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# A NEW GRAVITY MODEL FOR NAVIGATION CLOSE TO COMETS AND ASTEROIDS

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**The gravity of an irregularly shaped body, computed from a harmonic expansion of Legendre polynomials and associated functions, diverges inside a sphere of maximum radius that circumscribes the body. A spacecraft that is attempting a landing on an irregularly shaped body at a landing site that is near the minimum radius of the body will traverse the region from the sphere of maximum radius to the landing site. The spacecraft will experience considerable error in the computed gravitational acceleration if an harmonic expansion gravity model is used.**

**In this paper, a gravity model is described that computes the acceleration of the spacecraft directly from the mass distribution. Included in this model is provision for a variable density defined on the surface of the body and extending uniformly to the center of mass. The variable density is computed as a function of Legendre polynomials and associated functions. The principal advantage of this model is that the external gravity field is exact for the given shape of the body. When the model described in this paper is applied to the Earth, gravity anomolies are detected. Further analysis indicates that these anomolies explain the velocity perturbations observed in the orbit determination solutions obtained as the Near Earth Asteroid Rendezvous (NEAR) and Galileo spacecraft flew by the Earth.**



## I Introduction

The gravity of an irregularly shaped body may be computed from the harmonic expansion of Legendre polynomials and associated functions just as for bodies that are nearly spherical in shape. The harmonic coefficients are much greater for an irregularly shaped body and the unnormalized polynomial coefficients approach one in the limit of the most extremely irregularly shaped body, a dipole. However, the accuracy of the potential and acceleration computed from a harmonic expansion depends on the degree and order of the expansion and the ratio of the magnitude of the radius vector at the point of interest to the maximum radius of the gravitating body. The greater this ratio, the greater is the accuracy of the computed potential and acceleration. When the radius of interest for computing potential and acceleration is less than the maximum radius of the gravitating body, the harmonic expansion diverges. For an irregularly shaped body, a spacecraft may cross the spherical boundary defined by the maximum radius of the body, referred to as the Brillouin sphere, resulting in significant error near the surface.

A solution to this problem is to bypass the potential that the harmonic expansion is derived from and compute the acceleration directly from the mass distribution. Gravity models derived from point masses or polyhedra have proved useful. However, they fail to account for nonuniform density. The model described below computes the acceleration directly from the mass distribution and is exact for nonuniform density.

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## II Gravity Models

There are two classes of methods for improving the accuracy of a harmonic expansion gravity model. The first class relies on expansions in other systems of coordinates that more nearly approximate the shape of the body. For example, an irregularly shaped body may be approximated by a triaxial ellipsoid and an expansion in ellipsoidal coordinates may be expected to converge at points outside some reference ellipsoid. For an irregularly shaped body, like the asteroid Eros, it is difficult to define a mathematical function that will approximate Eros with sufficient accuracy. Also, truncation of the expansion and aliasing associated with nonuniform data coverage contribute additional error.

The second class of methods compute the gravitational acceleration directly from the mass distribution. A simple model, that avoids truncation and aliasing errors, represents the body as a cluster of point masses. The acceleration of a spacecraft may be determined by computing the acceleration by all the point masses comprising the body and summing. This task would be formidable, even with the most advanced high speed computers, and would need to be repeated at many points along the spacecraft trajectory. Furthermore, the problem of estimating the mass of each element would be hampered by the high correlation of the acceleration contributions of adjacent mass elements. Representation of the body by polyhedra[1] is one method that has been developed and was used to verify the Eros gravity model used by the NEAR mission. The potential and acceleration are computed for each polyhedron and summed. The polyhedra model assumed constant density and could not be adapted easily to the problem of variable density. A second method, developed for the NEAR landing and the subject of this paper, computed the acceleration of the spacecraft directly from the mass distribution and permitted solution for the density as a function of Legendre polynomials and associated functions. If the integration with respect to $r$ is performed analytically, the volume integral is reduced to a surface integral that is numerically tractable. The surface integral may be evaluated numerically and the density permitted to vary as a function of the spherical angles latitude and longitude.

## III Surface Density Model

The acceleration of a point mass or spacecraft is determined by summing or integrating the acceleration contribution of each mass element ($dm$) in a gravitationally attractive body[2]. In this paper, a gravitationally attracted point mass is referred to as a spacecraft. The mass of the spacecraft is assumed to be too small to affect the gravity field of the attractive body. The geometry is illustrated in Figure 1, The acceleration is

$$\mathbf{a} = G \int_V \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\, dm \tag{1}$$

where $V$ denotes the irregularly shaped body. The vector $\mathbf{r}$ is from the center of the coordinate system to the mass element and the vector $\mathbf{r}'$ is to the spacecraft. The mass element (dm) is defined by the Jacobian of spherical coordinates as

$$dm = \rho\, r^2 \cos\phi\; dr\; d\phi\; d\lambda$$

and the density $\rho$ is defined by an expansion of Legendre polynomials and associated functions as

$$\rho = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m(\sin\phi)[A_{nm}\ \cos m\lambda + B_{nm}\ \sin m\lambda] \tag{2}$$

where $\phi$ denotes the latitude and $\lambda$ the longitude of a point in the body $V$. The components of the vectors $\mathbf{r}$ and $\mathbf{r}'$ in Cartesian coordinates are

$$\mathbf{r} = (r\cos\lambda\cos\phi, r\sin\lambda\cos\phi, r\sin\phi)$$
$$\mathbf{r}' = (x', y', z')$$

Substituting the vector components into Equation 1 gives

$$\mathbf{a} = G \int_V \frac{(r\cos\lambda\cos\phi - x', r\sin\lambda\cos\phi - y', r\sin\phi - z')}{[(r\cos\lambda\cos\phi - x')^2 + (r\sin\lambda\cos\phi - y')^2 + (r\sin\phi - z')^2]^{\frac{3}{2}}}\, dm \tag{3}$$

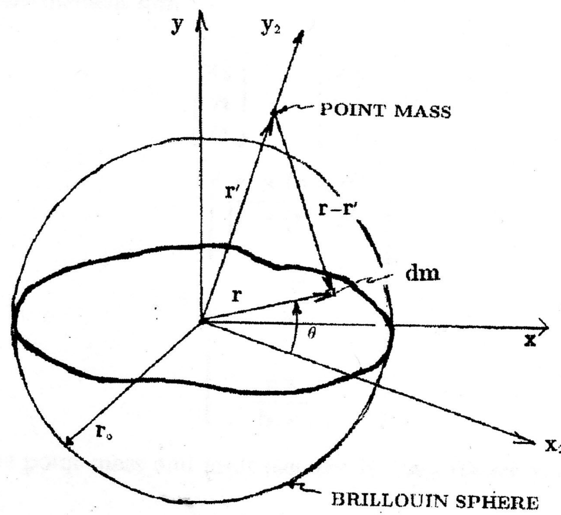


**Figure 1. Irregular Body Gravity Geometry**

Before integrating with respect to $r$, a rotation of coordinates may be defined that will simplify the integrand. The $y_2$ axis is placed through the location of the spacecraft ($\mathbf{r}'$) and the $x_2$ axis is placed in the plane defined by the mass element ($dm$) and $\mathbf{r}'$.

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4}$$

$$T = \begin{bmatrix} \dfrac{\mathbf{r}' \times (\mathbf{r} \times \mathbf{r}')}{|\mathbf{r}' \times (\mathbf{r} \times \mathbf{r}')|} \\ \\ \dfrac{\mathbf{r}'}{|\mathbf{r}'|} \\ \\ \dfrac{\mathbf{r} \times \mathbf{r}'}{|\mathbf{r} \times \mathbf{r}'|} \end{bmatrix} \tag{5}$$

The coordinates of the spacecraft $\mathbf{r}_2'$ and a mass element $\mathbf{r}_2$ in the rotated coordinate system are

$$\mathbf{r}_2' = (0,\ r',\ 0)$$
$$\mathbf{r}_2 = (r\cos\theta,\ r\sin\theta,\ 0)$$
$$\theta = 90 - \cos^{-1}\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{rr'}\right)$$

and

$$\mathbf{a} = G\int_V T^T \frac{[r\cos\theta,\ r\sin\theta - r',\ 0]^T}{[(r^2\cos^2\theta + (r\sin\theta - r')^2]^{\frac{3}{2}}}\, dm \tag{6}$$

Replacing the mass element $(dm)$ with $\rho\ r^2 \cos\phi dr d\phi d\lambda$ gives

$$\mathbf{a} = G \int_S \rho\ T^T \int_0^R \frac{[\cos\theta,\ \sin\theta - \frac{r'}{r},\ 0]^T}{\left((\cos^2\theta + (\sin\theta - \frac{r'}{r})^2\right)^{\frac{3}{2}}}\ dr\ \ \cos\phi\ d\phi\ d\lambda \tag{7}$$

where $R$ is the radius of the body as a function of $\lambda$ and $\phi$. The density $(\rho)$ and coordinate transformation $(T)$ factor out of the $r$ integration since they are only a function of latitude and longitude. Performing the $r$ integration gives

$$\mathbf{a} = G \int_S \rho\ T^T \begin{bmatrix} a_{2x}(R) - a_{2x}(0) \\ a_{2y}(R) - a_{2y}(0) \\ 0 \end{bmatrix} \cos\phi\ d\phi\ d\lambda \tag{8}$$

$$\begin{aligned} a_{2x}(r) = & \frac{r^2\cos^2\theta + r'\sin^2\theta(r\sin\theta - r') + \cos^2\theta(2r'^2 - 5rr'\sin\theta)}{\cos\theta\sqrt{r^2 + r'^2 - 2rr'\sin\theta}} \\ & + 3\cos\theta\sin\theta r'\ \ \ln\left[2r - 2r'\sin\theta + 2\sqrt{r^2 + r'^2 - 2rr'\sin\theta}\right] \\ a_{2y}(r) = & \frac{rr'\cos^2\theta + r^2\sin\theta - 5rr'\sin\theta^2 + 3r'^2\sin\theta}{\sqrt{r^2 + r'^2 - 2rr'\sin\theta}} \\ & - r'(1 - 3\sin^2\theta)\ \ \ln\left[2r - 2r'\sin\theta + 2\sqrt{r^2 + r'^2 - 2rr'\sin\theta}\right] \\ a_{2z} = & \ 0 \end{aligned}$$

The gravitational acceleration computed assuming constant density will generally not yield sufficient accuracy for navigation of a spacecraft. Therefore, the density at the surface is varied as a function of latitude and longitude. The density is assumed to be uniform from below the surface to the center of mass of the body. This assumption enables the density to be factored out of the r integration, simplifying the mathematics. The resultant mass distribution does not model reality inside the body, but provides an exact model of the external gravity field.

The integral given in Equation 8 may be evaluated by tiling the unit sphere with area patches that are nearly square and sum to exactly $4\pi$ over the unit sphere. The total acceleration is obtained by evaluating the integrand of Equation 8 at the center of each area patch, multiplying by the area and summing over all the area patches.

## IV Mass Distribution of an Irregularly Shaped Body

A cross section of the surface density gravity model is shown in Figure 2. The cross section of Eros is used as an example viewed looking down on the North Pole with longitude measured counterclockwise, or East, from the $x$ axis. The cross hatched segments have uniform density from below the surface defined by latitude and longitude to the center of mass. Each cross hatched segment may have a different density. In three dimensions, the cross hatched segments are actually pyramids with the apex at the center of figure and the base defined by rectangular patches whose sides are delimited by latitude and longitude. In the limit as the size of the area patches approaches zero, the gravity potential on the surface is exact. If the surface gravity potential is exact, the resulting external gravity field in vacuum is exact. The internal mass distribution is of little interest for navigation since there is an infinity of mass distributions that will yield the same external field. The mass distribution of the surface density model is selected because it is mathematically convenient.

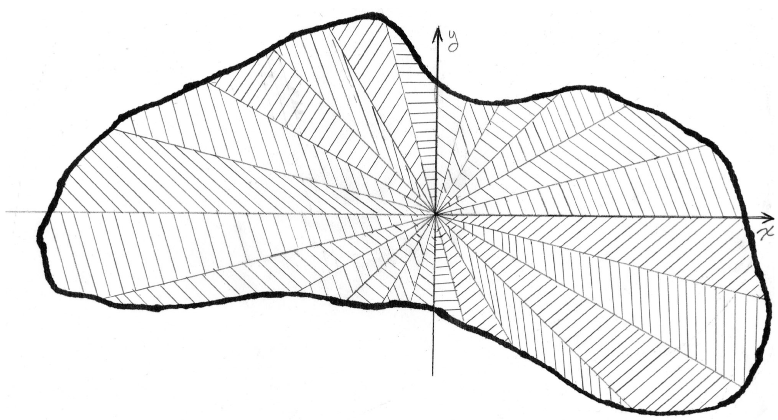


**Figure 2. Irregular Body Cross Section**

Consider the case of an irregularly shaped body with uniform density illustrated in Figure 3. The cross-hatched segments all have the same density. Centered at the origin are concentric spherical shells and a point mass at the center. If the densities of these shells are varied, there are no external gravity measurements that can distinguish the layered shells from a body that is uniform in density. This is a well-known property of a gravity field that satisfies Laplace's equation. The internal core of the Earth was determined by observing earthquake wave propagation and not by direct gravity measurement.

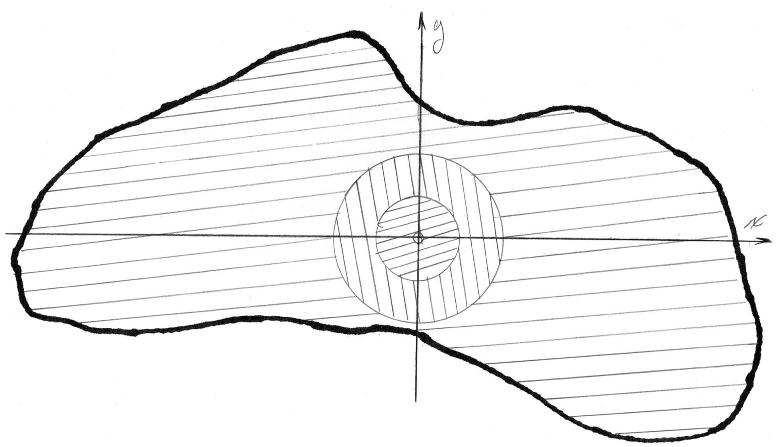


**Figure 3. Irregular Body with Uniform Density**

The geometry for a single point mass is illustrated in Figure 4. The point mass may have all the mass concentrated at a given location or be distributed in spherical shells as shown on the figure. The point mass distribution may be summed with the uniform density model shown in Figure 3, yielding an external gravity field with a wide variety of internal mass distributions. The pathological mass distribution model shown in Figure 2 may be used to represent any external gravity field as will be demonstrated later. Therefore, very little knowledge of the internal mass distribution of a body can be determined by observation of the external gravity field. The only way to determine the density at any point inside a body is to independently determine the volume surrounding the point, independently verify uniform density within the volume and separate the mass determination or gravity field from the rest of the body.

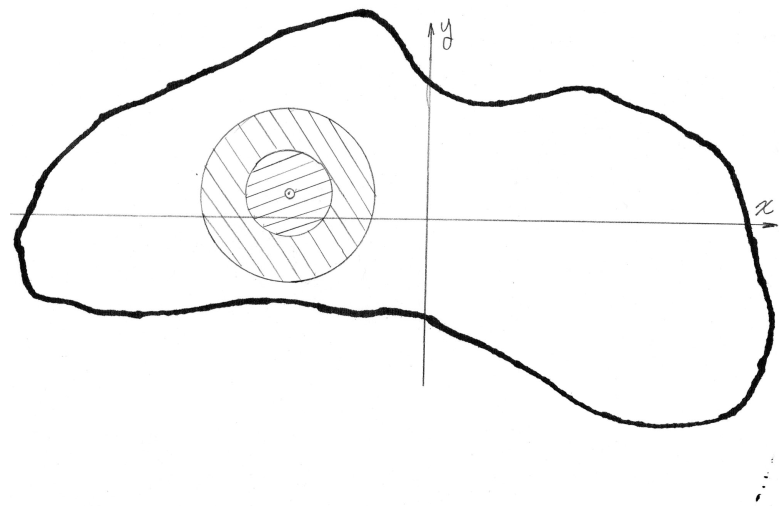


**Figure 4. Irregular Body with a Point Mass**

## V Numerical Verification of Surface Density Model

The uniform density gravity model described above was used to compute the Near Earth Asteroid Rendezvous (NEAR) landing trajectory[3]. Comparison of shape-derived and radio-tracking gravity harmonics indicated that Eros is uniform in density with the possibility of having a layered central core. However, there are small differences and these may be attributed to a small nonuniformity of the internal mass distribution. An exact gravity model may be obtained by computing the acceleration from a cluster of point masses that fill the shape of Eros. This approach is not easy to implement because of the large number of point masses required to represent a three-dimensional object. The approach used here is to represent a point mass as a surface density distribution. Each point mass would contribute a harmonic expansion as described above. The mass of the body is obtained by summing all the point masses associated with the volume elements as

$$m = \sum_{i=1}^{\infty} dm_i \tag{9}$$

where

$$dm_i = \rho_i dv_i$$

$$dv_i = r_i^2 \cos\phi_i \; dr_i d\phi_i d\lambda_i$$

The mass of the body is thus given by

$$m = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} P_n^m(\sin\phi_i)[A_{nm} \; \cos m\lambda_i + B_{nm} \; \sin m\lambda_i] r_i^2 \cos\phi_i \; dr_i d\phi_i d\lambda_i \tag{10}$$

When Equation 10 is applied to all the mass elements, $m$ sums to the total mass and convergence is guaranteed because the total mass is finite. The $A_{nm}$ and $B_{nm}$ coefficients for any $\lambda_i$ and $\phi_i$ apply for all $r_i$ and these may be obtained as part of the orbit determination solution for a spacecraft in orbit about the body.

A test to verify the veracity of the surface density model involves locating a point mass within some arbitrary shape and computing the surface density. Since any object may be replaced by an infinite cluster of point masses, verifying the surface density model for a single point mass verifies the model for the object. The acceleration of a spacecraft that is outside the body is simply the sum of the accelerations from all the point masses and has been shown above to be the sum of the harmonic expansions associated with all the point masses. The geometry is illustrated in Figure 5. The surface mass distribution is computed for two bodies. One has the shape of the asteroid Eros and the other is a sphere. Another sphere is defined that circumscribes the bodies and the acceleration of the point mass is computed at sample points that cover the larger sphere. A typical sample point is shown in the figure. The surface density model acceleration is also computed for each sample point. As a first guess, the bodies are assumed to have uniform density and the mass is the same as the point mass. The magnitude of the acceleration is computed for the point mass and the body of interest. Using a square root information filter, a least square solution is obtained for the surface density harmonic coefficients which give the magnitude of the acceleration of a spacecraft at the sample point. If the acceleration magnitudes of the point mass and gravity model are equal on the sampled sphere, the gradient of the potential is equal and a solution is obtained for the density harmonic coefficients.

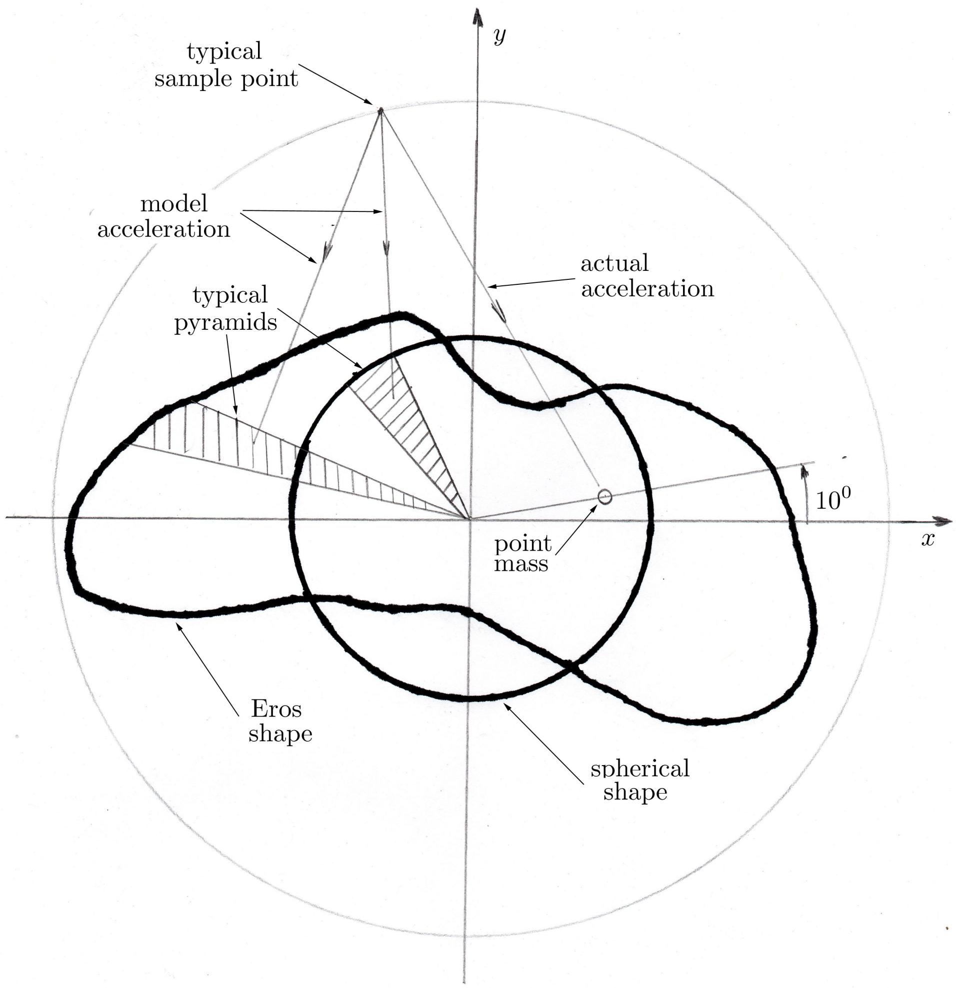


**Figure 5. Surface density of point mass**

The observable used for obtaining a solution for the density harmonic coefficients is defined by

$$a^2 = a_x^2 + a_y^2 + a_z^2 \tag{11}$$

and the required partial derivatives are

$$\frac{\partial a}{\partial(A_{nm}, B_{nm})} = \frac{\partial a}{\partial(a_x, a_y, a_z)} \frac{\partial(a_x, a_y, a_z)}{\partial \rho} \frac{\partial \rho}{\partial(A_{nm}, B_{nm})} \tag{12}$$

where

$$\frac{\partial a}{\partial \rho} = \frac{a_x}{a}\frac{\partial a_x}{\partial \rho} + \frac{a_y}{a}\frac{\partial a_y}{\partial \rho} + \frac{a_z}{a}\frac{\partial a_z}{\partial \rho}$$

$$\frac{\partial (a_x, a_y, a_z)}{\partial \rho} = G \int_S T^T \begin{bmatrix} a_{2x}(R) - a_{2x}(0) \\ a_{2y}(R) - a_{2y}(0) \\ 0 \end{bmatrix} \cos\phi d\phi d\lambda$$

$$\frac{\partial \rho}{\partial A_{nm}} = P_n^m(\sin\phi)\cos\, m\lambda$$

$$\frac{\partial \rho}{\partial B_{nm}} = P_n^m(\sin\phi)\sin\, m\lambda$$

The above partial derivatives and the difference between the observed acceleration and computed acceleration are packed into a square root information matrix, which is inverted after all the sample points have been processed to obtain the solution for the density harmonic coefficients.

The first test of the surface density theory involves computing the surface density distribution for a point mass offset from the center of the coordinate system on the surface of a sphere. The point mass is located at 10 degrees longitude, 10 degrees latitude and 6 km from the center of the sphere. The radius of the sphere is 8.43259 km which is sized to equal the volume of Eros. The geometry is illustrated in Figure 5. Accelerations of a spacecraft from the point mass and the surface density model are computed at various sample points that cover a sphere of radius 18 km. A plot of the surface density is shown in Figure 6. Since the gravity field is spherically symmetrical about the point mass, one would expect the contours of constant density on the sphere to be circles centered at the point on the surface of the sphere closest to the point mass. Figure 6 shows the maximum density at 10 degrees latitude and 10 degrees longitude and the contours of constant density are circles with a minimum density 180 degrees from the maximum density. The spacecraft acceleration from the surface density model and point mass are equal to very high precision.

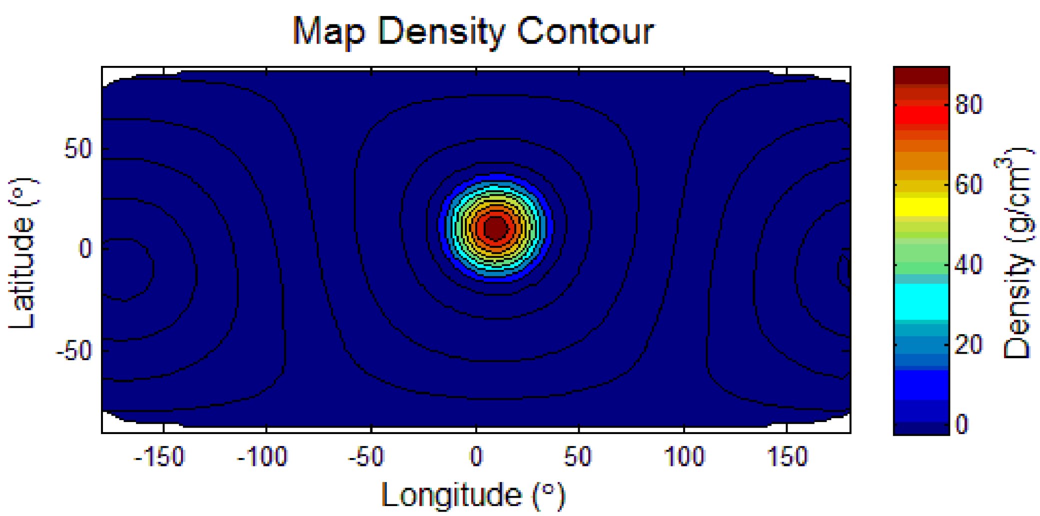


**Figure 6. Surface density on sphere from point mass**

The second test of the surface density theory involves computing the surface density for the same point mass as for the sphere but on the surface of the asteroid Eros. The geometry is also illustrated in Figure 4. A plot of the surface density, shown in Figure 7, is also obtained by sampling points on an 18 km sphere and fitting the surface density harmonic coefficients to the magnitude of the acceleration at each sample point on the same 18 km sphere. Since the gravity field is spherically symmetrical about the point mass, the surface density distribution is also symmetrical relative to the point mass. As a result, points on the surface of the body near the point mass will have greater density than points farther away. This observation is readily apparent from inspection of Figure 7. The surface density maximum occurs along the $y$ axis (longitude 90 deg and -90 deg) where the surface is closest to the point mass. Also, the region near the North Pole has elevated surface density because the point mass is at 10 deg latitude. At the ends of the asteroid, where the radius approachs 17 km, the surface density is small and even goes negative. Negative gravity is a mathematical artifact of the surface density model. Since the surface density model does not relate to actual mass distribution, the appearance of negative mass in regions where there is little or no mass does not affect the external gravity field. There are an infinite number of mass distributions that will yield a given external gravity field as discussed above. Some of these mass distributions are not physically realizable, at least with current physics, but will result in a valid external gravity field that may be used for spacecraft orbit determination.



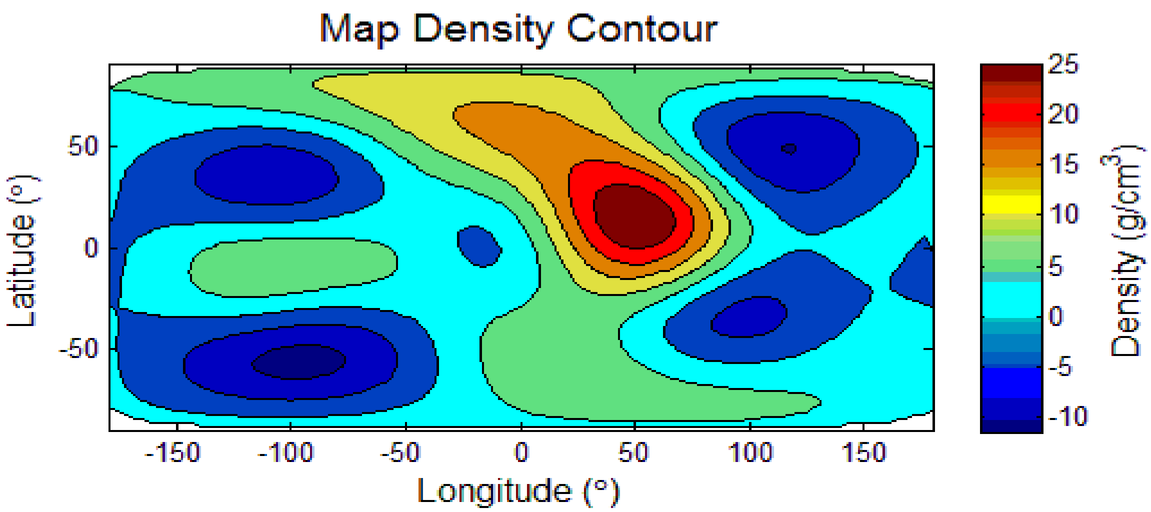


**Figure 7. Surface density on Eros Shape from point mass**

A more realistic test of the surface harmonic expansion may be obtained by processing real data from an orbiting spacecraft in an orbit determination program. A high quality set of data is available for the NEAR spacecraft orbiting the asteroid Eros. For one month, the spacecraft was in a 25 km polar orbit and nearly continuous Doppler data was obtained. Processing this data with the surface density gravity model would require some modification of existing orbit determination software and is therefore beyond the scope of this paper. However, we have a harmonic expansion of the gravity field obtained during Eros flight operations. This harmonic expansion closely replicated an independent gravity model obtained by integrating a laser altimetry derived shape model over the surface assuming constant density. This verification of the surface gravity model for constant density provided confidence in its use for Eros landing.

## VI Gravity Model Comparison for Eros and Earth

As a substitute for real data, the Eros harmonic expansion was used to compute acceleration at the sample points on the 18-km sphere defined above. These acceleration data points were processed in a square root information filter to obtain surface density harmonic coefficients. Figure 8 shows the result of this simulation and provides insight into the possible nonuniformity of the Eros mass distribution. Inspection of Figure 8 indicates that a large object is buried at 260 deg longitude and 0 deg latitude. This conclusion is a bit premature since further analysis indicates that the Eros shape model is shifted about 100 m along the $y$ axis. This apparent discrepancy was also observed when comparing the LIDAR shape model with the optical shape model derived independently by Peter Thomas[4]. It should also be noted that this discrepancy is small compared to the mean density of Eros and did not affect the Eros landing trajectory.

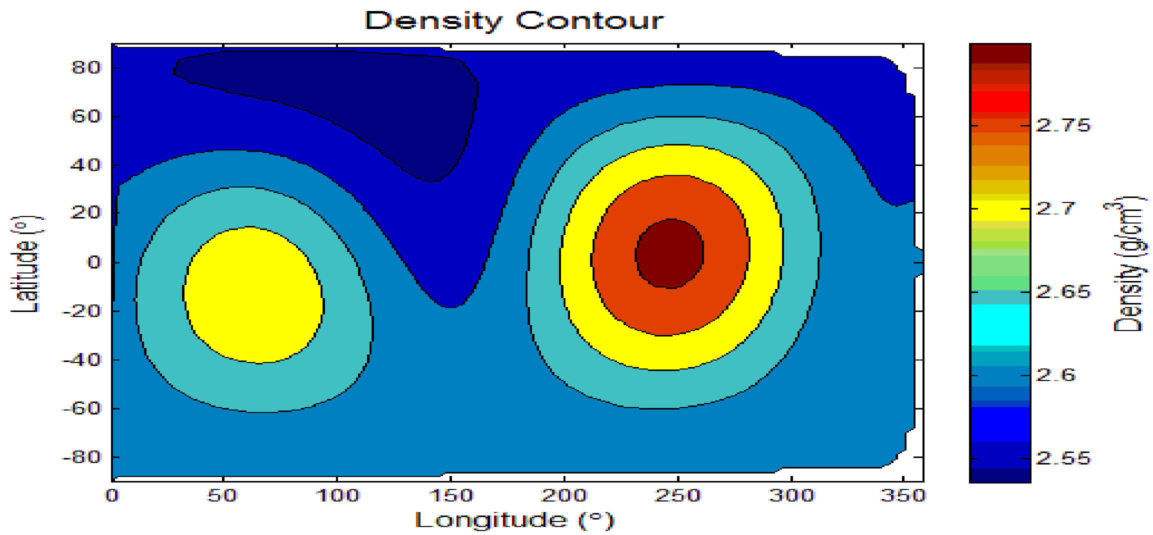


**Figure 8. Surface Density of Eros from Harmonic Expansion**

The same procedure applied to Eros above may be applied to the Earth. The Earth shape is computed from a model of the polar and equatorial radii with a mean equatorial radius of 6378 km. The sample points are computed on a 6478 km sphere 100 km above this Earth-shaped model. The sampled gravity was obtained from a 16 degree and order harmonic expansion that was used for the NEAR Earth flyby. The surface density is a fourth degree and order harmonic expansion. The surface density map in g/cm$^3$ is shown

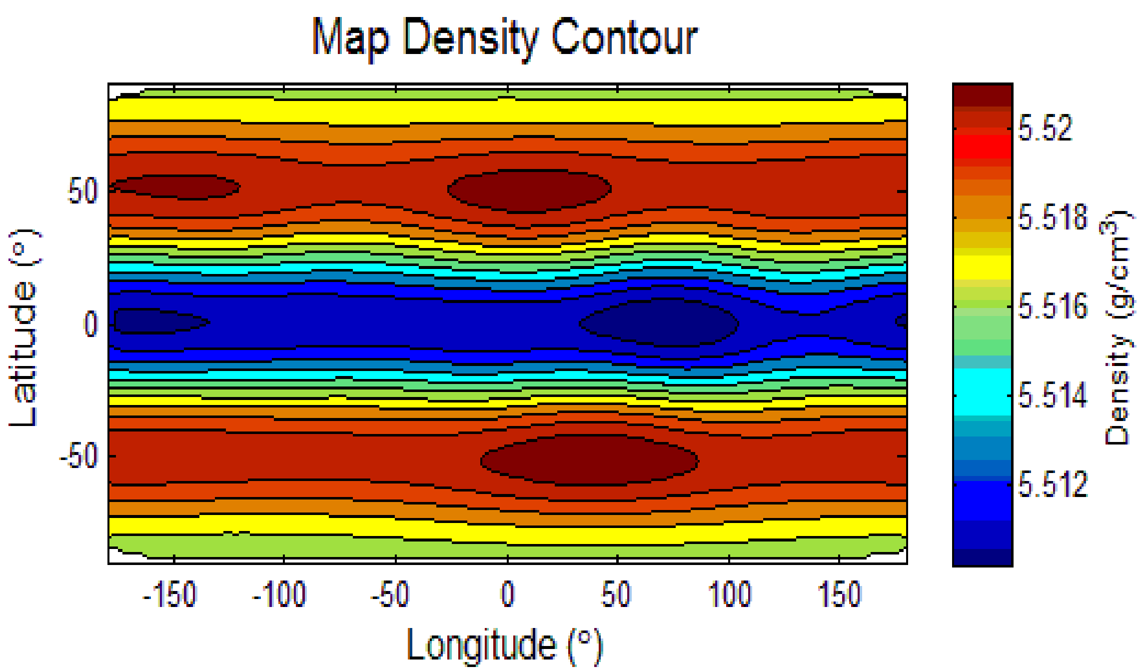


**Figure 9. Surface Density of Earth from Harmonic Expansion**

in Figure 9. The magnitude of the variation in density is much greater than would be expected from gravity measurements on the surface. However, these density variations are not large enough to be a problem for navigation of a spacecraft flying by the Earth at an altitude of about 450 km. If we moved Iceland into the middle of the Pacific Ocean, the spacecraft trajectory would not be significantly affected. However, a large gravity anomaly, as shown in Figure 9 at 50 degrees South latitude and 40 degrees East longitude, could result in a velocity perturbation of several mm/s, as was observed when the NEAR and Galileo spacecraft flew by the Earth.

The maxima and minima shown in Figure 9 may be attributed to several sources. The gravity harmonic expansion is only valid when evaluated at a point that is greater than a sphere of maximum radius for the harmonic expansion or above the surface for the surface density model. The assumption required by Laplace's equation is that the divergence is zero at any point that is assumed to be vacuum by the gravity model. Another problem is the truncation error. The harmonic expansion is exact for an infinite degree and order. The truncation error results in the potential being in error. If the divergence is not zero at any point that is assumed to be vacuum, it is not possible to find a mass distribution that will have an external field that matches the truncated expansion. Another source of error is caused by aliasing. Aliasing is a consequence of the estimation process. Truncated harmonic expansion coefficients and incomplete tracking data are reflected into the estimated harmonic coefficients resulting in a distorted gravity field. The surface density model does not have a problem with the divergence being zero, provided the defined surface is exact, because the theory is not based on a potential function. There is an error associated with mass outside the defined surface where the divergence is clearly not zero. This error would be small for a reasonably accurate shape model.

The source of the maxima and minima observed in Figure 9 can be attributed to any of the above sources. Further analysis is needed. However, the evidence seems to point at the Earth harmonic expansion. Previous studies have suggested that large mass concentrations could cause a velocity perturbation of a few millimeters per second. Since a gravity field is conservative, the outgoing velocity after a planetary encounter would equal the incoming velocity if the effects of other bodies are neglected. A spacecraft will appear to acquire or lose some energy from the gravity assist associated with the orbit about the Sun and mass concentrations associated with mass anomalies. These mass anomalies are not modeled by the orbit determination filter and result in an anomolus change in velocity.

## VII NEAR and Galileo Earth Flyby Gravity Anomaly

Consider the large mass concentration shown in the southern hemisphere in Figure 9. The peak density of 5.52180 g/cm$^3$ is 0.005278 g/cm$^3$ above the Earth's mean density of 5.516522 g/cm$^3$. Recall that this density variation extends from the surface to the center of the Earth. If we assume uniform density, the mass element may be raised 6.06 km to give the same mass. If we assume the mass is distributed over an area defined by a 420 km circle, the total volume of the mass concentration is $3.358\times10^6$ km$^3$. At the mean density of the Earth, the gravitational constant of the mass concentration is 1.243 km$^3$/s$^2$.

The NEAR spacecraft and Galileo spacecraft, along with other spacecraft, experienced an anomolous velocity perturbation as they flew by the Earth. One possible explanation for these anomolies is an error in the Earth's gravity field model. The anomolous mass concentration observed above may be attributed to an error in the harmonic expansion used to compute the Earth's gravity. The geometry of a typical Earth flyby is illustrated on Figure 10. The left side of Figure 10 shows the Earth approach geometry. The y axis is in the direction of the Earth velocity vector and the x-y plane is the ecliptic plane. The angle $\gamma$ is between the spacecraft approach asymptote or velocity vector relative to the sun and Earth's velocity vector. The approach velocity relative to the Earth is $V_{\infty i}$, computed from osculating hyperbolic orbit elements at closest approach, and the direction is defined by the angle $\alpha$. If we assume that $V_{\infty i}$=6.803 km/s, the Earth's velocity vector magnitude $V_e$ is 29.84 km/s and $\alpha = 100$ deg, the angle $\gamma$ is 13.1 deg and $V_{in}$=29.4 km/s. The angle $\gamma$ and $V_{in}$ are computed from the laws of sines and cosines. Alternatively, a value could be assumed for $\gamma$ and $\alpha$ computed. The values selected for this approximate analysis are close to the values obtained during the Earth flyby of the NEAR spacecaft and are typical of gravity assist trajectories.

The right side of Figure 10 shows the Earth encounter geometry. The approach hyperbola has an impact parameter $b$ of 12,845 km and the impact parameter angle $\theta$, commonly referred to as the B plane angle, is 180 deg for a retrograde orbit about Earth and 0 deg for a direct orbit. These two values of $\theta$, not shown in Figure 10, keep the approach hyperbola in the x-y plane ($\delta = 0$). The angles $\alpha$ and $\delta$ define the direction of the approach asymptote in the coordinate system shown in Figure 10. In Earth Mean Equator of 2000 coordinates the direction of the approach asymptote is defined by right ascension and declination. A retrograde orbit increases the orbital energy with respect to the Sun and a direct orbit decreases the orbital energy. The energy change is reversed if the Earth is approached from inside Earth's orbit. The angle $\phi$ =56.2 deg is the half angle of the hyperbola and is computed from $b$, $V_\infty$ and the gravitational constant ($\mu$= 398,600 km$^3$/s$^2$) for the Earth,

$$e = \sqrt{1 + \frac{b^2 V_\infty^4}{\mu^2}}$$

$$\phi = \cos^{-1} \frac{1}{e}$$

The approach velocity vector relative to the Earth is thus,

$$\mathbf{V}_{\infty i} = [-V_\infty \sin\alpha,\ V_\infty \cos\alpha,\ 0] \tag{13}$$

The departure velocity vector relative to the Earth for a retrograde flyby is given by,

$$\mathbf{V}_{\infty o} = [V_\infty \sin(180 - 2\phi - \alpha)\cos(\delta),\ V_\infty \cos(180 - 2\phi - \alpha)\cos(\delta),\ \sin(\delta)] \tag{14}$$

and for a direct flyby,

$$\mathbf{V}_{\infty o} = [-V_\infty \sin(180 - 2\phi + \alpha)\cos(\delta),\ V_\infty \cos(180 - 2\phi + \alpha)\cos(\delta),\ \sin(\delta)] \tag{15}$$

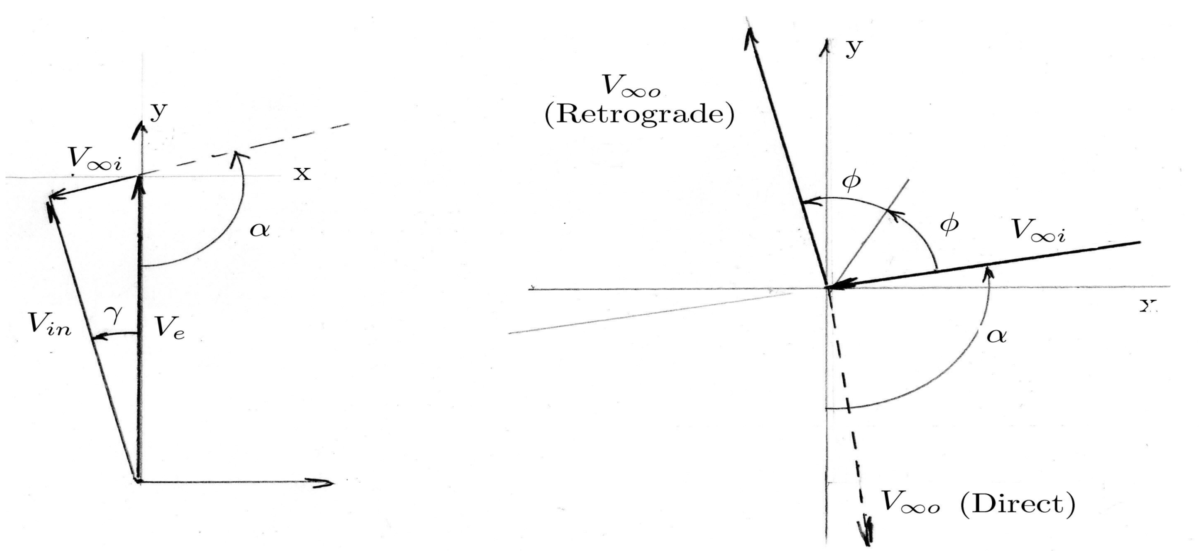


**Figure 10. Gravity Assist Geometry**

The velocity vector of the Earth relative to the Sun is given by

$$\mathbf{V}_e = [0,\ V_e,\ 0] \tag{16}$$

The approach velocity vector of the spacecraft relative to the Sun is obtained by summing Equations 13 and 16.

$$\mathbf{V}_{in} = [-V_\infty \sin\alpha,\ V_e + V_\infty \cos\alpha,\ 0] \tag{17}$$

The departure velocity vector for retrograde Earth flyby (Equations 14 plus 16) is

$$\mathbf{V}_{out} = [V_\infty \sin(180 - 2\phi - \alpha),\ V_e + V_\infty \cos(180 - 2\phi - \alpha),\ 0] \tag{18}$$

and for direct Earth flyby (Equations 15 plus 16),

$$\mathbf{V}_{out} = [-V_\infty \sin(180 - 2\phi + \alpha),\ V_e + V_\infty \cos(180 - 2\phi + \alpha),\ 0] \tag{19}$$

The velocity gain is simply the difference of the magnitudes

$$V_{gain} = V_{out} - V_{in} \tag{20}$$

The velocity gained ($V_{gain}$) computed from Equation 20 for the Earth flyby is 6.34 km/s. If we assume the velocity gained is proportional to the gravitational parameter, the mass concentration with a gravitational constant of 1.243 $km^3/s^2$ would contribute about 19.6 mm/s. Since the mass concentration is not at the center of mass of the Earth, the above result is only approximate. The above analytic formula also does not account for the Earth's rotation and solar tide. If the mass concentration is on the surface and the spacecraft flies directly over at an altitude of about 450 km, Equation 20 may be applied directly with b=450 km and the gravitational constant equal to 1.243 $km^3/s^2$ and the result is a perturbation of about 500 mm/s. A better result is to place the mass concentration at the center of mass of the pyramid shaped mass element about 2,290 km below the Earth's surface. Thus, for b=2,745 km, the velocity perturbation is 130 mm/s.

A much better analytic result may be obtained by integrating the equations of motion from several days before encounter until several days after encounter. The velocity gained may be obtained directly from the spacecraft velocity vectors relative to the Sun. The model includes the Sun's tidal acceleration and a mass concentration that is off the Earth spin axis and rotates at the Earth spin rate. The initial conditions are two-dimensional to be consistent with Equation 20. The Earth's equator is in the ecliptic plane and the mass concentration is placed 2,295 km below the Earth's equator. The Earth's gravity is reduced by the mass of the mass concentration. The spacecraft is timed to fly directly over the mass concentration at perigee. For the same geometry and parameters assumed for Equation 20, the velocity gained was 21.1 mm/s more than the result obtained with no mass concentration. This result is close to the anomolous perturbation of the spacecraft orbit observed by the author during the NEAR flyby. A better comparison would involve using the exact NEAR parameters and geometry and the velocity error would probably be much less since it is unlikely the NEAR spacecraft flew directly over the mass concentration used in this analysis.

## VI Conclusion

A gravity model has been developed that is based on the constant density gravity model developed for the NEAR mission to support navigation close to the asteroid Eros. An internal density distribution of a central body is defined that enables an exact computation of the acceleration of a spacecraft that is on or above the surface and achieves computational efficiency by converting a three dimensional volume integral into a surface integral. The internal density distribution may not be physically realizable, but the implied external gravity potential is exact. The surface density gravity model bypasses the potential function. Another important feature is the lack of singularities on or near the surface.

Several numerical experiments were defined to verify the veracity of the surface density model. First, a point mass was imbedded in shape models and the gravity model parameters determined by least square solution. These solutions verified that the surface density model is exact for a point mass and may be extended to a real body with an infinite number of point masses. Second, simulated data was generated from the Earth and Eros gravity model used during their respective flybys. These simulations revealed an apparent mass discrepancy that was attributed to inaccuracy in the coefficients of the harmonic expansion model used during flight operations to determine the orbit of the spacecraft. These simulations demonstrate that the so called gravity anomaly observed during the Galileo and NEAR Earth flybys were simply errors in the harmonic coefficients associated with aliasing and truncation. For a given defined shape, the surface density model is exact assuming the density is exact. The truncation error associated with the density calculation is assumed have a much smaller effect than the truncation error associated with gravity harmonics. This assumption requires further analysis.

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