Mod 6 Probability

Discrete
1) **Deterministic Experiment :**
An experiment whose outcome can be predicted with certainty.

2) **Random Experiment :**
Although all possible outcomes of an experiment may be known in advance, the exact outcome of the experiment cannot be predicted.
eg. Tossing a coin.

Random variable :-
For every random experiment a Fun X assigns one & only one real number $x \ (= X(s))$ to every element s of sample space S. Such funⁿ X is called as a random variable.
The random variable X can be discrete or continuous depending upon the nature of its domain.

a) **Discrete Random variable :-**
Let X be a random variable. If X takes finite or countably infinite values $x_1, x_2, x_3, \ldots x_n, \ldots$ then X is called a discrete random variable.

* **Probability Distribution of discrete random variable :**
Let X be a discrete random variable. Let $x_1, x_2, x_3, \ldots x_n, \ldots$ be the possible values of X. $p(x_i)$ is the probability of x_i.
$p(x_i), [i = 1, 2, 3, \ldots]$ must satisfy the following condition
1) $p(x_i) \geq 0$ for all i
2) $\Sigma p(x_i) = 1$

The funⁿ p is called the probability funⁿ or probability mass funⁿ or probability distribution funⁿ of the random variable X & the set of points (x_i, p_i) is called the probability distribution of X (here $p_i = p(x_i)$).

b] continuous Random Variable :-

Let X be a random variable. If X takes uncountably infinite values in a given interval then X is called a continuous random variable

eg. temperature of place, heights of students in a class.

* Probability density funⁿ of a continuous random variable :

A continuous funⁿ $y = f(x)$ such that -

i] $f(x)$ is integrable

ii] $f(x) \geq 0$

iii] $\int_a^b f(x)dx = 1$, if X lies in $[a,b]$

iv] $\int_\alpha^\beta f(x)dx = P(\alpha \leq X \leq \beta)$ where $a < \alpha < \beta < b$

is called probability density funⁿ of a continuous random variable X

Note :-

For a discrete random variable the probability at $X = c$ may not be zero but in a continuous random variable $P(X=c)$ is always zero because,

$$P(X=c) = \int_c^c f(x)dx = 0$$

* Properties of probability density funⁿ of continuous random variable:

1) $f(x) \geq 0, \quad -\infty < x < \infty$

2) $\int_{-\infty}^{\infty} f(x) = 1$

3) Probability that $\alpha \leq X \leq \beta$ is given by $P(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} f(x)\,dx$

Q.1) A discrete random variable A has following prop. distri. funⁿ.

x	-2	-1	0	1	2	3
$P(X=x)$	0.1	k	0.2	$2k$	0.3	$3k$

Fink
1) k
2) $P(X \geq 2)$
3) $P(-2 < X < 2)$
4) $P(X < 2)$

$\Rightarrow$ 1) $\sum p_i = 1 \Rightarrow 0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$

$6k = 0.4$

$k = 0.0667$

2) $P(X \geq 2) = P(X=2) + P(X=3)$

$= 0.3 + 3(0.0667)$

$= 0.5$

3) $P(-2 < X < 2) = P(X=-1) + P(X=0) + P(X=1)$

$= 0.0667 + 0.2 + 2(0.0667)$

$= 0.4$

4) $P(X < 2) = P(X=-2) + P(X=-1) + P(X=0) + P(X=1)$

$= 0.1 + 0.0667 + 0.2 + 0.0667(2)$

$= 0.5$

Date: YOUVA

Q.2] A continuous random variable has prob. dens. fun^n as below.

$$F(x) = \begin{cases} kx & , 0 \le x \le 1 \\ k & , 1 \le x \le 2 \\ 3k - kx & , 2 \le x \le 3 \\ 0 & , \text{else where} \end{cases}$$

1] Find k 2] Find $P(1 \le X \le 2.5)$ 3] $P(X \le 2)$

$\Rightarrow$

1]

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_0^3 f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 kx\,dx + \int_1^2 k\,dx + \int_2^3 (3k - kx)\,dx = 1$$

$$\Rightarrow \left. \frac{kx^2}{2} \right|_0^1 + \left. kx \right|_1^2 + \left. 3kx - \frac{kx^2}{2} \right|_2^3 = 1$$

$$\Rightarrow \frac{k}{2}(1-0) + k(2-1) + 3k(3-2) - \frac{k}{2}(9-4) = 1$$

$$\Rightarrow \frac{k}{2} + k + 3k - \frac{5k}{2} = 1$$

$$\Rightarrow 4k - 2k = 1$$

$$\Rightarrow 2k = 1$$

$$\Rightarrow k = 0.5$$

2] $P(1 \le X \le 2.5) = \int_1^{2.5} f(x)\,dx$

$$= \int_1^2 k\,dx + \int_2^{2.5} (3k - kx)\,dx$$

$$= \left. kx \right|_1^2 + \left. \left(3kx - \frac{kx^2}{2}\right) \right|_2^{2.5}$$

$$= 0.5(2-1) + 3\left(\frac{1}{2}\right)[2.5-2] - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)[6.25-4]$$

$$= \frac{1}{2} + \frac{3}{4} - \frac{2.25}{4}$$

$$= \frac{2.75}{4}$$

$$= 0.6875$$

3] $$P(X \leq 2) = \int_0^2 f(x)dx$$

$$= \int_0^1 kx\,dx + \int_1^2 k\,dx$$

$$= \frac{kx^2}{2}\Big|_0^1 + kx\Big|_1^2$$

$$= \frac{1}{4}[1-0] + \frac{1}{2}[2-1]$$

$$= \frac{1}{4} + \frac{1}{2}$$

$$= \frac{3}{4}$$

$$= 0.75$$

Mathematical expectation or Expected value :-

Let X be discrete random variable with $p_i = P(x_i)$ as probability distribution then the mathematical expectation or expected value is denoted by $E(X)$ and is given by -

$$\boxed{E(X) = \Sigma x_i p_i}$$, if it exists.

In case of continuous random variable x with probability density fun^n $f(x)$, the mathematical expectation is given as -

$$= 0.5(2-1) + 3\left(\frac{1}{2}\right)[2.5-2] - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)[6.25-4]$$

$$= \frac{1}{2} + \frac{3}{4} - \frac{2.25}{4}$$

$$= \frac{2.75}{4}$$

$$= 0.6875$$

3] $P(X \leq 2) = \int_0^2 f(x)\,dx$

$$= \int_0^1 kx\,dx + \int_1^2 k\,dx$$

$$= \left.\frac{kx^2}{2}\right|_0^1 + \left.kx\right|_1^2$$

$$= \frac{1}{4}[1-0] + \frac{1}{2}[2-1]$$

$$= \frac{1}{4} + \frac{1}{2}$$

$$= \frac{3}{4}$$

$$= 0.75$$

Mathematical expectation or Expected value :-

Let X be discrete random variable with $p_i = P(x_i)$ as probability distribution then the mathematical expectation or expected value is denoted by $E(X)$ and is given by -

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In case of continuous random variable x with probability density fun^n $f(x)$, the mathematical expectation is given as -

$$\boxed{E(X) = \int_{-\infty}^{\infty} x f(x)\,dx}$$, if it exists.

* Properties of expectation :
1) $E(X) = \Sigma x_i p_i$ is also called as mean (or average)
   i.e. $E(X) = \Sigma x_i p_i = \bar{X} = \mu$.

2) $E(k) = k$, where k is a constant.

3) $E(aX+b) = aE(X) + b$

4) $E(aX+bY) = aE(X) + bE(Y)$, where X & Y are random variab

5) In general $E(x^n) = \begin{cases} \Sigma x_i^n p_i & \text{, if } x \text{ is discrete random var} \\ \int_{-\infty}^{\infty} x^n f(x)\,dx & \text{, if } x \text{ is continuous ran var} \end{cases}$

6) $E(x^2) = \Sigma x_i^2 p_i = \int_{-\infty}^{\infty} x^2 f(x)\,dx$ is more frequently used.

here x is dis. ran. var.   here x is contin. ran. var.

7) Expectations are also considered as moments about origin.

8) If X & Y are two independent random variables then,
$$E(XY) = E(X) \cdot E(Y)$$

* Variance :-

The mathematical expectation is also treated as mean of the distribution.

There is one more parameter related to distribution, the variability

The variability of distribution reflects the fluctuations of random variable around some constants (mostly about mean or origin)

The most common expression of variability is variance (which is also called as dispersion) and it is denoted by σ_x^2

$$\boxed{var(X) = \sigma_x^2 = E(X^2) - [E(X)]^2}$$

Standard deviation of variable X is defined as positive square root of variance of X. And it is denoted by σ_x.

$$\boxed{\begin{aligned}\text{standard deviation of } X &= \sigma_x \\ &= \sqrt{var(x)} \\ &= \sqrt{E(X)^2 - [E(X)]^2}\end{aligned}}$$

* Properties of variance :-

1) Let K be any constant then $var(K) = 0$ i.e. the variance of constant is zero.

2) Let X be a random variable then we have $\boxed{var(aX+b) = a^2 var(X)}$ where 'a' & 'b' are constants.

3) Similarly, $var(aX+bY) = a^2 var(X) + b^2 var(Y)$, where X & Y are independent random variables & a, b are constants.

4) If X & Y are independent random variables, then $\boxed{var(X \pm Y) = var(X) + var(Y)}$ since, $var(-Y) = var(Y)$

Q. A discrete random variable has the probability density fn:

$x = x_i$	-2	-1	0	1	2	3
$f(x_i)$	0.2	k	0.1	$2k$	0.1	$2k$

Find k, mean, variance.

$\Rightarrow$

$\sum f_i = 1$

$\Rightarrow 0.2 + k + 0.1 + 2k + 0.1 + 2k = 1$

$\therefore 0.4 + 5k = 1$

$\therefore 5k = 0.6$

$\therefore k = 0.12$

Mean $= E(X) = \sum x_i f_i$ ... (i)

$Var(X) = E(X^2) - [E(X)]^2$

$= \sum x_i^2 f_i - [\sum x_i f_i]^2$ ... (ii)

x_i	$f(x_i)$	$x_i f_i$	x_i^2	$x_i^2 f_i$
-2	0.2	-0.4	4	0.8
-1	0.12	-0.12	1	0.12
0	0.1	0.0	0	0.0
1	0.24	0.24	1	0.24
2	0.1	0.2	4	0.4
3	0.24	0.72	9	2.16
		$\sum x_i f_i = 0.64$		$\sum x_i^2 f_i = 3.72$

$\therefore E(X) = \sum x_i f_i = 0.64$

$\therefore Var(X) = \sum x_i^2 f_i - [\sum x_i f_i]^2$

$= 3.72 - (0.64)^2$

$= 3.3104$

Q.

X	-2	-1	0	1	2	3
P(X)	0.1	3k	0.2	2k	0.3	5k

Find standard deviation.

⇒ $\Sigma p_i = 1$

$\Rightarrow 0.1 + 3k + 0.2 + 2k + 0.3 + 5k = 1$

$\therefore 0.6 + 10k = 1$

$\therefore 10k = 0.4$

$\therefore k = 0.04$

$\sigma_x = \sqrt{var(X)}$

$= \sqrt{E(X^2) - [E(X)]^2}$

$= \sqrt{\Sigma x_i^2 p_i - [\Sigma x_i p_i]^2}$

x_i	$P(x_i)$	$x_i p_i$	$x_i^2 p_i$
-2	0.1	-0.2	0.4
-1	0.12	-0.12	0.12
0	0.2	0	0
1	0.08	0.08	0.08
2	0.3	0.6	1.2
3	0.2	0.6	1.8
		$\Sigma x_i p_i = 0.96$	$\Sigma x_i^2 p_i = 3.6$

$\therefore \sigma_x = \sqrt{3.6 - (0.96)^2}$

$= 1.6365$

Q. 6 men & 5 women apply for a position in a company. 2 of the applicants are selected for an interview. How many women do you aspect in the interview?

⇒ Let X be the no. of women selected for interview.

X	0	1	2
$P(x_i)$	$3/11$	$6/11$	$2/11$

$$P(0) = \frac{{}^5C_0\,{}^6C_2}{{}^{11}C_2} = \frac{1 \times \frac{6\times5}{2}}{\frac{11\times10}{2}} = \frac{3}{11}$$

$$P(1) = \frac{{}^5C_1\,{}^6C_1}{{}^{11}C_2} = \frac{5\times6}{\frac{11\times10}{2}} = \frac{6}{11}$$

$$P(2) = \frac{{}^5C_2\,{}^6C_0}{{}^{11}C_2} = \frac{\frac{5\times4}{2}\times1}{\frac{11\times10}{2}} = \frac{2}{11}$$

$$E(X) = \Sigma x_i p_i$$

$$= 0\times\frac{3}{11} + 1\times\frac{6}{11} + 2\times\frac{2}{11}$$

$$= \frac{10}{11}$$

$$= 0.9090$$

* **Probability :-**

Let E be a random experiment and S be the sample space and A be an event associated with E.

Let $n(S)$ denote the no. of elements in S and $n(A)$ denotes the no. of elements in A. Then probability of occurance of event A is –

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{No. of favourable events}}{\text{Total no. of events.}}$$

Note :- 1) $P(A) \geq 0$

2) $P(S) = 1$

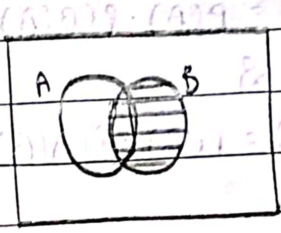
2) $P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n)$
where $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive events.

Properties of Probability :-

1) If $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive exhaustive events,
then, $P(A_1) + P(A_2) + \ldots + P(A_n) = 1$
$(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S)$

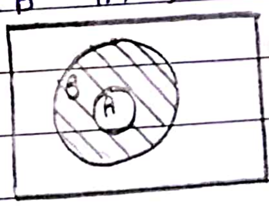
2) Let $A_1, A_2, A_3, \ldots, A_n$ be mutually exclusive exhaustive events and let B be any event then,
$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B)$.

3) For any two events A & B, $P(B) = P(A \cap B) + P(B \cap \bar{A})$



4) $P(\bar{A}) = 1 - P(A)$.

5) Let A & B be any two events & $A \subset B$ then,
$P(B) = P(A) + P(B \cap \bar{A})$



6) If $A \subseteq B$, then $P(A) \leq P(B)$.

7) Let A & B be any two events, then probability that atleast one of the events A & B will occur is
$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

8) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A)$
$+ P(A \cap B \cap C)$.

Note :- If A, B, C are mutually exclusive events then,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C).$$

* **Conditional Probability :-**

Let A & B be any two events then the conditional probability of an event B assuming that A has already occurred is denoted by $P(B|A)$.

$$\boxed{P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0}$$

* **Multiplication Theorem :-**

If A & B are two given events then probability that both of them will be occur is given by -

$$\boxed{\begin{array}{c} P(A \cap B) = P(A) \cdot P(B|A) \\ \& \\ P(A \cap B) = P(B) \cdot P(A|B) \end{array}}$$

Note : 1) If $A \subset B$ then, $P(B|A) = 1$

$A \cap B = A \Rightarrow P(A \cap B) = P(A)$

Proof : By definition, $P(B|A) = \frac{P(A \cap B)}{P(A)}$

$= \frac{P(A)}{P(A)} \quad \because A \subset B$

$= 1.$

2) If $A \subset B$, then $P(A) \leq P(A/B)$

3) If A & B are mutually exclusive events, then $P(B|A) = 0$.

$P(A) > P(B) \Rightarrow P(A/B) > P(B/A)$

If two events A & B are independent then,

$P(A \cap B) = P(A) \cdot P(B)$.

Q.1) A bag contains 4 yellow, 5 green and 6 orange marbles. If 2 marbles are drawn at random, Find the probability that both marbles are green.

$\Rightarrow$

$n(S) = 4 \text{ yellow} + 5 \text{ green} + 6 \text{ orange}$

$= 15$ marbles.

Let A be event of drawing first marble as green marble.

$\therefore P(A) = \frac{5}{15} = \frac{1}{3}$

Let B be event of drawing second marble also as green.

$\therefore P(B|A) = \frac{P(A \cap B)}{P(A)}$

$\therefore P(B|A) = \frac{4}{14} = \frac{2}{7}$

$\therefore$ The probability that both marbles are green is,

$P(A \cap B) = P(A) \cdot P(B|A)$

$= \frac{1}{3} \times \frac{2}{7}$

$= \frac{2}{21}$.

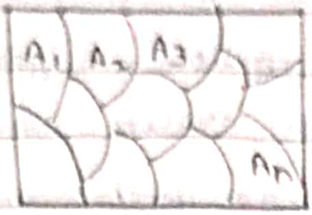
$= 0.09524$

- **Partition of a sample space :-**

If $A_1 \cap A_2 \cap A_3 \ldots \ldots \cap A_n = \phi$ &

$A_1 \cup A_2 \cup A_3 \ldots \ldots \cup A_n = S$

then, the set $\{A_1, A_2, \ldots\ldots, A_n\}$ is called partition of sample space.



- **Total Probability Theorem :-**

Let $A_1, A_2, \ldots\ldots, A_n$ be mutually exclusive and exhaustive events and B is another event associated with S then,

$$P(B) = P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + \ldots\ldots + P(A_n)\cdot P(B|A_n)$$

i.e. $$\boxed{P(B) = \sum_{i=1}^{n} P(A_i)\cdot P(B|A_i)}$$

- **Baye's Theorem :-**

Let $A_1, A_2, \ldots\ldots, A_n$ be mutually exclusive & exhaustive events and B is another event associated with A_i then,

$$\boxed{P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)\cdot P(B|A_i)}}$$

Q. 1 Two cards are drawn at random from a well shuffled pack of 52 cards. What is probability that second card is ace.

$\Rightarrow$

Let A_1 be event that first card drawn is ace.

Let A_2 be event that first card drawn is not ace.

$P(A_1) = \frac{4}{52} = \frac{1}{13}$ $\qquad$ $P(A_2) = \frac{48}{52} = \frac{12}{13}$

Let B be event that second card drawn is ace.

$P(B|A_1) = \frac{3}{51}$ & $P(B|A_2) = \frac{4}{51}$.

By Total Probability Theorem,

$$P(B) = \sum_{i=1}^{2} P(A_i) P(B|A_i)$$

$$= P(A_1) P(B|A_1) + P(A_2) P(B|A_2)$$

$$= \frac{4}{52} \times \frac{3}{51} + \frac{12}{13} \times \frac{4}{51}$$

$$= 0.07692.$$

Q.7 A certain test for a particular cancer is known to be 95% accurate. A person submits to test & result is positive. Suppose that person comes from population of 1,00,000 where 2000 people suffer from the disease. What is probability that person under test has that particular cancer?

→

Let A_1 be the event that the person has cancer.

A_2 be event that the person doesn't have cancer.

$$P(A_1) = \frac{2000}{100000} = 0.02$$

$$P(A_2) = \frac{98000}{100000} = 0.98$$

Let B be the event that test is positive ~~when a person has cancer~~

$P(B|A_1) = 0.95$

$P(B/A_2) = 0.05$

By Bayes Theorem.

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)}$$

$$= \frac{0.02 \times 0.95}{0.02 \times 0.95 + 0.98 \times 0.05}$$

$$= 0.279$$

Q.3) A football teams wins 60% its game when they have their first goal & 10% of its game when the opposite team scores the first goal. If the team scores the first goal about 30% of times what is the probability the team wins the game?

⇒

Let A_1 be the event first team scores the first goal.

$$P(A_1) = \frac{30}{100} = 0.3$$

Let A_2 be the event second team score the first goal.

$$P(A_2) = 1 - 0.3 = 0.7$$

Let B be the event that team wins.

$$P(B|A_1) = \frac{60}{100} = 0.6$$

$$P(B|A_1) = \frac{10}{100} = 0.1$$

By Total Probability Theorem,

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2)$$

$$= 0.3 \times 0.6 + 0.7 \times 0.1$$

$$= 0.18 + 0.07$$

$$= 0.25$$

Q 4) Among 4 coins there is one false coin with head on both sides. A coin is choosen at random & tossed four times. If head appears all four times, what is the probability that the false coin has been choosen?

⇒ Let A_1 be the event that true coin is chosen

$P(A_1) = \frac{3}{4}$

Let A_2 be the event that false coin is choosen

$P(A_2) = \frac{1}{4}$.

Let B be the event all 4 times head occurs

$P(B|A_1) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

$P(B|A_2) = 1$

By Baye's Theorem,

$$P(A_2|B) = \frac{P(A_2) \cdot P(B|A_2)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2)}$$

$$= \frac{1/4 \times 1}{3/4 \times 1/16 + 1/4 + 1}$$

$$= \frac{16}{19}$$

$$= 0.8421$$

* Moments :-
→ Moment generating funⁿ
→ Raw moments upto 4^{th} order
→

To understand the symmetry of the probability distribution we have to calculate the skewness of distribution and if we want to understand the flatness of the distribution then we have to calculate the Kurtosis of the distribution.

These parameters skewness & kurtosis both depends on the moments of the distribution.

These are two types of moments :

1] Moments about any arbitrary point.

2] Moments about central tendency (mean, median, mode)

1] Raw moments :-

The moments about any arbitrary points are called Raw moments.

Notation : μ', Rth raw moment is denoted by μ_r'

* Moments about arbitrary point 'a' :-

Let X be a random variable then the r^{th} raw moment about an arbitary point 'a' is given as -

$$\mu_r' = E[X-a]^r = \sum p_i(x_i-a)^r$$

... for discrete random variable X

$$\mu_r' = E[X-a]^r = \int_{-\infty}^{\infty} (x-a)^r f(x)\,dx$$

... for continuous random variable X.