SIAM J. CONTROL OPTIM.
Vol. 43, No. 2, pp. 685–696

NULL CONTROLLABILITY OF THE FREE SURFACE OF A LIQUID IN A CONTAINER*

VILMOS KOMORNIK†, PAOLA LORETI‡, AND LEONARDO MAZZINI§

Abstract. This paper deals with the well-posedness, boundary observability, and boundary controllability of a model of lateral sloshing in moving containers. It begins with a short review of the lateral sloshing theory, then the problem is solved by combining the Fourier method and the Hilbert uniqueness method of J.-L. Lions. At the end an application is discussed.

Key words. controllability, Fourier series, fluid model, free surface

AMS subject classifications. 93B05, 93C20, 42C15, 76B10

DOI. 10.1137/S0363012902405285

1. The control of sloshing in moving containers.

The sloshing equations in moving containers are used in many applications to study the motion of propellant fluids inside ships, aircraft, or space platforms; see, e.g., Silverman and Abramson [6]. The motion is often analyzed in order to demonstrate the stability of attitude control systems of the propellant carriers.

The behavior of the solutions depend very much on the proportion between the gravity forces and the surface tension terms, i.e., on the so-called Bond number defined by

$$\text{Bond number} = \frac{g\rho r^2}{\sigma}.$$

The equations we use are widely applied when the Bond number is much higher than one, as in the sloshing of water in a 1-g environment or in the sloshing of standard monomethile hydrazine propellant in accelerated rockets (with acceleration between $0,1$ g and 1 g) when the dimension (r) is larger than a half meter (as is often the case).

The model of lateral sloshing proposed in the paper has been used for many years in the sloshing of propellant inside satellite tanks in propulsed phases. The same model is used world-wide for the same purposes because the Bond number in such cases is typically higher than 10.

Today the sloshing in moving containers is best controlled by passive techniques like baffles and separators inside the tank. The control of the actual profile of the liquid phase has gotten less attention thus far, although it may become interesting in industrial applications to control the low frequency disturbances in spacecraft or ships that try to maintain a very quiet attitude. The generation of a pressure slope over a liquid phase is very far from being an impossible technical problem.

A simple implementation locates a number of valves connected to a high pressure air reservoir and a number of low pressure valves connected to a low pressure reservoir.

*Received by the editors April 8, 2002; accepted for publication (in revised form) January 14, 2004; published electronically August 4, 2004.
http://www.siam.org/journals/sicon/43-2/40528.html

†Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, 7 rue René Descartes, 67084 Strasbourg Cedex, France (komornik@math.u-strasbg.fr).

‡Dipartimento di Metodi e Modelli, Matematici per le Scienze Applicate, Università di Roma "La Sapienza," Via A. Scarpa 16, 00161 Roma, Italy (loreti@dmmm.uniroma1.it).

§Alenia Spazio S.p.A., via Saccomuro 24, 00100 Roma, Italy (l.mazzini@rmd250b.roma.alespazio.it).

The valves are distributed on the top of the tank and are controlled by a computer. A set of pressure sensors located in the tank in the gas phase provides the computer with information for commanding the valves in order to obtain the desired slope of pressure.

The aim of this paper is to provide exact controllability results via the Hilbert uniqueness method (HUM) technique for the problem of controlling the liquid motion in a container via the distribution of the pressures in the container itself. This result deviates from any classic approach because it applies directly to the partial differential equations that describe the motion of the liquid without using any preliminary discretization of the problem, while most of the research on this subject uses preliminary discretization.

Active control of the liquid itself inside its container may be achieved by controlling the pressure distribution above the fluid free surface.

At the beginning of the paper, we introduce the equations for studying the linearized motion of an incompressible fluid inside a moving container, then we establish a proper mathematical setting for this problem, and finally we provide necessary and sufficient conditions for the control of this motion.

This control problem may find proper applications where the frequency of motion of the fluid strongly interferes with the controller of the attitude of the carrier, making it necessary to actively control the dynamics of the fluids. This approach is new in engineering and in control theory for this particular subject.

1.1. The sloshing equations. Our treatment shall start with the basic Navier–Stokes equations written for an accelerating frame introducing the hypothesis of fluid incompressibility and fluid irrotational motion. Following the reference [1], we introduce the following quantities:

Ω	liquid free boundary,
h	tank height,
$Q = \Omega \times (-h, 0)$	tank interior,
$\Gamma = \partial\Omega \times [-h, 0]$	tank wet boundary,
$\overrightarrow{x} = (x, y, z) \in \Omega \times [-h, 0]$	position coordinate inside the tank.

The following fields are relevant to the physics of the problem:

$\delta(x, y, t)$	liquid free surface height,
$\overrightarrow{v}(x, y, z, t) \in \mathbb{R}^3$	liquid velocity in the accelerating frame,
$\overrightarrow{u} = \int \overrightarrow{v} dt$	liquid displacement,
$\phi(x, y, z, t)$	scalar potential of liquid velocity $\nabla\phi = v$,
$\psi(x, y, z, t)$	scalar potential of liquid displacement $\nabla\psi = u$,
$p(x, y, z, t)$	pressure of the liquid,
$p_0(x, y, t)$	ullage pressure above the liquid surface,
$\overrightarrow{n} \in \mathbb{R}^3$	normal to the container,
$\overrightarrow{V} \in \mathbb{R}^3$	velocity of the container walls in the accelerating frame,
$\overrightarrow{w}(t)$	container rigid motion translation vector,
$\overrightarrow{\theta}(t)$	container rigid motion rotation vector;

furthermore, g denotes the acceleration $+z$ of the reference frame or equivalent gravitational acceleration in the $-z$ direction.

From the classical Navier–Stokes equations written in a gravitational field (taken, for example, from reference [7]), introducing the hypotheses of incompressibility and irrotational motion with respect to that frame, we derive that the liquid velocity can be derived as gradient of a potential that satisfies the so-called Bernoulli equation:

$$\nabla \left(\partial_t \phi + \frac{1}{2} \nabla \phi^T \nabla \phi + \frac{p}{\rho} + gz \right) = 0.$$

The velocity is derived by the following statements:

$$\Delta \phi = 0, \qquad \nabla \phi = \overrightarrow{v}.$$

The potential is not unique, for a generic time function can be added to it leaving the same physical solution, and we can add as a gauge condition that the Bernoulli equation is written as

$$\partial_t \phi + \frac{1}{2} \nabla \phi^T \nabla \phi + \frac{p}{\rho} + gz = 0.$$

The boundary conditions for such equations are

$$\partial_n \phi = \overrightarrow{V} \cdot \overrightarrow{n} \quad \text{in} \quad \Gamma$$

and

$$p(x, y, 0, t) = p_0(x, y, t) \quad \text{in} \quad \Omega.$$

When the container is stationary in the accelerating frame and the ullage pressure is constant, we can derive the static equilibrium solution:

$$p = p_0 - \rho g z, \quad \phi = 0, \quad v = 0, \quad \delta = 0.$$

Now, let us consider a small rigid motion of the container and a small pressure variation in the ullage produced either by a control action or by external environmental conditions. We then have

$$\overrightarrow{V}(x, y, z, t) = \frac{d\overrightarrow{w}}{dt}(t) + \frac{d\overrightarrow{\theta}}{dt}(t) \times \overrightarrow{x}.$$

Since we consider rigid motion of the container, we have

$$\overrightarrow{u} = \nabla \psi = \nabla \int \phi dt,$$

$$\delta(x, y, t) = u_z(x, y, 0, t) = \partial_z \psi|_{z=0}.$$

We can finally write for the linearized problem in the displacement potential

$$\Delta \psi = 0 \quad \text{in} \quad Q \times [0, T] \tag{1.1}$$

with boundary conditions at the container wet wall

$$\partial_n \psi = \overrightarrow{n} \cdot (\overrightarrow{w} + \overrightarrow{\theta} \times \overrightarrow{x}) \quad \text{on} \quad \Gamma \times [0, T] \tag{1.2}$$

and boundary conditions at the free surface derived by the Bernoulli equation:

$$\partial^2_{tt}\psi + g\partial_z\psi + \frac{p_0}{\rho} = 0 \quad \text{on} \quad \Omega \times [0,T]. \tag{1.3}$$

The pressure inside the liquid and at the container boundary can be derived by the Bernoulli equation

$$\partial^2_{tt}\psi + g(\partial_z\psi + z) + \frac{p}{\rho} = 0. \tag{1.4}$$

Equations (1.1) and (1.2), (1.3) in a suitable functional setting provide a well-defined set of equations, as will be shown in the following.

It is quite common in practice to divide the potential into components:

$$\psi = \overrightarrow{\psi}_w \cdot \overrightarrow{w} + \overrightarrow{\psi}_\theta \cdot \overrightarrow{\theta} + \psi_K.$$

Here $\overrightarrow{\psi}_w$ satisfies

$$\begin{aligned} &\Delta\overrightarrow{\psi}_w = 0 \quad \text{in} \quad Q \times [0,T], \\ &\partial_n\overrightarrow{\psi}_w = \overrightarrow{n} \quad \text{in} \quad \Gamma \times [0,T], \\ &\partial^2_{tt}\overrightarrow{\psi}_w + g\partial_z\overrightarrow{\psi}_w = 0 \quad \text{in} \quad \Omega \times [0,T], \end{aligned}$$

while $\overrightarrow{\psi}_\theta$ satisfies

$$\begin{aligned} &\Delta\overrightarrow{\psi}_\theta = 0 \quad \text{in} \quad Q \times [0,T], \\ &\partial_n\overrightarrow{\psi}_\theta = \overrightarrow{n} \quad \text{in} \quad \Gamma \times [0,T], \\ &\partial^2_{tt}\overrightarrow{\psi}_\theta + g\partial_z\overrightarrow{\psi}_\theta = 0 \quad \text{in} \quad \Omega \times [0,T]. \end{aligned}$$

The last potential ψ_K has homogeneous boundary conditions at the wet container walls. So, ψ_K satisfies

$$\begin{aligned} &\Delta\psi_K = 0 \quad \text{in} \quad Q \times [0,T], \\ &\partial_n\psi_K = 0 \quad \text{in} \quad \Gamma \times [0,T], \end{aligned}$$

and

$$\partial^2_{tt}\psi_K + g\partial_z\psi_K + \frac{p_0}{\rho} = -\left(\overrightarrow{\psi}_w \cdot \frac{d^2\overrightarrow{w}}{dt^2} + g\partial_z\overrightarrow{\psi}_w \cdot \overrightarrow{w}\right) - \left(\overrightarrow{\psi}_\theta \cdot \frac{d^2\overrightarrow{\theta}}{dt^2} + g\partial_z\overrightarrow{\psi}_\theta \cdot \overrightarrow{\theta}\right)$$

in $\Omega \times [0,T]$.

While the first two vectorial potentials depend only on the shape of the container and can be solved a priori, the third scalar potential has homogeneous conditions at the wet boundary of the container and boundary conditions at the free surface containing all the external inputs, which are the ullage pressure distribution and the container motion variables with their second derivatives. These external inputs can be considered as control variables or external perturbations. It is to be remarked that controlling $\psi_K(x,y,z,t)$ to zero means that the motion of the fluid is rigid with the container. In the following we will focus our study on the production of a suitable controlling ullage pressure $p_0(x,y,t)$, so to produce $\psi_K(x,y,z,t) = 0$ for $t \geq T$ when the potential has been previously excited by a container motion occurring before $t = 0$, as per the classic paradigm of continuous controllability.

2. Well-posedness and boundary observability of the homogeneous problem. Let Ω be a bounded open domain in $\mathbb{R}^2$, having a sufficiently smooth boundary Γ. Given two positive numbers h and g, consider the following problem for an unknown function $\varphi(x,y,z,t)$:

$$\begin{cases} \Delta\varphi = 0 \quad \text{in} \quad \Omega\times(-h,0)\times\mathbb{R}, \\ \partial_\nu\varphi = 0 \quad \text{on} \quad \Gamma\times(-h,0)\times\mathbb{R}, \\ \partial_\nu\varphi = 0 \quad \text{on} \quad \Omega\times\{-h\}\times\mathbb{R}, \\ \frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial z} = 0 \quad \text{on} \quad \Omega\times\{0\}\times\mathbb{R}, \\ \varphi(x,y,0,0) = \varphi_0(x,y) \quad (x,y)\in\Omega, \\ \frac{\partial\varphi}{\partial t}(x,y,0,0) = \varphi_1(x,y), \quad (x,y)\in\Omega. \end{cases} \tag{2.1}$$

Let us introduce an orthonormal basis (e_n) of $L^2(\Omega)$, consisting of eigenfunctions of $-\Delta$ with homogeneous Neumann boundary conditions, corresponding to nonnegative eigenvalues λ_n, tending to ∞. Set

$$\omega_n = \sqrt{g\lambda_n\tanh(\lambda_n h)}.$$

Proposition 2.1. *Given $\varphi_0 \in L^2(\Omega)$ and $\varphi_1 \in (H^1(\Omega))'$, problem* (2.1) *has a unique solution of the form*

$$\varphi(x,y,z,t) = \sum_{n=1}^{\infty} e_n(x,y)\cosh[\lambda_n(z+h)]\big(a_n e^{i\omega_n t} + b_n e^{-i\omega_n t}\big), \tag{2.2}$$

with suitable complex coefficients a_n and b_n. The series converges in $L^2(\Omega)$ uniformly in z and t, and its termwise derivative with respect to t,

$$\sum_{n=1}^{\infty} e_n(x,y)\cosh[\lambda_n(z+h)]i\omega_n\big(a_n e^{i\omega_n t} - b_n e^{-i\omega_n t}\big), \tag{2.3}$$

converges in $(H^1(\Omega))'$ uniformly in z and t.

Furthermore,

$$\varphi(\cdot,\cdot,0,\cdot) \in C(\mathbb{R}; L^2(\Omega)) \cap C^1(\mathbb{R}; (H^1(\Omega))')$$

and

$$\|\varphi(\cdot,\cdot,0,t)\|^2_{L^2(\Omega)} + \left\|\frac{\partial\varphi}{\partial t}(\cdot,\cdot,0,t)\right\|^2_{(H^1(\Omega))'} \le \alpha\big(\|\varphi_0\|^2_{L^2(\Omega)} + \|\varphi_1\|^2_{(H^1(\Omega))'}\big)$$

with some constant α, independent of $t \in \mathbb{R}$ and of the particular choice of the initial data φ_0 and φ_1.

Proof. Thanks to the choice of the eigenfunctions $e_n(x,y)$ and of the numbers ω_n, the functions of the form (2.2) satisfy the boundary conditions (2.3), regardless of the particular choice of the coefficients a_n and b_n. Setting $h=0$ and $t=0$ in (2.2) and (2.3) and using the initial conditions, we obtain the equations

$$\varphi_0(x,y) = \sum_{n=1}^{\infty} e_n(x,y)\cosh(\lambda_n h)(a_n+b_n)$$

and

$$\varphi_1(x,y)=\sum_{n=1}^{\infty} e_n(x,y) i\omega_n \cosh(\lambda_n h)(a_n-b_n).$$

Since (e_n) is an orthogonal basis in both $L^2(\Omega)$ and $(H^1(\Omega))'$, these equations determine the coefficients a_n and b_n uniquely.

On the other hand, choosing the coefficients according to these equations, a straightforward computation shows the convergence of the series as stated in the proposition.

For the proof of the inequality we first obtain by a direct computation the following estimates:

$$\|\varphi(\cdot,\cdot,0,t)\|^2_{L^2(\Omega)} \le \sum_{n=1}^{\infty} \cosh(\lambda_n h)^2 \left| a_n e^{i\omega_n t} + b_n e^{-i\omega_n t}\right|^2$$

and

$$\left\| \frac{\partial\varphi}{\partial t}(\cdot,\cdot,0,t)\right\|^2_{(H^1(\Omega))'} \le \sum_{n=1}^{\infty} \cosh(\lambda_n h)^2 \frac{|\omega_n|^2}{\lambda_n} \left| a_n e^{i\omega_n t} + b_n e^{-i\omega_n t}\right|^2.$$

Since ${\omega_n}^2/\lambda_n$ converges to one, it follows that

$$\begin{aligned} c_1 \sum_{n=1}^{\infty} &\cosh(\lambda_n h)^2 \big(|a_n|^2+|b_n|^2\big) \\ &\le \|\varphi(\cdot,\cdot,0,t)\|^2_{L^2(\Omega)} + \left\| \frac{\partial\varphi}{\partial t}(\cdot,\cdot,0,t)\right\|^2_{(H^1(\Omega))'} \\ &\le c_2 \sum_{n=1}^{\infty} \cosh(\lambda_n h)^2 \big(|a_n|^2+|b_n|^2\big), \end{aligned}$$

with two positive constants c_1 and c_2 which do not depend on the particular choice of t. Hence the desired inequality follows with $\alpha = c_2/c_1$. □

Fix an arbitrary positive number T and set

$$\delta := \max_n \left| \frac{\sin(\omega_n T)}{\omega_n T} \right|.$$

Observe that $0 \le \delta < 1$.

We are going to establish the following inequalities.

PROPOSITION 2.2. *All solutions of* (2.1) *satisfy the inequalities*

$$\begin{aligned} (1-\delta)\sum_{n=1}^{\infty} \cosh^2(\lambda_n h)\big(|a_n|^2+|b_n|^2\big) &\le \frac{1}{T}\int_0^T \int_\Omega |\varphi(x,y,0,t)|^2 \, dx\, dy\, dt \\ &\le (1+\delta)\sum_{n=1}^{\infty} \cosh^2(\lambda_n h)\big(|a_n|^2+|b_n|^2\big). \end{aligned}$$

Proof. We obtain by a straightforward computation, using the orthonormality of the sequence (e_n), the equality

$$\int_\Omega |\varphi(x,y,0,t)|^2 \, dx\, dy = \sum_{n=1}^{\infty} \cosh^2(\lambda_n h) \left| a_n e^{i\omega_n t} + b_n e^{-i\omega_n t}\right|^2,$$

and then the equality

$$\int_0^T \int_\Omega |\varphi(x,y,0,t)|^2 \, dx \, dy \, dt \\ = \sum_{n=1}^{\infty} \cosh^2(\lambda_n h) \int_0^T |a_n|^2 + |b_n|^2 + 2\Re(a_n \overline{b_n} e^{2i\omega_n t}) \, dt.$$

The proof will be completed if we show for every n the estimate

$$\left| \int_0^T 2\Re(a_n \overline{b_n} e^{2i\omega_n t}) \, dt \right| \leq \delta T\big(|a_n|^2 + |b_n|^2\big).$$

This can be shown as follows:

$$\begin{aligned} \left| \int_0^T 2\Re(a_n \overline{b_n} e^{2i\omega_n t}) \, dt \right| &= \left| 2\Re \left(a_n \overline{b_n} \int_0^T e^{2i\omega_n t} \, dt \right) \right| \\ &= \left| 2\Re \left(a_n \overline{b_n} \frac{e^{2i\omega_n T} - 1}{2i\omega_n} \right) \right| \\ &= \left| 2\Re \left(a_n \overline{b_n} e^{i\omega_n T} \frac{\sin \omega_n T}{\omega_n} \right) \right| \\ &\leq 2|a_n| \cdot |b_n| \cdot \left| \frac{\sin(\omega_n T)}{\omega_n} \right| \\ &\leq \delta T\big(|a_n|^2 + |b_n|^2\big). \qquad \square \end{aligned}$$

Let us consider two examples.

Example 1. If Ω is a rectangle, say

$$\Omega = (0,a) \times (0,b),$$

then it is more convenient to arrange the eigenfunctions and eigenvalues in a double sequence

$$\begin{aligned} e_{m,n}(x,y) &= \frac{2}{\sqrt{ab}} \cos \frac{m\pi x}{a} \cdot \cos \frac{n\pi y}{b}, \\ \lambda_{m,n} &= \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right), \end{aligned}$$

where m, n are integers ranging from 0 to ∞, and to set

$$\omega_{m,n} = \sqrt{g\lambda_{m,n} \tanh(\lambda_{m,n} h)}.$$

Then the formula (2.2) is replaced by

$$\varphi(x,y,z,t) \\ = \frac{2}{\sqrt{ab}} \sum_{m,n=0}^{\infty} \cos \frac{m\pi x}{a} \cdot \cos \frac{n\pi y}{b} \cdot \cosh[\lambda_{m,n}(z+h)]\big(a_{m,n} e^{i\omega_{m,n} t} + b_{m,n} e^{-i\omega_{m,n} t}\big),$$

and the sums in the estimate of Proposition 2.1 are replaced by

$$\sum_{m,n=0}^{\infty} \cosh^2(\lambda_{m,n}h)\big(|a_{m,n}|^2 + |b_{m,n}|^2\big).$$

Example 2. If Ω is a disc of radius R, then it is more convenient to use polar coordinates (r, θ) and to arrange the eigenfunctions and eigenvalues in a double sequence by using the Bessel functions, as, e.g., in [2] or [3, pp. 138–141]. Then the solutions of (2.1) are given by the series

$$\varphi(r, \theta, z, t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} J_n(\lambda_{nk}r)\cosh[\lambda_{nk}(z+h)] \cdot \Big(A_{nk}(\theta)e^{i\omega_{nk}t} + B_{nk}(\theta)e^{-i\omega_{nk}t}\Big),$$

where for each fixed n,

- J_n denotes the Bessel function of order n,
- $R\lambda_{n1} < R\lambda_{n2} < \cdots$ are the (strictly) positive roots of its derivative J_n',
- $\omega_{nk} = \sqrt{g\lambda_{nk}\tanh(\lambda_{nk}h)}$,
- $A_{nk}(\theta)$ and $B_{nk}(\theta)$ are suitable linear combinations of the functions $\cos(k\theta)$ and $\sin(k\theta)$, depending on the initial data.

The sums in the estimate of Proposition 2.1 are replaced by

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\left(\int_0^R rJ_n(r)^2\,dr\right)\cdot\cosh^2(\lambda_{nk}h)\cdot\left(\int_0^{2\pi}|A_{nk}(\theta)|^2 + |B_{nk}(\theta)|^2\,d\theta\right).$$

3. Well posedness of the nonhomogeneous problem. Now let us consider the following nonhomogeneous version of problem (2.1) for an unknown function $\psi(x, y, z, t)$:

$$\begin{cases} \Delta\psi = 0 \quad \text{in} \quad \Omega\times(-h,0)\times\mathbb{R},\\ \partial_\nu\psi = 0 \quad \text{on} \quad \Gamma\times(-h,0)\times\mathbb{R},\\ \partial_\nu\psi = 0 \quad \text{on} \quad \Omega\times\{-h\}\times\mathbb{R},\\ \frac{\partial^2\psi}{\partial t^2} + g\frac{\partial\psi}{\partial z} = v \quad \text{on} \quad \Omega\times\{0\}\times\mathbb{R},\\ \psi(x,y,0,0) = \psi_0(x,y), \quad (x,y)\in\Omega,\\ \frac{\partial\psi}{\partial t}(x,y,0,0) = \psi_1(x,y), \quad (x,y)\in\Omega. \end{cases} \tag{3.1}$$

We are going to define the solutions of this problem by the method of transposition. Consider an arbitrary solution of (2.1). By a formal computation, we have for every real number T the following equalities:

$$\begin{aligned} 0 &= \int_0^T\int_{-h}^0\int_\Omega (\Delta\varphi)\psi - \varphi(\Delta\psi)\;dx\;dy\;dz\;dt\\ &= \int_0^T\int_{-h}^0\int_\Gamma (\partial_\nu\varphi)\psi - \varphi(\partial_\nu\psi)\;\Gamma\;dz\;dt\\ &\qquad + \int_0^T\int_\Omega \big((\partial_\nu\varphi)\psi - \varphi(\partial_\nu\psi)\big)(x,y,-h,t)\;dx\;dy\;dt\\ &\qquad + \int_0^T\int_\Omega \big((\partial_\nu\varphi)\psi - \varphi(\partial_\nu\psi)\big)(x,y,0,t)\;dx\;dy\;dt\\ &= \int_0^T\int_\Omega \left(\frac{\partial\varphi}{\partial z}\psi - \varphi\frac{\partial\psi}{\partial z}\right)(x,y,0,t)\;dx\;dy\;dt \end{aligned}$$

$$= \frac{1}{g}\int_0^T \int_\Omega \left(-\frac{\partial^2\varphi}{\partial t^2}\psi + \varphi\frac{\partial^2\psi}{\partial t^2} - \varphi v\right)(x,y,0,t)\ dx\ dy\ dt$$
$$= -\frac{1}{g}\int_0^T \int_\Omega (\varphi v)(x,y,0,t)\ dx\ dy\ dt + \frac{1}{g}\left[\int_\Omega \left(-\frac{\partial\varphi}{\partial t}\psi + \varphi\frac{\partial\psi}{\partial t}\right)(x,y,0,t)\ dx\ dy\right]_0^T.$$

Taking the initial conditions into account, it follows that

$$\int_\Omega \left(-\frac{\partial\varphi}{\partial t}\psi + \varphi\frac{\partial\psi}{\partial t}\right)(x,y,0,T)\ dx\ dy = \int_\Omega (-\varphi_1\psi_0 + \varphi_0\psi_1)\ dx\ dy + \int_0^T \int_\Omega (\varphi v)(x,y,0,t)\ dx\ dy\ dt.$$

Identifying $L^2(\Omega)$ with its dual $(L^2(\Omega))'$ as usual, we have the dense and continuous inclusions

$$(H^1(\Omega))' \subset (L^2(\Omega))' = L^2(\Omega) \subset H^1(\Omega).$$

Then we may rewrite the last identity in the following form:

$$\left\langle \left(\frac{\partial\psi}{\partial t}, -\psi\right)(\cdot,\cdot,0,T), \left(\varphi, \frac{\partial\varphi}{\partial t}\right)(\cdot,\cdot,0,T)\right\rangle_{L^2(\Omega)\times H^1(\Omega), L^2(\Omega)\times(H^1(\Omega))'} = \langle(\psi_1,-\psi_0),(\varphi_0,\varphi_1)\rangle_{L^2(\Omega)\times H^1(\Omega), L^2(\Omega)\times(H^1(\Omega))'} + (\varphi(\cdot,\cdot,0,\cdot),v)_{L^2(\Omega\times(0,T))}. \tag{3.2}$$

This leads to the following definition.

DEFINITION 3.1. *A solution of* (3.1) *is a function*

$$\psi(\cdot,\cdot,0,\cdot) \in C(\mathbb{R}; H^1(\Omega)) \cap C^1(\mathbb{R}; L^2(\Omega)),$$

satisfying (3.2) *for all* $\varphi_0 \in L^2(\Omega)$ *and* $\varphi_1 \in (H^1(\Omega))'$.

The definition is justified by the following claim.

PROPOSITION 3.2. *Given* $\psi_0 \in H^1(\Omega)$, $\psi_1 \in L^2(\Omega)$, *and* $v \in L^2_{loc}(\mathbb{R}; L^2\Omega))$ *arbitrarily, problem* (3.1) *has a unique solution.*

Proof. For any fixed T, the right-hand side of (3.2) defines a bounded linear form of

$$(\varphi_0,\varphi_1) \in L^2(\Omega) \times (H^1(\Omega))'.$$

Since the linear map

$$(\varphi_0,\varphi_1) \mapsto \left(\varphi, \frac{\partial\varphi}{\partial t}\right)(\cdot,\cdot,0,T)$$

is an automorphism of $L^2(\Omega)\times(H^1(\Omega))'$ onto itself, the existence of a unique couple

$$\left(\psi, \frac{\partial\psi}{\partial t}\right)(\cdot,\cdot,0,T) \in L^2(\Omega)\times(H^1(\Omega))'$$

satisfying (3.2) follows.

Since both sides of (3.2) change continuously with T, the solution also depends continuously of T, thereby completing the proof of Proposition 3.2. $\square$

4. Boundary controllability of liquid containers. A crucial idea in the HUM of Lions (see [4] and [5]) was the construction of suitable controls by solving a corresponding homogeneous dual problem. Using this duality approach, we establish in this section the following controllability result for problem (3.1).

THEOREM 4.1. *Given a positive number T and initial data $\psi_0 \in H^1(\Omega)$, $\psi_1 \in L^2(\Omega)$, there exists a function $v \in L^2(0,T;L^2(\Omega))$ such that the solution of* (3.1) *satisfies*

$$\left(\psi, \frac{\partial \psi}{\partial t}\right)(\cdot,\cdot,0,T) = 0.$$

Proof. Given $\varphi_0 \in L^2(\Omega)$ and $\varphi_1 \in (H^1(\Omega))'$ arbitrarily, solve the homogeneous problem (2.1) and then solve the following nonhomogeneous problem:

$$\begin{cases} \Delta\psi = 0 \quad \text{in} \quad \Omega\times(-h,0)\times\mathbb{R}, \\ \partial_\nu\psi = 0 \quad \text{on} \quad \Gamma\times(-h,0)\times\mathbb{R}, \\ \partial_\nu\psi = 0 \quad \text{on} \quad \Omega\times\{-h\}\times\mathbb{R}, \\ \frac{\partial^2\psi}{\partial t^2} + g\frac{\partial\psi}{\partial z} = \varphi \quad \text{on} \quad \Omega\times\{0\}\times\mathbb{R}, \\ \psi(x,y,0,T) = 0, \quad (x,y)\in\Omega, \\ \frac{\partial\psi}{\partial t}(x,y,0,T) = 0, \quad (x,y)\in\Omega. \end{cases} \tag{4.1}$$

(The well-posedness of this problem follows from Proposition 3.2 because the time 0 does not play any special role in this problem.) It is sufficient to prove that for a suitable choice of the initial data φ_0 and φ_1 we have

$$\psi(x,y,0,0) = \psi_0(x,y) \quad \text{and} \quad \frac{\partial\psi}{\partial t}(x,y,0,0) = \psi_1(x,y), \quad (x,y)\in\Omega.$$

Using the identity (3.2) of the preceding section, now we have

$$\langle(-\psi_1,\psi_0),(\varphi_0,\varphi_1)\rangle_{L^2(\Omega)\times H^1(\Omega),L^2(\Omega)\times(H^1(\Omega))'} = \int_0^T\int_\Omega |\varphi(x,y,0,t)|^2\,dx\,dy\,dt. \tag{4.2}$$

Thanks to Proposition 2.2, the right-hand side of this identity is a positive definite quadratic form of $(\varphi_0,\varphi_1)\in L^2(\Omega)\times(H^1(\Omega))'$. Applying the Lax–Milgram theorem (or simply the Riesz–Fréchet theorem), we conclude that the linear map

$$(\varphi_0,\varphi_1)\mapsto(-\psi_1,\psi_0)$$

maps $L^2(\Omega)\times(H^1(\Omega))'$ onto $L^2(\Omega)\times H^1(\Omega)$, and the theorem follows. ∎

5. Spectral boundary controllability of liquid containers. The controls applied in the preceding section are difficult to realize in practice. Here we present a more realistic variant: we apply simpler controls but we only look for partial controllability by eliminating a finite number of modes from the solution.

Choose a positive integer N and N functions $f_1(x,y),\dots,f_N(x,y)$ on Ω. Using controls of the form

$$v(x,y,t) := \sum_{j=1}^N (a_jt+b_j)f_j(x,y),$$

where we can act by choosing suitable constants a_j and b_j for $j = 1, \dots, N$, we try to drive the system to a final state satisfying the orthogonality conditions

$$\int_\Omega \psi(x, y, 0, T) e_n(x, y)\ dx\ dy = \int_\Omega \frac{\partial \psi}{\partial t}(x, y, 0, T) e_n(x, y)\ dx\ dy = 0$$

for $n = 1, \dots, N$.

Expanding the given functions $f_j(x, y)$ into Fourier series

$$f_j(x, y) = \sum_{n=1}^{\infty} f_{jn} e_n(x, y), \quad j = 1, \dots, N,$$

the control takes the form

$$v(x, y, t) := \sum_{n=1}^{\infty} \left(\sum_{j=1}^{N} f_{jn}(a_j t + b_j) \right) e_n(x, y).$$

Therefore the solution of (3.1) is given by the series

$$\psi(x, y, z, t) = \sum_{n=1}^{\infty} e_n(x, y) \cosh[\lambda_n(z + h)] g_n(t),$$

where the functions $g_n(t)$ are solutions to the initial-value problems

$$\begin{aligned} &g_n''(t) \cosh(\lambda_n h) + g_n(t) g \lambda_n \sinh(\lambda_n h) = \sum_{j=1}^{N} f_{jn}(a_j t + b_j), \\ &g_n(0) \cosh(\lambda_n h) = c_n, \\ &g_n'(0) \cosh(\lambda_n h) = d_n, \end{aligned}$$

where the constants c_n and d_n depend on the initial data.

We have to prove that for any given c_n and d_n there exist constants a_n and b_n such that

$$g_n(T) = g_n'(T) = 0, \quad n = 1, \dots, N. \tag{5.1}$$

Equivalently, we have to prove that if $c_n = d_n = 0$ for $n = 1, \dots, N$, then we only have the trivial solution $a_n = b_n = 0$ for $n = 1, \dots, N$.

In this homogeneous case we explicitly compute the solution: setting

$$\omega_n = \sqrt{g \lambda_n \tanh(\lambda_n h)}$$

as before, we have

$$g_n(t) = \sum_{j=1}^{N} f_{jn} \left(\frac{\omega_n t - \sin(\omega_n t)}{\omega_n^3} a_j + \frac{1 - \cos(\omega_n t)}{\omega_n^2} b_j \right)$$

for $n = 1, \dots, N$. It follows that the conditions (5.1) are equivalent to the following system of linear equations:

$$\begin{aligned} &\sum_{j=1}^{N} f_{jn} \big(\omega_n T - \sin(\omega_n T) \big) a_j + \sum_{j=1}^{N} f_{jn} \omega_n \big(1 - \cos(\omega_n T) \big) b_j = 0, \qquad n = 1, \dots, N, \\ &\sum_{j=1}^{N} f_{jn} \big(1 - \cos(\omega_n T) \big) a_j + \sum_{j=1}^{N} f_{jn} \omega_n \sin(\omega_n T) b_j = 0, \qquad n = 1, \dots, N. \end{aligned}$$

We arrive in this way at the following result.

THEOREM 5.1. *Assume that the determinant of the above system is different from zero for some $T > 0$. Then for any given initial data $\psi_0 \in H^1(\Omega)$, $\psi_1 \in L^2(\Omega)$, there exist constants $a_1, \dots, a_N$ and $b_1, \dots, b_N$ such that, applying the control*

$$v(x, y, t) := \sum_{j=1}^{N} (a_j t + b_j) f_j(x, y),$$

the solution of (3.1) *satisfies*

$$\int_\Omega \psi(x, y, 0, T) e_n(x, y)\ dx\ dy = \int_\Omega \frac{\partial \psi}{\partial t}(x, y, 0, T) e_n(x, y)\ dx\ dy = 0$$

for $n = 1, \dots, N$.

Example 3. Let us consider the simplest case where $N = 1$ and $f_1(x, y) = e_1(x, y)$. Then, setting $\alpha := T\omega_1$ for brevity, the determinant is equal to

$$\begin{vmatrix} \alpha - \sin\alpha & \omega_1(1 - \cos\alpha) \\ 1 - \cos\alpha & \omega_1 \sin\alpha \end{vmatrix} = 2\omega_1 \sin\alpha \left(\frac{\alpha}{2} - \tan\frac{\alpha}{2} \right).$$

It follows that the determinant vanishes if and only if $\alpha = T\omega_1$ is an integer multiple of π, i.e., if and only if

$$T = \frac{k\pi}{\sqrt{g\lambda_1 \tanh(\lambda_1 h)}}$$

for some positive integer k. For all other values of $T > 0$ the problem is controllable in the above sense.

Remark. From the point of view of applications, it is interesting to find control constants a_j and b_j of a moderate size: in some sense the size of these constants may be viewed as a measure of the cost of the control. (Furthermore, for large values of the controls our model may not be any more realistic.) This is equivalent to choosing functions $f_1, \dots f_N$ such that the inverse of the matrix of the above linear system has a sufficiently large norm. On the other hand, it is also useful to find relatively simple functions f_j, concentrated on some points of the domain Ω. These two, in a sense contradictory, requirements lead to an interesting optimization problem of a geometric nature, which can be studied in special cases if, for example, Ω is a rectangle or a disk.

REFERENCES

[1] H. N. ABRAMSON, *The Dynamic Behavior of Liquids in Moving Containers*, NASA-SP-106, 1966.

[2] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics* I, John Wiley & Sons, New York, 1989.

[3] D. JACKSON, *Fourier Series and Orthogonal Polynomials*, Carus Math. Monogr. 6, The Mathematical Association of America, Oberlin, OH, 1941.

[4] J.-L. LIONS, *Exact controllability, stabilization, and perturbations for distributed systems*, SIAM Rev., 30 (1988), pp. 1–68.

[5] J.-L. LIONS, *Contrôlabilité exacte et stabilisation de systèmes distribués*, Vol. 1, Masson, Paris, 1988.

[6] S. SILVERMAN AND H. N. ABRAMSON, *Lateral sloshing in moving containers*, in The Dynamic Behavior of Liquids in Moving Containers, H. N. Abramson, Technical Report NASA-SP-106, 1966, pp. 13–78.

[7] R. TEMAM, *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, 1983.