Mod I] Laplace Transformation

Laplace Transformation is an operator. It is used for solving differential equation with boundary values without finding particular integral.

$$\boxed{L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt}$$

★ **Generalised Rule for Integration by parts.**

$$\boxed{\int uv = (u)(v_1) - (u')(v_2) + (u'')(v_3) - (u''')(v_4) + \dots}$$

where u' : Derivative of u,
v_i : Integration of v_{i-1}
$v_2 : \int v_1$ and $v_3 : \int v_2$

eg. $\int_0^{\pi/2} \underbrace{\sin 3t}_{v} \underbrace{(t^3+2t+5)}_{u}\, dt$

$\Rightarrow$

$$I = \left[(t^3+2t+5)\left(\frac{-\cos 3t}{3}\right) - (3t^2+2)\left(\frac{-\sin 3t}{3^2}\right) + (6t)\left(\frac{\cos 3t}{3^3}\right) - 6\left(\frac{\sin 3t}{3^4}\right)\right]_0^{\pi/2}$$

Q.1] Find $L(t^2)$

$\Rightarrow$ By definition $L(t^2) = \int_0^\infty e^{-st}(t^2)\, dt$

$$L(t^2) = \left[(t^2)\left(\frac{-e^{-st}}{s}\right) - \int_0^\infty 2t \cdot \frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \left[-\frac{t^2 e^{-st}}{s} - (2t)\left(\frac{e^{-st}}{s^2}\right) - \int_0^\infty \frac{2 \cdot e^{-st}}{s^2}\right]_0^\infty$$

$e^{-\infty} = 0$

$e^{0} = 1$

$$= \left[-\frac{t^2 e^{-st}}{s} - (2t)\left(\frac{e^{-st}}{s^2}\right) + 2\left(\frac{e^{-st}}{-s^3}\right) \right]_0^{\infty}$$

$$= [0] - \left[\frac{-2}{s^3}\right]$$

$$= \frac{2}{s^3}$$

Q.2] Find Laplace of f(t) where $f(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ t, & 1 < t < 2 \\ 0, & t \geq 2 \end{cases}$

$\Rightarrow$ By definition,

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$L[f(t)] = \int_0^1 e^{-st}(0)\,dt + \int_1^2 e^{-st}(t)\,dt + \int_2^{\infty} e^{-st}(0)\,dt$$

$$= \int_1^2 \underset{v}{e^{-st}} \cdot \underset{u}{t}\, dt$$

$$= \left. t\left(\frac{e^{-st}}{-s}\right) - 1 \cdot \frac{e^{-st}}{(-s^2)} \right|_1^2$$

$$= \left[-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} \right] - \left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \right]$$

H.W

Q.1] Find $L[f(t)]$ for $f(t) = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & t > 2\pi \end{cases}$

Hint : $\int e^{ax} \cos bx\, dx = \frac{1}{a^2+b^2} e^{ax} (a\cos bx + b\sin bx)$

$\int e^{ax} \sin bx\, dx = \frac{1}{a^2+b^2} e^{ax} (a\sin bx - b\cos bx)$

Q.2] Find $L[f(t)]$ for $f(t) = \begin{cases} 0, & 0 < t < \pi \\ \sin^2(t-\pi), & t > \pi \end{cases}$

Hint : $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

* ## Linerity Property :-

If k_1 and k_2 are constants ,

$$\boxed{L[k_1 f_1(t) + k_2 f_2(t)] = k_1 L[f_1(t)] + k_2 L[f_2(t)]}$$

eg. $L[3t^2 + 5\sin 4t] = 3L[t^2] + 5L[\sin 4t]$

* ## Laplace Transform of standard functions :-

1] $\boxed{L[e^{at}] = \frac{1}{s-a}}$

Proof :-

By defination : $L[f(t)] = \int_0^\infty e^{-st} \cdot e^{at} \, dt$

$$L(e^{at}) = \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= 0 - \left[\frac{+1}{-(s-a)} \right]$$

$$= \frac{1}{s-a}$$

2] $\boxed{L[e^{-at}] = \frac{1}{s+a}}$

eg. $L[e^{5it}] = \frac{1}{s-5i}$

Note :- $\Gamma 0 = \infty$
$\Gamma(n+1) = n\Gamma n$

Note :- In above result, it is applicable for all values of a.
Like if $a=0$, then $\boxed{L[1] = \frac{1}{s}}$

3] $L[\sin at] = \frac{a}{s^2+a^2}$

4] $L[\cos at] = \frac{s}{s^2+a^2}$

5] $L[\sinh at] = \frac{a}{s^2-a^2}$

6] $L[\cosh at] = \frac{s}{s^2-a^2}$

7] $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$, [where $(n+1) > 0$]

eg. 1] $L(t^3) = \frac{\Gamma 4}{s^4} = \frac{3!}{s^4} = \frac{6}{s^4}$ $[\Gamma(n+1) = n!]$

2] $L(t^{3/2}) = \frac{\Gamma 5/2}{s^{5/2}} = \frac{3/2 \cdot 1/2 \cdot \sqrt{\pi}}{s^{5/2}} = \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}}$

3] $L[\cos 8t] = \frac{s}{s^2+64}$

4] $L[3t^2 + 7\sin 3t + 1] = 3L(t^2) + 7L(\sin 3t) + L(1)$

$= 3\frac{\Gamma 3}{s^3} + 7\left(\frac{3}{s^2+3^2}\right) + \frac{1}{s}$

$= \frac{6}{s^3} + \frac{21}{s^2+9} + \frac{1}{s}$.

→ **Steps to find Laplace Transform of given function.**

Step 1 : Simplify the given function and get it as the sum of standard funⁿ where laplace is known to us.

* gamma for negative funⁿ $\Rightarrow$ $\Gamma n = \frac{\Gamma n+1}{n}$

eg. $\Gamma -5/2 = \frac{\Gamma -3/2}{-5/2} = \frac{\Gamma -1/2}{(-5/2)(-3/2)} = \frac{\Gamma 1/2}{(-5/2)(-3/2)(-1/2)} = \frac{\sqrt{\pi}}{-15/8}$

Simplification can be done either by using standard rules and relations or by using properties of Laplace Tran.

Step 2 : Apply Laplace Transformation on both side of the equation.

Step 3 : Use Linearity property & write the values of Laplace Transform of each term on the R.H.S.

Q.1] Find $L(\sqrt{1+\sin t})$

$\Rightarrow$

$$\sqrt{1+\sin t} = \sqrt{(\cos t/2 + \sin t/2)^2} = \sin t/2 + \cos t/2$$

$$\therefore L(\sqrt{1+\sin t}) = L(\cos t/2) + L(\sin t/2)$$

$$= \left[\frac{s}{s^2+1/4}\right] + \frac{1/2}{s^2+1/4}$$

$$= \frac{2(2s+1)}{4s^2+1}$$

H.W Find Laplace Transform of :-

i] $\sin^4 t$ ii] $\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3$

* Properties of Laplace Transform.

We are going to study the following properties.

1] First shifting property.
2] Change of scale property.
3] Effect of multiplication by t
4] Effect of division by t
5] Laplace Transform of derivative

1] First Shifting Property :-

$$\boxed{\begin{array}{l} \text{If } L[f(t)] = \phi(s) \text{ then } L[e^{-at} f(t)] = \phi(s+a) \\ \text{or} \\ \text{If } L[f(t)] = \phi(s) \text{ then } L[e^{at} f(t)] = \phi(s-a) \end{array}}$$

eg. 1] $L[\sin 3t] = \dfrac{3}{s^2+9}$

$\therefore L[e^{-2t} \cdot \sin 3t] = \dfrac{3}{(s+2)^2+9} = \dfrac{3}{s^2+4s+4+9} = \dfrac{3}{s^2+4s+13}$

2] $L[e^{5t} \cdot t^3] = ?$

we know $L[t^3] = \dfrac{\overline{|4}}{s^4} = \dfrac{3!}{s^4} = \dfrac{6}{s^4}$

$\therefore L[e^{5t} \cdot t^3] = \dfrac{6}{(s-5)^4}$.

Q.1] Find $L[e^{-7t} \cdot t^{3/2}]$.

$\Rightarrow$

$L(t^{3/2}) = \dfrac{\overline{|5/2}}{s^{5/2}} = \dfrac{3/2 \cdot 1/2 \cdot \sqrt{\pi}}{s^{5/2}} = \dfrac{3}{4} \dfrac{\sqrt{\pi}}{s^{5/2}}$

By 1st shifting property,

$L[e^{-7t} \cdot t^{3/2}] = \dfrac{3}{4} \dfrac{\sqrt{\pi}}{(s+7)^{5/2}}$.

Q.2] Find $L[e^{2t} \cdot \sin^4 t]$

$\Rightarrow$

$\sin^4 t = (\sin^2 t)^2$

$= \left(\dfrac{1-\cos 2t}{2}\right)^2$

$$= \frac{1 + (\cos 2t)^2 - 2\cos 2t}{4}$$

$$= \frac{1}{4}\left[1 - 2\cos 2t + \cos^2 2t\right]$$

$$= \frac{1}{4}\left[1 - 2\cos 2t + \left[\frac{1+\cos 4t}{2}\right]\right]$$

$$\therefore \quad L(\sin^4 t) = \frac{1}{4}L(1) - \frac{1}{4}\times 2L(\cos 2t) + \frac{1}{8}L(1) + \frac{1}{8}L(\cos 4t)$$

$$= \frac{3}{8s} - \frac{1}{2}\times\frac{s}{s^2+4} + \frac{1}{8}\times\frac{s}{s^2+16}$$

By 1st shifting property,

$$L(e^{2t}\sin^4 t) = \frac{3}{8(s-2)} - \frac{(s-2)}{2[(s-2)^2+4]} + \frac{(s-2)}{8[(s-2)^2+16]}.$$

Q. 3] Find $L[e^{-2t}\cosh t \sin t]$.

$\Rightarrow$

$$e^{-2t}\cdot\cosh t\sin t = e^{-2t}\left[\frac{e^{t}+e^{-t}}{2}\right]\sin t$$

$$f(t) = \left[\frac{e^{-t}}{2} + \frac{e^{-3t}}{2}\right]\sin t$$

$$= \frac{1}{2}\left[e^{-t}\sin t + e^{-3t}\sin t\right]$$

$$\therefore L[e^{-2t}\cosh t\sin t] = \frac{1}{2}\left[L(e^{-t}\sin t) + L(e^{-3t}\sin t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s+1)^2+1} + \frac{1}{(s+3)^2+1}\right].$$

* Evaluation of Integral form (of basic formulas).

Q.1]

$$I = \int_0^\infty e^{-4t} (3t^3)\,dt$$

$\Rightarrow$ here, $f(t) = 3t^3$

$$L[3t^3] = 3\frac{\overline{|4}}{s^4} = \frac{18}{s^4}.$$

i.e. $\int_0^\infty e^{-4t} \cdot 3t^3\,dt = \frac{18}{s^4}$

As per the que. $\Rightarrow$ $s = 4$

$\therefore$ Put $s = 4$

$$\therefore \int_0^\infty e^{-4t} \cdot 3t^3\,dt = \frac{18}{(4)^4}.$$

Q.2] Evaluation, $\int_0^\infty e^{4t} [3t^3 + 5\sin 2t + 8]\,dt$

$\Rightarrow$ here, $f(t) = 3t^3 + 5\sin 2t + 8$

By Linearity Property, $L[f(t)] = 3L(t^3) + 5L(\sin 2t) + 8L(1)$

$$= 3\frac{\overline{|4}}{s^4} + 5\left(\frac{2}{s^2+4}\right) + 8\left(\frac{1}{s}\right)$$

Now, $\int_0^\infty e^{-5t} (3t^3 + 5\sin 2t + 8)\,dt = \frac{18}{s^4} + \frac{10}{s^2+4} + \frac{8}{s}$

on comparing with the question, $s = -4$

$$\therefore \int_0^\infty e^{4t} (3t^3 + 5\sin 2t + 8)\,dt = \frac{18}{(-4)^2} + \frac{10}{(-4)^2+4} + \frac{8}{(-4)}$$

$$= -\frac{183}{128}.$$

2] Change of Scale Property :-

$$\boxed{\text{If } L[f(t)] = \phi(s) \text{ then } L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)}$$

eg. $L[\sin t] = \dfrac{1}{s^2+1}$

now, $L[\sin 5t] = \dfrac{1}{5}\left[\dfrac{1}{(s/5)^2+1}\right] = \dfrac{1}{5}\left[\dfrac{25}{s^2+25}\right] = \dfrac{5}{s^2+5^2}$.

Q.1] If $L[f(t)] = \dfrac{1}{s\sqrt{s+1}}$ then find $L[f(9t)]$.

⇒ As $L[f(t)] = \dfrac{1}{s\sqrt{s+1}}$

then by change of scale property,

$$L[f(9t)] = \frac{1}{9}\left[\frac{1}{s/9\sqrt{s/9+1}}\right] = \frac{3}{s\sqrt{s+9}}$$

*** Q.2] If $L[\sin\sqrt{t}] = \dfrac{\pi}{2s}\dfrac{e^{-(1/4s)}}{\sqrt{s}}$, then find $L[\sin 2\sqrt{t}]$.

⇒ By change of scale property,

$$L[\sin 2\sqrt{t}] = L[\sin\sqrt{4t}] = \frac{1}{4}\left[\frac{\pi}{2\left(\frac{s}{4}\right)\sqrt{s/4}} \times e^{-1/(4\times s/4)}\right]$$

$$= \frac{1}{4}\left[\frac{4\pi}{s\sqrt{s}}e^{-1/s}\right]$$

$$= \frac{\pi}{s\sqrt{s}}e^{-1/s}.$$

3] **Effect of Multiplication By t :-**

$$\boxed{\text{If } L[f(t)] = \phi(s) \Rightarrow L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\phi(s)}$$

eg. $L[t^2 \cdot e^{5t}] = ?$

here, $f(t) = e^{5t} \Rightarrow L[f(t)] = \frac{1}{s-5}$

$$\therefore L[t^2 \cdot e^{5t}] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-5}\right)$$

$$= \frac{2}{(s-5)^3}.$$

Q.1 $L[te^{-5t}\sin 3t] = ?$

$\Rightarrow$ $L[\sin 3t] = \frac{3}{s^2+9}$.

By effect of multiplication by t,

$$L[t \sin 3t] = 3\left[(-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+9}\right)\right]$$

$$= 3\left[-1 \times \frac{-1}{(s^2+9)^2}(2s)\right]$$

$$= \frac{6s}{(s^2+9)^2}$$

By 1st shifting property,

$$L[te^{-5t}\sin 3t] = \frac{6(s+5)}{[(s+5)^2+9]^2}.$$

Q.2) Find $L[(1+te^{-t})^3]$

$\Rightarrow$

here, $f(t) = (1+te^{-t})^3$

$$= 1 + t^3 e^{-3t} + 3t^2 e^{-2t} + 3te^{-t}$$

$\therefore L[1 + t^{3}e^{-3t} + 3t^2 e^{-2t} + 3te^{-t}]$

$=$ By linearity property,

$L[1] + L[t^3 e^{-3t}] + 3L[t^2 e^{-2t}] + 3L[te^{-t}]$ ... (I)

$\therefore L[1] = \frac{1}{s}$ ... (a)

$\therefore L[t^3 e^{-3t}] \Rightarrow L[t^3] = \frac{\Gamma 4}{s^4}$

By 1st shifting property, $L[e^{-3t} \cdot t^3] = \frac{\Gamma 4}{(s+3)^4}$ ... (b)

$\therefore L[t^2 e^{-2t}] \Rightarrow L[t^2] = \frac{\Gamma 3}{s^3}$

By 1st shifting property, $L[e^{-2t} \cdot t^2] = \frac{\Gamma 3}{(s+2)^3}$ ... (c)

$\therefore L[t \cdot e^{-t}] \Rightarrow L[t] = \frac{\Gamma 2}{s^2}$

By 1st shifting property, $L[e^{-t} \cdot t] = \frac{\Gamma 2}{(6+1)^2}$ ... (d)

Putting a, b, c, d in (I),

$$= \frac{1}{s} + \frac{\Gamma 4}{(s+3)^4} + \frac{3\Gamma 3}{(s+2)^3} + \frac{3\Gamma 2}{(s+1)^2}$$

$$= \frac{1}{s} + \frac{6}{(s+3)^4} + \frac{6}{(s+2)^3} + \frac{3}{(s+1)^2}$$

H.W

Q.1] Find $L[t \cdot e^{-t} \cdot \cosh 2t]$

Q.2] Find $L[e^{-3t} \cdot \cosh 5t \cdot \sin 4t]$

Q.3] Find $L[\sin 2t \cdot \cos t \cdot \cosh 2t]$

Q.4] Find $L[e^{2t}(1+t)^2]$

Q.5] Show that, $L\left[\sinh(t/2)\sin\left(\frac{\sqrt{3}}{2}t\right)\right] = \frac{\sqrt{3}}{2}\frac{s}{(s^4+s^2+1)}$

*** Q.1] Evaluate. $\int_0^\infty e^{-3t}(\cosh 5t \cdot \sin 4t)\,dt$

$\Rightarrow$ Met 1 : $f(t) = \cosh 5t \cdot \sin 4t$

$= \frac{\sin 4t \cdot e^{5t}}{2} + \frac{\sin 4t\, e^{-5t}}{2}$

now, $L[\sin 4t] = \frac{4}{s^2+16}$

$L\left[\frac{e^{5t}\sin 4t}{2}\right] = \frac{1}{2}\left[\frac{4}{(s-5)^2+16}\right]$

$L\left[\frac{e^{-5t}\sin 4t}{2}\right] = \frac{1}{2}\left[\frac{4}{(s+5)^2+16}\right]$

$\therefore \int_0^\infty e^{-3t}\cosh 5t \sin 4t\,dt = \frac{1}{2}\left[\frac{4}{(s-5)^2+16} + \frac{4}{(s+5)^2+16}\right]$

so, $s = 3$

$= \frac{1}{2}\left[\frac{4}{(3-5)^2+16} + \frac{4}{(3+5)^2+16}\right]$

$= \frac{1}{2}\left[\frac{1}{4}\right]$

$= \frac{1}{8}.$

Met 2 : $\int_0^\infty e^{-3t}\left(\frac{e^{5t}+e^{-5t}}{2}\right)\sin 4t\,dt$

$$= \frac{1}{2}\int_0^\infty e^{2t}\sin 4t\,dt + \frac{1}{2}\int_0^\infty e^{-8t}\sin 4t\,dt$$

here, $s=-2$ here, $s=8$

$$= \frac{1}{2}\left[\frac{4}{s^2+16}\right] + \frac{1}{2}\left[\frac{4}{s^2+16}\right]$$

$$= \frac{1}{2}\left[\frac{4}{(-2)^2+16}\right] + \frac{1}{2}\left[\frac{4}{(8)^2+16}\right]$$

$$= \frac{1}{8}$$

Q.2] If $L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$, then evaluate $\int_0^\infty e^{-2t}\operatorname{erf}(2\sqrt{t})\,dt$

$\Rightarrow$

$$L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}},$$

By scale changing property,

$$L[\operatorname{erf}\sqrt{4t}] = \frac{1}{4}\left[\frac{1}{s/4\sqrt{s/4+1}}\right] = \frac{2}{s\sqrt{s+4}}$$

By standard formula, $s=2$

$$\therefore \int_0^\infty e^{-2t}\operatorname{erf}(2\sqrt{t})\,dt = \frac{2}{2\sqrt{2+4}} = \frac{1}{\sqrt{6}}.$$

Q.3] Find $L[t^2\cos 5t]$

$\Rightarrow$

$$f(t) = \cos 5t \quad \therefore \quad L[f(t)] = \frac{5}{s^2+25}.$$

By effect of multiplication by t.

$$L[t^2\cos 5t] = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2+25}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+25)(1) - s(2s)}{(s^2+25)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-s^2+25}{(s^2+25)^2}\right]$$

$$= \frac{(s^2+25)^2[-2s] - (-s^2+25)\cdot 2(s^2+25)\cdot(2s)}{(s^2+25)^4}$$

$$= \frac{-2s[s^4+2s^2+625] + 4s(s^4-625)}{(s^2+25)^4}$$

$$= \frac{2s^3-150s}{(s^2+25)^3}$$

Q.4] Find $L[t\sqrt{1+\sin 2t}]$

$\Rightarrow$

$$\sqrt{1+\sin 2t} = \sqrt{(\cos t+\sin t)^2} = \cos t+\sin t$$

$$L[t\cos t] + L[t\sin t] = (-1)^1\frac{d}{ds}\left(\frac{s}{s^2+1}\right) + (-1)^1\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= -1\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2} + \frac{-1}{(s^2+1)^2}(2s)\right]$$

$$= -1\left[\frac{s^2+1-2s^2-2s}{(s^2+1)^2}\right]$$

$$= \frac{s^2+2s-1}{(s^2+1)^2}.$$

P.T.O.

Q.5] Find $L[t \cdot e^{3t} \cdot \sinh 2t]$

$\Rightarrow$

Met.1 : $L\left[t \cdot e^{3t} \cdot \left(\frac{e^{2t} - e^{-2t}}{2}\right)\right] = L\left[t\left(\frac{e^{5t} - e^{t}}{2}\right)\right]$.

$$= \frac{1}{2} L[t \cdot e^{5t}] - \frac{1}{2} L[t \cdot e^{t}]$$

$$L[e^{5t} - e^{t}] = \frac{1}{s-5} - \frac{1}{s-1}$$

By effect of multiplication by t,

$$= \frac{1}{2} (-1)^1 \frac{d}{ds}\left(\frac{1}{s-5}\right) - \frac{1}{2} (-1)^1 \frac{d}{ds}\left(\frac{1}{s-1}\right)$$

$$= -\frac{1}{2}\left[\frac{-1}{(s-5)^2}\right] + \frac{1}{2}\left[\frac{-1}{(s-1)^2}\right]$$

$$= \frac{1}{2(s-5)^2} - \frac{1}{2(s-1)^2}.$$

Met.2 : $L[\sinh 2t] = \frac{2}{s^2-4}$

$$L[t \sinh 2t] = -\frac{d}{ds}\left(\frac{2}{s^2-4}\right) = \frac{4s}{(s^2-4)^2}$$

$$L[e^{3t} \cdot t \cdot \sinh 2t] = \frac{4(s-3)}{[(s-3)^2-4]^2}$$

[on simplifying, both answers would come same]

Q.6] Using Laplace Transform.
Evaluate. $\int_0^{\infty} e^{-2t} \cdot t^3 \sin t \, dt$

⇒ here, $f(t) = t^3 \sin t$

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L[t^3 \sin t] = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)$$

$$= -1 \frac{d^2}{ds^2}\left(\frac{-1 \times 2s}{(s^2+1)^2}\right)$$

$$= 2 \frac{d}{ds}\left[\frac{(s^2+1)^2(1) - (2(s^2+1)\cdot 2s)(s)}{(s^2+1)^4}\right]$$

$$= 2 \frac{d}{ds}\left[\frac{s^2+1-4s^2}{(s^2+1)^3}\right]$$

$$\cancel{= 2\left[\frac{(s^2+1)^3[2s-8] - (s^2-8s+1)\cdot 3(s^2+1)^2 s}{(s^2+1)^6}\right]} \quad -14s \quad -(14s)$$

$$= 2 \frac{d}{ds}\left[\frac{1-3s^2}{(s^2+1)^3}\right]$$

$$= 2\left[\frac{-6s(s^2+1)^3 - (1-3s^2)\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6}\right]$$

$$= 2\left[\frac{-6s(s^2+1)^2 - 6s(1-3s^2)}{(s^2+1)^5}\right]$$

$$= \frac{-12s}{(s^2+1)^5}\left[s^4+2s^2+1 \mp 3s^2+1\right]$$

$$= \frac{-12s}{s^2+1}\left[\cancel{5s^4+2s}\right](s^4 \mp 5s^2 + 2)$$

$$= -12s$$

here, $f(t) = t^3 \sin t$

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L[t^3 \sin t] = (-1)^3 \frac{d^3}{ds^3}\left[\frac{1}{s^2+1}\right]$$

$$= -1 \frac{d^2}{ds^2}\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$= 2\left[\frac{d}{ds}\left[\frac{(s^2+1)^2(1) - (2(s^2+1)\cdot 2s)s}{(s^2+1)^4}\right]\right]$$

$$= 2 \frac{d}{ds}\left[\frac{s^2+1-4s^2}{(s^2+1)^3}\right]$$

$$= 2 \frac{d}{ds}\left[\frac{1-3s^2}{(s^2+1)^3}\right]$$

$$= 2\left[\frac{-6s(s^2+1)^3 - (1-3s^2)(3(s^2+1)^2\cdot 2s)}{(s^2+1)^6}\right]$$

$$= 2\left[\frac{-6s(s^2+1) - 6s(1-3s^2)}{(s^2+1)^4}\right]$$

$$= \frac{-12s}{(s^2+1)^4}\left[s^2+1+1-3s^2\right]$$

$$= \frac{-12s}{(s^2+1)^4}\left[2-2s^2\right]$$

$$= \frac{24s(s^2-1)}{(s^2+1)^4}.$$

here, $s = 2$

$$\therefore = \frac{24 \times 2(4-1)}{(2^2+1)^4}$$

$$= \frac{144}{625}$$

$$= \underline{0.23}$$

H.W Q.1] Find $L[t \cdot \sinh 2t]$

Q.2] Find $L[t \cdot e^{-2t} \cdot \cosh 4t]$

Q.3] Evaluate $\int_0^\infty t \cdot e^{-2t} \cdot \cosh 4t \, dt$

Q.4] Evaluate $\int_0^\infty \frac{t^2 \sin 3t}{e^{2t}} \, dt$

4] Effect of division by t :-

$$\boxed{\text{If } L[f(t)] = \phi(s) \text{ then } L\left[\frac{1}{t} f(t)\right] = \int_s^\infty \phi(s) \, ds}$$

Q.1] Evaluate. $L\left[\frac{\sin 3t + \cosh t}{t}\right]$

$\Rightarrow$ $L[\sin 3t + \cosh t] = \frac{3}{s^2+9} + \frac{s}{s^2-1}$

$$L\left[\frac{1}{t}(\sin 3t + \cosh t)\right] = \int_s^\infty \left(\frac{3}{s^2+9} + \frac{s}{s^2-1}\right) ds$$

$$= \tan^{-1}\left(\frac{s}{3}\right) + \frac{1}{2} \log |s^2-1| \Big|_s^\infty$$

$$= \left(\frac{\pi}{2} + \infty\right) - \tan^{-1}\left(\frac{s}{3}\right) - \frac{1}{2} \log(s^2-1).$$

Note :- There is a form $L\left[\frac{f(t)}{t^2}\right]$.

It can be written as $L\left[\frac{f(t)/t}{t}\right]$.

Q.2] $L\left[\frac{\sin 3t}{t^2}\right] = ?$

$\Rightarrow$ $L[\sin 3t] = \frac{3}{s^2+9}$

$$L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty \frac{3}{s^2+9}\,ds = \frac{\pi}{2} - \tan^{-1}\frac{s}{3} = \cot^{-1}\frac{s}{3}$$

$$L\left[\frac{\sin 3t/t}{t}\right] = L\left[\frac{\sin 3t}{t^2}\right] = \int_s^\infty \cot^{-1}\frac{s}{3}$$

$$= \cancel{\cot^{-1}}$$

Q.3] Find $L\left[\frac{1}{t}\cdot e^{-t}\sin t\right]$.

$\Rightarrow$ $f(t) = \frac{\sin t}{t}$

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1} = \tan^{-1} s\Big|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

By 1^{st} shifting inverse $\Rightarrow$ $L\left[\frac{1}{t}\cdot e^{-t}\cdot\sin t\right] \Rightarrow s \Rightarrow s+1$

$\therefore$ $\cot^{-1}(s+1)$

Q.4] Find $L\left[\frac{1}{t}\sin t\cdot\sin 5t\right]$.

$\Rightarrow$ $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\therefore$ $\sin t\sin 5t = \frac{1}{2}[\underset{\downarrow\ \cos 4t}{\cos(-4t)} - \cos(6t)]$

$$= \frac{1}{2} L[\cos 4t - \cos 6t]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+16} - \frac{s}{s^2+36}\right]$$

$$\therefore L\left[\frac{1}{6}\sin t \sin 5t\right] = \frac{1}{2}\int_s^\infty \left(\frac{s}{s^2+16} - \frac{s}{s^2+36}\right) ds$$

$$= \frac{1}{2}\left[\frac{1}{2}\log(s^2+16) - \frac{1}{2}\log(s^2+36)\right|_s^\infty$$

$$= \frac{1}{4}\log\left|\frac{s^2+16}{s^2+36}\right|\Bigg|_s^\infty$$

$$= \frac{1}{4}\log\left(\frac{1+16/s^2}{1+36/s^2}\right)\Bigg|_s^\infty$$

$$= \frac{1}{4}\log\left(\frac{s^2+36}{s^2+16}\right)$$

Q.5] Find $L\left[\frac{1-\cos t}{t^2}\right]$

$\Rightarrow$ $L[1-\cos t] = \frac{1}{s} - \frac{s}{s^2+1}$

This implies, $L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[2\log s - \log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s^2 - \log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\log\left(\frac{s^2}{s^2+1}\right)\Bigg|_s^\infty$$

$$= \frac{1}{2}\log\left(\frac{1}{1+1/s^2}\right)\Bigg|_s^\infty$$

$$= -\frac{1}{2}\log\left(\frac{s^2}{s^2+1}\right)$$

now, $L\left[\frac{1-\cos t}{t^2}\right] = \frac{1}{2}\int_s^\infty \log\left(\frac{s^2+1}{s^2}\right)$

$$= \frac{1}{2}\int_s^\infty \left[\log(s^2+1) - \log(s^2)\right] ds$$

$$= \frac{1}{2}\left[s\log(1+s^2) - \int \frac{1}{s^2+1}\cdot 2s\cdot s\right]_0^\infty - \frac{2}{2}\left[s\log s - s\right]_0^\infty$$

$$= \frac{1}{2}\left[s\log(1+s^2) - \int\left(2 - \frac{2}{1+s^2}\right)ds\right] - (s\log s - s)_0^\infty$$

$$= \frac{1}{2}\left[s\log(1+s^2) - \cancel{2s} + 2\tan^{-1}s - 2s\log s + \cancel{2s}\right]_0^\infty$$

$$= \frac{1}{2}\left[s\log(1+s^2) - 2s\log s + 2\tan^{-1}s\right]_0^\infty$$

H.W

Q.1 Find $L\left[\frac{\sinh at}{t}\right]$

Q.2 Find $L\left[\frac{1}{t}e^{2t}\sin^3 t\right]$

Q.1 Prove that $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\,dt = \frac{1}{4}\log 5$

$\Rightarrow$

$\sin^2 t = \frac{1-\cos 2t}{2} \Rightarrow L[\sin^2 t] = \frac{1}{2}\left[L[1] - L[\cos 2t]\right]$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_s^{\infty}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \frac{1}{4}\left[\log\left(\frac{s^2+4}{s^2}\right)\right]$$

here, $s = 1$

$$\therefore \int_0^{\infty} \frac{e^{-t}\sin^2 t}{t}\, dt = \frac{1}{4}\log 5$$

H.W

Q.1] Evaluate $\int_0^{\infty} \frac{\cos 6t - \cos 4t}{t \cdot e^t}\, dt$

Q.2] Evaluate $\int_0^{\infty} \frac{\sin 2t + \sin 3t}{t \cdot e^t}\, dt$ $\left\{\text{Ans} = \frac{3\pi}{4}\right\}$

Q.3] Evaluate $\int_0^{\infty} e^{-t} \frac{\sin\sqrt{3}t}{t}\, dt$ $\left\{\text{Ans} = \frac{\pi}{3}\right\}$

5] Laplace Transform of derivative :-

Let $f(t)$ be any given funⁿ, then

$$L[f'(t)] = -f(0) + sL[f(t)]$$

$$L[f''(t)] = -f'(0) - sf(0) + s^2 L[f(t)]$$

$$L[f'''(t)] = -f''(0) - sf'(0) - s^2 f(0) + s^3 L[f(t)]$$

In general equ.

$$L[f^n(t)] = -f^{n-1}(0) - sf^{n-2}(0) - s^2 f^{n-3}(0) - \ldots - s^{n-1}f(0) + s^n L[f(t)]$$

6] Laplace Transform of Integral :-

$$\boxed{\text{If } L[f(t)] = \phi(s), \text{ then } L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}\phi(s)}$$

eg. $L\left[\int_0^t \cos 3u\,du\right] = ?$

$$L[\cos 3u] = \frac{s}{s^2+9}$$

$$L\left[\int_0^t \cos 3u\,du\right] = \frac{1}{s^2+9}$$

Corollary :- The generalised form of above result is

$$\boxed{L\left[\int_0^t\int_0^t\int_0^t \cdots \int_0^t f(u)\,(du)^n\right] = \frac{1}{s^n} L[f(t)].}$$

Q.1] $L\left[\int_0^t\int_0^t\int_0^t\int_0^t \cos 3u\,(du)^4\right] = ?$

$\Rightarrow$ $$L(\cos 3u) = \frac{s}{s^2+9}$$

$$\therefore L\left[\int_0^t\int_0^t\int_0^t\int_0^t \cos 3u\,(du)^4\right] = \frac{1}{s^4}\left[\frac{s}{s^2+9}\right] = \frac{1}{s^3(s^2+9)}$$

Q.2] If $f(t) = \begin{cases} t, & 0 \le t \le 3 \\ 6, & t > 3 \end{cases}$

then find $L[f(t)]$ and $L[f'(t)]$.

$\Rightarrow$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[f(t)] = \int_0^3 e^{-st}\, t\, dt + \int_3^\infty e^{-st} \cdot 6\, dt$$

$$= \left[t\left(\frac{e^{-st}}{-s}\right) - 1\cdot\left(\frac{e^{-st}}{s^2}\right)\right]_0^3 + 6\left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= \frac{-3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{6e^{-3s}}{s}$$

$$= \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)$$

Now, $L[f'(t)] = -f(0) + sL[f(t)]$

$$= -0 + s\left[\frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)\right]$$

$$= \frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right) \quad (\because f(t) = t,\ 0 \leq t \leq 3)$$

Q.3] Find $L\left[\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right]$.

$\Rightarrow$ $L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\, ds = \tan^{-1} s \Big|_s^\infty = \cot^{-1} s$

$f(0) = \frac{\sin 0}{0} =$ undefined

$$f(0) = \lim_{t\to 0} \frac{\sin t}{t} = \lim_{t\to 0} \frac{\cos t}{1} = 1$$

$$\therefore L\left[\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right] = -1 + s(\cot^{-1} s).$$

H.W Q.1] Find $L\left[\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)\right]$

Q.2] Find $L[f'(t)]$, if $f(t) = \begin{cases} 3, & 0 \leq t < 5 \\ 0, & t > 5 \end{cases}$

(fun^n value at point of discountinuity)

Q. 1] Find $L\left[\int_0^t u e^{-3u} \cdot \sin 4u \, du\right]$

$\Rightarrow$ here, $f(u) = u \cdot e^{-3u} \cdot \sin 4u$

$$L[\sin 4u] = \frac{4}{s^2+16}$$

$$L[u \sin u] = (-1)^1 \frac{d}{ds}\left[\frac{4}{s^2+16}\right] = \frac{8s}{(s^2+16)^2}$$

$$L[f(u)] = \frac{8(s+3)}{[(s+3)^2+16]^2}$$

$$L\left[\int_0^t f(u) du\right] = \frac{1}{s} \cdot \frac{8(s+3)}{[(s+3)^2+16]^2}$$

H.W If $\int_0^\infty e^{-2t} \sin(t+\alpha)\cos(t-\alpha)\,dt = \frac{3}{8}$,

then find α

Hint : 1] Evaluate $L[\sin(t+\alpha) \cdot \cos(t-\alpha)]$

2] Definition

3] Put $s=2$ i.e. evaluate integral on LHS

4] Ans in terms of α ($\alpha = \pi/4$)