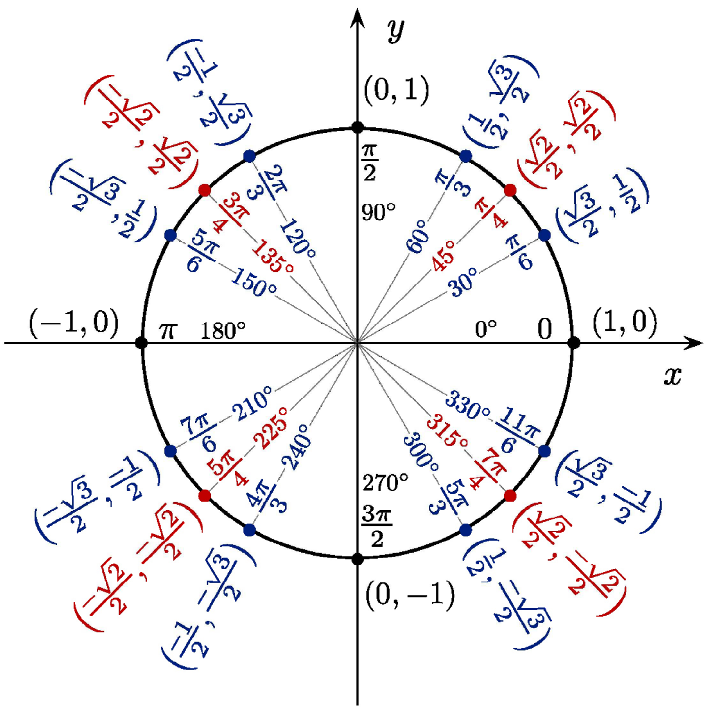
List of trigonometric identities

In mathematics, **trigonometric identities** are equalities that involve trigonometric functions and are true for every single value of the occurring variables. Geometrically, these are identities involving certain functions of one or more angles. They are distinct from triangle identities, which are identities involving both angles and side lengths of a triangle. Only the former are covered in this article.

These identities are useful whenever expressions involving trigonometric functions need to be simplified. An important application is the integration of non-trigonometric functions: a common technique involves first using the substitution rule with a trigonometric function, and then simplifying the resulting integral with a trigonometric identity.



Cosines and sines around the unit circle

Notation

Angles

This article uses Greek letters such as alpha (α), beta (β), gamma (γ), and theta (θ) to represent angles. Several different units of angle measure are widely used, including degrees, radians, and grads:

1 full circle $=$ 360 degrees = 2 π radians $=$ 400 grads.

The following table shows the conversions for some common angles:

Degrees	30°	60°	120°	150°	210°	240°	300°	330°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
Grads	33⅓ grad	66⅔ grad	133⅓ grad	166⅔ grad	233⅓ grad	266⅔ grad	333⅓ grad	366⅔ grad

Degrees	45°	90°	135°	180°	225°	270°	315°	360°
Radians	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
Grads	50 grad	100 grad	150 grad	200 grad	250 grad	300 grad	350 grad	400 grad

Unless otherwise specified, all angles in this article are assumed to be in radians, though angles ending in a degree symbol (°) are in degrees.

Trigonometric functions

The primary trigonometric functions are the sine and cosine of an angle. These are sometimes abbreviated sin(θ) and cos(θ), respectively, where θ is the angle, but the parentheses around the angle are often omitted, e.g., sin θ and cos θ.

The tangent (tan) of an angle is the ratio of the sine to the cosine:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Finally, the reciprocal functions secant (sec), cosecant (csc), and cotangent (cot) are the reciprocals of the cosine, sine, and tangent:

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

These definitions are sometimes referred to as ratio identities.

Inverse functions

The inverse trigonometric functions are partial inverse functions for the trigonometric functions. For example, the inverse function for the sine, known as the **inverse sine** ($\sin^{-1}$) or **arcsine** (arcsin or asin), satisfies

$$\sin(\arcsin x) = x \quad \text{for} \quad |x| \leq 1$$

and

$$\arcsin(\sin x) = x \quad \text{for} \quad |x| \leq \pi/2.$$

This article uses the notation below for inverse trigonometric functions:

Function	sin	cos	tan	sec	csc	cot
Inverse	arcsin	arccos	arctan	arcsec	arccsc	arccot

Pythagorean identity

The basic relationship between the sine and the cosine is the Pythagorean trigonometric identity:

$$\cos^2\theta + \sin^2\theta = 1$$

where $\cos^2\theta$ means $(\cos(\theta))^2$ and $\sin^2\theta$ means $(\sin(\theta))^2$.

This can be viewed as a version of the Pythagorean theorem, and follows from the equation $x^2 + y^2 = 1$ for the unit circle. This equation can be solved for either the sine or the cosine:

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} \quad \text{and} \quad \cos\theta = \pm\sqrt{1-\sin^2\theta}.$$

Related identities

Dividing the Pythagorean identity through by either $\cos^2\theta$ or $\sin^2\theta$ yields two other identities:

$$1+\tan^2\theta=\sec^2\theta \quad \text{and} \quad 1+\cot^2\theta=\csc^2\theta.$$

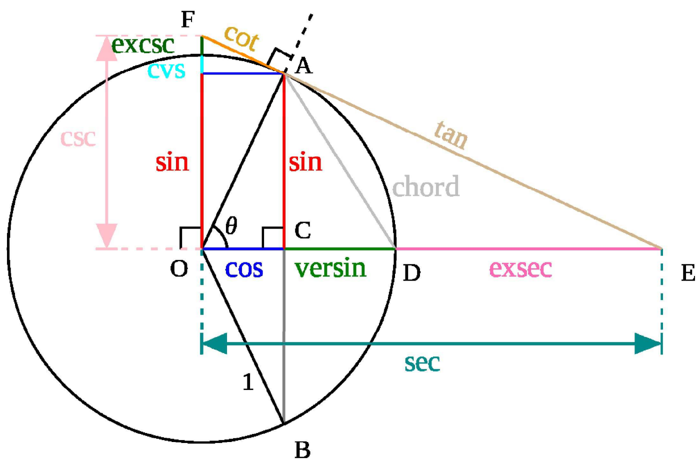
Using these identities together with the ratio identities, it is possible to express any trigonometric function in terms of any other (up to a plus or minus sign):

Each trigonometric function in terms of the other five.[1]

in terms of	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$\sin\theta=$	$\sin\theta$	$\pm\sqrt{1-\cos^2\theta}$	$\pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\csc\theta}$	$\pm\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\pm\frac{1}{\sqrt{1+\cot^2\theta}}$
$\cos\theta=$	$\pm\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\pm\frac{1}{\sqrt{1+\tan^2\theta}}$	$\pm\frac{\sqrt{\csc^2\theta-1}}{\csc\theta}$	$\frac{1}{\cos\theta}$	$\pm\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$
$\tan\theta=$	$\pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\pm\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\pm\frac{1}{\sqrt{\csc^2\theta-1}}$	$\pm\sqrt{\sec^2\theta-1}$	$\frac{1}{\cot\theta}$
$\csc\theta=$	$\frac{1}{\sin\theta}$	$\pm\frac{1}{\sqrt{1-\cos^2\theta}}$	$\pm\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\csc\theta$	$\pm\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\pm\sqrt{1+\cot^2\theta}$
$\sec\theta=$	$\pm\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\pm\sqrt{1+\tan^2\theta}$	$\pm\frac{\csc\theta}{\sqrt{\csc^2\theta-1}}$	$\sec\theta$	$\pm\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$
$\cot\theta=$	$\pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\pm\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\pm\sqrt{\csc^2\theta-1}$	$\pm\frac{1}{\sqrt{\sec^2\theta-1}}$	$\cot\theta$

Historic shorthands

The versine, coversine, haversine, and exsecant were used in navigation. For example the haversine formula was used to calculate the distance between two points on a sphere. They are rarely used today.



All of the trigonometric functions of an angle θ can be constructed geometrically in terms of a unit circle centered at O. Many of these terms are no longer in common use.

Name(s)	Abbreviation(s)	Value[2]
versed sine, versine	$\operatorname{versin}(\theta)$ $\operatorname{vers}(\theta)$ $\operatorname{ver}(\theta)$	$1-\cos(\theta)$
versed cosine, vercosine	$\operatorname{vercosin}(\theta)$	$1+\cos(\theta)$
coversed sine, coversine	$\operatorname{coversin}(\theta)$ $\operatorname{cvs}(\theta)$	$1-\sin(\theta)$
coversed cosine, covercosine	$\operatorname{covercosin}(\theta)$	$1+\sin(\theta)$
half versed sine, haversine	$\operatorname{haversin}(\theta)$	$\frac{1-\cos(\theta)}{2}$
half versed cosine, havercosine	$\operatorname{havercosin}(\theta)$	$\frac{1+\cos(\theta)}{2}$
half coversed sine, hacoversine cohaversine	$\operatorname{hacoversin}(\theta)$	$\frac{1-\sin(\theta)}{2}$
half coversed cosine, hacovercosine cohavercosine	$\operatorname{hacovercosin}(\theta)$	$\frac{1+\sin(\theta)}{2}$
exterior secant, exsecant	$\operatorname{exsec}(\theta)$	$\sec(\theta)-1$
exterior cosecant, excosecant	$\operatorname{excsc}(\theta)$	$\csc(\theta)-1$
chord	$\operatorname{crd}(\theta)$	$2\sin\left(\frac{\theta}{2}\right)$

Symmetry, shifts, and periodicity

By examining the unit circle, the following properties of the trigonometric functions can be established.

Symmetry

When the trigonometric functions are reflected from certain angles, the result is often one of the other trigonometric functions. This leads to the following identities:

Reflected in $\theta = 0$[3]	Reflected in $\theta = \pi/2$ (co-function identities)[4]	Reflected in $\theta = \pi$
$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = +\cos\theta$ $\tan(-\theta) = -\tan\theta$ $\csc(-\theta) = -\csc\theta$ $\sec(-\theta) = +\sec\theta$ $\cot(-\theta) = -\cot\theta$	$\sin(\frac{\pi}{2}-\theta) = +\cos\theta$ $\cos(\frac{\pi}{2}-\theta) = +\sin\theta$ $\tan(\frac{\pi}{2}-\theta) = +\cot\theta$ $\csc(\frac{\pi}{2}-\theta) = +\sec\theta$ $\sec(\frac{\pi}{2}-\theta) = +\csc\theta$ $\cot(\frac{\pi}{2}-\theta) = +\tan\theta$	$\sin(\pi-\theta) = +\sin\theta$ $\cos(\pi-\theta) = -\cos\theta$ $\tan(\pi-\theta) = -\tan\theta$ $\csc(\pi-\theta) = +\csc\theta$ $\sec(\pi-\theta) = -\sec\theta$ $\cot(\pi-\theta) = -\cot\theta$

Shifts and periodicity

By shifting the function round by certain angles, it is often possible to find different trigonometric functions that express the result more simply. Some examples of this are shown by shifting functions round by π/2, π and 2π radians. Because the periods of these functions are either π or 2π, there are cases where the new function is exactly the same as the old function without the shift.

Shift by π/2	**Shift by π** **Period for tan and cot**[5]	**Shift by 2π** **Period for sin, cos, csc and sec**[6]
$\sin(\theta + \frac{\pi}{2}) = +\cos\theta$	$\sin(\theta + \pi) = -\sin\theta$	$\sin(\theta + 2\pi) = +\sin\theta$
$\cos(\theta + \frac{\pi}{2}) = -\sin\theta$	$\cos(\theta + \pi) = -\cos\theta$	$\cos(\theta + 2\pi) = +\cos\theta$
$\tan(\theta + \frac{\pi}{2}) = -\cot\theta$	$\tan(\theta + \pi) = +\tan\theta$	$\tan(\theta + 2\pi) = +\tan\theta$
$\csc(\theta + \frac{\pi}{2}) = +\sec\theta$	$\csc(\theta + \pi) = -\csc\theta$	$\csc(\theta + 2\pi) = +\csc\theta$
$\sec(\theta + \frac{\pi}{2}) = -\csc\theta$	$\sec(\theta + \pi) = -\sec\theta$	$\sec(\theta + 2\pi) = +\sec\theta$
$\cot(\theta + \frac{\pi}{2}) = -\tan\theta$	$\cot(\theta + \pi) = +\cot\theta$	$\cot(\theta + 2\pi) = +\cot\theta$

Angle sum and difference identities

These are also known as the *addition and subtraction theorems* or *formulæ*. They were originally established by the 10th century Persian mathematician Abū al-Wafā' Būzjānī. One method of proving these identities is to apply Euler's formula. The use of the symbols $\pm$ and $\mp$ is described in the article plus-minus sign.

Sine	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ [7][8]
Cosine	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ [8][9]
Tangent	$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$ [8][10]
Arcsine	$\arcsin\alpha \pm \arcsin\beta = \arcsin\left(\alpha\sqrt{1-\beta^2} \pm \beta\sqrt{1-\alpha^2}\right)$ [11]
Arccosine	$\arccos\alpha \pm \arccos\beta = \arccos\left(\alpha\beta \mp \sqrt{(1-\alpha^2)(1-\beta^2)}\right)$ [12]
Arctangent	$\arctan\alpha \pm \arctan\beta = \arctan\left(\frac{\alpha \pm \beta}{1 \mp \alpha\beta}\right)$ [13]

Matrix form

The sum and difference formulae for sine and cosine can be written in matrix form as:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$$

This shows that these matrices form a representation of the rotation group in the plane (technically, the special orthogonal group *SO*(2)), since the composition law is fulfilled: subsequent multiplications of a vector with these two matrices yields the same result as the rotation by the sum of the angles.

Sines and cosines of sums of infinitely many terms

$$\sin\left(\sum_{i=1}^{\infty}\theta_i\right) = \sum_{\text{odd }k\ge 1} (-1)^{(k-1)/2} \sum_{\substack{A\subseteq\{1,2,3,\dots\}\\ |A|=k}} \left(\prod_{i\in A}\sin\theta_i \prod_{i\notin A}\cos\theta_i\right)$$

$$\cos\left(\sum_{i=1}^{\infty}\theta_i\right) = \sum_{\text{even }k\ge 0} (-1)^{k/2} \sum_{\substack{A\subseteq\{1,2,3,\dots\}\\ |A|=k}} \left(\prod_{i\in A}\sin\theta_i \prod_{i\notin A}\cos\theta_i\right)$$

In these two identities an asymmetry appears that is not seen in the case of sums of finitely many terms: in each product, there are only finitely many sine factors and cofinitely many cosine factors.

If only finitely many of the terms θ_i are nonzero, then only finitely many of the terms on the right side will be nonzero because sine factors will vanish, and in each term, all but finitely many of the cosine factors will be unity.

Tangents of sums

Let e_k (for k = 0, 1, 2, 3, ...) be the kth-degree elementary symmetric polynomial in the variables

$$x_i = \tan\theta_i$$

for i = 0, 1, 2, 3, ..., i.e.,

$$\begin{aligned} e_0 &= 1 \\ e_1 &= \sum_i x_i &&= \sum_i \tan\theta_i \\ e_2 &= \sum_{i<j} x_i x_j &&= \sum_{i<j} \tan\theta_i\tan\theta_j \\ e_3 &= \sum_{i<j<k} x_i x_j x_k &&= \sum_{i<j<k} \tan\theta_i\tan\theta_j\tan\theta_k \\ &\ \ \vdots &&\ \ \vdots \end{aligned}$$

Then

$$\tan\left(\sum_i \theta_i\right) = \frac{e_1 - e_3 + e_5 - \cdots}{e_0 - e_2 + e_4 - \cdots}.$$

The number of terms on the right side depends on the number of terms on the right side.

For example:

$$\tan(\theta_1+\theta_2) = \frac{e_1}{e_0-e_2} = \frac{x_1+x_2}{1\ -\ x_1x_2} = \frac{\tan\theta_1+\tan\theta_2}{1\ -\ \tan\theta_1\tan\theta_2},$$

$$\tan(\theta_1+\theta_2+\theta_3) = \frac{e_1-e_3}{e_0-e_2} = \frac{(x_1+x_2+x_3)\ -\ (x_1x_2x_3)}{1\ -\ (x_1x_2+x_1x_3+x_2x_3)},$$

$$\begin{aligned}\tan(\theta_1+\theta_2+\theta_3+\theta_4) &= \frac{e_1-e_3}{e_0-e_2+e_4} \\ &= \frac{(x_1+x_2+x_3+x_4)\ -\ (x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4)}{1\ -\ (x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)\ +\ (x_1x_2x_3x_4)},\end{aligned}$$

and so on. The case of only finitely many terms can be proved by mathematical induction.[14]

Secants and cosecants of sums

$$\sec\left(\sum_i \theta_i\right) = \frac{\prod_i \sec\theta_i}{e_0 - e_2 + e_4 - \cdots}$$

$$\csc\left(\sum_i \theta_i\right) = \frac{\prod_i \sec\theta_i}{e_1 - e_3 + e_5 - \cdots}$$

where e_k is the kth-degree elementary symmetric polynomial in the n variables $x_i = \tan\theta_i$, $i = 1, ..., n$, and the number of terms in the denominator and the number of factors in the product in the numerator depend on the number of terms in the sum on the left. The case of only finitely many terms can be proved by mathematical induction on the number of such terms. The convergence of the series in the denominators can be shown by writing the secant identity in the form

$$e_0 - e_2 + e_4 - \cdots = \frac{\prod_i \sec\theta_i}{\sec\left(\sum_i \theta_i\right)}$$

and then observing that the left side converges if the right side converges, and similarly for the cosecant identity.

For example,

$$\sec(\alpha+\beta+\gamma) = \frac{\sec\alpha\sec\beta\sec\gamma}{1 - \tan\alpha\tan\beta - \tan\alpha\tan\gamma - \tan\beta\tan\gamma}$$

$$\csc(\alpha+\beta+\gamma) = \frac{\sec\alpha\sec\beta\sec\gamma}{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}$$

Multiple-angle formulae

T_n is the nth Chebyshev polynomial	$\cos n\theta = T_n(\cos\theta)$ [15]
S_n is the nth spread polynomial	$\sin^2 n\theta = S_n(\sin^2\theta)$
de Moivre's formula, i is the imaginary unit	$\cos n\theta + i\sin n\theta = (\cos(\theta) + i\sin(\theta))^n$ [16]

Double-, triple-, and half-angle formulae

These can be shown by using either the sum and difference identities or the multiple-angle formulae.

Double-angle formulae[17][18]			
$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$	$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$	$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$	$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$
Triple-angle formulae[15][19]			
$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$	$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$	$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$	$\cot 3\theta = \frac{3\cot\theta - \cot^3\theta}{1 - 3\cot^2\theta}$
Half-angle formulae[20][21]			

$\sin\frac{\theta}{2} = \operatorname{sgn}\left(2\pi-\theta+4\pi\left\lfloor\frac{\theta}{4\pi}\right\rfloor\right)\sqrt{\frac{1-\cos\theta}{2}}$ $\left(\text{or } \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}\right)$	$\cos\frac{\theta}{2} = \operatorname{sgn}\left(\pi+\theta+4\pi\left\lfloor\frac{\pi-\theta}{4\pi}\right\rfloor\right)\sqrt{\frac{1+\cos\theta}{2}}$ $\left(\text{or } \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}\right)$	$\tan\frac{\theta}{2} = \csc\theta - \cot\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ $\tan\frac{\eta+\theta}{2} = \frac{\sin\eta+\sin\theta}{\cos\eta+\cos\theta}$ $\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) = \sec\theta + \tan\theta$ $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}$ $\tan\tfrac{1}{2}\theta = \frac{\tan\theta}{1+\sqrt{1+\tan^2\theta}}$ for $\theta \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$	$\cot\frac{\theta}{2} = \csc\theta + \cot\theta = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$

The fact that the triple-angle formula for sine and cosine only involves powers of a single function allows one to relate the geometric problem of a compass and straightedge construction of angle trisection to the algebraic problem of solving a cubic equation, which allows one to prove that this is in general impossible using the given tools, by field theory.

A formula for computing the trigonometric identities for the third-angle exists, but it requires finding the zeroes of the cubic equation $x^3 - \frac{3x+d}{4} = 0$, where x is the value of the sine function at some angle and d is the known value of the sine function at the triple angle. However, the discriminant of this equation is negative, so this equation has three real roots (of which only one is the solution within the correct third-circle) but none of these solutions is reducible to a real algebraic expression, as they use intermediate complex numbers under the cube roots, (which may be expressed in terms of real-only functions only if using hyperbolic functions).

Sine, cosine, and tangent of multiple angles

For specific multiples, these follow from the angle addition formulas, while the general formula was given by 16th century French mathematician Vieta.

$$\sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^k\theta\, \sin^{n-k}\theta\, \sin\left(\frac{1}{2}(n-k)\pi\right)$$
$$\cos n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^k\theta\, \sin^{n-k}\theta\, \cos\left(\frac{1}{2}(n-k)\pi\right)$$

In each of these two equations, the first parenthesized term is a binomial coefficient, and the final trigonometric function equals one or minus one or zero so that half the entries in each of the sums are removed. $\tan n\theta$ can be written in terms of $\tan\theta$ using the recurrence relation:

$$\tan\,(n{+}1)\theta = \frac{\tan n\theta + \tan\theta}{1 - \tan n\theta\,\tan\theta}.$$

$\cot n\theta$ can be written in terms of $\cot\theta$ using the recurrence relation:

$$\cot\,(n{+}1)\theta = \frac{\cot n\theta\,\cot\theta - 1}{\cot n\theta + \cot\theta}.$$

Chebyshev method

The Chebyshev method is a recursive algorithm for finding the n^{th} multiple angle formula knowing the $(n - 1)^{\text{th}}$ and $(n - 2)^{\text{th}}$ formulae.[22]

The cosine for nx can be computed from the cosine of $(n - 1)x$ and $(n - 2)x$ as follows:

$$\cos nx = 2 \cdot \cos x \cdot \cos(n-1)x - \cos(n-2)x$$

Similarly $\sin(nx)$ can be computed from the sines of $(n - 1)x$ and $(n - 2)x$

$$\sin nx = 2 \cdot \cos x \cdot \sin(n-1)x - \sin(n-2)x$$

For the tangent, we have:

$$\tan nx = \frac{H + K\tan x}{K - H\tan x}$$

where $H/K = \tan(n - 1)x$.

Tangent of an average

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = -\frac{\cos\alpha - \cos\beta}{\sin\alpha - \sin\beta}$$

Setting either α or β to 0 gives the usual tangent half-angle formulæ.

Viète's infinite product

$$\cos\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{4}\right) \cdot \cos\left(\frac{\theta}{8}\right) \cdots = \prod_{n=1}^{\infty} \cos\left(\frac{\theta}{2^n}\right) = \frac{\sin(\theta)}{\theta} = \operatorname{sinc}\theta.$$

Power-reduction formula

Obtained by solving the second and third versions of the cosine double-angle formula.

Sine	Cosine	Other
$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$	$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$	$\sin^2\theta\cos^2\theta = \frac{1 - \cos 4\theta}{8}$
$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$	$\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$	$\sin^3\theta\cos^3\theta = \frac{3\sin 2\theta - \sin 6\theta}{32}$
$\sin^4\theta = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}$	$\cos^4\theta = \frac{3 + 4\cos 2\theta + \cos 4\theta}{8}$	$\sin^4\theta\cos^4\theta = \frac{3 - 4\cos 4\theta + \cos 8\theta}{128}$
$\sin^5\theta = \frac{10\sin\theta - 5\sin 3\theta + \sin 5\theta}{16}$	$\cos^5\theta = \frac{10\cos\theta + 5\cos 3\theta + \cos 5\theta}{16}$	$\sin^5\theta\cos^5\theta = \frac{10\sin 2\theta - 5\sin 6\theta + \sin 10\theta}{512}$

and in general terms of powers of $\sin\theta$ or $\cos\theta$ the following is true, and can be deduced using De Moivre's formula, Euler's formula and binomial theorem.

	Cosine	Sine
if n is odd	$\cos^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\cos((n-2k)\theta)$	$\sin^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}(-1)^{\left(\frac{n-1}{2}-k\right)}\binom{n}{k}\sin((n-2k)\theta)$
if n is even	$\cos^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos((n-2k)\theta)$	$\sin^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}(-1)^{\left(\frac{n}{2}-k\right)}\binom{n}{k}\cos((n-2k)\theta)$

Product-to-sum and sum-to-product identities

The product-to-sum identities or prosthaphaeresis formulas can be proven by expanding their right-hand sides using the angle addition theorems. See beat (acoustics) and phase detector for applications of the sum-to-product formulæ.

Product-to-sum[23]
$\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2}$
$\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2}$
$\sin\theta\cos\varphi = \frac{\sin(\theta+\varphi)+\sin(\theta-\varphi)}{2}$
$\cos\theta\sin\varphi = \frac{\sin(\theta+\varphi)-\sin(\theta-\varphi)}{2}$
$\tan(\theta)\tan(\varphi) = \frac{cos(\theta-\varphi)-cos(\theta+\varphi)}{cos(\theta-\varphi)+cos(\theta+\varphi)}$
$\prod_{k=1}^{n}\cos\theta_k = \frac{1}{2^n}\sum_{e\in S}\cos(e_1\theta_1+\cdots+e_n\theta_n)$ where $S=\{1,-1\}^n$

Sum-to-product[24]
$\sin\theta\pm\sin\varphi = 2\sin\left(\frac{\theta\pm\varphi}{2}\right)\cos\left(\frac{\theta\mp\varphi}{2}\right)$
$\cos\theta+\cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$
$\cos\theta-\cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$

Other related identities

If *x*, *y*, and *z* are the three angles of any triangle, or in other words

$$\text{if } x+y+z=\pi=\text{half circle,}$$
$$\text{then } \tan(x)+\tan(y)+\tan(z)=\tan(x)\tan(y)\tan(z).$$

(If any of *x*, *y*, *z* is a right angle, one should take both sides to be ∞. This is neither +∞ nor −∞; for present purposes it makes sense to add just one point at infinity to the real line, that is approached by tan(θ) as tan(θ) either increases through positive values or decreases through negative values. This is a one-point compactification of the real line.)

$$\text{If } x+y+z=\pi=\text{half circle,}$$
$$\text{then } \sin(2x)+\sin(2y)+\sin(2z)=4\sin(x)\sin(y)\sin(z).$$

Hermite's cotangent identity

Charles Hermite demonstrated the following identity.[25] Suppose a_1, ..., a_n are complex numbers, no two of which differ by an integer multiple of π. Let

$$A_{n,k} = \prod_{\substack{1\leq j\leq n \\ j\neq k}} \cot(a_k - a_j)$$

(in particular, $A_{1,1}$, being an empty product, is 1). Then

$$\cot(z-a_1)\cdots\cot(z-a_n) = \cos\frac{n\pi}{2} + \sum_{k=1}^{n} A_{n,k}\cot(z-a_k).$$

The simplest non-trivial example is the case $n = 2$:

$$\cot(z-a_1)\cot(z-a_2) = -1+\cot(a_1-a_2)\cot(z-a_1)+\cot(a_2-a_1)\cot(z-a_2).$$

Ptolemy's theorem

$$\begin{aligned} &\text{If } w+x+y+z=\pi=\text{half circle,} \\ &\qquad \begin{aligned} \text{then }&\sin(w+x)\sin(x+y) \\ &= \sin(x+y)\sin(y+z) \\ &= \sin(y+z)\sin(z+w) \\ &= \sin(z+w)\sin(w+x) = \sin(w)\sin(y)+\sin(x)\sin(z). \end{aligned} \end{aligned}$$

(The first three equalities are trivial; the fourth is the substance of this identity.) Essentially this is Ptolemy's theorem adapted to the language of modern trigonometry.

Linear combinations

For some purposes it is important to know that any linear combination of sine waves of the same period or frequency but different phase shifts is also a sine wave with the same period or frequency, but a different phase shift. In the case of a non-zero linear combination of a sine and cosine wave[26] (which is just a sine wave with a phase shift of π/2), we have

$$a\sin x + b\cos x = \sqrt{a^2+b^2}\cdot\sin(x+\varphi)$$

where

$$\varphi = \begin{cases} \arcsin\left(\frac{b}{\sqrt{a^2+b^2}}\right) & \text{if } a \geq 0, \\ \pi - \arcsin\left(\frac{b}{\sqrt{a^2+b^2}}\right) & \text{if } a < 0, \end{cases}$$

or equivalently

$$\varphi = \operatorname{sgn}(b)\arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right)$$

or even

$$\varphi = \arctan\left(\frac{b}{a}\right) + \begin{cases} 0 & \text{if } a \geq 0, \\ \pi & \text{if } a < 0, \end{cases}$$

or using the atan2 function

$$\varphi = \operatorname{atan2}(b, a).$$

More generally, for an arbitrary phase shift, we have

$$a\sin x + b\sin(x+\alpha) = c\sin(x+\beta)$$

where

$$c = \sqrt{a^2 + b^2 + 2ab\cos\alpha},$$

and

$$\beta = \arctan\left(\frac{b\sin\alpha}{a + b\cos\alpha}\right) + \begin{cases} 0 & \text{if } a + b\cos\alpha \ge 0, \\ \pi & \text{if } a + b\cos\alpha < 0. \end{cases}$$

For the most general case, see Phasor addition.

Lagrange's trigonometric identities

These identities, named after Joseph Louis Lagrange, are:[27][28]

$$\sum_{n=1}^{N} \sin n\theta = \frac{1}{2}\cot\frac{\theta}{2} - \frac{\cos(N + \frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

$$\sum_{n=1}^{N} \cos n\theta = -\frac{1}{2} + \frac{\sin(N + \frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

A related function is the following function of *x*, called the Dirichlet kernel.

$$1 + 2\cos(x) + 2\cos(2x) + 2\cos(3x) + \cdots + 2\cos(nx) = \frac{\sin\left(\left(n + \frac{1}{2}\right)x\right)}{\sin(x/2)}.$$

Other sums of trigonometric functions

Sum of sines and cosines with arguments in arithmetic progression[29]:

$$\sin\varphi + \sin(\varphi + \alpha) + \sin(\varphi + 2\alpha) + \cdots$$
$$\cdots + \sin(\varphi + n\alpha) = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right) \cdot \sin\left(\varphi + \frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}.$$

$$\cos\varphi + \cos(\varphi + \alpha) + \cos(\varphi + 2\alpha) + \cdots$$
$$\cdots + \cos(\varphi + n\alpha) = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right) \cdot \cos\left(\varphi + \frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}.$$

For any *a* and *b*:

$$a\cos(x) + b\sin(x) = \sqrt{a^2 + b^2}\cos(x - \operatorname{atan2}(b, a))$$

where atan2(*y*, *x*) is the generalization of arctan(*y*/*x*) that covers the entire circular range.

$$\tan(x) + \sec(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

The above identity is sometimes convenient to know when thinking about the Gudermannian function, which relates the circular and hyperbolic trigonometric functions without resorting to complex numbers.

If *x*, *y*, and *z* are the three angles of any triangle, i.e. if $x + y + z = \pi$, then

$$\cot(x)\cot(y) + \cot(y)\cot(z) + \cot(z)\cot(x) = 1.$$

Certain linear fractional transformations

If $f(x)$ is given by the linear fractional transformation

$$f(x) = \frac{(\cos\alpha)x - \sin\alpha}{(\sin\alpha)x + \cos\alpha},$$

and similarly

$$g(x) = \frac{(\cos\beta)x - \sin\beta}{(\cos\beta)x + \sin\beta},$$

then

$$f(g(x)) = g(f(x)) = \frac{(\cos(\alpha+\beta))x - \sin(\alpha+\beta)}{(\sin(\alpha+\beta))x + \cos(\alpha+\beta)}.$$

More tersely stated, if for all α we let f_α be what we called f above, then

$$f_\alpha \circ f_\beta = f_{\alpha+\beta}.$$

If x is the slope of a line, then $f(x)$ is the slope of its rotation through an angle of $-\alpha$.

Inverse trigonometric functions

$$\arcsin(x) + \arccos(x) = \pi/2$$
$$\arctan(x) + \operatorname{arccot}(x) = \pi/2.$$
$$\arctan(x) + \arctan(1/x) = \begin{cases} \pi/2, & \text{if } x > 0 \\ -\pi/2, & \text{if } x < 0 \end{cases}$$

Compositions of trig and inverse trig functions

$\sin[\arccos(x)] = \sqrt{1-x^2}$	$\tan[\arcsin(x)] = \frac{x}{\sqrt{1-x^2}}$
$\sin[\arctan(x)] = \frac{x}{\sqrt{1+x^2}}$	$\tan[\arccos(x)] = \frac{\sqrt{1-x^2}}{x}$
$\cos[\arctan(x)] = \frac{1}{\sqrt{1+x^2}}$	$\cot[\arcsin(x)] = \frac{\sqrt{1-x^2}}{x}$
$\cos[\arcsin(x)] = \sqrt{1-x^2}$	$\cot[\arccos(x)] = \frac{x}{\sqrt{1-x^2}}$

Relation to the complex exponential function

$e^{ix} = \cos(x) + i\sin(x)$ [30] (Euler's formula),

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$$

$e^{i\pi} = -1$ (Euler's identity),

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$ [31]

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$ [32]

and hence the corollary:

$$\tan(x) = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})} = \frac{\sin(x)}{\cos(x)}$$

where $i^2 = -1$.

Infinite product formulae

For applications to special functions, the following infinite product formulae for trigonometric functions are useful:[33][34]

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right) \qquad \cos x = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2(n-\frac{1}{2})^2}\right)$$

$$\sinh x = x\prod_{n=1}^{\infty}\left(1+\frac{x^2}{\pi^2 n^2}\right) \qquad \cosh x = \prod_{n=1}^{\infty}\left(1+\frac{x^2}{\pi^2(n-\frac{1}{2})^2}\right)$$

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\cos\left(\frac{x}{2^n}\right) \qquad |\sin x| = \frac{1}{2}\prod_{n=0}^{\infty}\sqrt[2^{n+1}]{\left|\tan\left(2^n x\right)\right|}$$

Identities without variables

The curious identity

$$\cos 20^\circ\cdot\cos 40^\circ\cdot\cos 80^\circ = \frac{1}{8}$$

is a special case of an identity that contains one variable:

$$\prod_{j=0}^{k-1}\cos(2^j x) = \frac{\sin(2^k x)}{2^k\sin(x)}.$$

Similarly:

$$\sin 20^\circ\cdot\sin 40^\circ\cdot\sin 80^\circ = \frac{\sqrt{3}}{8}.$$

The same cosine identity in radians is

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8},$$

Similarly:

$$\tan 50^\circ\cdot\tan 60^\circ\cdot\tan 70^\circ = \tan 80^\circ.$$

$$\tan 40^\circ\cdot\tan 30^\circ\cdot\tan 20^\circ = \tan 10^\circ.$$

The following is perhaps not as readily generalized to an identity containing variables (but see explanation below):

$$\cos 24^\circ+\cos 48^\circ+\cos 96^\circ+\cos 168^\circ = \frac{1}{2}.$$

Degree measure ceases to be more felicitous than radian measure when we consider this identity with 21 in the denominators:

$$\cos\left(\frac{2\pi}{21}\right)+\cos\left(2\cdot\frac{2\pi}{21}\right)+\cos\left(4\cdot\frac{2\pi}{21}\right)$$
$$+\cos\left(5\cdot\frac{2\pi}{21}\right)+\cos\left(8\cdot\frac{2\pi}{21}\right)+\cos\left(10\cdot\frac{2\pi}{21}\right) = \frac{1}{2}.$$

The factors 1, 2, 4, 5, 8, 10 may start to make the pattern clear: they are those integers less than 21/2 that are relatively prime to (or have no prime factors in common with) 21. The last several examples are corollaries of a basic fact about the irreducible cyclotomic polynomials: the cosines are the real parts of the zeroes of those polynomials; the sum of the zeroes is the Möbius function evaluated at (in the very last case above) 21; only half of the zeroes are present above. The two identities preceding this last one arise in the same fashion with 21 replaced by 10 and 15, respectively.

Many of those curious identities stem from more general facts like the following[35]:

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$$

and

$$\prod_{k=1}^{n-1} \cos\left(\frac{k\pi}{n}\right) = \frac{\sin(\pi n/2)}{2^{n-1}}$$

Combining these gives us

$$\prod_{k=1}^{n-1} \tan\left(\frac{k\pi}{n}\right) = \frac{n}{\sin(\pi n/2)}$$

If n is an odd number ($n = 2m + 1$) we can make use of the symmetries to get

$$\prod_{k=1}^{m} \tan\left(\frac{k\pi}{2m+1}\right) = \sqrt{2m+1}$$

Computing π

An efficient way to compute π is based on the following identity without variables, due to Machin:

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

or, alternatively, by using an identity of Leonhard Euler:

$$\frac{\pi}{4} = 5\arctan\frac{1}{7} + 2\arctan\frac{3}{79}.$$

A useful mnemonic for certain values of sines and cosines

For certain simple angles, the sines and cosines take the form $\sqrt{n}/2$ for $0 \le n \le 4$, which makes them easy to remember.

$$\begin{array}{ccccccccc}
\sin 0 & = & \sin 0^\circ & = & \sqrt{0}/2 & = & \cos 90^\circ & = & \cos\left(\frac{\pi}{2}\right) \\
\sin\left(\frac{\pi}{6}\right) & = & \sin 30^\circ & = & \sqrt{1}/2 & = & \cos 60^\circ & = & \cos\left(\frac{\pi}{3}\right) \\
\sin\left(\frac{\pi}{4}\right) & = & \sin 45^\circ & = & \sqrt{2}/2 & = & \cos 45^\circ & = & \cos\left(\frac{\pi}{4}\right) \\
\sin\left(\frac{\pi}{3}\right) & = & \sin 60^\circ & = & \sqrt{3}/2 & = & \cos 30^\circ & = & \cos\left(\frac{\pi}{6}\right) \\
\sin\left(\frac{\pi}{2}\right) & = & \sin 90^\circ & = & \sqrt{4}/2 & = & \cos 0^\circ & = & \cos 0
\end{array}$$

Miscellany

With the golden ratio φ:

$$\cos\left(\frac{\pi}{5}\right) = \cos 36^\circ = \frac{\sqrt{5}+1}{4} = \frac{\varphi}{2}$$

$$\sin\left(\frac{\pi}{10}\right) = \sin 18^\circ = \frac{\sqrt{5}-1}{4} = \frac{\varphi-1}{2} = \frac{1}{2\varphi}$$

Also see exact trigonometric constants.

An identity of Euclid

Euclid showed in Book XIII, Proposition 10 of his *Elements* that the area of the square on the side of a regular pentagon inscribed in a circle is equal to the sum of the areas of the squares on the sides of the regular hexagon and the regular decagon inscribed in the same circle. In the language of modern trigonometry, this says:

$$\sin^2(18^\circ) + \sin^2(30^\circ) = \sin^2(36^\circ).$$

Ptolemy used this proposition to compute some angles in his table of chords.

Calculus

In calculus the relations stated below require angles to be measured in radians; the relations would become more complicated if angles were measured in another unit such as degrees. If the trigonometric functions are defined in terms of geometry, their derivatives can be found by verifying two limits. The first is:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

verified using the unit circle and squeeze theorem. The second limit is:

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0,$$

verified using the identity $\tan(x/2) = (1 - \cos x)/\sin x$. Having established these two limits, one can use the limit definition of the derivative and the addition theorems to show that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$. If the sine and cosine functions are defined by their Taylor series, then the derivatives can be found by differentiating the power series term-by-term.

$$\frac{d}{dx}\sin x = \cos x$$

The rest of the trigonometric functions can be differentiated using the above identities and the rules of differentiation:[36][37][38]

$$\begin{aligned}
\frac{d}{dx}\sin x &= \cos x, & \frac{d}{dx}\arcsin x &= \frac{1}{\sqrt{1-x^2}}\\
\frac{d}{dx}\cos x &= -\sin x, & \frac{d}{dx}\arccos x &= \frac{-1}{\sqrt{1-x^2}}\\
\frac{d}{dx}\tan x &= \sec^2 x, & \frac{d}{dx}\arctan x &= \frac{1}{1+x^2}\\
\frac{d}{dx}\cot x &= -\csc^2 x, & \frac{d}{dx}\operatorname{arccot} x &= \frac{-1}{1+x^2}\\
\frac{d}{dx}\sec x &= \tan x \sec x, & \frac{d}{dx}\operatorname{arcsec} x &= \frac{1}{|x|\sqrt{x^2-1}}\\
\frac{d}{dx}\csc x &= -\csc x \cot x, & \frac{d}{dx}\operatorname{arccsc} x &= \frac{-1}{|x|\sqrt{x^2-1}}
\end{aligned}$$

The integral identities can be found in "list of integrals of trigonometric functions". Some generic forms are listed below.

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

Implications

The fact that the differentiation of trigonometric functions (sine and cosine) results in linear combinations of the same two functions is of fundamental importance to many fields of mathematics, including differential equations and Fourier transforms.

Exponential definitions

Function	Inverse function[39]
$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$	$\arcsin x = -i \ln\left(ix + \sqrt{1 - x^2}\right)$
$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$	$\arccos x = -i \ln\left(x + \sqrt{x^2 - 1}\right)$
$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$	$\arctan x = \frac{i}{2}\ln\left(\frac{i + x}{i - x}\right)$
$\csc\theta = \frac{2i}{e^{i\theta} - e^{-i\theta}}$	$\operatorname{arccsc} x = -i \ln\left(\frac{i}{x} + \sqrt{1 - \frac{1}{x^2}}\right)$
$\sec\theta = \frac{2}{e^{i\theta} + e^{-i\theta}}$	$\operatorname{arcsec} x = -i \ln\left(\frac{1}{x} + \sqrt{1 - \frac{i}{x^2}}\right)$
$\cot\theta = \frac{i(e^{i\theta} + e^{-i\theta})}{e^{i\theta} - e^{-i\theta}}$	$\operatorname{arccot} x = \frac{i}{2}\ln\left(\frac{x - i}{x + i}\right)$
$\operatorname{cis}\theta = e^{i\theta}$	$\operatorname{arccis} x = \frac{\ln x}{i} = -i \ln x = \arg x$

Miscellaneous

Dirichlet kernel

The **Dirichlet kernel** $D_n(x)$ is the function occurring on both sides of the next identity:

$$1 + 2\cos(x) + 2\cos(2x) + 2\cos(3x) + \cdots + 2\cos(nx) = \frac{\sin\left[\left(n + \frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)}.$$

The convolution of any integrable function of period 2π with the Dirichlet kernel coincides with the function's nth-degree Fourier approximation. The same holds for any measure or generalized function.

Weierstrass substitution

If we set

$$t = \tan\left(\frac{x}{2}\right),$$

then[40]

$$\sin(x) = \frac{2t}{1+t^2} \text{ and } \cos(x) = \frac{1-t^2}{1+t^2} \text{ and } e^{ix} = \frac{1+it}{1-it}$$

where $e^{ix} = \cos(x) + i\sin(x)$, sometimes abbreviated to cis(x).

When this substitution of t for tan(x/2) is used in calculus, it follows that sin(x) is replaced by $2t/(1 + t^2)$, cos(x) is replaced by $(1 - t^2)/(1 + t^2)$ and the differential dx is replaced by $(2\,dt)/(1 + t^2)$. Thereby one converts rational functions of sin(x) and cos(x) to rational functions of t in order to find their antiderivatives.

Notes

[1] Abramowitz and Stegun, p. 73, 4.3.45

[2] Abramowitz and Stegun, p. 78, 4.3.147

[3] Abramowitz and Stegun, p. 72, 4.3.13–15

[4] The Elementary Identities (http://jwbales.home.mindspring.com/precal/part5/part5.1.html)

[5] Abramowitz and Stegun, p. 72, 4.3.9

[6] Abramowitz and Stegun, p. 72, 4.3.7–8

[7] Abramowitz and Stegun, p. 72, 4.3.16

[8] Weisstein, Eric W., " Trigonometric Addition Formulas (http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html)" from MathWorld.

[9] Abramowitz and Stegun, p. 72, 4.3.17

[10] Abramowitz and Stegun, p. 72, 4.3.18

[11] Abramowitz and Stegun, p. 80, 4.4.42

[12] Abramowitz and Stegun, p. 80, 4.4.43

[13] Abramowitz and Stegun, p. 80, 4.4.36

[14] Bronstein, Manuel (1989). "Simplification of real elementary functions". In G. H. Gonnet (ed.). *Proceedings of the ACM-SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation*. ISSAC'89 (Portland US-OR, 1989-07). New York: ACM. pp. 207–211. doi:10.1145/74540.74566. ISBN 0-89791-325-6.

[15] Weisstein, Eric W., " Multiple-Angle Formulas (http://mathworld.wolfram.com/Multiple-AngleFormulas.html)" from MathWorld.

[16] Abramowitz and Stegun, p. 74, 4.3.48

[17] Abramowitz and Stegun, p. 72, 4.3.24–26

[18] Weisstein, Eric W., " Double-Angle Formulas (http://mathworld.wolfram.com/Double-AngleFormulas.html)" from MathWorld.

[19] Abramowitz and Stegun, p. 72, 4.3.27–28

[20] Abramowitz and Stegun, p. 72, 4.3.20–22

[21] Weisstein, Eric W., " Half-Angle Formulas (http://mathworld.wolfram.com/Half-AngleFormulas.html)" from MathWorld.

[22] Ken Ward's Mathematics Pages, http://www.trans4mind.com/personal_development/mathematics/trigonometry/multipleAnglesRecursiveFormula.htm

[23] Abramowitz and Stegun, p. 72, 4.3.31–33

[24] Abramowitz and Stegun, p. 72, 4.3.34–39

[25] Warren P. Johnson, "Trigonometric Identities à la Hermite", *American Mathematical Monthly*, volume 117, number 4, April 2010, pages 311–327

[26] Proof at http://pages.pacificcoast.net/~cazelais/252/lc-trig.pdf

[27] Eddie Ortiz Muñiz (February 1953). "A Method for Deriving Various Formulas in Electrostatics and Electromagnetism Using Lagrange's Trigonometric Identities". *American Journal of Physics* **21** (2): 140. doi:10.1119/1.1933371.

[28] Alan Jeffrey and Hui-hui Dai (2008). "Section 2.4.1.6". *Handbook of Mathematical Formulas and Integrals* (4th ed.). Academic Press. ISBN 978-0-12-374288-9.

[29] Michael P. Knapp, Sines and Cosines of Angles in Arithmetic Progression (http://evergreen.loyola.edu/mpknapp/www/papers/knapp-sv.pdf)

[30] Abramowitz and Stegun, p. 74, 4.3.47

[31] Abramowitz and Stegun, p. 71, 4.3.2

[32] Abramowitz and Stegun, p. 71, 4.3.1

[33] Abramowitz and Stegun, p. 75, 4.3.89–90