Mod 4] Complex Variables

If by a rule or a set of rules, we can find one or more complex numbers W for every z (= x + iy) in a given domain, we say that W is a funⁿ of z.

Notation : $W = f(z) = u + iv$

here, z & W both are complex variables functions is called as complex funⁿ.

eg. $f(z) = z^2 = (x+iy)^2 = (x^2-y^2) + i\,2xy = u + iv$

- **z-plane & w-plane :**

A real funⁿ $y = f(x)$ can be represented by a curve in x-y plane is called as z-plane

But $w = f(z) = u(x,y) + iv(x,y)$ involves four variables x, y, u and v.

So, w cannot be represented on a single plane. Hence we need to use two planes, z plane & w plane.

- **Neighborhood of a point :**

consider the inequality, $|z - z_0| < \varepsilon$

i.e. a circle with centre at z_0 and radius ε

i.e. including point z_0 & excluding the boundary points of the circle.

The circlar region $|z - z_0| < \varepsilon$ is called as neighbourhood of a point z.

- **Limit of a funⁿ :**

Let $W = f(z)$ be a single values funⁿ of z defined in a bounded & closed domain D & let z approach z_0 along any path in D.

For a given $\varepsilon > 0$ ($\varepsilon \neq 0$) if we can find another small $\varepsilon > 0$, $\ni |f(z) - w_0| < \varepsilon$, $\forall z$ for which
$0 < |z - z_0| < \varepsilon$
then, we say that w_0 is limit of $f(z)$ as $z \to z_0$
i.e. $\lim_{z \to z_0} f(z) = w_0$

[z can approach z_0 along any path. the limit doesn't depend on the path]

* Continuity :

Let $w = f(z)$ be a single valued funⁿ defined in a bounded closed domain D.

$w = f(z)$ is said to be continuous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$

* Differentiability :

Let $w = f(z)$ be a single valued funⁿ of z defined in domain D.

$f(z)$ is said differentiable at any point z_0, if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ is unique as $z \to z_0$ along any path of the domain D.
$(x, y) \to (x_0, y_0)$

* Analytic Function :

Analytic means function is a complex funⁿ it is differentiable in the neighborhood of a point.

If a single valued funⁿ $w = f(z)$ is defined and differentiable at each point of domain D then it is called analytic or regular or holomorphic funⁿ of z in the domain D.

A funⁿ is said to be analytic at a point if it has a derivative at that point and in some neighborhood of that point as well as at every point in some neighborhood of that point.

- Singular Point :

If a funⁿ is not analytic at a point of the domain then that point is called as singular point of that funⁿ.

- CR Equations in Cartesian co-ordinates :

The necessary and sufficient conditions for a continuous single valued funⁿ –

$w = f(z) = u(x,y) + iv(x,y)$ to be analytic in region R are as follows :–

1) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous funⁿ of x & y in the region R.

2) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ($u_x = v_y$ & $u_y = -v_x$) at each point of R.

The conditions are known as Cauchy Riemann Equations.

Note :– a) If $f(z)$ is analytic, its derivative is given by anyone of the following –

i.e. $f'(z) = u_x + iv_x$,

$f'(z) = v_y + iv_x$, $(\because u_x = v_y)$

$f'(z) = v_y - iu_y$, $(\because v_x = -u_y)$

$f'(z) = u_x - iu_y$.

b] If $f(z)$ is analytic, then it can be differentiable in usual manner i.e. if $f(z) = z^2$

$\Rightarrow f'(z) = 2z$

eg. $f(z) = z^3 + 3z^2 + 5z + 3$

$f'(z) = 3z^2 + 6z + 5$

Q.1] Check whether $\cos z$ is differentiable at $z = i$.

$\Rightarrow$

here, $f(z) = \cos z$ and $z_0 = i$

By definition, $\lim\limits_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim\limits_{z \to i} \frac{\cos z - \cos i}{z - i}$

$= \lim\limits_{z \to i} \frac{-2\sin\left(\frac{z+i}{2}\right)\sin\left(\frac{z-i}{2}\right)}{z - i}$

$= \lim\limits_{z \to i} \frac{1}{2}\left[-2\sin\left(\frac{z+i}{2}\right)\right] \lim\limits_{z \to i}\left[\frac{\sin(z - i/2)}{z - i/2}\right]$

$= -2\sin i \times \frac{1}{2}$

$= -\sin i$

$\therefore$ $f(z)$ is differentiable at $z = i$

Q.2] Discuss the continuity of $\frac{z^2}{z^4 + 3z^2 + 1}$ at $z = e^{i\pi/4}$

$\Rightarrow$ Given funⁿ is continuous if $\lim\limits_{z \to e^{i\pi/4}} f(z) = f(e^{i\pi/4})$

At $z = e^{i\pi/4}$, $f(e^{i\pi/4}) = \frac{(e^{i\pi/4})^2}{(e^{i\pi/4})^4 + 3(e^{i\pi/4})^2 + 1}$

$$= \frac{i}{-1+3i+1} = \frac{1}{3}$$

now, $\lim_{z \to e^{i\pi/4}} f(z) = \dfrac{(e^{i\pi/4})^2}{(e^{i\pi/4})^4 + 3(e^{i\pi/4})^2 + 1} = \dfrac{1}{3}$

$\therefore \lim_{z \to e^{i\pi/4}} f(z) = f(e^{i\pi/4})$ Hence Proved

$\therefore$ The fun^n is continuous at $e^{i\pi/4}$.

Q.3] Show that $\lim_{z \to 0} \dfrac{xy}{x^2+y^2}$ doesn't exist

or Show that $f(z) = \dfrac{xy}{x^2+y^2}$ is not continuous at $z=0$

$\Rightarrow$ Let $x = ky$

$$\lim_{z \to 0} \frac{xy}{x^2+y^2} = \lim_{y \to 0} \frac{ky^2}{k^2y^2+y^2} = \frac{k}{k^2+1}$$

Thus, the limit depends upon the path

$\therefore \lim_{z \to 0} \dfrac{xy}{x^2+y^2}$ doesn't exist

$\therefore$ Hence we can also say that, $\dfrac{xy}{x^2+y^2}$ is not continuous at $z=0$.

Q.4] Discuss the continuity of $f(z) = \bar{z}/z$ at $z=0$.

$\Rightarrow$

Let $f(z) = \dfrac{\bar{z}}{z} = \dfrac{x-iy}{x+iy} \times \dfrac{x-iy}{x-iy} = \dfrac{(x-iy)^2}{x^2+y^2}$

$$= \frac{x^2-y^2}{x^2+y^2} - i\frac{2xy}{x^2+y^2}$$

Let $x = ky$.

$$\lim_{z\to 0} f(z) = \lim_{\substack{x\to 0\\ y\to 0}} f(z) = \lim_{y\to 0}\left[\frac{k^2y^2-y^2}{k^2y^2+y^2} - \frac{i(2ky^2)}{k^2y^2+y^2}\right]$$

$$= \frac{k^2-1}{k^2+1} - \frac{i2k}{k^2+1}$$

which depends upon k i.e. on the path

$\therefore$ Limit doesn't exist

$\Rightarrow$ $f(z)$ is not continuous at $z=0$.

[H.W]

Q.5] Show that $\lim_{z\to 0} \frac{x^2y}{x^4+y^2}$ doesn't exist [Hint : $x^2 = ky$]

* Conjugate Functions :-

If $f(z) = u+iv$ is an analytic funⁿ then the funⁿ u & v are called conjugate funⁿ.

Q.1] If $f(z) = z^2+3$ is analytic funⁿ then find conjugate funⁿ of $f(z)$.

$\Rightarrow$

$f(z) = z^2+3$

Replacing z by $x+iy$

$f(z) = (x+iy)^2+3$

$= x^2-y^2+3+i(2xy)$

$\therefore$ x^2-y^2+3 and $2xy$ are conjugate funⁿ.

now, $v_x = 0 \Rightarrow \boxed{u_y = 0}$

from (IV), $uu_x - vu_y = 0$ ... (VI)

v(V) + u(VI) $\Rightarrow$ $vuu_y + v^2v_y + u^2u_x - uvu_y = 0$

$(v^2+u^2)u_x = 0$

$c^2u_x = 0$

$\boxed{u_x = 0} \Rightarrow \boxed{v_y = 0}$

$\therefore$ Hence Proved.

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Note :- The formulas for differentiation of the complex fun^n s which are analytic are same as the correspond- formulas in calculas of real numbers.
i.e. $f(x) = \sin x \Rightarrow f'(x) = \cos x$
similarly, $f(z) = \sin z \Rightarrow f'(z) = \cos z$.

If $f(z)$ is an analytic fun^n with constant modulus, then prove that $f(z)$ is constant.

$\Rightarrow$

here, $f(z)$ is an analytic fun^n
$\Rightarrow$ CR equ.s are satisfied
i.e. $u_x = v_y$ ... (I) and $u_y = -v_x$ ... (II)
where $f(z) = u + iv$

Also, $|f(z)| = c$, $f(z) = u+iv$
$\Rightarrow u^2 + v^2 = c^2$ ... (III)

To prove that $f(z)$ is constant, i.e. $u_x = u_y = v_x = v_y = 0$
Partially differentiating equ. (III) wrt x,
$2uu_x + 2vv_x = 0 \Rightarrow uu_x + vv_x = 0$ (IV)

Partially differentiating equ. (III) wrt y,
$2uu_y + 2vv_y = 0 \Rightarrow uu_y + vv_y = 0$ (V)

now, v(IV) $- u$(V) $\Rightarrow vuu_x + v^2v_x - u^2u_y - uvv_y = 0$

$\Rightarrow v^2v_x - u^2v_y = 0$ $\quad \begin{bmatrix} u_x = v_y \\ u_y = -v_x \end{bmatrix}$

$v^2v_x + u^2v_x = 0$

$v_x(v^2+u^2) = 0$

$v_x c^2 = 0$

$\boxed{v_x = 0}$

Q.2) Determine whether following fun^n are analytic and if so find their derivatives

1) $x^2 - y^2 + 2ixy$ 2) ze^{2z}

$\Rightarrow$

$f(z) = u + iv$

$f(z)$ is analytic if 1) u_x, u_y, v_x, v_y are continuous

2) $u_x = v_y$ & $u_y = -v_x$

1) Here, $u = x^2 - y^2$

$\Rightarrow u_x = 2x$

$\Rightarrow u_y = -2y$

$v = 2xy$

$\Rightarrow v_x = 2y$

$\Rightarrow v_y = 2x$

here, u_x, u_y, v_x, v_y all are continuous and

$u_x = v_y$, $u_y = -v_x$

Hence, $f(z) = x^2 - y^2 + 2ixy$ is analytic fun^n.

$\therefore f'(z) = u_x + iv_x$

$= 2x + i2y$

2) Here, $ze^{2z} = (x+iy)e^{2(x+iy)}$

$= (x+iy)e^{2x} \cdot e^{2iy}$

$= e^{2x}(x+iy)(\cos 2y + i\sin 2y)$

$= e^{2x}[x\cos 2y + ix\sin 2y + iy\cos 2y - y\sin 2y]$

$= e^{2x}[x\cos 2y - y\sin 2y + i(x\sin 2y + y\cos 2y)]$

here, $u = e^{2x}[x\cos 2y - y\sin 2y]$

$\cancel{u_x} = \cancel{2e^{2x}}[\cancel{x\cos 2y} - \cancel{y\sin 2y}] + e^{2x}[\cancel{x(-\sin 2y)} + \cancel{\cos 2y} - \cancel{(0+0)}]$

$= \cancel{2e^{2x}x\cos 2y} - \cancel{2e^{2x}y\sin 2y} \cancel{-} \cancel{e^{2x}x\sin 2y} + \cancel{e^{2x}\cos 2y}$

$$u_x = 2e^{2x}[x\cos 2y - y\sin 2y] + e^{2x}[\cos 2y]$$
$$= [2e^{2x}x + e^{2x}]\cos 2y - 2ye^{2x}\sin 2y \quad \ldots (1)$$

$$u_y = e^{2x}[x(-2\sin 2y) - \sin 2y - y\cos 2y \cdot 2]$$
$$= -2xe^{2x}\sin 2y - e^{2x}[2y\cos 2y + \sin 2y] \quad \ldots (2)$$

here, $v = e^{2x}[x\sin 2y + y\cos 2y]$

$$v_x = 2e^{2x}[x\sin 2y + y\cos 2y] + e^{2x}[\sin 2y]$$
$$= \sin 2y[2xe^{2x} + e^{2x}] + 2y\,e^{2x}\cos 2y \quad \ldots (3)$$

$$v_y = e^{2x}[x\cos 2y \cdot 2 + \cos 2y + y(-\sin 2y)\cdot 2]$$
$$= 2xe^{2x}\cos 2y + e^{2x}[-2y\sin 2y + \cos 2y] \quad \ldots (4)$$

here, u_x, v_x, u_y, v_y all are continuous and

$u_x = v_y$ & $u_y = -v_x$

Hence, $f(z)$ is analytic fun^n.

$$\therefore f'(z) = u_x + iv_x$$
$$= (1) + i(3).$$

Note :- CR equations are only necessary conditions for a fun^n to be analytic i.e.

$f(z)$ is analytic at a point $\Rightarrow$ CR equ. are satisfied at that point

But, if CR equations are satisfied, than fun^n may or may not be analytic at that point.

★★★ Q. show that $f(z) = \frac{xy(y-ix)}{x^2+y^2}$, $z \neq 0$ is not analytic at the origin although CR equ. are satisfied.

⇒

here, $f(z) = \frac{xy^2}{x^2+y^2} - \frac{ix^2y}{x^2+y^2} = u + iv$

here, $u = \frac{xy^2}{x^2+y^2}$ and $v = \frac{yx^2(-1)}{x^2+y^2}$

$$u_x = \frac{(x^2+y^2)(y^2) + (xy^2)(2x)}{(x^2+y^2)^2} = \frac{y^2 + xy^2}{x^2+y^2}$$

$$= \frac{(x^2+y^2)(y^2) + 2x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{3x^2y^2 + y^4}{(x^2+y^2)^2}$$

or $u_x = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x-0} = 0$

$u_y = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y-0} = 0$

$v_x = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x-0} = 0$

$v_y = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y-0} = 0$

⇒ $u_x = v_y$ & $u_y = -v_x$

∴ CR equs are satisfied.

now, to check the analyticity at origin

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z-0}$$

$$= \lim_{\substack{x\to 0 \\ y\to 0}} \frac{\frac{xy(y-ix)}{x^2+y^2} - 0}{x+iy}$$

Let $z \to 0$ along $x = ky$

$$f'(0) = \lim_{y \to 0} \frac{ky^2 \cdot y(1-ik)}{(k^2+1)y^2 y(k+i)}$$

$$= \frac{k(1-ik)}{(k^2+1)(k+i)}$$

Thus, the limit depends upon the path (ie. depends upon k)

$\Rightarrow$ limit doesn't exist

$\Rightarrow$ $f(z)$ is not differentiable at $z=0$

$\Rightarrow$ $f(z)$ is not analytic at $z=0$

Q. Find the values of z for which the following funⁿ is not analytic. $z = e^{-v}(\cos u + i \sin u)$

$\Rightarrow$

$$z = e^{-v} \cdot (\cos u + i \sin u)$$

$$z = e^{-v} \cdot e^{iu}$$

$$z = e^{i(u+iv)}$$

$$z = e^{i\omega}$$

$$\log z = i\omega \qquad \omega = u+iv = f(z)$$

$$\omega = \frac{1}{i} \log z$$

$$\frac{d\omega}{dz} = \frac{1}{i} \times \frac{1}{z}$$

At $z=0$, this derivative becomes ∞

i.e. the derivative doesn't exist at $z=0$

$\Rightarrow$ ω is not analytic at $z=0$

i.e. given funⁿ is not analytic at $z=0$.

Q. Show that $\omega = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$ is an analytic funⁿ & find $\frac{d\omega}{dz}$ in terms of z.

$\Rightarrow$ $$w = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$$

here, $u = \frac{x}{x^2+y^2}$ $\qquad v = \frac{-y}{x^2+y^2}$

$$u_x = \frac{(x^2+y^2)-(x)(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = x\left[\frac{-1}{(x^2+y^2)^2}(2y)\right] = \frac{-2xy}{(x^2+y^2)^2}$$

$$v_x = -y\left[\frac{-1}{(x^2+y^2)^2}(2x)\right] = \frac{2xy}{(x^2+y^2)^2}$$

$$v_y = -\left[\frac{(x^2+y^2)-(y)(2y)}{(x^2+y^2)^2}\right] = -\left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$$

here, $u_x = v_y$ & $u_y = -v_x$ & all are continuous

$\therefore$ w is analytic funⁿ.

here, $w = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$

$$w = \frac{\cancel{x-iy}}{(\cancel{x-iy})(x+iy)}$$

$z = x+iy$

$\bar{z} = x-iy$

$|z| = \bar{z}z = (x+iy)(x-iy)$

$= (x^2+y^2)$

$$w = \frac{1}{(x+iy)}$$

$$w = \frac{1}{z}$$

$$\frac{dw}{dz} = \frac{-1}{z^2}.$$

* <u>Harmonic function</u> :-

Any funⁿ ϕ of x & y which has continuous partial derivatives of first & second order & which satisfies the equation $\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0}$ (*) is called as Harmonic funⁿ.

Here, equ. (*) is called as laplace Equation.

Note : 1] The real & imaginary parts u & v of an analytic funⁿ $f(z) = u+iv$ are Harmonic funⁿ.

2] If $f(z) = u+iv$ is analytic then u & v are harmonic funⁿ, but the converse need not be true.
i.e. if u & v both are Harmonic funⁿ, then $u+iv$ or $u-iv$ need not be an analytic funⁿ.

Q.1] $u = x^2-y^2$ & $v = \frac{-y}{x^2+y^2}$, show that u & v are Harmonic funⁿ but $u+iv$ is not an analytic funⁿ.

⇒

$u = x^2-y^2$

$u_x = 2x$ $\quad\quad$ $u_y = -2y$

$u_{xx} = 2$ $\quad\quad$ $u_{yy} = -2$

here, $u_{xx} + u_{yy} = 0$

$v = \frac{-y}{x^2+y^2}$

$v_x = -y\left(\frac{-1}{(x^2+y^2)^2}\right)(2x) = \frac{2xy}{(x^2+y^2)^2}$

$v_{xx} = \frac{2y(x^2+y^2)^2 - 2(x^2+y^2)2x(2xy)}{(x^2+y^2)^4}$

$= \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3}$

$= \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$

$v = \dfrac{-y}{x^2+y^2}$

$v_y = -\left[\dfrac{1(x^2+y^2) - y(2y)}{(x^2+y^2)^2}\right] = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$v_{yy} = \dfrac{2y(x^2+y^2)^2 - (y^2-x^2)(2(x^2+y^2)2y)}{(x^2+y^2)^4}$

$= \dfrac{2y(x^2+y^2) - 4y(y^2-x^2)}{(x^2+y^2)^3}$

$= \dfrac{-2y^3 + 6x^2y}{(x^2+y^2)^3}$

here $v_{xx} + v_{yy} = 0$

$\therefore$ u & v are Harmonic funⁿ.

but $u_x \neq v_y$ and $u_y \neq -v_x$

$\therefore$ $u+iv$ is not an analytic funⁿ.

Q.7 If $u(x,y)$ is a Harmonic funⁿ then prove that $f(z) = u_x - iu_y$ is an analytic funⁿ.

$\Rightarrow$

here, u is a Harmonic funⁿ

$\Rightarrow u_{xx} + u_{yy} = 0 \quad \ldots (I)$

also, $f(z) = u_x - iu_y$

$= U + iV$ where, $U = u_x$

& $V = -u_y$

$\therefore U_x = u_{xx} = -u_{yy}$ from (I)

$U_y = u_{xy}$

$V_x = -u_{yx}$

$V_y = -u_{yy}$

Thus, $U_x = V_y$

& $V_x = -U_y$ (provided $u_{xy} = u_{yx}$)

$\therefore U+iV = f(z)$ is an analytic funⁿ.

1) Check whether $u = x + e^{xy} + y + e^{-xy}$ is Harmonic.

2) Check whether $u = e^{x}\cos y + x^3 - 3xy$ is Harmonic.

* <u>Conjugate Harmonic funⁿ :-</u>

If $f(z) = u + iv$ is an analytic funⁿ so that u & v are Harmonic funⁿ, then u & v are called conjugate Harmonic funⁿ.

Each one is the conjugate Harmonic funⁿ of the other one.

• <u>Milne Thomson's Method :-</u>

- Type 1 : To find analytic funⁿ whose real part is given.

Type 2 : To find analytic funⁿ whose imaginary part is given.

Type 3 : when $u+v$ or $u-v$ is given to find $f(z) = u+iv$

• <u>Steps for Milne Thomson's Method :-</u>

<u>Type 1 :</u>

1) $f(z) = u + iv \Rightarrow f'(z) = u_x + iv_x$
$= u_x - iu_y$

2) Let $u_x = \phi_1(x,y)$ & $u_y = \phi_2(x,y)$
$\Rightarrow f'(z) = \phi_1(x,y) - i\phi_2(x,y)$

3) By M-T method. Let $f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

4) $f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c$

Q. Find analytic fun^n whose real part is –

$$u = \frac{x}{2}\log(x^2+y^2) - y\tan^{-1}\left(\frac{y}{x}\right) + \sin x\cosh y$$

$$\Rightarrow \quad u_x = \frac{1}{2}\log(x^2+y^2) + \frac{x}{2}\times\frac{1}{x^2+y^2}(2x) - y\times\frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right) + \cos x\cosh y$$

$$= \frac{1}{2}\log(x^2+y^2) + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + \cos x\cosh y$$

$$= \frac{1}{2}\log(x^2+y^2) + \cos x\cosh y + 1$$

$$u_y = \frac{x}{2}\times\frac{1}{x^2+y^2}(2y) - \left[\tan^{-1}\left(\frac{y}{x}\right) + y\cdot\frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right)\right] + \sin x(-\sinh y)$$

$$= \cancel{\frac{xy}{x^2+y^2}} - \tan^{-1}\left(\frac{y}{x}\right) - \cancel{\frac{xy}{x^2+y^2}} - \sin x\sinh y$$

$$= -\tan^{-1}\left(\frac{y}{x}\right) - \sin x\sinh y$$

now, $f'(z) = \frac{1}{2}\log(x^2+y^2) + \cos x\cosh y + 1 - i\left[\tan^{-1}\left(\frac{y}{x}\right) + \sin x\sinh y\right]$

By M.T methods,

$$f'(z) = \frac{1}{2}\log z^2 + \cos z + 1 - i\left[\tan^{-1}\left(\frac{0}{z}\right)^{0} + \sin z\,\sinh(0)^{0}\right]$$

$$\therefore f'(z) = \log z + \cos z + 1$$

$$\therefore f(z) = \sin z + z\log z - \cancel{z} + \cancel{z}$$

$$\therefore f(z) = \underline{\sin z + z\log z}.$$

Type 2:

1] $f(z) = u + iv \Rightarrow f'(z) = u_x + iv_x$
$= v_y + iv_x$

2] Let $v_y = \psi_1(x,y)$ & $v_x = \psi_2(x,y)$
$f'(z) = \psi_1(x,y) + i\psi_2(x,y)$

3] By M.T Method, let $f'(z) = \psi_1(z,0) + i\psi_2(z,0)$

4] $f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c$

where, $\psi_1 = v_y$ & $\psi_2 = v_x$

Q.] If $f(z) = u + iv$ and $v = e^{-x}(y\sin y + x\cos y)$, then find u.

$\Rightarrow$

Let $f(z) = u + iv$ be required funⁿ.
$f'(z) = u_x + iv_x = v_y + iv_x$
Since, $f(z)$ is an analytic
$\therefore$ CR equ's are satisfied.

$\psi_1(x,y) = v_y = e^{-x}(y\cos y + \sin y - x\sin y)$

$\psi_2(x,y) = v_x = -e^{-x}(y\sin y + x\cos y) + e^{-x}(\cos y)$

now, By Milne Thomson's Method,
$f'(z) = \psi_1(z,0) + i\psi_2(z,0)$
$= 0 + i(-e^{-z}(z) + e^{-z})$
$= i(e^{-z}(1-z))$

$f(z) = i\int(-e^{-z}(z) + e^{-z})\,dz$

$= i\left[(-z)(-e^{-z}) - (-1)(e^{-z}) + (\cancel{-e^{-z}})\right] + c$

$= i\left[z e^{-z} \right] + c$

Replacing z by $x+iy$.

$f(x+iy) = i\left((x+iy) e^{-(x+iy)} \right) + c$

$= i e^{-x}(x+iy)(\cos y - i\sin y) + c$

$= i e^{-x}\left[x\cos y - ix\sin y + iy\cos y + y\sin y \right] + c$

$= x e^{-x}\sin y - y e^{-x}\cos y + i e^{-x}(x\cos y + y\sin y) + c$

$\therefore u = x e^{-x}\sin y - y e^{-x}\cos y$

$= e^{-x}(x\sin y - y\cos y)$

Q.3) If $f(z) = u+iv$ is analytic & $u = \log\sqrt{x^2+y^2}$, find v.

$\Rightarrow$

Let $f(z) = u+iv$ be required funⁿ.

$f'(z) = u_x + iv_x = u_x - iu_y$

$\phi_1(x,y) = u_x = \dfrac{x}{x^2+y^2}$

$\phi_2(x,y) = -u_y = \dfrac{-y}{x^2+y^2}$

By Milne's Thomson Method.

$f'(z) = \phi_1(z,0) + (-i)\phi_2(z,0)$

$= \dfrac{z}{z^2} + i(0)$

$= \dfrac{1}{z}$

$$f'(z) = \frac{1}{z}$$

$$\therefore f(z) = \log z + c$$

$\sqrt{x^2+y^2}$ $\tan^{-1} y/x$

Replacing z by $x+iy$ or by polar form $(re^{i\theta})$

$$\therefore f(x+iy) = \log(x+iy) + c$$
$$= \log(re^{i\theta}) + c$$
$$= \log r + \log e^{i\theta} + c$$
$$= \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x} + c$$

$$\therefore v = \tan^{-1}(y/x)$$

Type 3: When $u+v$ or $u-v$ is given.

$$f(z) = u+iv \Rightarrow if(z) = iu - v$$

$$\therefore (1+i)f(z) = (u-v) + i(u+v)$$
$$= U + iV, \text{ where } U = u-v$$
$$\& \ V = u+v$$

Case 1: If $u-v$ is given –
i.e. $(1+i)f(z)$'s real part is given.
$\therefore$ we can find imaginary part of $(1+i)f(z)$ & hence the fun^n $f(z)$

Case 2: If $u+v$ is given –
i.e. $(1+i)f(z)$'s imaginary part is given.

Q.1] Find analytic funⁿ $f(z) = u+iv$ in terms of z if
$u-v = (x-y)(x^2+4xy+y^2)$

⇒

Let $f(z) = u+iv \Rightarrow if(z) = -v+iu$

$$(1+i)f(z) = (u-v) + i(u+v)$$
$$= U + iV$$
where, $U = u-v$ & $V = u+v$

here, real part of $(1+i)f(z)$ is given
∴ we need to find imaginary part of $(1+i)f(z)$

$$(1+i)f(z) = U+iV$$
$$(1+i)f'(z) = U_x + iV_x$$
$$= U_x - iU_y$$

$$u-v = (x-y)(x^2+4xy+y^2)$$

$$U_x = (x-y)(2x+4y) + 1(x^2+4xy+y^2)$$
$$= 3x^2+6xy-3y^2$$

$$U_y = (x-y)(4x+2y) - 1(x^2+4xy+y^2)$$
$$= 3x^2-6xy-3y^2$$

By Milne's Thomson Method,
$$(1+i)f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$
$$= 3z^2 - i\,3z^2$$
$$= (1-i)3z^2$$
integrating both sides,
$$\Rightarrow (1+i)f(z) = (1-i)z^3 + c$$

H.W
Q.
Q.3]
⇒

$$f(z) = \frac{(1-i)}{(1+i)} z^3 + c' \quad \text{where} \quad c' = \frac{c}{(1+i)}$$

$$f(z) = \frac{(1-i)}{(1+i)} \times \frac{(1-i)}{(1-i)} z^3 + c'$$

$$= \frac{(1-i)^2 z^3}{2} + c'$$

$$= \frac{(1-2i-1)z^3}{2} + c'$$

$$= -iz^3 + c'$$

HW

Q. Find the analytic funⁿ $f(z) = u+iv$ where

$$u-v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$$

Q.3 Find the analytic funⁿ $f(z) = u+iv$ where.

$$u+v = e^{x}(\cos y + \sin y) + \frac{x-y}{x^2+y^2}$$

$\Rightarrow$

Let $f(z) = u+iv$

$\Rightarrow if(z) = -v+iu$

$$\therefore (1+i)f(z) = (u-v) + i(u+v)$$
$$= U + iV$$
where, $U = u-v$ & $V = u+v$

here, imaginary part of $(1+i)f(z)$ is given.
$\therefore$ we need to find the real part of $(1+i)f(z)$

$$(1+i)f(z) = U + iV$$
$$(1+i)f'(z) = U_x + iV_x$$
$$= V_y + iV_x$$

$$u+v = e^x(\cos y+\sin y) + \frac{x-y}{x^2+y^2}$$

$$V_x = e^x(\cos y+\sin y) + \frac{(x^2+y^2)(1)-(x-y)(2x)}{(x^2+y^2)^2}$$

$$= e^x(\cos y+\sin y) + \frac{2xy-x^2+y^2}{(x^2+y^2)^2}$$

$$V_y = e^x(\cos y-\sin y) + \frac{(x^2+y^2)(-1)-(x-y)(2y)}{(x^2+y^2)^2}$$

$$= e^x(\cos y-\sin y) + \frac{y^2-x^2-2xy}{(x^2+y^2)^2}$$

By Milne Thomson's Method,

$$\Rightarrow (1+i)f'(z) = \phi_1(z,0) + i\phi_2(z,0)$$

$$= e^z - \frac{z^2}{(z^2)^2} + i\left[e^z - \frac{z^2}{(z^2)^2}\right]$$

$$= e^z(1+i) - \frac{1}{z^2}(1+i)$$

By integrating,

$$\Rightarrow (1+i)f(z) = e^z(1+i) + \frac{1}{z}(1+i) + c$$

$$\Rightarrow (1+i)f(z) = \left(e^z+\frac{1}{z}\right)(1+i) + c$$

$$\Rightarrow f(z) = e^z + \frac{1}{z} + c' \quad \text{where } c' = \frac{c}{(1+i)}$$

H.W -

Q. Find the analytic funⁿ $f(z) = u+iv$ such that,

$$u+v = \frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$$

Ans $\Rightarrow$ $f(z) = \frac{i}{1+i}(\cot z) + c$.

REDMI NOTE 9
AI QUAD CAMERA

- Orthogonal Curves :-

If $f(z) = u + iv$ is an analytic funⁿ, then the curves $u = c_1$ and $v = c_2$ intersect orthogonally.

- Orthogonal Trajectories :-

A curve which cuts every member of the given family of curves at right angle is called an orthogonal trajectory of the given family of curves.

eg. Family of straight lines passing through the origin i.e. $y = mx$ is set of orthogonal trajectory for family of circles centered at origin i.e. $x^2 + y^2 = r^2$.

Note :- To find orthogonal trajectories $u = c_1$ (or $v = c_2$), we can find the harmonic conjugate $v = c_2$ (or $u = c_1$) of u (or v).

Q.1 Find the orthogonal trajectories of the family of curves $e^x \cos y - xy = c$.

$\Rightarrow$

Let $u = e^x \cos y - xy$ (such that $f(z) = u + iv$ is analytic)

$f(z) = u + iv$

$\Rightarrow f'(z) = u_x + iv_x$

$= u_x - iu_y$

$u_x = e^x \cos y - y$ $\qquad$ $u_y = -e^x \sin y - x$

Let $\phi_1(x,y) = u_x = e^x \cos y - y$
$\phi_2(x,y) = u_y = -e^x \sin y - x$

$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$
$= e^z + iz$

integrating we get,
$f(z) = e^z + i\frac{z^2}{2}$

replacing z by $x+iy$,
$f(x+iy) = e^{x+iy} + \frac{i}{2}(x+iy)^2$

$= e^x(\cos y + i\sin y) + \frac{i}{2}(x^2+2xyi-y^2)$

$= e^x(\cos y - xy) + i(e^x \sin y + \frac{x^2-y^2}{2})$

$\therefore$ Required orthogonal trajectories are -
$\Rightarrow e^x \sin y + \frac{x^2-y^2}{2} = c$

Q.7 For fun^n $f(z) = z^3$, verify that families of curves $u = c_1$, $v = c_2$ cut orthogonally where c_1 & c_2 are constants & $f(z) = u+iv$.

$\Rightarrow$

To prove that $\left(\frac{dy}{dx}\right)_{u=c_1} \times \left(\frac{dy}{dx}\right)_{v=c_2} = -1$

$m_1 = \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} = -\frac{u_x}{u_y}$ ... (I)

$m_2 = \frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} = -\frac{v_x}{v_y}$ ... (II)

To prove that, $m_1 m_2 = -1$

here, $f(z) = z^3$

$= (x+iy)^3$

$= x^3 - 3xy^2 + i(3x^2y - y^3)$

here, $u = x^3 - 3xy^2$ $\quad$ $v = 3x^2y - y^3$

$u_x = 3x^2 - 3y^2$ $\quad$ $v_x = 6xy - \cancel{3y^2}$

$u_y = -6xy$ $\quad$ $v_y = 3x^2 - 3y^2$

from (I), $m_1 = \frac{-u_x}{+u_y} = \frac{-(3x^2-3y^2)}{(-6xy)} = \frac{3x^2-3y^2}{6xy}$

from (II), $m_2 = \frac{-v_x}{v_y} = \frac{-(6xy)}{(3x^2-3y^2)} = \frac{-6xy}{3x^2-3y^2}$

And, $m_1 \cdot m_2 = \frac{(3x^2-3y^2)}{6xy} \times \frac{-6xy}{(3x^2-3y^2)} = -1$

$\therefore m_1 m_2 = -1$

Hence proved

$\Rightarrow$ u & v cut orthogonally.

Q.5] Find the orthogonal trajectories of the family of curves $e^{-x}\cos y + xy = c$

$\Rightarrow$

Let $u = e^{-x}\cos y + xy$ (such that $f(z) = u+iv$ is analytic)

$f(z) = u + iv$

$\Rightarrow f'(z) = u_x + iv_x$

$= u_x - iu_y$

here, $u_x = -e^{-x}\cos y + y$

$u_y = -e^{-x}\sin y + x$

Let $\phi_1(x,y) = u_x = -e^{-x}\cos y + y$

$\phi_2(x,y) = u_y = -e^{-x}\sin y + x$

$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

$= -e^{-z} - iz$

$= -1(e^{-z} + iz)$

integrating both sides,

$f(z) = e^{-z} - \frac{iz^2}{2} + c$

Replace z by $x+iy$,

$f(x+iy) = e^{-(x+iy)} - \frac{i}{2}(x+iy)^2 + c$

$= e^{-x}(\cos y - i\sin y) - \frac{i}{2}(x^2 - y^2 + 2ixy)$

$= (e^{-x}\cos y + xy) - i\left(e^{-x}\sin y + \frac{x^2-y^2}{2}\right)$

$\therefore$ requried orthogonal trajectories are -

$\therefore -e^{-x}\sin y - \frac{(x^2-y^2)}{2} = c_2$.

H.W

(pg.no. 100) Q. Find analytic fun^n $f(z) = u+iv$ where $u+v = \frac{2\sin 2x}{e^{2y}+e^{-2y} - 2\cos 2x}$

$\Rightarrow$

Let $f(z) = u+iv$ be required analytic fun^n.

$if(z) = iu - v$

$\therefore (1+i)f(z) = (u-v) + i(u+v)$

$= U + iV$

where, $U = u-v$ & $V = u+v$

here, imaginary part of $(1+i)f(z)$ is given.

$\therefore$ we need to find the real part of $(1+i)f(z)$.

$$(1+i)f(z) = U + iV$$

$$\therefore \quad (1+i)f'(z) = U_x + iV_x$$

$$= V_y + iV_x \quad \ldots \text{(I)}$$