$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -(x+\pi)\sin nx\,dx + \int_{0}^{\pi}(x+\pi)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-(x+\pi)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[(x+\pi)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{n}(1)+0\right) - (0-0)\right] + \frac{1}{\pi}\left[\left(\frac{-2\pi}{n}(-1)^n + 0\right) - \left(\frac{-\pi}{n}(1)+0\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{2\pi}{n}(-1)^n + \frac{\pi}{n}\right]$$

$$= \frac{2}{n}\left[1-(-1)^n\right]$$

$\therefore$ F.S $\Rightarrow$ $f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi n^2}((-1)^n - 1)\cos nx + \frac{2}{n}[1-(-1)^n]\sin nx\right]$.

Q.7 Obtain F.S for $f(x) = \begin{cases} x + \pi/2, & -\pi < x < 0 \\ \pi/2 - x, & 0 < x < \pi \end{cases}$

Hence deduce that, i] $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$ ii] $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$

* Parseval Identity for $f(x)$ in $(-\pi, \pi)$:-

If $f(x)$ converges uniformly in open interval $-\pi$ to π, then F.S for $f(x)$ is given by -

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)}$$

And parseval's identity in $(-\pi, \pi)$ is given by -

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}[a_n^2 + b_n^2]}$$

$$\Rightarrow \quad f(-x) = \begin{cases} -x + \pi/2 , & -\pi < -x < 0 \\ \pi/2 - (-x) , & 0 < -x < \pi \end{cases}$$

$$f(-x) = \begin{cases} \pi/2 - x , & \pi > x > 0 \\ \pi/2 + x , & 0 > x > -\pi \end{cases}$$

$$f(-x) = f(x)$$

$\therefore$ It is an even funⁿ.

$\therefore b_n = 0$

now, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$

$$= \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2} - x\right)dx$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[(\pi^2 - \pi^2) - (0 - 0)\right]$$

$$= 0$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi}(\pi/2 - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi/2 - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 - \frac{(-1)^n}{n^2}\right) - \left(0 - \frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi n^2}\left[1 - (-1)^n\right]$$

$\therefore$ Fourier series $\Rightarrow$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[1-(-1)^n\right] \cos nx$$

now, at $x=0$, fun is discontinuous.

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2}$$

For deduction, put $x=0$ in F.S

$$\therefore \quad f(0) = \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[1-(-1)^n\right] \cos n(0)$$

$$\therefore \quad \frac{\pi}{2} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^2}$$

$$\therefore \quad \frac{\pi^2}{4} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \dots$$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Hence Proved.

now, To deduce, $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$

we need to use Parseval's Identity.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}[a_n^2+b_n^2]$$ ... Parsevals Identity.

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}[1-(-1)^n]\right)^2$$

$$= \frac{1}{\pi}\int_{0}^{\pi}[f(x)]^2dx = \frac{1}{2}\times\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]^2}{n^4}$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2}-x\right)^2dx = \frac{2}{\pi^2}\left[\frac{4}{1^4}+\frac{4}{3^4}+\frac{4}{5^4}+\cdots\right]$$

$$\frac{1}{\pi}\left[\frac{1}{3}\frac{\left(\frac{\pi}{2}-x\right)^3}{(-1)}\right]_0^{\pi} = \frac{8}{\pi^2}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right]$$

$$\frac{1}{\pi}\times\frac{-1}{3}\left[\left(\frac{-\pi}{2}\right)^3-\left(\frac{\pi}{2}\right)^3\right] = \frac{8}{\pi^2}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right]$$

$$\frac{-1}{3\pi}\left[\frac{-2\pi^3}{8}\right] = \frac{8}{\pi^2}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right]$$

$$\therefore \frac{\pi^4}{96} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$$

hence Proved.

H.W

Q. Find Fourier series of $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$,

hence deduce that, $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Parseval's Identity

* Fourier series in $(c, c+2l)$:

Form of FS in $(c, c+2l)$ i.e. in $(0, 2l)$ or $(-l, l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

For interval in $(0, 2l)$:	For interval in $(-l, l)$:
$a_0 = \frac{1}{2l}\int_0^{2l} f(x)dx$	$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx$
$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$	$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$
$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$	$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

* Parseval's identity in $f(x)$ in interval $(c, c+2l)$:

$$\boxed{\frac{1}{2l}\int_c^{c+2l} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}[a_n^2 + b_n^2]}$$

* Parseval's identity in $(0$ to $2l)$.

$$\frac{1}{2l}\int_0^{2l} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}[a_n^2 + b_n^2]$$

* Parseval's identity in $(-l$ to $l)$.

$$\frac{1}{2l}\int_{-l}^{l} (f(x))^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}[a_n^2 + b_n^2].$$

Q.1) Find FS for $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$ with period 2.

Hence deduce that, $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots$

⇒ here, $l=1$

∴ F.S ⇒ $f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\pi x) + b_n\sin(n\pi x)\right].$

$a_0 = \frac{1}{2(l)}\int_0^2 f(x)\,dx$

$= \frac{1}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx\right]$

$= \frac{1}{2}\left[\frac{\pi x^2}{2}\Big|_0^1 + \pi\left(2x - \frac{x^2}{2}\right)\Big|_1^2\right]$

$= \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) + \pi\left(\left(4 - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right)\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} + \pi\left(2 - \frac{3}{2}\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$

$= \frac{\pi}{2}.$

$a_n = \frac{1}{[l]}\int_0^2 f(x)\cos\left(\frac{n\pi x}{1}\right)dx$

$= \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$

$= \left[\pi x\left(\frac{\sin n\pi x}{n\pi}\right) - (\pi)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right)\right]_0^1 + \left[\pi(2-x)\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{(n\pi)^2}\right)\right]_1^2$

$= \left[\left(\frac{1}{n}\sin n\pi + \frac{1}{n^2\pi}\cos n\pi\right) - \left(0 + \frac{1}{n^2\pi}\cos 0\right)\right] +$

$\left[\left(0 - \frac{1}{n^2\pi}\cos 2n\pi\right) - \left(\frac{1}{n}\sin n\pi - \frac{1}{n^2\pi}\cos n\pi\right)\right]$

$= \left[\left(0 + \frac{1}{n^2\pi}(-1)^n\right) - \left(\frac{1}{n^2\pi}\right) - \left(\frac{1}{n^2\pi}\right) + \left(0 + \frac{(-1)^n}{n^2\pi}\right)\right]$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{[1]}\int_0^2 f(x)\sin\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_0^1 \pi x \sin n\pi x\, dx + \int_1^2 \pi(2-x)\sin n\pi x\, dx$$

$$= \left[\pi x\left(\frac{-\cos n\pi x}{n\pi}\right) - (\pi)\left(\frac{-\sin n\pi x}{(n\pi)^2}\right)\right]_0^1 +$$

$$\left[\pi(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-\pi)\left(\frac{-\sin n\pi x}{(n\pi)^2}\right)\right]_1^2$$

$$= \left[\left(-\frac{1}{n}\cos n\pi + \frac{1}{n^2\pi}\sin n\pi\right) - \left(0 + \frac{1}{n^2\pi}\sin 0\right)\right] +$$

$$\left[\left(0 - \frac{1}{n^2\pi}\sin 2n\pi\right) - \left(\frac{-1}{n}\cos n\pi - \frac{1}{n^2\pi}\sin n\pi\right)\right]$$

$$= \left[-\frac{(-1)^n}{n} + 0 - 0 + 0 + \frac{(-1)^n}{n} + 0\right]$$

$$= 0.$$

$$\therefore F.S \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2[(-1)^n - 1]}{n^2\pi}\cos n\pi x\right].$$

now,

$$\pi x = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos n\pi x$$

for deduction, put $x = 0$

$$\therefore 0 = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$-\frac{\pi}{2} = \frac{2}{\pi}\left[\frac{-2}{(1)^2} + 0 - \frac{2}{(3)^2} + 0 - \frac{2}{(5)^2} + \cdots\right]$$

$$-\frac{\pi^2}{4} = -2\left[\frac{1}{(1)^2} + \frac{1}{(3)^2} + \frac{1}{(5)^2} + \frac{1}{(7)^2} + \cdots\right]$$

$$\Rightarrow \pi^2/8 = 1/1^2 + 1/3^2 + 1/5^2 + \ldots$$ Hence Proved.

Q.3 Obtain F.S for $f(x) = 2x - x^2$ in $(0,3)$ and hence deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \infty = \frac{\pi^2}{12}$.

⇒ here, $l = \frac{3}{2}$.

So, let $f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi x}{3}\right) + b_n \sin\left(\frac{2n\pi x}{3}\right)\right]$

$$a_0 = \frac{1}{2(3/2)}\int_0^3 (2x - x^2)\,dx$$

$$= \frac{1}{3}\int_0^3 \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{1}{3}\left[\left(9 - \frac{27}{3}\right) - (0 - 0)\right]$$

$$= \frac{1}{3}\left[\cancel{\frac{-135}{4}}\right]$$

$$= \cancel{\frac{-15}{4}},\ 0$$

$$a_n = \frac{1}{3/2}\int_0^3 (2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[\frac{(2x - x^2)\sin\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}} - (2 - 2x)\left[\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right] + (-2)\left[\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{(2n\pi/3)^3}\right]\right]_0^3$$

$$= \frac{2}{3}\left\{\left[\frac{(-3)\sin 2n\pi}{2n\pi/3} + \frac{(-4)\cos 2n\pi}{(2n\pi/3)^2} + \frac{(2)\sin 2n\pi}{(2n\pi/3)^3}\right] - \left[0 + \frac{(-2)\cos 0}{(2n\pi/3)^2} + 0\right]\right\}$$

$$= \frac{2}{3}\left[\frac{-4\times 9(1)}{4n^2\pi^2} - \frac{2\times 9(1)}{2\cdot 4n^2\pi^2}\right]$$

$$= \frac{2}{3}\left[\frac{-2\times 9}{4n^2\pi^2}\right] \qquad \frac{2}{3}\left[\frac{-9}{n^2\pi^2} - \frac{9}{2n^2\pi^2}\right]$$

$$= \frac{-9}{3n^2\pi^2} \qquad = \frac{2}{3}\left[-\frac{3}{2}\left[\frac{9}{n^2\pi^2}\right]\right] = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{3/2}\int_0^3 (2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{2n\pi/3}\right) - (2-2x)\left(\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{(2n\pi/3)^2}\right)\right.$$

$$\left. + (-2)\left(\frac{\cos(2n\pi x/3)}{(2n\pi/3)^3}\right)\right]_0^3$$

$$= \frac{2}{3}\left\{\left[\frac{3\cdot\cos 2n\pi}{2n\pi/3} + \frac{(-4)\sin 2n\pi}{(2n\pi/3)^2} + \frac{(-2)\cos 2n\pi}{(2n\pi/3)^3}\right] - \left[0 + \frac{(2)\sin 0}{(2n\pi/3)^2} + \frac{(-2)\cos 0}{(2n\pi/3)^3}\right]\right\}$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi} + 0 - \cancel{\frac{2}{(2n\pi/3)^3}} - 0 + 0 + \cancel{\frac{2}{(2n\pi/3)^3}}\right]$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi}\right]$$

$$= \frac{3}{n\pi}.$$

$$\therefore F.S \Rightarrow f(x) = \sum_{n=1}^{\infty}\left[\frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]$$

Put $x=0$

$$\therefore 0 = \sum_{n=1}^{\infty}\frac{-9}{\pi^2}\cdot\frac{1}{n^2}(1)$$

$$\therefore \frac{-\pi^2}{9} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

put $x = 3/2$

$$\left[2\left(\frac{3}{2}\right) - \left(\frac{3}{2}\right)^2\right] = \sum_{n=1}^{\infty} \left[\frac{-9}{n^2\pi^2}\cos(n\pi) + \frac{3}{n\pi}\sin(n\pi)\right]$$

$$\left[3 - \frac{9}{4}\right] = \frac{-9}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{\cancel{3}}{4} \times \frac{-\pi^2}{\cancel{9}3} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Q.3) obtain F.S for $f(x) = |x|$, $-2 < x < 2$
Hence deduce that, $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

$\Rightarrow$

Here, $l = 2$

$\therefore$ F.S $\Rightarrow$ $|x| = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{2}\right) + b_n\sin\left(\frac{n\pi x}{2}\right)\right]$

here,

$|-x| = x = |x|$

$\Rightarrow f(x) = |x|$ is an even funⁿ.

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{l}\int_0^l f(x)\,dx$$

$$= \frac{1}{2}\int_0^2 |x|\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2$$

$$= 1.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{2}\int_{0}^{2} |x|\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[\frac{x\sin\left(\frac{n\pi x}{2}\right)}{n\pi/2} - (1)\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^2}\right)\right]_0^2$$

$$= \left[\frac{2\sin n\pi}{n\pi/2} + \frac{\cos n\pi}{(n\pi/2)^2}\right] - \left[0 + \frac{\cos 0}{(n\pi/2)^2}\right]$$

$$= 0 + \frac{4(-1)^n}{n^2\pi^2} - \frac{4(1)}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right].$$

$\therefore$ Since $f(x)$ is even fun^n

$\therefore b_n = 0$

$$F.S \Rightarrow \quad |x| = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]\cos\left(\frac{n\pi x}{2}\right)$$

To deduce, put $x = 0$

$$0 = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right](1)$$

$$-1 \times \frac{\pi^2}{4} = \frac{-2}{1^2} + 0 - \frac{2}{3^2} + 0 - \frac{2}{5^2} + \cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

* Fourier Transform :-

In real life problems not all funⁿ/events are periodic in nature. Fourier Transform deals with such non periodic funⁿ & hence it is very useful in practical applications.

If a funⁿ $f(x)$ is defined in $(-\infty, \infty)$, is piecewise continuous in each finite interval & is absolutely integrable in $(-\infty, \infty)$ then integral $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$ is called Fourier Transform of $f(x)$ & it is denoted by $F(s)$.

i.e. $$\boxed{F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx}$$

eg. $f(x) = x^2, \quad 0 < x < 2$

then, $$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{2} x^2 \cdot e^{isx} dx$$

* Fourier sine transform & cosine transform :-

If $f(x)$ is defined in $(0, \infty)$, we define its -

1) F.S.T by

$$\boxed{F_s(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx \, dx}$$

2) F.C.T by

$$\boxed{F_c(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx \, dx}$$

in nature. hence

conti- in $(-\infty,\infty)$ nsform

Q.1] Find the Fourier Transform of $f(x) = \begin{cases} 1 + x/a, & -a < x < 0 \\ 1 - x/a, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$

⇒

By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{0} \left(1 + \frac{x}{a}\right) e^{isx} dx + \int_{0}^{a} \left(1 - \frac{x}{a}\right) e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\left(1 + \frac{x}{a}\right) \frac{e^{isx}}{is} - \left(\frac{1}{a}\right)\left[\frac{e^{isx}}{(is)^2}\right] \right)_{-a}^{0} + \right.$$

$$\left. \left(\left(1 - \frac{x}{a}\right) \frac{e^{isx}}{is} - \left(\frac{-1}{a}\right)\left[\frac{e^{isx}}{(is)^2}\right] \right)_{0}^{a} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\left(\frac{1}{is} - \frac{1}{a(is)^2} \right) - \left(0 - \frac{e^{-isa}}{a(is)^2} \right) \right] + \right.$$

$$\left. \left[\left(0 + \frac{e^{isa}}{a(is)^2} \right) - \left(\frac{1}{(is)} + \frac{1}{a(is)^2} \right) \right] \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\cancel{\frac{1}{is}} - \frac{1}{a(is)^2} + \frac{e^{-isa}}{a(is)^2} + \frac{e^{isa}}{a(is)^2} - \cancel{\frac{1}{is}} - \frac{1}{a(is)^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{as^2} - \frac{e^{isa}}{as^2} - \frac{e^{-isa}}{as^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{as^2} - \frac{1}{as^2} \left[\cos as + \cancel{i \sin as} + \cos as - \cancel{i \sin as} \right] \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{as^2} - \frac{2\cos as}{as^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{as^2} \left[1 - \cos as \right].$$

Q.2] Find Fourier cosine transform of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

$\left[\int_0^\infty \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2} \right]$ imp

$\Rightarrow$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a \cos x \cos sx \, dx + \int_a^\infty (0) \cos sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a \frac{2 \cos x \cos sx}{2} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^a (\cos(s+1)x + \cos(s-1)x) \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{(s+1)} + \frac{\sin(s-1)x}{(s-1)} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{(s+1)} + \frac{\sin(s-1)a}{(s-1)} - 0 - 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{(s+1)} + \frac{\sin(s-1)a}{(s-1)} \right]$$

Q.3) Find Fourier sine transform of $\frac{1}{x}$.

$\Rightarrow$

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty \frac{\sin sx}{x} dx \right]$$

Put $x = \frac{\theta}{s} \Rightarrow \frac{1}{x} = \frac{s}{\theta}$

x	0	∞
θ	0	∞

$$dx = \frac{1}{s} d\theta$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty \frac{\cancel{s} \sin\theta}{\theta} \times \frac{d\theta}{\cancel{s}} \right]$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2}$$

$$= \sqrt{\frac{\pi}{2}}$$

* $I = \int f(x,a)\,dx$

$\Rightarrow \frac{dI}{da} = \int \frac{\partial (f(x,a))}{\partial a}\,dx$

$\Rightarrow I = \int \text{Answer}\,da.$ [DUIS]

*** Q.4] Find the Fourier transform of $f(x) = e^{-x^2}$.

$\Rightarrow$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{isx}\,dx$$

even fun^n $\uparrow$ odd fun^n $\uparrow$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} [\cos sx + i \sin sx]\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x^2} \cos sx\,dx + 0$$

Let $I(s) = \int_{0}^{\infty} e^{-x^2} \cos sx\,dx$

$$\frac{dI(s)}{ds} = \int_{0}^{\infty} x \cdot e^{-x^2} \cdot -\sin sx\,dx$$

$$= \frac{1}{2} \int_{0}^{\infty} \{e^{-x^2}(-2x)\} \sin sx\,dx$$

here, $\int e^{-x^2}(-2x)\,dx$

$= e^{-x^2}$

$$= \frac{1}{2}\left[\left(\sin sx\,(e^{-x^2})\right)_0^{\infty} - \int_0^{\infty} s \cdot \cos sx \cdot e^{-x^2}\,dx\right]$$