7] $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$ $\quad \because \left[L(\cosh at) = \frac{s}{s^2-a^2}\right]$

eg. $L^{-1}\left[\frac{1}{s^2-3^2}\right] = \frac{1}{3}\sinh 3t$

* **Methods for finding Inverse Laplace Transform :-**

1] Standard funⁿ
2] Using Shifting theorem
3] Using Partial Fractions
4] Using Convolution Theorem
5] Using derivatives.

eg. 1] $L^{-1}\left[\frac{1}{s^2-25}\right] = \frac{1}{5}\sinh 5t$

2] $L^{-1}\left[\frac{1}{s^{8/3}}\right] = \frac{t^{5/3}}{\Gamma 8/3} = \frac{t^{5/3}}{\frac{10}{9}\Gamma 2/3}$

3] $L^{-1}\left[\frac{s}{s^2-16}\right] = \cosh 4t$

4] $L^{-1}\left[\frac{1}{4s-5}\right] = \frac{1}{4}L^{-1}\left[\frac{1}{s-5/4}\right] = \frac{1}{4}e^{5/4t}$

* **Linearity property for inverse Laplace Transform.**

$$\boxed{L^{-1}[a\phi_1(s) + b\phi_2(s)] = aL^{-1}[\phi_1(s)] + bL^{-1}[\phi_2(s)]}$$

Q.1] $L^{-1}\left[\frac{2s+3}{s^2+9}\right] = 2L^{-1}\left[\frac{s}{s^2+9}\right] + L^{-1}\left[\frac{3}{s^2+9}\right]$

$= \underline{2\cos 3t + \sin 3t}$

Q.2] $L^{-1}\left[\frac{4s+15}{16s^2-25}\right] = \frac{1}{16}\left[4L^{-1}\left[\frac{s}{s^2-25/16}\right] + 15L^{-1}\left[\frac{1}{s^2-25/16}\right]\right]$

$= \frac{1}{16}\left[4\cosh\left(\frac{5}{4}t\right) + \frac{15}{5/4}L^{-1}\left[\frac{5/4}{s^2-25/16}\right]\right]$

$= \frac{1}{4}\cosh\left(\frac{5t}{4}\right) + \frac{15}{16}\times\frac{4}{5}\sinh\left(\frac{5t}{4}\right)$

$= \underline{\frac{1}{4}\cosh\left(\frac{5t}{4}\right) + \frac{3}{4}\sinh\left(\frac{5t}{4}\right)}.$

Q.3] $L^{-1}\left[\frac{(s+1)^2}{s^3}\right] = L^{-1}\left[\frac{s^2+2s+1}{s^3}\right]$

$= L^{-1}\left[\frac{1}{s} + \frac{2}{s^2} + \frac{1}{s^3}\right]$

$= L^{-1}\left[\frac{1}{s}\right] + 2L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s^3}\right]$

$= 1 + 2\cdot\frac{t}{\lceil 2} + \frac{t^2}{\lceil 3}$

$= \underline{1 + 2t + \frac{t^2}{2}}.$

2] <u>Use of shifting Theorem</u> :-

If $L[f(t)] = \phi(s) \Rightarrow L[e^{-at}f(t)] = \phi(s+a)$

i.e. $L^{-1}[\phi(s)] = f(t)$ then, $L^{-1}[\phi(s+a)] = e^{-at}f(t)$

i.e. $L^{-1}[\phi(s+a)] = e^{-at}L^{-1}[\phi(s)]$.

eg. 1] $L^{-1}\left[\frac{3}{(s+9)^2+9}\right]$

here, $L^{-1}\left[\frac{3}{s^2+9}\right] = \sin 3t$

$\Rightarrow L^{-1}\left[\frac{3}{(s+9)^2+9}\right] = e^{-9t}\sin 3t$

2] $L^{-1}\left[\frac{3s}{(s+9)^2+9}\right]$

here, $L^{-1}\left[\frac{3(s+9)-27}{(s+9)^2+9}\right]$

$= e^{-9t} L^{-1}\left[\frac{3s-27}{s^2+9}\right]$

$= e^{-9t}\left[3L^{-1}\left[\frac{s}{s^2+9}\right] - 9L^{-1}\left[\frac{3}{s^2+9}\right]\right]$

$= e^{-9t}\left[3\cos 3t - 9\sin 3t\right]$.

3] $L^{-1}\left[\frac{s+4}{(s+4)^2-25}\right]$

here, $L^{-1}\left[\frac{s}{s^2-25}\right] = \cosh 5t$

$\Rightarrow L^{-1}\left[\frac{s+4}{(s+4)^2-25}\right] = e^{-4t}\cosh 5t$

4] $L^{-1}\left[\frac{1}{(s-8)^4}\right] = e^{8t} L^{-1}\left[\frac{1}{s^4}\right]$

$= e^{8t}\frac{1}{\Gamma 4}t^3$

$= \frac{e^{8t}t^3}{3!}$

$= \frac{e^{8t}t^3}{6}$

5) $L^{-1}\left[\frac{5}{(s-2)^2+25}\right]$

$\Rightarrow L^{-1}\left[\frac{(s-2)+2}{(s-2)^2+25}\right] = e^{2t} L^{-1}\left[\frac{s+2}{(s)^2+25}\right]$

$= e^{2t}\left[L^{-1}\left[\frac{s}{s^2+25}\right] + \frac{2}{5} L^{-1}\left[\frac{5}{s^2+25}\right]\right]$

$= e^{2t}\left[\cosh 5t + \frac{2}{5}\sinh 5t\right]$

6) Find $L^{-1}\left[\frac{3s+2}{s^2-4s-5}\right]$

$\Rightarrow L^{-1}\left[\frac{3s+2}{s^2-4s+4-4-5}\right] = L^{-1}\left[\frac{3s+2}{(s-2)^2-9}\right]$

$= L^{-1}\left[\frac{3(s-2)+2+6}{(s-2)^2-9}\right]$

$= e^{2t} L^{-1}\left[\frac{3s+8}{s^2-9}\right]$

$= e^{2t}\left[3L^{-1}\left[\frac{s}{s^2-9}\right] + \frac{8}{3} L^{-1}\left[\frac{3}{s^2-9}\right]\right]$

$= e^{2t}\left[3\cosh 3t + \frac{8}{3}\sinh 3t\right]$

7) Find $L^{-1}\left[\frac{4s+3}{s^2-4s+13}\right]$

$\Rightarrow L^{-1}\left[\frac{4s+3}{s^2-4s+13}\right] = L^{-1}\left[\frac{4s+3}{s^2-4s+4-4+13}\right]$

$= L^{-1}\left[\frac{4s+3}{(s-2)^2+9}\right]$

$= L^{-1}\left[\frac{4(s-2)+3+8}{(s-2)^2+9}\right]$

$= e^{2t} L^{-1}\left[\frac{4s+11}{s^2+9}\right]$

$= e^{2t}\left[4L^{-1}\left[\frac{s}{s^2+9}\right] + \frac{11}{3} L^{-1}\left[\frac{3}{s^2+9}\right]\right]$

$$= e^{2t}\left[4\cos 3t + \frac{11}{3}\sin 3t\right]$$

8] Find $L^{-1}\left[\frac{s}{(2s+1)^2}\right]$

$\Rightarrow$ $L^{-1}\left[\frac{s}{4(s+1/2)^2}\right] = \frac{1}{4} L^{-1}\left[\frac{(s+1/2)-1/2}{(s+1/2)^2}\right]$

$$= \frac{1}{4} e^{-1/2 t} L^{-1}\left[\frac{s-1/2}{(s\cancel{+1/2})^2}\right]$$

$$= \frac{e^{-1/2 t}}{4}\left[L^{-1}\left[\frac{s}{s^2}\right] - \frac{1}{2} L^{-1}\left[\frac{1}{s^2}\right]\right]$$

$$= \frac{e^{-1/2 t}}{4}\left[1 - \frac{1}{2}\frac{t}{\lfloor 2}\right]$$

$$= \frac{e^{-1/2 t}}{8}\left[2-t\right]$$

* Partial Fractions.

1] $\frac{\cancel{\#}}{(s+a)(s+b)} = \frac{A}{(s+a)} + \frac{B}{(s+b)}$

2] $\frac{\cancel{s^2}}{(s+a)^2} = \frac{A}{(s+a)} + \frac{B}{(s+a)^2}$

3] $\frac{s^2}{(s+3)(s^2+3s+5)} = \frac{A}{(s+3)} + \frac{Bs+C}{(s^2+3s+5)}$

4] $\frac{s^2+3}{(s^2+9s+11)^2} = \frac{As+B}{(s^2+9s+11)} + \frac{Cs+D}{(s^2+9s+11)^2}$

Q.*1] Find $L^{-1}\left[\frac{s+29}{(s+4)(s^2+9)}\right]$

$\Rightarrow$ here, $\frac{s+29}{(s+4)(s^2+9)} = \frac{A}{(s+4)} + \frac{Bs+C}{(s^2+9)}$

$$= A(s^2+9) + (Bs+c)(s+4)$$

$$= As^2 + A9 + Bs^2 + 4Bs + Cs + 4C$$

$$s+29 = (A+B)s^2 + (4B+C)s + 9A + 4C$$

here, $A+B=0$

$4B+C+4A=1$

$4C+9A=29$

$\therefore A=+1,\ B=-1,\ C=5$

$$\therefore \frac{s+29}{(s+4)(s^2+9)} = \frac{1}{(s+4)} + \frac{-s+5}{(s^2+9)}$$

$$\therefore L^{-1}\left[\frac{s+29}{(s+4)(s^2+9)}\right] = L^{-1}\left[\frac{1}{(s+4)} - \frac{s-5}{s^2+9}\right]$$

$$= e^{-4t} - L^{-1}\left[\frac{s}{s^2+9}\right] + \frac{5}{3} L^{-1}\left[\frac{3}{s^2+9}\right]$$

$$= e^{-4t} - \cos 3t + \frac{5}{3}\sin 3t.$$

2) Find $L^{-1}\left[\frac{s+2}{s^2(s+3)}\right]$

$\Rightarrow$ here, $\dfrac{s+2}{(s+3)s^2} = \dfrac{A}{(s+3)} + \dfrac{Bs+C}{s^2}$

$= As^2 + (Bs+c)(s+3)$

$= As^2 + Bs^2 + 3Bs + Cs + 3C$

$s+2 = (A+B)s^2 + (3B+C)s + 3C$

here, $A+B=0 \Rightarrow A = -1/9$

$3B+C=1 \Rightarrow B = \frac{1}{3}(1-2/3) = 1/9$

$3C=2 \Rightarrow C = 2/3$

$$\therefore \frac{s+2}{(s+3)s^2} = \frac{-1/9}{(s+3)} + \frac{1/9\, s + 2/3}{s^2}$$

$$\therefore L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = -\frac{1}{9}L^{-1}\left[\frac{1}{s+3}\right] + \frac{1}{9}L^{-1}\left[\frac{s+6}{s^2}\right]$$

$$= -\frac{1}{9}e^{-3t} + \frac{1}{9}\left[L^{-1}\left(\frac{1}{s}\right) + 6L^{-1}\left(\frac{1}{s^2}\right)\right]$$

$$= -\frac{e^{-3t}}{9} + \frac{1}{9} + \frac{6\cdot t}{9}$$

Q.3] Find $L^{-1}\left[\frac{1}{(s+5)(s^2+2s+3)}\right]$

$\Rightarrow$ here, $\frac{1}{(s+5)(s^2+2s+3)} = \frac{A}{(s+5)} + \frac{Bs+C}{(s^2+2s+3)}$

now, $1 = A(s^2+2s+3) + (Bs+C)(s+5)$

$1 = As^2 + 2As + 3A + Bs^2 + B5s + Cs + 5C$

$1 = (A+B)s^2 + (2A+5B+C)s + 3A+5C$

here, $A+B = 0$

$2A+5B+C = 0$

$3A+5C = 1$

$\therefore A = 1/18 \quad B = -1/18 \quad C = 1/6$

$$\therefore L^{-1}\left[\frac{1}{(s+5)(s^2+2s+3)}\right] = L^{-1}\left[\frac{1}{18(s+5)} + \frac{-\frac{1}{18}s + \frac{1}{6}}{(s^2+2s+3)}\right]$$

$$= \frac{1}{18}L^{-1}\left[\frac{1}{s+5}\right] - \frac{1}{18}L^{-1}\left[\frac{s-3}{(s^2+2s+3)}\right]$$

$$= \frac{e^{-5t}}{18} - \frac{1}{18}L^{-1}\left[\frac{s+1-1-3}{(s+1)^2+2}\right]$$

$$= \frac{e^{-5t}}{18} - \frac{1}{18}L^{-1}\left[\frac{(s+1)-4}{(s+1)^2+2}\right]$$

$$= \frac{e^{-5t}}{18} - \frac{e^{-t}}{18} L^{-1}\left[\frac{s-4}{s^2+2}\right]$$

$$= \frac{e^{-5t}}{18} - \frac{e^{-t}}{18}\left[L^{-1}\left(\frac{s}{s^2+2}\right) - \frac{4}{\sqrt{2}}L^{-1}\left(\frac{\sqrt{2}}{s^2+2}\right)\right]$$

$$= \frac{e^{-5t}}{18} - \frac{e^{-t}}{18}\left[\cos\sqrt{2}t - 2\sqrt{2}\sin\sqrt{2}t\right].$$

Q.4] Find $L^{-1}\left[\frac{7s+9}{s^2+2s+9}\right]$

$\Rightarrow$ here, $\frac{7s+9}{(s^2+2s+1)+8} = \frac{7(s+1)-7+9}{(s+1)^2+8}$

$$\therefore L^{-1}\left[\frac{7(s+1)+2}{(s+1)^2+8}\right] = e^{-t}L^{-1}\left[\frac{7s+2}{s^2+8}\right]$$

$$= e^{-t}\left[7L^{-1}\left[\frac{s}{s^2+8}\right] + \frac{2}{\sqrt{8}}L^{-1}\left[\frac{\sqrt{8}}{s^2+8}\right]\right]$$

$$= e^{-t}\left[7\cos\sqrt{8}t + \frac{1}{\sqrt{2}}\sin\sqrt{8}t\right].$$

* **Convolution Theorem :-**

If $f_1(t)$ and $f_2(t)$ are two funⁿ,
then integral $\int_0^1 f_1(u) f_2(t-u)\,du$.

Notation : $f_1(t) * f_2(t)$

i.e. $\boxed{f_1(t) * f_2(t) = \int_0^t f_1(u) f_2(t-u)\,du}$

Statement :

let $L[f_1(t)] = \phi_1(s)$ & $L[f_2(t)] = \phi_2(s)$, then

$$\boxed{L^{-1}[\phi_1(s)\cdot\phi_2(s)] = \int_0^t f_1(u) f_2(t-2)\,du}$$

e.g. $L^{-1}\left[\frac{3}{(s^2+2s+1)(s^2+9)}\right]$

$\Rightarrow$ Let $\phi_1(s) = \frac{3}{s^2+9}$ and $\phi_2(s) = \frac{1}{s^2+2s+1}$

$\Rightarrow f_1(t) = L^{-1}(\phi_1(s))$ & $f_2(t) = L^{-1}\left[\frac{1}{s^2+2s+1}\right]$

$= L^{-1}\left[\frac{3}{s^2+9}\right]$ $\quad = L^{-1}\left[\frac{1}{(s+1)^2}\right]$

$= \sin 3t$ $\quad = e^{-t}.t$

By Convolution Theorem,

$$L^{-1}\left[\frac{3}{(s^2+2s+1)(s^2+9)}\right] = \int_0^t u.e^{-u} \sin 3(t-u)\, du$$

Q.1] $L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right]$

$\Rightarrow$ Let $\phi(s) = \frac{s}{(s^2+4)(s^2+9)} = \phi_1(s).\phi_2(s)$

$\therefore \phi_1(s) = \frac{s}{s^2+4}$ and $\phi_2(s) = \frac{1}{s^2+9}$

$L^{-1}[\phi_1(s)] = \cos 2t$ $\quad L^{-1}[\phi_2(s)] = \frac{1}{3}\sin 3t$

$$\therefore L^{-1}\left[\frac{s}{(s+4)(s^2+9)}\right] = \frac{1}{3}\int_0^t \cos 2u \cdot \sin 3(t-u)\, du$$

$$= \frac{1}{6}\int_0^t [(\sin[3t-u] + \sin[3t-5u])\, du]$$

$$= -\frac{1}{6}\int_0^t [\sin(u-3t) + \sin(5u-3t)]\, du$$

$$= -\frac{1}{6}\left[-\cos(u-3t) - \frac{\cos(5u-3t)}{5}\right]_0^t$$

$$= \frac{1}{6}\left[\left(\cos(-2t) + \frac{\cos(2t)}{5}\right) - \left(\cos(-3t) + \frac{\cos(-3t)}{5}\right)\right]$$

$$= \frac{1}{6}\left[\cos 2t + \frac{\cos 2t}{5} - \cos 3t - \frac{\cos 3t}{5}\right]$$

$$= \frac{1}{6}\left[\frac{6\cos 2t}{5} - \frac{6\cos 3t}{5}\right]$$

$$= \frac{1}{5}(\cos 2t - \cos 3t).$$

* **Steps for Convolution Theorem.**

To find $L^{-1}[\phi_1(s)\phi_2(s)]$

Step 1 : Find $f_1(u) = L^{-1}[\phi_1(s)]$

Step 2 : Find $f_2(u) = L^{-1}[\phi_2(s)]$

Step 3 : Find $L^{-1}[\phi_1(s)\phi_2(s)] = \int_0^t f_1(u) \cdot f_2(t-u)\,du$

Q.2] $L^{-1}\left[\frac{1}{s(s+4)^2}\right]$

$\Rightarrow$ $\phi_1(s) = \frac{1}{s}$ and $\phi_2(s) = \frac{1}{(s+4)^2}$

$L^{-1}[\phi_1(s)] = 1$ and $L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{(s+4)^2}\right] = e^{-4t} \cdot t$

$$L^{-1}\left[\frac{1}{s(s+4)^2}\right] = \int_0^t 1 \cdot (t-u)(e^{-4(t-u)}) \cdot du$$

$$= \int_0^t t \cdot e^{-4u(t-u)} - \int_0^t u \cdot e^{-4u(t-u)}$$

$$= \int_0^t t \cdot e^{4(u-t)}\, du - \int_0^t u \cdot e^{4(u-t)}$$

$$= t \cdot \frac{e^{4(u-t)}}{4} \Big|_0^t - e^{-4t} \int_0^t u \cdot e^{4u}$$

$$= t \cdot \frac{e^{4(u-t)}}{4} \Big|_0^t - e^{-4t}\left[\frac{u \cdot e^{4u}}{4} - \frac{e^{4u}}{16} \right] \Big|_0^t$$

$$= e^{-4t}\left[\frac{t \cdot e^{4u}}{4} - \frac{u \cdot e^{4u}}{4} + \frac{e^{4u}}{16} \right] \Big|_0^t$$

$$= e^{-4t}\left[\left(\cancel{\frac{t \cdot e^{4t}}{4}} - \cancel{\frac{t e^{4t}}{4}} + \frac{e^{4t}}{16} \right) - \left(\frac{t \cdot e^0}{4} - 0 + \frac{e^0}{16} \right) \right]$$

$$= e^{-4t}\left[\frac{e^{4t}}{16} - \frac{t}{4} + \frac{1}{16} \right]$$

$$= \frac{1}{16}\left[1 - e^{-4t}(4t+1) \right] \therefore$$

Q.3] Find $L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right]$ using convolution Theorem.

$\Rightarrow$ $L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$ $\qquad L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t$

$$L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right] = \int_0^t \cos 2u \cdot \cos(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos(u+t) + \cos(3u-t)]\, du$$

$$= \frac{1}{2}\left[\sin(u+t) + \frac{\sin(3u-t)}{3} \right]_0^t$$

$$= \frac{1}{2}\left[\left(\sin 2t + \frac{\sin 2t}{3}\right) - \left(\sin t + \frac{\sin(-t)}{3}\right) \right]$$

$$= \frac{1}{2}\left[\frac{4 \sin 2t}{3} - \frac{2 \sin t}{3} \right]$$

$$= \frac{1}{3}\left[2\sin 2t - \sin t\right].$$

*** Q.4] Evaluate $\int_0^\infty e^{-t}\cdot t \cdot \left(\int_0^t e^{-4u}\cos u\,du\right) dt$

$\Rightarrow$

$$L[\cos u] = \frac{s}{s^2+1}$$

$$L[e^{-4u}\cos u\,du] = \frac{s+4}{(s+4)^2+1}$$

$$L\left[\int_0^t e^{-4u}\cos u\,du\right] = \frac{1}{s}\left[\frac{s+4}{(s+4)^2+1}\right]$$

$$L\left[t\int_0^t e^{-4u}\cos u\,du\right] = -\frac{d}{ds}\left[\frac{(s+4)}{s[(s+4)^2+1]}\right]$$

$$= -\frac{d}{ds}\left[\frac{(s+4)}{s^3+8s^2+17s}\right]$$

$$= -\left[\frac{1(s^3+8s^2+17s) - (s+4)(3s^2+16s+17)}{(s^3+8s^2+17s)^2}\right]$$

$$= -\left[\frac{s^3+8s^2+17s-(3s^3+28s^2+81s+68)}{(s^3+8s^2+17s)^2}\right]$$

$$= -\left[\frac{-2s^3-20s^2-64s-68}{(s^3+8s^2+17s)^2}\right]$$

$$= \frac{2[s^3+10s^2+32s+34]}{(s^3+8s^2+17s)^2}$$

now, $\int_0^\infty e^{-t}\cdot t\int_0^t e^{-4u}\cos u\,du = \frac{2[s^3+10s^2+32s+34]}{(s^3+8s^2+17s)^2}$ at $s=1$

$$= \frac{144}{676}$$

$$= \frac{72}{338}.$$

★ Use of differentiation of $\phi(s)$:-

This method is used to find $L^{-1}[\phi(s)]$ if we know $L^{-1}[\phi'(s)]$ And also to find $L^{-1}[\phi'(s)]$ if we known $L^{-1}[\phi(s)]$.

★ If $L^{-1}[\phi'(s)] = f(t)$ then $L^{-1}[\phi(s)] = -\frac{1}{t} L^{-1}[\phi'(s)]$

i.e. If $L^{-1}[\phi(s)] = f(t)$ then $L^{-1}[\phi'(s)] = -t\, L^{-1}\phi(s)$

Also, If $L^{-1}[\phi''(s)] = f(t)$ then $L^{-1}[\phi(s)] = -\frac{1}{t^2} L^{-1}[\phi''(s)]$

i.e. $L^{-1}[\phi''(s)] = -t^2 L^{-1}[\phi(s)]$.

eg. $L^{-1}[\log(2s+3)]$

here, $\phi(s) = \log(2s+3)$

$\therefore \phi'(s) = \frac{2}{2s+3}$

now, $L^{-1}[\phi'(s)] = \frac{1}{2} L^{-1}\left[\frac{2}{s+3/2}\right]$

$= e^{-3/2 t}$

$\therefore L^{-1}[\phi(s)] = L^{-1}[\log(2s+3)] = -\frac{e^{-3/2 t}}{t}$

Q.1] Find $L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right]$

$\Rightarrow$ here, $\phi(s) = \log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)$

$$\therefore \phi'(s) = \frac{2s(\sqrt{s+b}) - (s^2+a^2)\cdot \frac{1}{2}(s+b)^{-1/2}}{(\sqrt{s+b})^2}$$

$$= \frac{\sqrt{s+b}\;[4s] - [s^2+a^2][s+b]^{-1/2}}{2(s+b)}$$

$$= \frac{4s - s^2 - a^2}{2\sqrt{s+b}}$$

$$= \frac{2s}{s^2+a^2} - \frac{1}{2(s+b)}$$

$$L^{-1}[\phi'(s)] = 2\cos at - \frac{1}{2}e^{-bt}$$

$$\Rightarrow L^{-1}[\phi(s)] = L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right] = -\frac{1}{t}\left[2\cos at - \frac{1}{2}e^{-bt}\right].$$

Q.2] Find $L^{-1}\left[\log\left(1+\frac{a^2}{s^2}\right)\right]$

$\Rightarrow$ here, $\phi(s) = \log\left(1+\frac{a^2}{s^2}\right) = \log(s^2+a^2) - \log(s^2)$

$$\therefore \phi'(s) = \frac{1}{s^2+a^2} \times 2s - \frac{2}{s}$$

$$L^{-1}[\phi'(s)] = 2\cos at - 2$$

$$L^{-1}[\phi(s)] = L^{-1}\left[\log\left(1+\frac{a^2}{s^2}\right)\right] = -\frac{1}{t}[2\cos at - 2]$$

Q.3] Find $L^{-1}[2\tanh^{-1}s]$

$\Rightarrow$

$$\tanh^{-1}s = \frac{1}{2}\log\left(\frac{1+s}{1-s}\right).$$

$$\Rightarrow 2\tanh^{-1}s = \log(1+s) - \log(1-s)$$

here, $\phi(s) = 2\tanh^{-1}s$

$$\therefore \phi'(s) = \frac{1}{s+1} - \frac{1(-1)}{1-s} = \frac{1}{s+1} - \frac{1}{s-1}$$

$$\therefore L^{-1}[\phi'(s)] = e^{-t} - e^{t}$$

$$\therefore L^{-1}[\phi(s)] = L^{-1}[2\tanh^{-1}s] = -\frac{1}{t}[e^{-t} - e^{t}].$$

H.W

1] Find $L^{-1}\left[\tan^{-1}\left(\frac{s+a}{b}\right)\right]$ Ans $= \frac{-1}{t}e^{-at}\sin bt$

2] Find $L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]$ Ans $= 2\sin t \sinh t$

Q.4] Find $L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right]$

$$\Rightarrow \frac{s+3}{(s^2+6s+13)^2} = \frac{s+3}{(s^2+6s+9+4)^2} = \frac{s+3}{((s+3)^2+4)^2}$$

$$\therefore L^{-1}\left[\frac{s+3}{((s+3)^2+4)^2}\right] = e^{-3t} L^{-1}\left[\frac{s}{(s^2+4)^2}\right] \quad \ldots \text{I}$$

$$= e^{-3t} L^{-1}\left[\frac{s}{(s^2+4)} \times \frac{1}{(s^2+4)}\right] \quad \ldots \text{II}$$

now, $\phi(s) = \frac{1}{s^2+4} \Rightarrow \phi'(s) = \frac{-2s}{(s^2+4)^2}$

But $L^{-1}[\phi'(s)] = -t\, L^{-1}[\phi(s)]$

$$L^{-1}[\phi(s)] = \frac{1}{2}\sin 2t$$

$$\Rightarrow L^{-1}[\phi'(s)] = -t\,L^{-1}[\phi(s)]$$

$$\text{i.e. } L^{-1}\left[\frac{-2s}{(s^2+4)^2}\right] = -t\cdot\frac{1}{2}\sin 2t$$

$$\Rightarrow L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \frac{t}{4}\sin 2t \quad \ldots \text{III}$$

substituting III in I, $L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right] = e^{-3t}\cdot\frac{t}{4}\sin 2t$.

H.W

1] Find $L^{-1}\left[\frac{s}{[s^2+a^2]^2}\right]$ using derivative of $\phi(s)$ method.

Q. Using convlution theorem, prove that

$$L^{-1}\left[\frac{1}{s}\log\left(\frac{s+a}{s+b}\right)\right] = \int_0^t \frac{e^{-bu}-e^{-au}}{u}\,du$$

$\Rightarrow$

$$\text{Let } \phi(s) = \frac{1}{s}\log\left(\frac{s+a}{s+b}\right) = \phi_1(s)\cdot\phi_2(s)$$

$$L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s}\right] = 1$$

$$L^{-1}[\phi_1(s)] = L^{-1}\left[\log\frac{(s+a)}{(s+b)}\right]$$

$$= L^{-1}[\log(s+a) - \log(s+b)]$$

$$\phi(s) = \log(s+a) - \log(s+b)$$

$$\phi'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L^{-1}[\phi'(s)] = e^{-at} - e^{-bt}$$

$$L^{-1}[\phi(s)] = -\frac{1}{t} L^{-1}[\phi'(s)]$$

$$L^{-1}\left[\log\frac{(s+a)}{(s+b)}\right] = -\frac{1}{t}\left[e^{-at} - e^{-bt}\right]$$

$$= \frac{1}{t}\left[e^{-bt} - e^{-at}\right]$$

By convolution Theorem,

$$L^{-1}\left[\frac{1}{s}\log\left(\frac{s+a}{s+b}\right)\right] = \int_0^t 1 \cdot \frac{(e^{-bu} - e^{-au})}{u}\,du$$