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Spacecraft Guidance Techniques for Maximizing Mission Success

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SPACECRAFT GUIDANCE TECHNIQUES FOR MAXIMIZING MISSION SUCCESS

by

Shane B. Robinson

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Abstract

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Traditional spacecraft guidance techniques have the objective of deterministically minimizing fuel consumption. These traditional approaches to guidance are developed independently of the navigation system, and without regard to stochastic effects. This work presents and demonstrates a new approach to guidance design. This new approach seeks to maximize the probability of mission success by minimizing the variance of trajectory dispersions subject to a fuel consumption constraint. The fuel consumption constraint is imposed by formulating the dynamics in terms of a steering command, and placing a constraint on the final time. Stochastic quadratic synthesis is then used to solve for the nominal control along with the estimator and feedback gains. This new approach to guidance is demonstrated by solving a simple Zermelo boat problem. This example shows that a significant reduction in terminal dispersions is possible with small increases to fuel budgeted for the maneuver.

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Public Abstract

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Traditional spacecraft guidance techniques have the objective of deterministically minimizing fuel consumption. These traditional approaches to guidance are developed independently of the navigation system, and without regard to stochastic effects. This work presents and demonstrates a new approach to guidance design. This new approach seeks to maximize the probability of mission success subject to a total fuel consumption constraint. A novel mechanism for imposing the fuel consumption constraint on a stochastic system is presented. Once the fuel constraint is imposed techniques from stochastic linear system theory are used to find a solution to the problem. This new approach to guidance is demonstrated by solving a simple Zermelo boat problem. This example shows a significant reduction in terminal dispersions is possible with small increases to fuel budgeted for the maneuver.

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Shane B. Robinson

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Chapter 1

Anatomy of Guidance Navigation and Control Systems on Spacecraft and Rockets

1.1 Introduction

The field of spacecraft guidance, navigation, and control (GNC) engineering consists of the study, development, analysis, and implementation of systems which govern the trajectory of spacecraft. These systems range from the sophisticated systems onboard large manned spacecraft such as the space shuttle (SS) or the international space station (ISS) to relatively simple passive systems found on some small spacecraft. GNC systems are studied at all phases of development from conceptual design through the eventual implementation of the system.

The objective of this dissertation is to detail the development of guidance techniques that attempt to maximize the probability of mission success rather than other more traditional parameters of optimization, such as fuel consumption. When traditional parameters of optimization are used the resulting optimization problem is deterministic in nature. This research attempts to apply optimization techniques which include probabilistic effects to the guidance problem.

A GNC system is made of subsystems which perform individual tasks. The tight integration of the individual subsystems can lead to difficulty in identifying the precise boundaries of each subsystem. Despite this fact, design of the individual subsystems is surprisingly independent. For example, current navigation systems do not utilize knowledge of the intended flight path to inform information gathering. Similarly current guidance systems do not account for the statistical behavior of the navigations system when making guidance decisions. This resulting situation is one where the goals of the navigation and guidance systems are disparate. The navigation system is focused on extraction of information from sensors without much consideration as the future flight path. Meanwhile, the guidance system is focused on minimizing some deterministic measure along the flight path without regard to the quality of the navigation estimate. While these goals are not necessarily mutually exclusive, they are also not necessarily mutually supporting. This work attempts to unify the design of these two systems into a structure where these systems work toward a single goal and offer mutual support.

1.2 Overview

The archetype for GN&C systems used in this work is shown in figure 1.1. In this paradigm the system is divided into several subsystems. Each subsystem has a very specific task. The sensors produce imperfect measurements of the true trajectory. The navigation subsystem is tasked with determining the value of parameters and variables of interest associated with the system. The targeting system produces a set of goals and constraints for a maneuver, or set of maneuvers. The guidance subsystem produces force and torque commands that, if implemented, will cause the spacecraft to follow a trajectory which complies with the information provided by the targeting system. These commands are then converted into specific actuation commands by the control subsystem. The actuators then execute these commands in an imperfect fashion. The loop is closed when the navigation system detects the effect of the maneuver by processing the measurements provided by the sensors.

The model of a GN&C system that is used in this work is shown in figure 1.1. The main components of the GN&C systems are:

- Physical Environment - The physical environment consists of the hardware and surrounding environment that makes up the physical system, as well as the laws of nature that govern the behavior of these physical components.

- Laws of Nature - These are the laws which govern the dynamic behavior of the system. Although the laws of nature are not known perfectly, modern scientists and engineers have highly accurate models for these laws.

- Sensors - A sensor is any component that provides measurements related to the state of the spacecraft to the flight computer. Common sensors include accelerometers, gyros,

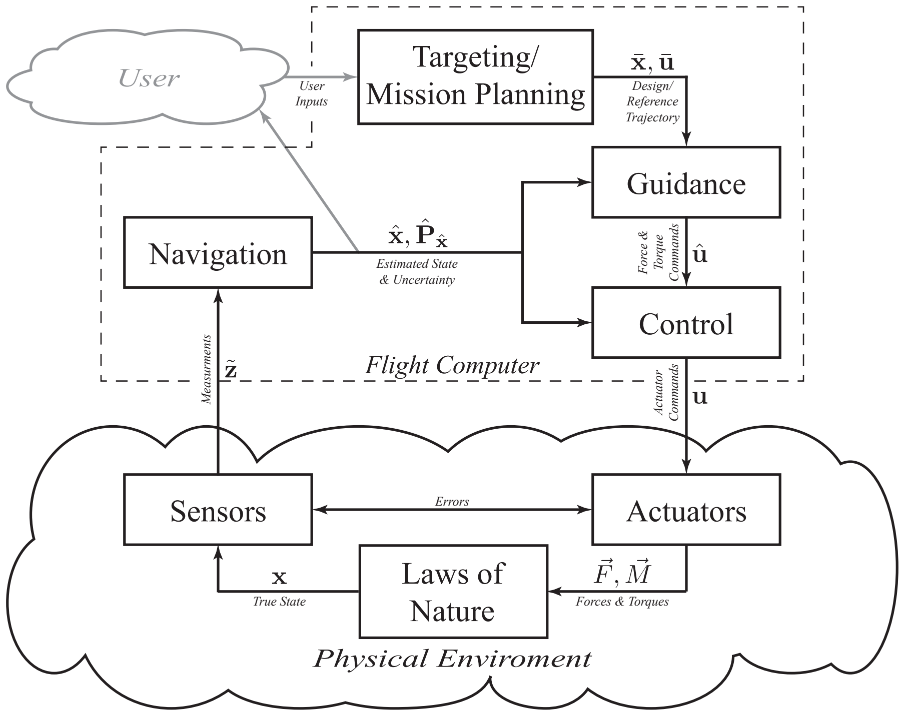

Fig. 1.1: The anatomy of a GN&C system.

radiometric devices that provide range and range-rate, and optical devices such as star cameras.

- Actuators - Actuators are devices used by the vehicle to interact with its environment. For spacecraft the most common actuators are rocket motors. Other common actuators are momentum wheels, control moment gyros (CMG), magnetic torquer-coils, and solar-sails.

- Flight Computer - The flight computer represents the software operating on-board the spacecraft.

- Navigation - The navigation system estimates the current state, or subset of the current state, of the spacecraft using the available measurements. Often, the navigation algorithm will also determine statistical information describing the confidence its the estimate.

- User - Nearly all space systems require the input of a human user. The user can either be a pilot on-board the spacecraft, or an Earth-based controller. Input from the user determines the final objectives of the mission or maneuver. The user might also impose constraints on the trajectory.

- Targeting/Mission Planning - The targeting system provides a set of targets and constraints that satisfy mission requirements.

- Guidance - The guidance system determines the forces and torques needed for the targets to be hit without violating constraints with the goal of optimizing some parameter of importance.

- Control - The control system is responsible for converting the desired force and torque commands coming from the guidance system to actuator commands.

1.3 GNC Subsystems

The targeting and mission planning subsystem selects the targets and constrains the maneuver and flight path to conform with mission requirements. In the past, this task has been the responsibility of human operators such as astronauts or ground controllers. As the level of autonomy continues to increase, the involvement of human operators in the process is decreasing.

In many situations, such a launch, these inputs are programed into the system a priori, and remain constant during the flight. For other situations, such as on orbital maneuvering, the targets are selected and uploaded to the vehicle during the mission. An example of this is the SS rendezvous procedure [1]. During rendezvous the shuttle make a series of maneuvers that successively bring the shuttle nearer the ISS. For each maneuver a new target and constraint set is selected which assures compliance with mission requirements. These targets are initially far away from the ISS, but progressively approach the ISS as successful maneuvers are executed.

The guidance system computes a trajectory which complies with mission constraints and targets. The guidance system then issues thrust commands to the vehicle, so that the vehicle follows the prescribed trajectory. As there are typically an infinite number of trajectories which comply with targets and other mission constraints, the guidance system selects a single trajectory by optimizing some metric of interest.

The navigation system uses receives measurements from the available sensors to produce an estimate of the state of the vehicle. Sometimes the navigation system does not estimate every state associated with the spacecraft, but only estimates certain states of interest. The navigation system is typically produces an estimate of the current state along with its statistical properties. A brief overview of the mathematics used to estimate the state can be found in the next chapter.

Finally, control refers to the manipulation of the forces, by way of steering controls, thrusters, etc., needed to track guidance commands while maintaining vehicle stability.

In this dissertation, targets and constraints are assumed to be proved as inputs to the system. The guidance system development, however, is central to this work and will be focused on in subsequent chapters. Navigation will be based on a relatively simple linearized Kalman filter and the control is assumed to be perfect, i.e., it is presumed that the guidance commands are executed perfectly.

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Chapter 2

Modern Spacecraft Navigation

The navigation system produces an estimate of the state, or subset of the state, of the spacecraft based on available information and measurements. This state estimate is produced by applying a mathematical construct referred to as an estimator. A simple estimator would naively convert the most recent measurement to the current state without regard for previous measurements, knowledge of the system behavior, or measurement errors. More sophisticated estimators take these factors (and associated statistics) into account. Often, these more sophisticated techniques are referred to as filters because they separate the signal from the noise and use only the useful information from measurements and obtain system knowledge.

Mathematical filters are analogous in many ways to mechanical filters used for cleansing of fluids. Both produce refined outputs when provided with corrupt inputs. In the case of a mechanical filter, physical contaminant are removed from the fluid. A mathematical filter performs a similar operation by using statistical information to extract a best possible estimate from a set of corrupt measurements and system information.

The most common filter used for in spacecraft navigation systems is the Kalman filter. Since it was first proposed by Rudolf E. Kalman [2], the value of Kalman filter has been demonstrated by robust development and application to nearly every field where data is extracted from measurements (and associated statistics) [3, 4]. The first application of the Kalman filter to space navigation occurred in late 1957 (3 years before Kalman's paper) by Peter Swerling [5–7]. However, the Kalman filter did not gain widespread acceptance in the new field of space navigation until the parallel work of Battin [8, 9] and Schmidt [10, 11]. Much of this early work surrounded the Apollo program and early ICBM development. Most notably, the Kalman filter was used as the navigation routine for the Apollo spacecraft [12]. Since this foundational work, the application of the Kalman filter has become ubiquitous within spacecraft navigation. The application of the Kalman filter to spacecraft navigation is the focus of numerous excellent texts [13–20]. The explanation of the Kalman filter provided in this document is intended to form a conceptual understanding of the filter, and is not intended to be mathematically rigorous. For more a more detailed development the reader is referred to any of the many excellent texts mentioned above.

The standard Kalman filter involves several key assumptions:

- Linear State Dynamics - The Kalman filter requires that the state dynamics be described by a system of first order linear differential equations.
- Linear Measurement Model - The measurements must be a linear function of the state variables.
- White Gaussian Noise - The noise in measurements and dynamics must be white gaussian noise. White noise [14, 17, 18] is not correlated in time. In other words the error at one time has absolutely no dependence on the error at any other time.

These assumptions can be very restrictive. Fortunately, space applications lend themselves well to these assumptions. For example, linearized models can describe the dynamics and measurements with surprising accuracy. Gaussian noise is also useful because often it accurately describes many processes in nature. The prevalence of Gaussian noise in processes found in nature is explained by Gauss' central limit theorem [14,17,18]. If needed, non-white (colored) noise can be included through the use of shaping filters (driven by white noise.)

2.1 Linear Batch Estimation

Before developing the Kalman filter, least-squares estimation will be explained. Since the true state, $\mathbf{x}$, is never known, a least squares estimator attempts to find an estimate of the state, $\hat{\mathbf{x}}$, by minimizing the norm[1] of the measurement error, $\mathbf{e}$. Minimizing the weighted norm, $|\mathbf{e}|_{\mathbf{W}}$, directly is difficult using analytic methods. However, an estimate that minimizes $|\mathbf{e}|_{\mathbf{W}}$ will also minimize $\frac{1}{2}\,|\mathbf{e}|^2_{\mathbf{W}}$. Minimization of $\frac{1}{2}\,|\mathbf{e}|^2_{\mathbf{W}}$ can be easily mechanized

[1] Here the weighted 2-norm or weighted Euclidean norm will be used. Moon [21] provides a discussion of estimating parameters using alternative norms.

using Gauss's principle of least squares [22]. Consider a vector of measurements $\tilde{\mathbf{z}}$ that are a linear mapping of the state estimate $\hat{\mathbf{x}}$ and the measurement error $\mathbf{e}$.

$$\tilde{\mathbf{z}} = \mathbf{H}\hat{\mathbf{x}} + \mathbf{e} \tag{2.1}$$

The sum of the errors squared can now be written as a function of the estimate.

$$J = \frac{1}{2}\left|\mathbf{e}\right|^2_{\mathbf{W}} = \frac{1}{2}\mathbf{e}^T\mathbf{W}\mathbf{e} = \frac{1}{2}\left(\tilde{\mathbf{z}}^T\mathbf{W}\tilde{\mathbf{z}} - 2\tilde{\mathbf{z}}^T\mathbf{W}\mathbf{H}\hat{\mathbf{x}} + \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{W}\mathbf{H}\hat{\mathbf{x}}\right) \tag{2.2}$$

where $\mathbf{W}$ is a symmetric positive definite weighting matrix. By setting the partial of J with respect to the estimate to zero, necessary conditions for an optimum can be established

$$\frac{\partial J}{\partial \hat{\mathbf{x}}} = \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{W}\mathbf{H} - \tilde{\mathbf{z}}^T\mathbf{W}\mathbf{H} = 0 \tag{2.3}$$

These necessary conditions can be rewritten as Gauss's famous normal[2]equations

$$\mathbf{H}^T\mathbf{W}\mathbf{H}\hat{\mathbf{x}} = \mathbf{H}^T\mathbf{W}\tilde{\mathbf{z}} \tag{2.4}$$

Sufficient conditions for a unique solution are provided when $\mathbf{H}^T\mathbf{W}\mathbf{H}$ is non-singular. This is equivalent to saying that the number of linearly independent measurements contained in $\tilde{\mathbf{z}}$ is greater than or equal to the number of elements in $\hat{\mathbf{x}}$. The solution to this equation produces the well known weighted pseudo-inverse of $\mathbf{H}$.

$$\hat{\mathbf{x}} = \left(\mathbf{H}^T\mathbf{W}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{W}\tilde{\mathbf{z}} \tag{2.5}$$

This provides a closed form expression for $\hat{\mathbf{x}}$. It should be noted that the required inverse is notoriously unstable from a numerical standpoint. As with all linear systems, it is therefore prudent to solve equation (2.4) directly [21] rather than attempt to naively compute the matrix inverse in equation (2.5).

[2]The label normal has reference to the deep geometric implications of these equations [15]. This equation requires that the estimation error $\mathbf{x} - \hat{\mathbf{x}}$ be orthogonal to range $\left(\mathbf{H}^T\right)$. This relationship is called the orthogonality principle. Moon [21] provides a insightful discussion of the orthogonality principle and its many consequences.

2.2 Linear Sequential Estimation

If a new measurement, $\tilde{\mathbf{z}}_n$, is added to the existing set of measurements, $\tilde{\mathbf{z}}$, the entire system of linear equations must be solved again. This process can be very computationally intensive, and as the number of measurements becomes large this process may not be feasible using computers available on-board the spacecraft. Thankfully, a recursive process has been developed to avoid the work of solving the entire system anew when additional measurements are added. To develop this begin by adding the new measurements to the system.

$$\begin{bmatrix} \tilde{\mathbf{z}} \\ \tilde{\mathbf{z}}_n \end{bmatrix} = \begin{bmatrix} \mathbf{H} \\ \mathbf{H}_n \end{bmatrix} \hat{\mathbf{x}} + \mathbf{e} \tag{2.6}$$

Presuming the weighting matrix for the norm of the error in this new system has a block diagonal structure, the least squares solution to this new system is given by

$$\hat{\mathbf{x}}_n = \left(\mathbf{H}^T\mathbf{W}\mathbf{H} + \mathbf{H}_n^T\mathbf{W}_n\mathbf{H}_n\right)^{-1}\left(\mathbf{H}^T\mathbf{W}\tilde{\mathbf{z}} + \mathbf{H}_n^T\mathbf{W}_n\tilde{\mathbf{z}}_n\right) \tag{2.7}$$

The sequential formulation works by leveraging the previously known inverse $\mathbf{P} = \left(\mathbf{H}^T\mathbf{W}\mathbf{H}\right)^{-1}$ to compute the new inverse $\mathbf{P}_n = \left(\mathbf{P} + \mathbf{H}_n^T\mathbf{W}_n\mathbf{H}_n\right)^{-1}$. The relationship between $\mathbf{P}$ and $\mathbf{P}_n$ is given by

$$\mathbf{P}^{-1} = \mathbf{P}_n^{-1} - \mathbf{H}_n^T\mathbf{W}_n\mathbf{H}_n \tag{2.8}$$

This can be substituted into equation 2.4

$$\left(\mathbf{P}_n^{-1} - \mathbf{H}_n^T\mathbf{W}_n\mathbf{H}_n\right)\hat{\mathbf{x}} = \mathbf{H}^T\mathbf{W}\tilde{\mathbf{z}} \tag{2.9}$$

This expression can now be used in 2.7

$$\hat{\mathbf{x}}_n = \mathbf{P}_n\left[\left(\mathbf{P}_n^{-1} - \mathbf{H}_n^T\mathbf{W}_n\mathbf{H}_n\right)\hat{\mathbf{x}} + \mathbf{H}_n^T\mathbf{W}_n\tilde{\mathbf{z}}_n\right] \tag{2.10}$$

which simplifies to

$$\hat{\mathbf{x}}_n = \hat{\mathbf{x}} + \underbrace{\mathbf{P}_n\mathbf{H}_n^T\mathbf{W}_n}_{\mathbf{K_n}}\left[\tilde{\mathbf{z}}_n - \mathbf{H}_n\hat{\mathbf{x}}\right] \tag{2.11}$$

This is known as the Kalman update equation, where $\mathbf{K}_n$ is the Kalman gain.

Repeated application of this technique yields a sequential linear estimation algorithm

$$\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_{i-1} + \mathbf{K}_i \left(\tilde{\mathbf{z}}_n - \mathbf{H}_i \hat{\mathbf{x}}_{i-1} \right) \tag{2.12}$$

where the Kalman gain is given by

$$\mathbf{K}_i = \mathbf{P}_i \mathbf{H}_i^T \mathbf{W}_i \tag{2.13}$$

with

$$\mathbf{P}_i^{-1} = \mathbf{P}_{i-1}^{-1} + \mathbf{H}_i^T \mathbf{W}_i \mathbf{H}_i \tag{2.14}$$

This set of equations is often referred to as the *information formulation* of the sequential linear estimator.

The matrix inversion lemma[3] can be used to rapidly compute the inverse with minimal computational effort. The matrix inversion lemma is applied to equation 2.14 to develop a more computationally efficient algorithm

$$\mathbf{P}_i = \mathbf{P}_{i-1} - \mathbf{P}_{i-1} \mathbf{H}_i^T \left(\mathbf{H}_i \mathbf{P}_{i-1} \mathbf{H}_i^T + \mathbf{W}_i^{-1} \right)^{-1} \mathbf{H}_i \mathbf{P}_{i-1} \tag{2.15}$$

This expression can now be substituted into the expression for the Kalman gain

$$\mathbf{K}_i = \mathbf{P}_{i-1} \mathbf{H}_i^T \left[\mathbf{I} - \left(\mathbf{H}_i \mathbf{P}_{i-1} \mathbf{H}_i^T + \mathbf{W}_i^{-1} \right)^{-1} \mathbf{H}_i \mathbf{P}_{i-1} \mathbf{H}_i^T \right] \mathbf{W}_i \tag{2.16}$$

which simplifies to

$$\mathbf{K}_i = \mathbf{P}_{i-1} \mathbf{H}_i^T \left(\mathbf{H}_i \mathbf{P}_{i-1} \mathbf{H}_i^T + \mathbf{W}_i^{-1} \right)^{-1} \tag{2.17}$$

[3] The Sherman-Morrison-Woodbury matrix inversion lemma, often referred to simply as the the matrix inversion lemma, states, in part, the following

$$\left(\mathbf{A}_{n\times n} + \mathbf{B}_{n\times m} \mathbf{C}_{m\times m} \mathbf{D}_{m\times n} \right)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1} \mathbf{B} \left(\mathbf{D} \mathbf{A}^{-1} \mathbf{B} + \mathbf{C}^{-1} \right)^{-1} \mathbf{D} \mathbf{A}^{-1}$$

Of course this is only valid when $\mathbf{A}^{-1}$ and $\mathbf{C}^{-1}$ exist.

This result can be used to simplify equation 2.15

$$\mathbf{P}_i = \left(\mathbf{I} - \mathbf{K}_i\mathbf{H}_i\right)\mathbf{P}_{i-1} \tag{2.18}$$

Equations 2.12, 2.17, and 2.18 together form the *covariance formulation* of the sequential least squares estimation algorithm.

Here it should be noted that equation 2.18 is known to suffer from numerical difficulties after a large number of successive applications of the recursion. This difficulty can be overcome by using the Joseph form [23] of the same equation

$$\mathbf{P}_i = \left(\mathbf{I} - \mathbf{K}_i\mathbf{H}_i\right)\mathbf{P}_{i-1}\left(\mathbf{I} - \mathbf{K}_i\mathbf{H}_i\right)^T + \mathbf{K}_i\mathbf{W}_i\mathbf{K}_i^T \tag{2.19}$$

which is known to be much more numerically stable [14, 15].

2.3 Kalman Filter

The Kalman filter applies the ideas of recursive linear estimation to linear dynamic systems where the states change over time according to the equation

$$\dot{\mathbf{x}}\left(t\right) = \mathbf{F}\left(t\right)\mathbf{x}\left(t\right) + \mathbf{B}\left(t\right)\mathbf{u}\left(t\right) + \mathbf{G}\left(t\right)\mathbf{w}\left(t\right) \tag{2.20}$$

where $\mathbf{w}\left(t\right)$ is a continuous Gaussian white noise process denoted $\mathscr{N}\left(\mathbf{0}, \mathbf{Q}\left(t\right)\delta\left(t\right)\right)$ or zero mean with a power spectral density $\mathbf{Q}\left(t\right)$. The measurements are taken at discrete times k

$$\tilde{\mathbf{z}} = \mathbf{H}_k\mathbf{x}\left(t_k\right) + \mathbf{v}_k \tag{2.21}$$

where $\mathbf{v}_k$ is a discrete Gaussian white noise denoted $\mathscr{N}\left(\mathbf{0}, \mathbf{R}_k\right)$. Here is assumed that $\mathbf{w}\left(t\right)$ is not correlated with $\mathbf{v}_k$.

In the Kalman filter, the state estimate is propagated using the system dynamics and measurement model

$$\dot{\hat{\mathbf{x}}}\left(t\right) = \mathbf{F}\left(t\right)\hat{\mathbf{x}}\left(t\right) + \mathbf{B}\left(t\right)\mathbf{u}\left(t\right) \tag{2.22}$$

$$\hat{\mathbf{z}}_k = \mathbf{H}_k\hat{\mathbf{x}}\left(t_k\right) \tag{2.23}$$

where the error in the estimate is given by

$$\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t) \tag{2.24}$$

Time differentiation is used to develop a differential equation for the state estimation error

$$\dot{\mathbf{e}}(t) = \mathbf{F}(t)\,\mathbf{e}(t) - \mathbf{G}(t)\,\mathbf{w}(t) \tag{2.25}$$

Note that the control is canceled from the equation. Formally,[4] the solution to this differential equation is given by

$$\mathbf{e}(t) = \mathbf{\Phi}(t, t_0)\,\mathbf{e}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\,\mathbf{G}(\tau)\,\mathbf{w}(\tau)\,d\tau \tag{2.26}$$

The statistics of the estimation error can now be found using the expectation operator

$$E[\mathbf{e}(t)] = \mathbf{\Phi}(t, t_0)\,E[\mathbf{e}(t_0)] \tag{2.27}$$

Therefore, if the state estimate is initialized with an unbiased estimate, the estimation error continues to be unbiased. The state error covariance is also found using the expectation operator and assuming that $\delta\mathbf{x}(t)$ and $\mathbf{w}(t)$ are uncorrected.

$$\mathbf{P}(t) = E\left[\mathbf{e}(t)\,\mathbf{e}^T(t)\right] = \mathbf{\Phi}(t, t_0)\underbrace{E\left[\mathbf{e}(t_0)\,\mathbf{e}^T(t_0)\right]}_{\mathbf{P}(t_0)}\mathbf{\Phi}^T(t, t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\,\mathbf{G}(\tau)\,\mathbf{Q}(t)\,\mathbf{G}^T(\tau)\,\mathbf{\Phi}^T(t, \tau)\,d\tau \tag{2.28}$$

[4]For a more mathematically rigorous presentation of the solution to stochastic differential equations the reader is referred to [14, 18, 24] where the required stochastic integrals are evaluated using Itō calculus.

This equation can now be differentiated using the Leibniz integral rule[5] and the properties of the state transition matrix[6]

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\,\mathbf{\Phi}(t,t_0)\,\mathbf{P}(t_0)\,\mathbf{\Phi}^T(t,t_0) + \mathbf{\Phi}(t,t_0)\,\mathbf{P}(t_0)\,\mathbf{\Phi}^T(t,t_0)\,\mathbf{F}^T(t) + \mathbf{G}(\tau)\,\mathbf{Q}(t)\,\mathbf{G}^T(\tau)\,\mathbf{\Phi}^T + \mathbf{F}(t)\int_{t_0}^{t}\mathbf{\Phi}(t,\tau)\,\mathbf{G}(\tau)\,\mathbf{Q}(t)\,\mathbf{G}^T(\tau)\,\mathbf{\Phi}^T(t,\tau)\,d\tau + \int_{t_0}^{t}\mathbf{\Phi}(t,\tau)\,\mathbf{G}(\tau)\,\mathbf{Q}(t)\,\mathbf{G}^T(\tau)\,\mathbf{\Phi}^T(t,\tau)\,d\tau\,\mathbf{F}(t) \quad (2.29)$$

Using equation 2.28 to replace the integrals results in the continuous time Riccati equation

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{F}^T(t) + \mathbf{G}(t)\,\mathbf{Q}(t)\,\mathbf{G}^T(t) \quad (2.30)$$

Between measurements, this equation can be used to propagate the covariance of the state estimation error. When a measurement is encountered, the propagation halts and the estimate is updated in the same way it was updated during linear sequential estimation. Thus the Kalman filter equations can be summarized as follows:

Between measurements propagate the estimate and the error covariance by integrating the appropriate differential equations

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\,\hat{\mathbf{x}}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \quad (2.31)$$

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{F}^T(t) + \mathbf{G}(t)\,\mathbf{Q}(t)\,\mathbf{G}^T(t) \quad (2.32)$$

When a measurement is provided, update the estimate and covariance using the update equations

$$\hat{\mathbf{x}}^+(t_i) = \hat{\mathbf{x}}^-(t_i) + \mathbf{K}_i\left(\tilde{\mathbf{z}}_i - \mathbf{H}_i\hat{\mathbf{x}}^-(t_i)\right) \quad (2.33)$$

[5] The Leibniz integral rule [25] is given by

$$\frac{d}{dt}\int_{f(t)}^{g(t)}\mathbf{A}(t,\sigma)\,d\sigma = \mathbf{A}(t,g(t))\,\dot{g}(t) - \mathbf{A}(t,f(t))\,\dot{f}(t) + \int_{f(t)}^{g(t)}\frac{\partial\mathbf{A}(t,\sigma)}{\partial t}d\sigma$$

[6] The following properties of the state transition matrix are helpful [25]

$$\mathbf{\Phi}(t_0,t_0) = \mathbf{I} \qquad \frac{\partial\mathbf{\Phi}(t,t_0)}{\partial t} = \mathbf{F}(t)\,\mathbf{\Phi}(t,t_0) \qquad \frac{\partial\mathbf{\Phi}(t,t_0)}{\partial t_0} = -\mathbf{\Phi}(t,t_0)\,\mathbf{F}(t)$$

$$\mathbf{K}_i = \mathbf{P}^-(t_i)\,\mathbf{H}_i^T\left[\mathbf{H}_i\mathbf{P}^-(t_i)\,\mathbf{H}_i^T + \mathbf{R}_i^{-1}\right]^{-1} \tag{2.34}$$

$$\mathbf{P}^+(t_i) = (\mathbf{I} - \mathbf{K}_i\mathbf{H}_i)\,\mathbf{P}^-(t_i)\,(\mathbf{I} - \mathbf{K}_i\mathbf{H}_i)^T + \mathbf{K}_i\mathbf{R}_i\mathbf{K}_i^T \tag{2.35}$$

2.4 The Linearized Kalman Filter

The standard Kalman filter requires that the system dynamics be linear, however the Kalman filter is often applied to nonlinear systems

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t) + \mathbf{G}(\tau)\,\mathbf{w}(t) \tag{2.36}$$

$$\tilde{\mathbf{z}}_i = h_i(\mathbf{x}, t) + \mathbf{v}_i \tag{2.37}$$

By linearizing the system about a nominal, $\bar{\mathbf{x}}$, $\bar{\mathbf{u}}$, the state dynamics are now expressed as dispersions from the nominal

$$\delta\mathbf{x} = \mathbf{x} - \bar{\mathbf{x}} \tag{2.38}$$

$$\delta\hat{\mathbf{x}} = \hat{\mathbf{x}} - \bar{\mathbf{x}} \tag{2.39}$$

$$\mathbf{e} = \delta\hat{\mathbf{x}} - \delta\mathbf{x} = \hat{\mathbf{x}} - \mathbf{x} \tag{2.40}$$

$$\delta\mathbf{u} = \mathbf{u} - \bar{\mathbf{u}} \tag{2.41}$$

The dispersed states and the error covariance are now propagated by linear dynamics

$$\delta\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\,\delta\hat{\mathbf{x}}(t) + \mathbf{B}(t)\,\delta\mathbf{u}(t) \tag{2.42}$$

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{F}^T(t) + \mathbf{G}(t)\,\mathbf{Q}(t)\,\mathbf{G}^T(t) \tag{2.43}$$

where

$$\mathbf{F}(t) = \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}, \bar{\mathbf{u}}, t} \tag{2.44}$$

$$\mathbf{B}(t) = \left.\frac{\partial f}{\partial \mathbf{u}}\right|_{\bar{\mathbf{x}}, \bar{\mathbf{u}}, t} \tag{2.45}$$

Similarly, the update is executed by linearizing the measurement model about the nominal

$$\delta\hat{\mathbf{x}}^{+}\left(t_i\right) = \delta\hat{\mathbf{x}}^{-}\left(t_i\right) + \mathbf{K}_i\left(\tilde{\mathbf{z}}_i - \mathbf{H}_i\delta\hat{\mathbf{x}}^{-}\left(t_i\right)\right) \tag{2.46}$$

$$\mathbf{K}_i = \mathbf{P}^{-}\left(t_i\right)\mathbf{H}_i^T\left[\mathbf{H}_i\mathbf{P}^{-}\left(t_i\right)\mathbf{H}_i^T + \mathbf{R}_i^{-1}\right]^{-1} \tag{2.47}$$

$$\mathbf{P}^{+}\left(t_i\right) = \left(\mathbf{I} - \mathbf{K}_i\mathbf{H}_i\right)\mathbf{P}^{-}\left(t_i\right)\left(\mathbf{I} - \mathbf{K}_i\mathbf{H}_i\right)^T + \mathbf{K}_i\mathbf{R}_i\mathbf{K}_i^T \tag{2.48}$$

where

$$\mathbf{H}_i = \left.\frac{\partial h_i}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}},t} \tag{2.49}$$

The linearized Kalman filter works well as long as the state is close to the nominal. For situations where the linearized Kalman filter does not work well, other approaches such as extended Kalman filters, unscented filters, or particle filters may be used. For this work the linearized Kalman filter will be used.

Chapter 3

Modern Spacecraft Guidance

This chapter presents a review of the development of modern spacecraft guidance. First some brief comments are made regarding the infancy and initial development of spacecraft guidance. Next, a survey of deterministic guidance approaches is given. Finally, a discussion of stochastic guidance approaches are presented.

3.1 Historical Roots

This section is intended to give a brief history of the ideas that shaped modern spacecraft guidance. For a detailed historical narrative regarding the events leading up to spaceflight the reader is referred to Ehricke [26]. Here only the details pertinent to the development of guidance systems are presented.

The basic idea of steering a vessel through the sky to reach a destination or target, is an ancient idea. It is impossible to say exactly where or when this idea was first proposed. Perhaps the desire to explore the heavens is rooted in the deep connection between the motion of heavily bodies and daily life. The celestial cycles are central to nearly every aspect of human life. Most notably, the periodic motions found in the sun-earth-moon system are so fundamental to our lives that they define our basic reckoning of time. As the ability to predict seasonal changes is key to agrarian success, it is reasonable to presume that humankind's first serious study of the heavenly motions was primarily a pragmatic concern. It is equally reasonable to believe that the first human investigations of celestial motion were motivated by the majesty of starry sky, and the spiritual connection[1] one feels when observing it. However, humankind's inquisitive nature transformed a purely pragmatic or spiritual concern into astronomy, the first true science. This in turn drove the development of applied mathematics.

[1]Immanuel Kant (1724 - 1804) is credited with saying, "There are two things which fill my mind with never-ending awe: the starred sky above us and the moral law within us."

Surprisingly sophisticated astronomy occurred in many ancient cultures throughout the world. It should come as no surprise that the earliest recorded ideas regarding spaceflight come from the ancient Greeks. Ovid recounts the story of Phaëton [27] which shows the tragic consequences of failed guidance. In this story Phaëton's loses control of the sun chariot while trying to guide it across the sky. Zeus must then destroy Phaëton to prevent the destruction of the Earth. These mythological ideas quickly gave way to Lucian's [28] tale of a ship which travels to the moon.

Newton's Principia moved the idea of space flight from the realms of mythology, and placed the notion of space flight on firm scientific footing. Although rockets had existed in China for centuries, Newton [29] suggested that they be used for traveling through empty space. Together with this idea, Newton endowed the world with his three laws of motion, his inverse-square law of gravity, and the mathematics of calculus. Newton also showed that his laws could by used to explain Kelpler's empirical laws governing the motion of celestial bodies. The magnitude of this advancement cannot be overstated. This theoretical framework laid out by Newton continues to underpin the dynamics of modern spaceflight.

Although Newton laid out many of the physical laws and mathematical tools needed, the many problems of building and operating a spacecraft remained unsolved. The insurmountable technical challenges of building a spacecraft assured that no reputable scientists seriously pursued the idea of actually building and flying a space ship. Not long after Newton, Voltaire [30] described the basic guidance problem thusly: "Our voyager was very familiar with the laws of gravity and with all the other attractive and repulsive forces. He utilized them so well that, whether with the help of a ray of sunlight or some comet, he jumped from globe to globe like a bird vaulting itself from branch to branch." Thus, from the time of Newton to the beginning of the 20th century the topic of spaceflight was the preview of fiction. Famous writers such as Wells [31], Tolstoy, and Jules Vern [32] penned famous science fiction vividly describing what spaceflight might be like. These spaceflight fantasies did little to advance the technical state-of-the-art beyond Newton, but served greatly to popularize the idea of space flight and provide an air of the practical to what had previously only been theoretical.

The spacecraft guidance problem was next examined by the great father figures of modern rocketry. The great Russian spaceflight pioneer Konstantin Tsiolkovsky [33] realized that automatic guidance would relieve the crew of the task of steering the rocket, and was a necessity especially during accent to orbit. Tsiolkovsky's guidance scheme involved projecting a picture of the sun onto a sphere inside the spacecraft. The vehicle would then track the sun in a so-call "sun-flower" guidance technique. In the same work Tsiolkovsky presented what he called the "formula of aviation." This fundamental relation is now known as the "Tsiolkovsky rocket equation"

$$\Delta V = V_{\text{exit}} \ln\left(\frac{m_i}{m_f}\right) \tag{3.1}$$

or

$$\frac{\Delta m}{m_i} = 1 - \exp\left(-\frac{\Delta V}{V_{\text{exit}}}\right) \tag{3.2}$$

where ΔV is the change in velocity, V_{exit} is the exit velocity of the exhaust, and $m_i - m_f = \Delta m$ is the change is mass from the initial to the final states.

Although Tsiolkovsky never tied this law to his guidance scheme, this equation became central to the development of subsequent guidance laws. The rocket equation explains the exponential relation between the change in velocity and the change in mass. Note that as the change in velocity becomes large, the change in mass approaches the total value of the initial mass leaving very little mass for a payload. That is, as $\Delta V \to \infty$ then $\Delta m \to m_i$. This equation explains why the overwhelming fraction of the mass of a modern rocket is fuel which is expelled before the spacecraft reaches orbit. In practice, the rocket equation establishes a relationship between mission requirements (ΔV), and spacecraft capability (Δm). Thus, by making Δm as small as possible, the capability of the rocket is increased, and a more capable rocket can perform a wider range of missions. Despite serving no role in Tsiolkovsky's sun-flower guidance, the idea of minimizing Δm is the main motivating factor in the development of nearly all modern guidance laws. In other words, nearly all modern guidance laws seek to steer the spacecraft in such a way that the use of propellant is minimized.

Independent of Tsiolkovsky, the american Robert H. Goddard proposed his method [34,35] of reaching extreme altitudes with a sounding rocket. The Goddard problem[2] (maximizing the altitude of a sounding rocket) was the first example of using a guidance law (thrust program) to maximize some quantity of interest. Goddard realized that a guidance law should be selected based on an optimization problem. That is, a guidance law should select the best possible trajectory from the set of admissible trajectories.

The Romanian engineer Hermann Oberth also realized the value of selecting the best possible trajectory based on some criteria. In his book [42] he proposes a "synergic ascent" to orbit. Oberth's synergic ascent demanded minimizing the accent energy by deflecting the trajectory from the vertical toward the horizontal and passing through the atmosphere at an optimum velocity.

Walter Hohmann, a german engineer, was able to show [43] that a cotangential ellipse is the most economical (in terms of ΔV) means of transferring between coplanar circular orbits. Hohmann also realized that these cotangential transfer orbits required significant time to execute. To remedy this issue, he also investigated "fast" transfer ellipses which intersected the circular orbits at an angle. Hohmann's trajectory optimization work focused on impulsive maneuvers that do not require steering laws due to their impulsive nature. Nevertheless, the motivation used by Hohmann is the same motivation that is frequently used when finding steering laws for finite maneuvers.

Together, Goddard and Oberth canonized the term "optimal" to mean energy efficient in the context of orbital ascent trajectories. Similarly, Hohmann's work served to make the term "optimal" synonymous with most economical in the context of of space maneuvers. This occurrence was natural, as cost and capability are the primary drivers when building systems and performing conceptual trade studies. It will be shown in the following section that these ideas has a powerful influence over the development of modern guidance laws.

All of the work mentioned above happened prior to the 1930s. The spaceflight enthusiasts who engaged in this work were supported by relatively small grants and limited private support. During this time, the success of the few experiments that had been conducted, led

[2] The Goddard problem has since been extensively studied, and possesses a variety of solutions for a variety of situations [36–41].

to the common belief that the development of the spaceship was only a few years away[3] [26]. Vernher von Braun was the first to realize the futility of these efforts. It was von Braun who understood that the funding and resources to develop a spacecraft where so great that only with the aid of a large government would anyone be capable of actually building a spaceship. In 1932 Dr. Walter Dornberger, a young artillery captain in the German army, recruited von Braun to help build rocket weapons for the German army [44]. The German army was interested in building long range rocket weapons, as they were one of the few weapons not prohibited by the Versailles Treaty of 1919. Previous to this point, unguided rockets such as the Congreve and Hale rockets had found military use, but were notoriously inaccurate [45]. Thus the only country to seriously consider development of a rocket at this time was Germany. With the support of the German army and the passion of von Braun, the development of the guided spacecraft became synonymous with the development of the first long-range guided missile, the V-2.

The first two development prototypes (Model A-2) were launched in 1934 and reached an altitude of 6500 ft. [26, 44]. These rockets were stabilized by a heavy gyroscope in the nose of the rocket. Owing to the weight of the gyroscope, no payload could be carried. From these initial prototypes, von Braun learned of the central role the guidance, navigation, and control would play in spaceflight. Von Braun realized that in order to hit its target, the missile must follow a prescribed trajectory during powered flight with greater accuracy than even a human pilot could provide.

In America, Robert Goddard was also working on developing a rocket which would follow a pitch program. Goddard's system provided steering by using gyroscopes to control exhaust vanes. In the spring of 1935 he successfully flew rockets which transitioned from vertical to horizontal flight. Goddard's rockets were build on a much more limited scale than those built by the military-funded German team. Goddard was successful in reaching an altitude of 4800 feet and covering a distance of 2.5 miles [46].

In 1937 the missile program was moved to Peenemuende and work on the first missile with an active guidance system (Model A-4) began [44]. Von Braun placed such emphasis on

[3] Ehricke [26] made the following comment regarding gross underestimation of the cost and effort needed to build a functional spacecraft: "In fact, those familiar with modern missile projects know that underestimation of schedules and budgets is not restricted to the period [of the 1930s]."

the development of the guidance, navigation, and control system that he personally oversaw the development of a guidance system for V-2. The V-2 had to follow a carefully optimized pitch profile that would rotate the axis of the vehicle from the vertical to an angle of 43° with respect to the horizontal. The orientation of the vehicle was sensed by two small gyroscopes. Once the orientation of the vehicle was determined, the V-2 was steered by using graphite exhaust veins. In addition, aerodynamic control surfaces were used in the transonic regions of the flight. A ground signal, a so called "guidance beam," was used to help control lateral dispersions of the rocket. Finally, the cutoff command was signaled from the ground based on Doppler measurements of the missile's velocity. This primitive guidance system signaled the birth of modern launch vehicle and spacecraft guidance.

In the ensuing years the field of space vehicle guidance rapidly progressed. The initial guidance of the V-2 rocket was focused on following a predetermined set of commands, augmented with input from the ground. After the end of World War II, von Braun and his team began work on rockets for the United States government at the Red Stone arsenal in Huntsville Alabama.[4] Although the sophistication of these rockets greatly increased, the basic motive of maximizing capability (range and payload) of the rocket subject to other mission requirements (such as accuracy) remained central to guidance development. Thus, the steering laws used to guide the vehicles were derived heavily from mission planning and conceptual design tools. A variety of these guidance laws will be discussed in the following sections, and in most cases it is apparent that the objective of the guidance law is to conserve fuel, and thus maximize the capability of the rocket.

Derek F. Lawden, who made significant early contributions the mathematics of trajectory optimization, made the following comment regarding the use of fuel as the optimization parameter, "[There] is the problem of computing the most convenient trajectory to be followed by a space shipm [sic] undertaking a particular interplanetary journey. Among the

[4]It is interesting to note, after World War II the federal government awarded a $25,000 contract to the famous father of behavior sciences, B. F. Skinner. Skinner had developed an "organic homing device" with the support of the General Mills Company. The "organic homing device" consisted of a sacrificial pigeon which was mounted in the nose of the missile. Missile steering commands were then provided by the pigeon who was mounted in a harness and was looking out a small window. Skinner later described the review and cancellation of his pigeon guidance by a defense committee saying, "By this time we had begun to realize that a pigeon was more easily controlled than a physical scientist on a committee. It was very difficult to convince the latter that the former was an orderly system" [47]. Needless to say, Skinners guidance scheme was passed by in favor of systems more amenable to mathematic analysis.

various possible trajectories satisfying the conditions imposed by such factors as the maximum motor thrust available, the maximum allowable time for the journey, the minimum distance of approach to the sun and so on, the most satisfactory trajectory will clearly be that requiring the least expenditure of fuel. *In the early stages, it will be necessary to relax all conditions to the greatest degree possible in order to achieve fuel economy, while looking forward hopefully to the day when atomic drives become a reality and the necessity for fuel economy becomes less pressing.* In such circumstances, the calculation of trajectories of least fuel expenditure will have to be carried out under a variety of conditions imposed by such considerations as time schedules and the safety of passengers. For example, it may be a requirement that certain orbits be avoided because of their passing through regions of high meteor or cosmic ray intensity" (emphasis added) [48]. This initial motivation to optimized trajectories to minimize fuel consumption has continued unchanged to present day.

Although the precise reasons that guidance laws were developed from fuel optimal mission planning and conceptual design are unknown, it was a natural occurrence for several reasons. First, fundamentally guidance involves selecting a single "best" solution from a family of admissible solutions to the boundary value problem. In mission planning and conceptual design tools "best" was taken to mean economical (fuel savings). It was natural for the inertia of this definition of optimal to carryover into guidance development. Second, when the need for a guidance law developed, mission planning and conceptual design tools were already well developed. These tools were understood and trusted by the technical community. Third, the mathematical framework available at the time (calculus of variations) was ideal for optimizing with respect to control effort. Optimizing with respect to fuel consumption resulted in simple and elegant solutions that could be implemented on primitive flight computers. Optimizing with respect to other variables is often considerably more difficult. Regardless of the reasons, modern spacecraft guidance is a descendent of these mission planning and conceptual design tools. Presently, in the context of spacecraft guidance the term optimal, continues to be synonymous with minimizing fuel.

3.2 Deterministic Guidance Schemes

This section contains a summary of some common modern spacecraft guidance laws which are deterministic in nature. That is to say, the laws in this section make no attempt to account for the stochastic nature of the system. Although the motivation for these laws is discussed, the details of development and performance analysis are not presented here. Citations to relevant literature are provided for the reader who wishes to explore the details of each law.

Although the details vary, spacecraft guidance has traditionally been a framed as the deterministic solution which optimizes a boundary value problem with respect to fuel consumption. The techniques outlined in this section each work within this paradigm.

As has been noted previously, these feedback laws are rooted in the theory of trajectory optimization. Lawden presents an excellent history of early trajectory optimization work [49]. The modern concept of trajectory optimization was first explored by Oberth [42], Hohmann [43], and Goddard [34, 35]. Foundational work in trajectory optimization was performed by Lawden who presented his theory of primer vectors [50]. Lion also performed work on primer vector theory [51, 52]. These results verified Hohmann's proposal of an impulsive cotangential transfer ellipses being an optimal solution to the problem of transferring between two circular orbits. It is important to note that primer vector theory shows impulsive trajectories are fuel optimal solutions for a gravitation field when no dissipative forces such as friction or aerodynamic forces are included in the analysis. Thus, for space applications the fuel optimal trajectory is always a series of impulsive burns modeled by the Dirac delta function

$$\vec{a}_T = \overrightarrow{\Delta V}\delta\left(t - t_{\text{burn}}\right) \tag{3.3}$$

Note that, in this application,[5] the Dirac delta function has the units $1/\text{sec}$, such that

$$\int_{-\infty}^{\infty} \vec{a}_T dt = \overrightarrow{\Delta V} \tag{3.4}$$

Clearly, burns prescribed by primer vector theory are not realizable due to their infinite magnitude. In many cases, the acceleration loading is limited not only by finite capacity,

[5] The Dirac delta function is defined to have the same units as the inverse of its argument.

but also by a need to avoid injury to crew, payload, or vehicle. It is important to keep in mind, that any exoatmospheric maneuver that contains finite burns is *not* truly fuel optimal. As will be seen in the following sections, guidance (steering) laws provide a mechanism for performing finite burns which achieve the same result as an impulsive trajectory, while carefully optimizing the fuel consumed for the finite burn.

3.2.1 Fly-the-Wire and Explicit Guidance

Broadly speaking there are two guidance paradigms [53, 54]. The first paradigm, is called "fly-the-wire" guidance. Fly-the-wire guidance works by computing a trajectory before launch. This a priori trajectory is the "wire." The vehicle follows this trajectory in both time and space to arrive at the target. When the vehicle deviates from the wire, the GN&C system is tasked with detecting the deviation from the precomputed trajectory and issuing commands that will return the vehicle to the precomputed trajectory. Fly-the-wire systems are believed to be the guidance paradigm employed by the majority of the Soviet ICBMs. Fly-the-wire systems require substantial calculations to be done before launch, but only require minimal calculation to be done during flight. For example, some Soviet systems require that the system be programmed with fifty to one hundred constants prior to launch. These same soviet systems oftentimes use only analog computers for calculations during flight. Fly-the-wire systems also require that the propulsion system be throttle-able, which is one of the reasons that the Soviet ICBM fleet uses liquid chemical propulsion.

The second guidance paradigm is call "explicit" guidance. In the explicit guidance paradigm the flight computer continually estimates its current state, and then calculates the required steering commands required to hit the target. Guidance systems for missiles in the US ICBM fleet generally tend toward the explicit guidance paradigm. Explicit guidance greatly reduces the need for detailed calculation of a trajectory prior to launch, but greatly increases the computational burden of calculations performed on the vehicle during flight. With the exception of delta guidance, the guidance schemes in this document generally tend toward the explicit guidance paradigm.

3.2.2 Delta Guidance

Delta guidance is a fly the wire technique based on a Taylor expansion. Begin by writing the thrust command, $\vec{F}$, as a function of the position, $\vec{r}$, velocity, $\vec{v}$, and the target time, t. The Taylor series for this function will then reveal the guidance law.

$$\vec{F} \approx \overline{\vec{F}}_0 + \begin{bmatrix} \frac{\partial \vec{V}_R}{\partial \vec{r}} & \frac{\partial \vec{V}_R}{\partial \vec{v}} & \frac{\partial \vec{V}_R}{\partial t} \end{bmatrix} \begin{bmatrix} \delta \vec{r} \\ \delta \vec{v} \\ \delta t \end{bmatrix} + \ldots \tag{3.5}$$

The quantities $\overline{\vec{F}}_0$, $\frac{\partial \vec{V}_R}{\partial \vec{r}}$, $\frac{\partial \vec{V}_R}{\partial \vec{r}}$, and $\frac{\partial \vec{V}_R}{\partial t}$ are computed before launch in a process often called targeting which can require substantial computational effort. Finding an appropriate function for $\vec{F}$ can also be a difficult task. In addition to the linear terms shown, some higher order terms might also be included in the guidance law. For a more detailed explantation of delta guidance the reader is referred to [54, 55].

Battin also develops Delta guidance [13, 56] and, claims to have "invented" it together with Hal Laning at the MIT instrumentation laboratory. Battin utilizes the function

$$\vec{F}\left(\vec{r}, \vec{v}, \vec{r}_T\right) = \left(\vec{v} \times \vec{r}\right) \cdot \left[\vec{v} \times \left(\vec{r}_T - \vec{r}\right)\right] + \mu \vec{r}_T \cdot \left(\frac{\vec{r}_T}{r_T} - \frac{\vec{r}}{r}\right) \tag{3.6}$$

where μ is the gravitational parameter of the earth, and $\vec{r}_T$ is the location vector of the target. Regarding the motivation for his guidance law, Battin states, "I am unable to recall from whence the expression came. Since at that time neither Hal nor I were celestial mechanists (nor acquainted with any), the mystery is all the more puzzling."

The motivation for the delta guidance is simply to follow a predetermined path. Any attempt at optimization must be accomplished during the targeting phase when the nominal trajectory is computed. The onboard calculation only drives the missile back to the nominal trajectory without attempting to optimize the process in any way.

3.2.3 The Powered Flight Maneuver Equation and Cross-Product Steering

Cross product steering is a guidance technique which makes use of the concept of "velocity-to-be-gained"

$$\vec{v}_g = \vec{v}_r - \vec{v} \tag{3.7}$$

where $\vec{v}$ is the velocity of the vehicle, and $\vec{v}_r$ is the velocity required to meet mission objectives. Here, the required velocity is assumed to be a known of function of the vehicle position, $\vec{r}$, and the current time, t. Clearly, any steering law that drives $\vec{v}_g$ to zero will be successful. The particular steering law that is chosen can greatly effect the amount of fuel that is used.

A differential equation for $\vec{v}_g$ can be developed by a clever substitution

$$\dot{\vec{v}} = \vec{g}\left(\vec{r}\right) + \vec{F} \tag{3.8}$$

where $\vec{g}$ is the acceleration due to gravity. Recalling that $\vec{v}_r$ is a function of $\vec{r}$ and t, the time derivative of the velocity-to-be-gained can be written

$$\dot{\vec{v}}_g = \frac{\partial \vec{v}_r}{\partial t} + \frac{\partial \vec{v}_r}{\partial \vec{r}}\vec{v} - \vec{g}\left(\vec{r}\right) - \vec{F} \tag{3.9}$$

Now the relation $\vec{v} = \vec{v}_r - \vec{v}_g$ is substituted into this relation

$$\dot{\vec{v}}_g = \underbrace{\frac{\partial \vec{v}_r}{\partial t} + \frac{\partial \vec{v}_r}{\partial \vec{r}}\vec{v}_r}_{\vec{g}(\vec{r})} - \frac{\partial \vec{v}_r}{\partial \vec{r}}\vec{v}_g - \vec{g}\left(\vec{r}\right) - \vec{F} \tag{3.10}$$

The substitution of $\vec{g}\left(\vec{r}\right)$ into the above equation is justified by noting that $\vec{v}_r$ is a coasting trajectory

$$\dot{\vec{v}}_r\left(\vec{r}, t\right) = \vec{g}\left(\vec{r}\right) \tag{3.11}$$

$$\frac{\partial \vec{v}_r}{\partial t} + \frac{\partial \vec{v}_r}{\partial \vec{r}}\underbrace{\frac{\partial \vec{r}}{\partial t}}_{\vec{v}_r} = \vec{g}\left(\vec{r}\right) \tag{3.12}$$

The gravity terms now cancel, resulting in a differential equation for powered flight that is independent of gravity. Battin labels the partial derivatives with the term $\mathbf{C}^\star$

$$\dot{\vec{v}}_g = -\underbrace{\frac{\partial \vec{v}_r}{\partial \vec{r}}}_{\mathbf{C}^\star}\vec{v}_g - \vec{F} \tag{3.13}$$

Sometimes $\mathbf{C}^\star$ is labeled as $\mathbf{Q}$ and the resulting guidance scheme is referred to as Q-guidance [13, 56].

A naive approach would simply be to align the engine thrust with $\vec{v}_g$.

$$\vec{F} = F_{\text{thrust}}\hat{e}_{\vec{v}_g} \tag{3.14}$$

Assuming that the engine provides sufficient thrust to overcome any growth in $\vec{v}_g$, this steering law will eventually drive $\vec{v}_g$ to zero. This guidance law, can be shown to be optimal for the case where the gravity field is constant with respect to position. However, real gravity field requires some additional insight to develop an optimal guidance law.

Battin realized that an effective way to drive $\vec{v}_g$ to zero would be to align the time derivative of the velocity-to-be-gained, $\dot{\vec{v}}_g$, with the $\vec{v}_g$ vector. This is equivalent to the statement that the cross-product of the these two vectors must be zero

$$\dot{\vec{v}}_g \times \vec{v}_g = 0 \tag{3.15}$$

This approach, called "cross-product steering" [57], is very simple and deeply insightful. Aligning the time derivative of a vector with the vector itself has two important effects. First, the constraint $\dot{\vec{v}}_g \| \vec{v}_g$ renders the vector $\vec{v}_g$ irrotational. Second, if it is presumed that $\left|\dot{\vec{v}}_g\right|$ is constant, or very nearly constant, with respect to direction, then aligning the $\dot{\vec{v}}_g$ with $\vec{v}_g$ results in the most rapid decrease (or increase) of $|\vec{v}_g|$. Both of these have the effect reducing the amount of fuel that need be used to execute the maneuver. Martin [58, 59] showed that cross-product steering is a fuel optimal steering law for constant gravity fields. Additional study by Laning, Battin and others [60, 61] showed that cross-product steering is very nearly optimal for non-constant gravity fields.

The steering law is devised by substituting equation 3.13 into equation 3.15.

$$\left(-\mathbf{C}^{\star}\vec{v}_g - \vec{F}\right) \times \vec{v}_g = 0 \tag{3.16}$$

This equation is then expanded, and this equation is once again crossed with $\vec{v}_g$

$$\left[\left(-\mathbf{C}^{\star}\vec{v}_g\right) \times \vec{v}_g\right] \times \vec{v}_g - \left(\vec{F} \times \vec{v}_g\right) \times \vec{v}_g = 0 \tag{3.17}$$

Lagrange's formula[6] can now be applied

$$\left[\left(-\mathbf{C}^{\star}\vec{v}_g\right) \cdot \vec{v}_g\right] \vec{v}_g - v_g^2 \left(-\mathbf{C}^{\star}\vec{v}_g\right) - \left(\vec{F} \cdot \vec{v}_g\right) \vec{v}_g + v_g^2 \vec{F} = 0 \tag{3.18}$$

the steering command can now be written

$$\vec{F} = \left[\vec{F} \cdot \hat{e}_{\vec{v}_g} + \left(\mathbf{C}^{\star}\vec{v}_g\right) \cdot \hat{e}_{\vec{v}_g}\right] \hat{e}_{\vec{v}_g} - \mathbf{C}^{\star}\vec{v}_g \tag{3.19}$$

The dot product containing $\vec{F}$ can be found by dotting this equation with itself

$$\vec{F} \cdot \vec{F} = F^2 = \left[\vec{F} \cdot \hat{e}_{\vec{v}_g} + \left(\mathbf{C}^{\star}\vec{v}_g\right) \cdot \hat{e}_{\vec{v}_g}\right]^2 - 2\left[\vec{F} \cdot \hat{e}_{\vec{v}_g} + \left(\mathbf{C}^{\star}\vec{v}_g\right) \cdot \hat{e}_{\vec{v}_g}\right] \left(\mathbf{C}^{\star}\vec{v}_g\right) \cdot \hat{e}_{\vec{v}_g} + \left(\mathbf{C}^{\star}\vec{v}_g\right) \cdot \left(\mathbf{C}^{\star}\vec{v}_g\right) \tag{3.20}$$

which can be rewritten as

$$\left(\vec{F} \cdot \hat{e}_{\vec{v}_g}\right)^2 = F^2 + \left[\left(\mathbf{C}^{\star}\vec{v}_g\right) \cdot \hat{e}_{\vec{v}_g}\right]^2 - \left|\mathbf{C}^{\star}\vec{v}_g\right|^2 \tag{3.21}$$

By substituting the positive value for this dot-product into equation 3.19 the final guidance law is revealed

$$\vec{F} = \left[\sqrt{F^2 + \left[\left(\mathbf{C}^{\star}\vec{v}_g\right) \cdot \hat{e}_{\vec{v}_g}\right]^2 - \left|\mathbf{C}^{\star}\vec{v}_g\right|^2} + \left(\mathbf{C}^{\star}\vec{v}_g\right) \cdot \hat{e}_{\vec{v}_g}\right] \hat{e}_{\vec{v}_g} - \mathbf{C}^{\star}\vec{v}_g \tag{3.22}$$

where F is the magnitude of the thrust available.

[6] $\left(\vec{a} \times \vec{b}\right) \times \vec{c} = \left(\vec{a} \cdot \vec{c}\right) \vec{b} - \left(\vec{b} \cdot \vec{c}\right) \vec{a}$

The main difficulty with cross-product steering is finding an expression for the matrix $\mathbf{C}^{\star}$. Battin and others [13, 62] present analytical expressions for $\mathbf{C}^{\star}$ for several common orbital transfer maneuvers, and also provide techniques for estimating the total burn time required for a maneuver. Battin also shows how to modify cross product steering to allow for variable time of arrival [63].

3.2.4 Powered Descent Guidance

The objective of powered descent guidance is to land a vehicle on the surface of a planet (or moon). This section will focus on "soft" landing techniques, and not discuss "hard" landing or "impact" landing techniques, as they are less applicable to human space flight. The most prominent example of powered descent guidance is provided by the Apollo moon landings. Cherry [64–67] pioneered the development of powered decent guidance. Later Klumpp [68–71] advanced the technique and produce the actual guidance laws that were used by Apollo. More recently, the approach used by Cherry and Klumpp was revisited by Sostaric [72, 73] to develop powered decent guidance for the Constellation program. The details of these developments can be tedious, and are more focused on appropriate selection of boundary conditions. The development presented here is given by D'Souza [74] and is focused on optimization of fuel consumption. Battin [13] presents a development similar to the one found here.

Before proceeding with the development, it is important to note the stated objective of this guidance law. The earliest work seems content with finding an admissible solution to the problem, but the focus quickly becomes minimal fuel consumption. Cherry [64] is careful to select "fuel-optimizing functions" for his guidance law. Cherry [65] also states, "[T]he need to optimize the guidance program, to minimize the fuel consumed during a powered maneuver ... is inescapable." Klumpp refers to the "cost in fuel" [68, 69]. Sostaric goes to great length [73] to show the fuel-optimal features of the guidance law. Battin and D'Souza have the stated objective of minimizing fuel consumption.

Begin with the minimum control effort functional

$$J = \Gamma t_f + \frac{1}{2}\int_{t_0}^{t_f} \vec{a}_T^T \vec{a}_T dt \tag{3.23}$$

The objective is to minimize this function subject to the conditions

$$\dot{\vec{r}} = \vec{v} \tag{3.24}$$

$$\dot{\vec{v}} = \vec{a} + \vec{g} \tag{3.25}$$

$$\vec{r}\,(t_0) = \vec{r}_0 \tag{3.26}$$

$$\vec{v}\,(t_0) = \vec{v}_0 \tag{3.27}$$

$$\vec{r}\,(t_f) = 0 \tag{3.28}$$

$$\vec{v}\,(t_f) = 0 \tag{3.29}$$

The development is omitted here, but the calculus of variations can be used to solve this problem and find the guidance law which minimizes the above functional.

$$\vec{a}_T = -\frac{4\vec{v}}{t_{go}} - \frac{6\vec{r}}{t_{go}^2} - \vec{g} \tag{3.30}$$

where $t_{go} = t_f - t$ is a root of the quartic polynomial

$$\left(\Gamma + \frac{1}{2}g^2\right) t_{go}^4 - 2\vec{v}\cdot\vec{v}t_{go}^2 - 12\vec{v}\cdot\vec{r}t_{go} - 18\vec{r}\cdot\vec{r} = 0$$

D'Souza notes "that a guidance law which provides for a minimum time to landing can be obtained quite easily by setting Γ to a large positive number. In most cases, however, when it is desired to minimize the acceleration (and thus the propellant consumed), Γ can be set to zero."

3.2.5 The Bilinear Linear Tangent Law and Powered Explicit Guidance

The bilinear tangent law is a fuel optimal guidance law that serves as the basis for the Powered Explicit Guidance (PEG) scheme used by the space shuttle for all on-orbit maneuvers. The bilinear tangent technique was first mentioned by Lawden [48, 50, 75]. Subsequently, this technique has been well studied and extensive literature exists. References [76–80] provide and excellent introduction to the topic of powered explicit guidance.

A variety of derivations exist, but all are based on the calculus of variations. The derivation shown here relies on the penalty function, and has a zero value for the Lagrangian. Some derivations utilize a Lagrangian of $L = 1$ to minimize the time of flight, and thus the fuel consumption.

The objective of the bilinear tangent law, is to minimize the function

$$J = \phi\left[\vec{r}\left(t_f\right), \vec{v}\left(t_f\right), t_f\right] \tag{3.31}$$

subject to the following constraints

$$\begin{bmatrix} \dot{\vec{r}} \\ \dot{\vec{v}} \end{bmatrix} = \begin{bmatrix} \vec{v} \\ \vec{g}\left(\vec{r}, t\right) + \frac{F}{m}\hat{u} \end{bmatrix} \tag{3.32}$$

$$0 = \psi\left[\vec{r}\left(t_f\right), \vec{v}\left(t_f\right), t_f\right] \tag{3.33}$$

where $\hat{u}$ is unit vector in the direction of the thrust, and ψ is the terminal constraints for the trajectory.

The Hamiltonian for the system is given by

$$H = \begin{bmatrix} \vec{\lambda}_r \\ \vec{\lambda}_v \end{bmatrix}^T \begin{bmatrix} \dot{\vec{r}} \\ \dot{\vec{v}} \end{bmatrix} = \vec{\lambda}_r \cdot \vec{v} + \vec{\lambda}_v \cdot \vec{g}\left(\vec{r}, t\right) + \vec{\lambda}_v \cdot \frac{F}{m}\hat{u} \tag{3.34}$$

The Euler equation is used to find the time derivative of the Lagrange multipliers

$$\begin{bmatrix} \dot{\vec{\lambda}}_r \\ \dot{\vec{\lambda}}_v \end{bmatrix} = -\begin{bmatrix} \frac{\partial H}{\partial \vec{r}} & \frac{\partial H}{\partial \vec{v}} \end{bmatrix}^T = \begin{bmatrix} -\left[\frac{\partial \vec{g}(\vec{r},t)}{\partial \vec{r}}\right]^T \vec{\lambda}_v \\ -\vec{\lambda}_r \end{bmatrix} = -\begin{bmatrix} \mathbf{0} & \frac{\partial \vec{g}(\vec{r},t)}{\partial \vec{r}} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{\lambda}_r \\ \vec{\lambda}_v \end{bmatrix} \tag{3.35}$$

Note that the transpose may be removed from the partial when the gravity can be expressed in terms of a potential function. Since gravity is known to be a conservative force, it is also known to be expressible as a potential function, and the aforementioned partial is equivalent to the negative Hessian of the potential function, which is symmetric by definition.

Finally, the transversality constraints are used to enforce the boundary constraints

$$\frac{\partial \phi}{\partial \vec{r}\,(t_f)} + \nu^T \frac{\partial \psi}{\partial \vec{r}\,(t_f)} = \vec{\lambda}_r^T\,(t_f) \tag{3.36}$$

$$\frac{\partial \phi}{\partial \vec{v}\,(t_f)} + \nu^T \frac{\partial \psi}{\partial \vec{v}\,(t_f)} = \vec{\lambda}_v^T\,(t_f) \tag{3.37}$$

$$\frac{\partial \phi}{\partial t_f} + \nu \cdot \frac{\partial \psi}{\partial t_f} + H\,(t_f) = 0 \tag{3.38}$$

Together, these equations form the necessary conditions for an optimal solution.

Pontryagin's maximum principle [81–83] can be applied directly to the Hamiltonian in equation 3.34 to find the optimal control program. Pontryagin's principle requires that the control, $\hat{u}$, be selected from admissible values such that the Hamiltonian is maximized. This is equivalent to maximizing the dot product

$$\underset{\hat{u}}{\text{maximize}} \left[\vec{\lambda}_v \cdot \frac{F}{m}\hat{u}\right] = \lambda_v \frac{F}{m} \tag{3.39}$$

Two things are learned from this problem. First, the thrust force, F, should always be maximized. Second, the thrust force should point in the direction of $\vec{\lambda}_v$. Thus the fuel optimal steering law is given by

$$\vec{a}_t = \frac{F}{m} \underbrace{\frac{\vec{\lambda}_v}{\lambda_v}}_{\hat{u}} \tag{3.40}$$

where F represents the maximum available thrust, and the Lagrange multiplier $\vec{\lambda}_v$ is identified to be Lawden's primer vector.

Now the functional form of $\vec{\lambda}_v$ is found using the necessary condition from equation 3.35

$$\ddot{\vec{\lambda}}_v = \frac{\partial \vec{g}\,(\vec{r}, t)}{\partial \vec{r}} \vec{\lambda}_v \tag{3.41}$$

For simple gravity fields, this equation has closed form solutions. The most simple case is to assume that the gravity field is constant ($\vec{g} = \text{constant}$). For a constant gravity field, the solution is

$$\vec{\lambda}_v = \vec{\lambda}_{v_f} + (t_f - t)\,\dot{\vec{\lambda}}_v \tag{3.42}$$

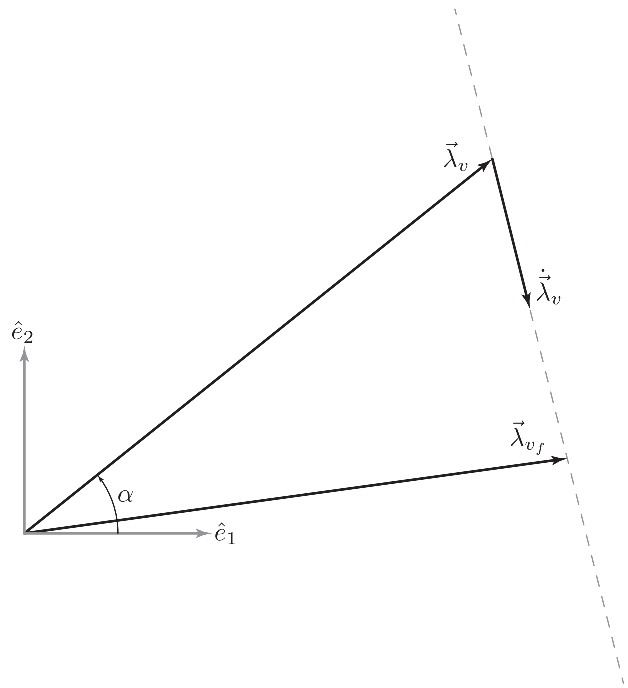

Fig. 3.1: Geometry of the primer vector in the bilinear tangent law.

where $\vec{\lambda}_{v_f}$ and $\dot{\vec{\lambda}}_v$ are constants. Clearly $\vec{\lambda}_v$ is a linear combination of $\vec{\lambda}_{v_f}$ and $\dot{\vec{\lambda}}_v$ and thus resides in the plane spanned by $\vec{\lambda}_{v_f}$ and $\dot{\vec{\lambda}}_v$. This geometry is shown in figure 3.1. The angle α in figure 3.1 represents the in-plane steering command and is expressed as

$$\tan\alpha = \frac{\vec{\lambda}_{v_f} \cdot \hat{e}_2 + (t_f - t)\,\dot{\vec{\lambda}}_v \cdot \hat{e}_2}{\vec{\lambda}_{v_f} \cdot \hat{e}_1 + (t_f - t)\,\dot{\vec{\lambda}}_v \cdot \hat{e}_1} \tag{3.43}$$

This relation gives rise to the name "bilinear tangent law". The constants $\vec{\lambda}_{v_f}$, $\dot{\vec{\lambda}}_v$, and t_f can be found by applying the transversality conditions, and solving the boundary value problem.

The fuel consumption can be minimized by minimizing the burn duration

$$\phi\left[\vec{r}\,(t_f)\,,\vec{v}\,(t_f)\,,t_f\right] = t_f \tag{3.44}$$

subject to 5 or less constraints on the final position and velocity

$$0 = \psi\left[\vec{r}\,(t_f)\,,\vec{v}\,(t_f)\right] \tag{3.45}$$

PEG uses these boundary conditions and then solves the appropriate system of equations to find, $\vec{\lambda}_{v_f}$, $\dot{\vec{\lambda}}_v$, and t_f such that the terminal boundary conditions are met.

It is worth noting that Lawden proposed using the bilinear tangent law to optimize trajectories for variables other than fuel consumption [84]. Lawden noted that *any* parameter (e.g., rocket range, maximum altitude, or orbital eccentricity) which is purely a function of the final state can be optimized by applying the bilinear tangent steering law. Consider the case where there are no constraints on the final position or velocity, and the final time is permitted to vary. In other words, optimize the penalty function, $\phi\left[\vec{r}\left(t_f\right), \vec{v}\left(t_f\right)\right]$, without any constraint on the terminal state. The transversality conditions then become

$$\frac{\partial \phi}{\partial \vec{r}\left(t_f\right)} = \vec{\lambda}_r^T\left(t_f\right) = \dot{\vec{\lambda}}_v^T \tag{3.46}$$

$$\frac{\partial \phi}{\partial \vec{v}\left(t_f\right)} = \vec{\lambda}_v^T\left(t_f\right) = \vec{\lambda}_{v_f}^T \tag{3.47}$$

Thus, the primer vector is given by

$$\vec{\lambda}_v^T = \frac{\partial \phi}{\partial \vec{v}\left(t_f\right)} + \left(t_f - t\right)\frac{\partial \phi}{\partial \vec{r}\left(t_f\right)} \tag{3.48}$$

or the steering law can be given by

$$\tan\left(\alpha\right) = \frac{\frac{\partial \phi}{\partial \vec{v}_2\left(t_f\right)} + \left(t_f - t\right)\frac{\partial \phi}{\partial \vec{r}_2\left(t_f\right)}}{\frac{\partial \phi}{\partial \vec{v}_1\left(t_f\right)} + \left(t_f - t\right)\frac{\partial \phi}{\partial \vec{r}_1\left(t_f\right)}} \tag{3.49}$$

where the subscripts denote the individual components of the vectors corresponding to the basis shown in figure 3.1. It is interesting to note, that a *linear* tangent law results if and only if ϕ is not a function of $\vec{r}_1$. Despite Lawden's suggestion, all known applications of the bilinear tangent law are motivated by a desire to limit fuel consumption.

Lawden's idea can be extended slightly by proposing a hybrid penalty function

$$\phi = \Gamma t_f + \varphi\left[\vec{r}\left(t_f\right), \vec{v}\left(t_f\right), t_f\right] \tag{3.50}$$

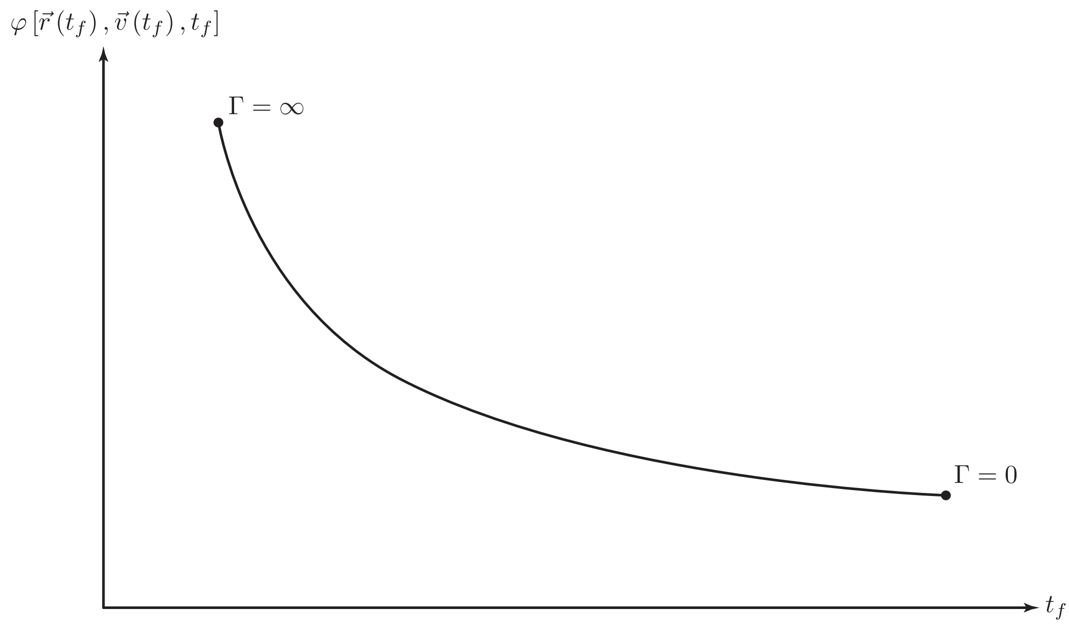

Fig. 3.2: Geometry of the primer vector in the bilinear tangent law.

As was shown earlier, this will result in the bilinear tangent law of equation 3.42. This penalty function, is a weighted combination of the burn time, t_f, (fuel consumption) and some other function of the terminal state. The curve in figure 3.2 shows the Pareto optimal curve associated with this objective function.

3.2.6 The Inverse Optimal Control Problem

Although the Inverse Optimal Control problem was originally proposed by Letov [85, 86]. Kalman was the first to make substantial progress on the inverse optimal control problem [87]. Kalman's study was confined to linear time invariant systems with a single control variable, and with a quadratic optimization parameter. Soon after Kalman's initial study Anderson [88] extended the results to linear time invariant systems with multiple control variables, and a quadratic performance index. More recently, other authors have [89] extended the theory to include control derivatives.

This research shows a link between linear controller design aimed at reducing sensitivities to plant parameters, and linear controllers resulting from the application of optimal control theory. Anderson [88] states, "In essence ... a design where sensitivity to plant parameter variation is reduced is also a design which is optimal with regard to some performance index." This relationship between optimal control solutions and sensitivity to plant parameter variations is one of the reasons that the results of optimal control are so frequently applied to stochastic systems with success.

3.3 Stochastic Guidance Schemes

There are several approaches to guidance which attempt to account for the stochastic nature of the system. Here the word stochastic refers to the treatment of the state estimate supplied to the guidance law, and not to the steering command produced by the guidance law.

3.3.1 Relationship between Time-of-Flight and the Probability of Mission Failure

Roche [90] showed that an increase in the time-of-flight corresponds to an increase in the risk of component failure. Essentially, he claimed that the longer a machine (spacecraft) operated, the greater the probability of its failure. Roche reasoned that if a rocket had a certain probability of component failure during a specified time frame, then the same rocket would have a smaller probability of failure given a smaller time frame. Although Roche was primarily concerned with component failure, his view that mission success should be optimized subject to the available resources is fundamental to this thesis.

Roche's proposal of minimizing the probability of component failure was simply to minimize the time of flight. During the mission planning phase it is desirable to find fuel-optimal trajectories. The cost of carrying the additional mass associated with unneeded fuel on the spacecraft is so large that every possible reduction in fuel during mission planning results in substantial cost savings and/or increased capability. Once a mission launches, mission success replaces issues such as fuel savings or capability augmentation as the primary metric for decision making. An important factor in the decision making process is that fuel not used to execute maneuvers will be discarded or returned to Earth unused. After launch the cost to the mission is fixed, regardless of the amount of fuel consumed to execute required maneuvers. Therefore, during operations, it is desirable to consider fuel available as a constraint on the system rather than a parameter that should be optimized by the flight computer. In these situations, rather than minimizing the fuel consumed by the maneuver, it may be more appropriate to minimize parameters that could pose a more immediate danger to the mission, such as the time-of-flight, or the number and the complexity of impulses required to execute a maneuver.

Roche's conclusion that the probability of mission failure is minimized by minimizing the time-of-flight is focused on component failure and does not account for the complexities of state dispersion dynamics, or the behavior of a closed loop GN&C system. Nevertheless, his work clearly showed the role that resources, such a fuel, should play in the guidance scheme. Roche also realized the value of maximizing mission success probability.

3.3.2 Quadratic Synthesis and the Certainty and Equivalence Principle

Many guidance and control schemes make tacit claims regarding stochastic optimality by invoking the certainty and equivalence principle [82, 91–95]. GNC systems designed using this approach are often said to be designed using *quadratic synthesis*.

The certainty and equivalence principle states that the statistically optimum GNC system is produced by joining a statistically optimal estimator with a deterministically optimal feedback law. That is to say, a deterministic feedback law which optimizes some performance index J when supplied with perfect state knowledge will also optimize

$$J = E\left[\phi\left(\mathbf{x}_f\right) + \int_{t_i}^{t_f} L dt\right] \tag{3.51}$$

when supplied with a state estimate from a statistical optimal estimator. In this framework, the feedback law treats the current estimate as if it were perfect. Meanwhile, the state estimator only provides the best possible estimate.

The practical consequence of the certainty and equivalence principle is that the state estimator and the feedback law may be designed independent of one another. These independently designed subsystems can then be combined to form the optimal closed loop control system. Thus, the certainty and equivalence principle provides solid rational for designing the estimator and control system separately. While this can be helpful, and greatly simplifies the design process, it is important to note that it is only applicable in the narrow confines of the following assumptions

1. The state dynamics are governed by linear differential equations.
2. Measurements are a linear function of the state.
3. The cost function J is a quadratic with respect to the state and control.
4. The estimator provides a full-state estimate.
5. The state is controllable.
6. The state is observable.

For space applications the linearity conditions are typically considered to be met once the dynamics and measurement have been linearized about a given nominal or reference trajectory. The quadratic cost function condition is also amenable to may common objectives in space flight. In the case the the cost function is not quadratic, it is often sufficient to provide the Taylor series of the cost function truncated after the quadratic term [93].

3.3.3 Trajectory Desensitization

Trajectory desensitization is the process of finding a nominal trajectory that minimized the effect of unknown model parameters on the outcome. That is, the trajectory is desensitized with respect to the unknown parameters. Fundamental work regarding trajectory desensitization was done by Kahne and and his colleagues [96–99]. The main idea is to add the sensitivity of the trajectory to unknown system parameters to the optimization index for the classical quadratic problem from the calculus of variations.

$$J = \int_{t_0}^{t_f} \left[\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{s}^T\mathbf{B}\mathbf{s} + \mathbf{u}^T\mathbf{C}\mathbf{u}\right] dt \tag{3.52}$$

The sensitivity vector, $\mathbf{s}$, is the partial of the state with respect to the unknown parameter.

$$\mathbf{s} = \frac{\partial \mathbf{x}}{\partial \alpha} \tag{3.53}$$

The guidance policy can now be determined by solving the problem using the calculus of variations with appropriate selection of the weights. This approach has been used to aid in development of guidance for landing on Mars, and other space navigation problems [100,101]. More recently, this work has been extended by attempting to include terms relating to the estimator into the cost function [102, 103].

3.3.4 Mission Success Maximization

Denham [104–107] and Schmidt [108–110] both explored optimization of statistical cost functions. In particular Denham focused on optimizing statistical quantities relating only to the terminal state using the calculus of variations. The technique is based on minimizing the first two terms of a Taylor expansion of the expectation operator applied to a penalty function, ϕ.

$$J = E\left[\phi\left(\mathbf{x}_f\right)\right] + \frac{1}{2}E\left[\left(\mathbf{x}_f - \bar{\mathbf{x}}_f\right)^T \frac{\partial^2 \phi}{\partial \mathbf{x}^2}\left(\mathbf{x}_f - \bar{\mathbf{x}}_f\right)\right] \tag{3.54}$$

Denham's approach also involves taking a Taylor series expansion of the Hamiltonian. By doing this, Denham is successful in finding conditions which are solely a function of statistics of the terminal state. Unfortunately, the necessary conditions developed by Denham are expansive and difficult to solve.

Schmidt's work is focused on a more general performance index which requires the inclusion of a path integral term in the cost function

$$J = E\left[\phi\left(\mathbf{x}_f\right)\right] + E\left[\int_{t_0}^{t_f} L\left(\mathbf{x}, \mathbf{u}, t\right) dt\right] \tag{3.55}$$

Inclusion of the integral term allows Schmidt to avoid the expansion of the Hamiltonian, but the resulting cost is a functional of the entire trajectory, not just the final state. Schmidt's necessary conditions are significantly simpler than those of Denham. Via appropriate weighting of the integral term Schmidt's technique can function effectively the same as Denham's approach.

Both Denham and Schmidt's approaches are intended to be used a priori. As outputs, these techniques provide both an open loop control, and a schedule of linear feedback gains.

This work makes use of the methods developed by Schmidt, which will be presented in a subsequent chapter.

3.4 Summary

This chapter presented a brief overview of modern spacecraft guidance techniques. The chapter contains an overview of key historical conditions that fostered the development of spacecraft guidance. Brief summaries of both common deterministic and stochastic guidance schemes are included. The common features of these schemes is the desire to minimize a quantity connected to fuel consumption. The certainty and equivalence principle was presented along with other approaches to stochastic guidance. In the next three chapters of this dissertation the details of three stochastic optimization approaches will be discussed.

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Chapter 4

Neighboring Optimal Control

This chapter develops the theory of neighboring optimal control [82, 111, 112]. Neighboring optimal is distinguished by using an expansion of the augmented cost function or the Hamiltonian. The subsequent two chapters will present alternative approaches which use other expansions.

Consider the classical Bolza [113–116] problem with free final time from the calculus of variations.

$$\begin{aligned}
\text{minimize:} \quad & J = \phi\left(\mathbf{x}_f, t_f\right) + \int_{t_i}^{t_f} L\left(\mathbf{x}, \mathbf{u}, t\right) dt \\
\\
\text{subject to:} \quad & \dot{\mathbf{x}} = \boldsymbol{f}\left(\mathbf{x}, \mathbf{u}, t\right) \\
& \mathbf{0} = \boldsymbol{\psi}\left(\mathbf{x}_f, t_f\right) \\
& t_i \text{ and } \mathbf{x}_i \text{ are specified}
\end{aligned}$$

The constraints are adjoined to the cost function

$$J' = \underbrace{\phi\left(\mathbf{x}_f, t_f\right) + \boldsymbol{\nu}^T \boldsymbol{\psi}\left(\mathbf{x}_f, t_f\right)}_{\Phi} + \underbrace{\int_{t_0}^{t_f} \left[\underbrace{L\left(\mathbf{x}, \mathbf{u}, t\right) + \boldsymbol{\lambda}^T \boldsymbol{f}\left(\mathbf{x}, \mathbf{u}, t\right)}_{H} - \boldsymbol{\lambda}^T \dot{\mathbf{x}} \right] dt}_{I} \tag{4.1}$$

The first variation[1] of this equation can be found with the techniques and relations show in the appendix.

$$dJ' = \frac{\partial \Phi}{\partial t_f} dt_f + \frac{\partial \Phi}{\partial \mathbf{x}_f} d\mathbf{x}_f + \frac{\partial \Phi}{\partial \boldsymbol{\nu}} d\boldsymbol{\nu} + \left[\left(H - \boldsymbol{\lambda}^T \dot{\mathbf{x}} \right) dt \right]_{t_i}^{t_f} + \int_{t_i}^{t_f} \left[\frac{\partial H}{\partial \mathbf{x}} \delta \mathbf{x} + \frac{\partial H}{\partial \mathbf{u}} \delta \mathbf{u} + \left(\frac{\partial H}{\partial \boldsymbol{\lambda}} - \dot{\mathbf{x}}^T \right) \delta \boldsymbol{\lambda} - \boldsymbol{\lambda}^T \delta \dot{\mathbf{x}} \right] dt \quad (4.2)$$

where d indicates a differential and δ indicates a variation. Integrate the final term of the integrand by parts

$$dJ' = \frac{\partial \Phi}{\partial t_f} dt_f + \frac{\partial \Phi}{\partial \mathbf{x}_f} d\mathbf{x}_f + \overbrace{\frac{\partial \Phi}{\partial \boldsymbol{\nu}}}^{\boldsymbol{\psi}^T = \mathbf{0}} d\boldsymbol{\nu} + \left[\overbrace{\left(H - \boldsymbol{\lambda}^T \dot{\mathbf{x}} \right)}^{L} dt - \boldsymbol{\lambda}^T \delta \mathbf{x} \right]_{t_i}^{t_f} + \int_{t_i}^{t_f} \left[\left(\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^T \right) \delta \mathbf{x} + \frac{\partial H}{\partial \mathbf{u}} \delta \mathbf{u} + \left(\overbrace{\frac{\partial H}{\partial \boldsymbol{\lambda}}}^{\boldsymbol{f}^T} - \dot{\mathbf{x}}^T \right) \delta \boldsymbol{\lambda} \right] dt \quad (4.3)$$

The indicated substitutions along with the consequences of specified initial conditions ($dt_i = 0$ and $\delta \mathbf{x}_i = \mathbf{0}$) allow for a simpler expression to be written.

$$dJ' = \frac{\partial \Phi}{\partial t_f} dt_f + \frac{\partial \Phi}{\partial \mathbf{x}_f} d\mathbf{x}_f + L_f dt_f - \boldsymbol{\lambda}_f^T \delta \mathbf{x}_f + \int_{t_i}^{t_f} \left[\left(\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^T \right) \delta \mathbf{x} + \frac{\partial H}{\partial \mathbf{u}} \delta \mathbf{u} \right] dt \quad (4.4)$$

The differential and variance of $\mathbf{x}_f$ are related by $d\mathbf{x}_f = \delta \mathbf{x}_f + \dot{\mathbf{x}}_f dt_f$

$$dJ' = \underbrace{\left(\frac{\partial \Phi}{\partial t_f} + \frac{\partial \Phi}{\partial \mathbf{x}_f} \dot{\mathbf{x}}_f + L_f \right)}_{0} dt_f + \underbrace{\left(\frac{\partial \Phi}{\partial \mathbf{x}_f} - \boldsymbol{\lambda}_f^T \right)}_{\mathbf{0}} \delta \mathbf{x}_f + \int_{t_i}^{t_f} \left[\underbrace{\left(\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^T \right)}_{\mathbf{0}} \delta \mathbf{x} + \underbrace{\frac{\partial H}{\partial \mathbf{u}}}_{\mathbf{0}} \delta \mathbf{u} \right] dt \quad (4.5)$$

The fundamental theorem of the calculus of variations [120] states that in order for the variation to be zero, $dJ' = 0$, it is necessary that the all indicated quantities be zero. These

[1] In the literature this is often called the "first variation." However, the correct terminology is "first differential" [112, 117–119].

relations then become the well known Euler-Lagrange necessary conditions.

$$\dot{\mathbf{x}} = \boldsymbol{f}\left(\mathbf{x}, \mathbf{u}, t\right) \tag{4.6}$$

$$\mathbf{x}_i \text{ is specified} \tag{4.7}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}^T} \tag{4.8}$$

$$\boldsymbol{\lambda}_f = \frac{\partial \Phi}{\partial \mathbf{x}_f^T} \tag{4.9}$$

$$\mathbf{0} = \frac{\partial H}{\partial \mathbf{u}} \tag{4.10}$$

$$\mathbf{0} = \boldsymbol{\psi}\left(\mathbf{x}_f, t_f\right) \tag{4.11}$$

$$0 = \frac{\partial \Phi}{\partial t_f} + \underbrace{\overbrace{\frac{\partial \Phi}{\partial \mathbf{x}_f}}^{\boldsymbol{\lambda}_f^T} \dot{\mathbf{x}}_f + L_f}_{H_f} = \Omega \tag{4.12}$$

For fixed final time problems, $dt_f = 0$, and the final relation is no longer a necessary condition. Solving these equations gives quantities for the nominal trajectory, $\bar{\mathbf{u}}$, $\bar{\mathbf{x}}$, and $\bar{t}_f$.

The second differential of the cost function can be used to find the control for a path in the neighborhood of the optimal solution [82,112,121–125]. This approach is referred to as a neighboring optimal control (NOC) problem. The development of this technique is initiated by finding the second differential of the cost function. Begin with the first differential from equation 4.3 without the requirement that the initial state, $\mathbf{x}$, lie on the curve $\bar{\mathbf{x}}$, but allow it be in the vicinity. That is, permit a perturbation from the nominal initial condition, $\delta \mathbf{x}_i \neq \mathbf{0}$. However, the initial time is not allowed to vary, $dt_i = 0$.

$$\begin{aligned} dJ' = \frac{\partial \Phi}{\partial t_f} dt_f + \frac{\partial \Phi}{\partial \mathbf{x}_f} d\mathbf{x}_f + \frac{\partial \Phi}{\partial \boldsymbol{\nu}} d\boldsymbol{\nu} + \left[L dt - \boldsymbol{\lambda}^T \delta \mathbf{x} \right]_{t_i}^{t_f} \\ + \int_{t_i}^{t_f} \left[\left(\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^T \right) \delta \mathbf{x} + \frac{\partial H}{\partial \mathbf{u}} \delta \mathbf{u} + \left(\boldsymbol{f}^T - \dot{\mathbf{x}}^T \right) \delta \boldsymbol{\lambda} \right] dt \end{aligned} \tag{4.13}$$

or

$$dJ' = \Omega dt_f + \left(\frac{\partial \Phi}{\partial \mathbf{x}_f} - \boldsymbol{\lambda}_f^T\right)\delta\mathbf{x}_f + \frac{\partial \Phi}{\partial \boldsymbol{\nu}} d\boldsymbol{\nu} + \boldsymbol{\lambda}_i^T \delta\mathbf{x}_i$$
$$+ \int_{t_i}^{t_f} \left[\left(\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^T\right)\delta\mathbf{x} + \frac{\partial H}{\partial \mathbf{u}}\delta\mathbf{u} + \left(\boldsymbol{f}^T - \dot{\mathbf{x}}^T\right)\delta\boldsymbol{\lambda}\right] dt \quad (4.14)$$

where Ω is defined in equation 4.12. It is important to note that all of the indicated partial derivatives are evaluated along the nominal trajectory defined by $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$.

The differential of dJ' is now taken

$$d^2J' = dt_f\left(\frac{\partial \Omega}{\partial t_f}dt_f + \frac{\partial \Omega}{\partial \mathbf{x}_f}d\mathbf{x}_f + \frac{\partial \Omega}{\partial \boldsymbol{\nu}}d\boldsymbol{\nu} + \frac{\partial \Omega}{\partial \mathbf{u}_f}d\mathbf{u}_f\right) + \delta\mathbf{x}_f^T\left(\frac{\partial^2 \Phi}{\partial \mathbf{x}_f^T \partial t_f}dt_f + \frac{\partial^2 \Phi}{\partial \mathbf{x}_f^T \partial \mathbf{x}_f}d\mathbf{x}_f\right)$$
$$- \delta\mathbf{x}_f^T d\boldsymbol{\lambda}_f + d\boldsymbol{\nu}^T \frac{\partial^2 \Phi}{\partial \boldsymbol{\nu}^T \partial \mathbf{x}_f}\delta\mathbf{x}_f + \delta\mathbf{x}_i^T d\boldsymbol{\lambda}_i$$
$$+ \int_{t_i}^{t_f}\left[\delta\mathbf{x}^T\left(\frac{\partial H}{\partial \mathbf{x}^T\partial \mathbf{x}}\delta\mathbf{x} + \frac{\partial H}{\partial \mathbf{x}^T\partial \mathbf{u}}\delta\mathbf{u} + \frac{\partial H}{\partial \mathbf{x}^T\partial \boldsymbol{\lambda}}\delta\boldsymbol{\lambda} + \delta\dot{\boldsymbol{\lambda}}\right)\right.$$
$$\left. + \delta\mathbf{u}^T\left(\frac{\partial H}{\partial \mathbf{u}^T\partial \mathbf{x}}\delta\mathbf{x} + \frac{\partial H}{\partial \mathbf{u}^T\partial \mathbf{u}}\delta\mathbf{u} + \frac{\partial H}{\partial \mathbf{u}^T\partial \boldsymbol{\lambda}}\delta\boldsymbol{\lambda}\right) + \delta\boldsymbol{\lambda}^T\left(\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\delta\mathbf{x} + \frac{\partial \boldsymbol{f}}{\partial \mathbf{u}}\delta\mathbf{u} - \delta\dot{\mathbf{x}}\right)\right] dt \quad (4.15)$$

Note that several terms form a quadratic structure, and the $\delta\mathbf{x}^T\delta\dot{\boldsymbol{\lambda}}$ term can be integrated by parts to produce

$$d^2J' = dt_f\left(\frac{\partial \Omega}{\partial t_f}dt_f + \frac{\partial \Omega}{\partial \mathbf{x}_f}d\mathbf{x}_f + \frac{\partial \Omega}{\partial \boldsymbol{\nu}}d\boldsymbol{\nu} + \frac{\partial \Omega}{\partial \mathbf{u}_f}d\mathbf{u}_f\right) + \delta\mathbf{x}_f^T\left(\frac{\partial^2 \Phi}{\partial \mathbf{x}_f^T \partial t_f}dt_f + \frac{\partial^2 \Phi}{\partial \mathbf{x}_f^T \partial \mathbf{x}_f}d\mathbf{x}_f\right)$$
$$- \delta\mathbf{x}_f^T d\boldsymbol{\lambda}_f + d\boldsymbol{\nu}^T \frac{\partial^2 \Phi}{\partial \boldsymbol{\nu}^T \partial \mathbf{x}_f}d\mathbf{x}_f + \delta\mathbf{x}_i^T d\boldsymbol{\lambda}_i + \left[\delta\mathbf{x}^T\delta\boldsymbol{\lambda}\right]_{t_i}^{t_f}$$
$$+ \int_{t_i}^{t_f}\left[\begin{bmatrix} \delta\mathbf{x}^T & \delta\mathbf{u}^T \end{bmatrix}\begin{bmatrix} \frac{\partial H}{\partial \mathbf{x}^T\partial \mathbf{x}} & \frac{\partial H}{\partial \mathbf{x}^T\partial \mathbf{u}} \\ \frac{\partial H}{\partial \mathbf{u}^T\partial \mathbf{x}} & \frac{\partial H}{\partial \mathbf{u}^T\partial \mathbf{u}} \end{bmatrix}\begin{bmatrix} \delta\mathbf{x} \\ \delta\mathbf{u} \end{bmatrix}\right.$$
$$\left. + \left(\delta\mathbf{x}^T\frac{\partial^2 H}{\partial \mathbf{x}^T\partial \boldsymbol{\lambda}} + \delta\mathbf{u}^T\frac{\partial^2 H}{\partial \mathbf{u}^T\partial \boldsymbol{\lambda}} - \delta\dot{\mathbf{x}}^T\right)\delta\boldsymbol{\lambda} + \delta\boldsymbol{\lambda}^T\left(\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\delta\mathbf{x} + \frac{\partial \boldsymbol{f}}{\partial \mathbf{u}}\delta\mathbf{u} - \delta\dot{\mathbf{x}}\right)\right] dt \quad (4.16)$$

Use the following partials[2]

$$\frac{\partial^2 H}{\partial \mathbf{x}^T \partial \boldsymbol{\lambda}} = \frac{\partial \boldsymbol{f}}{\partial \mathbf{x}^T} \tag{4.17}$$

$$\frac{\partial^2 H}{\partial \mathbf{u}^T \partial \boldsymbol{\lambda}} = \frac{\partial \boldsymbol{f}}{\partial \mathbf{u}^T} \tag{4.18}$$

to simplify the expression. In addition, note that some terms can be identified as zero assuming first order variational dynamics, and a specified initial time ($d\boldsymbol{\lambda}_i = \delta\boldsymbol{\lambda}_i$).

$$\begin{aligned} d^2 J' = {} & dt_f \left(\frac{\partial \Omega}{\partial t_f} dt_f + \frac{\partial \Omega}{\partial \mathbf{x}_f} d\mathbf{x}_f + \frac{\partial \Omega}{\partial \boldsymbol{\nu}} d\boldsymbol{\nu} + \frac{\partial \Omega}{\partial \mathbf{u}_f} d\mathbf{u}_f \right) + \delta \mathbf{x}_f^T \left(\frac{\partial^2 \Phi}{\partial \mathbf{x}_f^T \partial t_f} dt_f + \frac{\partial^2 \Phi}{\partial \mathbf{x}_f^T \partial \mathbf{x}_f} d\mathbf{x}_f \right) \\ & - \delta \mathbf{x}_f^T \left(d\boldsymbol{\lambda}_f - \delta \boldsymbol{\lambda}_f \right) + d\boldsymbol{\nu}^T \frac{\partial^2 \Phi}{\partial \boldsymbol{\nu}^T \partial \mathbf{x}_f} d\mathbf{x}_f + \delta \mathbf{x}_i^T \underbrace{\left(d\boldsymbol{\lambda}_i - \delta \boldsymbol{\lambda}_i \right)}_{\mathbf{0}} \\ & + \int_{t_i}^{t_f} \left(\begin{bmatrix} \delta \mathbf{x}^T & \delta \mathbf{u}^T \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \mathbf{x}^T \partial \mathbf{x}} & \frac{\partial H}{\partial \mathbf{x}^T \partial \mathbf{u}} \\ \frac{\partial H}{\partial \mathbf{u}^T \partial \mathbf{x}} & \frac{\partial H}{\partial \mathbf{u}^T \partial \mathbf{u}} \end{bmatrix} \begin{bmatrix} \delta \mathbf{x} \\ \delta \mathbf{u} \end{bmatrix} + 2 \delta \boldsymbol{\lambda}^T \underbrace{\left(\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}} \delta \mathbf{x} + \frac{\partial \boldsymbol{f}}{\partial \mathbf{u}} \delta \mathbf{u} - \delta \dot{\mathbf{x}} \right)}_{\mathbf{0}} \right) dt \end{aligned} \tag{4.19}$$

[2]The transpose in the denominator of the partial derivative simply indicates a transpose of the indicated partial derivative. For example

$$\frac{\partial H}{\partial \mathbf{x}^T} = \left[\frac{\partial H}{\partial \mathbf{x}} \right]^T$$

and

$$\frac{\partial^2 H}{\partial \mathbf{x}^T \mathbf{u}} = \frac{\partial}{\partial \mathbf{u}} \left(\left[\frac{\partial H}{\partial \mathbf{x}} \right]^T \right) = \left(\frac{\partial^2 H}{\partial \mathbf{u}^T \mathbf{x}} \right)^T$$

For example the Hessian of a scalar function is denoted

$$\frac{\partial^2 H}{\partial \mathbf{x}^T \partial \mathbf{x}}$$

In some texts the partial with the transpose is referred to as the *left hand* partial, and the partial without the transpose is referred to as the *right hand* partial.

Require that first differential of the terminal constraint be zero. As the terminal constraint is a algebraic relation for t_f and $\mathbf{x}_f$, the first differential is

$$0 = d\boldsymbol{\psi} = \frac{\partial \boldsymbol{\psi}}{\partial t_f} dt_f + \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}_f} d\mathbf{x}_f \tag{4.20}$$

$$= \frac{\partial \boldsymbol{\psi}}{\partial t_f} dt_f + \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}_f} \overbrace{\left(\delta \mathbf{x}_f + \dot{\mathbf{x}}_f dt_f\right)}^{d\mathbf{x}_f} \tag{4.21}$$

$$= \left(\frac{\partial \boldsymbol{\psi}}{\partial t_f} + \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}_f} \dot{\mathbf{x}}_f\right) dt_f + \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}_f} \delta \mathbf{x}_f \tag{4.22}$$

Consider the terms associated with the terminal constraint from the second order expansion of the cost function

$$d\boldsymbol{\nu}^T \left(\frac{\partial \Omega}{\partial \boldsymbol{\nu}} dt_f + \frac{\partial^2 \Phi}{\partial \boldsymbol{\nu}^T \partial \mathbf{x}_f} d\mathbf{x}_f\right) = d\boldsymbol{\nu}^T \left[\overbrace{\frac{\partial \boldsymbol{\psi}}{\partial t_f}}^{\frac{\partial \Omega}{\partial \boldsymbol{\nu}}} dt_f + \overbrace{\frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}_f}}^{\frac{\partial^2 \Phi}{\partial \boldsymbol{\nu}^T \partial \mathbf{x}_f}} \overbrace{\left(\delta \mathbf{x}_f + \dot{\mathbf{x}}_f dt_f\right)}^{d\mathbf{x}_f}\right] \tag{4.23}$$

$$= d\boldsymbol{\nu}^T \underbrace{\left[\left(\frac{\partial \boldsymbol{\psi}}{\partial t_f} + \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}_f} \dot{\mathbf{x}}_f\right) dt_f + \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}_f} \delta \mathbf{x}_f\right]}_{d\boldsymbol{\psi} = \mathbf{0}} \tag{4.24}$$

Since the first differential of the terminal constraint is required to be zero, the associated terms are now removed from the second order expansion.

$$d^2 J' = dt_f \left(\frac{\partial \Omega}{\partial t_f} dt_f + \frac{\partial \Omega}{\partial \mathbf{x}_f} d\mathbf{x}_f + \overbrace{\frac{\partial \Omega}{\partial \mathbf{u}_f} d\mathbf{u}_f}^{\mathbf{0}}\right) + \delta \mathbf{x}_f^T \left(\frac{\partial^2 \Phi}{\partial \mathbf{x}_f^T \partial t_f} dt_f + \frac{\partial^2 \Phi}{\partial \mathbf{x}_f^T \partial \mathbf{x}_f} d\mathbf{x}_f\right)$$
$$- \delta \mathbf{x}_f^T \left(d\boldsymbol{\lambda}_f - \delta \boldsymbol{\lambda}_f\right) + \int_{t_i}^{t_f} \begin{bmatrix} \delta \mathbf{x}^T & \delta \mathbf{u}^T \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \mathbf{x}^T \partial \mathbf{x}} & \frac{\partial H}{\partial \mathbf{x}^T \partial \mathbf{u}} \\ \frac{\partial H}{\partial \mathbf{u}^T \partial \mathbf{x}} & \frac{\partial H}{\partial \mathbf{u}^T \partial \mathbf{u}} \end{bmatrix} \begin{bmatrix} \delta \mathbf{x} \\ \delta \mathbf{u} \end{bmatrix} dt \tag{4.25}$$

Consider the term associated with the terminal control

$$\frac{\partial \Omega}{\partial \mathbf{u}_f} d\mathbf{u}_f = \underbrace{\left(\overbrace{\frac{\partial}{\partial \mathbf{u}_f}\left(\frac{\partial \Phi}{\partial t_f}\right)}^{\mathbf{0}} + \overbrace{\frac{\partial H_f}{\partial \mathbf{u}_f}}^{\mathbf{0}}\right)}_{\mathbf{0}} d\mathbf{u}_f \tag{4.26}$$

where Φ is not a function of the control, and the necessary conditions require that the partial of the Hamiltonian be zero. Use the relation between differentials and variations

$$d\mathbf{x}_f = \delta\mathbf{x}_f + \dot{\mathbf{x}}_f dt_f \tag{4.27}$$

$$d\boldsymbol{\lambda}_f = \delta\boldsymbol{\lambda}_f + \underbrace{\dot{\boldsymbol{\lambda}}_f}_{-\frac{\partial H}{\partial \mathbf{x}_f^T}} dt_f \tag{4.28}$$

to further simplify the expressions

$$\begin{aligned} d^2J' = {} & dt_f \left[\left(\frac{\partial \Omega}{\partial t_f} + \frac{\partial \Omega}{\partial \mathbf{x}_f}\dot{\mathbf{x}}_f \right) dt_f + \frac{\partial \Omega}{\partial \mathbf{x}_f}\delta\mathbf{x}_f \right] \\ & + \delta\mathbf{x}_f^T \left[\left(\overbrace{\frac{\partial^2 \Phi}{\partial \mathbf{x}_f^T \partial t_f} + \frac{\partial H}{\partial \mathbf{x}_f^T} + \frac{\partial^2 \Phi}{\partial \mathbf{x}_f^T \partial \mathbf{x}_f}\dot{\mathbf{x}}_f}^{\frac{\partial \Omega}{\partial \mathbf{x}^T}} \right) dt_f + \frac{\partial^2 \Phi}{\partial \mathbf{x}_f^T \partial \mathbf{x}_f}\delta\mathbf{x}_f \right] \\ & + \int_{t_i}^{t_f} \begin{bmatrix} \delta\mathbf{x}^T & \delta\mathbf{u}^T \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \mathbf{x}^T \partial \mathbf{x}} & \frac{\partial H}{\partial \mathbf{x}^T \partial \mathbf{u}} \\ \frac{\partial H}{\partial \mathbf{u}^T \partial \mathbf{x}} & \frac{\partial H}{\partial \mathbf{u}^T \partial \mathbf{u}} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x} \\ \delta\mathbf{u} \end{bmatrix} dt \end{aligned} \tag{4.29}$$

where

$$\frac{\partial H}{\partial \mathbf{x}_f^T} = \frac{\partial L}{\partial \mathbf{x}_f^T} + \left(\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}_f^T} \right)^T \overbrace{\frac{\partial \Phi}{\partial \mathbf{x}_f}}^{\boldsymbol{\lambda}_f} \tag{4.30}$$

The second differential of the cost function can now be written succinctly as a quadratic expression

$$\begin{aligned} d^2J' = {} & \begin{bmatrix} \delta\mathbf{x}_f^T & dt_f \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \Phi}{\partial \mathbf{x}_f^T \partial \mathbf{x}_f} & \frac{\partial \Omega}{\partial \mathbf{x}^T} \\ \frac{\partial \Omega}{\partial \mathbf{x}_f} & \frac{\partial \Omega}{\partial t_f} + \frac{\partial \Omega}{\partial \mathbf{x}_f}\dot{\mathbf{x}}_f \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_f \\ dt_f \end{bmatrix} \\ & + \int_{t_i}^{t_f} \begin{bmatrix} \delta\mathbf{x}^T & \delta\mathbf{u}^T \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \mathbf{x}^T \partial \mathbf{x}} & \frac{\partial H}{\partial \mathbf{x}^T \partial \mathbf{u}} \\ \frac{\partial H}{\partial \mathbf{u}^T \partial \mathbf{x}} & \frac{\partial H}{\partial \mathbf{u}^T \partial \mathbf{u}} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x} \\ \delta\mathbf{u} \end{bmatrix} dt \end{aligned} \tag{4.31}$$

where

$$\frac{\partial\Omega}{\partial\mathbf{x}} = \frac{\partial^2\Phi}{\partial t_f\partial\mathbf{x}_f} + \underbrace{\frac{\partial L}{\partial\mathbf{x}_f} + \overbrace{\frac{\partial\Phi}{\partial\mathbf{x}}}^{\boldsymbol{\lambda}_f^T}\frac{\partial\boldsymbol{f}}{\partial\mathbf{x}_f}^T}_{\frac{\partial H}{\partial\mathbf{x}_f}} + \dot{\mathbf{x}}_f^T\overbrace{\frac{\partial^2\Phi}{\partial\mathbf{x}_f^T\partial\mathbf{x}_f}}^{\frac{\partial\boldsymbol{\lambda}}{\partial\mathbf{x}_f}} \tag{4.32}$$

It is important to recall that this expression relies on the first order constraints

$$\delta\dot{\mathbf{x}} = \frac{\partial\boldsymbol{f}}{\partial\mathbf{x}}\delta\mathbf{x} + \frac{\partial\boldsymbol{f}}{\partial\mathbf{u}}\delta\mathbf{u} \tag{4.33}$$

$$d\boldsymbol{\psi} = \mathbf{0} \tag{4.34}$$

The neighboring optimal control problem now consists of finding $\delta\mathbf{u}$ such that the quantity d^2J is minimized.

$$\begin{aligned} \text{minimize:} \quad & d^2J' \\ \text{subject to:} \quad & \delta\dot{\mathbf{x}} = \frac{\partial\boldsymbol{f}}{\partial\mathbf{x}}\delta\mathbf{x} + \frac{\partial\boldsymbol{f}}{\partial\mathbf{u}}\delta\mathbf{u} \\ & \mathbf{0} = d\boldsymbol{\psi} \\ & t_i \text{ and } \delta\mathbf{x}_i \text{ are specified} \end{aligned}$$

The solution to this problem is well known [82,112,126] and follows directly from the calculus of variations. The details of this solution are not difficult, and not shown here. The solution consists of the following the necessary conditions

$$\delta\dot{\mathbf{x}} = \frac{\partial\boldsymbol{f}}{\partial\mathbf{x}}\delta\mathbf{x} + \frac{\partial\boldsymbol{f}}{\partial\mathbf{u}}\delta\mathbf{u} \tag{4.35}$$

$$\mathbf{0} = d\boldsymbol{\psi} \tag{4.36}$$

$$\delta\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial\mathbf{x}^T\partial\mathbf{x}}\delta\mathbf{x} - \frac{\partial H}{\partial\mathbf{x}^T\partial\mathbf{u}}\delta\mathbf{u} - \frac{\partial\boldsymbol{f}}{\partial\mathbf{x}^T}\delta\boldsymbol{\lambda} \tag{4.37}$$

$$\mathbf{0} = \frac{\partial H}{\partial\mathbf{u}^T\partial\mathbf{x}}\delta\mathbf{x} + \frac{\partial H}{\partial\mathbf{u}^T\partial\mathbf{u}}\delta\mathbf{u} + \frac{\partial\boldsymbol{f}}{\partial\mathbf{x}^T}\delta\boldsymbol{\lambda} \tag{4.38}$$

subject to the terminal boundary conditions

$$\begin{bmatrix} \delta\boldsymbol{\lambda}_f \\ d\boldsymbol{\psi} \\ d\Omega \end{bmatrix} = \begin{bmatrix} \frac{\partial^2\Phi}{\partial\mathbf{x}_f^T\partial\mathbf{x}_f} & \frac{\partial\boldsymbol{\psi}}{\partial\mathbf{x}_f^T} & \frac{\partial\Omega}{\partial\mathbf{x}_f^T} \\ \frac{\partial\boldsymbol{\psi}}{\partial\mathbf{x}_f} & \mathbf{0} & \frac{\partial\boldsymbol{\psi}}{\partial t_f} + \frac{\partial\boldsymbol{\psi}}{\partial\mathbf{x}_f}\boldsymbol{f}_f \\ \frac{\partial\boldsymbol{\Omega}}{\partial\mathbf{x}_f} & \frac{\partial\boldsymbol{\psi}}{\partial t_f^T} + \boldsymbol{f}_f^T\frac{\partial\boldsymbol{\psi}}{\partial\mathbf{x}_f^T} & \frac{\partial\Omega}{\partial t_f} + \frac{\partial\Omega}{\partial\mathbf{x}_f}\boldsymbol{f}_f \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_f \\ d\boldsymbol{\nu} \\ dt_f \end{bmatrix} \tag{4.39}$$

Once again, it is important to remember that all partials are evaluated along the nominal trajectory, $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$.

Methods for finding a solution to the two point boundary value problem are also well documented in the literature [82,111,122,127–130]. These methods are not covered here for the sake of brevity.

It is important to note that considerable advantage has been found in indexing the feedback gains by the “time-to-go” ($t_{go} = t_f - t$). Indexing the gains this way has been shown to greatly improve the effectiveness of neighboring optimal control [82,112,125,131]. This approach also insures that the system does not “run out” of control gains as it nears the terminal state. This technique of indexing with respect to t_{go} is an artifact of solving a boundary value problem with free final time and given initial time.

In summary, the performance index can be represented as a truncated variational Taylor series expansion to second order about the optimal trajectory

$$J' \approx \bar{J}' + dJ' + \frac{1}{2!}d^2J' \tag{4.40}$$

where $\bar{J}'$ represents the cost evaluated at the optimal value, and dJ' and d^2J' are the first and second order changes in the cost as the trajectory departs from the optimal value. Optimization theory requires that $dJ' = 0$ be a necessary condition for an optimal value. Thus finding dJ' and equating it to zero results in necessary conditions for an optimal trajectory. The second order term, d^2J' is then minimized by linearizing the dynamics about the nominal trajectory.

The resulting control consists of two terms: an open loop (nominal) term, and a feedback term which accounts for the state perturbation. The quadratic structure of the second order term coupled with linear dynamics insures that the feedback term is linear with respect to the state perturbation.

$$\mathbf{u}(t) = \bar{\mathbf{u}}(t_{go}) + \overbrace{\mathbf{C}(t_{go})\underbrace{(\mathbf{x}(t_{go}) - \bar{\mathbf{x}}(t_{go}))}_{\delta\mathbf{x}}}^{\delta\mathbf{u}} \tag{4.41}$$

For most applications $\bar{\mathbf{u}}$, $\bar{\mathbf{x}}$, and $\mathbf{C}$ are computed a priori and implemented using table lookups during flight. These table lookups are indexed using t_{go}.

The neighboring optimal control approach uses the second order term to minimize the original cost function when the state is perturbed. It is important to note that the neighboring optimal control approach requires that the nominal trajectory be compliant with the classical Euler-Lagrange equations. If the nominal trajectory does not conform the classical Euler-Lagrange equations, the term d^2J' is significantly more complex, and closed form solutions for the feedback law may not result.

Bryson and Jardin make the observation [111, 132, 133] that the feedback gain matrix from neighboring optimal control is simply the partials of the nominal control function with respect to the current state at the same t_{go}.

$$\mathbf{C}(t_{go}) = \left.\frac{\partial\bar{\mathbf{u}}}{\partial\mathbf{x}}\right|_{t_{go}} \tag{4.42}$$

Bryson and Jardin also demonstrate that for some problems, it is easier to compute these feedback gains by evaluating this partial derivative along a known nominal trajectory rather than by solving a boundary value problem.

Application of the neighboring optimal control and classical quadratic synthesis to stochastic systems both tacitly rely on the certainty and equivalence principle [82, 91–95]. However, the certainty and equivalence principle states only applies to a narrow range of systems.

The certainty and equivalence principle states that the expectation of the cost function is minimized by connecting the optimal (Kalman) estimator, with the optimal deterministic controller from the calculus of variations. However, this is only true when the following conditions are met.

1. The cost function has quadratic criteria.

2. The state dynamics are linear.

3. The measurements are linear in the state.

4. The estimator is a full-state estimator.

5. The state is propagated using a mathematical model and not with model replacement.[3]

6. The noise signals corrupting the dynamics, measurement, and control are additive Gaussian white noise processes.

This tells us that if the stander neighboring optimal control problem is augmented with linear measurements and additive Gaussian white noise errors, the expectation of the quadratic performance index, d^2J', can be minimized by applying a standard Kalman filter, and the neighboring optimal controller.

This allows for a statement regarding the expectation of the original cost function from neighboring optimal control

$$E\left[J'\right] \approx \underbrace{E\left[\bar{J}'\right]}_{\bar{J}'} + \underbrace{E\left[dJ'\right]}_{\boldsymbol{\lambda}_i^T \delta \mathbf{x}_i} + \frac{1}{2!} E\left[d^2 J'\right] \tag{4.43}$$

where $\boldsymbol{\lambda}_i^T \delta \mathbf{x}_i$ is a fixed quantity that accounts for the sensitivity of the cost due to a perturbation in the state at the initial time.

Since the certainty and equivalence principle states that $E\left[d^2J'\right]$ is minimized, the quantity $E\left[J'\right]$ is also made smaller. It is important to emphasize that the application of the certainty and equivalence principle to most GNC systems requires that $\bar{J}'$ and $E\left[d^2J'\right]$ be minimized independently. That is, $\bar{J}'$ is minimized by selection of a deterministic nominal trajectory, and then $E\left[d^2J'\right]$is minimized by applying linear control and estimation theory in the linear neighborhood of the trajectory found in the first step. Later in this document, stochastic quadratic synthesis will show that, in some cases, this reasoning ignores the coupling between $\bar{J}'$ and $E\left[d^2J'\right]$ via the nominal trajectory used to find $\bar{J}'$. Nevertheless, traditional quadratic synthesis serves a central role in modern GNC development.

[3] In navigation context, *model replacement* refers to propagation of the state through direct integration of sensor data. Most commonly gyros and accelerometers are integrated directly to propagate the state. This process is sometimes referred to as *dead reckoning*. In other words, measurements are used to replace the dynamic model.

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Chapter 5

Classical Quadratic Synthesis

This chapter presents an alternative to neighboring optimal control. In the previous chapter neighboring optimal control was developed using an expansion of the constrained cost function. In this chapter classical quadratic synthesis is developed by expanding the unconstrained cost function. As will be seen, this difference is equivalent to taking the expansion of the Lagrangian rather than the Hamiltonian.

As was done in the previous chapter for neighboring optimal control, this approach to quadratic synthesis views the system as the superposition of an unperturbed, non-linear nominal system with a perturbed linear system. Well known results from linear system theory (linear estimator and linear feedback) are applied to the linear portion of the problem, while the optimal nominal trajectory is generated using results from the classical solution to the Bolza problem from the calculus of variations. This nominal, or reference, trajectory is the point about which system is linearized.

5.1 Linear Perturbation Model

The nominal trajectory is denoted by a bar over the vector and a dispersed state, denoted with a δ. The true state, control, and measurement are the sum of the nominal and the dispersion.

$$\mathbf{u} = \bar{\mathbf{u}} + \delta\mathbf{u} \tag{5.1}$$

$$\mathbf{x} = \bar{\mathbf{x}} + \delta\mathbf{x} \tag{5.2}$$

$$\mathbf{y} = \bar{\mathbf{y}} + \delta\mathbf{y} \tag{5.3}$$

These quantities are subject to nonlinear equations describing the dynamics and the measurement

$$\dot{\mathbf{x}} = \boldsymbol{f}\left(\mathbf{x}, \mathbf{u}, t\right) + \mathbf{w} \tag{5.4}$$

$$\mathbf{y} = \boldsymbol{h}\left(\mathbf{x}, \mathbf{u}, t\right) + \boldsymbol{\nu} \tag{5.5}$$

These nonlinear equations are driven by additive white Gaussian noise with the following properties

$$E\left[\mathbf{w}\right] = \mathbf{0}, \qquad E\left[\mathbf{w}\left(t\right)\mathbf{w}^T\left(\tau\right)\right] = \mathbf{Q}\delta\left(t-\tau\right) \tag{5.6}$$

$$E\left[\boldsymbol{\nu}\right] = \mathbf{0}, \qquad E\left[\boldsymbol{\nu}\left(t\right)\boldsymbol{\nu}^T\left(\tau\right)\right] = \mathbf{R}\delta\left(t-\tau\right) \tag{5.7}$$

The differential equation governing the trajectory dispersion can be found by taking a Taylor expansions of f about the nominal

$$\dot{\mathbf{x}} = \dot{\bar{\mathbf{x}}} + \delta\dot{\mathbf{x}} = \boldsymbol{f}\left(\bar{\mathbf{x}} + \delta\mathbf{x}, \bar{\mathbf{u}} + \delta\mathbf{u}, t\right) + \mathbf{w} \approx \underbrace{\boldsymbol{f}\left(\bar{\mathbf{x}}, \bar{\mathbf{u}}, t\right)}_{\dot{\bar{\mathbf{x}}}} + \underbrace{\overbrace{\left.\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}}^{\mathbf{F}} \delta\mathbf{x} + \overbrace{\left.\frac{\partial \boldsymbol{f}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}}^{\mathbf{G}} \delta\mathbf{u} + \ldots + \mathbf{w}}_{\delta\dot{\mathbf{x}}} \tag{5.8}$$

Since $\delta\mathbf{x}$ is small, a very good approximation of the dispersed dynamics can be provided by keeping the linear term from the Taylor series, and discarding all higher order terms. This results in linear dispersion dynamics

$$\delta\dot{\mathbf{x}} = \mathbf{F}\delta\mathbf{x} + \mathbf{G}\delta\mathbf{u} + \mathbf{w} \tag{5.9}$$

The measurement dispersion equation is also found using the Taylor series.

$$\mathbf{y} = \bar{\mathbf{y}} + \delta\mathbf{y} = \boldsymbol{h}\left(\bar{\mathbf{x}} + \delta\mathbf{x}, \bar{\mathbf{u}} + \delta\mathbf{u}, t\right) + \boldsymbol{\nu} \approx \underbrace{\boldsymbol{h}\left(\bar{\mathbf{x}}, \bar{\mathbf{u}}, t\right)}_{\bar{\mathbf{y}}} + \underbrace{\overbrace{\left.\frac{\partial \boldsymbol{h}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}}^{\mathbf{H}} \delta\mathbf{x} + \overbrace{\left.\frac{\partial \boldsymbol{h}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}}^{\mathbf{D}} \delta\mathbf{u} + \ldots + \boldsymbol{\nu}}_{\delta\mathbf{y}} \tag{5.10}$$

Again, owing to the smallness of $\delta\mathbf{x}$, the dispersed measurement equation is given by keeping only the linear terms from the Taylor series.

$$\delta\mathbf{y} = \mathbf{H}\delta\mathbf{x} + \mathbf{D}\delta\mathbf{u} + \boldsymbol{\nu} \tag{5.11}$$

At this point two key assumptions are made. These assumptions are applied to the linear system, and are motivated from the results of linear system theory.

1. The dispersed control is assumed to be a linear function of the dispersed state.

$$\delta \mathbf{u} = -\mathbf{C}\delta\hat{\mathbf{x}} + \boldsymbol{\eta} \tag{5.12}$$

   where the feedback gains,$\mathbf{C}$, will be selected as part of the optimization process. The control execution error, $\boldsymbol{\eta}$, is a white Gaussian noise process with

$$E\left[\boldsymbol{\eta}\right] = \mathbf{0} \tag{5.13}$$

$$E\left[\boldsymbol{\eta}\left(t\right)\boldsymbol{\eta}^{T}\left(\tau\right)\right] = \mathbf{U}\delta\left(t-\tau\right) \tag{5.14}$$

2. The state estimate, $\delta\hat{\mathbf{x}}$, is generated by a linear unbiased estimator. An unbiased estimator is an estimator whose estimation error

$$\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x} = \delta\hat{\mathbf{x}} - \delta\mathbf{x} \tag{5.15}$$

   is expected to be zero

$$E\left[\mathbf{e}\right] = \mathbf{0} \tag{5.16}$$

   The linear unbiased estimator has the following form

$$\delta\dot{\hat{\mathbf{x}}} = \mathbf{F}\delta\hat{\mathbf{x}} + \mathbf{G}\delta\mathbf{u} + \mathbf{K}\left(\delta\mathbf{y} - \mathbf{H}\delta\hat{\mathbf{x}} - \mathbf{D}\delta\mathbf{u}\right) \tag{5.17}$$

   where the estimation gain, $\mathbf{K}$, will be found as part of the optimization procedure.

At this point the first statistical moments of the estimate and the estimation error can be computed. Equations 5.11 and 5.12 are now substituted into the equation 5.17

$$\delta\dot{\hat{\mathbf{x}}} = \mathbf{F}\delta\hat{\mathbf{x}} - \mathbf{G}\mathbf{C}\delta\hat{\mathbf{x}} + \mathbf{G}\boldsymbol{\eta} + \mathbf{K}\left(\mathbf{H}\underbrace{\left(\delta\mathbf{x} - \delta\hat{\mathbf{x}}\right)}_{-\mathbf{e}} + \boldsymbol{\nu}\right) \tag{5.18}$$

The expectation of this equation is now taken

$$E\left[\delta\dot{\hat{\mathbf{x}}}\right] = \frac{d}{dt}E\left[\delta\hat{\mathbf{x}}\right] = \left(\mathbf{F}-\mathbf{G}\mathbf{C}\right)E\left[\delta\hat{\mathbf{x}}\right] + \mathbf{G}\underbrace{E\left[\boldsymbol{\eta}\right]}_{\mathbf{0}} - \mathbf{K}\mathbf{H}\underbrace{E\left[\mathbf{e}\right]}_{\mathbf{0}} + \mathbf{K}\underbrace{E\left[\boldsymbol{\nu}\right]}_{\mathbf{0}} \tag{5.19}$$

Assuming that the initial estimate is expected to be zero, $E\left[\delta\hat{\mathbf{x}}\left(t_0\right)\right] = 0$, it is clear that

$$E\left[\delta\hat{\mathbf{x}}\left(t\right)\right] = 0 \tag{5.20}$$

for all t. This result can be substituted into equation 5.16 to find the expectation of the true dispersion

$$E\left[\mathbf{e}\right] = \underbrace{E\left[\delta\hat{\mathbf{x}}\right]}_{\mathbf{0}} - E\left[\delta\mathbf{x}\right] = 0$$

it follows directly that

$$E\left[\delta\mathbf{x}\left(t\right)\right] = 0 \tag{5.21}$$

for all t. The expected value of the control dispersion can be found by taking the expectation of equation 5.12

$$E\left[\delta\mathbf{u}\right] = -\mathbf{C}\underbrace{E\left[\delta\hat{\mathbf{x}}\right]}_{\mathbf{0}} + \underbrace{E\left[\boldsymbol{\eta}\right]}_{\mathbf{0}} \tag{5.22}$$

clearly

$$E\left[\delta\mathbf{u}\left(t\right)\right] = 0 \tag{5.23}$$

for all t.

These assumptions allow for the construction of the augmented linear state and the augmented linear state dynamics. The augmented state vector is formed by concatenating the dispersed state with its estimate. Begin by substituting equation 5.11 and 5.12 into equations 5.9 and 5.17. This is conveniently expressed using the augmented state

$$\underbrace{\begin{bmatrix} \delta\dot{\mathbf{x}} \\ \delta\dot{\hat{\mathbf{x}}} \end{bmatrix}}_{\dot{\mathbf{z}}} = \underbrace{\begin{bmatrix} \mathbf{F} & -\mathbf{G}\mathbf{C} \\ \mathbf{K}\mathbf{H} & \mathbf{F}-\mathbf{G}\mathbf{C}-\mathbf{K}\mathbf{H} \end{bmatrix}}_{\boldsymbol{\mathcal{F}}} \underbrace{\begin{bmatrix} \delta\mathbf{x} \\ \delta\hat{\mathbf{x}} \end{bmatrix}}_{\mathbf{z}} + \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{G} \\ \mathbf{0} & \mathbf{K} & \mathbf{G} \end{bmatrix}}_{\boldsymbol{\mathcal{G}}} \underbrace{\begin{bmatrix} \mathbf{w} \\ \boldsymbol{\nu} \\ \boldsymbol{\eta} \end{bmatrix}}_{\boldsymbol{\xi}} \tag{5.24}$$

The covariance associated with the augmented state is

$$\mathbf{P} = \begin{bmatrix} \mathbf{X} & \mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}}^T \\ \mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}} & \hat{\mathbf{X}} \end{bmatrix} = E\left[\mathbf{z}\mathbf{z}^T\right] \tag{5.25}$$

The differential equations for $\mathbf{P}$ are well known [14, 15, 18, 134].

$$\dot{\mathbf{P}} = \boldsymbol{\mathcal{F}}\mathbf{P} + \mathbf{P}\boldsymbol{\mathcal{F}}^T + \boldsymbol{\mathcal{G}} \begin{bmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{U} \end{bmatrix} \boldsymbol{\mathcal{G}}^T \tag{5.26}$$

Applying the mapping

$$\begin{bmatrix} \mathbf{e} \\ \delta\hat{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} -\mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x} \\ \delta\hat{\mathbf{x}} \end{bmatrix} \tag{5.27}$$

to equation 5.24 gives the differential equations for the estimation error

$$\underbrace{\begin{bmatrix} \dot{\mathbf{e}} \\ \delta\dot{\hat{\mathbf{x}}} \end{bmatrix}}_{\dot{\mathbf{z}}'} = \underbrace{\begin{bmatrix} \mathbf{F} - \mathbf{K}\mathbf{H} & \mathbf{0} \\ -\mathbf{K}\mathbf{H} & \mathbf{F} - \mathbf{G}\mathbf{C} \end{bmatrix}}_{\boldsymbol{\mathcal{F}}'} \underbrace{\begin{bmatrix} \mathbf{e} \\ \delta\hat{\mathbf{x}} \end{bmatrix}}_{\mathbf{z}'} + \underbrace{\begin{bmatrix} -\mathbf{I} & \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} & \mathbf{G} \end{bmatrix}}_{\boldsymbol{\mathcal{G}}'} \underbrace{\begin{bmatrix} \mathbf{w} \\ \boldsymbol{\nu} \\ \boldsymbol{\eta} \end{bmatrix}}_{\boldsymbol{\xi}} \tag{5.28}$$

Thus

$$\dot{\mathbf{P}}' = \boldsymbol{\mathcal{F}}'\mathbf{P}' + \mathbf{P}'\boldsymbol{\mathcal{F}}'^T + \boldsymbol{\mathcal{G}}' \begin{bmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{U} \end{bmatrix} \boldsymbol{\mathcal{G}}'^T \tag{5.29}$$

where

$$\mathbf{P}' = \begin{bmatrix} \mathbf{E} & \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}^T \\ \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} & \hat{\mathbf{X}} \end{bmatrix} = E\left[\mathbf{z}'\mathbf{z}'^T\right] \tag{5.30}$$

The covariance propagation in equations 5.29 can be written as three separate matrix equations.

$$\dot{\mathbf{E}} = (\mathbf{F} - \mathbf{KH})\,\mathbf{E} + \mathbf{E}\,(\mathbf{F} - \mathbf{KH})^T + \mathbf{Q} + \mathbf{KRK}^T \tag{5.31}$$

$$\dot{\mathbf{P}}_{\hat{\mathbf{x}}\mathbf{e}} = (\mathbf{F} - \mathbf{GC})\,\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} + \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\,(\mathbf{F} - \mathbf{KH})^T - \mathbf{KHE} + \mathbf{KRK}^T \tag{5.32}$$

$$\dot{\hat{\mathbf{X}}} = (\mathbf{F} - \mathbf{GC})\,\hat{\mathbf{X}} + \hat{\mathbf{X}}\,(\mathbf{F} - \mathbf{GC})^T - \mathbf{KHP}_{\hat{\mathbf{x}}\mathbf{e}}^T - \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\mathbf{H}^T\mathbf{K}^T + \mathbf{KRK}^T + \mathbf{GUG} \tag{5.33}$$

This is important because the estimation error, $\mathbf{e}$, and its covariance, $\mathbf{E}$, can be propagated from an initial condition without knowledge of the control gains, $\mathbf{C}$. The relationship between $\mathbf{P}$ and $\mathbf{P}'$ will also be useful

$$\begin{bmatrix} \mathbf{X} & \mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}}^T \\ \mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}} & \hat{\mathbf{X}} \end{bmatrix} = \begin{bmatrix} -\mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{E} & \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}^T \\ \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} & \hat{\mathbf{X}} \end{bmatrix} \begin{bmatrix} -\mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^{-T} \tag{5.34}$$

$$= \begin{bmatrix} -\mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{E} & \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}^T \\ \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} & \hat{\mathbf{X}} \end{bmatrix} \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \end{bmatrix} \tag{5.35}$$

$$= \begin{bmatrix} \mathbf{E} - \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} - \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}^T + \hat{\mathbf{X}} & \hat{\mathbf{X}} - \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}^T \\ \hat{\mathbf{X}} - \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} & \hat{\mathbf{X}} \end{bmatrix} \tag{5.36}$$

5.2 Expansion of the Performance Index

Consider the cost function before constraints are used to augment it

$$J = \phi\left(\mathbf{x}_f, t_f\right) + \int_{t_i}^{t_f} L\left(\mathbf{x}, \mathbf{u}, t\right) dt \tag{5.37}$$

where the final time is free. The performance index is now expanded to second order using the Gateaux [120] derivative

$$J \approx \underbrace{\bar{\phi} + \int_{t_i}^{t_f} \bar{L} dt}_{\bar{J}} + \underbrace{\frac{\partial \phi}{\partial \mathbf{x}_f} \delta \mathbf{x}_f + \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial \mathbf{x}} \delta \mathbf{x} + \frac{\partial L}{\partial \mathbf{u}} \delta \mathbf{u} \right] dt}_{dJ}$$
$$+ \underbrace{\frac{1}{2} \delta \mathbf{x}_f^T \frac{\partial^2 \phi}{\partial \mathbf{x}_f^T \partial \mathbf{x}_f} \delta \mathbf{x}_f + \frac{1}{2} \int_{t_i}^{t_f} \left(\begin{bmatrix} \delta \mathbf{x}^T & \delta \mathbf{u}^T \end{bmatrix} \begin{bmatrix} \frac{\partial^2 L}{\partial \mathbf{x}^T \partial \mathbf{x}} & \frac{\partial^2 L}{\partial \mathbf{x}_f^T \partial \mathbf{u}} \\ \frac{\partial^2 L}{\partial \mathbf{u}^T \partial \mathbf{x}} & \frac{\partial^2 L}{\partial \mathbf{u}^T \partial \mathbf{u}} \end{bmatrix} \begin{bmatrix} \delta \mathbf{x} \\ \delta \mathbf{u} \end{bmatrix} \right) dt}_{\frac{1}{2!} d^2 J} \quad (5.38)$$

Note that this expansion is about a time correlated trajectory. That is, the final time for the nominal trajectory is allowed to vary, but the final time for the perturbed trajectory, $\delta \mathbf{x}$, is not permitted to differ from that of the nominal trajectory $\bar{\mathbf{x}}$. This is a key difference between this approach, and the neighboring optimal control approach. Now take the expectation[1]

$$E[J] \approx \underbrace{E\left[\bar{\phi} + \int_{t_i}^{t_f} \bar{L} dt \right]}_{\bar{J}} + \underbrace{\frac{\partial \phi}{\partial \mathbf{x}_f} \overbrace{E[\delta \mathbf{x}_f]}^{\mathbf{0}} + \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial \mathbf{x}} \overbrace{E[\delta \mathbf{x}]}^{\mathbf{0}} + \frac{\partial L}{\partial \mathbf{u}} \overbrace{E[\delta \mathbf{u}]}^{\mathbf{0}} \right] dt}_{dJ}$$
$$\underbrace{+ \frac{1}{2} \mathrm{tr} \left(\overbrace{\frac{\partial^2 \phi}{\partial \mathbf{x}_f^T \partial \mathbf{x}_f}}^{\mathbf{S}_f} \overbrace{E\left[\delta \mathbf{x}_f \delta \mathbf{x}_f^T \right]}^{\mathbf{X}_f} \right) + \frac{1}{2} \int_{t_i}^{t_f} \mathrm{tr} \left(\begin{bmatrix} \overbrace{\frac{\partial^2 L}{\partial \mathbf{x}^T \partial \mathbf{x}}}^{\mathbf{A}} & \overbrace{\frac{\partial^2 L}{\partial \mathbf{x}_f^T \partial \mathbf{u}}}^{\mathbf{N}^T} \\ \underbrace{\frac{\partial^2 L}{\partial \mathbf{u}^T \partial \mathbf{x}}}_{\mathbf{N}} & \underbrace{\frac{\partial^2 L}{\partial \mathbf{u}^T \partial \mathbf{u}}}_{\mathbf{B}} \end{bmatrix} E \left\{ \begin{bmatrix} \delta \mathbf{x} \\ \delta \mathbf{u} \end{bmatrix} \begin{bmatrix} \delta \mathbf{x}^T & \delta \mathbf{u}^T \end{bmatrix} \right\} \right) dt}_{\frac{1}{2!} d^2 J}$$

$$(5.40)$$

Note that the last expectation on the right hand side can be written as

$$E \left\{ \begin{bmatrix} \delta \mathbf{x} \\ \delta \mathbf{u} \end{bmatrix} \begin{bmatrix} \delta \mathbf{x}^T & \delta \mathbf{u}^T \end{bmatrix} \right\} = \begin{bmatrix} E\left[\delta \mathbf{x} \delta \mathbf{x}^T \right] & E\left[\delta \mathbf{x} \delta \mathbf{u}^T \right] \\ E\left[\delta \mathbf{u} \delta \mathbf{x}^T \right] & E\left[\delta \mathbf{u} \delta \mathbf{u}^T \right] \end{bmatrix} \quad (5.41)$$

[1] Note that

$$E\left[\mathbf{a}^T \mathbf{A} \mathbf{c} \right] = E\left[\mathrm{tr}\left(\mathbf{a}^T \mathbf{A} \mathbf{c} \right) \right] = E\left[\mathrm{tr}\left(\mathbf{A} \mathbf{c} \mathbf{a}^T \right) \right] = \mathrm{tr}\left(\mathbf{A} E\left[\mathbf{c} \mathbf{a}^T \right] \right) \quad (5.39)$$

where

$$E\left[\delta\mathbf{x}\delta\mathbf{x}^T\right] = \mathbf{X} \tag{5.42}$$

$$E\left[\delta\mathbf{u}\delta\mathbf{x}^T\right] = -\mathbf{C}\overbrace{E\left[\delta\hat{\mathbf{x}}\delta\mathbf{x}^T\right]}^{\mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}}} + \overbrace{E\left[\boldsymbol{\eta}\delta\mathbf{x}^T\right]}^{\mathbf{0}} \tag{5.43}$$

$$E\left[\delta\mathbf{u}\delta\mathbf{u}^T\right] = \mathbf{C}\overbrace{E\left[\delta\hat{\mathbf{x}}\delta\hat{\mathbf{x}}^T\right]}^{\hat{\mathbf{X}}}\mathbf{C}^T - \mathbf{C}\overbrace{E\left[\delta\hat{\mathbf{x}}\boldsymbol{\eta}^T\right]}^{\mathbf{0}} - \overbrace{E\left[\boldsymbol{\eta}\delta\hat{\mathbf{x}}^T\right]}^{\mathbf{0}}\mathbf{C}^T + \overbrace{E\left[\boldsymbol{\eta}\boldsymbol{\eta}^T\right]}^{\mathbf{U}} \tag{5.44}$$

Now the integrand of the second order term becomes

$$\begin{aligned} &\operatorname{tr}\left(\begin{bmatrix}\mathbf{A} & \mathbf{N}^T \\ \mathbf{N} & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{X} & -\mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}}^T\mathbf{C}^T \\ -\mathbf{C}\mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}} & \mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T + \mathbf{U}\end{bmatrix}\right) \\ &\qquad = \operatorname{tr}\left(\mathbf{A}\mathbf{X} - \mathbf{N}^T\mathbf{C}\mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}}\right) + \operatorname{tr}\left(\mathbf{B}\mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T - \mathbf{N}\mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}}^T\mathbf{C}^T + \mathbf{B}\mathbf{U}\right) \end{aligned} \tag{5.45}$$

and the cost function can now be written as

$$E[J] \approx \phi + \int_{t_i}^{t_f} L dt + \frac{1}{2}\operatorname{tr}\left(\mathbf{S}_f\mathbf{X}_f\right) + \frac{1}{2}\int_{t_i}^{t_f}\left[\operatorname{tr}\left(\mathbf{A}\mathbf{X} - \mathbf{N}^T\mathbf{C}\mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}}\right) + \operatorname{tr}\left(\mathbf{B}\mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T - \mathbf{N}\mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}}^T\mathbf{C}^T + \mathbf{B}\mathbf{U}\right)\right] dt \tag{5.46}$$

The original cost function is extracted from the expectation operator because is is a deterministic quantity computed about the nominal trajectory.

5.3 An Explicit Expression for the Stochastic Cost

Often, it is convenient to separate the cost due to the initial dispersion, from the cost due to cumulative effects of noise on the system over the time frame of interest. This goal is accomplished by using integration by parts

$$\mathbf{S}_f\mathbf{X}_f - \mathbf{S}_i\mathbf{X}_i = \int_{t_i}^{t_f}\frac{d\left(\mathbf{S}\mathbf{X}\right)}{dt}dt = \int_{t_i}^{t_f}\left(\dot{\mathbf{S}}\mathbf{X} + \mathbf{S}\dot{\mathbf{X}}\right)dt \tag{5.47}$$

The trace operator is now applied

$$\operatorname{tr}\left(\mathbf{S}_f\mathbf{X}_f\right) = \operatorname{tr}\left(\mathbf{S}_i\mathbf{X}_i\right) + \int_{t_i}^{t_f}\operatorname{tr}\left(\dot{\mathbf{S}}\mathbf{X} + \mathbf{S}\dot{\mathbf{X}}\right)dt \tag{5.48}$$

This expression is placed in the second order expansion for the cost function

$$E\left[J\right]-\bar{J}=\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_i\mathbf{X}_i\right)+\frac{1}{2}\int_{t_i}^{t_f}\Big[\mathrm{tr}\left(\dot{\mathbf{S}}\mathbf{X}+\mathbf{S}\dot{\mathbf{X}}+\mathbf{A}\mathbf{X}-\mathbf{N}^T\mathbf{C}\hat{\mathbf{X}}\right)$$
$$+\mathrm{tr}\left(\mathbf{B}\mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T-\mathbf{N}\hat{\mathbf{X}}\mathbf{C}^T+\mathbf{B}\mathbf{U}\right)\Big]\,dt \quad (5.49)$$

The differential equation for $\dot{\mathbf{X}}$ can be found using equation 5.26

$$\dot{\mathbf{X}}=\mathbf{F}\mathbf{X}+\mathbf{X}\mathbf{F}^T-\mathbf{G}\mathbf{C}\hat{\mathbf{X}}-\hat{\mathbf{X}}\mathbf{C}^T\mathbf{G}^T+\mathbf{Q}+\mathbf{G}\mathbf{U}\mathbf{G}^T \quad (5.50)$$

This relation and equation 5.96 are substituted into the cost function

$$E\left[J\right]-\bar{J}=\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_i\mathbf{X}_i\right)+\frac{1}{2}\int_{t_i}^{t_f}\Big[\mathrm{tr}\Big(-\mathbf{F}\mathbf{S}\mathbf{X}-\mathbf{S}\mathbf{F}\mathbf{X}+\mathbf{C}^T\mathbf{B}\mathbf{C}\mathbf{X}-\mathbf{A}\mathbf{X}$$
$$+\mathbf{S}\mathbf{F}\mathbf{X}+\mathbf{S}\mathbf{X}\mathbf{F}^T-\mathbf{S}\mathbf{G}\mathbf{C}\hat{\mathbf{X}}-\mathbf{S}\hat{\mathbf{X}}\mathbf{C}^T\mathbf{G}^T+\mathbf{S}\mathbf{Q}+\mathbf{S}\mathbf{G}\mathbf{U}\mathbf{G}^T+\mathbf{A}\mathbf{X}-\mathbf{N}^T\mathbf{C}\hat{\mathbf{X}}\Big)$$
$$+\mathrm{tr}\left(\mathbf{B}\mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T-\mathbf{N}\hat{\mathbf{X}}\mathbf{C}^T+\mathbf{B}\mathbf{U}\right)\Big]\,dt \quad (5.51)$$

where the substitution from equation 5.36 has been made. Canceling like terms gives the following equation.

$$E\left[J\right]-\bar{J}=\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_i\mathbf{X}_i\right)+\frac{1}{2}\int_{t_i}^{t_f}\Big[\mathrm{tr}\left(\mathbf{C}^T\mathbf{B}\mathbf{C}\mathbf{X}-\mathbf{S}\mathbf{G}\mathbf{C}\hat{\mathbf{X}}-\mathbf{S}\hat{\mathbf{X}}\mathbf{C}^T\mathbf{G}^T+\mathbf{S}\mathbf{Q}+\mathbf{S}\mathbf{G}\mathbf{U}\mathbf{G}^T-\mathbf{N}^T\mathbf{C}\hat{\mathbf{X}}\right)$$
$$+\mathrm{tr}\left(\mathbf{B}\mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T-\mathbf{N}\hat{\mathbf{X}}\mathbf{C}^T+\mathbf{B}\mathbf{U}\right)\Big]\,dt \quad (5.52)$$

Since the trace is a linear operator it can now be applied to each individual term of the integrand. With the trace now applied to each term the invariance of the trace operator with respect to the transpose and cyclic permutations can be used to collect like terms

$$E\left[J\right]-\bar{J}=\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_i\mathbf{X}_i\right)+\frac{1}{2}\int_{t_i}^{t_f}\Big[\mathrm{tr}\left(\mathbf{S}\mathbf{Q}+\mathbf{S}\mathbf{G}\mathbf{U}\mathbf{G}^T\right)+\mathrm{tr}\left(\mathbf{C}^T\mathbf{B}\mathbf{C}\left(\mathbf{X}+\hat{\mathbf{X}}\right)\right)+\mathrm{tr}\left(\mathbf{B}\mathbf{U}\right)$$
$$-2\mathrm{tr}\left(\mathbf{C}^T\mathbf{N}\hat{\mathbf{X}}\right)-2\mathrm{tr}\left(\mathbf{C}^T\mathbf{G}^T\mathbf{S}\hat{\mathbf{X}}\right)\Big]\,dt \quad (5.53)$$

Since the dimensions of matrices for many terms are the same, the linearity of the trace operator can be used to combine terms

$$E\left[J\right]-\bar{J}=\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_i\mathbf{X}_i\right)+\frac{1}{2}\int_{t_i}^{t_f}\left[\mathrm{tr}\left(\mathbf{SQ}+\mathbf{SGUG}^T+\mathbf{C}^T\mathbf{BC}\left(\mathbf{X}+\hat{\mathbf{X}}\right)-2\mathbf{C}^T\overbrace{\left(\mathbf{G}^T\mathbf{S}+\mathbf{N}\right)}^{\mathbf{BC}}\hat{\mathbf{X}}\right)\right.$$
$$\left.+\mathrm{tr}\left(\mathbf{BU}\right)\right]dt \quad (5.54)$$

Two terms can now be combined

$$E\left[J\right]-\bar{J}=\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_i\mathbf{X}_i\right)+\frac{1}{2}\int_{t_i}^{t_f}\left[\mathrm{tr}\left(\mathbf{S}\left(\mathbf{Q}+\mathbf{GUG}^T\right)+\mathbf{C}^T\mathbf{BC}\overbrace{\left(\mathbf{X}-\hat{\mathbf{X}}\right)}^{\mathbf{E}}\right)+\mathrm{tr}\left(\mathbf{BU}\right)\right]dt \quad (5.55)$$

The cost or the dispersed control about the nominal can now be found without ever having to numerically compute the covariances using equation 5.29. All that are needed to complete the calculation are the parameters **E** and **S** which are computed computed about the nominal for the estimator and the controller.

5.4 Solution to the Classical Quadratic Synthesis Problem

The classical approach breaks the problem into two separate optimization problems. The first optimization problem is the deterministic portion of the cost subject to the deterministic constraints. The solution to this problem is the nominal trajectory used for the second portion of the problem. The deterministic optimization problem is summarized as follows.

$$\text{minimize:} \quad \phi\left(\bar{\mathbf{x}}_f,t_f\right)+\int_{t_i}^{t_f}L\left(\bar{\mathbf{x}},\bar{\mathbf{u}},t\right)dt$$

$$\begin{aligned}\text{subject to:} \quad & \dot{\bar{x}}=\boldsymbol{f}\left(\bar{\mathbf{x}},\bar{\mathbf{u}},t\right)\\ & \mathbf{0}=\boldsymbol{\psi}\left(\bar{\mathbf{x}}_f,t_f\right)\\ & t_i \text{ and } \bar{\mathbf{x}}_i \text{ are specified}\end{aligned}$$

This is the well known Bolza problem with variable final time whose well known solution was presented earlier. The necessary conditions are

$$\dot{\bar{x}} = \boldsymbol{f}\left(\bar{\mathbf{x}}, \bar{\mathbf{u}}, t\right) \tag{5.56}$$

$$\bar{\mathbf{x}}_i \text{ is specified} \tag{5.57}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \bar{\mathbf{x}}^T} \tag{5.58}$$

$$\boldsymbol{\lambda}_f = \frac{\partial \Phi}{\partial \bar{\mathbf{x}}_f^T} \tag{5.59}$$

$$\mathbf{0} = \frac{\partial H}{\partial \bar{\mathbf{u}}} \tag{5.60}$$

$$\mathbf{0} = \boldsymbol{\psi}\left(\bar{\mathbf{x}}_f, t_f\right) \tag{5.61}$$

$$0 = \frac{\partial \Phi}{\partial t_f} + \underbrace{\overbrace{\frac{\partial \Phi}{\partial \mathbf{x}_f}}^{\boldsymbol{\lambda}_f^T} \dot{\bar{x}}_f + L_f}_{H_f} = \Omega \tag{5.62}$$

The last necessary condition is only needed for free terminal time problems. The Hamiltonian is given by

$$H = L + \boldsymbol{\lambda}^T \boldsymbol{f} \tag{5.63}$$

The partials in the above equations are all evaluated along the nominal trajectory, $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$.

Once the classical deterministic problem is solved, the second optimization problem is

$$\text{minimize:}\quad \frac{1}{2}\mathrm{tr}\left(\mathbf{S}_f\mathbf{X}_f\right) + \frac{1}{2}\int_{t_i}^{t_f}\left[\mathrm{tr}\left(\mathbf{A}\mathbf{X} - \mathbf{N}^T\mathbf{C}\mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}}\right) + \mathrm{tr}\left(\mathbf{B}\mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T - \mathbf{N}\mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}}^T\mathbf{C}^T + \mathbf{B}\mathbf{U}\right)\right]dt$$

$$\begin{aligned}
\text{subject to:}\quad & \mathbf{X} = \mathbf{E} - \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} - \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}^T + \hat{\mathbf{X}} \\
& \mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}} = \hat{\mathbf{X}} - \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} \\
& \dot{\mathbf{E}} = \left(\mathbf{F} - \mathbf{K}\mathbf{H}\right)\mathbf{E} + \mathbf{E}\left(\mathbf{F} - \mathbf{K}\mathbf{H}\right)^T + \mathbf{Q} + \mathbf{K}\mathbf{R}\mathbf{K}^T \\
& \dot{\mathbf{P}}_{\hat{\mathbf{x}}\mathbf{e}} = \left(\mathbf{F} - \mathbf{G}\mathbf{C}\right)\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} + \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\left(\mathbf{F} - \mathbf{K}\mathbf{H}\right)^T - \mathbf{K}\mathbf{H}\mathbf{E} + \mathbf{K}\mathbf{R}\mathbf{K}^T \\
& \dot{\hat{\mathbf{X}}} = \left(\mathbf{F} - \mathbf{G}\mathbf{C}\right)\hat{\mathbf{X}} + \hat{\mathbf{X}}\left(\mathbf{F} - \mathbf{G}\mathbf{C}\right)^T - \mathbf{K}\mathbf{H}\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}^T - \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\mathbf{H}^T\mathbf{K}^T + \mathbf{K}\mathbf{R}\mathbf{K}^T + \mathbf{G}\mathbf{U}\mathbf{G} \\
& \left(\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\right)_i = \mathbf{0}, \quad \hat{\mathbf{X}}_i = \mathbf{0} \quad \mathbf{E}_i = \mathbf{E}_0 \\
& t_i,\ t_f,\ \text{and } \mathbf{X}_i \text{ are specified} \\
& \text{Required partials are evaluated at } \bar{\mathbf{x}} \text{ and } \bar{\mathbf{u}}
\end{aligned}$$

The estimation gain can now be found by taking the differential of this cost function with respect to the estimation gain $\mathbf{K}$.

$$dJ_2 \approx \mathrm{tr}\left(\mathbf{S}_f\delta\mathbf{X}_f\right) + \int_{t_i}^{t_f}\left[\mathrm{tr}\left(\mathbf{A}\delta\mathbf{X} - \mathbf{N}^T\mathbf{C}\delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}}\right) + \mathrm{tr}\left(\mathbf{B}\mathbf{C}\delta\hat{\mathbf{X}}\mathbf{C}^T - \mathbf{N}\delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}}^T\mathbf{C}^T\right)\right]dt$$

where

$$\delta\mathbf{X} = \delta\mathbf{E} - \delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} - \delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}^T + \delta\hat{\mathbf{X}} \tag{5.64}$$

$$\delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{x}} = \delta\hat{\mathbf{X}} - \delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} \tag{5.65}$$

and

$$\delta\dot{\mathbf{E}} = \left(\mathbf{F} - \mathbf{K}\mathbf{H}\right)\delta\mathbf{E} + \delta\mathbf{E}\left(\mathbf{F} - \mathbf{K}\mathbf{H}\right)^T - \delta\mathbf{K}\mathbf{H}\mathbf{E} - \mathbf{E}\mathbf{H}^T\delta\mathbf{K}^T + \delta\mathbf{K}\mathbf{R}\mathbf{K}^T + \mathbf{K}\mathbf{R}\delta\mathbf{K}^T \tag{5.66}$$

$$\delta\dot{\mathbf{P}}_{\hat{\mathbf{x}}\mathbf{e}} = -\mathbf{KH}\delta\mathbf{E} + (\mathbf{F}-\mathbf{GC})\,\delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} + \delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\,(\mathbf{F}-\mathbf{KH})^T - \delta\mathbf{KHE} - \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\mathbf{H}^T\delta\mathbf{K}^T + \delta\mathbf{KRK}^T + \mathbf{KR}\delta\mathbf{K}^T \quad (5.67)$$

$$\delta\dot{\hat{\mathbf{X}}} = (\mathbf{F}-\mathbf{GC})\,\delta\hat{\mathbf{X}} + \delta\hat{\mathbf{X}}\,(\mathbf{F}-\mathbf{GC})^T - \mathbf{KH}\delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} - \delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\mathbf{H}^T\mathbf{K}^T - \delta\mathbf{KHP}_{\hat{\mathbf{x}}\mathbf{e}} - \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\mathbf{H}^T\delta\mathbf{K}^T + \delta\mathbf{KRK}^T + \mathbf{KR}\delta\mathbf{K}^T \quad (5.68)$$

By hypothesis, select the estimator gain to be the Kalman gain

$$\mathbf{K} = \mathbf{EH}^T\mathbf{R}^{-1} \quad (5.69)$$

The Kalman gain invokes the orthogonality principle [21]. That is

$$\mathbf{e}^T\hat{\mathbf{x}} = 0$$

This can be seen by examining the differential equation for $\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}$ taken from equation 5.29 and substituting in the Kalman gain

$$\begin{aligned}
\dot{\mathbf{P}}_{\hat{\mathbf{x}}\mathbf{e}} &= -\mathbf{KHE} + (\mathbf{F}-\mathbf{GC})\,\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} + \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\,(\mathbf{F}-\mathbf{KH})^T + \mathbf{KRK}^T && (5.70)\\
&= -\mathbf{EH}^T\mathbf{R}^{-1}\mathbf{HE} + (\mathbf{F}-\mathbf{GC})\,\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} + \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\left(\mathbf{F}-\mathbf{EH}^T\mathbf{R}^{-1}\mathbf{H}\right)^T + \mathbf{EH}^T\mathbf{R}^{-1}\mathbf{HE} && (5.71)\\
&= (\mathbf{F}-\mathbf{GC})\,\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} + \mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\left(\mathbf{F}-\mathbf{EH}^T\mathbf{R}^{-1}\mathbf{H}\right)^T && (5.72)
\end{aligned}$$

Clearly, if the initial error and the initial estimate are uncorrelated ($\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\,(t_0) = 0$), they remain uncorrected for all time

$$\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\,(t) = 0 \quad (5.73)$$

It follows directly that

$$\mathbf{X} = \hat{\mathbf{X}} + \mathbf{E} \quad (5.74)$$

Substitution of this result along with the Kaman gain into the equations 5.66-5.68 gives

$$\delta\dot{\mathbf{E}} = \left(\mathbf{F} - \mathbf{E}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)\delta\mathbf{E} + \delta\mathbf{E}\left(\mathbf{F} - \mathbf{E}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)^T \tag{5.75}$$

$$\delta\dot{\mathbf{P}}_{\hat{\mathbf{x}}\mathbf{e}} = -\mathbf{E}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\delta\mathbf{E} + (\mathbf{F} - \mathbf{G}\mathbf{C})\,\delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} + \delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\left(\mathbf{F} - \mathbf{E}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)^T + \mathbf{E}\mathbf{H}^T\delta\mathbf{K}^T \tag{5.76}$$

$$\delta\dot{\hat{\mathbf{X}}} = (\mathbf{F} - \mathbf{G}\mathbf{C})\,\delta\hat{\mathbf{X}} + \delta\hat{\mathbf{X}}\,(\mathbf{F} - \mathbf{G}\mathbf{C})^T - \mathbf{E}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} - \delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{E} + \delta\mathbf{K}\mathbf{H}\mathbf{E} + \mathbf{E}\mathbf{H}^T\delta\mathbf{K}^T \tag{5.77}$$

Assume that the initial covariance is a given quantity (constant)

$$\delta\mathbf{E}_i = \left(\delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}}\right)_i = \delta\hat{\mathbf{X}}_i = \mathbf{0} \tag{5.78}$$

it follows directly from equation 5.75 that the variation in the error covariance is zero for all time

$$\delta\mathbf{E} = \mathbf{0} \tag{5.79}$$

This leads to the conclusion that the variance of the Kalman gain is also zero for all time

$$\delta\mathbf{K} = \delta\mathbf{E}\mathbf{H}^T\mathbf{R}^{-1} = \mathbf{0} \tag{5.80}$$

Substituting this into equation 5.76 shows that

$$\delta\mathbf{P}_{\hat{\mathbf{x}}\mathbf{e}} = \mathbf{0} \tag{5.81}$$

Finally, this result is used in equation 5.77 to show

$$\delta\hat{\mathbf{X}} = \mathbf{0} \tag{5.82}$$

Since the Kalman gain causes all the variations in the covariance terms to be zero, the variation in the cost function is also zero when the Kalman estimator is used.

$$dJ_2 \approx 0 \tag{5.83}$$

Thus the Kalman estimator is shown to be the optimal estimator.

Substituting the Kalman gain into the equation 5.29 results in significant simplification of the differential equations

$$\dot{\mathbf{E}} = \mathbf{FE} + \mathbf{EF}^T - \mathbf{EH}^T\mathbf{R}^{-1}\mathbf{HE} + \mathbf{Q} \tag{5.84}$$

$$\dot{\hat{\mathbf{X}}} = (\mathbf{F} - \mathbf{GC})\,\hat{\mathbf{X}} + \hat{\mathbf{X}}\,(\mathbf{F} - \mathbf{GC})^T + \mathbf{EH}^T\mathbf{R}^{-1}\mathbf{HE} + \mathbf{GUG} \tag{5.85}$$

The stochastic portion of the optimization program can now be written as

$$\begin{aligned}\text{minimize:}\quad & \operatorname{tr}(\mathbf{S}_f\mathbf{E}_f) + \operatorname{tr}\left(\mathbf{S}_f\hat{\mathbf{X}}_f\right) + \int_{t_i}^{t_f}\left[\operatorname{tr}\left(\mathbf{AE} + \mathbf{A}\hat{\mathbf{X}} - \mathbf{N}^T\mathbf{C}\hat{\mathbf{X}}\right)\right.\\ & \left.+\operatorname{tr}\left(\mathbf{BC}\hat{\mathbf{X}}\mathbf{C}^T - \mathbf{N}\hat{\mathbf{X}}\mathbf{C}^T + \mathbf{BU}\right)\right] dt \end{aligned} \tag{5.86}$$

$$\begin{aligned}\text{subject to:}\quad & \dot{\mathbf{E}} = \mathbf{FE} + \mathbf{EF}^T - \mathbf{EH}^T\mathbf{R}^{-1}\mathbf{HE} + \mathbf{Q}\\ & \dot{\hat{\mathbf{X}}} = (\mathbf{F} - \mathbf{GC})\,\hat{\mathbf{X}} + \hat{\mathbf{X}}\,(\mathbf{F} - \mathbf{GC})^T + \mathbf{EH}^T\mathbf{R}^{-1}\mathbf{HE} + \mathbf{GUG}\\ & \hat{\mathbf{X}}_i = \mathbf{0}\\ & t_i,\ t_f,\ \text{and } \mathbf{E}_i\ \mathbf{X}_i \text{ are specified}\\ & \text{Required partials are evaluated at } \bar{\mathbf{x}} \text{ and } \bar{\mathbf{u}}\end{aligned}$$

This problem is amenable to a solution via the calculus of variations. Finding this solution is greatly aided by using the Hamiltonian

$$\mathcal{H} = \operatorname{tr}\left(\mathbf{AE} + \mathbf{A}\hat{\mathbf{X}} - \mathbf{N}^T\mathbf{C}\hat{\mathbf{X}}\right) + \operatorname{tr}\left(\mathbf{BC}\hat{\mathbf{X}}\mathbf{C}^T - \mathbf{N}\hat{\mathbf{X}}\mathbf{C}^T + \mathbf{BU}\right) + \operatorname{tr}\left(\mathbf{\Lambda}\dot{\mathbf{E}}\right) + \operatorname{tr}\left(\mathbf{S}\dot{\hat{\mathbf{X}}}\right) \tag{5.87}$$

or

$$\begin{aligned}\mathcal{H} = {} & \operatorname{tr}(\mathbf{AE}) + \operatorname{tr}\left(\left(\mathbf{A} - \mathbf{N}^T\mathbf{C} - \mathbf{C}^T\mathbf{N} + \mathbf{C}^T\mathbf{BC}\right)\hat{\mathbf{X}}\right) + \operatorname{tr}(\mathbf{BU})\\ & + \operatorname{tr}\left(\mathbf{\Lambda FE} + \mathbf{\Lambda EF}^T - \mathbf{\Lambda EH}^T\mathbf{R}^{-1}\mathbf{HE} + \mathbf{\Lambda Q}\right)\\ & + \operatorname{tr}\left(\mathbf{S}\,(\mathbf{F} - \mathbf{GC})\,\hat{\mathbf{X}} + \mathbf{S}\hat{\mathbf{X}}\,(\mathbf{F} - \mathbf{GC})^T + \mathbf{SEH}^T\mathbf{R}^{-1}\mathbf{HE} + \mathbf{SGUG}\right)\end{aligned} \tag{5.88}$$

The necessary conditions are given by

$$\dot{\mathbf{\Lambda}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{E}} \qquad , \qquad \mathbf{\Lambda}(t_f) = \mathbf{S}_f \tag{5.89}$$

$$\dot{\mathbf{S}} = -\frac{\partial \mathcal{H}}{\partial \hat{\mathbf{X}}} \qquad , \qquad \mathbf{S}(t_f) = \mathbf{S}_f \tag{5.90}$$

and

$$\mathbf{0} = \frac{\partial \mathcal{H}}{\partial \mathbf{C}} \tag{5.91}$$

These partials are relatively easy to evaluate

$$\dot{\mathbf{S}} = -(\mathbf{F}-\mathbf{GC})^T\mathbf{S} - \mathbf{S}(\mathbf{F}-\mathbf{GC}) + \mathbf{C}^T\mathbf{N} + \mathbf{N}^T\mathbf{C} - \mathbf{C}^T\mathbf{BC} - \mathbf{A} \tag{5.92}$$

$$\dot{\mathbf{\Lambda}} = -\left(\mathbf{F}-\mathbf{EH}^T\mathbf{R}^{-1}\mathbf{H}\right)^T\mathbf{\Lambda} - \mathbf{\Lambda}\left(\mathbf{F}-\mathbf{EH}^T\mathbf{R}^{-1}\mathbf{H}\right) - \mathbf{SEH}^T\mathbf{R}^{-1}\mathbf{H} - \mathbf{H}^T\mathbf{R}^{-1}\mathbf{HES} - \mathbf{A} \tag{5.93}$$

$$\mathbf{0} = 2\underbrace{\left(-\mathbf{N}+\mathbf{BC}-\mathbf{G}^T\mathbf{S}\right)}_{\mathbf{0}}\hat{\mathbf{X}} \tag{5.94}$$

The final term in the last equation is zero because $\hat{\mathbf{X}}$ is known to be positive definite. Now the control can be solved and used to simplify the expression for $\dot{\mathbf{S}}$

$$\mathbf{C} = \mathbf{B}^{-1}\left(\mathbf{G}^T\mathbf{S}+\mathbf{N}\right) \tag{5.95}$$

$$\dot{\mathbf{S}} = -\mathbf{FS} - \mathbf{SF} + \left(\mathbf{G}^T\mathbf{S}+\mathbf{N}\right)^T\mathbf{B}^{-1}\left(\mathbf{G}^T\mathbf{S}+\mathbf{N}\right) - \mathbf{A} \tag{5.96}$$

which is the result from classical optimal control of linear systems with quadratic criteria.

5.5 A Simple Example

Insight into the character of systems designed using the certainty and equivalence principle can be gained by using it to solve a simple one-dimensional problem. Consider the following optimization problem

$$\text{minimize:} \quad E\left[\frac{1}{2}S_f x\left(t_f\right)^2 + \frac{1}{2}\int_{t_0}^{t_f}\left[Ax^2 + Bu^2\right]dt\right]$$

$$\text{subject to:} \quad \dot{x} = -\frac{1}{\tau}x + u + w$$

$$y = x + \nu$$

where the statistical properties are given by

$$E\left[w\right] = \mathbf{0}, \qquad E\left[w\left(t\right)w\left(t'\right)\right] = Q\delta\left(t - t'\right) \tag{5.97}$$

$$E\left[\nu\right] = \mathbf{0}, \qquad E\left[\nu\left(t\right)\nu\left(t'\right)\right] = R\delta\left(t - t'\right) \tag{5.98}$$

$$E\left[x\left(t_0\right)\right] = \mathbf{0}, \qquad E\left[x\left(t_0\right)x\left(t_0\right)\right] = X_0 \tag{5.99}$$

The solution to this problem is found using the certainty and equivalence principle. This tells us that the solution is the optimal linear estimator (the Kalman-Bucy filter)

$$\dot{\hat{x}} = -\frac{1}{\tau}\hat{x} + u + K\left(z - \hat{x}\right) \tag{5.100}$$

with initial conditions

$$\hat{x}\left(t_0\right) = 0$$

The certainty and equivalence principle also states the the optimal controller is the quadratic terminal controller from linear system theory.

$$u = -C\hat{x}$$

The feedback and estimation gains are given by

$$K = \frac{P}{R} \tag{5.101}$$

$$C = \frac{S}{B} \tag{5.102}$$

where P, and S are found using the differential Riccati equations

$$\dot{P} = -\frac{2}{\tau}P - \frac{1}{R}P^2 + Q \tag{5.103}$$

$$\dot{S} = \frac{2}{\tau}S + \frac{1}{B}S^2 - A \tag{5.104}$$

subject to the boundary conditions

$$P\left(t_0\right) = X_0 \tag{5.105}$$

$$S\left(t_f\right) = S_f \tag{5.106}$$

these equations have known analytic solutions

$$P = R\left(\frac{1}{2\tau_P} - \frac{1}{\tau}\right) + \frac{\frac{R}{\tau_P}\left[X_0 - R\left(\frac{1}{2\tau_P} - \frac{1}{\tau}\right)\right]}{\left[X_0 + R\left(\frac{1}{2\tau_P} + \frac{1}{\tau}\right)\right]e^{\frac{t-t_0}{\tau_P}} - \left[X_0 - R\left(\frac{1}{2\tau_P} - \frac{1}{\tau}\right)\right]} \tag{5.107}$$

$$S = B\left(\frac{1}{2\tau_S} - \frac{1}{\tau}\right) + \frac{\frac{B}{\tau_S}\left[S_f - B\left(\frac{1}{2\tau_S} - \frac{1}{\tau}\right)\right]}{\left[S_f + B\left(\frac{1}{2\tau_S} + \frac{1}{\tau}\right)\right]e^{\frac{t_f-t}{\tau_S}} - \left[S_f - B\left(\frac{1}{2\tau_S} - \frac{1}{\tau}\right)\right]} \tag{5.108}$$

where the time constants are given by

$$\frac{1}{\tau_P} = 2\sqrt{\frac{Q}{R} + \frac{1}{\tau^2}} \tag{5.109}$$

$$\frac{1}{\tau_S} = 2\sqrt{\frac{A}{B} + \frac{1}{\tau^2}} \tag{5.110}$$

Analytic expressions for the gains can now be written as a function of time

$$K = \left(\frac{1}{2\tau_P} - \frac{1}{\tau}\right) + \frac{\frac{1}{\tau_P}\left[X_0 - R\left(\frac{1}{2\tau_P} - \frac{1}{\tau}\right)\right]}{\left[X_0 + R\left(\frac{1}{2\tau_P} + \frac{1}{\tau}\right)\right]e^{\frac{t-t_0}{\tau_P}} - \left[X_0 - R\left(\frac{1}{2\tau_P} - \frac{1}{\tau}\right)\right]} \tag{5.111}$$

$$C = \left(\frac{1}{2\tau_S} - \frac{1}{\tau}\right) + \frac{\frac{1}{\tau_S}\left[S_f - B\left(\frac{1}{2\tau_S} - \frac{1}{\tau}\right)\right]}{\left[S_f + B\left(\frac{1}{2\tau_S} + \frac{1}{\tau}\right)\right]e^{\frac{t_f-t}{\tau_S}} - \left[S_f - B\left(\frac{1}{2\tau_S} - \frac{1}{\tau}\right)\right]} \tag{5.112}$$

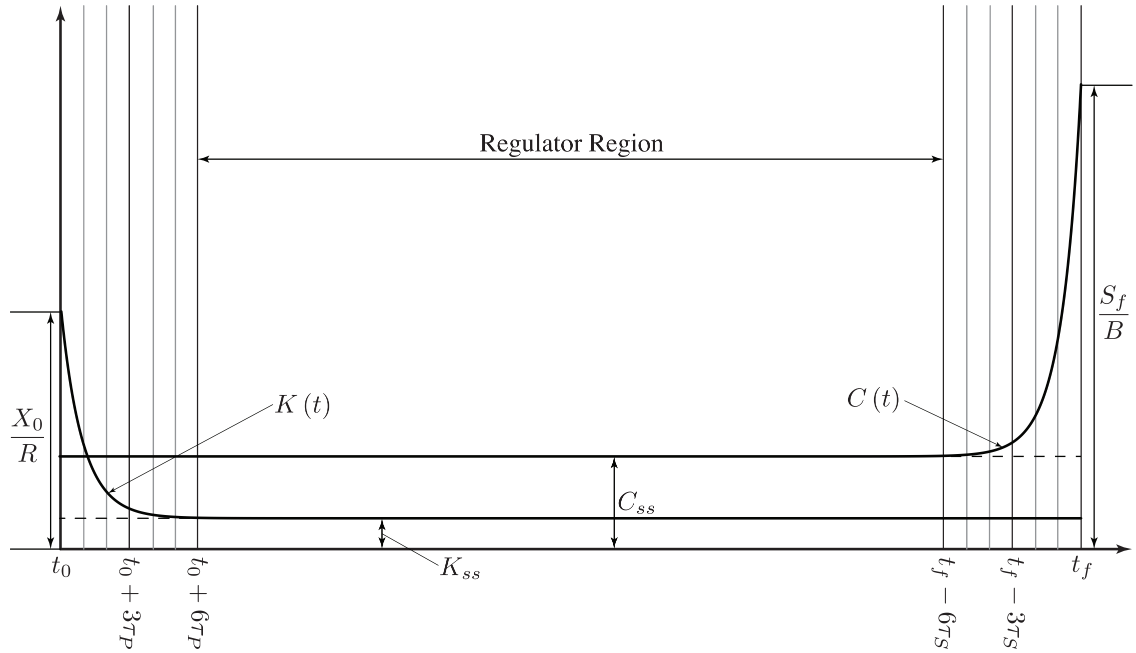

Fig. 5.1: The time history of the control and estimation gains.

These have the following limits at the boundary conditions

$$\lim_{t\to t_0} K = \frac{X_0}{R} \tag{5.113}$$

$$\lim_{t\to t_f} C = \frac{S_f}{B} \tag{5.114}$$

Near these boundary conditions a transient exists. Like most functions involving exponentials this transient dies away within a few time constants of the boundary. The steady state values can be found by taking the appropriate limit

$$K_{ss} = \lim_{t\to\infty} K = \frac{1}{2\tau_P} - \frac{1}{\tau} \tag{5.115}$$

$$C_{ss} = \lim_{t\to-\infty} C = \frac{1}{2\tau_S} - \frac{1}{\tau} \tag{5.116}$$

Although these gains are dependent on several parameters, these equations can be used to provide a general characterization of the gains as shown in figure 5.1.

Figure 5.1 clearly shows that the navigation gains experience an initial transient and quickly approach a steady state value. Similarly, the feedback gains begin at a steady state value and experience a transient near the terminal time. Between these two transients, there is a region where the gains are nearly static and the entire GNC system is essentially behaving as a regulator.

Linear covariance analysis techniques can now be applied to analyze the statistical behavior of the system.

The augmented state dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau} & -C \\ K & -\frac{1}{\tau} - C - K \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} w \\ \nu \end{bmatrix} \tag{5.117}$$

can be written in terms of the navigation error, $e = x - \hat{x}$ as

$$\begin{bmatrix} \dot{e} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau} - K & 0 \\ K & -\frac{1}{\tau} - C \end{bmatrix} \begin{bmatrix} e \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 1 & -K \\ 0 & K \end{bmatrix} \begin{bmatrix} w \\ \nu \end{bmatrix} \tag{5.118}$$

The covariance equations for this state are well known.

$$\begin{aligned} \begin{bmatrix} \dot{P} & \dot{N} \\ \dot{N} & \dot{\hat{X}} \end{bmatrix} = & \begin{bmatrix} -\frac{1}{\tau} - K & 0 \\ K & -\frac{1}{\tau} - C \end{bmatrix} \begin{bmatrix} P & N \\ N & \hat{X} \end{bmatrix} + \begin{bmatrix} P & N \\ N & \hat{X} \end{bmatrix} \begin{bmatrix} -\frac{1}{\tau} - K & 0 \\ K & -\frac{1}{\tau} - C \end{bmatrix}^T \\ & + \begin{bmatrix} 1 & -K \\ 0 & K \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} 1 & -K \\ 0 & K \end{bmatrix}^T \end{aligned} \tag{5.119}$$

which can be written as

$$\begin{aligned} \begin{bmatrix} \dot{P} & \dot{N} \\ \dot{N} & \dot{\hat{X}} \end{bmatrix} = & \begin{bmatrix} -2\left(\frac{1}{\tau} + K\right) P & KP - \left(\frac{1}{\tau} + K\right) N - \left(\frac{1}{\tau} + C\right) N \\ KP - \left(\frac{1}{\tau} + K\right) N - \left(\frac{1}{\tau} + C\right) N & 2KN - 2\left(\frac{1}{\tau} + C\right) \hat{X} \end{bmatrix} \\ & + \begin{bmatrix} Q + K^2 R & -K^2 R \\ -K^2 R & R \end{bmatrix} \end{aligned} \tag{5.120}$$

The expression for the Kaman gain can be substituted into the off diagonal equation

$$\dot{N} = KP - \left(\frac{1}{\tau} + K\right) N - \left(\frac{1}{\tau} + C\right) N - K^2 R \tag{5.121}$$

$$= \frac{P}{R} P - \left(\frac{1}{\tau} + K\right) N - \left(\frac{1}{\tau} + C\right) N - \frac{P^2}{R^2} R \tag{5.122}$$

$$= -\left(\frac{2}{\tau} + K + C\right) N \tag{5.123}$$

Since the initial navigation error is uncorrelated with the initial estimate, $N\left(t_0\right) = E\left[e\left(t_0\right)\hat{x}\left(t_0\right)\right] = 0$, $N(t)$ must be zero. Now the differential equations for the diagonal elements reduce to

$$\dot{P} = -2\left(\frac{1}{\tau} + K\right) P + Q + K^2 R \tag{5.124}$$

$$\dot{\hat{X}} = -2\left(\frac{1}{\tau} + C\right) \hat{X} + R \tag{5.125}$$

where the first equation was previously solved analytically. The second equation might be solved by using an integrating factor, but the algebra associated with an analytic solution quickly expands and becomes unwieldy. However, the steady state values that are reached during the regulation phase can be found without difficulty

$$P_{ss} = K_{ss} R = \left(\frac{1}{2\tau_P} - \frac{1}{\tau}\right) R \tag{5.126}$$

$$\hat{X}_{ss} = \frac{R}{2\left(\frac{1}{\tau} + C_{ss}\right)} = R\tau_S \tag{5.127}$$

Since the true state is given by $x = \hat{x} + e$, the variance of the true state is

$$E\left[x^2\right] = X = \hat{X} + P \tag{5.128}$$

where the state error is uncorrelated to the state estimate. The steady state value

$$X_{ss} = \left(\tau_s + \frac{1}{2\tau_P} - \frac{1}{\tau}\right) R \tag{5.129}$$

The key characteristics of these equations are shown in figure 5.2.

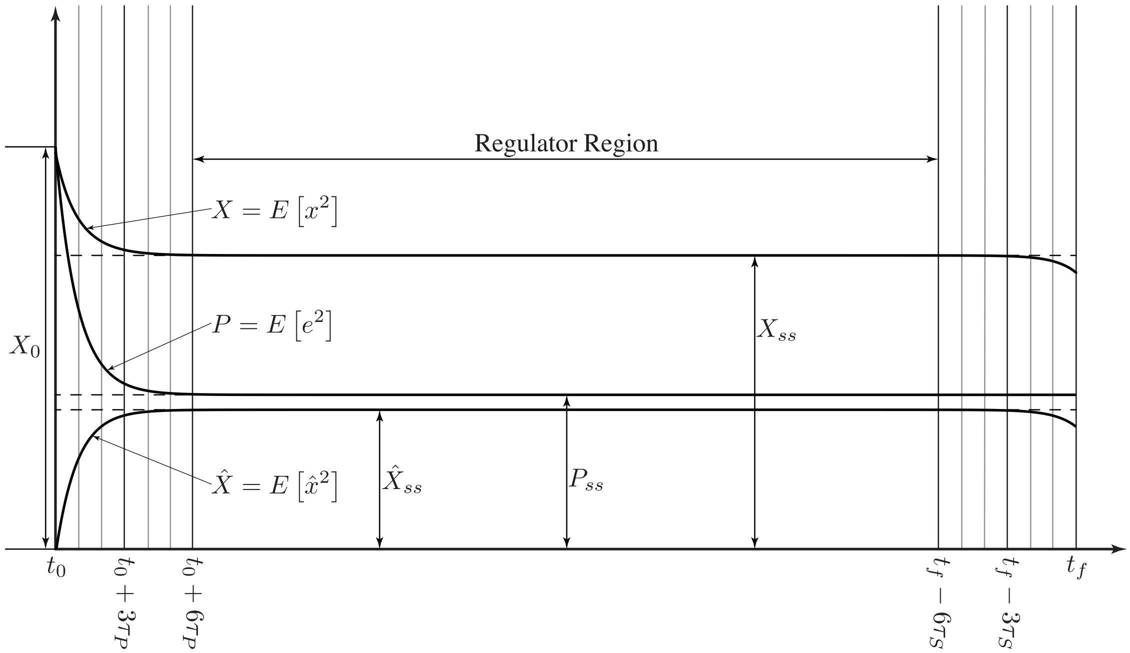

Fig. 5.2: Time history of the dispersions.

An expression for the cost can can now be written

$$J = E\left[\frac{1}{2}S_f x\left(t_f\right)^2 + \frac{1}{2}\int_{t_0}^{t_f}\left[Ax^2 + Bu^2\right]dt\right] \tag{5.130}$$

$$= \frac{1}{2}S_f \underbrace{E\left[x\left(t_f\right)^2\right]}_{X_f} + \frac{1}{2}\int_{t_0}^{t_f}\left(A\underbrace{E\left[x^2\right]}_{X} + B\underbrace{E\left[u^2\right]}_{C^2\hat{X}}\right)dt \tag{5.131}$$

Although a closed form expression is not readily available, the needed numeric integration is not difficult to perform.

This simple example show the characteristic behavior of quadratically synthesized systems. The response can be divided into three distinct regions. First, there is a transient near the initial time associated with filter starting up. Second, there is a region where the system operates in a steady state fashion. Finally, near the terminal time there is a transient associated with the quadratic terminal controller. It is important to remember that this example is a linear time invariant system. However, broadly speaking, these same transient are still present in time varying systems. While many of these distinct characteristics and analytic solutions may not exist for time varying linear systems, the theory can still be applied to generate the optimal solution.

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Chapter 6

Stochastic Quadratic Synthesis

This chapter contains a development of Schmidt's approach to optimizing statistical parameters for a stochastic nonlinear system [108–110]. Stochastic quadratic synthesis was developed in the late 60's, but was never pursued by GNC system designers. The stochastic approach optimizes the cost function directly, rather than breaking the problem into subproblems, as was done in the classical approach.

The optimal open-loop control, $\bar{\mathbf{u}}$, and the optimal feedback control gains, $\mathbf{C}$, are found by solving the following calculus of variations problem. The cost function used for this problem is equation 5.46 which was developed in the previous chapter. This cost function contains two parts: the zeroth-order (deterministic) term and second-order (stochastic) terms. When the Kalman gain is utilized the cost reduces to equation 5.86.

$$\begin{aligned}
\text{minimize:}\quad & \phi\left(\bar{\mathbf{x}}_f\right) + \int_{t_0}^{t_f} L\left(\bar{\mathbf{x}}, \bar{\mathbf{u}}, t\right) dt \\
& + \frac{1}{2}\mathrm{tr}\left[\mathbf{S}_f\left(\mathbf{E}\left(t_f\right) + \hat{\mathbf{X}}\left(t_f\right)\right)\right] \\
& + \frac{1}{2}\mathrm{tr}\left[\int_{t_0}^{t_f}\left(\mathbf{A}\left(\mathbf{E} + \hat{\mathbf{X}}\right) - \mathbf{N}\mathbf{C}\hat{\mathbf{X}}\right.\right. \\
& \qquad\qquad \left.\left. - \hat{\mathbf{X}}\mathbf{C}^T\mathbf{N}^T + \mathbf{B}\mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T + \mathbf{B}\mathbf{U}\right) dt\right]
\end{aligned}$$

$$\begin{aligned}
\text{subject to:}\quad & \dot{\bar{\mathbf{x}}} = f\left(\bar{\mathbf{x}}, \bar{\mathbf{u}}, t\right) \quad \bar{\mathbf{x}}_i = \text{given} \\
& \dot{\mathbf{E}} = \mathbf{F}\mathbf{E} + \mathbf{E}\mathbf{F}^T + \mathbf{Q} - \mathbf{E}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{E} \quad \mathbf{E}_i = \text{given} \\
& \dot{\hat{\mathbf{X}}} = \left(\mathbf{F} - \mathbf{G}\mathbf{C}\right)\hat{\mathbf{X}} + \hat{\mathbf{X}}\left(\mathbf{F} - \mathbf{G}\mathbf{C}\right)^T + \mathbf{E}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{E} + \mathbf{G}\mathbf{U}\mathbf{G} \quad \hat{\mathbf{X}}_i = \text{given}
\end{aligned}$$

where the Kalman gain has been utilized; and where all the partials are evaluated along (and can be viewed as functions of) $\bar{\mathbf{x}}$, and $\bar{\mathbf{u}}$.

This optimization problem can be solved by adjoining constraints to the performance index with appropriate Lagrange multipliers (and the appropriate inner products[1]) to form the Hamiltonian of the system

$$H = L + \frac{1}{2}\mathrm{tr}\left[\mathbf{A}\left(\mathbf{E}+\hat{\mathbf{X}}\right) - \mathbf{N}\mathbf{C}\hat{\mathbf{X}} - \hat{\mathbf{X}}\mathbf{C}^T\mathbf{N}^T + \mathbf{B}\mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T + \mathbf{B}\mathbf{U}\right] + \boldsymbol{\lambda}^T\mathbf{f} + \frac{1}{2}\mathrm{tr}\left[\boldsymbol{\Lambda}\dot{\mathbf{E}}\right] + \frac{1}{2}\mathrm{tr}\left[\mathbf{S}\dot{\hat{\mathbf{X}}}\right] \quad (6.1)$$

Recall that the cost function can be written in the following way[2]

$$\begin{aligned} J =& \phi\left(\bar{\mathbf{x}}_f\right) + \frac{1}{2}\mathrm{tr}\left[\mathbf{S}_f\mathbf{E}\left(t_f\right)\right] + \frac{1}{2}\mathrm{tr}\left[\mathbf{S}_f\hat{\mathbf{X}}\left(t_f\right)\right] \\ &- \boldsymbol{\lambda}\left(t_f\right)^T\bar{\mathbf{x}}\left(t_f\right) + \boldsymbol{\lambda}\left(t_0\right)^T\bar{\mathbf{x}}\left(t_0\right) \\ &- \frac{1}{2}\mathrm{tr}\left[\boldsymbol{\Lambda}\left(t_f\right)\mathbf{E}\left(t_f\right)\right] + \frac{1}{2}\mathrm{tr}\left[\boldsymbol{\Lambda}\left(t_0\right)\mathbf{E}\left(t_0\right)\right] \\ &- \frac{1}{2}\mathrm{tr}\left[\mathbf{S}_f\left(t_f\right)\hat{\mathbf{X}}\left(t_f\right)\right] + \frac{1}{2}\mathrm{tr}\left[\mathbf{S}_f\left(t_0\right)\hat{\mathbf{X}}\left(t_0\right)\right] \\ &+ \int_{t_0}^{t_f}\left(H + \dot{\boldsymbol{\lambda}}^T\bar{\mathbf{x}} + \frac{1}{2}\mathrm{tr}\left[\dot{\boldsymbol{\Lambda}}\mathbf{E}\right] + \frac{1}{2}\mathrm{tr}\left[\dot{\mathbf{S}}\hat{\mathbf{X}}\right]\right)dt \end{aligned} \quad (6.2)$$

[1] For vectors the inner product used here, is the Euclidian inner product or dot product

$$\mathbf{a}^T\mathbf{b} = \sum_{i=1}^{n} a_i b_i$$

For matrices the inner product used here is the Frobenius product which is component-wise equivalent to the Euclidian product of the

$$\mathrm{tr}\left[\mathbf{A}^T\mathbf{B}\right] = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ij}$$

Both the dot product and the Frobedius product are examples of the Hilbert-Schmidt inner product. The Hilbert-Schmidt inner product is fundamental to much of estimation theory. The connection between the Hilbert-Schmidt inner product and the Kalman filter is the orthogonal relationship between the estimate and the error [21].

[2] This is easily shown by using integration by parts to integrate portions of the cost function once the constraints are adjoined.

Now consider the first variation in $\mathcal{J}$ due to variations in the controls $\bar{\mathbf{u}}$ and $\mathbf{C}$ for fixed times t_0 and t_f

$$\begin{aligned}\partial J = &\left[\left(\frac{\partial \phi}{\partial \mathbf{x}} - \boldsymbol{\lambda}^T\right)\partial \bar{\mathbf{x}}\right]_{t_f} + \frac{1}{2}\mathrm{tr}\left[\left(\mathbf{S}_f - \boldsymbol{\Lambda}\right)\partial \mathbf{E}\right]_{t_f} + \frac{1}{2}\mathrm{tr}\left[\left(\mathbf{S}_f - \mathbf{S}\right)\partial \hat{\mathbf{X}}\right]_{t_f} \\ &+ \boldsymbol{\lambda}(t_0)^T \partial \bar{\mathbf{x}}(t_0) + \frac{1}{2}\mathrm{tr}\left[\boldsymbol{\Lambda}(t_0)\partial \mathbf{E}(t_0)\right] + \frac{1}{2}\mathrm{tr}\left[\mathbf{S}(t_0)\partial \hat{\mathbf{X}}(t_0)\right] \\ &+ \int_{t_0}^{t_f}\left(\frac{\partial H}{\partial \bar{\mathbf{u}}}\partial \bar{\mathbf{u}} + \mathrm{tr}\left[\frac{\partial H}{\partial \mathbf{C}}\partial \mathbf{C}\right] + \left(\frac{\partial H}{\partial \bar{\mathbf{x}}} + \dot{\boldsymbol{\lambda}}^T\right)\partial \bar{\mathbf{x}}\right. \\ &\left. + \mathrm{tr}\left[\left(\frac{\partial H}{\partial \mathbf{E}} + \frac{1}{2}\dot{\boldsymbol{\Lambda}}\right)\partial \mathbf{E}\right] + \mathrm{tr}\left[\left(\frac{\partial H}{\partial \hat{\mathbf{X}}} + \frac{1}{2}\dot{\mathbf{S}}\right)\partial \hat{\mathbf{X}}\right]\right) dt\end{aligned} \tag{6.3}$$

The Lagrange multiplier functions are selected to cause the associated terms to be stationary

$$\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \bar{\mathbf{x}}} \qquad \boldsymbol{\lambda}^T(t_f) = \left.\frac{\partial \phi}{\partial \bar{\mathbf{x}}}\right|_{\bar{\mathbf{x}}_f} \tag{6.4}$$

$$\dot{\boldsymbol{\Lambda}} = -2\frac{\partial H}{\partial \mathbf{E}} \qquad \boldsymbol{\Lambda}(t_f) = \mathbf{S}_f \tag{6.5}$$

$$\dot{\mathbf{S}} = -2\frac{\partial H}{\partial \hat{\mathbf{X}}} \qquad \mathbf{S}(t_f) = \mathbf{S}_f \tag{6.6}$$

where

$$\mathbf{S}_f = \left.\frac{\partial^2 \phi}{\partial \mathbf{x}_f^T \mathbf{x}_f}\right|_{\bar{\mathbf{x}}_f} \tag{6.7}$$

Selection of these necessary conditions reduces the variation of the cost function to

$$\begin{aligned}\partial J = &\boldsymbol{\lambda}(t_0)^T \partial \bar{\mathbf{x}}(t_0) + \frac{1}{2}\mathrm{tr}\left[\boldsymbol{\Lambda}(t_0)\partial \mathbf{E}(t_0)\right] + \frac{1}{2}\mathrm{tr}\left[\mathbf{S}(t_0)\partial \hat{\mathbf{X}}(t_0)\right] \\ &+ \int_{t_0}^{t_f}\left(\frac{\partial H}{\partial \bar{\mathbf{u}}}\partial \bar{\mathbf{u}} + \mathrm{tr}\left[\frac{\partial H}{\partial \mathbf{C}}\partial \mathbf{C}\right]\right) dt\end{aligned} \tag{6.8}$$

The terms associated with the initial conditions are zero due to the initial conditions being given quantities. The stationary point can now be found by making the control partials zero

$$\frac{\partial H}{\partial \bar{\mathbf{u}}} = \mathbf{0} \tag{6.9}$$

$$\frac{\partial H}{\partial \mathbf{C}} = \mathbf{0} \tag{6.10}$$

Before the indicated partials are taken, the entire Hamiltonian is written by substituting $\dot{\mathbf{E}}$ and $\dot{\hat{\mathbf{X}}}$ in equations 5.84 and 5.85 into equation 6.1

$$H = L + \boldsymbol{\lambda}^T \mathbf{f} + \frac{1}{2}\mathrm{tr}\Big[\mathbf{AE} + \mathbf{A}\hat{\mathbf{X}} - \mathbf{NC}\hat{\mathbf{X}} - \hat{\mathbf{X}}\mathbf{C}^T\mathbf{N}^T + \mathbf{BC}\hat{\mathbf{X}}\mathbf{C}^T + \mathbf{BU} + \mathbf{S}\left(\mathbf{F} - \mathbf{GC}\right)\hat{\mathbf{X}} + \mathbf{S}\hat{\mathbf{X}}\left(\mathbf{F} - \mathbf{GC}\right)^T + \mathbf{SEH}^T\mathbf{R}^{-1}\mathbf{HE} + \mathbf{SGUG} + \boldsymbol{\Lambda}\mathbf{FE} + \boldsymbol{\Lambda}\mathbf{EF}^T + \boldsymbol{\Lambda}\mathbf{Q} - \boldsymbol{\Lambda}\mathbf{EH}^T\mathbf{R}^{-1}\mathbf{HE}\Big] \tag{6.11}$$

Taking the indicated matrix partials

$$\dot{\boldsymbol{\Lambda}} = -2\frac{\partial H}{\partial \mathbf{E}} = -\mathbf{A}^T - \mathbf{S}^T\mathbf{EH}^T\mathbf{R}^{-1}\mathbf{H} - \mathbf{H}^T\mathbf{R}^{-1}\mathbf{HES}^T - \mathbf{F}^T\boldsymbol{\Lambda}^T - \boldsymbol{\Lambda}^T\mathbf{F} + \boldsymbol{\Lambda}^T\,\mathbf{EH}^T\mathbf{R}^{-1}\mathbf{H} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{HE}\boldsymbol{\Lambda}^T \tag{6.12}$$

$$\dot{\mathbf{S}} = -2\frac{\partial H}{\partial \hat{\mathbf{X}}} = -\mathbf{A}^T + \mathbf{C}^T\mathbf{N}^T + \mathbf{NC} - \mathbf{C}^T\mathbf{BC} - \left(\mathbf{F} - \mathbf{GC}\right)^T\mathbf{S}^T - \mathbf{S}^T\left(\mathbf{F} - \mathbf{GC}\right) \tag{6.13}$$

$$\mathbf{0} = \frac{\partial H}{\partial \mathbf{C}} = \frac{1}{2}\left(-\mathbf{N}^T\hat{\mathbf{X}} - \mathbf{N}^T\hat{\mathbf{X}} + \mathbf{B}^T\mathbf{C}\hat{\mathbf{X}} + \mathbf{BC}\hat{\mathbf{X}} - \mathbf{G}^T\mathbf{S}^T\hat{\mathbf{X}} - \mathbf{G}^T\mathbf{S}\hat{\mathbf{X}}\right) \tag{6.14}$$

Since, $\mathbf{S}_f$, $\mathbf{A}$, and $\mathbf{B}$ are symmetric matrices, $\boldsymbol{\Lambda}$ and $\mathbf{S}$ are symmetric matrices.

$$\dot{\boldsymbol{\Lambda}} = -\left(\mathbf{F} - \mathbf{EH}^T\mathbf{R}^{-1}\mathbf{H}\right)^T\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\left(\mathbf{F} - \mathbf{EH}^T\mathbf{R}^{-1}\mathbf{H}\right) - \mathbf{SEH}^T\mathbf{R}^{-1}\mathbf{H} - \mathbf{H}^T\mathbf{R}^{-1}\mathbf{HES} - \mathbf{A} \tag{6.15}$$

$$\dot{\mathbf{S}} = -\left(\mathbf{F} - \mathbf{GC}\right)^T\mathbf{S} - \mathbf{S}\left(\mathbf{F} - \mathbf{GC}\right) + \mathbf{C}^T\mathbf{N}^T + \mathbf{NC} - \mathbf{C}^T\mathbf{BC} - \mathbf{A} \tag{6.16}$$

$$\mathbf{0} = \left(-\mathbf{N}^T + \mathbf{BC} - \mathbf{G}^T\mathbf{S}\right)\hat{\mathbf{X}} \tag{6.17}$$

and since $\hat{\mathbf{X}}$ is known to be positive definite, this last relation can be solved for $\mathbf{C}$.

$$\mathbf{C} = \mathbf{B}^{-1}\left(\mathbf{G}^T\mathbf{S} + \mathbf{N}^T\right) \tag{6.18}$$

This is identical to the linear feedback gain produced by classical quadratic synthesis about a particular given nominal trajectory. The requirement that $\mathbf{B}$ be invertible places requires that the Hessian of L with respect to $\bar{\mathbf{u}}$ be full rank.

The differential equations for $\dot{\mathbf{S}}$ and $\dot{\hat{\mathbf{X}}}$ can be simplified by substitution of $\mathbf{C}$.

$$\dot{\mathbf{S}} = -\mathbf{F}^T\mathbf{S} - \mathbf{S}\mathbf{F} + \left(\mathbf{G}^T\mathbf{S} + \mathbf{N}^T\right)^T \mathbf{B}^{-1}\left(\mathbf{G}^T\mathbf{S} + \mathbf{N}^T\right) - \mathbf{A} \tag{6.19}$$

or

$$\dot{\mathbf{S}} = -\mathbf{F}^T\mathbf{S} - \mathbf{S}\mathbf{F} + \mathbf{C}^T\mathbf{B}\mathbf{C} - \mathbf{A} \tag{6.20}$$

together with

$$\begin{aligned}\dot{\hat{\mathbf{X}}} = \left[\mathbf{F} - \mathbf{G}\mathbf{B}^{-1}\left(\mathbf{G}^T\mathbf{S} + \mathbf{N}^T\right)\right]\hat{\mathbf{X}} + \hat{\mathbf{X}}\left[\mathbf{F} - \mathbf{G}\mathbf{B}^{-1}\left(\mathbf{G}^T\mathbf{S} + \mathbf{N}^T\right)\right]^T \\ + \mathbf{E}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{E} + \mathbf{G}\mathbf{U}\mathbf{G}\end{aligned} \tag{6.21}$$

The remaining necessary conditions are found by executing the required vector partials

$$\begin{aligned}\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \bar{\mathbf{x}}} = -\frac{\partial L}{\partial \bar{\mathbf{x}}} - \boldsymbol{\lambda}^T\mathbf{F} - \frac{1}{2}\frac{\partial}{\partial \bar{\mathbf{x}}}\mathrm{tr}\Big[\mathbf{A}\mathbf{E} + \mathbf{A}\hat{\mathbf{X}} - \mathbf{N}\mathbf{C}\hat{\mathbf{X}} \\ - \hat{\mathbf{X}}\mathbf{C}^T\mathbf{N}^T + \mathbf{B}\left(\mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T + \mathbf{U}\right) + \mathbf{S}\left(\mathbf{F} - \mathbf{G}\mathbf{C}\right)\hat{\mathbf{X}} \\ + \mathbf{S}\hat{\mathbf{X}}\left(\mathbf{F} - \mathbf{G}\mathbf{C}\right)^T + \left(\mathbf{S} - \boldsymbol{\Lambda}\right)\mathbf{E}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{E} + \mathbf{S}\mathbf{G}\mathbf{U}\mathbf{G} \\ + \boldsymbol{\Lambda}\left(\mathbf{F}\mathbf{E} + \mathbf{E}\mathbf{F}^T + \mathbf{Q}\right)\Big]\end{aligned} \tag{6.22}$$

$$\begin{aligned}\mathbf{0} = \frac{\partial H}{\partial \bar{\mathbf{u}}} = \frac{\partial L}{\partial \bar{\mathbf{u}}} + \boldsymbol{\lambda}^T\mathbf{G} + \frac{1}{2}\frac{\partial}{\partial \bar{\mathbf{u}}}\mathrm{tr}\Big[\mathbf{A}\mathbf{E} + \mathbf{A}\hat{\mathbf{X}} - \mathbf{N}\mathbf{C}\hat{\mathbf{X}} \\ - \hat{\mathbf{X}}\mathbf{C}^T\mathbf{N}^T + \mathbf{B}\left(\mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T + \mathbf{U}\right) + \mathbf{S}\left(\mathbf{F} - \mathbf{G}\mathbf{C}\right)\hat{\mathbf{X}} \\ + \mathbf{S}\hat{\mathbf{X}}\left(\mathbf{F} - \mathbf{G}\mathbf{C}\right)^T + \left(\mathbf{S} - \boldsymbol{\Lambda}\right)\mathbf{E}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{E} + \mathbf{S}\mathbf{G}\mathbf{U}\mathbf{G} \\ + \boldsymbol{\Lambda}\left(\mathbf{F}\mathbf{E} + \mathbf{E}\mathbf{F}^T + \mathbf{Q}\right)\Big]\end{aligned} \tag{6.23}$$

In these last equations the indicated partials are not readily expressed in a useful manner using the notation from linear algebra. It bears noting, that these partials vanish in the case of a linear system with linear measurements, and the familiar deterministic result is left. Thus the partial derivatives of the trace operator, account for the interaction of the statistical quantities with the nonlinearities of the system.

6.1 Summary of the Equations

This section presents a summary of the equations and optimization approach for the three key techniques presented presented in this chapter and the preceding two chapters.

6.1.1 Neighboring Optimal Control

Neighboring Optimal Control relies on a two stage optimization process to minimize a second order expansion of the Bolza *augmented* cost function.

$$J' \approx \bar{J}' + dJ' + \frac{1}{2}d^2J' \tag{6.24}$$

A two stage optimization approach is used:

1. The nominal trajectory is found using the deterministic solution from the calculus of variations. This is accomplished by solving the Euler-Legrange equations

$$\dot{\bar{\mathbf{x}}} = f\left(\bar{\mathbf{x}}, \bar{\mathbf{u}}, t\right) \tag{6.25}$$

$$\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \bar{\mathbf{x}}} \tag{6.26}$$

$$\mathbf{0} = \frac{\partial H}{\partial \bar{\mathbf{u}}} \tag{6.27}$$

   subject to appropriate boundary conditions. Presuming a minimum exists, this has the following consequences:

   (a) $dJ' = 0$

   (b) $d^2J' > 0$

   (c) $\bar{J}'$ is a constant.

2. Once the nominal trajectory has been selected, the certainty and equivalence principle is used to minimize the expectation of the cost function about the nominal (assume zero-mean noise).

$$E\left[J'\right] \approx \underbrace{E\left[\bar{J}'\right]}_{\text{const.}} + \underbrace{E\left[dJ'\right]}_{0} + \underbrace{\frac{1}{2}E\left[d^2J'\right]}_{\text{minimize}} \tag{6.28}$$

The certainty and equivalence principle states that this is minimized by the Kalman gain and a quadratic terminal controller. The weights for the controller are the Hessian of the *Hamiltonian*.

6.1.2 Classical Quadratic Synthesis

Classical Quadratic Synthesis relies on a two stage optimization process to minimize a second order expansion of the Bolza *un-augmented* cost function. The expectation of the expansion is taken (assume zero-mean noise).

$$E[J] \approx \underbrace{E\left[\bar{J}\right]}_{\text{minimize}} + \underbrace{E[dJ]}_{0} + \underbrace{\frac{1}{2}E\left[d^2 J\right]}_{\text{minimize}} \tag{6.29}$$

A two stage optimization approach is used:

1. The nominal is found my minimizing $E\left[\bar{J}\right]$ using the deterministic solution. In order to generate a valid nominal trajectory, the constraints are appended only to this term. The minimum is found by solving the Euler-Lagrange equations

$$\dot{\bar{\mathbf{x}}} = f\left(\bar{\mathbf{x}}, \bar{\mathbf{u}}, t\right) \tag{6.30}$$

$$\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \bar{\mathbf{x}}} \tag{6.31}$$

$$\mathbf{0} = \frac{\partial H}{\partial \bar{\mathbf{u}}} \tag{6.32}$$

   subject to appropriate boundary conditions. Presuming a minimum exists, this has the following consequences:

   (a) $d\bar{J} = 0$

   (b) $d^2\bar{J} > 0$

   (c) $\bar{J}$ is a constant.

2. The Certainty and Equivalence principle states that $E\left[d^2 J\right]$ is minimized by using the Kalman gain and a quadratic terminal controller. The weights for the controller are the Hessian of the *Lagrangian*.

6.1.3 Stochastic Quadratic Synthesis

Stochastic Quadratic Synthesis relies on a single stage optimization process to minimize a second order expansion of the Bolza *un-augmented* cost function. The expectation of the expansion is taken (assume zero-mean noise)

$$E\left[J\right] \approx E\left[\bar{J}\right] + \underbrace{E\left[dJ\right]}_{0} + \frac{1}{2} E\left[d^2 J\right] \tag{6.33}$$

Single stage optimization approach:

- Append the dynamic constraints for nominal trajectory $\left(\dot{\mathbf{x}}\right)$ and for the estimate state covariance $\left(\hat{\mathbf{X}}\right)$ and error covariance $\left(\mathbf{E}\right)$ to the expanded cost using Lagrange multipliers.
- Schmidt (and Denham) showed that the Kaman gain and the quadratic terminal controller minimize the cost.
- Necessary conditions are found using the calculus of variations.

$$\begin{aligned}\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \bar{\mathbf{x}}} = -\frac{\partial L}{\partial \bar{\mathbf{x}}} - \boldsymbol{\lambda}^T \mathbf{F} - \frac{1}{2}\frac{\partial}{\partial \bar{\mathbf{x}}} \mathrm{tr}\Big[&\mathbf{AE} + \mathbf{A}\hat{\mathbf{X}} - \mathbf{NC}\hat{\mathbf{X}} \\ &- \hat{\mathbf{X}}\mathbf{C}^T\mathbf{N}^T + \mathbf{B}\left(\mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T + \mathbf{U}\right) + \mathbf{S}\left(\mathbf{F} - \mathbf{GC}\right)\hat{\mathbf{X}} \\ &+ \mathbf{S}\hat{\mathbf{X}}\left(\mathbf{F} - \mathbf{GC}\right)^T + \left(\mathbf{S} - \boldsymbol{\Lambda}\right)\mathbf{EH}^T\mathbf{R}^{-1}\mathbf{HE} + \mathbf{SGUG} \\ &+ \boldsymbol{\Lambda}\left(\mathbf{FE} + \mathbf{EF}^T + \mathbf{Q}\right)\Big]\end{aligned} \tag{6.34}$$

$$\begin{aligned}\mathbf{0} = \frac{\partial H}{\partial \bar{\mathbf{u}}} = \frac{\partial L}{\partial \bar{\mathbf{u}}} + \boldsymbol{\lambda}^T \mathbf{G} + \frac{1}{2}\frac{\partial}{\partial \bar{\mathbf{u}}} \mathrm{tr}\Big[&\mathbf{AE} + \mathbf{A}\hat{\mathbf{X}} - \mathbf{NC}\hat{\mathbf{X}} \\ &- \hat{\mathbf{X}}\mathbf{C}^T\mathbf{N}^T + \mathbf{B}\left(\mathbf{C}\hat{\mathbf{X}}\mathbf{C}^T + \mathbf{U}\right) + \mathbf{S}\left(\mathbf{F} - \mathbf{GC}\right)\hat{\mathbf{X}} \\ &+ \mathbf{S}\hat{\mathbf{X}}\left(\mathbf{F} - \mathbf{GC}\right)^T + \left(\mathbf{S} - \boldsymbol{\Lambda}\right)\mathbf{EH}^T\mathbf{R}^{-1}\mathbf{HE} + \mathbf{SGUG} \\ &+ \boldsymbol{\Lambda}\left(\mathbf{FE} + \mathbf{EF}^T + \mathbf{Q}\right)\Big]\end{aligned} \tag{6.35}$$

6.2 A Simple Example

Schmidt offers a simple example [108,109] that illustrates the difference between the classical and stochastic approaches to quadratic synthesis. Here the example has been slightly modified to more clearly demonstrate the differences between the approaches. Consider the case of an exponentially correlated random variable.

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -x_1 x_2 + u \\ 0 \end{bmatrix}}_{\boldsymbol{f}(\mathbf{x})} + \underbrace{\begin{bmatrix} w \\ 0 \end{bmatrix}}_{\mathbf{w}} \tag{6.36}$$

with the linear measurement model

$$y = \underbrace{x_1}_{h(\mathbf{x})} + \nu \tag{6.37}$$

The various boundary conditions and system parameters are shown in table 6.1.

The cost function to be minimized is

$$J = \frac{1}{2}\left[\mathbf{x}(t_f) - \mathbf{x}_f\right]^T \mathbf{S}_f \left[\mathbf{x}(t_f) - \mathbf{x}_f\right] + \frac{1}{2}\int_{t_0}^{t_f} u^2 dt$$

6.2.1 Solution via Classical Quadratic Synthesis

The traditional solution begins by finding a nominal trajectory that meets the boundary constraints (hits the target exactly) and minimizes the cost. This is done without regard for the stochastic nature of the system. The Hamiltonian for the system is given by

$$H = \boldsymbol{\lambda}^T \begin{bmatrix} -x_1 x_2 + u \\ 0 \end{bmatrix} + \frac{1}{2}u^2 \tag{6.38}$$

The necessary conditions are given by

$$\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \mathbf{x}} = -\boldsymbol{\lambda}^T \begin{bmatrix} -x_2 & -x_1 \\ 0 & 0 \end{bmatrix} \tag{6.39}$$

Table 6.1: Boundary conditions and system parameter values for a simple example.

Parameter	Value
Initial Truth State	$\mathbf{x}(t_i) \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\right)$
Initial Time	$t_i = 0$
Final Time	$t_f = 10$
Process Noise	$\mathbf{w} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{Q}}\right)$
Measurement Noise	$\nu \sim \mathcal{N}\left(0, \underbrace{1}_{\mathbf{R}}\right)$
Penalty Weights	$\mathbf{S}_f = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$
Desired Terminal State	$\mathbf{x}_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
Initial Estimation Error Variance	$\mathbf{E}(t_i) = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

$$\mathbf{0} = \frac{\partial H}{\partial u} = \boldsymbol{\lambda}^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u \tag{6.40}$$

From these two equations we learn that

$$\dot{\lambda}_1 = x_2 \lambda_1 \tag{6.41}$$

$$u = -\lambda_1 \tag{6.42}$$

Since x_2 is a constant the control is now found

$$u = u_0 e^{x_2 t} \tag{6.43}$$

with the cost

$$J = \frac{u_0^2}{4x_2}\left[e^{2x_2 t_f} - e^{2x_2 t_0}\right] \tag{6.44}$$

By applying the boundary conditions $x_1(t_f) = x_f$ and $x_1(t_0) = x_0$, the quantity u_0 can be found

$$u_0 = x_2 \frac{x_0 e^{-x_2 t_f} - x_f e^{-x_2 t_0}}{\sinh\left[-x_2 (t_f - t_0)\right]} \tag{6.45}$$

Applying this control

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \bar{x}(t_0) \frac{\sinh\left[b\left(t_f - t\right)\right]}{\sinh\left[b\left(t_f - t_0\right)\right]} + \bar{x}(t_f) \frac{\sinh\left[b(t - t_0)\right]}{\sinh\left[b\left(t_f - t_0\right)\right]} \\ b \end{bmatrix} \tag{6.46}$$

where b is a constant.

Using the numeric values from table 6.1 the nominal optimal control is given by

$$\bar{u} = \frac{e^t}{\sinh(10)} \tag{6.47}$$

The deterministic cost function evaluated along the nominal trajectory is

$$\bar{J} = \frac{1}{1 - e^{-20}} \tag{6.48}$$

Note that the approximate control yields a result nearer unity that exact control. This can be used to generate the nominal optimal trajectory

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\sinh(t)}{\sinh(10)} \\ 1 \end{bmatrix} \tag{6.49}$$

The weighting matrix for the dispersed cost is given by the partials of the Lagrangian with respect to the state and the control

$$\frac{\partial^2 L}{\partial \mathbf{x}^T \partial \mathbf{x}} = \mathbf{A} = \mathbf{0} \tag{6.50}$$

$$\frac{\partial^2 L}{\partial \mathbf{u}^T \partial \mathbf{x}} = \mathbf{N} = \mathbf{0} \tag{6.51}$$

$$\frac{\partial^2 L}{\partial \mathbf{u}^T \partial \mathbf{u}} = \mathbf{B} = 1 \tag{6.52}$$

The partials of the dynamics are given by

$$\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}} = \mathbf{F} = \begin{bmatrix} -x_2 & -x_1 \\ 0 & 0 \end{bmatrix} \tag{6.53}$$

For the classical approach, these partials are evaluated along the nominal trajectory

$$\mathbf{F}_{\mathrm{c}} = \begin{bmatrix} -1 & -\frac{\sinh(t)}{\sinh(10)} \\ 0 & 0 \end{bmatrix} \tag{6.54}$$

For this problem, the remaining partials are not a function of the nominal state

$$\frac{\partial \boldsymbol{f}}{\partial \mathbf{u}} = \mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{6.55}$$

and

$$\frac{\partial \boldsymbol{h}}{\partial \mathbf{x}} = \mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{6.56}$$

Using these partials, the construction of the filter and controller gains is accomplished by numerically solving the needed Riccati equations subject to the boundary conditions.

In summary, the classical solution has two boundary value problems. The first, is the well known solution from the calculus of variations,

$$\dot{\bar{\mathbf{x}}} = \begin{bmatrix} -\bar{x}_1\bar{x}_2 + \bar{u} \\ 0 \end{bmatrix} \qquad \bar{\mathbf{x}}_i = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \ \bar{\mathbf{x}}_f = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \tag{6.57}$$

$$\dot{\boldsymbol{\lambda}}^T = -\boldsymbol{\lambda}^T \begin{bmatrix} -\bar{x}_2 & -\bar{x}_1 \\ 0 & 0 \end{bmatrix} \tag{6.58}$$

$$\bar{u} = -\lambda_1 \tag{6.59}$$

Once the solution to this boundary value problem is found, the boundary problem corresponding the linearized optimal control problem can be determined

$$\dot{\mathbf{S}} = -\mathbf{F}^T\mathbf{S} - \mathbf{S}\mathbf{F} + \mathbf{S}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{S} \qquad \mathbf{S}\left(t_f\right) = \mathbf{S}_f = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \tag{6.60}$$

$$\dot{\mathbf{E}} = \mathbf{F}\mathbf{E} + \mathbf{E}\mathbf{F}^T + \mathbf{Q} - \mathbf{E}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{E} \qquad \mathbf{E}\left(t_i\right) = \mathbf{E}_i = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \tag{6.61}$$

The uncoupled nature of this last boundary problem ensures that the solutions is trivial.

6.2.2 Solution via Stochastic Quadratic Synthesis

The solution via stochastic quadratic synthesis requirers only one boundary value problem be solved. This boundary value problem consists of the same equations used before, with the exception the the classical Euler-Lagrange equations are replaced by equations 6.22 and 6.23. These necessary conditions can be used to construct the following boundary value problem

$$\dot{\bar{\mathbf{x}}} = \begin{bmatrix} -\bar{x}_1\bar{x}_2 + \bar{u} \\ 0 \end{bmatrix} \qquad \bar{\mathbf{x}}_i = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \ \bar{\mathbf{x}}_f = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \tag{6.62}$$

$$\dot{\mathbf{E}} = \mathbf{FE} + \mathbf{EF}^T + \mathbf{Q} - \mathbf{E}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{E} \quad \mathbf{E}_i = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \tag{6.63}$$

$$\dot{\hat{\mathbf{X}}} = (\mathbf{F} - \mathbf{GC})\,\hat{\mathbf{X}} + \hat{\mathbf{X}}\,(\mathbf{F} - \mathbf{GC})^T + \mathbf{E}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{E} \quad \hat{\mathbf{X}}_i = \mathbf{0} \tag{6.64}$$

$$\dot{\boldsymbol{\lambda}}^T = -\boldsymbol{\lambda}^T\mathbf{F} - \frac{1}{2}\frac{\partial}{\partial \bar{\mathbf{x}}}\mathrm{tr}\left[\boldsymbol{\Lambda}\dot{\mathbf{E}} + \mathbf{S}\dot{\hat{\mathbf{X}}}\right] \quad \boldsymbol{\lambda}^T = \mathbf{0} \tag{6.65}$$

$$\mathbf{0} = u + \boldsymbol{\lambda}^T\mathbf{G} + \frac{1}{2}\frac{\partial}{\partial u}\mathrm{tr}\left[\boldsymbol{\Lambda}\dot{\mathbf{E}} + \mathbf{S}\dot{\hat{\mathbf{X}}}\right] \tag{6.66}$$

$$\begin{aligned}\dot{\boldsymbol{\Lambda}} = &-\left(\mathbf{F} - \mathbf{E}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)^T \boldsymbol{\Lambda} - \boldsymbol{\Lambda}\left(\mathbf{F} - \mathbf{E}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &- \mathbf{SE}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{ES} \quad \boldsymbol{\Lambda}_f = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\end{aligned} \tag{6.67}$$

$$\dot{\mathbf{S}} = -\mathbf{F}^T\mathbf{S} - \mathbf{SF} + \mathbf{C}^T\mathbf{C} \quad \mathbf{S}_f = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \tag{6.68}$$

$$\mathbf{C} = \mathbf{G}^T\mathbf{S} \tag{6.69}$$

Once this problem is solved by finding $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$, the resulting control and states can be used to calculate the dispersed cost directly using equation 5.55.

6.2.3 Results

For this simple example the example the analytic solution is available for use with the classical quadratic synthesis approach, but numerics must be used to solve the stochastic quadratic synthesis. The solutions are shown in figures 6.1 through 6.9.

It is important to remember that this example makes no attempt to minimize a trajectory dispersion subject to control (fuel) constraint. These goals will be addressed later. Nevertheless, this example provides insight into how the selection of the nominal trajectory affects the behavior of the closed loop GN&C system. This interaction between the nominal trajectory and the closed loop GN&C system clearly demonstrates how a singleminded tendency to minimize the control could conflict with the desire to minimize dispersions. This example also shows how increasing the nominal control can, surprisingly, decrease the expected total control. As has been repeatedly stated, conventional GN&C design is frequently naive to these interactions.

Figures 6.1 and 6.2 show the control and nominal state histories. Since the dynamic model is an exponentially correlated random variable, the dynamics have a natural tendency toward zero in the absence of control. Thus, it should come as no surprise that both solutions have an initial steady state behavior near zero. As t increases the control and states begin to diverge from zero. The classical behavior is largely what one would expect; as the terminal time approaches control is gradually applied until the terminal boundary condition is met at precisely the terminal time. This ensures that, for the nominal, absolutely no fuel is wasted. Only the bare minimum of control required to meet the terminal constraints is applied. Meanwhile, the stochastic control applies higher thrust levels, initially in the opposite direction! This results in a state time-history which initially moves farther from the target. From a classical perspective this effort seems largely wasted given that the dynamics naturally return x_1 to zero. At the terminal time, the nominal thrust for the stochastic case is nearly double that of the classical case. Again, when viewed from the classical standpoint, the large magnitude of the stochastic control profile seems ill advised. However, it will be seen that this control results in smaller state dispersions than its classical counterpart.

The navigation estimation error is shown in figure 6.3 and the accompanying estimation gains are shown in figure 6.4. It is interesting to note that initially the estimation error for x_1 is larger for the stochastic case. Meanwhile the estimation error for x_2 is smaller for the stochastic case. However, these relationships are reversed as the terminal time nears. It is also interesting to note that the positive correlation between the states becomes negative as the terminal time nears. This change in the sense of the correlation is expected, and corresponds to the change in sign of the nominal state. This indicates that the unexpected state excursion mentioned earlier has the effect of providing a better estimate of the value of x_2. As the terminal time nears, further improvement in the estimate of x_2 has a diminishing re-

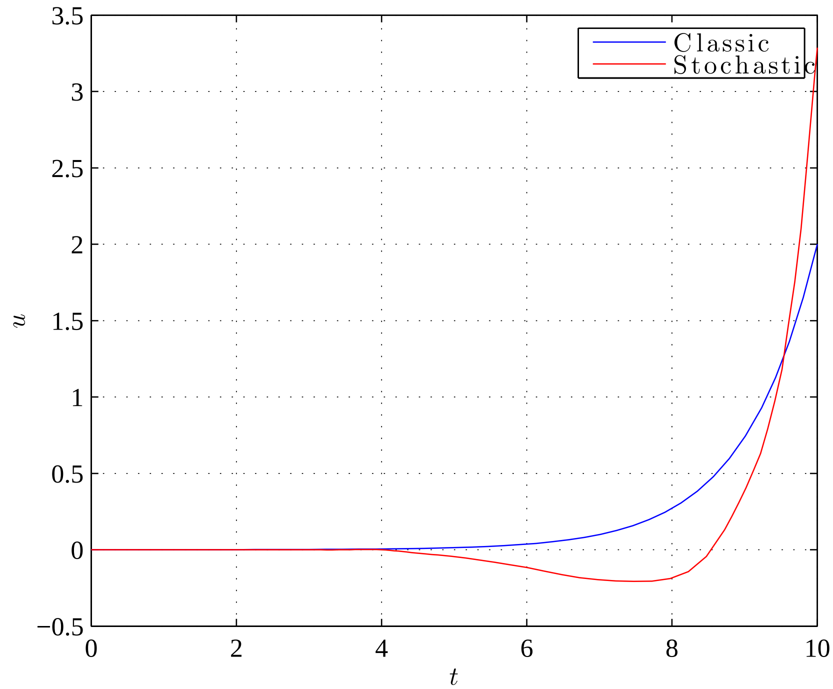

Fig. 6.1: The control history for the example problem.

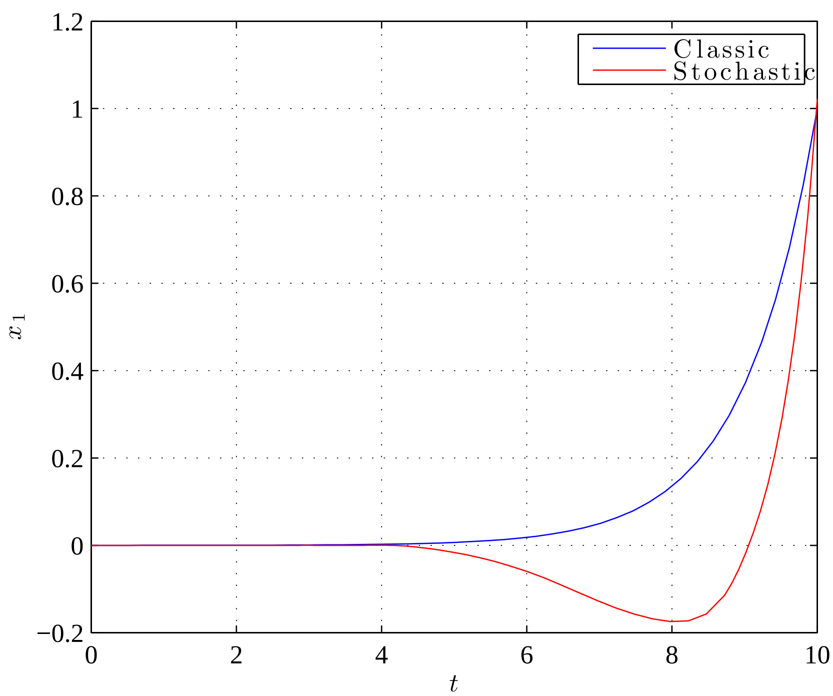

Fig. 6.2: The nominal state history for the example problem.

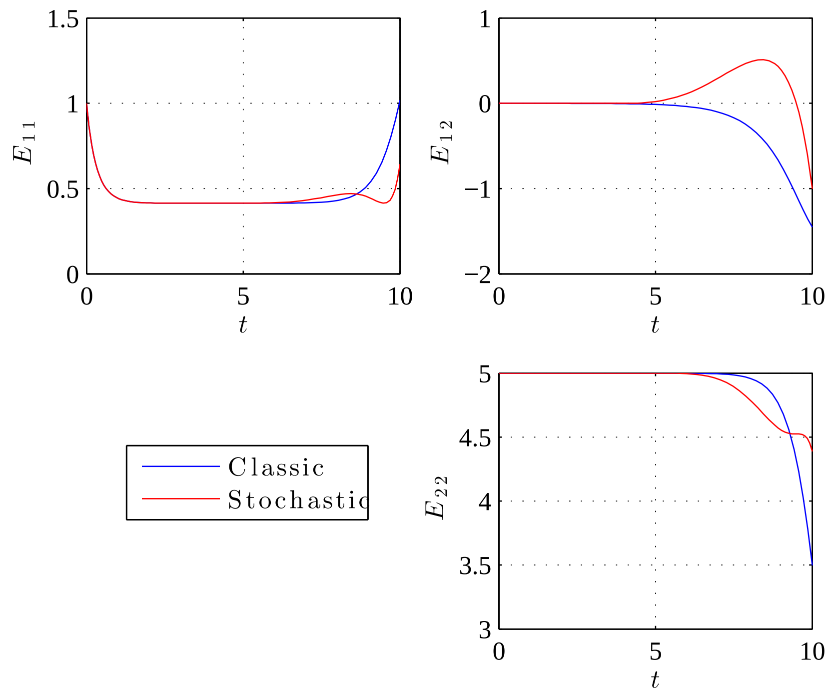

Fig. 6.3: The estimation error covariance for the example problem.

turn. At this point, the filter begins to focus on improving the estimate of x_1. The estimator gains[3] confirm these observations. The estimator gains indicate that the stochastic estimator initially places more emphasis on learning about the state than the classical approach. At some point, the filter gains from the classical approach become significantly larger than the stochastic approach. This is an indication, that the filter from the stochastic approach seeks state information earlier, and then wishes to reduce its sensitivity to stochastic in the measurement as the terminal time nears.

Figures 6.5 and 6.6 show the solution to the Ricatti equation for the controller and the control gains respectively. The feedback gains for the x_1 state are indistinguishable for both cases. It is surprising, that the feedback gains for the x_2 state are smaller for the stochastic approach. This is especially unexpected in light of the fact that the stochastic filter has a higher quality estimate to use for the x_2 state. In other words, the stochastic approach

[3]Recall that large estimator gains indicate that the filter is applying the measurement information to the corresponding state estimate. In other words the estimator gains represent the sensitivity of the state estimate with respect to the measurement at any given time. Large values indicate a strong dependence, and small values indicate the lack of dependence.

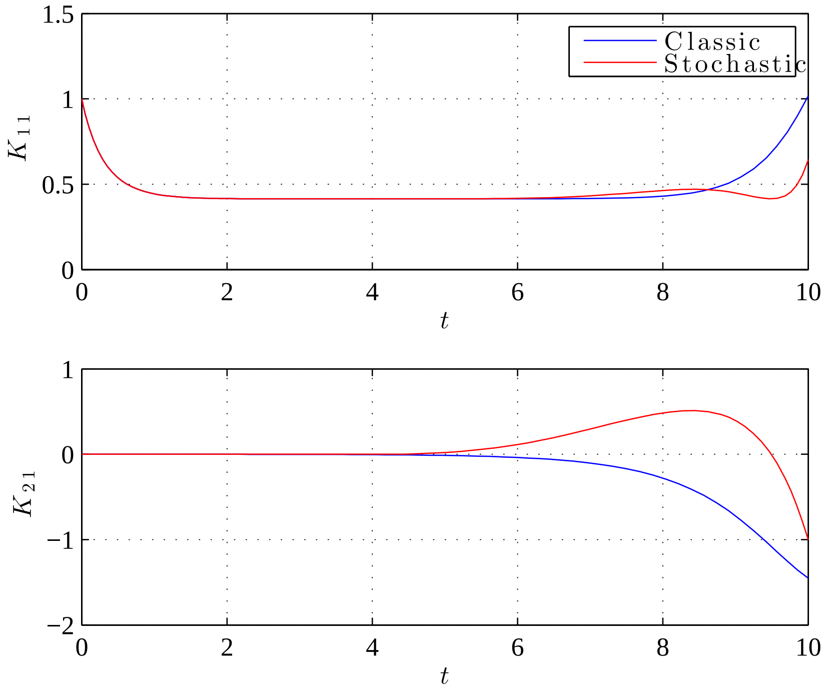

Fig. 6.4: The estimator gains for the example problem.

has a higher quality estimate of x_2, but is less eager to make control decisions using this information. As the terminal time nears, the gains on x_2 for both cases approach the same value, and become indistinguishable.

Note that the feedback gains for the stochastic case depart from zero well after control has begun to be applied. In other words, the initial application of control is nearly all open-loop for the stochastic case. The classical case has less open loop control, but is more sensitive to feedback. Unsurprisingly, this difference in control gains has a significant effect on the control dispersion that will be shown later.

The true dispersions are shown in figure 6.7. As expected, the dispersions of the x_2 state are constant and identical for both cases. Once again, the change in sense of correlation is expected, and and corresponds to the change in sign of $\bar{x}_2$. It is interesting to note that the true state dispersions are actually larger for the stochastic case during a small portion of the trajectory. After this this initial jump, the state dispersions for x_1 in the stochastic case quickly return the the steady state value. This is explained by noting that when u is near zero the homogenous dynamics dominate the state behavior.

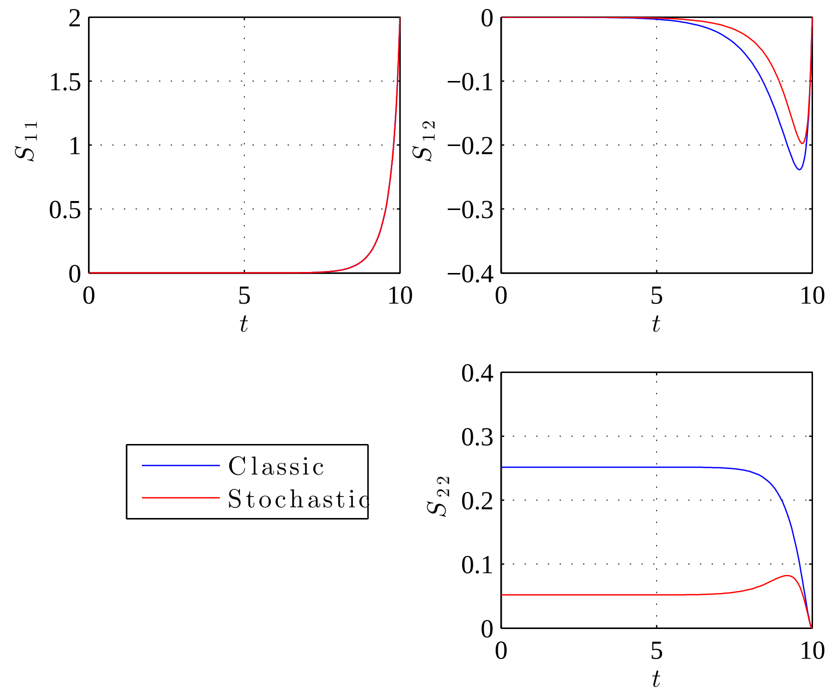

Fig. 6.5: The solution to the optimal control Riccati equation for the example problem.

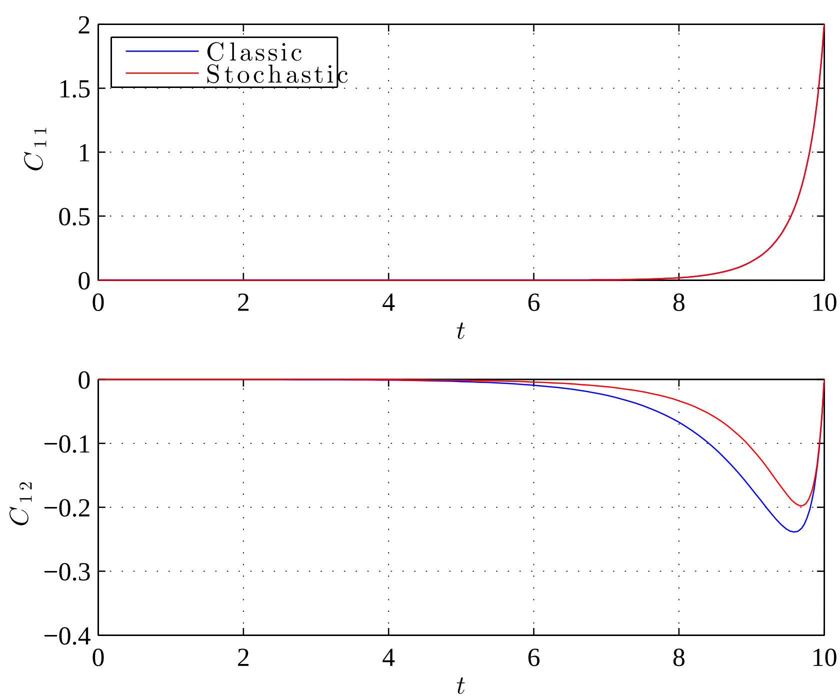

Fig. 6.6: The feedback gains for the example problem.

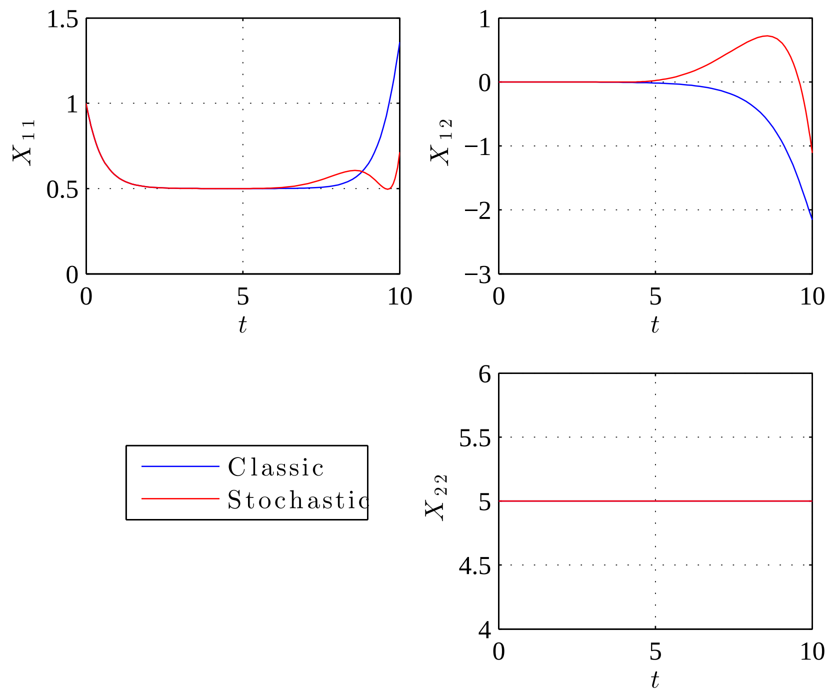

Fig. 6.7: The true state dispersions for the example problem.

This initial jump in state dispersions corresponds to the negative excursion of $\bar{x}_1$. The state dispersions then briefly return to the steady state value as $\bar{x}_1$ returns to zero. Finally, the dispersions increase as $\bar{x}_1$ approaches departs 0 and approaches the final value of 1. This behavior is caused by the system dynamics being less sensitive to dispersions in x_2 when x_1 is near the origin.

The effect is even more pronounced in the estimator dispersions. The increase in estimation dispersions for x_2 shows that the initial application of negative control, allows the estimator to acquire a better estimate of x_2, and thus a better dynamic model for x_1. Next, the natural dynamics to decrease the dispersions in x_1. Thus, the stochastic approach uses some control to learn about x_2. Later, the focus shifts from learning about x_2 to using that knowledge to reduce the state dispersions.

The control dispersions are shown in figure 6.9. Clearly the control dispersions are substantially less for the stochastic case. The effect of the smaller control dispersions is shown in table 6.2. Despite the substantially larger nominal cost, these smaller control dispersions result in a significantly smaller expected overall cost. This demonstrates that

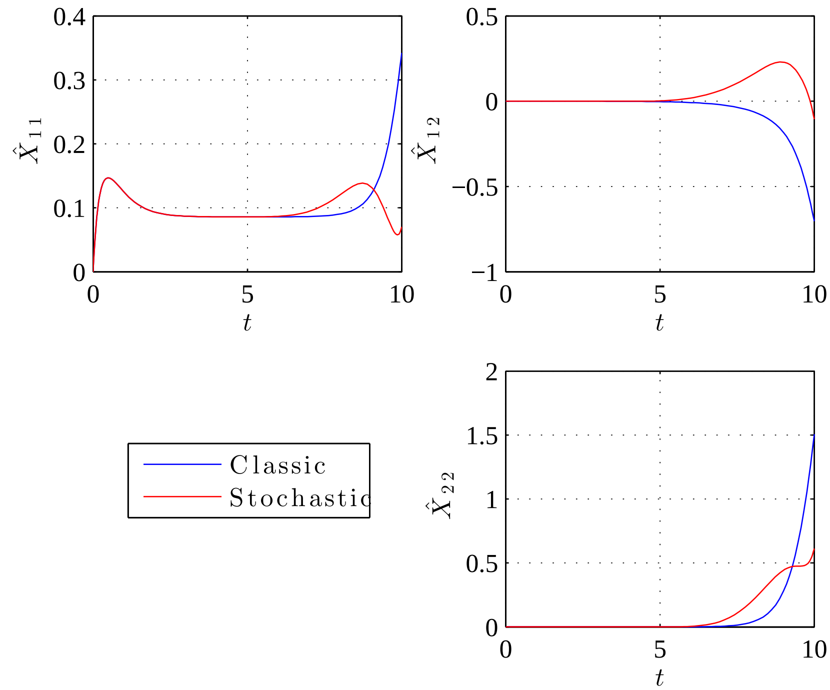

Fig. 6.8: The estimated state dispersion for the example problem.

Table 6.2: Breakdown of the contributions to cost.

	$\bar{J}$	$E\left[d^2J\right]$	$E\left[J\right]$
Classic	1	1.51	2.51
Stochastic	1.32	0.72	2.03

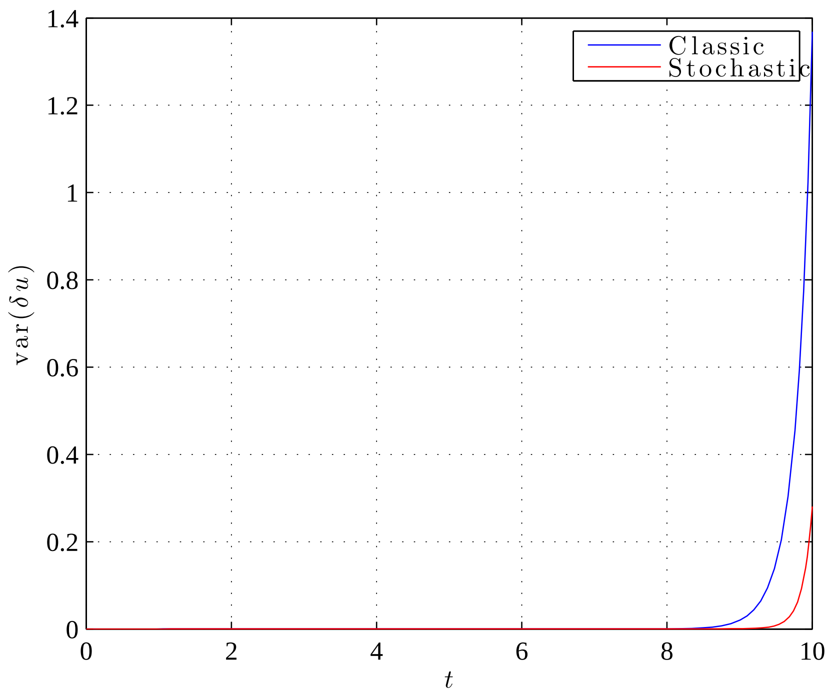

Fig. 6.9: The control dispersion for the example problem.

the increase in nominal cost, can be more than offset my the reduction in the expected value of the cost. The stochastic quadratic synthesis approach is able to leverage the dynamics of the problem to reduce the dispersions.

Although this example is focused on a cost related to fuel, it will be shown in the next chapter that this technique can be used for fuel-limited trajectories where minimizing the terminal dispersion is the primary objective.

Chapter 7

Fuel-Limited Mission Success Guidance

Thesis statement: A guidance law which attempts to minimize trajectory dispersions subject to a fuel constraint can be developed using the calculus of variations.

The thesis of this work revolves around maximizing mission success subject to a fuel constraint. The strategy for accomplishing this will be outlined in this chapter. The famous Zermelo boat problem will be used to demonstrate the proposed approach.

7.1 Limiting Fuel Use

The main difficulty in developing a fuel-limited guidance strategy is to find a way to limit the fuel use. The stochastic behavior of traditional feedback guidance makes it impossible to place a constraint on the fuel use. For example, a typical feedback guidance scheme often consists of the superposition of an open-loop (deterministic) component with a close-loop (stochastic) component. In this situation, the open-loop control is the nominal control, and the closed-loop control accounts for excursions from the nominal trajectory. For the oft used linear feedback laws this looks like

$$\mathbf{u} = \underbrace{\bar{\mathbf{u}}}_{\text{nominal}} + \underbrace{\mathbf{C}\hat{\mathbf{x}}}_{\text{feedback}} \tag{7.1}$$

and the fuel consumed is found by integrating the norm of the control

$$\Delta V = \int_{t_i}^{t_f} |\mathbf{u}|\, dt = \int_{t_i}^{t_f} |\bar{\mathbf{u}} + \mathbf{C}\hat{\mathbf{x}}|\, dt \tag{7.2}$$

Unfortunately, this integral is not well defined when $\hat{\mathbf{x}}$ is a random variable. It thus follows that ΔV is also a random variable. In the case, where $\hat{\mathbf{x}}$ is a Gaussian random variable, the integral is unbounded. It follows directly that ΔV is unbounded. While it is possible, to find statistical information regarding ΔV, current frameworks do not allow for ΔV to be bounded.

The inability to mathematically bound fuel use for dispersed trajectories is at odds with the limited capacity of physical fuel tanks. In practice, difficulties caused by this contradiction are avoided by carrying sufficient fuel to assure that, in practice, the fuel tank is never completely empty.

Several measures are taken to assure sufficient fuel is always available. First, by utilizing fuel optimal guidance onboard it is tacitly implied that the vehicle will consume as little fuel as possible while still meeting any other objectives. This argument for efficiency underpins classical guidance theory. Second, detailed statistical analysis is used to compute the amount of fuel that is needed to reach mission objectives with near certainty. Finally, engineering margin is applied to insure that the fuel tank is never completely empty, even in the worst possible situation.

One way to bound fuel use is to provide steering angle commands (rather than thrust commands) to a fixed-thrust rocket

$$\begin{bmatrix} \theta \\ \phi \end{bmatrix} = \begin{bmatrix} \bar{\theta} \\ \bar{\phi} \end{bmatrix} + \mathbf{C}\hat{\mathbf{x}} \tag{7.3}$$

The fuel consumed during a fixed time interval is found by integrating the thrust (presumed constant) over time

$$\Delta V = \int_{t_i}^{t_f} \left| F_{tr} \hat{\mathbf{u}} \left(\theta, \phi \right) \right| dt = F_{tr} \left(t_f - t_i \right) \tag{7.4}$$

where the thrust magnitude is given by F_{tr} and the thrust direction unit vector, $\hat{\mathbf{u}}$, is a function of the steering angles θ and ϕ. Since ΔV is not a function of the steering angles, the fuel-limited problem is equivalent to a time-limited problem. Thus, for fixed thrust rockets, the fuel can be limited by simply limiting the final time. This is the approach taken here for limiting the fuel consumption. Since the thrust is constant, this approach bounds fuel regardless of the control dispersions. In addition, writing the dynamics in terms of steering angles is often very useful for spaceflight applications.

7.2 Proposed Fuel-Limited Mission Success Guidance Scheme

The proposed guidance scheme relies on the stochastic quadratic synthesis presented in the previous chapter. The scheme is outlined as follows:

1. Select a dynamic model in such a way that the fuel use is only a function of time. This makes a fuel limit equivalent to a final time limit. One convenient way of accomplishing this, is by using steering angles as the control. Most powered flight rocket problems are immediately amenable to this as these problems are typically formulated with steering angles as the control.

2. Apply the techniques of stochastic quadratic synthesis to minimize the following cost function
$$J = E\left[\frac{1}{2}\left(\mathbf{x}\left(t_f\right) - \bar{\mathbf{x}}_f\right)^T \mathbf{S}_f\left(\mathbf{x}\left(t_f\right) - \bar{\mathbf{x}}_f\right) + \frac{1}{2}\int_{t_i}^{t_f} \varepsilon \mathbf{u}^T \mathbf{u} dt\right]$$
where ε is a small positive scalar[1] chosen during the optimization process.

3. The resulting open loop control, $\bar{\mathbf{u}}$, and feedback gains, $\mathbf{C}$, are implemented as a linear feedback law.
$$\mathbf{u}\left(t\right) = \bar{\mathbf{u}}\left(t\right) + \mathbf{C}\hat{\mathbf{x}} \tag{7.5}$$
Since $\mathbf{u}$ is a steering angle, its magnitude has no effect on the fuel use. This linear feedback law allows the controls and feedback gains to be computed a priori rather than real time. Computing these values beforehand and implementing them via gain scheduling, avoids any potential convergence problems during flight.

By formulating the problem in terms of steering angles, it is possible to place a bound on the fuel used by simply placing a bound on the final time. By formulating the dynamics and implementing the control in this manner the machinery of stochastic quadratic synthesis can be used to minimize a terminal dispersion subject to a fuel (time) constraint.

Any reduction in the trajectory dispersions at the target leads to an increase in the probability of a successful outcome. In other words, as the dispersions at the target point are minimized, the probability of successfully hitting the target and completing the mission

[1] Setting $\varepsilon = 0$ results in an undefined solution to the quadratic cost optimal control problem used in quadratic synthesis [82, 135].

is maximized. Thus, given the proper formulation of dynamics, maximizing the probability of mission success subject to a fuel constraint is equivalent to minimizing the dispersions subject to a time constraint.

The remainder of this chapter shows how this approach can be applied to minimize dispersions for the famous Zermelo boat problem. The classical deterministic minimum time (fuel) solutions to the Zermelo problem will be reviewed. This is followed by the optimal stochastic solution found using mission success guidance.

7.3 The Zermelo Boat Problem and its Classical Solutions

The Zermelo boat problem is a famous problem in which the calculus of variations is applied to a simple guidance type problem. The problem was first posed and solved by Zermelo in 1930 and 1931 [136, 137]

> "In an infinite plane in which the wind distribution is given by a vector field as a function of place and time, a vehicle moves at a constant airspeed relative to the surrounding air mass. How must the vehicle be controlled in order to move in the shortest possible time from a starting point to a given destination?" [137]

The Zermelo navigation problem serves as one of the key problems in the study of trajectory optimization and guidance. Among other reasons, the popularity of the problem is caused by its simplicity, practical utility, and the availability of closed form solutions. In the field of guidance the Zermelo problem is both directly, and indirectly applicable [132, 133, 138]. It is directly applicable to trajectory optimization of sea-craft and aircraft. For spacecraft the application is indirect. In certain cases, the solution to the Zermelo problem is a linear tangent steering law. Since optimal trajectories for powered rocket flight in a uniform gravity field are also known to be governed by a linear tangent steering law, the study of one problem can yield insight into the solution of the other problem. Many of the most well regarded texts in optimal control and guidance theory make frequent use of various versions of Zermelo's problem [82, 111, 112, 127]. In addition, Zermelo's problem is frequently used as a central example in important papers in the field of guidance [100, 139, 140].

7.3.1 Arbitrary Flow Field

Zermelo solved the problem for a 3-dimensional flow field [82,137]. In three dimensions, the Zermelo problem is

$$\begin{aligned}
\underset{\theta(t)}{\text{minimize}} \quad & \phi = t_f + \Gamma\varphi\left[t_f, \vec{r}\left(t_f\right)\right] && (7.6)\\
\text{subject to} \quad & \dot{\vec{r}} = \vec{w}\left(\vec{r}\right) + V\hat{u} && (7.7)\\
& 1 = \hat{u}^T\hat{u} && (7.8)\\
& \vec{r}_0 = \vec{r}\left(t_0\right) && (7.9)\\
& 0 = \psi\left[t_f, \vec{r}\left(t_f\right)\right] && (7.10)
\end{aligned}$$

where Γ is a non-negative weighting factor, and $\vec{r}\left(t\right)$ is the position at time t, $\vec{w}$ is the flow field, and $\hat{u}$ is a unit vector. The Hamiltonian is

$$H = \lambda^T\left(\vec{w}\left(\vec{r}\right) + V\hat{u}\right) + \mu\left(1 - \hat{u}^T\hat{u}\right) \tag{7.11}$$

Given the optimal control, the Hamiltonian is a constant with respect to time since it is not an explicit function of time [82,112].

The Euler-Lagrange equations are

$$\begin{aligned}
\dot{\lambda}^T &= -\frac{\partial H}{\partial \vec{r}} = -\,\lambda^T\frac{\partial \vec{w}}{\partial \vec{r}} && (7.12)\\
0 &= \frac{\partial H}{\partial \hat{u}} = V\lambda^T - 2\mu\hat{u}^T && (7.13)
\end{aligned}$$

Together with the transversality conditions

$$\begin{aligned}
0 &= \left[\frac{\partial \phi}{\partial t_f} + \nu^T\frac{\partial \psi}{\partial t_f}\right]_{t=t_f} + H\left(t_f\right) && (7.14)\\
0 &= \left[\frac{\partial \phi}{\partial \vec{r}\left(t_f\right)} + \nu^T\frac{\partial \psi}{\partial \vec{r}\left(t_f\right)}\right]_{t=t_f} - \lambda^T\left(t_f\right) && (7.15)
\end{aligned}$$

The steering law can be found by finding a solution to equation 7.13

$$\hat{u} = \pm \frac{\lambda}{|\lambda|} \tag{7.16}$$

$$2\mu = \pm V |\lambda| \tag{7.17}$$

The appropriate sign can be chosen based on equation 7.14. For the minimum time problem $\left(\phi = t_f, \quad \psi [\vec{r}(t_f)] \right)$ equation 7.14 indicates that the Hamiltonian must be negative at the final time. Thus, the negative sign in the above equations is the appropriate sign.

Despite Zermelo posing the problem for aircraft traveling through winds in three dimensions, the problem is more frequently presented as a boat traveling through a field of currents in two dimensions. This is often referred to as the Zermelo boat problem. First, the solution to this problem for an arbitrary flow field will be shown. Afterwards, more specialized problems will be examined.

The general problem is

$$\underset{\theta(t)}{\text{minimize}} \quad \phi = t_f + \Gamma\varphi [t_f, x(t_f), y(t_f)] \tag{7.18}$$

$$\text{subject to} \quad \dot{x} = w_x(x, y) + V \cos [\theta(t)] \tag{7.19}$$

$$\dot{y} = w_y(x, y) + V \sin [\theta(t)] \tag{7.20}$$

$$x_0 = x(t_0) \tag{7.21}$$

$$y_0 = y(t_0) \tag{7.22}$$

$$0 = \psi [t_f, x(t_f), y(t_f)] \tag{7.23}$$

where $\Gamma \geq 0$ is a weighting factor, w is the flow field (current) as a function of position, and $V \geq 0$ is the boat's speed relative to the water. The problem is to select $\theta(t)$ in order to minimize ϕ subject to the boundary conditions. The necessary conditions for the problem can be found using the calculus of variations. The Hamiltonian is given by

$$H = \lambda_x [w_x + V \cos(\theta)] + \lambda_y [w_y + V \sin(\theta)] \tag{7.24}$$

Since the Hamiltonian is not an explicit function of time it is known to be constant with respect to time along optimal trajectories.

The necessary conditions for an optimal solution can now be found by appropriate differentiation of the Hamiltonian. First the Euler-Lagrange equations are found

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = -\lambda_x \frac{\partial w_x}{\partial x} - \lambda_y \frac{\partial w_y}{\partial y} \tag{7.25}$$

$$\dot{\lambda}_y = -\frac{\partial H}{\partial y} = -\lambda_x \frac{\partial w_x}{\partial y} - \lambda_y \frac{\partial w_y}{\partial y} \tag{7.26}$$

The necessary conditions for the optimal control, θ, are

$$0 = \frac{\partial H}{\partial \theta} = V\left[-\lambda_x \sin(\theta) + \lambda_y \cos(\theta)\right] \tag{7.27}$$

and this expression quickly reduces to the following steering law.

$$\tan(\theta) = \frac{\lambda_y}{\lambda_x} \tag{7.28}$$

Finally, the transversality constraints are given by

$$0 = \left[\frac{\partial \phi}{\partial t_f} + \nu^T \frac{\partial \psi}{\partial t_f}\right]_{t=t_f} + H(t_f) \tag{7.29}$$

$$0 = \left[\frac{\partial \phi}{\partial x(t_f)} + \nu^T \frac{\partial \psi}{\partial x(t_f)}\right]_{t=t_f} - \lambda_x(t_f) \tag{7.30}$$

$$0 = \left[\frac{\partial \phi}{\partial y(t_f)} + \nu^T \frac{\partial \psi}{\partial y(t_f)}\right]_{t=t_f} - \lambda_y(t_f) \tag{7.31}$$

where ν is a vector containing constant Lagrangian multipliers associated with the terminal constraint vector $\boldsymbol{\psi}$.

The original version of Zermelo's problem minimized only the final time, subject to a terminal constraint.

$$\underset{\theta(t)}{\text{mimimize}} \quad \phi = t_f \tag{7.32}$$

$$\text{subject to} \quad \dot{x} = w_x\left(x, y\right) + V\cos\left[\theta\left(t\right)\right] \tag{7.33}$$

$$\dot{y} = w_y\left(x, y\right) + V\sin\left[\theta\left(t\right)\right] \tag{7.34}$$

$$x\left(t_0\right) = x_0 \tag{7.35}$$

$$y\left(t_0\right) = y_0 \tag{7.36}$$

$$x\left(t_f\right) = x_f \tag{7.37}$$

$$y\left(t_f\right) = y_f \tag{7.38}$$

The necessary conditions for this problem are the same as the previous problem with $\Gamma = 0$.

Given that the Hamiltonian is a constant along an optimal trajectory, the transversality condition connected with the final time reduces to

$$H = -1 \tag{7.39}$$

By solving equations 7.24, 7.28 and 7.39 expressions can be found for the Lagrange multipliers

$$\lambda_x = \frac{-\cos\left(\theta\right)}{V + w_x\cos\left(\theta\right) + w_y\sin\left(\theta\right)} \tag{7.40}$$

$$\lambda_y = \frac{-\sin\left(\theta\right)}{V + w_x\cos\left(\theta\right) + w_y\sin\left(\theta\right)} \tag{7.41}$$

These equations can be used to develop a steering law. First, time differentiate equation 7.28 with respect to time and substitute equations 7.25-7.26 and 7.40-7.41 into the resulting expression yields a steering law.

$$\dot{\theta} = \frac{\partial w_y}{\partial x}\sin^2\left(\theta\right) + \left(\frac{\partial w_x}{\partial x} - \frac{\partial w_y}{\partial y}\right)\cos\left(\theta\right)\sin\left(\theta\right) - \frac{\partial w_x}{\partial y}\cos^2\left(\theta\right) \tag{7.42}$$

This is the steering law first shown by Zermelo. The boundary conditions may now be satisfied by selecting an appropriate value for $\theta(t_0)$ and t_f, such that the terminal boundary conditions are met. For simple flow fields, an analytic solution to this boundary value problem can be developed.[2]

7.3.2 Linearly Varying Flow Field

The most famous example of the Zermelo boat problem, is where the flow field is known to vary linearly in only one direction

$$w_x = py \tag{7.43}$$

$$w_y = 0 \tag{7.44}$$

In this situation the differential equations of motion reduce to

$$\dot{x} = py + V\cos(\theta) \tag{7.45}$$

$$\dot{y} = V\sin(\theta) \tag{7.46}$$

together with the steering law given by

$$\dot{\theta} = -p\cos^2(\theta) \tag{7.47}$$

The value of $\theta_0 = \theta(t_0)$ can be found by direct integration of these three differential equations. This process is facilitated by changing the independent variable from t to θ, using equation 7.47

$$dt = \frac{d\theta}{-p\cos^2(\theta)} \tag{7.48}$$

[2] Another interesting example relates to Snell's law from optics. If the flow field is only a function of the y-axis component $(w_x(y),\ w_y(y))$ this problem reduces to an analog of Snell's law [141] from optics

$$\text{constant} = \frac{\cos\theta}{V + w_x(y)\cos(\theta) + w_y(y)\sin(\theta)}$$

which is amenable to direct integration

$$t_{go} = t_f - t = \frac{1}{p}\left[\tan(\theta) - \tan(\theta_f)\right] \tag{7.49}$$

or

$$\tan(\theta) = \tan(\theta_f) + p t_{go} \tag{7.50}$$

where $\theta_f = \theta(t_f)$ and $\theta = \theta(t)$. This is a linear tangent steering law, which is the same structural form found in space guidance applications. This same change of variables can be used to directly integrate the equations of motion. Begin by directly integrating y with respect to θ

$$f_1(\theta, \theta_f) = \frac{V}{p}\left[\frac{1}{\cos(\theta_f)} - \frac{1}{\cos(\theta)}\right] + y_f - y = 0 \tag{7.51}$$

This result can be used to integrate x with respect to θ

$$f_2(\theta, \theta_f) = y_f\left[\tan(\theta_f) - \tan(\theta)\right] + \frac{V}{2p}\left[\frac{\tan(\theta)}{\cos(\theta)} - 2\frac{\tan(\theta)}{\cos(\theta_f)} + \frac{\tan(\theta_f)}{\cos(\theta_f)} - \ln\left|\frac{\tan(\theta_f) + \frac{1}{\cos(\theta_f)}}{\tan(\theta) + \frac{1}{\cos(\theta)}}\right|\right] + x_f - x = 0 \tag{7.52}$$

For given values of x and y, equations 7.51 and 7.52 can be solved to find θ and θ_f. The solution to the boundary value problem is completed, by finding t_{go} using equation 7.49.

The solution to these equations 7.51 and 7.52 must be found using numerical methods. A simple Newton-Raphson root finding technique works very well for this purpose, if a good initial guess is provided. The partials needed to execute a Newton-Raphson routine are

$$\frac{\partial f_1}{\partial \theta} = -\frac{V}{p}\frac{\tan(\theta)}{\cos(\theta)} \tag{7.53}$$

$$\frac{\partial f_1}{\partial \theta_f} = \frac{V}{p}\frac{\tan(\theta_f)}{\cos(\theta_f)} \tag{7.54}$$

$$\frac{\partial f_2}{\partial \theta} = -y_f\frac{1}{\cos^2(\theta)} + \frac{V}{p\cos(\theta)}\left[\tan^2(\theta) - \frac{1}{\cos(\theta)\cos(\theta_f)}\right] \tag{7.55}$$

$$\frac{\partial f_2}{\partial \theta_f} = y_f\frac{1}{\cos^2(\theta_f)} + \frac{V}{p\cos(\theta_f)}\left[\frac{1}{\cos^2(\theta_f)} - \tan(\theta)\tan(\theta_f)\right] \tag{7.56}$$

The form of the optimal control law can be verified by examining the necessary conditions provided by Euler-Lagrange equations

$$\dot{\lambda}_x = 0 \tag{7.57}$$

$$\dot{\lambda}_y = -p\lambda_x \tag{7.58}$$

These differential equations can be easily solved

$$\lambda_x = \text{const.} \tag{7.59}$$

$$\lambda_y = \lambda_{y_f} + (t_f - t)\lambda_x p \tag{7.60}$$

These expressions are substituted into equation 7.28 revealing the same linear tangent steering law found in equation 7.50

$$\tan(\theta) = \underbrace{\frac{\lambda_{y_f}}{\lambda_x}}_{\tan(\theta_f)} + \underbrace{(t_f - t)}_{t_{go}} p \tag{7.61}$$

Table 7.1 and figures 7.1 and 7.2 show some examples of solutions to this boundary value problem.

Another version of the Zermelo boat problem is the case where t_f is fixed, and a function of the terminal state is to be extremelized.

$$\underset{\theta(t)}{\text{minimize}} \quad \phi = \varphi\left[x(t_f), y(t_f)\right] \tag{7.62}$$

$$\text{subject to} \quad \dot{x} = w_x(x, y) + V\cos\left[\theta(t)\right] \tag{7.63}$$

$$\dot{y} = w_y(x, y) + V\sin\left[\theta(t)\right] \tag{7.64}$$

$$x_0 = x(t_0) \tag{7.65}$$

$$y_0 = y(t_0) \tag{7.66}$$

$$0 = \psi\left[x(t_f), y(t_f)\right] \tag{7.67}$$

Table 7.1: Example solutions to Zermelo's boat problem in linearly varying current. A-D are numeric solutions, and E is an analytic solution that can be used to check coded equations for accuracy. Trajectories associated with columns A-D are shown in figure 7.1.

	A	B	C	D	E
$V =$	$1\,\frac{\mathrm{m}}{\mathrm{s}}$	$1\,\frac{\mathrm{m}}{\mathrm{s}}$	$1\,\frac{\mathrm{m}}{\mathrm{s}}$	$1\,\frac{\mathrm{m}}{\mathrm{s}}$	$1\,\frac{\mathrm{m}}{\mathrm{s}}$
$p =$	$0.01\,\frac{1}{\mathrm{s}}$	$0.01\,\frac{1}{\mathrm{s}}$	$0.01\,\frac{1}{\mathrm{s}}$	$0.01\,\frac{1}{\mathrm{s}}$	$\frac{1}{100}\,\frac{1}{\mathrm{s}}$
$x =$	$0\,\mathrm{m}$	$250\,\mathrm{m}$	$500\,\mathrm{m}$	$600\,\mathrm{m}$	$0\,\mathrm{m}$
$x_f =$	$500\,\mathrm{m}$	$500\,\mathrm{m}$	$500\,\mathrm{m}$	$500\,\mathrm{m}$	$100\left(1+\frac{1}{\sqrt{2}}+\sqrt{6}-\frac{1}{2}\ln\frac{2-\sqrt{3}}{1+\sqrt{2}}\right)\,\mathrm{m}$ $\approx 525.5762\,\mathrm{m}$
$y =$	$100\,\mathrm{m}$	$100\,\mathrm{m}$	$100\,\mathrm{m}$	$100\,\mathrm{m}$	$100\,\mathrm{m}$
$y_f =$	$50\,\mathrm{m}$	$50\,\mathrm{m}$	$50\,\mathrm{m}$	$50\,\mathrm{m}$	$100\left(\sqrt{2}-1\right)\,\mathrm{m}$ $\approx 41.4214\,\mathrm{m}$
$\theta =$	$43.9400°$	$14.7553°$	$-127.2954°$	$-123.2751°$	$45°$
$\theta_f =$	$-58.0318°$	$-49.3190°$	$-150.3755°$	$139.1191°$	$-60°$
$t_{go} =$	$256.5983\,\mathrm{s}$	$142.6764\,\mathrm{s}$	$74.4265\,\mathrm{s}$	$238.9444\,\mathrm{s}$	$100\left(1+\sqrt{3}\right)\,\mathrm{s}$ $\approx 273.2051\,\mathrm{s}$

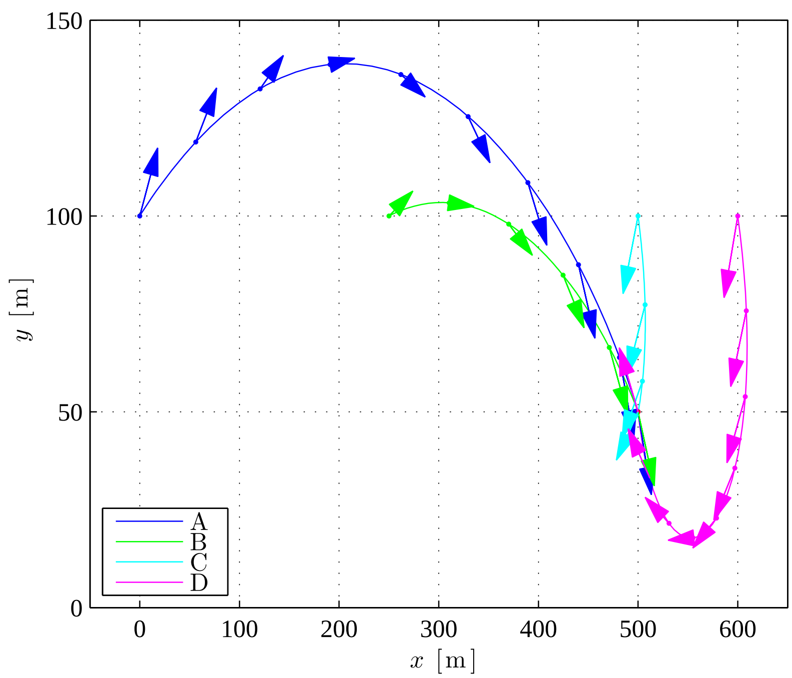

Fig. 7.1: Trajectories for example solutions to Zermelo's boat problem in linearly varying current. These trajectories correspond with the indicated columns of table 7.1.

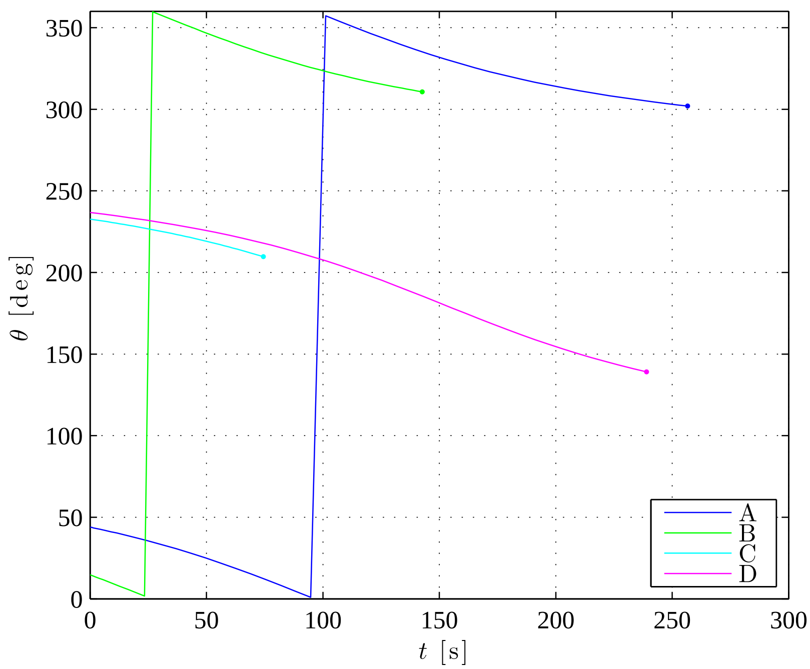

Fig. 7.2: Control trajectories for example solutions to Zermelo's boat problem in linearly varying current. These controls correspond with the indicated columns of table 7.1.

This problem has the same Euler-Lagrange equations as the last problem, but the transversality conditions used to find the final time are not applicable. The remaining transversality conditions are

$$0 = \left[\frac{\partial \phi}{\partial x\left(t_f\right)} + \nu^T \frac{\partial \psi}{\partial x\left(t_f\right)}\right]_{t=t_f} - \lambda_x\left(t_f\right) \tag{7.68}$$

$$0 = \left[\frac{\partial \phi}{\partial y\left(t_f\right)} + \nu^T \frac{\partial \psi}{\partial y\left(t_f\right)}\right]_{t=t_f} - \lambda_y\left(t_f\right) \tag{7.69}$$

This version of the Zermelo boat problem can also be solved for the case of the linear flow field. For example, consider the following maximization problem

$$\underset{\theta(t)}{\text{maximize}} \quad \phi = x\left(t_f\right) \tag{7.70}$$

$$\text{subject to} \quad \dot{x} = py + V\cos\left[\theta\left(t\right)\right] \tag{7.71}$$

$$\dot{y} = V\sin\left[\theta\left(t\right)\right] \tag{7.72}$$

$$x_0 = x\left(t_0\right) \tag{7.73}$$

$$y_0 = y\left(t_0\right) \tag{7.74}$$

$$0 = y\left(t_f\right) \tag{7.75}$$

Conveniently, the solution to this problem shares many common necessary conditions with the minimum time problem solved earlier. Equations 7.47-7.52 are valid for this problem also, but must be applied in a slightly different manner. For this problem, equations 7.49 and 7.51 must be solved to find θ and θ_f.

7.3.3 Constant Flow Field

Another popular version of the Zermelo problem [112] is the case where the flow field is taken to be a constant. As before, it is assumed that the reference frame is chosen such that the x-axis is aligned with the direction of the flow and t_f is free.

$$\underset{\theta(t)}{\text{minimize}} \quad \phi = t_f + \Gamma\varphi\left[t_f, x\left(t_f\right), y\left(t_f\right)\right] \tag{7.76}$$

$$\text{subject to} \quad \dot{x} = V_p + V\cos\left[\theta\left(t\right)\right] \tag{7.77}$$

$$\dot{y} = V\sin\left[\theta\left(t\right)\right] \tag{7.78}$$

$$x_0 = x\left(t_0\right) \tag{7.79}$$

$$y_0 = y\left(t_0\right) \tag{7.80}$$

$$0 = \psi\left[t_f, x\left(t_f\right), y\left(t_f\right)\right] \tag{7.81}$$

The Hamiltonian for this problem is

$$H = \lambda_x \left[V_p + V \cos(\theta)\right] + \lambda_y V \sin(\theta) \tag{7.82}$$

The Euler-Lagrange equations are

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0 \tag{7.83}$$

$$\dot{\lambda}_y = -\frac{\partial H}{\partial y} = 0 \tag{7.84}$$

$$0 = \frac{\partial H}{\partial \theta} = -\lambda_x V \sin(\theta) + \lambda_y V \cos(\theta) \tag{7.85}$$

These equations imply that the control is constant

$$\tan(\theta) = \frac{\lambda_x}{\lambda_y} = \text{const.} \tag{7.86}$$

This constant control can be used to directly integrate the dynamics

$$x_f = \left[V_p + V \cos(\theta)\right] \overbrace{(t_f - t)}^{t_{go}} + x(t) \tag{7.87}$$

$$y_f = \left[V \sin(\theta)\right] \overbrace{(t_f - t)}^{t_{go}} + y(t) \tag{7.88}$$

This system of equations can be used to solve directly for t_{go} and θ. Although not needed for the solution, the transversality conditions are found by taking the same partials as before.

$$0 = \left[\frac{\partial \phi}{\partial t_f} + \nu^T \frac{\partial \psi}{\partial t_f}\right]_{t=t_f} + H(t_f) \tag{7.89}$$

$$0 = \left[\frac{\partial \phi}{\partial x(t_f)} + \nu^T \frac{\partial \psi}{\partial x(t_f)}\right]_{t=t_f} - \lambda_x(t_f) \tag{7.90}$$

$$0 = \left[\frac{\partial \phi}{\partial y(t_f)} + \nu^T \frac{\partial \psi}{\partial y(t_f)}\right]_{t=t_f} - \lambda_y(t_f) \tag{7.91}$$

Due to the simplicity of these equations , the resulting boundary value problem is typically comparatively easy to solve. In the worst case scenario, the solution can be found

Table 7.2: Example solutions to Zermelo's boat problem in constant current. A, B, and D are numeric solutions. C and E are analytic solution that can be used to check coded equations for accuracy. Boundary conditions associated with columns A-D are the same as the corresponding columns shown in figure 7.1.

	A	B	C	D	E
$V =$	$1\frac{\mathrm{m}}{\mathrm{s}}$	$1\frac{\mathrm{m}}{\mathrm{s}}$	$1\frac{\mathrm{m}}{\mathrm{s}}$	$1\frac{\mathrm{m}}{\mathrm{s}}$	$1\frac{\mathrm{m}}{\mathrm{s}}$
$V_p =$	$0.5\frac{\mathrm{m}}{\mathrm{s}}$	$0.5\frac{\mathrm{m}}{\mathrm{s}}$	$\frac{1}{2}\frac{\mathrm{m}}{\mathrm{s}}$	$0.5\frac{\mathrm{m}}{\mathrm{s}}$	$\frac{1}{2}\frac{\mathrm{m}}{\mathrm{s}}$
$x =$	$0\,\mathrm{m}$	$250\,\mathrm{m}$	$500\,\mathrm{m}$	$600\,\mathrm{m}$	$25\left(18-\sqrt{2}\right)\,\mathrm{m}$ $\approx 414.6447\,\mathrm{m}$
$x_f =$	$500\,\mathrm{m}$	$500\,\mathrm{m}$	$500\,\mathrm{m}$	$500\,\mathrm{m}$	$500\,\mathrm{m}$
$y =$	$100\,\mathrm{m}$	$100\,\mathrm{m}$	$100\,\mathrm{m}$	$100\,\mathrm{m}$	$100\,\mathrm{m}$
$y_f =$	$50\,\mathrm{m}$	$50\,\mathrm{m}$	$50\,\mathrm{m}$	$50\,\mathrm{m}$	$50\,\mathrm{m}$
$\theta =$	$-8.5623°$	$-16.9373°$	$-120°$	$-166.3559°$	$-45°$
$t_{go} =$	$335.8287\,\mathrm{s}$	$171.6297\,\mathrm{s}$	$100/\sqrt{3}\,\mathrm{s}$ $\approx 57.7350\,\mathrm{s}$	$211.9633\,\mathrm{s}$	$50\sqrt{2}\,\mathrm{s}$ $\approx 70.7107\,\mathrm{s}$

by a numerical search for θ on the interval $[0, 2\pi)$. Table 7.2 shows the solutions for many of the same boundary conditions examined earlier for the linearly varying flow field.

7.4 Statistical GN&C Analysis

To facilitate the study of GNC systems it is useful to select a baseline problem for study. For this work, the specific problem selected is related to problem A from table 7.1. Consider a river whose bank is defined by the x-axis with a linear flow field flowing in the direction of the x-axis. The nominal starting position, problem parameters, deterministic optimal solution, and desired terminal condition, are taken from case A in table 7.1. Now however, the cost function is given by

$$J = \frac{1}{2}\left[\mathbf{x}\left(t_f\right) - \mathbf{x}_f\right]^T \mathbf{S}_f \left[\mathbf{x}\left(t_f\right) - \mathbf{x}_f\right] + \frac{1}{2}\int_{t_0}^{t_f} \varepsilon u^2 dt \tag{7.92}$$

where ε is a small number $\left(1 \times 10^{-3}\right)$, and

$$\mathbf{S}_f = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.93}$$

For this problem, the true initial position is dispersed with a bivariate normal distribution

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 100 \end{bmatrix} \mathrm{m}, \begin{bmatrix} 10^2 & 0 \\ 0 & 10^2 \end{bmatrix} \mathrm{m}^2 \right) \tag{7.94}$$

The gradient of the flow field is also taken to be an unknown parameter given by

$$p \sim \mathcal{N}\left(0.01\,\frac{1}{\mathrm{s}}, 0.001^2\,\frac{1}{\mathrm{s}^2}\right) \tag{7.95}$$

The true system dynamics are give by

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{p} \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} py + V\cos(\theta) \\ V\sin(\theta) \\ 0 \end{bmatrix}}_{f(\mathbf{x},\theta)} + \mathbf{w} \tag{7.96}$$

where $\mathbf{w}$ is a zero-mean Guassian white noise process

$$\mathbf{w} = \mathcal{N}\left(\mathbf{0}, \underbrace{\mathrm{diag}\left(\begin{bmatrix} 1^2 & 1^2 & 0^2 \end{bmatrix} \right) \frac{\mathrm{m}^2}{\mathrm{s}^2}}_{\mathbf{Q}} \right) \tag{7.97}$$

The navigation system on the boat receives a single line of sight measurement to a beacon located at the origin. This angle measurement is shown in figure 7.3 and is given by

$$\alpha = -\cos^{-1}\left[\frac{y}{\sqrt{x^2+y^2}}\right] + \nu \tag{7.98}$$

where the negative sign resolves the quadrant ambiguity. The noise ν is given by

$$\nu = \mathcal{N}\left(0, \underbrace{0.015^2}_{\mathbf{R}}\ \mathrm{radians}\right) \tag{7.99}$$

For this problem, the measurements are continuous. A Kalman-Bucy filter [18, 23, 82, 142, 143] is employed to continually estimate the current position and the flow field gradient.

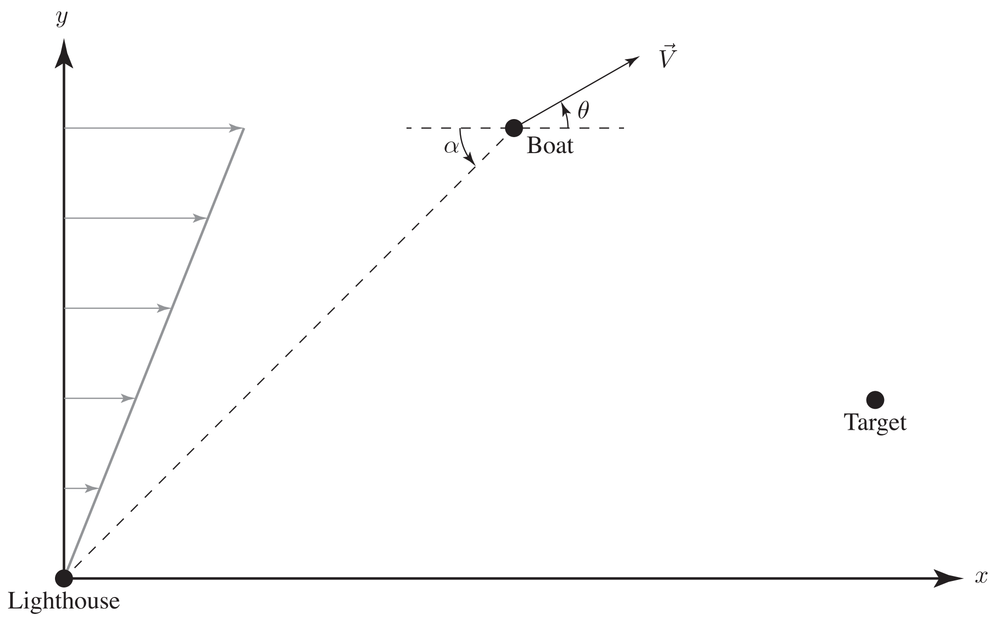

Fig. 7.3: Geometry of the Zermelo boat problem.

The linearized dynamics for this filter are given by

$$\underbrace{\begin{bmatrix} \delta\dot{\hat{x}} \\ \delta\dot{\hat{y}} \\ \delta\dot{\hat{p}} \end{bmatrix}}_{\dot{\hat{\mathbf{x}}}} = \underbrace{\begin{bmatrix} 0 & \bar{p} & \bar{y} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} \delta\hat{x} \\ \delta\hat{y} \\ \delta\hat{p} \end{bmatrix}}_{\hat{\mathbf{x}}} + \underbrace{\begin{bmatrix} -V\sin\left(\bar{\theta}\right) \\ V\cos\left(\bar{\theta}\right) \\ 0 \end{bmatrix}}_{\mathbf{G}} \delta\theta + \mathbf{K}\left[\alpha - \hat{\alpha}\right] \tag{7.100}$$

The filter covariance is propagated by

$$\dot{\mathbf{P}} = \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^T + \mathbf{Q} - \mathbf{K}\mathbf{R}\mathbf{K}^T \tag{7.101}$$

with the Kalman gain

$$\mathbf{K} = \mathbf{P}\mathbf{H}^T\mathbf{R}^{-1} \tag{7.102}$$

The required measurement partials are given by

$$\mathbf{H} = \frac{\partial\phi}{\partial\mathbf{x}} = \begin{bmatrix} \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{bmatrix} \tag{7.103}$$

where this partial is evaluated along the nominal.

The feedback controller gain is found by integrating the Riccati equation

$$\dot{\mathbf{S}} = -\mathbf{F}^T\mathbf{S} - \mathbf{S}\mathbf{F} + \varepsilon\mathbf{C}^T\mathbf{C} \tag{7.104}$$

backwards from the final value of $\mathbf{S}_f$. Recall that ε is the small weighting factor introduced into the cost function. The feedback gains are given by

$$\mathbf{C} = \frac{1}{\varepsilon}\mathbf{G}^T\mathbf{S} \tag{7.105}$$

Case A from table 7.1 and the parameters listed above were used to generate a solution using classical quadratic synthesis and stochastic quadratic synthesis. Since the problem is formulated with the steering angle as the control, the fuel use is directly controlled by selection of the final time. For the given case, the deterministic minimum final time is already known to be 256.5983 seconds. This minimum time (fuel) case provides the nominal trajectory for the classical quadratic synthesis problem, and serves as a baseline against which the solutions from stochastic quadratic synthesis are compared. Several cases of stochastic quadratic synthesis are considered. In each case, the fuel available (final time) is a percentage increase (0.25%, 0.5%, 1%, 2%, 3%, 4%, 5% , 10%, 25%) over the classical case. Each case was considered for two levels of sensor noise. The first level of sensor noise $\left(\mathbf{R} = (0.015\,\mathrm{rad})^2\right)$ represents a nominal sensor, and the second level of sensor noise $\left(\mathbf{R} = (0.075\,\mathrm{rad})^2\right)$ represents a poor sensor.

The solutions to the stochastic quadratic synthesis problem were found using gradient descent techniques. This minimization process required significant manual effort, as the gradient descent techniques are prone to converge to various local minima, and the solutions are quite sensitive to the parameters used in the optimization process and the required numeric integration. Some of this sensitivity is attributable to the boundary layers found in equations 7.101 and 7.104. These boundary layers are responsible for creating the strong transient responses at both ends of the trajectory.

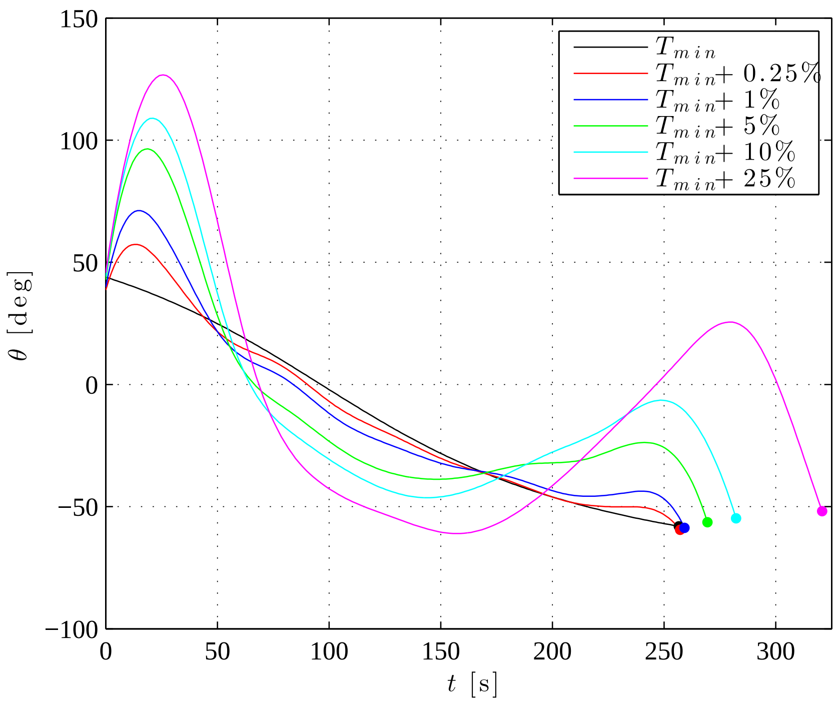

Fig. 7.4: Control for the example case of the Zermelo boat problem.

Figures 7.4 and 7.5 show the control and the state trajectory solutions respectively. There was not a distinguishable difference between the nominal control and nominal trajectory for different sensor noise levels. Thus, these figures only show the shared nominal values for each final time. These figures do show that the nominal control and nominal trajectory are sensitive to changes in the final time. With only a small increase in the final time, significant changes occur in the control while the associated changes to the state trajectory are less dramatic.

The changes in trajectory and control result in significant changes to the variance of the final state. This is shown in table 7.3 and figure 7.6. Note that only a small increase in fuel used results in an significant decrease in the variance of the final state! While continued addition of fuel provides further tightening of the terminal dispersions, the gains are modest when compared to the initial gains.

It should be emphasized that the data in figure 7.6 and table 7.3 strongly support the thesis of this work. These results clearly show that the probability of mission success can be significantly increased by consumption of all available fuel. The Pareto optimal front shown

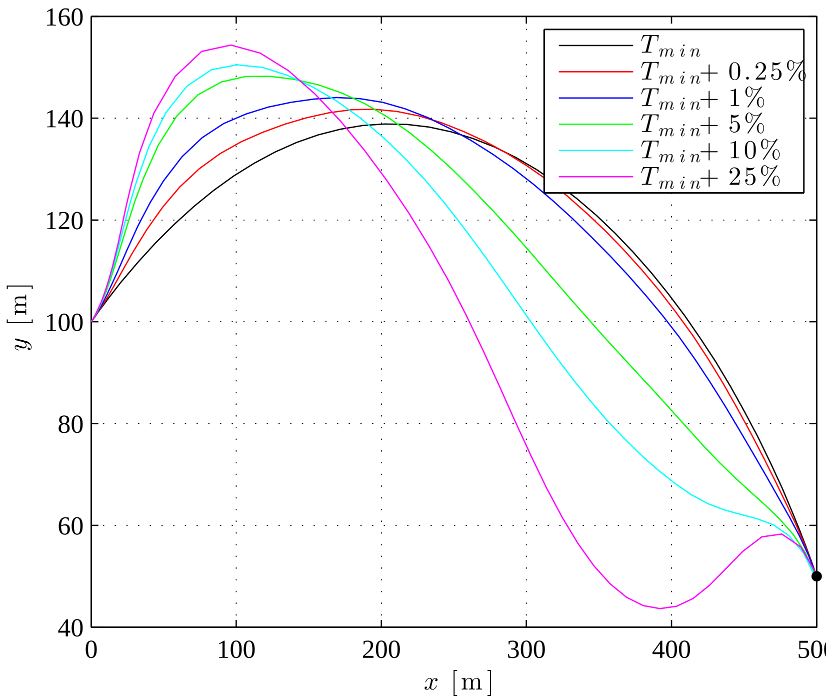

Fig. 7.5: Control for the example case of the Zermelo boat problem.

Table 7.3: Breakdown of variance of the terminal state.

		t_f	$E\left[(\mathbf{x}(t_f)-\mathbf{x}_f)^T(\mathbf{x}(t_f)-\mathbf{x}_f)\right]$ (%chng.) $\mathbf{R}=(0.015\,\text{rad})^2$	$\mathbf{R}=(0.075\,\text{rad})^2$
Classic	t_{min}	256.5983	314.0241 (–)	592.5981 (–)
Stochastic	$t_{min}+0.25\%$	257.2398	179.5820(-42.8%)	494.9273(-16.5%)
	$t_{min}+0.5\%$	257.8813	136.0867(-56.7%)	443.8088 (-25.1%)
	$t_{min}+1\%$	259.8813	97.1672 (-69.1%)	390.3054 (-34.1%)
	$t_{min}+2\%$	261.7303	67.5590 (-78.5%)	345.0623 (-41.8%)
	$t_{min}+3\%$	264.2962	54.0300 (-82.8%)	322.0920 (-45.6%)
	$t_{min}+4\%$	266.8622	45.6507 (-85.5%)	305.9061 (-48.4%)
	$t_{min}+5\%$	269.4282	39.7961 (-87.3%)	294.0033 (-50.4%)
	$t_{min}+10\%$	282.2581	26.0478 (-91.7%)	256.0682 (-56.8%)
	$t_{min}+25\%$	320.7479	13.9445 (-95.6%)	194.7098 (-67.1%)

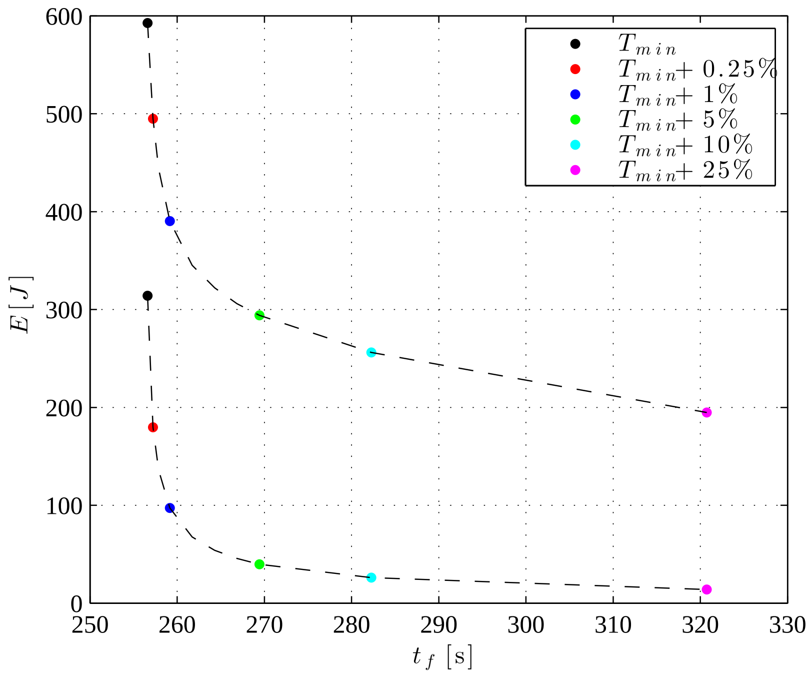

Fig. 7.6: Pareto optimal front for mission success guidance applied to the Zermelo boat problem.

in figure 7.6 shows the dependence of the final dispersion with respect to the fuel available for consumption. Providing such information to mission planners allows them to understand the significant reductions in dispersions that can be achieved for a modest penalty in fuel consumption.

Although outside the scope of this thesis, it is probable that an effect similar to the one observed in the example in section 6.2.1 may also exist here. That is, by permitting additional fuel use, the control may be able to reduce the expectation of total fuel used. This is due to the fact that the large dispersions from the classical quadratic synthesis approach will be unacceptable, and additional maneuvers will need to be executed to reduce these dispersions. It is quite probable that the fuel needed to execute these "clean-up" maneuvers is greater than the additional fuel that would be expended executing the original maneuver with fuel-limited mission success.

Given the dramatic results, some insight may be gained by examining the filter error covariance and the true trajectory dispersions. The sum of the position elements of the error covariance is shown in figure 7.7 and 7.8. The significant improvement in the estimation error

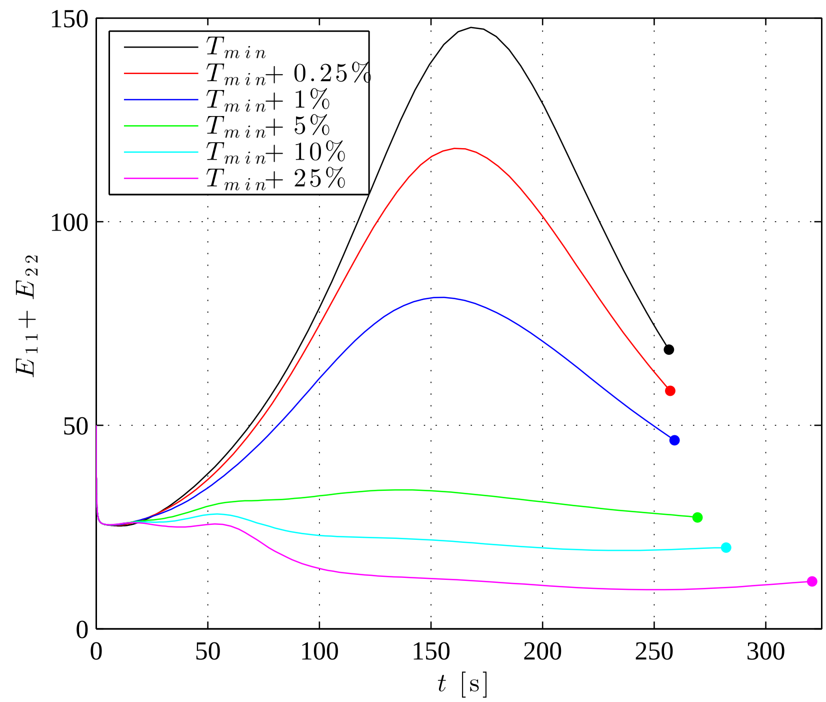

Fig. 7.7: Position estimator error covariance for $\mathbf{R} = (0.015\,\text{rad})^2$.

for fuel-limited mission success guidance is clearly visible. The element of the estimation error covariance corresponding to p is shown in figures 7.9 and 7.10. Examination of these plots shows that much of the reduction in state estimation error is tied to improved estimates of p. This behavior mirrors what was observed in the example from section 6.2.1. Mission success guidance uses some of the additional fuel to learn about the dynamics of the system. The improved dynamic model then allows for much better informed feedback control for the remainder of the maneuver.

The drastic change between figures 7.9 and 7.10 is purely a function of measurement quality, and is indicative of the value of an accurate sensor. The small estimation error shown in figure 7.9 shows how an accurate sensor allows an improved estimate of p is hugely benificial. The improvements shown in figure 7.9 show that the sensor works with the control law to produce a better state estimate. Similarly, the poor quality of the sensor used to produce the results shown in figure 7.10 shows that the sensor quality places limits on the reduction of dispersions provided by the control law.

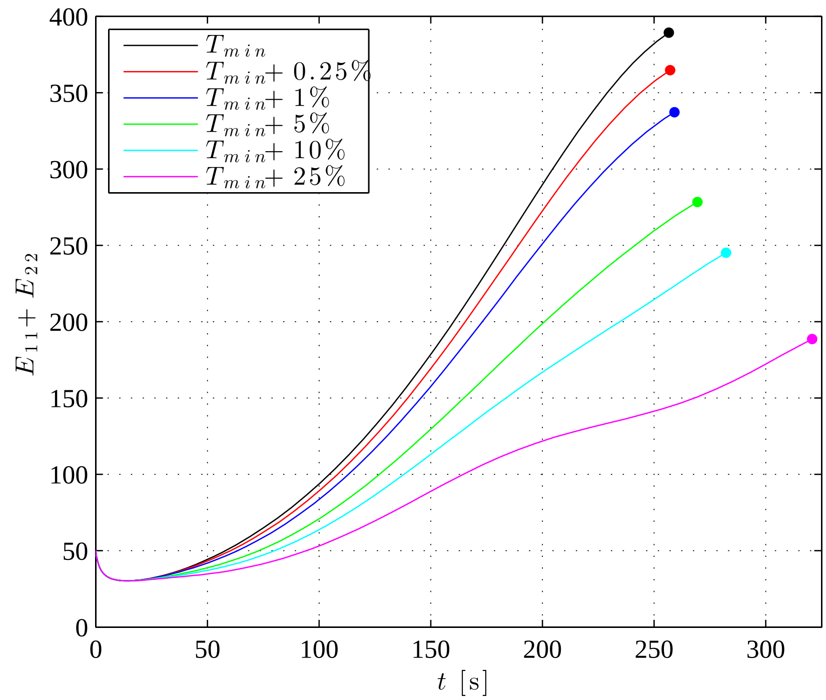

Fig. 7.8: Position estimator error covariance for $\mathbf{R} = (0.075\,\mathrm{rad})^2$.

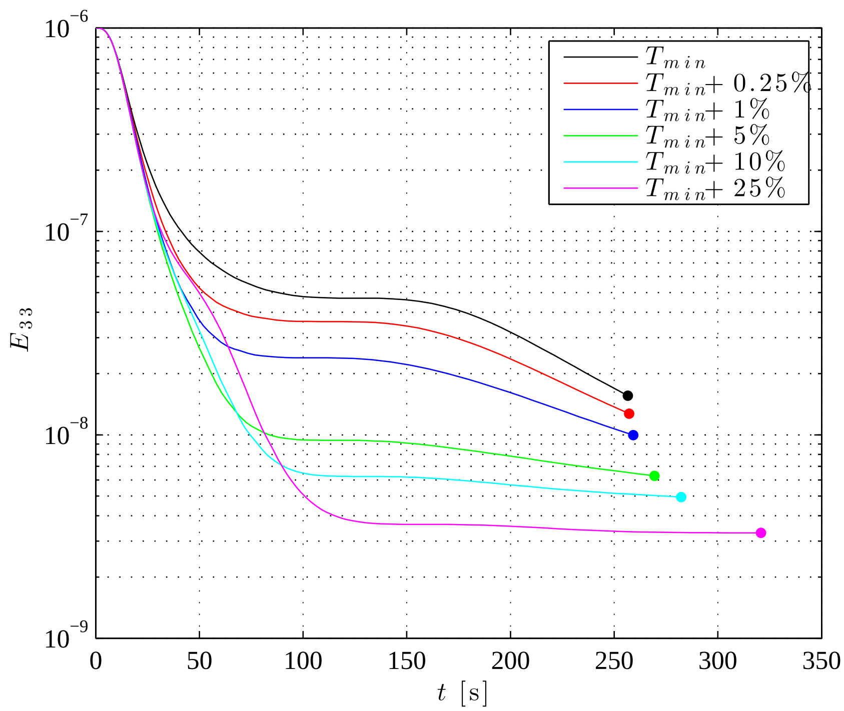

Fig. 7.9: Parameter estimator error covariance for $\mathbf{R} = (0.015\,\mathrm{rad})^2$.

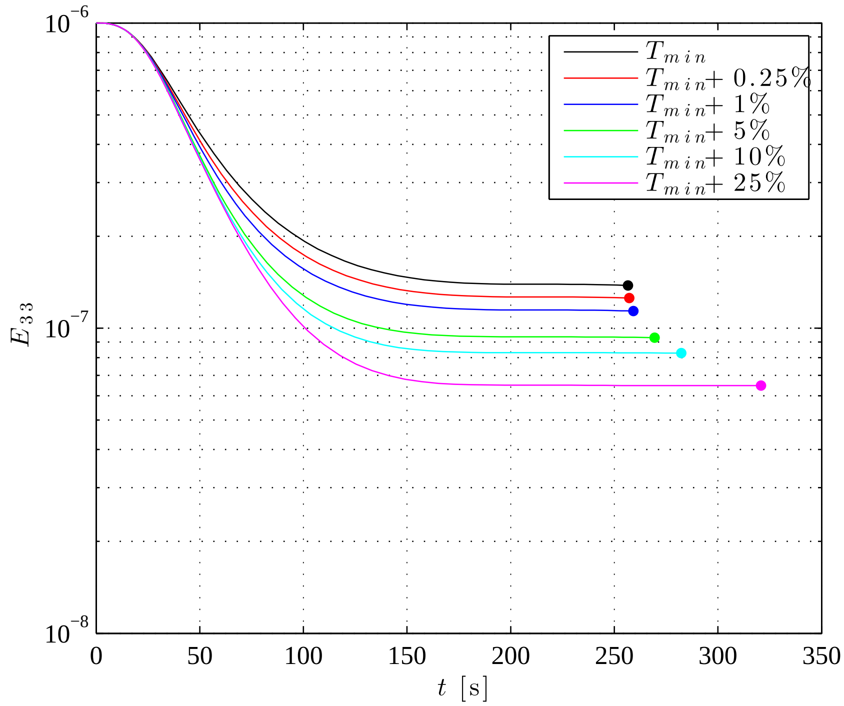

Fig. 7.10: Parameter estimator error covariance for $\mathbf{R} = (0.075\,\text{rad})^2$.

Figure 7.11 shows the feedback gains. Note that the feedback gains are a function of the nominal control and nominal trajectory. The feedback gains are not an explicit function of $\mathbf{R}$. Since the nominal trajectory is effectively the same for both values of $\mathbf{R}$ examined here, the feedback gains are also the same for both values of $\mathbf{R}$. Note that the control gains for p is much more sensitive that the control gains for the position states. The dominance of p in computing the feedback control aids in understanding the importance of having a good estimate of p. In fact, the feedback gains indicate that position feedback is only important in the last moments (boundary layer) before the terminal time.

The estimator gains are shown in figures 7.12 and 7.13. These gains show that the estimator focuses on learning about p (directly) during the initial portion of the maneuver. The time when the estimation gains corresponding to p are largest is the same time that the feedback gains corresponding to p are largest. Later during the maneuver the the estimator gains for p become much smaller, and the estimator gains little new information regarding p.

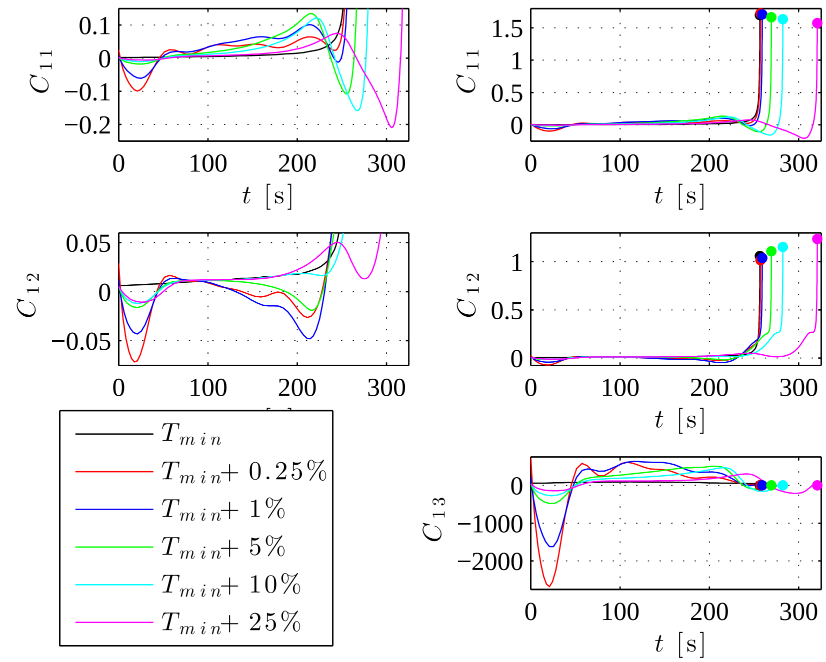

Fig. 7.11: Feedback gains for the Zermelo Boat Problem.

For the Zermelo boat problem, the estimation gains are a function of the nominal trajectory, and not an explicit function of control. Since the nominal trajectories are not drastically different from each other, the estimation gains are also relatively similar. Meanwhile, the feedback gains are an explicit function of the nominal trajectory *and* the control. This helps to explain the large control gain discrepancy between the stochastic and classical solutions. The large initial control gains for p couple with the initial estimation gains for p to point the estimator's initial focus toward obtaining an accurate estimate of p. These large initial gains on control insure that any estimation error in p is present in the control, which in turn will manifest itself in the measurement. While this same feedback loop is present in the classical solution, the stochastic solution greatly accentuates the behavior resulting in improved estimates of p.

This is somewhat analogous to the driver of a car rapidly swerving off course for a moment, or briefly exercising the brakes to learn about road conditions. Presuming this deviation in control is brief, it does not have a overly large impact on the nominal trajectory of the vehicle. However, the driver uses this opportunity to learn about the road conditions,

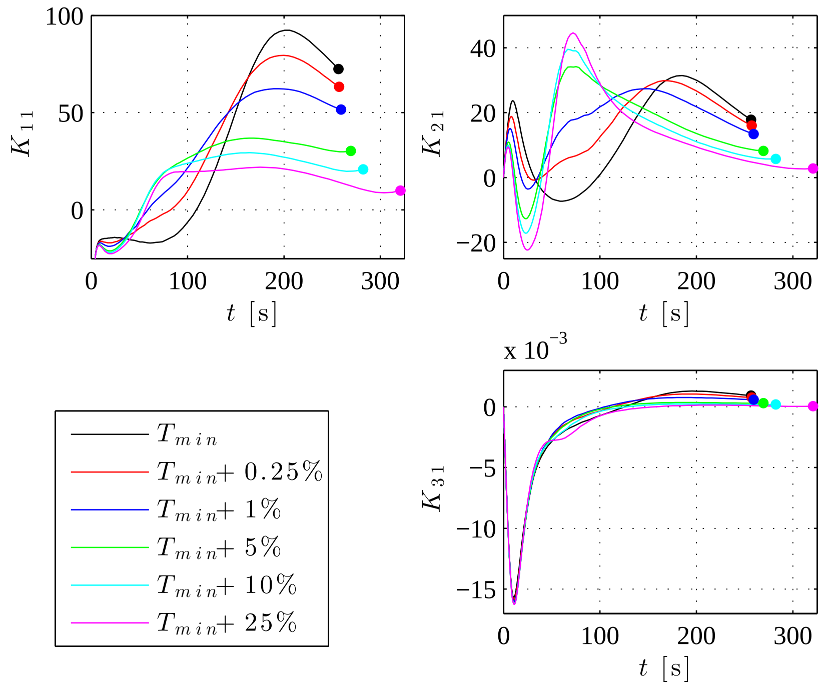

Fig. 7.12: Estimator gains for $\mathbf{R} = (0.015\,\text{rad})^2$.

and the response of the car. Later, the driver will use the information gained during the initial experiment to make better informed decisions regarding steering and braking. Similarly, in the Zermelo boat problem, the large initial excursions from fuel optimal steering allow the estimator to learn about the flow rate in the river, without a drastic change to the nominal mission. Later this improved information allows the feedback controller to produce better informed control.

Figures 7.14 and 7.15 show the true position state dispersion and the estimator state dispersions. Note that some stochastic cases allow the true dispersions to grow larger during the maneuver, but rapidly shrink these dispersions near the end of the maneuver. Comparing figures 7.14 and 7.15 against figures 7.7 and 7.8 shows that the even when the true dispersion is larger, the estimator has a better estimate of the dispersed state. Thus as the terminal time approaches control may be applied in a more precise way to remove the true state dispersions.

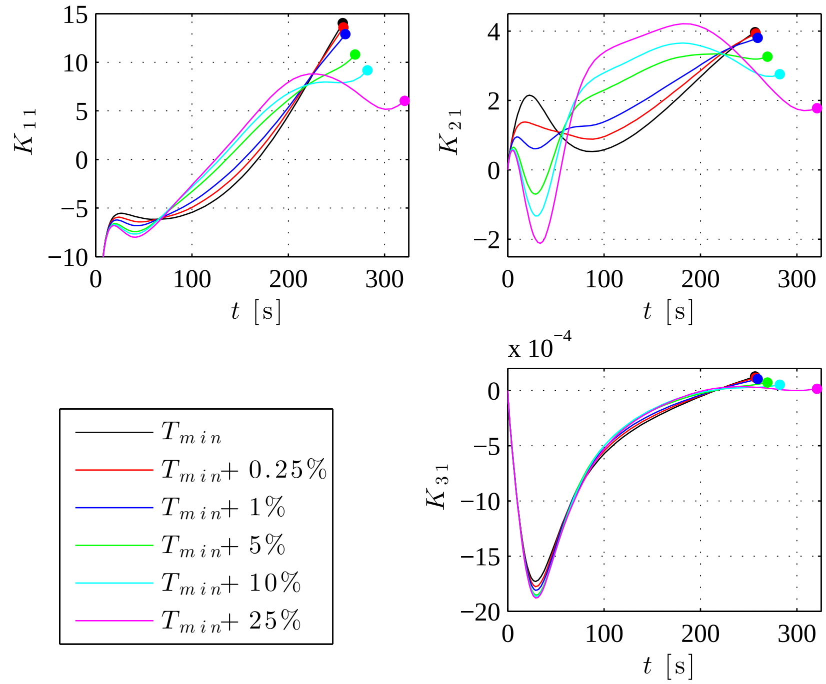

Fig. 7.13: Estimator gains for $\mathbf{R} = (0.075\,\mathrm{rad})^2$.

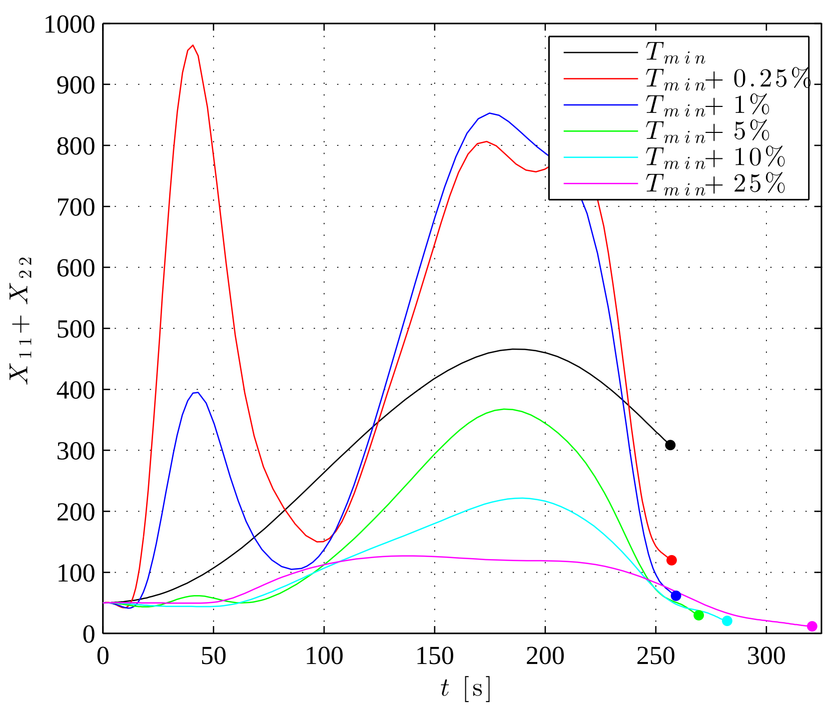

Fig. 7.14: True state dispersion covariance for $\mathbf{R} = (0.015\,\mathrm{rad})^2$.

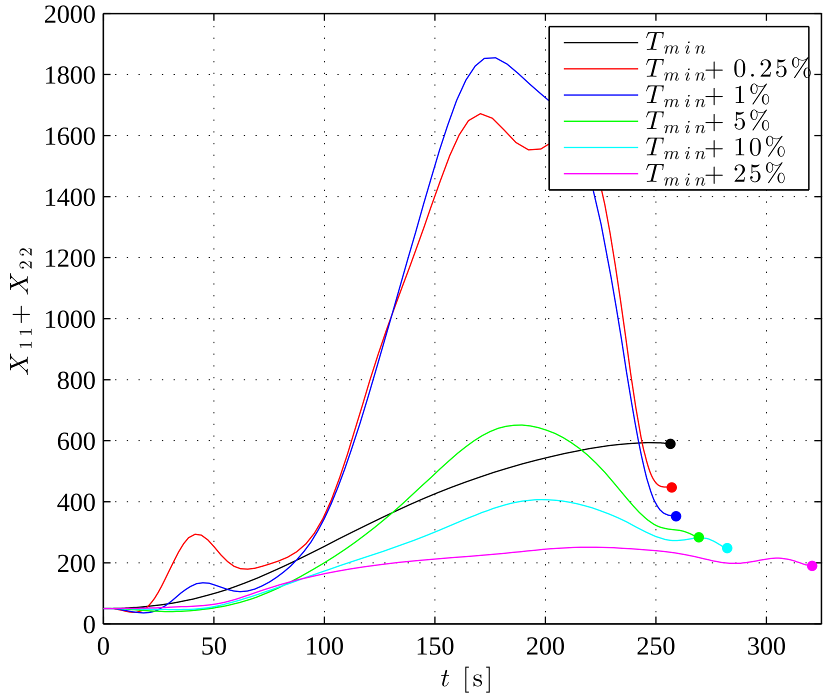

Fig. 7.15: True state dispersion covariance for $\mathbf{R} = (0.075\,\text{rad})^2$.

7.5 Summary

A guidance scheme that maximizes mission success subject to a fuel constraint has been proposed and demonstrated. The guidance law is formulated in terms of a control based on a steering command. Such a formulation allow for the total fuel use to be limited by imposing a limit on the terminal time. Stochastic quadratic synthesis can then by applied to produce a solution to the problem. The solution consists of a nominal control, and an associated nominal trajectory. In the linear region surrounding this nominal trajectory the Kalman gain provides the optimal estimator. A linear feedback law is provided by solving the Riccati equation associated with the quadratic cost terminal controller.

This scheme is demonstrated by applying it to a Zermelo boat problem with linearly varying current. This showed that the variance of the terminal state can be dramatically reduced by allowing for the consumption of only a small amount of extra fuel. A Pareto optimal curve was constructed to show the relationship between the fuel available for use and the achievable terminal dispersion, i.e., the probability of mission success.

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Chapter 8

Conclusion

The work in this dissertation is focused on developing and demonstrating a new guidance paradigm. This involved questioning the goal of guidance, examining its interaction with other subsystems, considering stochastic effects, and ultimately proposing a new framework for guidance system design. This new approach to guidance was demonstrated using the Zermelo boat problem as an illustrative example. This research has several novel aspects:

- Extension of stochastic quadratic synthesis techniques to account for control corrupted by an additive gaussian quite noise process.

- A mechanism for using a steering angle control to bound fuel use for stochastic problems was presented. This formulation permits fuel consumption to be tied directly to the final time. Furthermore, this approach also allows for the implementation of the guidance law via linear state feedback and gain scheduling.

- These techniques were used to create an approach to guidance which minimizes dispersions, subject to a fuel limit. By minimizing dispersions, this guidance approach maximizes the probability of mission success.

- Mission success guidance was used to solve a Zermeo boat type problem. It was demonstrated that a dramatic reduction in dispersions is possible for a minimal increase in fuel budgeted for the maneuver. Further improvement is possible when additional fuel is available. This information can be used to create a Pareto optimal front relating fuel consumption to mission success.

This document is intended to develop and demonstrate this concept. To this end, the example problems were chosen with an emphasis on simplicity. The use of simple problems is advantageous as it allows clearer insight into the characteristics of the approach used to solve the problem. Now that mission success guidance has been developed and demonstrated on the simple Zermelo problem, a logical next step would be to solve other more sophisticated problems using the same technique. Examples of potential problems include: accent guidance, angles only spacecraft rendezvous, and pinpoint guidance for powered planetary descent. The suite of missions available for this analysis might be increased by expanding the technique to include discrete impulsive maneuvers and discrete measurements.

Before these techniques can be desirable for more widespread use, it is important to find a better method for finding numerical solutions to the stochastic quadratic synthesis problem. Finding many of the solutions in this work proved to be a difficult and temperamental task. Even solving the simplest problem proved nearly impossible to automate, and required substantial manual input. It may be fruitful to solve many of these problems by using direct methods, or higher order indirect methods.

Additional study is needed to fully understand the interactions of the systems involved. This work suggests the current GN&C design approach of segregating the design does not properly account for the interaction of the guidance and navigation systems. The currently accepted approach to GN&C design is based on the certainty and equivalence principle, but violates some of the fundamental assumptions of the principle. Here it is shown how a unified design can provide improved statistical performance. Additional study may find similar results by questioning the other assumptions inherent in the use of the certainty and equivalence principle. For example, the effect of reduced order estimators might be studied to determine an appropriate set of filter states.

It is hoped that mission success guidance will be used in the future to solve even more sophisticated problems, and ultimately will prove useful to engineers who design closed loop GN&C systems for spacecraft.

References

[1] Goodman, J. L., "Introduction to Space Shuttle Rendezvous Guidance, Navigation, and Control," Tech. Rep. JSC-49686, NASA - Lyndon B. Johnson Space Center, November 2009.

[2] Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems," *Transactions of the ASME. Series D, Journal of Basic Engineering*, Vol. 82, 1960, pp. 35–45.

[3] Sorenson, H. W., "Least-squares estimation: from Gauss to Kalman," *Spectrum, IEEE*, Vol. 7, No. 7, July 1970, pp. 63–68.

[4] Kailath, T., "A View of Three Decades of Linear Filtering Theory," *Information Theory, IEEE Transactions on*, Vol. 20, No. 2, March 1974, pp. 146–181.

[5] Swerling, P., "Comment on 'A Statistical Optimizing Navigation Procedure for Space Flight'," *AIAA Journal*, Vol. 1, No. 8, August 1963, pp. 1968.

[6] Swerling, P., "A Proposed Stagewise Differential Correction Procedure for Satellite Tracking and Prediction," Tech. Rep. P-1292, The Rand Corporation, January 1958.

[7] Swerling, P., "First Order Error Propagation in a Stagewise Smoothing Procedure for Satellite Observations," *The Journal of the Astronautical Sciences*, 1959.

[8] Battin, R. H., "A Statistical Optimizing Navigation Procedure for Space Flight," *ARS Journal*, Vol. 32, No. 11, November 1962, pp. 1681–1696.

[9] Battin, R. H., *Astronautical Guidance*, McGraw-Hill, New York, 1964.

[10] McLean, J. D., Schmidt, S. F., and McGee, L. A., "Optimal Filtering and Linear Prediction applied to a Midcourse Navigation System for the Circumlunar Mission," Technical Note NASA-TN-D-1208, NASA, March 1962.

[11] McGee, L. A. and Schmidt, S. F., "Discovery fo the Kalman Filter as a Practical Tool for Aerospace and Industry," Technical Memorandum NASA-TM-86847, NASA - Ames Research Center, November 1985.

[12] Battin, R. H., "Some Funny Things Happened on the Way to the Moon," *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 1, January-February 2002, pp. 1–7.

[13] Battin, R. H., *An Introduction to the Mathematics and Methods of Astodynamics*, AIAA Education Series, American Institude of Aeronautics and Astonautics, Reston, revised ed., 1999.

[14] Maybeck, P. S., *Stochastic Models, Estimation, and Control*, Vol. 1 of *Mathematics in Science and Engineering*, Academic Press, New York, 1979.

[15] Crassidis, J. L. and Junkins, J. L., *Optimal Estimation of Dynamic Systems*, Applied Mathematics and Nonlinear Sciences Series, Chapman and Hall/CRC, Boca Raton, 2nd ed., 2004.

[16] Gelb, A., Kasper, Jr., J. F., Raymond A. Nash, J., Price, C. F., and Arthur A. Southerland, J., *Applied Optimal Estimation*, The MIT Press, Cambridge, 1974.

[17] Lear, W. M., "Kalman Filtering Techniques," Tech. Rep. JSC-13960, NASA - Lyndon B. Johnson Space Center, Houston, Texas, April 1978.

[18] Jazwinski, A. H., *Stochastic Processes and Filtering Theory*, Dover Publications, Mineola, 2007.

[19] Tapley, B. D., Schutz, B. E., and Born, G. H., *Statistical Orbit Determination*, Elsevier Academic Press, Boston, 2004.

[20] Moyer, T. D., *Formulation for Observed and Computed Values of Deep Space Network Data Types for Navigation*, Vol. 3, Wiley-Interscience, Hoboken, 2005.

[21] Moon, T. K. and Stirling, W. C., *Mathematical Methods and Algorithms for Signal Processing*, Prentice Hall, Upper Saddle River, 2000.

[22] Gauss, C. F., *Theory of the Motion of the Heaveanly Bodies about the Sun in Conic Sections, a Translation of Theoria Motus*, Dover Publications, New York, 1963.

[23] Bucy, R. S. and Joseph, P. D., *Filtering for Stochastic Processes with Applications to Guidance*, Interscience Publishers, New York, 1968.

[24] Øksendal, B., *Stochastic Differential Equations*, Springer, Berlin, 2010.

[25] Rugh, W. J., *Linear System Theory*, Prentice Hall, Upper Saddle River, 1996.

[26] Ehricke, K. A., *Space Flight*, Vol. 1: Environment and Celestial Mechanics, D. Van Nostrand Company, Princeton, 1960.

[27] Ovid, *Metamorphoses*, Oxford University Press, New York, 2009.

[28] Lucian of Samosata, *Trips to the Moon*, Echo Library, London, 2007.

[29] Newton, I., *Philosophiae Naturalis Principia Mathematica*, University of California Press, Ewing, 1999.

[30] Voltaire, *Micromegas*, Firestone Books, Los Angeles, 2012.

[31] Wells, H. G., *The War of the Worlds*, Tribeca Books, New York, 2013.

[32] Verne, J. G., *From Earth to the Moon, and A Trip Around It*, Forgotten Books, London, 2008.

[33] Tsiolkovsky, K. E., *Sobraniye Sochineniy, Tom II: Reaktivnyye Letatel'nyye Apparaty*, Izdatel'stvo Akademii Nauk, Moscow, 1954, Konstantin Tsiolkovsky, Collected Works of K. E. Tsiolkovskiy, Volume II: Reactive Flying Machines, NASA Technical Translation F. 237.

[34] Goddard, R. H., "A Method of Reaching Extreme Altitudes," *Smithsonian Miscellaneous Collections*, Vol. 71, No. 2, 1919.

[35] Goddard, R. H., "A Method of Reaching Extreme Altitudes," *Nature*, Vol. 105, No. 2652, August 1920, pp. 809–811.

[36] Tsien, H. S. and Evans, R. C., "Optimum Thrust Programming for a Sounding Rocket," *Journal of the American Rocket Society*, Vol. 21, No. 5, 1951, pp. 99–107.

[37] Leitmann, G., "Optimum Thrust Programming for High Altitude Rockets," *Aeronautical Engineering Review*, Vol. 16, No. 6, June 1957, pp. 63–66.

[38] Bryson, Jr., A. E. and Ross, S. E., "Optimum Rocket Trajectories with Aerodynamic Drag," *Journal of Jet Propulsion*, Vol. 28, No. 7, July 1958, pp. 465–469.

[39] Bejczy, A. K., "The Reverse Goddard Problem," *AIAA Guidance, Control, and Flight Mechanics Conference*, No. AIAA-69-868, August 1969.

[40] Tsiotras, P. and Kelley, H. J., "Goddard Problem with Constrained Time of Flight," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 2, March-April 1992, pp. 289–296.

[41] Seywald, H. and Cliff, E. M., "Goddard Problem in Presence of a Dynamic Pressure Limit," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 4, August 1993, pp. 776–781.

[42] Oberth, H., *Wege zur Raumschiffahrt*, R. Oldenbourg, München, 1929, Ways to Spaceflight, NASA Technical Translation F-622, 1970.

[43] Hohmann, W., *Die Erreichbarkeit der Himmelskörper*, R. Oldenbourg, München, 1925, The attainability of heavenly bodies, NASA Technical Translation F. 44, 1960.

[44] Dornberger, W., *V-2*, The Viking Press, New York, 1954.

[45] von Braun, W. and Ordway, III, F. I., *The Rockets' red glare*, Anchor Press, Garden City, 1976.

[46] Baker, D., *The Rocket: the History and Development of Rocket and Missile Technology*, Crown Publishers, New York, 1978.

[47] Humphries, C., *Superdove: How the Pigeon Took Manhattan ... And the World*, Smithsonian Books, New York, 2008.

[48] Lawden, D. F., "Dynamic Problems of Interplanetary Flight," *The Aeronautical Quarterly*, Vol. 6, No. 3, August 1955, pp. 165–180.

[49] Lawden, D. F., "Rocket Trajectory Optimization: 1950-1963," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 4, July-August 1991, pp. 705–711.

[50] Lawden, D. F., *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963.

[51] Lion, P. M., "A Primer on the Primer," NASA Tech. Rep. NASA-PM-67-21, Princeton University, 1968, This is taken from "First Compilation of Papers on Trajectory Analysis and Guidance Theory" NASA-PM-67-21 1968 pp.87-103.

[52] Lion, P. M. and Handelsman, M., "Primer Vector on Fixed-Time Impulsive Trajectories," *AIAA Journal*, Vol. 6, No. 1, January 1968, pp. 127–132.

[53] Mackenzie, D., *Inventing Accuracy: A Historical Sociology of Nuclear Missile Guidance*, The MIT Press, Cambridge, 1990.

[54] Siouris, G. M., *Missile Guidance and Control Systems*, Springer, New York, 2004.

[55] Pittman, Jr., G. R., editor, *Inertial Guidance*, John Wiley and Sons Inc., New York, 1962.

[56] Battin, R. H., "Space Guidance Evolution - A Personal Narative," *Journal of Guidance, Control, and Dynamics*, Vol. 5, No. 2, March-April 1982, pp. 97–110.

[57] Corey, D. A., "The Development and Implementation of the Cross Product Guidance Equations - 310," Contractor Rep. NASA-CR-153717, Bellcomm Inc., March 1967.

[58] Martin, F. H., *Closed-Loop Near-Optimum Steering for a Class of Space Mission*, Ph.D. thesis, Massachusetts Institute of Technology, May 1965.

[59] Martin, F. H., "Closed-Loop Near-Optimum Steering for a Class of Space Missions," *AIAA Journal*, Vol. 4, No. 11, November 1966, pp. 1920–1927.

[60] Laning, J. H. and Battin, R. H., "Optimum Fuel Trajectories for a Q-Type Guidance System," Engineering Memorandum E-520, MIT Instrumentation Laboratory, February 1956.

[61] Sokkappa, B. G., "On Optimal Steering to Achieve "Required Velocity"," Tech. Rep. R-491, MIT Instrumentation Laboratory, April 1965.

[62] Moore, W. E., "Lambert's Problem and Cross-Product Steering," Tech. Rep. No. 67-8, Analytical Mechanics Associates, 1967.

[63] Battin, R. H., "A Comparison of Fixed and Variable Time of Arrival Navigation for Interplanetary Flight," Tech. Rep. R-283, MIT Instrumentation Laboratory, May 1960.

[64] Cherry, G. W., "E Guidance - A General Explicit Optimizing Guidance Law for Rocket-Propelled Spaceflight," Tech. Rep. R-456, MIT Instrumentation Laboratory, August 1964.

[65] Cherry, G. W., "A General, Explicit, Optimizing Guidance Law For Rocket-Propelled Spaceflight," *AIAA/ION Astrodynamics Guidance and Control Conference*, No. AIAA-1964-638, American Institute of Aeronautics and Astronautics/Institute of Navigation, August 1964.

[66] Cherry, G. W., "A Class of Unified Explicit Methods for Steering Throttleable and Fixed-Thrust Rockets," *Guidance and Control - II: A Selection of Technical Papers basded manily on American Institute of Aeronautics and Astonautics held at Cambridge, Massachusetts, August 12-14, 1963*, edited by R. C. Langford and C. J. Mundo, Vol. 13 of *Progress in Astonautics and Aeronautics*, American Institute of Aeronautics and Astronautics, Academic Press, Reston, 1964, pp. 689–726.

[67] Cherry, G. W., "A Unified Explicit Technique for Performing Orbital Insertion, Soft Landing, and Rendezvous with a Throttleable Rocket-Propelled Space Vehicle," *AIAA Guidance and Control Conference*, No. AIAA-1963-335, American Institute of Aeronautics and Astronautics, August 1963.

[68] Klumpp, A. R., "A Manually Retargeted Automatic Landing System for LM," Tech. Rep. R-539 Rev. 1, MIT Instrumentation Laboratory, August 1967.

[69] Klumpp, A. R., "A Manually Retargeted Automatic Landing System for Lunar Module (LM)," *Journal of Spacecraft and Rockets*, Vol. 5, No. 2, February 1968, pp. 129–138.

[70] Klumpp, A. R., "Apollo Lunar-Decent Guidance," Tech. Rep. R-695, Draper Laboratory, 1971.

[71] Klumpp, A. R., "Apollo Lunar Descent Guidance," *Automatica*, Vol. 10, No. 2, March 1974, pp. 133–146.

[72] Sostaric, R. R. and Rea, J., "Powered Descent Guidance Methods For The Moon and Mars," *AIAA Guidance, Navigation, and Control Conference and Exhibit*, August 2005.

[73] Sostaric, R. R., "Powered Descent Trajectory Guidance and Some Considerations for Lunar Landing," *AAS Guidance and Control Conference*, No. AAS-07-051, American Astonautical Society, February 2007.

[74] D'Souza, C. N., "An Optimal Guidance Law for Planetary Landing," *AIAA Guidance, Navigation, and Control Conference*, No. AIAA-97-3709, 1997.

[75] Lawden, D. F., "Optimal Programming of Rocket Thrust Direction," *Astronautica Acta*, Vol. 1, No. 1, January 1954, pp. 41–56.

[76] McHenry, R. L., Brand, T. J., Long, A. D., Cockrell, B. F., and Thidodeau, III, J. R., "Space Shuttle Ascent Guidance, Navigation, and Control," *The Journal of the Astronautical Sciences*, Vol. 27, No. 1, January-March 1979, pp. 1–38.

[77] Brand, T. J., Brown, D. W., and Higgins, J. P., "Space Shuttle G&N Equation Document," Tech. Rep. C-4108, The Charles Stark Draper Laboratory, April 1974.

[78] Fill, T., "Introduction to Bi-Linear Tangent Steering for Shuttle Ascent Aborts," EGB Memo EGB-89-108, Charles Stark Draper Laboratory, May 1989.

[79] McHenry, T. L., "Introduction to Powered Flight Guidance Development Concepts," Tech. Rep., Charles Stark Draper Laboratory, March 1991.

[80] McHenry, R. L., "Near Fuel Optimal Powered Flight Guidance Laws," Tech. rep., Charles Stark Draper Laboratory, 1991.

[81] Pontryagin, L. S., Boltyanskij, V. G., Gamkrelidze, R. V., and Mishchenko, E. F., *Mathematical Theory of Optimal Processes*, Interscience, New York, 1962.

[82] Bryson, Jr., A. E. and Ho, Y.-C., *Applied Optimal Control: Optimization, Estimation, and Control*, Taylor and Francis, New York, revised ed., 1975.

[83] Leitmann, G., editor, *Optimization Techniques With Applications to Aerospace Systems*, Vol. 5 of *Mathematics in Science and Engineering*, Academic Press, New York, 1962.

[84] Lawden, D. F., "Optimal Rocket Trajectories," *Journal of Jet Propulsion*, Vol. 27, No. 12, December 1957, pp. 1263.

[85] Letov, A. M., "Analytical Controller Design II," *Automation and Remote Control*, Vol. 21, No. 5, May 1960, pp. 561–568.

[86] Rekasius, Z. V. and Hsia, T. C., "On an Inverse Problem in Optimal Control," *IEEE Transactions on Automatic Control*, Vol. 9, No. 4, October 1964, pp. 370–375.

[87] Kalman, R. E., "When is a Linear Control System Optimal?" *Journal of Basic Engineering*, Vol. 86, No. 1, 1964, pp. 51.

[88] Anderson, B. D. O., "The Inverse Problem of Optimal Control," Tech. Rep. 6560-3, Stanford Electronics Laboratories, April 1966.

[89] Kreindler, E. and Jameson, A., "Optimality of Linear Control Systems," *IEEE Transactions on Automatic Control*, Vol. 17, No. 3, June 1972, pp. 349–351.

[90] Roche, R., "Selection of the Optimum Abort Trajectory," Apollo Guidance and Navigation Note 71, Bissett-Berman Corp., May 1963.

[91] Gunckel, II, T. L. and Franklin, G. F., "A General Solution for Linear Sampled-Data Control," *Transactions of the ASME. Series D, Journal of Basic Engineering*, Vol. 85, No. 2, June 1963, pp. 197–203.

[92] Joseph, P. D. and Tou, J. T., "On Linear Control Theory," *Transactions of the American Institute of Electrical Engineers, Part II: Applications and Industry*, Vol. 80, No. 4, September 1961, pp. 193–196.

[93] Potter, J. E., "A Guidance-Navigation Separation Theorem," *AIAA/ION Astrodynamics Guidance and Control Conference*, No. AIAA-66-653, American Institute of Aeronautics and Astronautics/Institute of Navigation, August 1964.

[94] Skelton, R. E., *Dynamic Systems Control: Linear Systems Analysis and Synthesis*, John Wiley and Sons Inc., New York, 1988.

[95] Stengel, R. F., *Optimal Control and Estimation*, Dover Publications, New York, 1994, originally published as STOCHASTIC OPTIMAL CONTROL; Theory and Application, J. Wiley and Sons, New York, 1986.

[96] Kahne, S. J., "Low-Sensitivity Design of Optimal Linear Control Systems," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-4, No. 3, May 1968, pp. 374–379.

[97] Lamont, G. B. and Kahne, S. J., "Comparisons of Sensitivity Improvement Techniques for Linear Optimal Control Systems," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-5, No. 2, March 1969, pp. 142–151.

[98] Neuman, C. P. and Sood, A. K., "Sensitivity Functions for Linear Discrete Systems with Applications," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-8, No. 6, November 1972, pp. 764–770.

[99] Ciric, V. V., *Design of Minimum Sensitivity Control Systems*, Ph.D. thesis, Rice University, April 1969.

[100] Seywald, H. and Kumar, R. R., "Desensitized Optimal Trajectories," *AAS/AIAA Space Flight Mechanics Meeting*, No. AAS-96-107, 1996.

[101] Seywald, H., "Desensitized Optimal Trajectories with Control Constraints," *AAS/AIAA Space Flight Mechanics Meeting*, No. AAS-03-147, 2003.

[102] Zimmer, S. J., *Reducing Spacecraft Uncertainty Through Indirect Trajectory*, Ph.D. thesis, The University of Texas at Austin, December 2005.

[103] Zimmer, S. J., Ocampo, C. A., and Bishop, R. H., "Reducing Orbit Covariance for Continuous Thrust Spacecraft Transfers," *Aerospace and Electronic Systems, IEEE Transactions on*, Vol. 46, No. 2, April 2010, pp. 771–791.

[104] Denham, W. F., "Choosing the Nominal Path for a Dynamic System with Random Forcing Functions to Optimize the Statistical Performance," Tech. Rep. TR-449, Cruft Lab, Division of Engineering and Applied Physics, Harvard University, July 1964.

[105] Denham, W. F., "Choosing a Nominal System Path to Maximize the Probability of Reaching a Given Region of State Space," Tech. Rep. NASA-CR-83074, NASA, January 1965.

[106] Denham, W. F. and Mohan, J. J., "The Probability of Mission Success (POMS) Program," Contractor Rep. NASA-CR-84815, NASA, 1965.

[107] Denham, W. F. and Mohan, J. J., "Choosing Trajectory Parameters to Maximize Mission Probability of Success," *AIAA Journal*, Vol. 4, No. 7, July 1966, pp. 1293–1298.

[108] Schmidt, G. T., *A New Technique for Identification and Control of Systems With Unknown Parameters*, Ph.D. thesis, Massachusetts Institute of Technology, October 1970.

[109] Schmidt, G. T., "Close-Loop Control of Stochastic Nonlinear Systems," Tech. Rep. R-676, Draper Laboratory, October 1970.

[110] Schmidt, G. T., "A New Technique for Closed Loop Control of Discrete Stochastic Nonlinear Systems," *IEEE Conference on Decision and Control*, December 1971, pp. 643–649.

[111] Bryson, Jr., A. E., *Dynamic Optimization*, Addison Wesley Longman, Menlo Park, 1999.

[112] Hull, D. G., *Optimal Control Theory for Applications*, Springer, New York, 2003.

[113] Bolza, O., "Über den Anormalen Fall beim Lagrangeschen und Mayerschen Problem mit gemischten Bedingungen und variablen Endpunkten," *Mathematische Annalen*, Vol. 74, No. 3, 1913, pp. 430–446.

[114] Bliss, G. A., "The Problem of Bolza in the Calculus of Variations," *The Annals of Mathematics*, Vol. 33, No. 2, 1932, pp. 261–274.

[115] Bliss, G. A., *Lectures on the Calculus of Variations*, Chicago University Press, Chicago, 1946.

[116] Bolza, O., *Lectures on the Calculus of Variations*, Chelsea Publishing Company, New York, 3rd ed., 1973.

[117] Hull, D. G., "Differentials, Variations, and Taylor Series," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 1, January-February 2005, pp. 169–172.

[118] Hull, D. G., "Equations for Approximate Solutions Using Variational Calculus," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 5, September-October 2000, pp. 898–900.

[119] Hull, D. G., "Variational Calculus and Approximate Solutions of Algebraic Equations," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 3, May-June 2001, pp. 630–632.

[120] Logan, J. D., *Applied Mathematics*, Wiley-Interscience, Hoboken, 3rd ed., 2006.

[121] Breakwell, J. V., Speyer, J. L., and Bryson, Jr., A. E., "Optimization and Control of Nonlinear Systems Using the Second Variation," *Journal of the Society for Industrial and Applied Mathematics Series A Control*, Vol. 1, No. 2, 1963, pp. 193–223.

[122] Kelley, H. J., Kopp, R. E., and Moyer, H. G., "Successive Approximation Techniques for Trajectory Optimization," *IAS Vehicle Systems Optimization Symposium*, November 1961.

[123] Kelley, H. J., Kopp, R. E., and Moyer, H. G., "A Trajectory Optimization Technique Based Upon The Theory of the Second Variation," *AIAA Astrodynamics Conference*, August 1963.

[124] Lee, A. Y. and Bryson, Jr., A. E., "Neighboring Extremals of Dynamic Optimization Problems with Parameter Variations," *Optimal Control Applications and Methods*, Vol. 10, No. 1, January/March 1989, pp. 39–52.

[125] Speyer, J. L. and Bryson, Jr., A. E., "A Neighboring Optimum Feedback Control Scheme Based on Estimated Time-to-Go with Application to Re-Entry Flight Paths," *AIAA Journal*, Vol. 6, No. 5, May 1968, pp. 769–776.

[126] Wood, L. J., "Second-Order Optimality Conditions for the Bolza Problem with Both Endpoints Variable," *Journal of Aircraft*, Vol. 11, No. 4, April 1974, pp. 212–221.

[127] Ben-Asher, J. Z., *Optimal Control Theory with Aerospace Applications*, American Institute of Aeronautics and Astonautics, Reston, 2010.

[128] Gottlieb, R. G., "Rapid Convergence to Optimum Solutions Using a Min-H Strategy," *AIAA Journal*, Vol. 5, No. 2, February 1967, pp. 322–329.

[129] Gottlieb, R. G. and Fowler, W. T., "Improved Secant Method Applied to Boost Trajectory Optimization," *Journal of Spacecraft and Rockets*, Vol. 14, No. 2, February 1977, pp. 96–101.

[130] Miele, A. and Prichard, R. E., "Gradient Methods in Control Theory Part 2 - Sequential Gradient-Restoration Algorithm," Aero-Astronautics Rep. AAR-62, Rice University, 1969.

[131] Wood, L. J. and Bryson, Jr., A. E., "Second-Order Optimality Conditions for Variable End Time Terminal Control Problems," *AIAA Journal*, Vol. 11, No. 9, September 1973, pp. 1241–1246.

[132] Jardin, M. R. and Bryson, Jr., A. E., "Neighboring Optimal Aircraft Guidance in Winds," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 4, July-August 2001, pp. 710–715.

[133] Jardin, M. R., "Analytical Solutions for Minimum-Time Neighboring Optimal Aircraft Guidance in Winds," *AIAA Guidance, Navigation, and Control Conference and Exhibit*, No. AIAA-2008-6977, American Institute of Aeronautics and Astronautics, 2008.

[134] Lear, W. M., "Kalman Filtering Techniques," Tech. Rep. JSC-20688, NASA - Lyndon B. Johnson Space Center, Houston, Texas, September 1985.

[135] Athans, M. and Falb, P. L., *Optimal Control: An Introduction to the Theory and Its Applications*, McGraw-Hill, New York, 1966.

[136] Zermelo, E., "Über die Navigation in der Luft als Problem der Variationsrechnung," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. 39, 1930, pp. 44–48.

[137] Zermelo, E., "Über das Navigationsproblem bei ruhender oder veränderlicher Windverteilung," *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 11, No. 2, 1931, pp. 114–124.

[138] Jardin, M. R. and Bryson, A. E., "Methods for Computing Minimum-Time Paths in Strong Winds," *Journal of Guidance, Control, and Dynamics*, Vol. 35, No. 1, January-February 2012, pp. 165–171.

[139] Kelley, H. J., "Gradient Theory of Optimal Flight Paths," *ARS Journal*, Vol. 30, No. 10, October 1960, pp. 947–954.

[140] Kelley, H. J., "Guidance Theory and Extremal Fields," *Automatic Control, IRE Transactions on*, Vol. 7, No. 5, October 1962, pp. 75–82.

[141] Feynman, R. P., *The Feynman Lectures on Physics*, Vol. 1, Addison Wesley, San Francisco, 2nd ed., August 2005.

[142] Kalman, R. E. and Bucy, R. S., "New Results in Linear Filtering and Prediction Theory," *Journal of basic Engineering*, Vol. 83, No. 3, March 1961, pp. 95–108.

[143] Falb, P. L., "Infinite-Dimensional Filtering: The Kalman-Bucy filter in Hilbert space," *Information and Control*, Vol. 11, No. 1–2, July-August 1967, pp. 102–137.

Appendix

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Taylor Approximations, Displacements, Differentials, and Variations

In order to clearly approach the approximations of functions and functionals, it is useful to clearly distinguish the difference between displacements, differentials, and variations [112, 117–119]. Clear distinction of these ideas has great practical import in developing the necessary conditions for the calculus of variations. Briefly stated, a displacement (denoted by Δ) is a total (including terms of all orders) change in a variable. The differential (denoted by d) is the familiar differential from calculus, meaning an infinitesimal change in a variable. Meanwhile, the variation (denoted by δ) is the variation from the calculus of variations and denotes only the change in indicated variable independent of other effects. It will be seen that for certain problems these quantities are equivalent, and in other cases they are quite distinct.

Functions

Consider a vector valued function $\mathbf{y} = \boldsymbol{f}(\mathbf{x})$. Use a Taylor series to express a displacement of the independent variable, $\Delta\mathbf{y}$, in terms of a displacement, $\Delta\mathbf{x}$, from a nominal point, $\bar{\mathbf{x}}$

$$\underbrace{\overbrace{\boldsymbol{f}_k(\bar{\mathbf{x}}+\Delta\mathbf{x})}^{\bar{\mathbf{y}}_k+\Delta\mathbf{y}_k} - \overbrace{\boldsymbol{f}_k(\bar{\mathbf{x}})}^{\bar{\mathbf{y}}_k}}_{\Delta\mathbf{y}_k} = \frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}}\Delta\mathbf{x} + \frac{1}{2!}\Delta\mathbf{x}^T\frac{\partial^2 \boldsymbol{f}_k}{\partial \mathbf{x}^T\partial \mathbf{x}}\Delta\mathbf{x} + \ldots \tag{1}$$

The subscripts denote that this equation applies to each element of $\mathbf{y}$. This is done to allow for the use of linear algebra notation rather than using summations over various indexes, Einstein notation. For this work, the small displacement, $\Delta\mathbf{x}$, is considered to be a constant

$$\Delta\mathbf{x} = d\mathbf{x} + \frac{1}{2!}\underbrace{d^2\mathbf{x}}_{\mathbf{0}} + \frac{1}{3!}\underbrace{d^3\mathbf{x}}_{\mathbf{0}}\ldots \tag{2}$$

That is, the displacement the independent variable is composed only of a first order part. The Taylor series for the displacement in the dependent variable now becomes

$$\Delta\mathbf{y}_j = \underbrace{\frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}}d\mathbf{x}}_{d\mathbf{y}_k} + \frac{1}{2!}\underbrace{d\mathbf{x}^T\frac{\partial^2 \boldsymbol{f}_k}{\partial \mathbf{x}^T\partial \mathbf{x}}d\mathbf{x}}_{d^2\mathbf{y}_k} + \ldots \tag{3}$$

Thus, the displacement of the independent variable is composed of higher order terms. These terms (excluding the inverse of the factorial) are the differentials of $\mathbf{y}$.

Algebraic Equations

Consider the algebraic equation

$$\boldsymbol{\psi}\left(\mathbf{x},\mathbf{y}\right)=\mathbf{0} \tag{4}$$

where $\mathbf{x}$ is the independent variable ($\Delta\mathbf{x}=d\mathbf{x}$), and $\mathbf{y}$ is the independent variable. The Taylor series is taken about $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$.

$$\begin{aligned}\mathbf{0}=&\overbrace{\boldsymbol{\psi}_k\left(\bar{\mathbf{x}}+\Delta\mathbf{x},\bar{\mathbf{y}}+\Delta\mathbf{y}\right)}^{0}-\overbrace{\boldsymbol{\psi}_k\left(\bar{\mathbf{x}},\bar{\mathbf{y}}\right)}^{0}=\frac{d\boldsymbol{\psi}_k}{d\mathbf{x}}\Delta\mathbf{x}+\frac{d\boldsymbol{\psi}_k}{d\mathbf{y}}\Delta\mathbf{y}\\&+\frac{1}{2!}\left(\Delta\mathbf{x}^T\frac{d^2\boldsymbol{\psi}_k}{d\mathbf{x}^Td\mathbf{x}}\Delta\mathbf{x}+\Delta\mathbf{y}^T\frac{d^2\boldsymbol{\psi}_k}{d\mathbf{y}^Td\mathbf{x}}\Delta\mathbf{x}+\Delta\mathbf{x}^T\frac{d^2\boldsymbol{\psi}_k}{d\mathbf{x}^Td\mathbf{y}}\Delta\mathbf{y}+\Delta\mathbf{y}^T\frac{d^2\boldsymbol{\psi}_k}{d\mathbf{y}^Td\mathbf{y}}\Delta\mathbf{y}\right)+\ldots\end{aligned} \tag{5}$$

Now substitute in $\Delta\mathbf{y}=d\mathbf{y}+\frac{1}{2!}d^2\mathbf{y}+\ldots$ and $\Delta\mathbf{x}=d\mathbf{x}$ Higher order terms are also discarded

$$\mathbf{0}=\underbrace{\frac{d\boldsymbol{\psi}_k}{d\mathbf{x}}d\mathbf{x}+\frac{d\boldsymbol{\psi}_k}{d\mathbf{y}}d\mathbf{y}}_{d\boldsymbol{\psi}_k=0}+\frac{1}{2!}\underbrace{\left(\begin{bmatrix} d\mathbf{x}^T & d\mathbf{y}^T\end{bmatrix}\begin{bmatrix}\frac{d^2\boldsymbol{\psi}_k}{d\mathbf{x}^Td\mathbf{x}} & \frac{d^2\boldsymbol{\psi}_k}{d\mathbf{x}^Td\mathbf{y}}\\ \frac{d^2\boldsymbol{\psi}_k}{d\mathbf{y}^Td\mathbf{x}} & \frac{d^2\boldsymbol{\psi}_k}{d\mathbf{y}^Td\mathbf{y}}\end{bmatrix}\begin{bmatrix} d\mathbf{x}\\ d\mathbf{y}\end{bmatrix}+\frac{d\boldsymbol{\psi}_k}{d\mathbf{y}}d^2\mathbf{y}\right)}_{d^2\boldsymbol{\psi}_k=0}+\ldots \tag{6}$$

Each element of the expansion is claimed to be zero by simply neglecting the higher order terms. This can be proved by induction. The first order term is found to be zero by neglecting all higher order terms. Once the first term is known to be zero, the second order term is found to be zero by repeating the process. These expressions can now be used to find $d\mathbf{y}$ and $d^2\mathbf{y}$.

Differential Equations

Consider a nominal solution, $\bar{\mathbf{x}}$, and an nearby solution, $\mathbf{x}$, to the differential equation $\dot{\mathbf{x}}=\boldsymbol{f}\left(\mathbf{x},t\right)$. Where these two solutions are related by

$$\mathbf{x}\left(t+\Delta t\right)=\bar{\mathbf{x}}\left(t\right)+\Delta\mathbf{x} \tag{7}$$

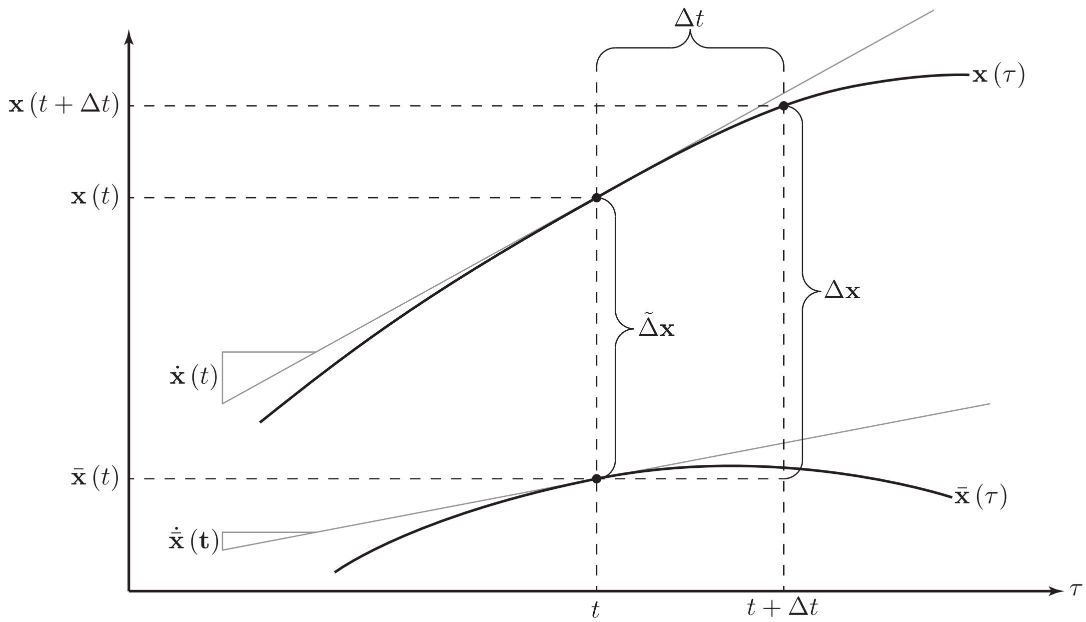

Fig. 1: The geometry of time correlated and non-time correlated displacements.

or

$$
\begin{aligned}
\Delta\mathbf{x} &= \mathbf{x}\left(t+\Delta t\right) - \bar{\mathbf{x}}\left(t\right) && (8)\\
&= \mathbf{x}\left(t+\Delta t\right) + \underbrace{\mathbf{x}\left(t\right) - \mathbf{x}\left(t\right)}_{\mathbf{0}} - \bar{\mathbf{x}}\left(t\right) && (9)\\
&= \underbrace{\mathbf{x}\left(t\right) - \bar{\mathbf{x}}\left(t\right)}_{\tilde{\Delta}\mathbf{x}} + \mathbf{x}\left(t+\Delta t\right) - \mathbf{x}\left(t\right) && (10)
\end{aligned}
$$

where the time correlated displacement is given by $\tilde{\Delta}\mathbf{x}$. The relations in this figure are shown in figure 1. A Taylor expansion of $\mathbf{x}\left(t+\Delta t\right)$ is taken about the time t

$$
\begin{aligned}
\Delta\mathbf{x} &= \tilde{\Delta}\mathbf{x} + \mathbf{x}\left(t\right) + \dot{\mathbf{x}}\Delta t + \frac{1}{2!}\ddot{\mathbf{x}}\left(\Delta t\right)^2 + \ldots - \mathbf{x}\left(t\right) && (11)\\
&= \tilde{\Delta}\mathbf{x} + \dot{\mathbf{x}}\Delta t + \frac{1}{2!}\ddot{\mathbf{x}}\left(\Delta t\right)^2 + \ldots && (12)
\end{aligned}
$$

A Taylor series of $\boldsymbol{f}$ can be used to develop and expression for the dynamics of $\dot{\mathbf{x}}$

$$\begin{aligned}
\dot{\mathbf{x}}_k(t) = \dot{\bar{\mathbf{x}}}_k(t) + \tilde{\Delta}\dot{\mathbf{x}}_k &= \boldsymbol{f}_k(\mathbf{x}(t), t) && (13)\\
&= \boldsymbol{f}_k\left(\bar{\mathbf{x}} + \tilde{\Delta}\mathbf{x}, t\right) && (14)\\
&= \underbrace{\boldsymbol{f}_k(\bar{\mathbf{x}}, t)}_{\dot{\bar{\mathbf{x}}}_k(t)} + \frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}}\tilde{\Delta}\mathbf{x} + \frac{1}{2!}\tilde{\Delta}\mathbf{x}^T \frac{\partial^2 \boldsymbol{f}_k}{\partial \mathbf{x}^T \partial \mathbf{x}}\tilde{\Delta}\mathbf{x} + \ldots && (15)
\end{aligned}$$

This can now be used for $\dot{\mathbf{x}}$ in the last equation

$$\Delta\mathbf{x} = \tilde{\Delta}\mathbf{x} + \left(\dot{\bar{\mathbf{x}}}(t) + \frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\tilde{\Delta}\mathbf{x} + \ldots\right)\Delta t + \frac{1}{2!}\left(\ddot{\bar{\mathbf{x}}}(t) + \ldots\right)(\Delta t)^2 + \ldots \tag{16}$$

Into this expression substitute the differential expansions for the dependent and independent variables.

$$\begin{aligned}
\underbrace{d\mathbf{x} + \frac{1}{2!}d^2\mathbf{x} + \ldots}_{\Delta\mathbf{x}} = \underbrace{\delta\mathbf{x} + \frac{1}{2!}\delta^2\mathbf{x} + \ldots}_{\tilde{\Delta}\mathbf{x}} + \left[\dot{\bar{\mathbf{x}}}(t) + \frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\underbrace{\left(\delta\mathbf{x} + \frac{1}{2!}\delta^2\mathbf{x} + \ldots\right)}_{\tilde{\Delta}\mathbf{x}} + \ldots\right]\underbrace{\left(dt + \frac{1}{2!}d^2t + \ldots\right)}_{\Delta t}\\
+ \frac{1}{2!}\ddot{\bar{\mathbf{x}}}\left(\underbrace{dt + \frac{1}{2!}d^2t + \ldots}_{\Delta t}\right)^2 + \ldots
\end{aligned} \tag{17}$$

The first order terms from this equation give the desired relation between a differential and a variation

$$d\mathbf{x} = \delta\mathbf{x} + \dot{\bar{\mathbf{x}}}dt \tag{18}$$

the second order terms gives

$$d^2\mathbf{x} = \delta^2\mathbf{x} + \dot{\bar{\mathbf{x}}}d^2t + \frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\delta\mathbf{x} + \ddot{\bar{\mathbf{x}}}dt^2 \tag{19}$$

Since $\mathbf{x}(t)$ could be any differentiable function of time, a general relationship between the variation and the differential is established.

$$d(\cdot) = \delta(\cdot) + \dot{\overline{(\cdot)}}\,dt \tag{20}$$

A Taylor expansion is used to develop and expression for $\delta\dot{\mathbf{x}}$

$$\tilde{\Delta}\dot{\mathbf{x}}_i = \frac{\partial \boldsymbol{f}_i}{\partial \mathbf{x}}\tilde{\Delta}\mathbf{x} + \frac{1}{2!}\tilde{\Delta}\mathbf{x}^T \frac{\partial^2 \boldsymbol{f}_i}{\partial \mathbf{x}^T \partial \mathbf{x}}\tilde{\Delta}\mathbf{x} + \dots \tag{21}$$

Substitute in the differential expansions for the dependent and independent variables.

$$\underbrace{\delta\dot{\mathbf{x}}_i + \frac{1}{2!}\delta^2\dot{\mathbf{x}}_i + \dots}_{\tilde{\Delta}\dot{\mathbf{x}}_i} = \frac{\partial \boldsymbol{f}_i}{\partial \mathbf{x}}\underbrace{\delta\mathbf{x} + \frac{1}{2!}\delta^2\mathbf{x} + \dots}_{\tilde{\Delta}\mathbf{x}} + \frac{1}{2!}\underbrace{\left(\delta\mathbf{x}^T + \frac{1}{2!}\delta^2\mathbf{x}^T + \dots\right)}_{\tilde{\Delta}\mathbf{x}^T}\frac{\partial^2 \boldsymbol{f}_i}{\partial \mathbf{x}^T \partial \mathbf{x}}\underbrace{\left(\delta\mathbf{x} + \frac{1}{2!}\delta^2\mathbf{x} + \dots\right)}_{\tilde{\Delta}\mathbf{x}} + \dots \tag{22}$$

Truncated to first order to get an expression for the variational dynamics.

$$\delta\dot{\mathbf{x}}_i = \frac{\partial \boldsymbol{f}_i}{\partial \mathbf{x}}\delta\mathbf{x} \tag{23}$$

Differential of an Integral

Consider the the integral

$$\boldsymbol{I} = \int_{t_i}^{t_f} \boldsymbol{f}(\mathbf{x}, \tau)\, d\tau \tag{24}$$

A displacement can be written as

$$\Delta\boldsymbol{I} = \int_{t_i+\Delta t_i}^{t_f+\Delta t_f} \boldsymbol{f}(\mathbf{x}, \tau)\, d\tau - \int_{t_i}^{t_f} \boldsymbol{f}(\bar{\mathbf{x}}, \tau)\, d\tau \tag{25}$$

this can also be written as

$$\Delta\boldsymbol{I} = \int_{t_i+\Delta t_i}^{t_i} \boldsymbol{f}(\mathbf{x}, \tau)\, d\tau + \int_{t_i}^{t_f} \left[\boldsymbol{f}(\mathbf{x}, \tau) - \boldsymbol{f}(\bar{\mathbf{x}}, \tau)\right] d\tau + \int_{t_f}^{t_f+\Delta t_f} \boldsymbol{f}(\mathbf{x}, \tau)\, d\tau \tag{26}$$

Consider the first integral, and use a Taylor series to expand the integrand about t_i.

$$\int_{t_i+\Delta t_i}^{t_i} \boldsymbol{f}\left(\mathbf{x},\tau\right)d\tau = \int_{t_i+\Delta t_i}^{t_i}\left[\boldsymbol{f}\left(\mathbf{x}_i,t_i\right)+\dot{\boldsymbol{f}}_i\left(\tau-t_i\right)+\ldots\right]d\tau \tag{27}$$

$$= -\boldsymbol{f}\left(\mathbf{x}_i,t_i\right)\Delta t_i - \frac{1}{2!}\dot{\boldsymbol{f}}_i\Delta t_i^2 - \ldots \tag{28}$$

$$= -\left[\boldsymbol{f}\left(\bar{\mathbf{x}}\left(t_i\right),t_i\right)+\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\tilde{\Delta}\mathbf{x}_i+\ldots\right]\Delta t_i - \frac{1}{2!}\left[\dot{\boldsymbol{f}}\left(\bar{\mathbf{x}}\left(t_i\right),t_i\right)+\ldots\right]\Delta t_i^2 - \ldots \tag{29}$$

$$= -\boldsymbol{f}\left(\bar{\mathbf{x}}\left(t_i\right),t_i\right)\Delta t_i - \frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\tilde{\Delta}\mathbf{x}_i\Delta t_i - \frac{1}{2!}\dot{\boldsymbol{f}}\left(\bar{\mathbf{x}}\left(t_i\right),t_i\right)\Delta t_i^2 - \ldots \tag{30}$$

where a Taylor series is used to expand $\boldsymbol{f}$ about $\bar{\mathbf{x}}\left(t_i\right)$. Replace the displacements with the appropriate series representations

$$\begin{aligned}\int_{t_i+\Delta t_i}^{t_i} \boldsymbol{f}\left(\mathbf{x},\tau\right)d\tau = -\boldsymbol{f}\left(\bar{\mathbf{x}}\left(t_i\right),t_i\right)\overbrace{\left(dt_i+\frac{1}{2!}d^2t_i+\ldots\right)}^{\Delta t_i} - \frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\overbrace{\left(\delta\mathbf{x}_i+\frac{1}{2!}\delta^2\mathbf{x}_i+\ldots\right)}^{\tilde{\Delta}\mathbf{x}_i}\overbrace{\left(dt_i+\frac{1}{2!}d^2t_i+\ldots\right)}^{\Delta t_i} \\ - \frac{1}{2!}\dot{\boldsymbol{f}}\left(\bar{\mathbf{x}}\left(t_i\right),t_i\right)\left(\overbrace{dt_i+\frac{1}{2!}d^2t_i+\ldots}^{\Delta t_i}\right)^2 - \ldots\end{aligned} \tag{31}$$

The first and second order terms can be written as

$$\int_{t_i+\Delta t_i}^{t_i} \boldsymbol{f}\left(\mathbf{x},\tau\right)d\tau = -\boldsymbol{f}\left(\bar{\mathbf{x}}\left(t_i\right),t_i\right)dt_i - \frac{1}{2!}\left[\boldsymbol{f}\left(\bar{\mathbf{x}}\left(t_i\right),t_i\right)d^2t_i + 2\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\delta\mathbf{x}_i dt_i + \dot{\boldsymbol{f}}\left(\bar{\mathbf{x}}\left(t_i\right),t_i\right)dt_i^2\right] - \ldots \tag{32}$$

The same approach can be used to develop an expression for the third integral term

$$\int_{t_f}^{t_f+\Delta t_f} \boldsymbol{f}\left(\mathbf{x},\tau\right)d\tau = \boldsymbol{f}\left(\bar{\mathbf{x}}\left(t_f\right),t_f\right)dt_f + \frac{1}{2!}\left[\boldsymbol{f}\left(\bar{\mathbf{x}}\left(t_f\right),t_f\right)d^2t_f + 2\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\delta\mathbf{x}_f dt_f - \dot{\boldsymbol{f}}\left(\bar{\mathbf{x}}\left(t_f\right),t_f\right)dt_f^2\right] - \ldots \tag{33}$$

Now consider the second integral. Use a Taylor series to expand the integrand about $\bar{\mathbf{x}}$

$$\int_{t_i}^{t_f} \left[\boldsymbol{f}_k\left(\mathbf{x},\tau\right) - \boldsymbol{f}_k\left(\bar{\mathbf{x}},\tau\right)\right] d\tau = \int_{t_i}^{t_f} \left[\boldsymbol{f}_k\left(\bar{\mathbf{x}},\tau\right) + \frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}}\tilde{\Delta}\mathbf{x} + \frac{1}{2!}\tilde{\Delta}\mathbf{x}^T \frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}^T \partial \mathbf{x}}\tilde{\Delta}\mathbf{x} + \ldots - \boldsymbol{f}_k\left(\bar{\mathbf{x}},\tau\right)\right] d\tau \tag{34}$$

$$= \int_{t_i}^{t_f} \left[\frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}}\tilde{\Delta}\mathbf{x} + \frac{1}{2!}\tilde{\Delta}\mathbf{x}^T \frac{\partial^2 \boldsymbol{f}_k}{\partial \mathbf{x}^T \partial \mathbf{x}}\tilde{\Delta}\mathbf{x} + \ldots\right] d\tau \tag{35}$$

Using the expansion of $\tilde{\Delta}\mathbf{x}$ this can be written as

$$\begin{aligned}\int_{t_i}^{t_f} \left[\boldsymbol{f}_k\left(\mathbf{x},\tau\right) - \boldsymbol{f}_k\left(\bar{\mathbf{x}},\tau\right)\right] d\tau = \int_{t_i}^{t_f} \Bigg[& \frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}} \overbrace{\left(\delta\mathbf{x} + \frac{1}{2!}\delta^2\mathbf{x} + \ldots\right)}^{\tilde{\Delta}\mathbf{x}} \\ & + \frac{1}{2!}\overbrace{\left(\delta\mathbf{x}^T + \frac{1}{2!}\delta^2\mathbf{x}^T + \ldots\right)}^{\tilde{\Delta}\mathbf{x}^T} \frac{\partial^2 \boldsymbol{f}_k}{\partial \mathbf{x}^T \partial \mathbf{x}} \overbrace{\left(\delta\mathbf{x} + \frac{1}{2!}\delta^2\mathbf{x} + \ldots\right)}^{\tilde{\Delta}\mathbf{x}} + \ldots \Bigg] d\tau\end{aligned} \tag{36}$$

Gathering the second order terms

$$\int_{t_i}^{t_f} \left[\boldsymbol{f}_k\left(\mathbf{x},\tau\right) - \boldsymbol{f}_k\left(\bar{\mathbf{x}},\tau\right)\right] d\tau = \int_{t_i}^{t_f} \underbrace{\left[\frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}}\delta\mathbf{x} + \frac{1}{2!}\left(\frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}}\delta^2\mathbf{x} + \delta\mathbf{x}^T \frac{\partial^2 \boldsymbol{f}_k}{\partial \mathbf{x}^T \partial \mathbf{x}}\delta\mathbf{x}\right) + \ldots\right]}_{\delta \boldsymbol{f}} d\tau \tag{37}$$

The expressions for each of the three integrals can now be combined

$$\begin{aligned}\Delta \boldsymbol{I} = & \overbrace{\left[\boldsymbol{f}\left(\bar{\mathbf{x}},\tau\right) d\tau\right]_{dt_i}^{dt_f} + \int_{t_i}^{t_f} \left[\frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}}\delta\mathbf{x}\right] d\tau}^{d\boldsymbol{I}} \\ & + \frac{1}{2!}\underbrace{\left(\left[\boldsymbol{f}\left(\bar{\mathbf{x}},\tau\right) d^2\tau + 2\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\delta\mathbf{x}d\tau - \dot{\boldsymbol{f}}\left(\bar{\mathbf{x}},\tau\right) d\tau^2\right]_{dt_i}^{dt_f} + \int_{t_i}^{t_f} \left[\frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}}\delta^2\mathbf{x} + \delta\mathbf{x}^T \frac{\partial^2 \boldsymbol{f}_k}{\partial \mathbf{x}^T \partial \mathbf{x}}\delta\mathbf{x}\right] d\tau\right)}_{d^2\boldsymbol{I}} + \ldots\end{aligned} \tag{38}$$

The first and second differentials for an integral are then

$$d\boldsymbol{I} = \left[\boldsymbol{f}\left(\bar{\mathbf{x}}, \tau\right) d\tau\right]_{dt_i}^{dt_f} + \int_{t_i}^{t_f} \left[\frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}} \delta\mathbf{x}\right] d\tau \tag{39}$$

$$d^2\boldsymbol{I} = \left[\boldsymbol{f}\left(\bar{\mathbf{x}}, \tau\right) d^2\tau + 2\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}} \delta\mathbf{x} d\tau - \dot{\boldsymbol{f}}\left(\bar{\mathbf{x}}, \tau\right) d\tau^2\right]_{dt_i}^{dt_f} + \int_{t_i}^{t_f} \left[\frac{\partial \boldsymbol{f}_k}{\partial \mathbf{x}} \delta^2\mathbf{x} + \delta\mathbf{x}^T \frac{\partial^2 \boldsymbol{f}_k}{\partial \mathbf{x}^T \partial \mathbf{x}} \delta\mathbf{x}\right] d\tau \tag{40}$$

Curriculum Vitae

Shane B. Robinson

EDUCATION

Ph.D: Mechanical Engineering with and Aerospace Emphasis
Utah State University, Logan, UT. 2008-2014

- Emphasis on linear covariance analysis, guidance, navigation, and control
- Dissertation: Spacecraft Guidance Techniques for Maximizing Mission Success

Bachelor of Science: Mechanical Engineering with and Aerospace Emphasis
Utah State University, Logan, UT. 2006-2008

Associates: Pre-Engineering
Snow College, Ephraim, UT. 2005-2006

PAPERS AND PRESENTATIONS

- Robinson, S. B., and Christian, J. A., "Pattern Design for 3D Point Matching," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, Submitted April 2013.
- Christian, J. A., Robinson, S. B., D'Souza, C. N., and Ruiz, J. P., "Cooperative Relative Navigation of Spacecraft at Close Range Using Flash LIDARs," *Journal of Guidance, Control, and Dynamics*, Vol. 37, No. 2, March-April 2013, pp. 452–465.
- Moesser, T. J., Geller, D. K., and Robinson, S. B., "Guidance and Navigation Linear Covariance Analysis for Lunar Powered Decent," *AAS/AIAA Astrodynamics Specialist Conference*, No. AAS 11-532, August 2011.
- Robinson, S. B. and Geller, D. K., "A Simple Strategy for Computing Time Optimal Earth-Return Trajectories for Lunar and Cislunar Flight," *AIAA Guidance, Navigation, and Control Conference*, American Institute of Aeronautics and Astronautics, Toronto, Ontario, Canada, August 2010.
- Robinson, S. B. and Geller, D. K., "A Simple Numerical Procedure for Computing Multi-Impulse Lunar Escape Sequences With Minimal Time-of-Flight," *AAS/AIAA Space Flight Mechanics Meeting*, No. AAS-10-166, American Astronautical Society/American Institute of Aeronautics and Astronautics, San Diego, California, February 2010.

- Robinson, S. B. and Geller, D. K., "A Simple Targeting Procedure for Lunar Trans-Earth Injection," *AIAA Guidance, Navigation, and Control Conference*, No. AIAA-2009-6107, American Institute of Aeronautics and Astronautics, Chicago, Illinois, August 2009.

- Geller, D. K., McInroy, J., Robinson, S. B., and Schmidt, J., "Autonomous Quality Space Imagery For LEO/GEO Space Operations," *AIAA/AAS Astrodynamic Specialist Conference and Exhibit*, No. AIAA-2008-7209, American Astronautical Society/American Institute of Aeronautics and Astronautics, Honolulu Hawaii, August 2008.

EXPERIENCE

Aerospace Engineer, *GN&C Autonomous Flight Systems Branch, NASA Johnson Space Center*, Houston, TX. Aug 2013–Present

Currently, I am the On-Orbit Guidance Lead for Orion MPCV Orbit MODE team. This includes responsibility for the development of the guidance flight software for all exoatmospheric phases of flight. Also, I actively participates in the development and testing of flash LIDAR and associated image processing software for use during rendezvous, proximity, and docking operations.

Graduate Research Assistant, *Mechanical and Aerospace Engineering*, Logan, Utah, May 2007–July 2013

My duties included conducting general guidance, navigation, control, and flight mechanics research. I developed guidance targeting and navigation algorithms for an on-orbit inspection spacecraft. I also developed and performed analysis of on-board guidance and targeting for contingency operations or the Orion spacecraft. Particular emphasis was given to performing analysis using linear covariance techniques.

Graduate Co-Op, *GN&C Autonomous Flight Systems Branch, NASA Johnson Space Center, Houston, TX. Jan 2012–May 2012*

I generated a best estimate trajectory for the STORRM DTO flown on STS-134. Identified several anomalous behaviors of the STORRM VNS. I developed a new approach for the design of retro-reflector target patterns. I also, created a novel pose estimation scheme using total least squares to process flash lidar data.

Summer Intern, *GN&C Autonomous Flight Systems Branch, NASA Johnson Space Center, Houston, TX. June 2011–July 2011*

I worked on generating a best estimate trajectory for the STORRM DTO flown on STS-134. Primary focus was placed on developing a Kalman smoothing tool and associated support software. These tools were developed with the goal of processing the data gathered by a flash lidar instrument during rendezvous and proximity operations with the ISS.

Research Affiliate, *GN&C Section, Jet Propultion Laboratory, Pasadena, CA. Jan 2010–Dec 2010*

I executed a cross validation of a JPL institutional navigational analysis tool with the USU linear covariance tool. Also developed a concept for accounting for the dispersive effects caused by the interaction of the guidance and navigations systems for the JPL tool.

GN&C Lead for USU Chimaera Rocket Project, *Mechanical and Aerospace Engineering, Logan, Utah, Sept 2007–June 2009*

As part of NASA's University Student Launch Initiative Competition, I lead the Chimaera team from USU in designing a closed loop guidance navigation and control system, which would precisely control apogee of the rocket by asymptotically dissipating surplus energy. This design included hardware selection, integration, software and simulation development, and the underlying theoretical work. The system earned the award "best payload" by the competition judges in 2008 and 2009, and was cited as the principle reason the Chimaera team won the competition in 2008 and 2009. Subsequent team victories relied heavily on this work.

Summer Intern, *GN&C Section, Jet Propultion Laboratory, Pasadena, CA. June 2008–Aug 2008 and June 2009–Aug 2009*

As part of a summer internship I conducted a parametric trade study which evaluated radiometric tracking baseline performance for the lunar orbit. In addition I developed a linear mapping of delivery error to variance in fuel use.

AWARDS

2005 Utah Governor's Scholar, Presidential Scholarship at Utah State University, NASA Rocky Mountain Space Grant Fellow, 2012 NASA Outstanding Co-Op Award

RELEVANT COURSES

Spacecraft Navigation (Geller), Optimal Spacecraft Guidance (Geller), Advanced Astrodynamics (Geller), Stochastic Estimation with Aerospace Applications (Geller), Compressible Fluid Flow (Whitmore), Propultion (Whitmore), Linear Multivariable Control (Chen), Spacecraft Attitude Control (Fullmer), Nonlinear Control (Fullmer), Complex Optimization (Gunther), Math Methods for Signals and Systems (Moon).