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Robust $\mathscr{H}_\infty$ static output-feedback design for time-invariant discrete-time polytopic systems from parameter-dependent state-feedback gains

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Abstract— **This paper investigates the problem of designing $\mathscr{H}_\infty$ robust output-feedback controllers for uncertain discrete-time linear systems with time-invariant parameters lying in polytopic domains. The computation procedure is based on linear matrix inequalities and is performed in two steps. The first stage designs a *parameter-dependent* state-feedback controller, that is used as an input parameter for the second stage, which synthesizes the desired *robust* $\mathscr{H}_\infty$ output-feedback gain. The conditions are based on parameter-dependent Lyapunov functions and can deal with uncertainties in the output matrix of the system. Numerical examples illustrate the advantages of the proposed method when compared to other techniques available in the literature.**

I. INTRODUCTION

One of the main purposes in control theory is to provide methods to synthesize stabilizing controllers for dynamic systems, and several approaches are available in the literature. The most popular approach is probably the one based on the Lyapunov theory. The existence of a Lyapunov function satisfying certain conditions, which can be assessed by applying widely known and well developed numerical methods, is directly related to the computation of a stabilizing controller, possibly considering some performance criterion [1–4]. The existence conditions may be cast as matrix inequalities and a feasible solution provides the Lyapunov matrix and the desired controllers.

In the case of state-feedback design for precisely known systems, the matrix inequality conditions are linear. Therefore, the synthesis conditions can be solved by convex optimization procedures based on Linear Matrix Inequalities (LMIs) [5, 6]. Problems involving LMIs can be programmed through simple interfaces (LMI Control Toolbox, YALMIP) and solved by efficient algorithms (SeDuMi) [7–9].

For uncertain systems, however, the state-feedback synthesis conditions cannot be formulated directly as LMIs. One simple approach is to consider synthesis conditions for each vertex of the polytope and then to determine a single Lyapunov matrix satisfying all the conditions. Such approach guarantees the quadratic stability of the system [10], though the conditions are only sufficient and, in general, conservative. To reduce the conservativeness one can employ parameter-dependent Lyapunov matrices [11, 12], but the conditions do not cast as LMIs straightforwardly. Concerning discrete-time uncertain systems, it is possible to synthesize robust controllers by inserting slack variables in the LMI conditions that provide the gain and the parameter-dependent Lyapunov matrix [13]. The conditions obtained are less conservative and contain the quadratic stability condition as a particular case.

In practice, state-feedback controllers are difficult to implement since the states may be not available for feedback. On the other hand, output-feedback controllers are easy to implement, but the synthesis procedure is more complex [14]. Even in the case of precisely known systems, the synthesis conditions are Bilinear Matrix Inequalities (BMIs) which are known to be NP-hard [15]. In general, to obtain LMI conditions it is necessary to impose constraints, resulting on conservativeness. In the case of uncertain systems, one cannot use the same parameterizations applied to the state-feedback case unless some structural constraints are imposed on the system matrices. For instance, to impose that the output matrix is precisely known [14, 16, 17].

The method proposed in this paper is capable to synthesize a robust output-feedback controller even when the output matrix is uncertain, being similar to the strategy adopted in [18, 19] (continuous-time) and [20] (discrete-time). In this paper, the method is applied to time-invariant discrete-time systems and consists of two stages: in the first stage a parameter-dependent state-feedback controller is synthesized; the result is then used as an input for the second stage, in which the robust output-feedback controller is determined. Extensions to cope with $\mathscr{H}_\infty$ state-feedback and output-feedback control are also developed. The immediate advantage of this method when compared to the one presented in [20] is that the system does not need to be robustly stabilizable by a robust state-feedback controller.

The paper is organized as follows. In Section II the preliminary results and the notation are presented. The main results of the paper are given in Section III. In Section IV numerical examples illustrate the advantages of the approach and Section V concludes the paper.

II. PRELIMINARIES

Consider the discrete-time time-invariant linear system described by

$$\begin{aligned} x(k+1) &= A(\alpha)x(k) + B_1(\alpha)w(k) + B_2(\alpha)u(k), \\ z(k) &= C_1(\alpha)x(k) + D_1(\alpha)w(k) + D_2(\alpha)u(k), \\ y(k) &= C_2(\alpha)x(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^r$ the exogenous input, $u(k) \in \mathbb{R}^m$ the control input, $z(k) \in \mathbb{R}^p$ the controlled output and $y(k) \in \mathbb{R}^q$ is the measured output. The system matrices $A(\alpha) \in \mathbb{R}^{n\times n}$, $B_1(\alpha) \in \mathbb{R}^{n\times r}$, $B_2(\alpha) \in \mathbb{R}^{n\times m}$, $C_1(\alpha) \in$

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$\mathbb{R}^{p\times n}$, $C_2(\alpha)\in\mathbb{R}^{q\times n}$, $D_1(\alpha)\in\mathbb{R}^{p\times r}$ and $D_2(\alpha)\in\mathbb{R}^{p\times m}$ are not precisely known and belong to the polytope

$$\begin{aligned}\mathscr{D}=\big\{&(A,B_1,B_2,C_1,C_2,D_1,D_2)(\alpha):\\&(A,B_1,B_2,C_1,C_2,D_1,D_2)(\alpha)=\\&\sum_{i=1}^{N}\alpha_i(A,B_1,B_2,C_1,C_2,D_1,D_2)_i,\ \alpha\in\Lambda_N\big\},\end{aligned}$$

where $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_N)'$ is the vector of time-invariant uncertain parameters lying in the unit simplex Λ_N given by

$$\Lambda_N=\big\{\delta\in\mathbb{R}^N:\sum_{i=1}^{N}\delta_i=1,\ \delta_i\geq 0,\ i=1,\dots,N\big\}. \tag{2}$$

This uncertainty model, known as *polytopic model*, is largely used in the literature. Matrices A_i, B_{1i}, B_{2i}, C_{1i}, C_{2i}, D_{1i} and D_{2i} are known as the vertices of the system and are given *a priori*.

For a fixed α, the transfer function of the open-loop system from the input vector $w(k)$ to the controlled output vector $z(k)$ is denoted by

$$H(\alpha,\xi)=C_1(\alpha)(\xi\mathbf{I}-A(\alpha))^{-1}B_1(\alpha)+D_1(\alpha) \tag{3}$$

and the $\mathscr{H}_\infty$ norm, for a fixed α, is given by

$$||H(\alpha)||_\infty=\max_{\omega\in[-\pi,\pi]}\sigma_{max}\big(H(\alpha,\exp(j\omega))\big), \tag{4}$$

being σ_{max} the maximum singular value. The maximum value of $\mathscr{H}_\infty$ norm that an uncertain system may assume is known as the worst-case $\mathscr{H}_\infty$ norm or the optimal $\mathscr{H}_\infty$ guaranteed cost and is given by

$$\max_{\alpha\in\Lambda_N}||H(\alpha)||_\infty. \tag{5}$$

Lemma 1: $A(\alpha)$ is Schur stable for all $\alpha\in\Lambda_N$ and the inequality $||H(\alpha)||_\infty^2<\gamma^2$ holds if, and only if, there exists a symmetric parameter-dependent matrix $P(\alpha)$ such that

$$\begin{bmatrix}P(\alpha) & A(\alpha)'P(\alpha) & \mathbf{0} & C_1(\alpha)'\\ \star & P(\alpha) & P(\alpha)B_1(\alpha) & \mathbf{0}\\ \star & \star & \gamma^2\mathbf{I} & D_1(\alpha)'\\ \star & \star & \star & \mathbf{I}\end{bmatrix}>0 \tag{6}$$

is feasible $\forall\alpha\in\Lambda_N$, being $\star$ the symmetric block in the LMI.

The result in Lemma 1 is also known (for α fixed) as the Bounded Real Lemma [5]. The worst-case $\mathscr{H}_\infty$ norm of system (1) can be calculated by solving the following optimization problem

$$\gamma^*=\min_{P(\alpha)}\gamma \quad \text{s.t. (6) hold}$$

using, for instance, the convergent LMI relaxations based on polynomial approximations for $P(\alpha)$ proposed in [21] (see also [22]).

The problem investigated in this paper is to determine a gain K associated to the static output-feedback control law $u(k)=Ky(k)$ such that the closed-loop system is robustly stable for all $\alpha\in\Lambda_N$ with a prescribed $\mathscr{H}_\infty$ guaranteed cost. The robust LMIs of Lemma 1 are explored in the sequel for state and output-feedback design by considering particular structures for matrix $P(\alpha)$, yielding finite dimensional LMI relaxations.

III. MAIN RESULTS

The following theorem presents a condition to obtain a parameter-dependent state-feedback stabilizing gain for system (1).

Theorem 1: If there exist symmetric matrices $P_i\in\mathbb{R}^{n\times n}$, matrices $G_i\in\mathbb{R}^{n\times n}$, $Z_i\in\mathbb{R}^{m\times n}$, $i=1,\dots,N$, such that the following LMIs are verified

$$\begin{bmatrix}P_i & A_iG_i+B_{2i}Z_i\\ \star & G_i+G_i'-P_i\end{bmatrix}>0,\quad i=1,\dots,N \tag{7}$$

$$\begin{gathered}\begin{bmatrix}P_i+P_j & A_iG_j+A_jG_i+B_{2i}Z_j+B_{2j}Z_i\\ \star & G_i+G_j+G_i'+G_j'-P_i-P_j\end{bmatrix}>0,\\ i=1,\dots,N-1\ ,\quad j=i+1,\dots,N\end{gathered} \tag{8}$$

then the system (1) is stabilizable by the parameter-dependent control law $u(k)=Z(\alpha)G(\alpha)^{-1}x(k)$, where $Z(\alpha)$ and $G(\alpha)$ are given by

$$Z(\alpha)=\sum_{i=1}^{N}\alpha_iZ_i\ ,\quad G(\alpha)=\sum_{i=1}^{N}\alpha_iG_i\ ,\quad \alpha\in\Lambda_N. \tag{9}$$

Proof: Multiply (7) by α_i^2 and sum for $i=1,\dots,N$. Multiply (8) by $\alpha_i\alpha_j$ and sum for $i=1,\dots,N-1$, $j=i+1,\dots,N$. Summing the results one has

$$\begin{bmatrix}P(\alpha) & A_{cl}(\alpha)G(\alpha)\\ \star & G(\alpha)+G(\alpha)'-P(\alpha)\end{bmatrix}>0, \tag{10}$$

with $A_{cl}(\alpha)=A(\alpha)+B_2(\alpha)Z(\alpha)G(\alpha)^{-1}$. Multiply (10) on the left by $[\mathbf{I}\quad -A_{cl}(\alpha)']$ and on the right by the respective transpose, obtaining $A_{cl}(\alpha)P(\alpha)A_{cl}(\alpha)'-P(\alpha)<0$, which proves the stability of the closed-loop system with the parameter-dependent state-feedback gain $Z(\alpha)G(\alpha)^{-1}$. ∎

Theorem 1 is concerned only with the stabilization of the system. In the following, a synthesis condition for a state-feedback parameter-dependent stabilizing gain with an $\mathscr{H}_\infty$ guaranteed cost is presented.

Theorem 2: If there exist symmetric matrices $P_i\in\mathbb{R}^{n\times n}$, matrices $G_i\in\mathbb{R}^{n\times n}$, $Z_i\in\mathbb{R}^{m\times n}$, $i=1,\dots,N$ and $\gamma>0$ such that the following LMIs are verified

$$\begin{bmatrix}P_i & A_iG_i+B_{2i}Z_i & \mathbf{0} & B_{1i}\\ \star & G_i+G_i'-P_i & G_i'C_{1i}'+Z_i'D_{2i}' & \mathbf{0}\\ \star & \star & \gamma^2\mathbf{I} & D_{1i}\\ \star & \star & \star & \mathbf{I}\end{bmatrix}>0,\ i=1,\dots,N \tag{11}$$

$$\begin{gathered}\begin{bmatrix}P_i+P_j & A_iG_j+A_jG_i+B_{2i}Z_j+B_{2j}Z_i & \mathbf{0} & B_{1i}+B_{1j}\\ \star & G_i+G_j+G_i'+G_j'-P_i-P_j & G_i'C_{1j}'+G_j'C_{1i}'+Z_i'D_{2j}'+Z_j'D_{2i}' & \mathbf{0}\\ \star & \star & 2\gamma^2\mathbf{I} & D_{1i}+D_{1j}\\ \star & \star & \star & 2\mathbf{I}\end{bmatrix}>0,\\ i=1,\dots,N-1\qquad j=i+1,\dots,N\ ,\end{gathered} \tag{12}$$

then the state-feedback parameter-dependent control law given by $u(k)=Z(\alpha)G(\alpha)^{-1}x(k)$, with $Z(\alpha)$ and $Z(\alpha)$ as in (9), stabilizes system (1) with an $\mathscr{H}_\infty$ guaranteed cost given by γ.

Proof: Multiply (11) by α_i^2 and sum for $i=1,\ldots,N$. Multiply (12) by $\alpha_i\alpha_j$ and sum for $i=1,\ldots,N-1$, $j=i+1,\ldots,N$. Summing the results one has

$$\begin{bmatrix} P(\alpha) & A(\alpha)G(\alpha)+B_2(\alpha)Z(\alpha) & \mathbf{0} & B_1(\alpha) \\ \star & G(\alpha)+G(\alpha)'-P(\alpha) & G(\alpha)'C_1(\alpha)'+Z(\alpha)'D_2(\alpha)' & \mathbf{0} \\ \star & \star & \gamma^2\mathbf{I} & D_1(\alpha) \\ \star & \star & \star & \mathbf{I} \end{bmatrix} > 0. \quad (13)$$

Let $A_{cl}(\alpha) = A(\alpha) + B_2(\alpha)Z(\alpha)G(\alpha)^{-1}$ and $C_{cl}(\alpha) = C_1(\alpha)+D_2(\alpha)Z(\alpha)G(\alpha)^{-1}$. Inequality (13) then yields

$$\begin{bmatrix} P(\alpha) & A_{cl}(\alpha)G(\alpha) & \mathbf{0} & B_1(\alpha) \\ \star & G(\alpha)+G(\alpha)'-P(\alpha) & G(\alpha)'C_{cl}(\alpha)' & \mathbf{0} \\ \star & \star & \gamma^2\mathbf{I} & D_1(\alpha) \\ \star & \star & \star & \mathbf{I} \end{bmatrix} > 0. \quad (14)$$

Hence, $G(\alpha)+G(\alpha)' > P(\alpha) > 0$ and, consequently, the inequality

$$(P(\alpha)-G(\alpha))'P(\alpha)^{-1}(P(\alpha)-G(\alpha)) \geq 0$$

holds. Thus, $G(\alpha)'P(\alpha)^{-1}G(\alpha) \geq G(\alpha)+G(\alpha)'-P(\alpha)$, implying that

$$\begin{bmatrix} P(\alpha) & A_{cl}(\alpha)G(\alpha) & \mathbf{0} & B_1(\alpha) \\ \star & G(\alpha)'P(\alpha)^{-1}G(\alpha) & G(\alpha)'C_{cl}(\alpha)' & \mathbf{0} \\ \star & \star & \gamma^2\mathbf{I} & D_1(\alpha) \\ \star & \star & \star & \mathbf{I} \end{bmatrix} > 0. \quad (15)$$

Multiplying on the right by $\mathrm{diag}(\mathbf{I},G(\alpha)^{-1}P(\alpha),\mathbf{I},\mathbf{I})$ and on the left by its transpose, one has

$$\begin{bmatrix} P(\alpha) & A_{cl}(\alpha)P(\alpha) & \mathbf{0} & B_1(\alpha) \\ \star & P(\alpha) & P(\alpha)C_{cl}(\alpha)' & \mathbf{0} \\ \star & \star & \gamma^2\mathbf{I} & D_1(\alpha) \\ \star & \star & \star & \mathbf{I} \end{bmatrix} > 0 \quad (16)$$

which is the Bounded Real Lemma shown in Lemma 1 for the dual system $(A(\alpha)',C_1(\alpha)',B_1(\alpha)',D_1(\alpha)')$. ∎

Note that the gain provided by Theorem 2 is parameter-dependent. If a robust gain is needed, the following result can be used.

Corollary 1: If there exist symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, $i=1,\ldots,N$, matrices $G \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{m\times n}$ and $\gamma > 0$ such that the LMIs given in (11) and (12) are verified with $G_i = G$, $Z_i = Z$, $i=1,\ldots,N$, then system (1) is stabilizable with the robust control law $u(k) = ZG^{-1}x(k)$.

The condition of Corollary 1 was originally published in [16] and will be used in the numerical experiments for numerical comparisons. In the following, the conditions for the synthesis of a robust output-feedback gain with an $\mathscr{H}_\infty$ guaranteed cost are presented. The conditions use as inputs matrices Z_i and G_i obtained from the synthesis of the parameter-dependent state-feedback gain.

Theorem 3: Consider matrices $Z_i \in \mathbb{R}^{m\times n}$ and $G_i \in \mathbb{R}^{n\times n}$ solutions of Theorem 1 or Theorem 2. If there exist symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, matrices $F_i \in \mathbb{R}^{n\times n}$, $H_i \in \mathbb{R}^{p\times p}$, $i=1,\ldots,N$, matrices $R \in \mathbb{R}^{m\times m}$, $L \in \mathbb{R}^{m\times q}$ and $\gamma > 0$ such that the following LMIs are verified

$$\begin{bmatrix} G_i'P_iG_i & G_i'A_i'F_i+Z_i'B_{2i}'F_i & \mathbf{0} & G_i'C_{1i}'H_i+Z_i'D_{2i}'H_i & G_i'C_{2i}'L'-Z_i'R' \\ \star & F_i+F_i'-P_i & F_i'B_{1i} & \mathbf{0} & F_i'B_{2i} \\ \star & \star & \gamma^2\mathbf{I} & D_{1i}'H_i & \mathbf{0} \\ \star & \star & \star & H_i+H_i'-\mathbf{I} & H_i'D_{2i} \\ \star & \star & \star & \star & -R-R' \end{bmatrix} > 0, i=1,\ldots,N \quad (17)$$

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \mathbf{0} & \Theta_{14} & \Theta_{15} \\ \star & \Theta_{22} & \Theta_{23} & \mathbf{0} & \Theta_{25} \\ \star & \star & 3\gamma^2\mathbf{I} & \Theta_{34} & \mathbf{0} \\ \star & \star & \star & \Theta_{44} & \Theta_{45} \\ \star & \star & \star & \star & -3(R+R') \end{bmatrix} > 0,$$

$$i=1,\ldots,N\ , \quad j\neq i\ , \quad j=1,\ldots,N, \quad (18)$$

with

$$\begin{aligned} \Theta_{11} =\ & G_i'P_iG_j+G_j'P_iG_i+G_i'P_jG_i, \\ \Theta_{12} =\ & G_i'A_i'F_j+G_j'A_i'F_i+G_i'A_j'F_i \\ & +Z_i'B_{2i}'F_j+Z_j'B_{2i}'F_i+Z_i'B_{2j}'F_i, \\ \Theta_{14} =\ & G_i'C_{1i}'H_j+G_j'C_{1i}'H_i+G_i'C_{1j}'H_i \\ & +Z_i'D_{2i}'H_j+Z_j'D_{2i}'H_i+Z_i'D_{2j}'H_i, \\ \Theta_{15} =\ & (G_i'C_{2i}'+G_i'C_{2j}'+G_j'C_{2i}')L'-(2Z_i'+Z_j')R', \\ \Theta_{22} =\ & 2F_i+F_j+2F_i'+F_j'-2P_i-P_j, \\ \Theta_{23} =\ & F_i'B_{1i}+F_i'B_{1j}+F_j'B_{1i}, \\ \Theta_{25} =\ & F_i'B_{2i}+F_i'B_{2j}+F_j'B_{2i}, \\ \Theta_{34} =\ & D_{1i}'H_i+D_{1i}'H_j+D_{1j}'H_i, \\ \Theta_{44} =\ & 2H_i+H_j+2H_i'+H_j'-3\mathbf{I}, \\ \Theta_{45} =\ & H_i'D_{2i}+H_i'D_{2j}+H_j'D_{2i}, \end{aligned}$$

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \mathbf{0} & \Upsilon_{14} & \Upsilon_{15} \\ \star & \Upsilon_{22} & \Upsilon_{23} & \mathbf{0} & \Upsilon_{25} \\ \star & \star & 6\gamma^2\mathbf{I} & \Upsilon_{34} & \mathbf{0} \\ \star & \star & \star & \Upsilon_{44} & \Upsilon_{45} \\ \star & \star & \star & \star & -6(R+R') \end{bmatrix} > 0,$$

$$i=1,\ldots,N-2, j=i+1,\ldots,N-1, k=j+1,\ldots,N \quad (19)$$

with

$$\begin{aligned} \Upsilon_{11} =\ & G_j'P_iG_k+G_k'P_iG_j+G_i'P_jG_k \\ & +G_k'P_jG_i+G_i'P_kG_j+G_j'P_kG_i, \\ \Upsilon_{12} =\ & G_j'A_i'F_k+G_k'A_i'F_j+G_i'A_j'F_k+G_k'A_j'F_i \\ & +G_i'A_k'F_j+G_j'A_k'F_i+Z_j'B_{2i}'F_k+Z_k'B_{2i}'F_j \\ & +Z_i'B_{2j}'F_k+Z_k'B_{2j}'F_i+Z_i'B_{2k}'F_j+Z_j'B_{2k}'F_i \\ \Upsilon_{14} =\ & G_j'C_{1i}'H_k+G_k'C_{1i}'H_j+G_i'C_{1j}'H_k+G_k'C_{1j}'H_i \\ & +G_i'C_{1k}'H_j+G_j'C_{1k}'H_i+Z_j'D_{2i}'H_k+Z_k'D_{2i}'H_j \\ & +Z_i'D_{2j}'H_k+Z_k'D_{2j}'H_i+Z_i'D_{2k}'H_j+Z_j'D_{2k}'H_i \end{aligned}$$

$$
\begin{aligned}
\Upsilon_{15} = \;& (G_i'C_{2j}' + G_j'C_{2i}' + G_i'C_{2k}' + G_k'C_{2i}' + \\
& G_j'C_{2k}' + G_k'C_{2j}')L' - 2(Z_i' + Z_j' + Z_k')R' \\
\Upsilon_{22} = \;& 2(F_i + F_j + F_k + F_i' + F_j' + F_k' - P_i - P_j - P_k) \\
\Upsilon_{23} = \;& F_i'B_{1j} + F_j'B_{1i} + F_i'B_{1k} + F_k'B_{1i} + F_j'B_{1k} + F_k'B_{1j} \\
\Upsilon_{25} = \;& F_i'B_{2j} + F_j'B_{2i} + F_i'B_{2k} + F_k'B_{2i} + F_j'B_{2k} + F_k'B_{2j} \\
\Upsilon_{34} = \;& D_{1i}'H_j + D_{1j}'H_i + D_{1i}'H_k + D_{1k}'H_i + D_{1j}'H_k + D_{1k}'H_j \\
\Upsilon_{44} = \;& 2(H_i + H_j + H_k + H_i' + H_j' + H_k') - 6\mathbf{I} \\
\Upsilon_{45} = \;& H_i'D_{2j} + H_j'D_{2i} + H_i'D_{2k} + H_k'D_{2i} + H_j'D_{2k} + H_k'D_{2j}
\end{aligned}
$$

then the robust output-feedback control law given by $u(k) = R^{-1}Ly(k)$ stabilizes system (1) with an $\mathscr{H}_\infty$ guaranteed cost given by γ.

Proof: Multiply (17) by α_i^3 and sum for $i = 1, \ldots, N$. Multiply (18) by $\alpha_i^2\alpha_j$ and sum for $i = 1, \ldots, N$, $i \neq j$, $j = 1, \ldots, N$. Multiply (19) by $\alpha_i\alpha_j\alpha_k$ and sum for $i = 1, \ldots, N-2$, $j = i+1, \ldots, N-1$, $k = j+1, \ldots, N$. Summing the results one has

$$
\begin{bmatrix}
G(\alpha)'P(\alpha)G(\alpha) & (G(\alpha)'A(\alpha)' + Z(\alpha)'B_2(\alpha)')F(\alpha) & \mathbf{0} & (G(\alpha)'C_1(\alpha)' + Z(\alpha)'D_2(\alpha)')H(\alpha) & G(\alpha)'C_2(\alpha)'L' - Z(\alpha)'R' \\
\star & F(\alpha) + F(\alpha)' - P(\alpha) & F(\alpha)'B_1(\alpha) & \mathbf{0} & F(\alpha)'B_2(\alpha) \\
\star & \star & \gamma^2\mathbf{I} & D_1(\alpha)'H(\alpha) & \mathbf{0} \\
\star & \star & \star & H(\alpha) + H(\alpha)' - \mathbf{I} & H(\alpha)'D_2(\alpha) \\
\star & \star & \star & \star & -(R + R')
\end{bmatrix} > 0 \quad (20)
$$

Multiplying on the right by $\mathrm{diag}(G(\alpha)^{-1}, \mathbf{I}, \mathbf{I}, \mathbf{I}, \mathbf{I})$ and on the left by its transpose, one gets

$$
\begin{bmatrix}
P(\alpha) & (A(\alpha)' + (G(\alpha)^{-1})'Z(\alpha)'B_2(\alpha)')F(\alpha) & \mathbf{0} & (C_1(\alpha)' + (G(\alpha)^{-1})'Z(\alpha)'D_2(\alpha)')H(\alpha) & C_2(\alpha)'L' - (G(\alpha)^{-1})'Z(\alpha)'R' \\
\star & F(\alpha) + F(\alpha)' - P(\alpha) & F(\alpha)'B_1(\alpha) & \mathbf{0} & F(\alpha)'B_2(\alpha) \\
\star & \star & \gamma^2\mathbf{I} & D_1(\alpha)'H(\alpha) & \mathbf{0} \\
\star & \star & \star & H(\alpha) + H(\alpha)' - \mathbf{I} & H(\alpha)'D_2(\alpha) \\
\star & \star & \star & \star & -(R + R')
\end{bmatrix} > 0 \quad (21)
$$

The multiplication of (21) on the left by $R_1(\alpha)$ and on the right by $R_1(\alpha)'$ with

$$
R_1(\alpha) = \begin{bmatrix}
\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & S(\alpha)' \\
\mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0}
\end{bmatrix} \quad (22)
$$

and $S(\alpha) = R^{-1}LC_2(\alpha) - Z(\alpha)G(\alpha)^{-1}$ results in

$$
\begin{bmatrix}
P(\alpha) & A_{cl}(\alpha)'F(\alpha) & \mathbf{0} & C_{cl}(\alpha)'H(\alpha) \\
\star & F(\alpha) + F(\alpha)' - P(\alpha) & F(\alpha)'B_1(\alpha) & \mathbf{0} \\
\star & \star & \gamma^2\mathbf{I} & D_1(\alpha)'H(\alpha) \\
\star & \star & \star & H(\alpha) + H(\alpha)' - \mathbf{I}
\end{bmatrix} > 0 \quad (23)
$$

with $A_{cl}(\alpha) = A(\alpha) + B_2(\alpha)R^{-1}LC_2(\alpha)$ and $C_{cl}(\alpha) = C_1(\alpha) + D_2(\alpha)R^{-1}LC_2(\alpha)$.

The feasibility of (23) guarantees that $F(\alpha) + F(\alpha)' > P(\alpha) > 0$, $H(\alpha) + H(\alpha)' > \mathbf{I} > 0$ and, consequently

$$
(P(\alpha) - F(\alpha))'P(\alpha)^{-1}(P(\alpha) - F(\alpha)) \geq 0 \;, \quad (I - H(\alpha))'(I - H(\alpha)) \geq 0
$$

Thus, (23) implies that

$$
\begin{bmatrix}
P(\alpha) & A_{cl}(\alpha)'F(\alpha) & \mathbf{0} & C_{cl}(\alpha)'H(\alpha) \\
\star & F(\alpha)'P(\alpha)^{-1}F(\alpha) & F(\alpha)'B_1(\alpha) & \mathbf{0} \\
\star & \star & \gamma^2\mathbf{I} & D_1(\alpha)'H(\alpha) \\
\star & \star & \star & H(\alpha)'H(\alpha)
\end{bmatrix} > 0 \quad (24)
$$

Multiplying on the right by $\mathrm{diag}(\mathbf{I}, F(\alpha)^{-1}P(\alpha), \mathbf{I}, H(\alpha)^{-1})$ and on the left by its transpose one gets

$$
\begin{bmatrix}
P(\alpha) & A_{cl}(\alpha)'P(\alpha) & \mathbf{0} & C_{cl}(\alpha)' \\
\star & P(\alpha) & P(\alpha)B_1(\alpha) & \mathbf{0} \\
\star & \star & \gamma^2\mathbf{I} & D_1(\alpha)' \\
\star & \star & \star & \mathbf{I}
\end{bmatrix} > 0, \quad (25)
$$

which is the Bounded Real Lemma, presented in Lemma 1. ■

Note that, if one imposes $S(\alpha) = \mathbf{0}$ in matrix $R_1(\alpha)$ given by (22), the dynamic matrix of the closed-loop system will be given by $A(\alpha) + B_2(\alpha)Z(\alpha)G(\alpha)^{-1}$, which means that the Lyapunov matrix $P(\alpha)$ resulting from the conditions stated on Theorem 3 certificates the stability of the closed-loop system both for the state and output-feedback gains.

The conditions of Theorem 3 can be adapted to provide a robust state-feedback gain, as stated in Corollary 2.

Corollary 2: If there exist symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, matrices $F_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, N$, matrices $R \in \mathbb{R}^{m\times m}$, $L \in \mathbb{R}^{m\times n}$ and $\gamma > 0$ such that the LMIs (17), (18) and (19) are verified with $C_{2i} = \mathbf{I}_n$, $i = 1, \ldots, N$, then the system is stabilizable by the robust state-feedback control law $u(k) = R^{-1}Lx(k)$.

The robust output-feedback stabilizing gain obtained from Theorem 3, and consequently the $\mathscr{H}_\infty$ cost of the resultant closed-loop system, depends on the matrices Z_i and G_i that compose the parameter-dependent state-feedback gain provided by the first stage. Different gains may be obtained when solving the condition of Theorem 2 in the first stage for a given value of γ, resulting on different values for the minimized limiar γ^* yielded in the second stage. Therefore, one can perform a linear search on γ, used as an input parameter in the first stage, in order to obtain better values of γ^*, which is an output parameter from the second stage.

Different results may also be provided by using parameter-dependent state-feedback gains obtained from any other method or performance criterion. In other words, instead of matrices G_i and Z_i that generate a parameter-dependent state-feedback gain $K(\alpha)$ given by

$$K(\alpha) = Z(\alpha)G(\alpha)^{-1}\ ,\ Z(\alpha) \text{ and } G(\alpha) \text{ as in (9),}$$

assuring an $\mathscr{H}_\infty$ guaranteed cost γ from Theorem 2, one could use G_i and Z_i from Theorem 1, or from $\mathscr{H}_2$ norm minimization, pole location or real positivity, provided that $A(\alpha)+B_2(\alpha)Z(\alpha)G(\alpha)^{-1}$ is stable for all $\alpha \in \Lambda_N$. Note that $K(\alpha) = Z(\alpha)G^{-1}$ or $K(\alpha) = ZG(\alpha)^{-1}$ could provide alternative parameter-dependent gains. As a consequence, different robust output-feedback stabilizing gains with distinct $\mathscr{H}_\infty$ guaranteed costs could be obtained from Theorem 3.

IV. NUMERICAL EXPERIMENTS

The numerical complexity associated to an optimization problem based on LMIs can be estimated from the number V of scalar variables and the number L of LMI rows. The worst-case $\mathscr{H}_\infty$ norm of the closed-loop system, denoted by $||H(\alpha)||_\infty$, is also given for comparison. The values of γ used in Theorem 2 were chosen, by line search, as the ones that provided the smaller guaranteed costs at the second stage (i.e. Theorem 3). All the experiments have been performed in an Intel Core 2 T5500 (1.66 GHz), 1GB RAM (981 MHz), Windows XP SP2, using Yalmip [8] and SeDuMi [9] under Matlab 7.1.0.

Example I: Consider a system with $n = 2$ and $N = 2$, whose matrices are given by

$$\left[\, A_1 \,\vdots\, A_2 \,\right] = \begin{bmatrix} 0.4 & 0.7 & \vdots & 0.9 & 0.6 \\ 0.7 & 0.4 & \vdots & -0.7 & -1.3 \end{bmatrix},\ B_{11} = B_{12} = \begin{bmatrix} 0.7 \\ 0.6 \end{bmatrix}$$

$$\left[\, B_{21} \,\vdots\, B_{22} \,\right] = \begin{bmatrix} 0.5 & \vdots & 0.4 \\ 2.1 & \vdots & 0.2 \end{bmatrix},\ \left[\, D_{21} \,\vdots\, D_{22} \,\right] = \left[\, 0.8 \,\vdots\, -0.9 \,\right]$$

$$C_{11} = C_{12} = \begin{bmatrix} 1.3 & 0 \end{bmatrix},\ C_{21} = C_{22} = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D_{11} = D_{12} = 0.$$

Firstly, the synthesis of a robust state-feedback gain is investigated, by considering $C_2 = \mathbf{I}$. Using Corollary 1 ([16, Theorem 10]) no solution has been obtained. On the other hand, the procedure on two steps (i.e. Theorem 2 with $\gamma = 83.84$ and then Corollary 2) yielded $K_{sf} = [-0.5878\ 0.4063]$ and $\gamma^* = 6.64$, which is quite close to the worst-case value $||H(\alpha)||_\infty = 6.52$ for this gain.

Considering the output-feedback case, the methods from [23, 24] failed. Since $C_{21} = C_{22} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, the sufficient conditions proposed in [16] can be applied, but they also fail. On the other hand, a feasible solution for this example has been obtained through Theorem 3 initialized with G_i and Z_i obtained from Theorem 2 for $\gamma = 93.46$. The resulting robust output-feedback gain is $K_{out} = R^{-1}L = -0.9257$, $\gamma^* = 17.72$. Again, $||H(\alpha)||_\infty = 16.64$ shows that the guaranteed cost γ^* is quite accurate.

Example II: The system, modeled as in [25], has two masses and two springs, as shown in Figure 1. The positions of the masses m_1 and m_2 are given by x_1 and x_2, respectively, the stiffness constants of the springs are k_1 and k_2 and the friction forces, related to the viscous friction coefficient c_0, are given by f_1 and f_2. The parameters of the system are given by $m_1 = 2$, $m_2 = 1$, $1 \le k_1 \le 4$, $k_2 = 0.5$, $1 \le c_0 \le 4$. Note that the parameters k_1 and c_0 are uncertain.

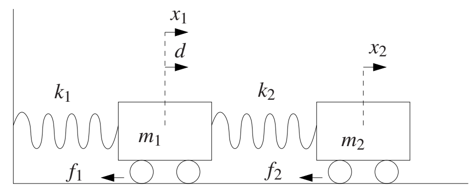

Fig. 1. Mass-spring system.

The system presents four states (the positions and the speed of the masses) and the transfer function considered is from the input force d on the mass m_1 to the error signal $e = x_2$. The discretization of the system is performed using Euler first-order approximation and a sampling time of $0.1s$. The dynamic matrix of the discretized system is given by

$$A = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ \frac{-0.1(k_1+k_2)}{m_1} & \frac{0.1k_2}{m_1} & 1-\frac{0.1c_0}{m_1} & 0 \\ \frac{0.1k_2}{m_2} & \frac{-0.1k_2}{m_2} & 0 & 1-\frac{0.1c_0}{m_2} \end{bmatrix}.$$

The vertices A_i are constructed by taking all the possible combinations of the lower and higher values of the uncertain parameters k_1 and c_0, resulting in this case in a polytope of $N = 4$ vertices. The remaining system matrices do not present uncertain parameters and are given by

$$B_1 = \begin{bmatrix} 0 \\ 0.1 \\ 0.1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{0.1}{m_1} \\ 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}', C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}'$$

and $D_1 = D_2 = 0$. Neither the methods from [23, 24] nor the sufficient conditions from [16], with the similarity transformation T given by

$$\tilde{x} = Tx\ ,\quad T = [C_2'(C_2C_2')^{-1} \quad C_2^{\perp}]\ ,\quad C_2C_2^{\perp} = 0 \qquad (26)$$

in order to obtain $\tilde{C}_2 = [\,\mathbf{I} \quad \mathbf{0}\,]$, provided a feasible solution.

Table I shows the results obtained with the proposed method in two situations: using G_i, Z_i from Theorem 1 or from Theorem 2 in the first stage as input matrices for Theorem 3 and, again, it can be noted that the guaranteed costs are close to the worst-case norm. The corresponding output-feedback gains are

$$K_{1out} = [-1.9133 \quad 1.9953]\ ,\quad K_{2out} = [-12.9923 \quad 6.0045]$$

TABLE I

OUTPUT-FEEDBACK DESIGN FOR EXAMPLE II.

Theorems	γ^*	$\|\|H(\alpha)\|\|_\infty$	V	L	Time (s)
Th. 2, $\gamma = 31$	13.14	12.00	121	100	0.2
Th. 1	8.54	8.08	120	80	0.1

Example III: The discretized version of the linearized dynamic equation of the VTOL helicopter [26] is analyzed, using first Euler approximations and sampling time of $0.01s$. The discretized system is given by

$$A_i = \begin{bmatrix} 0.999634 & 0.100271 & 0.000188 & -0.004555 \\ 0.000482 & 0.989900 & 0.000024 & -0.040208 \\ 0.001002 & a_{32} & 0.992930 & a_{34} \\ 0 & 0 & 0.010000 & 1 \end{bmatrix},$$

$$B_{1i} = 0.01\mathbf{I},\ B_{2i} = \begin{bmatrix} -0.004422 & -0.001761 \\ b_{21} & 0.075922 \\ 0.055200 & -0.044900 \\ 0 & 0 \end{bmatrix},\ D_{1i} = \mathbf{0}$$

$$C_{1i} = 0.1\mathbf{I},\ C_{2i} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ D_{2i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}'$$

The uncertainties of the system (polytope with $N = 8$ vertices) are $a_{32} \in [-0.006319, 0.013681]$, $a_{34} = [0.012200,\ 0.016200]$ and $b_{21} \in [0.027446, 0.043446]$. The methods [23, 24] and [16] (with the similarity transformation given in (26)) fail, while with G_i and Z_i from Theorem 2, $\gamma = 1.22$, Theorem 3 provided

$$K_{out} = \begin{bmatrix} 0.3475 & -0.1403 \\ -1.3929 & -0.5802 \end{bmatrix}, \qquad \gamma^* = 2.67.$$

V. CONCLUSION

A new method to synthesize a robust $\mathscr{H}_\infty$ output-feedback stabilizing gain for uncertain systems was proposed. The method consists of two stages: a parameter-dependent state-feedback gain is calculated in the first stage and then used as an input for the second stage, which synthesizes a robust output-feedback gain with an $\mathscr{H}_\infty$ guaranteed cost. The technique may also be extended for the calculation of robust state-feedback gains. Numerical experiments show the efficiency of the method, finding solutions in cases where other techniques fail. Future works include the investigation of polynomial Lyapunov functions and time-varying parameters.

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