Mod. 3] Fourier Series & Fourier Transform

* Trigometric Series :

$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ where all a's and b's are constants is called as Trigonmetric series.

The exapansion of any periodic function $f(x)$ (satisfying Dirichlet's conditions) in the form of the above series is called as Fourier series.

* Dirichlet's conditions :-

A funⁿ $f(x)$ defined in the interval $c_1 \le x \le c_2$ can be expressed as Fourier series if in the interval -

i] $f(x)$ and its integrals are finite & single valued.

ii] $f(x)$ has discontinuities finite in number.

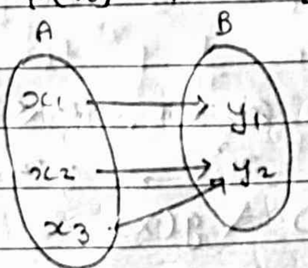
iii] $f(x)$ has finite number of maxima & minima.

These conditions are known as Dirichlet's conditions.

* Single-valued function :-

eg. $f(x) = x^2$

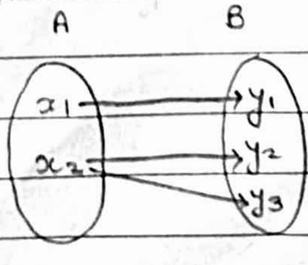
$f(x) : A \longrightarrow B$



* Multiple-valued function :-

eg. $f(x) = \sqrt{x}$

$f(x) : A \longrightarrow B$



1] i] **Even fun^n :-** If $f(-x) = f(x)$

$\Rightarrow f(x)$ is an even fun^n.

eg. $f(x) = \cos x$, x^2, $|x|$

ii] **Odd fun^n :-** If $f(-x) = -f(x)$

$\Rightarrow f(x)$ is an odd fun^n.

eg. $f(x) = \sin x$, x^3

iii] **Neither even nor odd fun^n :-**

eg. $\log x$, e^x, 10^x

eg. $f(x) = x+1$ & $f(-x) = -x+1$
$= -(x-1)$

Q. Check whether fun^n is even or odd.

$$g(x) = \begin{cases} \cos x, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$$

$\Rightarrow$

For $g(-x) = \begin{cases} \cos(-x), & -2 \le -x \le 0 \\ (-x)^2, & 0 < -x \le 2 \end{cases}$

$= \begin{cases} \cos x, & -2 \le -x \le 0 \\ x^2, & 0 < -x \le 2 \end{cases}$

$= \begin{cases} \cos x, & 0 \le x \le 2 \\ x^2, & -2 \le x \le 0 \end{cases}$

here, $g(-x) \neq g(x)$

$\therefore$ This is neither even nor odd.

$g(x) = f_1(x) \cdot f_2(x)$

$f_1(x)$	$f_2(x)$	$g(x)$
E	O	O
O	E	O
E	E	E
O	O	E

imp 2] If n is an integer, then i] $\sin n\pi = 0$

ii] $\cos n\pi = (-1)^n$

iii] $\cos 2n\pi = 1$

iv] $\sin 2n\pi = 0$

$(\cos n\pi \cos\pi \mp \sin n\pi \sin\pi)$ v] $\cos(n \pm 1)\pi = -\cos n\pi$

$(\sin n\pi \cos\pi \pm \cos n\pi \sin\pi)$ vi] $\sin(n \pm 1)\pi = 0$

vii] $\sin(2n\pi + x) = \sin x$

viii] $\cos(2n\pi + x) = \cos x$

3] $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even} \end{cases}$$

4] $$\int_c^{c+2\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0, \quad (n \neq 0)$$

$$\int_c^{c+2\pi} \sin nx\,dx = \left[\frac{-\cos nx}{n}\right]_c^{c+2\pi} = 0, \quad (n \neq 0)$$

5] a] $$\int_{-\pi}^{\pi} \cos nx\,dx = 2\int_0^{\pi} \cos nx\,dx$$

b] $$\int_{-\pi}^{\pi} \sin nx\,dx = 0$$

c] $$\int_{-\pi}^{\pi} \underset{(O)}{\sin mx}\,\underset{(E)}{\cos nx}\,dx = 0, \quad \forall\, m, n \ (m \neq n)$$

d] $$\int_{-\pi}^{\pi} \underset{(E)}{\cos mx}\,\underset{(E)}{\cos nx}\,dx = 2\int_0^{\pi} \cos mx \cos nx\,dx$$

e] $$\int_{-\pi}^{\pi} \underset{(O)}{\sin mx}\,\underset{(O)}{\sin nx}\,dx = 2\int_0^{\pi} \sin mx \sin nx\,dx$$

6] $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

* **Fourier series in $(0, 2\pi)$ & $(-\pi, \pi)$:-**

Fourier series in $(c, c+2\pi)$ is -

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

here, a_0, a_n, b_n are called fourier coefficients of $f(x)$

$$a_0 = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Case 1 : If $c = 0$, the interval becomes $(0, 2\pi)$ and F.S of $f(x)$ is given by $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

where, $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

Q. Expand $f(x) = x^2$ as F.S in $(0, 2\pi)$.

$\Rightarrow$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ ... (I)

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{8\pi^3}{3} - 0\right] = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\, dx = \frac{1}{\pi}\left[x^2 \cdot \frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left[\frac{-\sin nx}{n^3}\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(0 - 4\pi\left[\frac{-1}{n^2}\right] + 0\right) - (0 - 0 + 0)\right]$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\, dx = \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left[\frac{\cos nx}{n^3}\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left((2\pi)^2\left(\frac{-1}{n}\right) - 0 + 2\left(\frac{1}{n^3}\right)\right) - \left(0 - 0 + 2\left(\frac{1}{n^3}\right)\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \cancel{\frac{2}{n^3}} - \cancel{\frac{2}{n^3}}\right]$$

$$= -\frac{4\pi}{n}$$

Putting in (I)

$$x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx + \left(\frac{-4\pi}{n}\sin nx\right)\right)$$

$$x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n}.$$

Q. Expand $f(x) = e^{-x}$ in F.S in $(0, 2\pi)$

$\Rightarrow$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{2\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} = \frac{-1}{2\pi}\left[e^{-2\pi} - e^{0}\right]$$

$$= \frac{1}{2\pi}\left[1 - e^{-2\pi}\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx = \frac{1}{\pi}\left[e^{-x}\left(\cancel{\frac{\sin nx}{n}}\right) - e^{-x}(-1)\left(\cancel{\frac{-\cos nx}{n^2}}\right) + \right.$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left[-1\cos nx + n\sin nx\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{1+n^2}(-1+0) - \frac{1}{1+n^2}(-1+0)\right]$$

$$= \frac{1}{\pi}\left[\frac{1-e^{-2\pi}}{1+n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left[-1\sin nx - n\cos nx\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{1+n^2}(0-n) - \frac{1}{1+n^2}(0-n)\right]$$

$$= \frac{n}{\pi}\left[\frac{1-e^{-2\pi}}{1+n^2}\right].$$

$\therefore$ F.S $\Rightarrow$

$$e^{-x} = \frac{1}{2\pi}\left[1-e^{-2\pi}\right] + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\left(\frac{1-e^{-2\pi}}{1+n^2}\right)\cos nx + \frac{n}{\pi}\left(\frac{1-e^{-2\pi}}{1+n^2}\right)\sin nx\right].$$

H.W

Q.1] Expand $f(x) = x \cdot \sin x$ in F.S in $(0, 2\pi)$.

Hint: $a_n = \frac{2}{n^2-1}$, if $n \neq 1$, $b_n = 0$, if $n \neq 1$

Calculate a_1 & b_1 by putting $n=1$.

Q.2] Obtain b_5 and a_0 for $f(x) = \begin{cases} \sin x, & 0 \leq x \leq \pi \\ 0, & \pi \leq x \leq 2\pi \end{cases}$

Hint: $a_0 = \frac{1}{\pi}$, $b_5 = 0$

Q.3] Find F.S for $f(x) = \frac{3x^2 - 6x\pi + 2\pi^2}{12}$ in $(0, 2\pi)$

Q.4] Find a_3, b_n and a_0 for $f(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ 2\pi - x, & \pi \leq x \leq 2\pi \end{cases}$

Q.5] Obtain F.S for $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$

Also derive the value of b_5, a_3.

Q.3] $\Rightarrow$ $a_0 = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{3x^2 - 6x\pi + 2\pi^2}{12} \right) dx$

$= \frac{1}{24\pi} \left[\frac{3x^3}{3} - \frac{6x^2\pi}{2} + 2\pi^2 x \right]_0^{2\pi}$

$= \frac{1}{24\pi} \left[(8\pi^3 - 3(4\pi^3) + 4\pi^3) - (0 + 0 + 0) \right]$

$= 0$

P.T.O

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(3x^2 - 6x\pi + 2\pi^2)}{12} \cos nx \, dx$$

$$= \frac{1}{12\pi}\left[(3x^2 - 6x\pi + 2\pi^2)\left(\frac{\sin nx}{n}\right) - (6x - 6\pi)\left(\frac{-\cos nx}{n^2}\right) + (6)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[\left(\frac{0 + 6x - 6\pi + 0}{n^2}\right) - \left(\frac{6x - 6\pi}{n^2}\right)\right]$$

$$= \frac{1}{12\pi}\left[\frac{12\cancel{\pi} - \cancel{6}\pi - 0 + \cancel{6}\pi}{n^2}\right]$$

$$= \frac{1}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(3x^2 - 6x\pi + 2\pi^2)}{12} \sin nx \, dx$$

$$= \frac{1}{12\pi}\left[(3x^2 - 6x\pi + 2\pi^2)\left(-\frac{\cos nx}{n}\right) - (6x - 6\pi)\left(\frac{-\sin nx}{n^2}\right) + (6)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(3(4\pi^2) - 6\pi(2\pi) + 2\pi^2)\left[\frac{-1}{n}\right] + 6\left[\frac{1}{n^3}\right] - 2\pi^2\left(\frac{-1}{n}\right) - 6\left(\frac{1}{n^3}\right)\right]$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi^2}{n} + \frac{6}{n^3} + \frac{2\pi^2}{n} - \frac{6}{n^3}\right]$$

$$= 0$$

$$\therefore f(x) = 0 + \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos(nx) + 0 \sin(nx)\right).$$

* Parseval's Identity :-

If $f(x)$ converges uniformly in $(c, c+2l)$ then,

$$\boxed{\frac{1}{2l}\int_{c}^{c+2l}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)}$$

This is known as Parseval's Identity for $f(x)$ in $(c, c+2l)$

case i : If $l=\pi$, the interval is $(c, c+2\pi)$ and
$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

For $(0, 2\pi)$ $\quad \frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$

Q. 1] Find the Fourier Series for $f(x) = \begin{cases} x, & 0<x<\pi \\ 2\pi - x, & \pi<x<2\pi \end{cases}$

in $f(x) \rightarrow (0, 2\pi)$. Hence deduce that, $\frac{\pi^2}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$

$\Rightarrow$

let $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ ... (I)

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_0 = \frac{1}{2\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}\left[\pi^2 - 0\right] + \frac{1}{2}\left[(4\pi(2\pi) - 4\pi^2) - \left[(4\pi^2) - \pi^2\right]\right]\right]$$

$$= \frac{1}{4\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - 3\pi^2\right] = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cdot\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} +$$

$$\frac{1}{\pi}\left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] + \left[\frac{-1}{n^2} + \frac{(-1)^n}{n^2}\right]\right\}$$

$$= \frac{-2}{n^2\pi}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0 & \text{, if } n \text{ is even} \\ \frac{-4}{n^2\pi} & \text{, if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cdot\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} +$$

$$\frac{1}{\pi}\left[(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[\left(-\frac{\pi(-1)^n}{n}\right) - (0)\right] + \left[-\frac{\pi}{n}(-1)^n\right]\right\}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n}\right]$$

$$= 0$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{-2}{\pi n^2}(1-(-1)^n)\cos nx + 0\right)$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(1-(-1)^n)\cos nx}{n^2}\right].$$

now, $f(x) = \frac{\pi}{2} - \frac{2}{\pi}\left[\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} + \frac{2}{5^2}\cos 5x + \dots\right]$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{3^2\pi}\cos 3x - \frac{4}{5^2\pi}\cos 5x$$

To prove that, $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\dots$

we need to use Parsenal's Identity,

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2) \dots \text{(II)}$$

now, $\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = \frac{1}{2\pi}\left[\int_0^{\pi} x^2 dx + \int_{\pi}^{2\pi}(2\pi-x)^2 dx\right]$

$$= \frac{1}{2\pi}\left[\left[\frac{x^3}{3}\right]_0^{\pi} + \left.\frac{(2\pi-x)^3}{3(-1)}\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{\pi^3}{3} - 0\right) + \left(\frac{-1}{3}\right)\left[(0)^3 - (\pi^3)\right]\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= \frac{\pi^2}{3}.$$

now, $\frac{\pi^2}{3} = \left(\frac{\pi}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\left[\frac{-2}{\pi n^2}(1-(-1)^n)\right]^2 + 0^2\right)$

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{1}{2}\left(\frac{4}{\pi^2}\right)\sum_{n=1}^{\infty}\frac{(1-(-1)^n)^2}{n^4}.$$

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{2}{\pi^2}\left[\frac{2^2}{1^4} + \frac{2^2}{3^4} + \frac{2^2}{5^4} + \frac{2^2}{7^4} + \dots\right]$$

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{2}{\pi^2}\left[\frac{4}{1^4} + \frac{4}{3^4} + \frac{4}{5^4} + \dots\right]$$

$$\frac{\pi^2}{12} = \frac{8}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right]$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$

Hence Proved.

Q.3] $f(x) = \frac{3x^2 - 6\pi x + 2\pi^2}{12}$, for $x \in (0, 2\pi)$, Deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(from p.g.no. 50 continue)

$\Rightarrow$ Fourier Series : $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$

i.e. $\frac{3x^2 - 6\pi x + 2\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$

For deduction here, substitution can be done.
Put $x = 0$ in above series,

$$\therefore \frac{3(0)^2 - 6\pi(0) + 2\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n(0)$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Hence Proved.

Q.3] Obtain F.S for $f(x) = \frac{1}{2}(\pi - x)$ in $(0, 2\pi)$ with period 2π & hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

$\Rightarrow$ here, $f(x) = \frac{1}{2}(\pi - x)$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\,dx$$

$$= \frac{1}{4\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(2\pi^2 - \frac{4\pi^2}{2}\right) - (0 - 0)\right]$$

$$= \frac{1}{4\pi}\left[0\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\cos nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{(+1)}{n^2} - \left(-\frac{(+1)}{n^2}\right)\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{\pi(1)}{n} - 0\right) - \left(-\frac{\pi(1)}{n} - 0\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n}\right]$$

$$= \frac{1}{n}$$

$$\therefore F.S = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx.$$

$$\frac{1}{2}(\pi - x) = \frac{1}{1}\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots$$

For deduction, put $x = \frac{\pi}{2}$ in above equ.

$$\Rightarrow \frac{1}{2}\left(\frac{\pi}{2}\right) = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7}$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Q.4] Obtain FS of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $(0, 2\pi)$.

Also, deduce that, $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty$

$\Rightarrow$

$$f(x) = \frac{1}{4}(\pi - x)^2$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 dx$$

$$= \frac{1}{8\pi} \left. \frac{(\pi - x)^3}{3(-1)} \right|_0^{2\pi}$$

$$= \frac{-1}{24\pi}\left[(-\pi)^3 - (\pi)^3\right]$$

$$= \frac{-1}{24\pi} \times -2\pi^3$$

$$= \frac{\pi^2}{12}.$$

$$\cancel{a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \cos nx \, dx}$$

$$\cancel{= \frac{1}{4\pi}\left[\left(2(\pi - x)^2(-1)\right)\left[\frac{\sin nx}{n}\right] - (+2)^{(\pi - x)(-1)}\left[\frac{-\cos nx}{n^2}\right]\right]_0^{2\pi}}$$

$$\cancel{= \frac{1}{4\pi}\left[\frac{-1}{n^2} - \left(\frac{+2}{n^2}\right)\right]}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi-x)^2 \cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left[\frac{\sin nx}{n}\right] - (2(\pi-x)(-1))\left[\frac{-\cos nx}{n^2}\right] + (-2)\left[\frac{-\sin nx}{n^3}\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[+2(-\pi)\left(\frac{-1}{n^2}\right) - \left[2(\pi)\left(\frac{-1}{n^2}\right)\right]\right]$$

$$= \frac{1}{4\pi}\left[\frac{4\pi}{n^2}\right]$$

$$= \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi-x)^2 \sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left[\frac{-\cos nx}{n}\right] - (2(\pi-x)(-1))\left[\frac{-\sin nx}{n^2}\right] + (-2)\left[\frac{\cos nx}{n^3}\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left[(-\pi)^2\left(\frac{-1}{n}\right) + 0 + (-2)\left(\frac{1}{n^3}\right)\right] - \left[\pi^2\left(\frac{-1}{n}\right) + 0 + (-2)\left(\frac{1}{n^3}\right)\right]\right]$$

$$= \frac{1}{4\pi}[0]$$

$$= 0$$

$\therefore$ F.S $\Rightarrow$ $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$

$$\therefore \frac{1}{4}(\pi-x)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx \quad \ldots \text{(I)}$$

For deduction, put $x=0$ in equ. (I)

$$\Rightarrow \frac{1}{4}\pi^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}(1)$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{\pi^2}{12} + \left(\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots\right)$$

$$\Rightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \quad \text{(II)}$$

now, put $x=\pi$ in equ. (I)

$$\therefore \frac{1}{4}(\pi-\pi)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos n\pi$$

$$\therefore -\frac{\pi^2}{12} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \dots$$

$$\therefore \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots \quad \dots (III)$$

now, adding equ. (II) & (III),

$$\therefore \frac{\pi^2}{6} + \frac{\pi^2}{12} = 2 + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \dots$$

$$\therefore \frac{3\pi^2}{4 \cdot 12} = 2\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right]$$

$$\therefore \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

Hence Proved.

H.W

1] Obtain F.S $f(x) = \cos px$ where p is not an integer and $x \in (0, 2\pi)$ and deduce that,

1] $\pi \operatorname{cosec} \pi x = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{p+n} + \frac{1}{p-n}\right]$

2] $\pi \cot 2\pi p = \frac{1}{2p} + p \sum_{n=1}^{\infty} \frac{1}{p^2-n^2}$

Q.4] Find the F.S $f(x) = \left(\frac{\pi - x}{2}\right)^2$ and

Continue

Prove that, $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$

$\Rightarrow$ for $f(x) = \left(\frac{\pi - x}{2}\right)^2$

F.S $\Rightarrow$ $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$.

For proving the following, we have to use parseval's identity –

$$\frac{1}{2\pi}\int_0^{2\pi} (f(x))^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

L.H.S $\Rightarrow$ $\frac{1}{32\pi}\int_0^{2\pi} \left(\frac{\pi - x}{\cancel{2}}\right)^4 dx$

$= \frac{1}{32\pi}\left[\frac{1}{5}\left(\frac{\pi - x}{\cancel{2}}\right)^5 \times \frac{1}{(-1)}\right]_0^{2\pi}$

$= \frac{1}{32\pi}\left[\frac{1}{5}\left(\frac{-\pi}{\cancel{2}}\right)^5 \times \frac{1}{(-1)} - \left[\frac{1}{5}\left(\frac{\pi}{\cancel{2}}\right)^5 \times \frac{1}{(-1)}\right]\right]$

$= \frac{1}{32\pi}\left[\frac{\pi^5}{5\cancel{32}} + \frac{\pi^5}{5\cancel{32}}\right]$

$= \frac{\pi^4}{80}$

R.H.S $\Rightarrow$ $a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \left(\frac{\pi^2}{12}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\left(\frac{1}{n^2}\right)^2 + 0^2\right)$

$\therefore \frac{\pi^4}{80} = \frac{\pi^4}{144} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$

$\therefore \frac{\pi^4}{80} = \frac{\pi^4}{144} + \frac{1}{2}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots\right]$

$\therefore \frac{\cancel{128}\,\cancel{64}\,\pi^4}{\underset{10}{\cancel{80}} \times \underset{9}{\cancel{144}}\,\cancel{72}} \times \cancel{2}^1 = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$

$\therefore \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$

* Fourier series in $(-\pi, \pi)$:-

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

→ If $f(x)$ is odd fun^n,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{O} \underbrace{\cos nx}_{E}}_{O} dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{O} \underbrace{\sin nx}_{O}}_{E} dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx\, dx$$

i.e. If $f(x)$ is odd then, $a_0 = 0$ & $a_n = 0$.

→ If $f(x)$ is even fun^n,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi} \int_{0}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{E} \underbrace{\cos nx}_{E}}_{E} dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{E} \underbrace{\sin nx}_{O}}_{O} dx = 0$$

i.e. If $f(x)$ is even then, $b_n = 0$.

* Step for fourier series in $(-\pi, \pi)$

Step 1 – Check whether funⁿ is even or odd.

Step 2 – a] If $f(x)$ is neither even nor odd –

then, Calculate all 3 coefficients in $(-\pi, \pi)$.

b] If $f(x)$ is odd –

Show that funⁿ is odd & state that $a_0 = 0$, $a_n = 0$.

Calculate b_n & substitute in F.S.

c] If $f(x)$ is even –

Show that funⁿ is even & state that $b_n = 0$

Calculate a_0, a_n & substitute in F.S.

Q. Find F.S for $f(x) = \begin{cases} x+\pi, & 0 \le x \le \pi \\ -x-\pi, & -\pi \le x \le 0 \end{cases}$

$\Rightarrow$

Consider $f(-x) = \begin{cases} -x+\pi, & 0 \le -x \le \pi \\ -(-x)-\pi, & -\pi \le -x \le 0 \end{cases}$

$= \begin{cases} -x+\pi, & 0 \ge x \ge -\pi \\ x-\pi, & \pi \ge x \ge 0 \end{cases}$

$= \begin{cases} -x+\pi, & -\pi \le x \le 0 \\ x-\pi, & 0 \le x \le \pi \end{cases}$

$\neq f(x)$ also $\neq f(-x)$

$\therefore f(x)$ is neither even nor odd.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} -(x+\pi)\,dx + \int_{0}^{\pi} (x+\pi)\,dx \right]$$

$$= \frac{1}{2\pi}\left[-\left[\frac{x^2}{2}+\pi x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}+\pi x\right]_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\left[0+0-\left(\frac{\pi^2}{2}-\pi^2\right)\right] + \left[\left(\frac{\pi^2}{2}+\pi^2\right)-\left(0+0\right)\right]\right]$$

$$= \frac{1}{2\pi}\left[\frac{-\pi^2}{2}+\frac{3\pi^2}{2}\right]$$

$$= \frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -(x+\pi)\cos nx\,dx + \int_{0}^{\pi}(x+\pi)\overset{\cos}{\cancel{\sin}}\,nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[-(x+\pi)\left(\frac{\sin nx}{n}\right)-(-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \right.$$

$$\left.\left[(x+\pi)\left(\frac{\overset{\sin}{\cancel{+\cos}}\,nx}{n}\right)-(1)\left(\frac{\overset{-\cos}{\cancel{\sin}}\,nx}{n^2}\right)\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\left(0-\frac{1}{n^2}\right)-\left(0-\frac{(-1)^n}{n^2}\right)\right) + \left(\cancel{\left(2\pi\left(\frac{-(-1)^n}{n}\right)\right)}-\cancel{\left(\pi\left(\frac{-1}{n}\right)+0\right)}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{n^2}+\frac{(-1)^n}{n^2}+\cancel{\frac{-2\pi(-1)^n}{n}+\frac{\pi}{n}}\right]$$

$$+\left(\left(0+\frac{(-1)^n}{n^2}\right)-\left(0+\frac{1}{n^2}\right)\right)$$

$$= \frac{1}{\pi}\left[\frac{-1}{n^2}+\frac{(-1)^n}{n^2}+\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n-1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -(x+\pi)\sin nx\,dx + \int_{0}^{\pi}(x+\pi)\sin nx\,dx\right]$$