Stability Analysis and H_∞ Controller Design of Discrete-Time Fuzzy Large-Scale Systems Based on Piecewise Lyapunov Functions

Hongbin Zhang, *Member, IEEE*, and Gang Feng, *Senior Member, IEEE*

***Abstract*—This paper is concerned with stability analysis and H_∞ decentralized control of discrete-time fuzzy large-scale systems based on piecewise Lyapunov functions. The fuzzy large-scale systems consist of J interconnected discrete-time Takagi–Sugeno (T–S) fuzzy subsystems, and the stability analysis is based on Lyapunov functions that are piecewise quadratic. It is shown that the stability of the discrete-time fuzzy large-scale systems can be established if a piecewise quadratic Lyapunov function can be constructed, and moreover, the function can be obtained by solving a set of linear matrix inequalities (LMIs) that are numerically feasible. The H_∞ controllers are also designed by solving a set of LMIs based on these powerful piecewise quadratic Lyapunov functions. It is demonstrated via numerical examples that the stability and controller synthesis results based on the piecewise quadratic Lyapunov functions are less conservative than those based on the common quadratic Lyapunov functions.**

***Index Terms*—Discrete-time, fuzzy large-scale system, H_∞ control, linear matrix inequalities (LMIs), stability, Takagi–Sugeno (T–S) model.**

I. Introduction

One of the foremost challenges to system theory brought forth for present-day technological, environmental, and societal processes is to overcome the increasing size and complexity of the relevant mathematical models. The methodologies of large-scale systems provide a technique through the manipulation of system structure in some way to overcome this problem. There were considerable interests in the research of large-scale systems in past years [1]. Many methods have been presented to investigate the stability and stabilization of large-scale systems [2]–[5].

During the past several years, fuzzy systems of the Takagi–Sugeno (T–S) model [6] have attracted great interests from the control community. The method of the T–S fuzzy model suggests an efficient method to represent complex nonlinear systems by fuzzy sets and fuzzy reasoning. Recently, the issue of stability and stabilization of fuzzy systems has been studied extensively [7]–[26], [37]–[40]. In particular, the stability analysis and stabilization of fuzzy large-scale systems for both discrete and continuous cases were discussed in [22] and [23], where the fuzzy large-scale system consists of J subsystems S_i, $i = 1, 2, \ldots, J$. The sufficient stability conditions were derived based on asymptotic stability of the existence of a common positive definite matrix P_i satisfying the Lyapunov equation or the LMI for each subsystem S_i.

However, a common quadratic Lyapunov function might not exist for many fuzzy systems, particularly for highly nonlinear complex systems. In fact, it has been known that many stable fuzzy systems do not admit such a common Lyapunov function [27], where a number of examples have been demonstrated. Instead, taking into consideration the information of membership functions, Johansson *et al.* [27] presented a stability result of continuous-time fuzzy system using a piecewise quadratic Lyapunov function, which is continuous across region boundaries of the state space. It is also demonstrated in the paper that the piecewise Lyapunov function is a much richer class of Lyapunov function candidates than the common Lyapunov function candidates, and thus, it is able to deal with a larger class of fuzzy dynamic systems. Similar work can be seen in [32], [33], and [34], where the authors consider the stability analysis and controller synthesis for both continuous- and discrete-time fuzzy dynamic systems based on piecewise quadratic Lyapunov functions. In fact, the common Lyapunov function is a special case of the more general piecewise Lyapunov function.

One of the most important requirements for a control system is the so-called robustness. In the past 20 years, considerable attention has been paid to the problems of robust stabilization and robust performance of uncertain dynamical systems. Since the pioneering work on the so-called H_∞ optimal control theory [35], there has been a considerable progress in H_∞ control theory. In [26], we consider the H_∞ control problem for continuous-time fuzzy large-scale systems based on the common Lyapunov function. However, to our best knowledge, the problem of H_∞ control for the discrete-time fuzzy large-scale systems remains to be investigated. In this paper, we address the issue of stability analysis and H_∞ control of discrete-time fuzzy large-scale systems based on piecewise quadratic Lyapunov functions. A stability condition is given in terms of linear matrix inequalities (LMIs) by using a vector Lyapunov function. The vector Lyapunov function is a sum of J piecewise quadratic functions, with each of which being constructed for each

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H. Zhang is with the Center for Nonlinear and Complex Systems, School of Electronic Engineering, University of Electronic Science and Technology of China, Chengdu 610054, China (e-mail: zhanghb@uestc.edu.cn).

G. Feng is with the Department of Manufacturing Engineering and Engineering Management, City University of Hong Kong, Kowloon, Hong Kong.

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subsystem, respectively, by using their structural information in the rule base. The major advantage of the proposed approach is that the piecewise quadratic stability analysis is much more powerful than the existing approaches based on the common Lyapunov function. Then, based on these powerful piecewise quadratic Lyapunov functions, we develop a new constructive H_∞ controller design method for the discrete-time fuzzy large-scale systems.

The rest of this paper is organized as follows. System descriptions and preliminaries are presented in Section II. Stability analysis of discrete-time fuzzy systems is presented in Section III. H_∞ decentralized controller design for such systems is considered in Section IV. In Section V, simulation examples are provided to demonstrate the effectiveness of our results. Finally, conclusions are given in Section VI.

Notations: The notations used are fairly standard. We denote by $z^T z$ or $\|z\|^2$ the square norm of a vector $z \in R^k$. The notation $l_2[0,T]$ is used for vector-valued functions, i.e., we say that $z:[0,N] \rightarrow R^n$ is in $l_2[0,N]$ if $\sum_{k=0}^{N} z^T(k)z(k) < \infty$ and its l_2 norm is defined as $\|z\|_2 = [\sum_{k=0}^{\infty} \|z\|^2]^{1/2}$. The symbol "*" in a matrix $A \in R^{n\times n}$ stands for the transposed elements in the symmetric positions.

II. System Descriptions and Preliminaries

Takagi and Sugeno have proposed a fuzzy model to represent complex nonlinear systems. This fuzzy dynamic model is described by fuzzy IF–THEN rules which represent local linear input–output relations of a nonlinear system, and in fact, it is proved that the Takagi–Sugeno fuzzy model is a universal approximator. Consider a fuzzy large-scale system S consisting of J interconnected fuzzy subsystems S_i, $i = 1, 2, \ldots, J$. The ith fuzzy subsystem S_i is described by the following T–S fuzzy model:

Rule j : IF $x_{i1}(k)$ is M_{i1j} and $\cdots$ and $x_{ig}(k)$ is M_{igj}, THEN

$$\begin{aligned} x_i(k+1) &= A_{ij}x_i(k) + B_{ij}u_i(k) \\ &\quad + D_{ij}v_i(k) + \sum_{\substack{n=1\\ n\neq i}}^{J} C_{ni}x_n(k) \\ y_i(k) &= H_{ij}x_i(k) + G_{ij}u_i(k) \end{aligned} \tag{1}$$

where $x_{i1}(k), x_{i2}(k), \ldots, x_{ig}(k)$ are the premise variables, $M_{ipj}(p = 1, 2, \ldots, g)$ are the fuzzy sets, $x_i(k) \in R^{g_i}$ is the state vector of the ith fuzzy subsystem, $u_i(k) \in R^{p_i}$ is the control signal, $v_i(k) \in R^{q_i}$ is the disturbance which belongs to $l_2[0,\infty)$, $(A_{ij}, B_{ij}, D_{ij}, H_{ij}, G_{ij})$ is the jth local model of the ith fuzzy subsystem, C_{ni} is the interconnection between the nth and ith subsystems, and r_i is the number of fuzzy implications. Through the use of "fuzzy blending," the final output of the ith fuzzy subsystem is inferred as follows:

$$\begin{aligned} x_i(k+1) &= \frac{\sum_{j=1}^{r_i} w_{ij}(k)\left[A_{ij}x_i(k) + B_{ij}u_i(k) + D_{ij}v_i(k)\right]}{\sum_{j=1}^{r_i} w_{ij}(k)} \\ &\quad + \sum_{\substack{n=1\\ n\neq i}}^{J} C_{ni}x_n(k) \\ &= \sum_{j=1}^{r_i} h_{ij}(k)\left[A_{ij}x_i(k) + B_{ij}u_i(k) + D_{ij}v_i(k)\right] \\ &\quad + \sum_{\substack{n=1\\ n\neq i}}^{J} C_{ni}x_n(k) \\ y_i(k) &= \frac{\sum_{j=1}^{r_i} w_{ij}(t)\left[H_{ij}x_i(k) + G_{ij}u_i(k)\right]}{\sum_{j=1}^{r_i} w_{ij}(k)} \\ &= \sum_{j=1}^{r_i} h_{ij}(k)\left[H_{ij}x_i(k) + G_{ij}u_i(k)\right] \end{aligned} \tag{2}$$

with

$$w_{ij}(k) = \prod_{p=1}^{g} M_{jpi}\left(x_{ip}(k)\right), \qquad h_{ij}(k) = \frac{w_{ij}(k)}{\sum_{j=1}^{r_i} w_{ij}(k)} \tag{3}$$

in which $M_{ipj}(x_{ip}(k))$ is the grade of membership function of $x_{ip}(k)$ in M_{ipj}. It is assumed that $w_{ij}(k) \geq 0$ for all k, $j = 1, 2, \ldots, r_i$. Therefore, the normalized membership function $h_{ij}(k)$ satisfies

$$h_{ij}(k) \geq 0, \qquad \sum_{j=1}^{r_i} h_{ij}(k) = 1, \quad \text{for all } k\text{'s}. \tag{4}$$

To end this section, we state the following lemmas which are useful for proving the main results.

Lemma 1: Given three matrices $A \in R^{m\times n}$, $B \in R^{m\times n}$, and $M \in R^{n\times n}$ and two positive definite matrices $P \in R^{m\times m}$ and $Q \in R^{n\times n}$ such that

$$A^T PA - Q + M < 0 \qquad B^T PB - Q + M < 0$$

then

$$A^T PB + B^T PA - 2Q + 2M < 0.$$

This lemma can be seen as an extension of [21, Lemma 3].

III. Piecewise Quadratic Stability

In this section, we consider the stability analysis of the large-scale fuzzy systems described in the last section. Following the idea of Zhang *et al.* [26] and Johansson *et al.* [27], we can rewrite the ith fuzzy subsystem in the form as (5), shown at the bottom of the next page, with $0 < h_{im} \leq 1$, $\sum_{m\in M(il)} h_{im} = 1$. For each cell, the set $M(il)$ contains the indices for the subsystem matrices used in the interpolation within that cell. For a crisp subspace, $M(il)$ contains a single element.

We consider a discrete-time piecewise quadratic Lyapunov function candidate for subsystem S_i as the following form:

$$V_i(k) = x_i^T P_{il} x_i, \qquad x \in X_{il}, l \in L_i. \tag{6}$$

The stability condition for the fuzzy large-scale system without control input and external disturbance can be summarized in the following theorem.

Theorem 1 (Stability Analysis): The fuzzy large-scale system composed of J fuzzy subsystems as (5) with $u_i \equiv v_i \equiv 0$, $i = 1, 2, \ldots, J$, is asymptotically stable if there exist symmetric positive definite matrices P_{il}, $P_{i0} \geq P_{il}$ for all $l \in L_i$ and positive constant ε_i satisfying (7), shown at the bottom of the page, for $i = 1, 2, \ldots, J$, $l \in L_i$, $l' \in L_i$, $m \in M(il')$.

Proof: Consider the following piecewise quadratic Lyapunov function $V(k)$ for the discrete-time fuzzy large-scale system:

$$V(k) = \sum_{i=1}^{J} V_i(k) = \sum_{i=1}^{J} x_i^T P_{il} x_i, \quad \text{for} \quad x_i \in X_{il}, l \in L_i. \tag{8}$$

Assuming that $x_i(k) \in X_{il'}$ and $x_i(k+1) \in X_{il}$, then we have (9), shown at the bottom of the page. Noticing that $P_{i0} \geq P_{il}$ for all $l \in L_i$, we have (10), shown at the bottom of the next page.

Using Schur's complement and Lemma 1, we know that LMIs (7) imply that

$$\Delta V(k) < 0, \qquad \text{for all } x_i(k) \neq 0,\ i = 1, 2, \ldots, J$$

and thus, it follows that the open-loop fuzzy large-scale system is asymptotically stable.

The aforementioned conditions are LMIs in the variables P_{il}, $l \in L_i$, and P_{i0}. A solution to those inequalities ensures $V(k)$ defined in (8) to be a Lyapunov function for the fuzzy large-scale system. When $l = l'$, the LMIs in (7) guarantee that the function decreases along all subsystems' trajectories within each cell partition. When $l \neq l'$, the LMI in (8) guarantees that

$$S_i : \begin{cases} x_i(k+1) = \sum\limits_{m \in M(il)} h_{im}(k) \left[A_{im} x_i(k) + B_{im} u_i(k) + D_{im} v_i(k) \right] + \sum\limits_{\substack{n=1 \\ n \neq i}}^{J} C_{ni} x_n(k), \qquad x_i \in X_{il} \\ y_i(k) = \sum\limits_{m \in M(il)} h_{im}(k) \left[H_{im} x_i(k) + G_{im} u_i(k) \right] \end{cases} \tag{5}$$

$$\begin{bmatrix} A_{im}^T P_{il} A_{im} + 2 \sum\limits_{\substack{n=1 \\ n \neq i}}^{J} \left(\varepsilon_n C_{in}^T C_{in} + C_{in} P_{n0} C_{in} \right) - P_{il'} & A_{im}^T P_{il} \\ P_{il} A_{im} & -\varepsilon_i I \end{bmatrix} < 0 \tag{7}$$

$$\begin{aligned}
\Delta V(k) &= \sum_{i=1}^{J} \Delta V_i(k) = \sum_{i=1}^{J} \left[x_i^T(k+1) P_{il} x_i(k+1) - x_i^T(k) P_{il'} x_i(k) \right] \\
&= \sum_{i=1}^{J} \left\{ \left[\sum_{m \in M(il')} h_{im}(k) A_{im} x_i(k) + \sum_{\substack{n=1 \\ n \neq i}}^{J} C_{ni} x_n(k) \right]^{\mathrm{T}} P_{il} \left[\sum_{m' \in M(il')} h_{im'}(k) A_{im'} x_i(k) + \sum_{\substack{n=1 \\ n \neq i}}^{J} C_{ni} x_n(k) \right] - x_i^T(k) P_{il'} x_i(k) \right\} \\
&= \sum_{i=1}^{J} \left\{ \left[\sum_{m \in M(il')} h_{im}(k) A_{im} x_i(k) \right]^{\mathrm{T}} P_{il} \left[\sum_{m' \in M(il')} h_{im'}(k) A_{im'} x_i(k) \right] \right. \\
&\qquad \left. + 2 \sum_{m \in M(il')} h_{im} x_i^T(k) A_{im}^T P_{il} \sum_{\substack{n=1 \\ n \neq i}}^{J} C_{ni} x_n(k) + \sum_{\substack{n=1 \\ n \neq i}}^{J} x_n^T(k) C_{ni}^T P_{il} \sum_{\substack{n=1 \\ n \neq i}}^{J} C_{ni} x_n(k) - x_i^T(k) P_{il'} x_i(k) \right\} \\
&\leq \sum_{i=1}^{J} \left\{ \left[\sum_{m \in M(il')} h_{im}(k) A_{im} x_i(k) \right]^{\mathrm{T}} P_{il} \left[\sum_{m' \in M(il')} h_{im'}(k) A_{im'} x_i(k) \right] + \varepsilon_i \sum_{\substack{n=1 \\ n \neq i}}^{J} x_n^T(k) C_{ni}^T \sum_{\substack{n=1 \\ n \neq i}}^{J} C_{ni} x_n(k) \right. \\
&\qquad + \varepsilon_i^{-1} \left[\sum_{m \in M(il')} h_{im}(k) x_i^T(k) A_{im}^T P_{il} P_{il} \sum_{m' \in M(il')} h_{im'}(k) A_{im'} x_i(k) \right] + \sum_{\substack{n=1 \\ n \neq i}}^{J} x_n^T(k) C_{ni}^T P_{il} \sum_{\substack{n=1 \\ n \neq i}}^{J} C_{ni} x_n(k) \\
&\qquad \left. - x_i^T(k) P_{il'} x_i(k) \right\}
\end{aligned} \tag{9}$$

the function decreases when the states of the subsystem transit from one cell partition to another. ■

Remark 1: It should be noted that when $P_{il} = P_{il'} = P_{i0} = P_i$, then Theorem 1 reduces to the common Lyapunov-function-based stability condition as in [22]. In fact, the common Lyapunov function is a special case of the more general piecewise Lyapunov function.

Remark 2: Due to the discrete nature of the system, the transitions could occur between nonadjacent cell partitions in one step. Thus, every cell partition pair (l, l') has to be computed in (7). The transitions may not occur between every cell partition pair for some special systems. How to refine the transition pairs for such systems' stability analysis is our future work.

IV. Piecewise H_∞ Performance Analysis and H_∞ Decentralized State-Feedback Controller Design

In this section, we first analyze the H_∞ disturbance attenuation performance for the open-loop discrete-time fuzzy large-scale system. Considering the discrete-time fuzzy large-scale system composed of J subsystems as (5) without control input, the ith subsystem S_i can be rewritten as (11), shown at the bottom of the page.

Now, we give the definition of H_∞ disturbance attenuation performance of a discrete-time fuzzy large-scale system as follows.

Definition 1: Given a constant $\gamma > 0$, the open-loop discrete-time fuzzy large-scale system composed of J subsystems as (11) is said to be stable with γ-disturbance attenuation if it is asymptotically stable and the output satisfies

$$\|y\|_2 < \gamma \|v\|_2 \tag{12}$$

for all nonzero $v(k) \in l_2[0, \infty)$ under the zero initial condition. Here, $y(k) = [y_1^T(k), y_2^T(k), \ldots, y_J^T(k)]^{\mathrm{T}}$, $v(k) = [v_1^T(k), v_2^T(k), \ldots, v_J^T(k)]^{\mathrm{T}}$.

$$\begin{aligned}
\Delta V(k) \le \sum_{i=1}^{J} \Bigg\{ & \left[\sum_{m\in M(il')} h_{im}(k) A_{im} x_i(k)\right]^{\mathrm{T}} P_{il} \left[\sum_{m'\in M(il')} h_{im'}(k) A_{im'} x_i(k)\right] \\
& + 2\sum_{\substack{n=1\\n\neq i}}^{J} \varepsilon_n x_i^T(k) C_{in}^T C_{in} x_i(k) + \varepsilon_i^{-1} \left[\sum_{m\in M(il')} h_{im}(k) x_i^T(k) A_{im}^T P_{il} P_{il} \sum_{m'\in M(il')} h_{im'}(k) A_{im'} x_i(k)\right] \\
& + 2\sum_{\substack{n=1\\n\neq i}}^{J} x_i^T(k) C_{in}^T P_{n0} C_{in} x_i(k) - x_i^T(k) P_{il'} x_i(k) \Bigg\} \\
= \sum_{i=1}^{J} \Bigg\{ & \sum_{m\in M(il')} \sum_{m'\in M(il')} h_{im} h_{im'} \Bigg[x_i^T(k) A_{im}^T P_{il} A_{im'} x_i^T(k) + 2\sum_{\substack{n=1\\n\neq i}}^{J} x_i^T(k) \left(\varepsilon_n C_{in}^T C_{in} + C_{in}^T P_{n0} C_{in}\right) x_i^T(k) \\
& + \varepsilon_i^{-1} x_i^T(k) A_{im}^T P_{il} P_{il} A_{im'} x_i(k) - x_i^T(k) P_{il'} x_i(k) \Bigg] \Bigg\} \\
= \sum_{i=1}^{J} \Bigg\{ & \sum_{m\in M(il')} h_{im}^2 x_i^T(k) \Bigg[A_{im}^T P_{il} A_{im} + 2\sum_{\substack{n=1\\n\neq i}}^{J} \left(\varepsilon_n C_{in}^T C_{in} + C_{in}^T P_{n0} C_{in}\right) + \varepsilon_i^{-1} A_{im}^T P_{il} P_{il} A_{im} - P_{il'} \Bigg] x_i(k) \\
& + \sum_{\substack{m\in M(il')\\m'\in M(il')\\m'<m}} h_{im} h_{im'} x_i^T(k) \Bigg[A_{im}^T P_{il} A_{im'} + A_{im'}^T P_{il} A_{im} + 4\sum_{\substack{n=1\\n\neq i}}^{J} \left(\varepsilon_n C_{in}^T C_{in} + C_{in}^T P_{n0} C_{in}\right) \\
& + \varepsilon_i^{-1} A_{im}^T P_{il} P_{il} A_{im'} + \varepsilon_i^{-1} A_{im'}^T P_{il} P_{il} A_{im} - 2P_{ilq'} \Bigg] x_i(k) \Bigg\}
\end{aligned} \tag{10}$$

$$S_i : \begin{cases} x_i(k+1) = \sum\limits_{m\in M(il)} h_{im}(k)\left[A_{im} x_i(k) + D_{im} v_i(k)\right] + \sum\limits_{\substack{n=1\\n\neq i}}^{J} C_{ni} x_n(k), \qquad x_i \in X_{il} \\ y_i(k) = \sum\limits_{m\in M(il)} h_{im}(k)\left[H_{im} x_i(k)\right] \end{cases} \tag{11}$$

Then, we are ready to present the following H_∞ performance analysis result.

Theorem 2 (H_∞ Performance Analysis): The fuzzy large-scale system composed of J fuzzy subsystems as (5) with $u_i \equiv 0$, $i = 1, 2, \ldots, J$ is stable with γ-disturbance attenuation performance if there exist symmetric positive definite matrices P_{il}, $P_{i0} \geq P_{il}$ for all $l \in L_i$ and positive constants ε_{i1} and ε_{i2} satisfying

$$\begin{bmatrix} (1,1) & A_{im}^T P_{il} D_{im} \\ D_{im}^T P_{il} A_{im} & -\gamma^2 I + D_{im}^T P_{il} D_{im} + \varepsilon_{i2}^{-1} D_{im}^T P_{il} D_{im} \end{bmatrix} < 0 \tag{13}$$

for $i = 1, 2, \ldots, J$, $l \in L_i$, $l' \in L_i$, $m \in M(il')$, where

$$\begin{aligned}(1,1) = & A_{im}^T P_{il} A_{im} + \varepsilon_{i1}^{-1} A_{im}^T P_{il} A_{im} \\ & + 2 \sum_{\substack{n=1 \\ n \neq i}}^{J} \left(C_{in}^T P_{n0} C_{in} + \varepsilon_{n1} C_{in}^T P_{n0} C_{in} + \varepsilon_{n2} C_{in}^T P_{n0} C_{in} \right) \\ & + H_{im}^T H_{im} - P_{il'}.\end{aligned} \tag{14}$$

Proof: Under zero initial condition and $u_i(k) \equiv 0$, $i = 1, 2, \ldots, J$, we have (15), shown at the bottom of the page.

We know that

$$\begin{aligned}\sum_{i=1}^{J} & \left\{ \left(\sum_{\substack{n=1 \\ n \neq i}}^{J} x_n^T(k) C_{ni}^T \right) P_{il} \left(\sum_{\substack{n=1 \\ n \neq i}}^{J} C_{ni} x_n(k) \right) \right\} \\ & \leq \sum_{i=1}^{J} \left\{ 2 \sum_{\substack{n=1 \\ n \neq i}}^{J} x_n^T(k) C_{ni}^T P_{il} C_{ni} x_n(k) \right\} \\ & \leq \sum_{i=1}^{J} \left\{ 2 \sum_{\substack{n=1 \\ n \neq i}}^{J} x_n^T(k) C_{ni}^T P_{i0} C_{ni} x_n(k) \right\} \\ & = \sum_{i=1}^{J} \left\{ 2 \sum_{\substack{n=1 \\ n \neq i}}^{J} x_i^T(k) C_{in}^T P_{n0} C_{in} x_n(k) \right\}.\end{aligned} \tag{16}$$

$$\begin{aligned}
\sum_{k=0}^{N-1} & \left\{ y^T(k) y(k) - \gamma^2 v^T(k) v(k) \right\} \leq \sum_{k=0}^{N-1} \left\{ y^T(k) y(k) - \gamma^2 v^T(k) v(k) + V(k+1) - V(k) \right\} \\
= & \sum_{k=0}^{N-1} \sum_{i=1}^{J} \Bigg\{ \left[\sum_{m \in M(il')} h_{im}(k) H_{im} x_i(k) \right]^{\mathrm{T}} \left[\sum_{m' \in M(il')} h_{im'}(k) H_{im'} x_i(k) \right] - \gamma^2 v_i^T(k) v_i(k) \\
& + \left[\sum_{m \in M(il')} h_{im}(k) \left[A_{im} x_i(k) + D_{im} v_i(k) \right] + \sum_{\substack{n=1 \\ n \neq i}}^{J} C_{ni} x_n(k) \right]^{\mathrm{T}} \\
& \times P_{il} \left[\sum_{m' \in M(il')} h_{im'}(k) \left[A_{im'} x_i(k) + D_{im'} v_i(k) \right] + \sum_{\substack{n=1 \\ n \neq i}}^{J} C_{ni} x_n(k) \right] - x_i^T(k) P_{il'} x_i(k) \Bigg\} \\
= & \sum_{k=0}^{N-1} \sum_{i=1}^{J} \Bigg\{ \sum_{m \in M(il')} \sum_{m' \in M(il')} h_{im}(k) h_{im'}(k) \left[x_i^T(k) H_{im}^T H_{im'} x_i(k) \right] - \gamma^2 v_i^T(k) v_i(k) \\
& + \sum_{m \in M(il')} \sum_{m' \in M(il')} h_{im}(k) h_{im'}(k) \left[x_i^T(k) A_{im}^T P_{il} A_{im'} x_i(k) \right] \\
& + \left(\sum_{\substack{n=1 \\ n \neq i}}^{J} x_n^T(k) C_{ni}^T \right) P_{il} \left(\sum_{\substack{n=1 \\ n \neq i}}^{J} C_{ni} x_n(k) \right) + \sum_{m \in M(il')} \sum_{m' \in M(il')} h_{im}(k) h_{im'}(k) \left[v_i^T(k) D_{im}^T P_{il} D_{im'} v_i(k) \right] \\
& + 2 \sum_{m \in M(il')} h_{im}(k) x_i^T(k) A_{im}^T P_{il} \left(\sum_{\substack{n=1 \\ n \neq i}}^{J} C_{ni} x_n(k) \right) \\
& + 2 \sum_{m \in M(il')} \sum_{m' \in M(il')} h_{im}(k) h_{im'}(k) \left[x_i^T(k) A_{im}^T P_{il} D_{im'} v_i(k) \right] \\
& + 2 \sum_{m \in M(il')} h_{im} \left(\sum_{\substack{n=1 \\ n \neq i}}^{J} x_n^T(k) C_{ni}^T \right) P_{il} D_{im} v_i(k) - x_i^T(k) P_{il'} x_i(k) \Bigg\}
\end{aligned} \tag{15}$$

We also know that (17) and (18), shown at the bottom of the page.

By substituting (16)–(18) into (15), we get that

$$\sum_{k=0}^{N-1}\left\{y^T(k)y(k)-\gamma^2 v^T(k)v(k)\right\} \leq \sum_{k=0}^{N-1}\sum_{i=1}^{J}\sum_{m\in M(il')}\sum_{m'\in M(il')} h_{im}(k)h_{im'}\bar{x}_i^T(k) \times\left(\bar{A}_{im}^T\bar{P}_{il}\bar{A}_{im'}-\hat{P}_{il'}+M\right)\bar{x}_i^T(k) \tag{19}$$

where $\bar{x}_i(k)$, M, $\bar{A}_{im}$, $\bar{P}_{il}$, and $\hat{P}_{il'}$ are shown at the bottom of the page.

Applying Lemma 1, we have

$$\sum_{k=0}^{N-1}\left\{y^T(k)y(k)-\gamma^2 v^T(k)v(k)\right\} \leq \sum_{k=0}^{N-1}\sum_{i=1}^{J}\sum_{m\in M(il')} h_{im}^2\bar{x}_i^T(k) \times\left(\bar{A}_{im}^T\bar{P}_{il}\bar{A}_{im}-\hat{P}_{il'}+M\right)\bar{x}_i^T(k). \tag{20}$$

Thus, $\sum_{k=0}^{N-1}\{y^T(k)y(k)-\gamma^2 v^T(k)v(k)\}<0$ holds for all $N>0$ if LMIs in (13) hold. It implies that $\|y\|_2<\gamma\|v\|_2$ for any nonzero $v\in l_2[0,\infty)$. Hence, the open-loop discrete-time fuzzy large-scale system is asymptotically stable, and the H_∞ disturbance attenuation performance in (12) is guaranteed for a prescribed γ.

$$\begin{aligned}
&\sum_{i=1}^{J}\left\{2\sum_{m\in M(il')} h_{im}(k)x_i^T(k)A_{im}^TP_{il}\left(\sum_{\substack{n=1\\n\neq i}}^{J}C_{ni}x_n(k)\right)\right\}\\
&\quad\leq\sum_{i=1}^{J}\left\{\varepsilon_{i1}\sum_{\substack{n=1\\n\neq i}}^{J}x_n^T(k)C_{ni}^TP_{il}\sum_{\substack{n=1\\n\neq i}}^{J}C_{ni}x_n(k)+\varepsilon_{i1}^{-1}\sum_{m\in M(il')}h_{im}(k)x_i^T(k)A_{im}^TP_{il}P_{il}^{-1}P_{il}\sum_{m'\in M(il')}h_{im'}(k)A_{im'}x_i(k)\right\}\\
&\quad\leq\sum_{i=1}^{J}\left\{2\sum_{\substack{n=1\\n\neq i}}^{J}\varepsilon_{i1}x_n^T(k)C_{ni}^TP_{i0}C_{ni}x_n(k)+\varepsilon_{i1}^{-1}\sum_{m\in M(il')}h_{im}(k)x_i^T(k)A_{im}^TP_{il}\sum_{m'\in M(il')}h_{im'}(k)A_{im'}x_i(k)\right\}\\
&\quad=\sum_{i=1}^{J}\left\{2\sum_{\substack{n=1\\n\neq i}}^{J}\varepsilon_{n1}x_i^T(k)C_{in}^TP_{n0}C_{in}x_i(k)+\varepsilon_{i1}^{-1}\sum_{m\in M(il')}h_{im}(k)x_i^T(k)A_{im}^TP_{il}\sum_{m'\in M(il')}h_{im'}(k)A_{im'}x_i(k)\right\}
\end{aligned}\tag{17}$$

$$\begin{aligned}
&\sum_{i=1}^{J}\left\{2\sum_{m\in M(il')} h_{im}(k)x_i^T(k)A_{im}^TP_{il}(\sum_{\substack{n=1\\n\neq i}}^{J}C_{ni}x_n(k))\right\}\\
&\quad\leq\sum_{i=1}^{J}\left\{\varepsilon_{i2}\sum_{\substack{n=1\\n\neq i}}^{J}x_n^T(k)C_{ni}^TP_{il}\sum_{\substack{n=1\\n\neq i}}^{J}C_{ni}x_n(k)+\varepsilon_{i2}^{-1}\sum_{m\in M(il')}h_{im}(k)v_i^T(k)D_{im}^TP_{il}P_{il}^{-1}P_{il}\sum_{m'\in M(il')}h_{im'}(k)D_{im'}v_i(k)\right\}\\
&\quad\leq\sum_{i=1}^{J}\left\{2\sum_{\substack{n=1\\n\neq i}}^{J}\varepsilon_{i2}x_n^T(k)C_{ni}^TP_{i0}C_{ni}x_n(k)+\varepsilon_{i2}^{-1}\sum_{m\in M(il')}h_{im}(k)v_i^T(k)D_{im}^TP_{il}\sum_{m'\in M(il')}h_{im'}(k)D_{im'}v_i(k)\right\}\\
&\quad=\sum_{i=1}^{J}\left\{2\sum_{\substack{n=1\\n\neq i}}^{J}\varepsilon_{n2}x_i^T(k)C_{in}^TP_{n0}C_{in}x_i(k)+\varepsilon_{i2}^{-1}\sum_{m\in M(il')}h_{im}(k)v_i^T(k)D_{im}^TP_{il}\sum_{m'\in M(il')}h_{im'}(k)D_{im'}v_i(k)\right\}
\end{aligned}\tag{18}$$

$$\bar{x}_i(k)=\begin{bmatrix}x_i^T(k) & v_i^T(k)\end{bmatrix}^{\mathrm{T}}$$

$$M=\begin{bmatrix}2\sum\limits_{\substack{n=1\\n\neq i}}^{J}\left(C_{in}^TP_{n0}C_{in}+\varepsilon_{n1}C_{in}^TP_{n0}C_{in}+\varepsilon_{n2}C_{in}^TP_{n0}C_{in}\right) & 0\\ 0 & 0\end{bmatrix}$$

$$\bar{A}_{im}=\begin{bmatrix}A_{im} & D_{im}\\ H_{im} & 0\\ A_{im} & 0\\ 0 & D_{im}\end{bmatrix}\quad \bar{P}_{il}=\begin{bmatrix}P_{il} & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & \varepsilon_{i1}^{-1}P_{il} & 0\\ 0 & 0 & 0 & \varepsilon_{i2}^{-1}P_{il}\end{bmatrix}\quad \hat{P}_{il'}=\begin{bmatrix}P_{il'} & 0\\ 0 & \gamma^2 I\end{bmatrix}$$

Next, we address the piecewise H_∞ decentralized controller synthesis problem for the discrete-time fuzzy large-scale system. The objective is to design a suitable controller for the discrete-time fuzzy large-scale system with a guaranteed H_∞ disturbance attenuation performance.

According to the decentralized control scheme, we consider the following piecewise controller for the ith subsystem S_i:

$$u_i(k) = K_{il} x_i(k), \qquad x_i \in X_{il}, l \in L_i. \tag{21}$$

Substituting (21) into (5) yields the ith closed-loop subsystem $\bar{S}_i$ in (22), shown at the bottom of the page, where $A^c_{im} = A_{im} + B_{im}K_{il}$, $H^c_{im} = H_{im} + G_{im}K_{il}$.

Then, we have the following H_∞ controller design result. ■

Theorem 3 (H_∞ Decentralized Controller Design): The closed-loop discrete-time fuzzy large-scale system composed of J fuzzy subsystems as (22) is stable with γ-disturbance attenuation performance if there exist matrix $Y_{il'}$, symmetric positive definite matrices R_i, R_{i0} $(R_{i0} - R_{i0}R^{-1}_{il}R_{i0} \ge 0)$, and positive constants ε_{i1} and ε_{i2} satisfying (23), shown at the bottom of the page, for $i = 1, 2, \ldots, J$, $l \in L_i$, $l' \in L_i$, $m \in M(il')$,where

$$\begin{aligned}
\hat{R}_{i0} &= \mathrm{diag}[R_{10}, \ldots, R_{n0,n\neq i}, \ldots, R_{J0}]\\
\hat{R}_{\varepsilon i1} &= \mathrm{diag}[\varepsilon_{11}R_{10}, \ldots, \varepsilon_{n1}R_{n0,n\neq i}, \ldots, \varepsilon_{J1}R_{J0}]^{\mathrm{T}}\\
\hat{R}_{\varepsilon i2} &= \mathrm{diag}[\varepsilon_{12}R_{10}, \ldots, \varepsilon_{n2}R_{n0,n\neq i}, \ldots, \varepsilon_{J2}R_{J0}]^{\mathrm{T}}\\
\hat{C}_i &= [R_{10}C^T_{i1}, \ldots, R_{n0,n\neq i}C^T_{in,n\neq i}, \ldots, R_{J0}C^T_{iJ}]^{\mathrm{T}}\\
\hat{C}_{\varepsilon i1} &= [\varepsilon_{11}R_{10}C^T_{i1}, \ldots, \varepsilon_{n1}R_{n0,n\neq i}C^T_{in,n\neq i}, \ldots, \varepsilon_{J1}R_{J0}C^T_{iJ}]^{\mathrm{T}}\\
\hat{C}_{\varepsilon i2} &= [\varepsilon_{12}R_{10}C^T_{i1}, \ldots, \varepsilon_{n2}R_{n0,n\neq i}C^T_{in,n\neq i}, \ldots, \varepsilon_{J2}R_{J0}C^T_{iJ}]^{\mathrm{T}}.
\end{aligned}$$

Moreover, the piecewise state-feedback law for each subsystem is given by

$$K_{il'} = Y_{il'}R^{-1}_{il'}, \qquad l' \in L_i, \quad i = 1, 2, \ldots, J. \tag{24}$$

Proof: Based on the result in Theorem 2 and its proof, we learn that the fuzzy large-scale system composed of J subsystems as (22) is globally stable with γ-disturbance attenuation if there exist symmetric positive definite matrices P_{il}, $P_{i0} \ge P_{il}$ for all $l \in L_i$, and positive constants ε_{i1} and ε_{i2}, satisfying the following inequality:

$$\begin{bmatrix} (1,1) & (A^c_{im})^T P_{il}D_{im} \\ D^T_{im}P_{il}(A^c_{im}) & -\gamma^2 I + D^T_{im}P_{il}D_{im} + \varepsilon^{-1}_{i2}D^T_{im}P_{il}D_{im} \end{bmatrix} < 0 \tag{25}$$

for $i = 1, 2, \ldots, J$, $l \in L_i$, $l' \in L_i$, $m \in M(il')$, where

$$\begin{aligned}
(1,1) = {} & (A^c_{im})^T P_{il}(A^c_{im}) + \varepsilon^{-1}_{i1}(A^c_{im})^T P_{il}(A^c_{im})\\
& + 2\sum_{\substack{n=1\\n\neq i}}^{J}\left(C^T_{in}P_{n0}C_{in} + \varepsilon_{n1}C^T_{in}P_{n0}C_{in} + \varepsilon_{n2}C^T_{in}P_{n0}C_{in}\right)\\
& + (H^c_{im})^T(H^c_{im}) - P_{il'}.
\end{aligned} \tag{26}$$

We will show that the inequality (23) implies (25). Multiplying the both sides of (23) with

$$\mathrm{diag}\{P_{il}, I, P_{il}, P_{il}, P_{il}, I, \widehat{P}_{i0}, \widehat{P}_{i0}, \widehat{P}_{i0}\}$$

and noticing the fact that $R_{il} = P^{-1}_{il}$, $R_{i0} = P^{-1}_{i0}$, $Y_{il'} = K_{il'}P^{-1}_{il'}$, and $\hat{P}_{i0} = \mathrm{diag}[P_{10}, \ldots, P_{n0,n\neq i}, \ldots, P_{J0}]$, we have (27), shown at the bottom of the next page, where

$$\begin{aligned}
\hat{P}_{\varepsilon i1} &= \mathrm{diag}[\varepsilon_{11}P_{10}, \ldots, \varepsilon_{n1}P_{n0,n\neq i}, \ldots, \varepsilon_{J1}P_{J0}]^{\mathrm{T}}\\
\hat{P}_{\varepsilon i2} &= \mathrm{diag}[\varepsilon_{12}P_{10}, \ldots, \varepsilon_{n2}P_{n0,n\neq i}, \ldots, \varepsilon_{J2}P_{J0}]^{\mathrm{T}}\\
\tilde{C}_i &= [C^T_{i1}P_{10}, \ldots, C^T_{in,n\neq i}P_{n0,n\neq i}, \ldots, C^T_{iJ}P_{J0}]^{\mathrm{T}}\\
\tilde{C}_{\varepsilon i1} &= [\varepsilon_{11}C^T_{i1}P_{10}, \ldots, \varepsilon_{n1}C^T_{in,n\neq i}P_{n0,n\neq i}, \ldots, \varepsilon_{J1}C^T_{iJ}P_{J0}]^{\mathrm{T}}\\
\tilde{C}_{\varepsilon i2} &= [\varepsilon_{12}C^T_{i1}P_{10}, \ldots, \varepsilon_{n2}C^T_{in,n\neq i}P_{n0,n\neq i}, \ldots, \varepsilon_{J2}C^T_{iJ}P_{J0}]^{\mathrm{T}}.
\end{aligned}$$

$$\bar{S}_i : \begin{cases} x_i(k+1) = \sum\limits_{m\in M(il)} h_{im}(k)\left[A^c_{im}x_i(k) + D_{im}v_i(k)\right] + \sum\limits_{\substack{n=1\\n\neq i}}^{J} C_{ni}x_n(k), & x_i \in X_{il}\\ y_i(k) = \sum\limits_{m\in M(il)} h_{im}(k)\left[H^c_{im}x_i(k)\right] \end{cases} \tag{22}$$

$$\begin{bmatrix}
-R_{il'} & 0 & R_{il'}A^T_{im} + Y^T_{il'}B^T_{im} & 0 & R_{il'}A^T_{im} + Y^T_{il'}B^T_{im} & R_{il'}H^T_{im} + Y^T_{il'}G^T_{im} & \hat{C}^T_i & \hat{C}^T_{\varepsilon i1} & \hat{C}^T_{\varepsilon i2}\\
* & -\gamma^2 I & 0 & D^T_{im} & D^T_{im} & 0 & 0 & 0 & 0\\
* & * & -\varepsilon_{i1}R_{il} & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & -\varepsilon_{i2}R_{il} & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & -R_{il} & 0 & 0 & 0 & 0\\
* & * & * & * & * & -I & 0 & 0 & 0\\
* & * & * & * & * & * & -\frac{1}{2}\hat{R}_{i0} & 0 & 0\\
* & * & * & * & * & * & * & -\frac{1}{2}\hat{R}_{\varepsilon i1} & 0\\
* & * & * & * & * & * & * & * & -\frac{1}{2}\hat{R}_{\varepsilon i2}
\end{bmatrix} < 0 \tag{23}$$

Using Schur's complement, we know that (27) implies (25). Thus, we have shown that (23) implies (25). Therefore, it can be concluded that the closed-loop discrete-time fuzzy large-scale system composed of J fuzzy subsystems as (22) is globally stable with a guaranteed H_∞ performance, and thus, the proof is completed. ■

Remark 3: Due to the physical configuration and high dimensionality of large-scale systems, a centralized control is neither economically feasible nor even necessary. The decentralized control scheme attempts to avoid difficulties in complexity of design, debugging, data gathering, and storage requirements. Therefore, the decentralized scheme is preferred in control design of the large-scale interconnected systems. In general, due to the interactions among subsystems, it is difficult to design an H_∞ decentralized controller for nonlinear interconnected systems. In this paper, a new decentralized H_∞ control scheme is developed for discrete-time fuzzy large-scale systems based on piecewise Lyapunov functions. This thus suggests an approach to the more difficult H_∞ decentralized controller design for general discrete-time nonlinear interconnected systems.

V. Numerical Examples

Example 1: First, we consider the stability of the following discrete-time fuzzy large-scale system without input, output, or disturbance. Consider a fuzzy large-scale system S composed of three fuzzy subsystems S_i as follows.

Subsystem S_i:

Rule 1) IF $x_{i1}(k)$ is M_{i11}, THEN

$$x_i(k+1) = A_{i1}x_i(k) + \sum_{\substack{n=1\\n\neq i}}^{J} C_{ni}x_n(k).$$

Rule 2) IF $x_{i1}(k)$ is M_{i12}, THEN

$$x_i(k+1) = A_{i2}x_i(k) + \sum_{\substack{n=1\\n\neq i}}^{J} C_{ni}x_n(k).$$

Rule 3) IF $x_{i1}(k)$ is M_{i13}, THEN

$$x_i(k+1) = A_{i3}x_i(k) + \sum_{\substack{n=1\\n\neq i}}^{J} C_{ni}x_n(k)$$

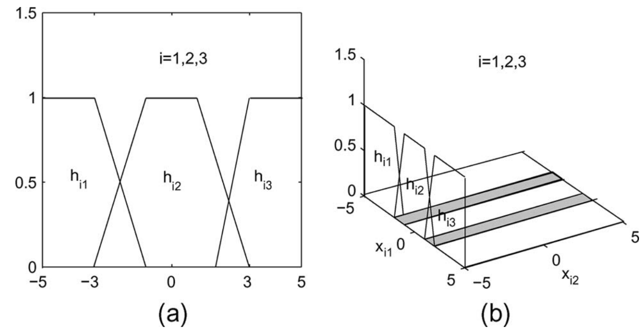

Fig. 1. (a) Normalized membership functions. (b) Cell partitions of the state space.

with $i = 1, 2, 3$. Where $x_i(k) = [x_{i1}(k), x_{i2}(k)]^{\mathrm{T}}$

$$A_{11} = \begin{bmatrix} 1 & 0.5 \\ -0.3 & 0.8 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 1 & 0.1 \\ -0.1 & 0.8 \end{bmatrix}$$
$$A_{13} = \begin{bmatrix} 0.9 & -0.35 \\ 0.2 & 0.8 \end{bmatrix} \quad A_{21} = \begin{bmatrix} 0.7 & 0.1 \\ -0.2 & 0.7 \end{bmatrix}$$
$$A_{22} = \begin{bmatrix} 0.5 & -0.2 \\ 0.2 & 0.5 \end{bmatrix} \quad A_{23} = \begin{bmatrix} 0.5 & -0.1 \\ -0.2 & 0.7 \end{bmatrix}$$
$$A_{31} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.7 \end{bmatrix} \quad A_{32} = \begin{bmatrix} 0.8 & -0.1 \\ 0.1 & 0.8 \end{bmatrix}$$
$$A_{33} = \begin{bmatrix} 0.9 & 0.1 \\ -0.2 & 0.7 \end{bmatrix} \quad C_{21} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$
$$C_{31} = \begin{bmatrix} -0.01 & 0 \\ 0 & 0 \end{bmatrix} \quad C_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0.01 \end{bmatrix}$$
$$C_{32} = \begin{bmatrix} 0 & 0 \\ 0 & -0.01 \end{bmatrix} \quad C_{13} = \begin{bmatrix} 0 & 0.01 \\ 0 & 0 \end{bmatrix}$$
$$C_{23} = \begin{bmatrix} -0.01 & 0 \\ 0 & 0 \end{bmatrix}.$$

The normalized membership functions and cell partitions of each subsystem are the same and shown in Fig. 1.

It is noted that there is no solution to the stability analysis approach based on the common quadratic Lyapunov function, although the simulation results shown in Figs. 2–4 indicate that this fuzzy large-scale system is stable. That is, if a common positive definite matrix P_i is used in Theorem 1, then there is no feasible solution to those LMIs. In fact, by using the MATLAB LMI toolbox, one can easily verify that there exists no common positive definite matrix P_1 for the isolated subsystem 1 to show its stability. Therefore, the method in [22] cannot be used to show the stability of this fuzzy large-scale system. However,

$$\begin{bmatrix}
-P_{il'} & 0 & (A_{im}^c)^T P_{il} & 0 & (A_{im}^c)^T P_{il} & (H_{im}^c)^T & \tilde{C}_i^T & \tilde{C}_{\varepsilon i1}^T & \tilde{C}_{\varepsilon i2}^T \\
* & -\gamma^2 I & 0 & D_{im}^T P_{il} & D_{im}^T P_{il} & 0 & 0 & 0 & 0 \\
* & * & -\varepsilon_{i1}P_{il} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\varepsilon_{i2}P_{il} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -P_{il} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -I & 0 & 0 & 0 \\
* & * & * & * & * & * & -\frac{1}{2}\hat{P}_{i0} & 0 & 0 \\
* & * & * & * & * & * & * & -\frac{1}{2}\hat{P}_{\varepsilon i1} & 0 \\
* & * & * & * & * & * & * & * & -\frac{1}{2}\hat{P}_{\varepsilon i2}
\end{bmatrix} < 0 \tag{27}$$

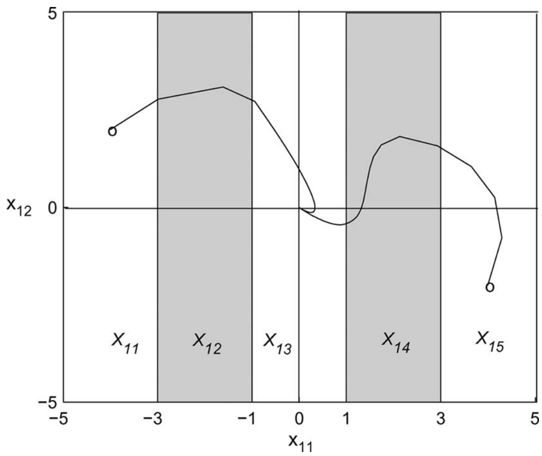

Fig. 2. Trajectories from two initial conditions and cell partitions of subsystem S_1.

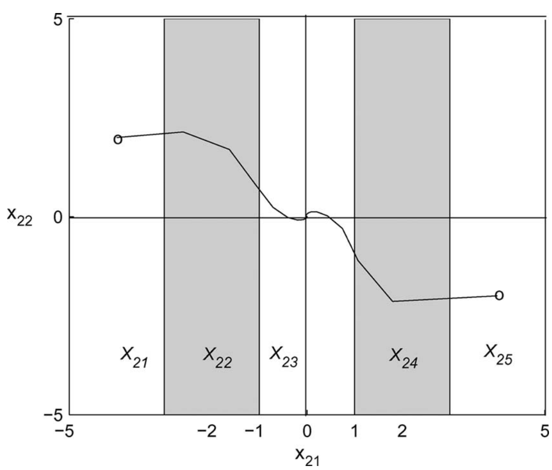

Fig. 3. Trajectories from two initial conditions and cell partitions of subsystem S_2.

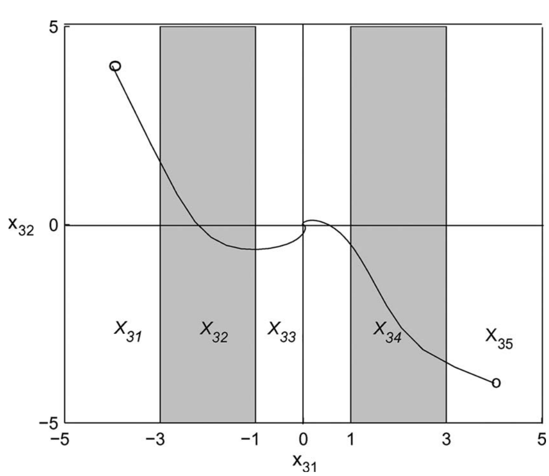

Fig. 4. Trajectories from two initial conditions and cell partitions of subsystem S_3.

based on the proposed approach, we can have the following cell partitions for the three subsystems.

Subsystem S_i:

$$x_{i1} = [-5, -3), [-3, -1), [-1, 1), [1, 3), [3, 5]$$

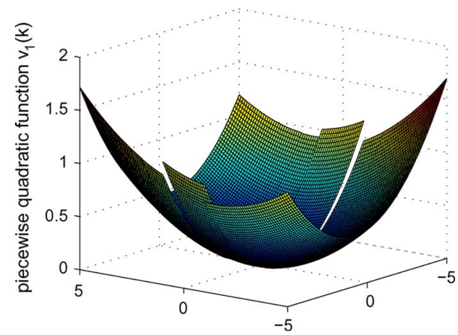

Fig. 5. Surface of piecewise quadratic function $V_1(k)$.

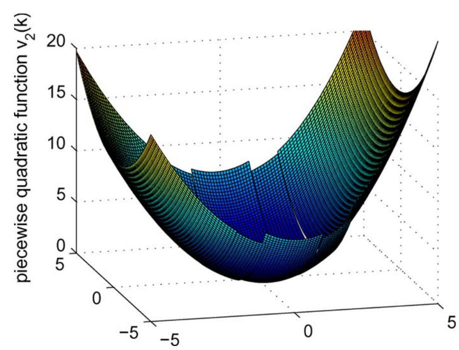

Fig. 6. Surface of piecewise quadratic function $V_2(k)$.

which are denoted by X_{i1}, X_{i2}, X_{i3}, X_{i4}, and X_{i5}, respectively, where $i = 1, 2, 3$. Using Theorem 1, a piecewise quadratic Lyapunov function $V(k)$ can be found so that the asymptotic stability of the origin of the fuzzy large-scale system is ensured. Furthermore, the solutions to those LMIs are given as

$$P_{11} = \begin{bmatrix} 0.0200 & 0.0062 \\ 0.0062 & 0.0356 \end{bmatrix} \quad P_{12} = \begin{bmatrix} 0.0200 & 0.0061 \\ 0.0061 & 0.0355 \end{bmatrix}$$

$$P_{13} = \begin{bmatrix} 0.0204 & 0.0074 \\ 0.0074 & 0.0425 \end{bmatrix} \quad P_{14} = \begin{bmatrix} 0.0195 & 0.0052 \\ 0.0052 & 0.0385 \end{bmatrix}$$

$$P_{15} = \begin{bmatrix} 0.0195 & 0.0052 \\ 0.0052 & 0.0387 \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} 0.4941 & -0.0143 \\ -0.0143 & 0.2641 \end{bmatrix} \quad P_{22} = \begin{bmatrix} 0.4831 & -0.0095 \\ -0.0095 & 0.2578 \end{bmatrix}$$

$$P_{23} = \begin{bmatrix} 0.6291 & 0.0210 \\ 0.0210 & 0.2845 \end{bmatrix} \quad P_{24} = \begin{bmatrix} 0.5673 & -0.0101 \\ -0.0101 & 0.2564 \end{bmatrix}$$

$$P_{25} = \begin{bmatrix} 0.6945 & -0.0337 \\ -0.0337 & 0.2645 \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} 0.1282 & -0.0584 \\ -0.0584 & 0.1273 \end{bmatrix} \quad P_{32} = \begin{bmatrix} 0.1133 & -0.0371 \\ -0.0371 & 0.0968 \end{bmatrix}$$

$$P_{33} = \begin{bmatrix} 0.1261 & -0.0262 \\ -0.0262 & 0.1054 \end{bmatrix} \quad P_{34} = \begin{bmatrix} 0.1009 & -0.0148 \\ -0.0148 & 0.0977 \end{bmatrix}$$

$$P_{35} = \begin{bmatrix} 0.1006 & -0.0124 \\ -0.0124 & 0.1242 \end{bmatrix}$$

and $\varepsilon_1 = 12.4900$, $\varepsilon_2 = 0.7695$, and $\varepsilon_3 = 2.6112$. The surfaces of the corresponding piecewise quadratic functions $V_1(k)$, $V_2(k)$, and $V_3(k)$ are shown in Figs. 5–7, respectively.

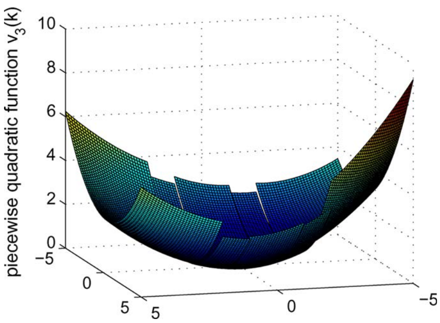

Fig. 7. Surface of piecewise quadratic function $V_3(k)$.

It is observed that this example does not admit a common quadratic Lyapunov function but admits a piecewise quadratic Lyapunov function $V(k)$, and thus, it can be expected that the proposed stability analysis approach based on the piecewise quadratic Lyapunov function is less conservative and more powerful than the existing approaches based on the common quadratic Lyapunov function.

Example 2: To illustrate the controller synthesis, we consider the following problem of balancing double-inverted pendulums connected by a spring [26]. This system composed of two interconnected subsystems can be described by the following:

$$\begin{aligned}\dot{x}_{11}(t) &= x_{12}\\ \dot{x}_{12}(t) &= \frac{m_1 g r}{J_1}\sin(x_{11}) - \frac{k}{J_1}x_{11} + \frac{u_1}{J_1} + \frac{k}{J_1}x_{21} + \frac{v_1}{J_1}\\ \dot{x}_{21}(t) &= x_{22}\\ \dot{x}_{22}(t) &= \frac{m_2 g r}{J_2}\sin(x_{21}) - \frac{k}{J_2}x_{21} + \frac{u_2}{J_2} + \frac{k}{J_2}x_{11} + \frac{v_2}{J_2}\end{aligned} \tag{28}$$

where $x_{i1}(t)$ is the angular displacement of the ith pendulum from the vertical reference. Each pendulum may be positioned by a torque input $u_i(t)$ applied by a servomotor at its base. $v_1(t) = 10\sin(4\pi t)\cdot e^{-0.02t}$ and $v_2(t) = 10\sin(2\pi t)\cdot e^{-0.02t}$ are the torque disturbances. It is assumed that both $x_{i1}(t)$ and $\dot{x}_{i1}(t)$ (angular position and velocity, respectively) are available to the ith controller for $i = 1, 2$. The end masses of the pendulums are $m_1 = 2$ kg and $m_2 = 2.5$ kg, the moments of inertia are $J_1 = 2$ kg and $J_2 = 2.5$ kg, respectively, the constant of the connecting torsional spring is $k = 2$ N · m/rad, the pendulum height is $r = 1$ m, the gravitational acceleration is $g = 9.81$ m/s^2, and the torsional spring is relaxed when the pendulums are all in the upright position. Thus, the origin $x_{11} = x_{12} = x_{21} = x_{22} = 0$ is the equilibrium point of this system. With a sampling period $T = 0.005$ s, the discrete-time nonlinear dynamic equation of subsystem S_i can be written as (29), shown at the bottom of the page. Here, the objective is to drive the angular positions to zero with a given disturbance attenuation γ. To apply the proposed fuzzy decentralized control approach, we represent the discrete-time nonlinear large-scale system by a fuzzy model via local approximation in fuzzy partition spaces [7]. Linearizing the subsystem at three points $x_{i1}(k) = 0, \pm 88^\circ$, we obtain the following three-rule fuzzy model for subsystem S_i.

Subsystem S_i:

Rule 1) IF $x_{i1}(k)$ is about -88°, THEN

$$\begin{aligned} x_i(k+1) &= A_{i1}x_i(k) + B_{i1}u_i(k)\\ &\quad + D_{i1}v_i(k) + \sum_{n=1,n\neq i}^{2} C_{ni}x_n(k)\\ y_i(k) &= H_{i1}x_i(k).\end{aligned}$$

Rule 2) IF $x_{i1}(k)$ is about zero, THEN

$$\begin{aligned} x_i(k+1) &= A_{i2}x_i(k) + B_{i2}u_i(k)\\ &\quad + D_{i2}v_i(k) + \sum_{n=1,n\neq i}^{2} C_{ni}x_n(k)\\ y_i(k) &= H_{i2}x_i(k).\end{aligned}$$

Rule 3) IF $x_{i1}(k)$ is about $+88^\circ$, THEN

$$\begin{aligned} x_i(k+1) &= A_{i3}x_i(k) + B_{i3}u_i(k)\\ &\quad + D_{i3}v_i(k) + \sum_{n=1,n\neq i}^{2} C_{ni}x_n(k)\\ y_i(k) &= H_{i3}x_i(k)\end{aligned}$$

where

$$A_{11} = A_{13} = \begin{bmatrix} 1.0000 & 0.0050\\ 0.02620 & 1.0000\end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 1.0000 & 0.0050\\ 0.0441 & 1.0000\end{bmatrix}$$

$$A_{21} = A_{23} = \begin{bmatrix} 1.0000 & 0.0050\\ 0.0272 & 1.0000\end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 1.0000 & 0.0050\\ 0.0451 & 1.0000\end{bmatrix}$$

$$B_{11} = B_{12} = B_{13} = \begin{bmatrix} 0\\ 0.0025\end{bmatrix}$$

$$B_{21} = B_{22} = B_{23} = \begin{bmatrix} 0\\ 0.0020\end{bmatrix}$$

$$S_i : \begin{cases} x_{i1}(k+1) = x_{i1}(k) + Tx_{i2}(k)\\ x_{i2}(k+1) = x_{i2}(k) + T\left[\frac{m_i g r}{J_i}\sin\left(x_{i1}(k)\right) - \frac{k}{J_i}x_{i1}(k) + \frac{u_i}{J_i} + \sum_{n=1,n\neq i}^{2}\frac{k}{J_i}x_{n1}(k) + \frac{v_i(k)}{J_i}\right]\\ y_i(k) = x_{i1}(k).\end{cases} \tag{29}$$

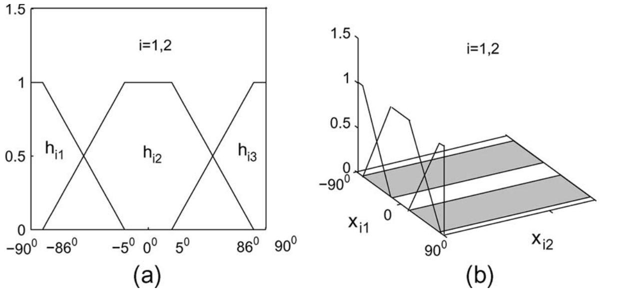

Fig. 8. (a) Normalized membership functions. (b) Cell partitions of the state space.

$$C_{21} = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0050 & 0.0000 \end{bmatrix} \quad C_{12} = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0040 & 0.0000 \end{bmatrix}$$

$$D_{11} = D_{12} = D_{13} = \begin{bmatrix} 0.0000 \\ 0.0025 \end{bmatrix}$$

$$D_{21} = D_{22} = D_{23} = \begin{bmatrix} 0.0000 \\ 0.0020 \end{bmatrix}$$

$$H_{11} = H_{12} = H_{13} = H_{21} = H_{22} = H_{23} = [\,1.0000 \quad 0.0000\,].$$

The normalized membership functions for Rules 1)–3) of the two subsystems are chosen to be same and shown in Fig. 8. The working range of state is $x_{i1} = [-90°, 90°]$, $i = 1, 2$. According to membership functions of the two subsystems, we can have the following cell partitions for the two subsystems.

Subsystem S_i:

$$x_{i1} = [-1.5708, -1.5010), [-1.501, -0.0873), [-0.0873, 0.0873), [0.0873, 1.5010), [1.5010, 1.5708]$$

which are denoted by X_{i1}, X_{i2}, X_{i3}, X_{i4}, and X_{i5}, respectively, where $i = 1, 2$.

We set the disturbance attenuation $\gamma = 0.01$. Using Theorem 3, the following solutions to LMIs (24) can be found when $\varepsilon_{i1} = \varepsilon_{i2} = 100$, where $i = 1, 2$:

$$R_{11} = R_{15} = \begin{bmatrix} 0.0145 & -0.0561 \\ -0.0561 & 0.5547 \end{bmatrix}$$

$$R_{12} = R_{14} = \begin{bmatrix} 0.0145 & -0.0555 \\ -0.0555 & 0.5047 \end{bmatrix}$$

$$R_{13} = \begin{bmatrix} 0.0145 & -0.0560 \\ -0.0560 & 0.5458 \end{bmatrix}$$

$$R_{21} = R_{25} = \begin{bmatrix} 0.0145 & -0.0557 \\ -0.0557 & 0.5517 \end{bmatrix}$$

$$R_{22} = R_{24} = \begin{bmatrix} 0.0144 & -0.0551 \\ -0.0551 & 0.5023 \end{bmatrix}$$

$$R_{23} = \begin{bmatrix} 0.0145 & -0.0557 \\ -0.0557 & 0.5429 \end{bmatrix}$$

$$Y_{11} = Y_{15} = [0.1646 - 150.2584]$$

$$Y_{12} = Y_{14} = [0.0972 - 131.2850]$$

$$Y_{13} = [0.0612 - 146.3731]$$

$$Y_{22} = Y_{25} = [0.1989 - 187.1641]$$

$$Y_{22} = Y_{24} = [0.1143 - 163.7319]$$

$$Y_{23} = [0.0705 - 182.3645].$$

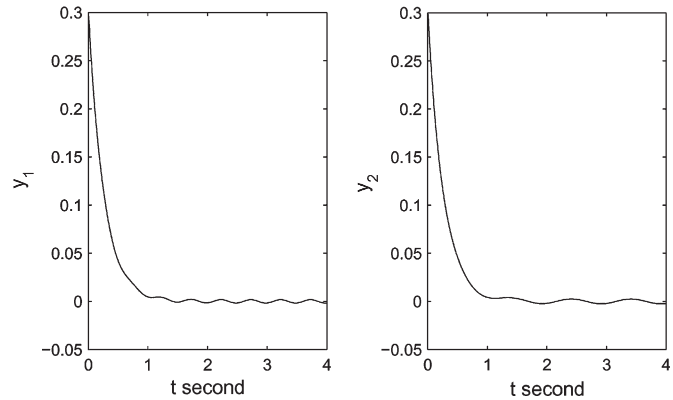

Fig. 9. Output responses of the closed-loop system with the presence of disturbances.

Subsequently, we can obtain the feedback gains

$$K_{11} = K_{15} = [-1690 - 442]$$

$$K_{12} = K_{14} = [-1938 - 511]$$

$$K_{13} = [-1732 - 453]$$

$$K_{22} = K_{25} = [-2117 - 553]$$

$$K_{22} = K_{24} = [-2424 - 639]$$

$$K_{23} = [-2169 - 567].$$

When the initial condition is $x_1(0) = [0.3, 0]^{\mathrm{T}}$, $x_2(0) = [0.3, 0]^{\mathrm{T}}$, Fig. 9 shows the output responses of the closed-loop discrete-time nonlinear large-scale system with the presence of disturbances. It can be observed that the piecewise controller proposed in this paper based on the fuzzy dynamic model not only stabilizes the original nonlinear large-scale system but also effectively attenuates the disturbances as expected.

VI. Conclusion

In this paper, a new method is developed to test stability of discrete-time fuzzy large-scale systems based on piecewise quadratic Lyapunov functions. It is shown that the stability result based on the piecewise quadratic Lyapunov functions is less conservative than that based on the common quadratic Lyapunov functions. Moveover, based on these powerful piecewise Lyapunov functions, a new procedure is developed to design H_∞ controller for discrete-time fuzzy large-scale systems. Two examples are presented to demonstrate the advantage of the proposed approach.

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Hongbin Zhang (S'06–M'06) received the B.Eng. degree in aerocraft design from Northwestern Polytechnical University, Xi'an, China, in 1999, and the M.Eng. and Ph.D. degrees in circuits and systems from the University of Electronic Science and Technology of China, Chengdu, in 2002 and 2006, respectively.

He has been with the School of Electrical Engineering, University of Electronic Science and Technology of China, since 2002, where he is currently an Associate Professor and with the Center for Nonlinear and Complex Systems. His current research interests include fuzzy control, stochastic control, and time-delay control systems.

Gang Feng (SM'95) received the B.Eng. and M.Eng. degrees in automatic control (of electrical engineering) from the Nanjing Aeronautical Institute, Nanjing, China, in 1982 and 1984, respectively, and the Ph.D. degree in electrical engineering from The University of Melbourne, Melbourne, Australia, in 1992.

He was a Lecturer/Senior Lecturer with the School of Electrical Engineering, The University of New South Wales, Sydney, Australia, from 1992 to 1999. He was a Visiting Fellow with the National University of Singapore, Singapore, in 1997, and with the Aachen Technology University, Aachen, Germany, in 1997–1998. He has been with City University of Hong Kong, Kowloon, Hong Kong, since 2000, where he is currently a Professor with the Department of Manufacturing Engineering and Engineering Management. His current research interests include piecewise linear systems, robot networks, and intelligent systems and control. He is an Associate Editor for the *Journal of Control Theory and Applications*.

Prof. Feng is an Associate Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL and the IEEE TRANSACTIONS ON FUZZY SYSTEMS, and he was an Associate Editor for the IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS—PART C and the Conference Editorial Board of the IEEE Control System Society. He was awarded an Alexander von Humboldt Fellowship in 1997–1998 and the 2005 IEEE TRANSACTIONS ON FUZZY SYSTEMS Outstanding Paper Award in 2007.