PERGAMON

Planetary and Space Science 48 (2000) 965–971

Planetary and Space Science

www.elsevier.nl/locate/planspasci

Estimating asteroid density distributions from shape and gravity information ☆

D.J. Scheeres[a,*], B. Khushalani[a], R.A. Werner[b]

[a]Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2140, USA

[b]Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA 91109-8099, USA

Received 23 September 1999; accepted 23 November 1999

Abstract

A least-squares approach for estimating the internal density distribution of an asteroid is presented and applied to a simple polyhedron asteroid shape. The method assumes that the asteroid gravity field is measured to a specified degree and order and that a polyhedral model of the asteroid is available and has been discretized into a finite number of constant density polyhedra. The approach is derived using several basic properties of spherical harmonic gravitational expansions and can explicitly accommodate a fully correlated covariance matrix for the estimated gravity field. For an asteroid shape discretized into M constant density polyhedra and a gravity field measured to degree and order N, the least-squares problem is under-determined if $M > (N+1)^2$ and is over-determined if $M < (N+1)^2$. For both cases a singular-value decomposition (SVD) approach will yield solutions. We apply our approach to a number of ideal test situations using an asteroid shape consisting of 508 tetrahedra. We show that the under-determined case is sensitive to non-uniform density distributions. The over-determined case shows very good performance independent of the initial density distribution guess.

Keywords: Eros; NEAR; Polyhedral gravity field; Density distribution

1. Introduction

Two of the main products of the Near Earth Asteroid Rendezvous (NEAR) mission to asteroid 433 Eros (Farquhar, 1995) will be the gravity field, measured at least to degree and order 12 (Yeomans et al., 1997), and the shape, measured with both the imager and lidar instruments (Veverka et al., 1997; Zuber et al., 1997). By combining the total mass and the asteroid volume it is, of course, possible to extract the bulk density of the body. But to determine the distribution of that density is a non-trivial problem which cannot be uniquely determined in general. In this paper we present a method for estimating the internal density distribution of an asteroid given its gravity field and shape. This approach relies on the discretization of the asteroid shape into a finite number of constituent polyhedra, and then uses a least-squares approach to assign densitites to these polyhedra. When the number of polyhedra is less than $(N+1)^2$, where N is the degree of the gravity field, this assignment procedure is over-determined and the resulting density assignment is unique (assuming a given set of weights). Thus, using such a procedure reduces the problem of finding the density distribution to finding the regions of different density distributions. Using this approach it should be possible to gain additional insight into the internal structure of asteroids, comets, or any irregularly shaped body for which we have a gravity field and a shape. This approach makes no distinction between the computation of density variations or the identification of voids within the asteroid.

There is an important additional application that becomes available once the internal density distribution of an asteroid is approximated — the ability to evaluate the gravity field close to the surface of the body. For a general body which has a major to minor axis ratio greater than $\sqrt{2}$ (which includes almost all asteroid shapes measured to date) the spherical harmonic gravity field description diverges once one moves within the circumscribing sphere about that body. It can be shown that this divergence cannot be remedied by discarding higher-order coefficients, as the coefficients themselves at all orders are no longer

☆Poster 24.18-P, presented at the Asteroids, Comets, Meteors Conference, Cornell, 1999.

* Corresponding author. Fax: 1-734-763-0578.
E-mail address: scheeres@umich.edu (D.J. Scheeres).

0032-0633/00/$ - see front matter
PII: S0032-0633(00)00064-7

properly defined. The polyhedron-based gravity field that we derive, however, is free of this divergence and can be used to model the gravity field close to and on the surface, enabling both studies of surface ejecta and analysis of spacecraft trajectories that closely approach or land on the asteroid's surface.

2. Gravity field descriptions

2.1. Spherical harmonic gravity field description

The usual form for a spherical harmonic gravity potential is (Kaula, 1966)

$$U = \frac{GM}{r}\left[1 + \sum_{l=1}^{N}\sum_{m=0}^{l}\left(\frac{R}{r}\right)^{l} \times P_{lm}(\sin\theta)(C_{lm}\cos m\lambda + S_{lm}\sin m\lambda)\right], \quad (1)$$

where G is the universal gravitation constant, M is the total asteroid mass, l is the degree, m is the order of the gravity field, N is the maximum degree of the expansion, P_{lm} is the associated Legendre polynomial, R is an arbitrary radius scaling factor used in conjunction with the gravity coefficients, r is the distance of the test particle from the center of the expansion, θ is the latitude of the test particle, λ is the longitude of the test particle, and C_{lm} and S_{lm} are the gravity coefficients themselves. Following the estimation of the gravity field, these coefficients are then specified up to degree and order N.

Defining the coefficient $C_{00} = 1$ gives a total number of gravity coefficients, given a maximum degree and order of N, of $(N+1)^2$, with $2n+1$ coefficients at each degree n. Generally, a covariance matrix P_C is also identified with the gravity coefficients, which states the uncertainty of their measurement. This uncertainty grows with increasing degree, the total mass and first few degrees of the field having the least uncertainty (Sanso and Rummel, 1989).

2.2. Polyhedron gravity field description

An alternate way to specify the gravitational potential of an arbitrary shape is by discretizing the shape into a finite number of polyhedra and then assigning a density (mass) to each. This approach works since the gravitational potential of a polyhedron is known in the closed form and can be used directly to evaluate the gravitational attraction of the resulting field on a test particle (Werner and Scheeres, 1997). The main advantage of this formulation is that the derived gravitational potential suffers no divergence up to the surface of the polyhedron. In fact, as a by-product of computing the gravitational acceleration on a test particle it is possible to determine if the test particle is inside or outside the polyhedron by checking the value of the Laplacian of the potential, $\nabla^2 U$, a useful property when concerned with motion close to the surface of the body. The main disadvantage of this approach is that the closed-form polyhedron potential result holds only for constant density polyhedra.

In Werner (1997), a closed-form algorithm for generating the spherical harmonic coefficients of an arbitrary tetrahedron is given. By applying this result repeatedly and summing up the resulting spherical harmonic potentials, it is possible to generate the exact spherical harmonic coefficients of an arbitrary polyhedral shape. This result lies at the heart of our approach.

3. Estimating density distribution

3.1. Measured data

We assume as input the measured spherical harmonic coefficients of the asteroid up to degree and order N and the correlated covariance matrix of these coefficients. Also, we assume that a polyhedral model of the asteroid shape exists, which is comprised of an arbitrary number of tetrahedra, N_T. These tetrahedra can be grouped into arbitrary collections of polyhedra. In general the groupings of these polyhedra will either be based on some physical reasoning or to ensure uniform tiling of the asteroid volume. The number of these polyhedra is assumed to be p, and that $1 \leqslant p \leqslant N_T$.

3.2. Spherical harmonic expansion of the polyhedron field

Using the results from Werner (1997) it becomes possible to compute the spherical harmonic coefficients of each of the p polyhedra.

$$U_i(r,\theta,\lambda) = \frac{GV_i\rho_i}{r}\sum_{l=0}^{N}\sum_{m=0}^{l}\left(\frac{R}{r}\right)^{l} P_{lm}(\sin\theta)(C^i_{lm}\cos m\lambda + S^i_{lm}\sin m\lambda), \quad (2)$$

where U_i is the gravity potential of the ith polyhedron, V_i is the volume of that polyhedron, ρ_i is its density, and C^i_{lm} and S^i_{lm} are its spherical harmonic coefficients. Since the volume and the coefficients are computed directly from the geometry of the polyhedron, the only unknown value in this potential is the density ρ_i. It should be noted that we choose a common point for the spherical harmonic expansion of all the polyhedra that coincides with the expansion center used for the measured gravity field (usually chosen to be the center of mass of the asteroid, as determined using the first degree and order gravity coefficients).

Since the potentials are all expanded about a common point, we can sum them together using the same coordinate

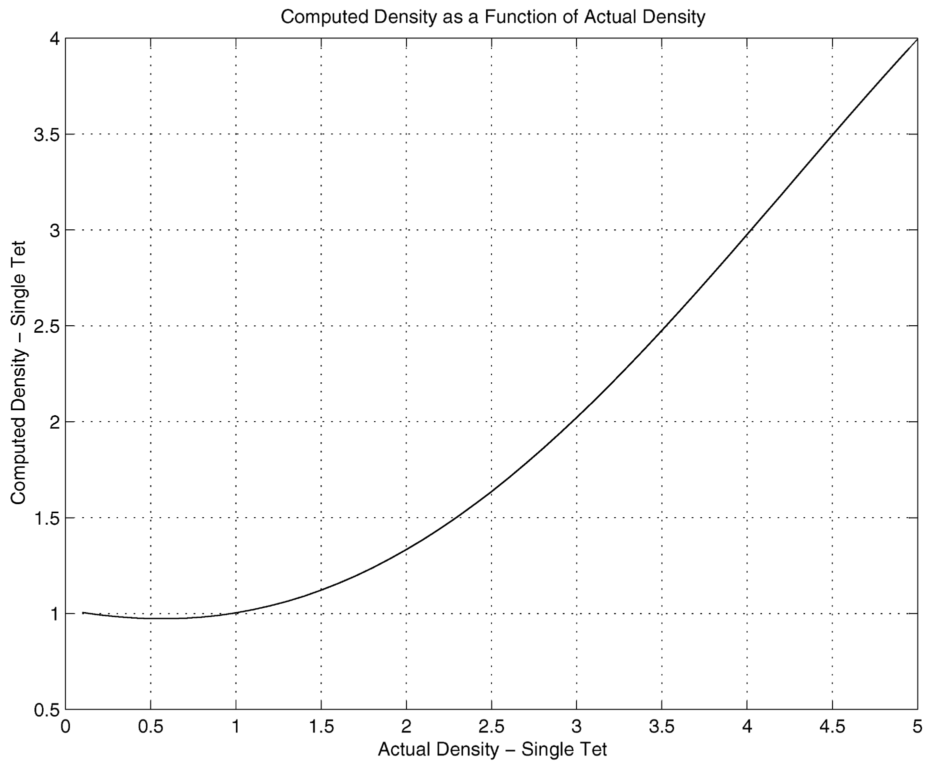

Fig. 1. Computed density of the variable tetrahedron as a function of its actual value for the under-determined case.

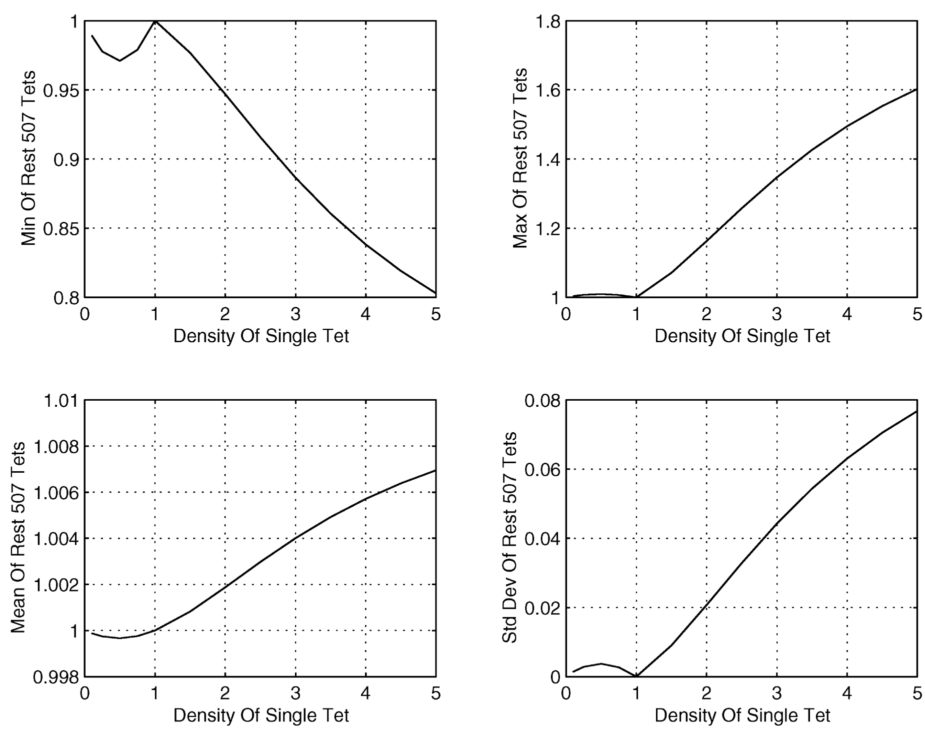

Fig. 2. Statistics of the computed densities of the 507 non-varying tetrahedra as a function of the actual value of the variable tetrahedron for the under-determined case.

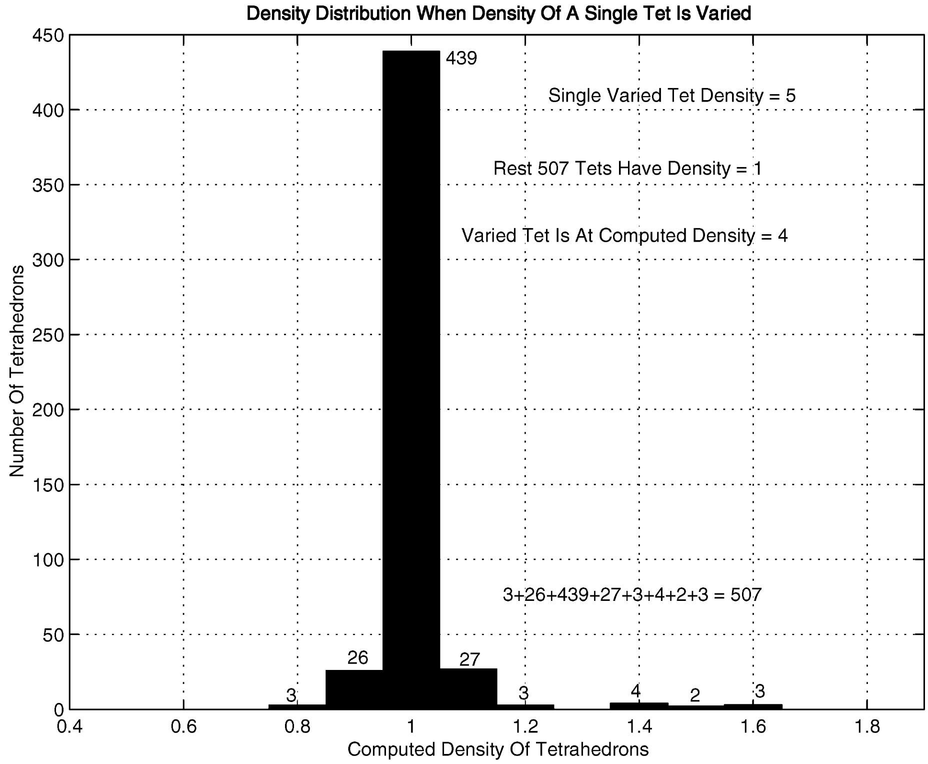

Fig. 3. Histogram showing the number of tetrahedra with a given value of density for the case when the variable tetrahedron has a value of 5 g/cm^3.

system for each separate polyhedron

$$U_{\text{poly}}(r, \theta, \lambda) = \sum_{i=1}^{p} U_i(r, \theta, \lambda). \tag{3}$$

To completely specify the gravity potential we must then supply the p densities ρ_i. The accuracy of this final potential will be commensurate with the accuracy to which the chosen shape discretization matches the actual internal topology of the asteroid's density distribution.

3.3. Conditions to satisfy

Given the above results, we ideally wish to choose the separate density values to satisfy the condition

$$U(r, \theta, \lambda) = U_{\text{poly}}(r, \theta, \lambda), \tag{4}$$

where U is the measured (true) gravity field potential, or minimize the difference between these two potentials.

Under the conditions we have here the spherical harmonic expansion of a given potential is unique, which yields the set of relations we wish to satisfy

$$MC_{lm} = \sum_{i=1}^{p} V_i \rho_i C_{lm}^i, \tag{5}$$

$$MS_{lm} = \sum_{i=1}^{p} V_i \rho_i S_{lm}^i, \tag{6}$$

which is valid for all degrees and orders of l and m up to the measured degree and order of the gravity field N.

For convenience, we reformulate the problem and group the coefficients into one vector $[\tilde{C}_{lmn}]$, defined so that

$$\tilde{C}_{l00} = C_{l0}, \tag{7}$$

$$\tilde{C}_{l01} = \text{does not exist}, \tag{8}$$

$$\tilde{C}_{lm0} = C_{lm}, \tag{9}$$

$$\tilde{C}_{lm1} = S_{lm}, \tag{10}$$

with the ordering

$$[\tilde{C}_{lmn}] = [C_{00}, C_{10}, C_{11}, S_{11}, \ldots, C_{NN}, S_{NN}]. \tag{11}$$

Given this notation the conditions to satisfy can be restated as

$$\left[\frac{V_i}{M}\tilde{C}_{lmn}^i\right][\rho_i] = [\tilde{C}_{lmn}] \tag{12}$$

having the following matrix dimensions:

$$[(N+1)^2 \times p] \cdot [p \times 1] = [(N+1)^2 \times 1].$$

Given this definition, the covariance of the measured gravity field coefficients can be specified in a natural form as a matrix $P_{\tilde{C}}$.

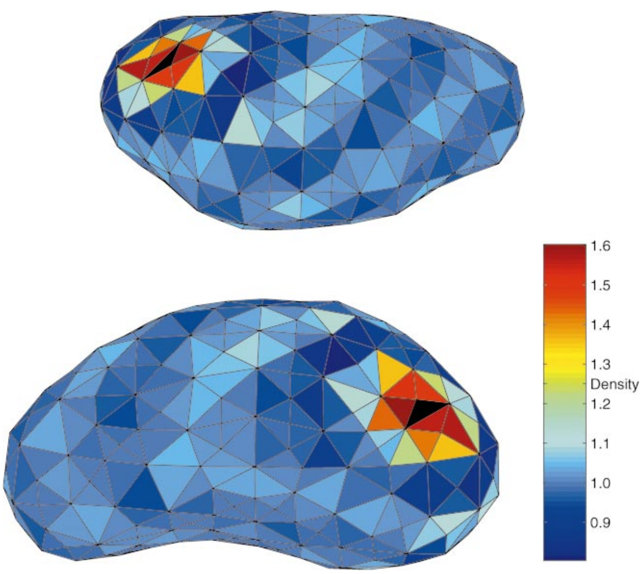

Fig. 4. Two views of the computed local density distribution for the case when the variable tetrahedron has a value of 5 g/cm^3.

3.4. Least-squares formulation

We wish to choose the density vector $\rho = [\rho_i]$ to minimize the functional

$$J = \frac{1}{2}\left(\left[\frac{V_i}{M}\tilde{C}^i_{lmn}\right][\rho_i] - [\tilde{C}_{lmn}]\right)^{\mathrm{T}}$$

$$\times P_{\tilde{C}}^{-1}\left(\left[\frac{V_i}{M}\tilde{C}^i_{lmn}\right][\rho_i] - [\tilde{C}_{lmn}]\right). \tag{13}$$

Computing the partial of J with respect to the density vector and solving the resulting equation yields

$$\left[\frac{V_i}{M}\tilde{C}^i_{lmn}\right]^{\mathrm{T}} P_{\tilde{C}}^{-1}\left[\frac{V_i}{M}\tilde{C}^i_{lmn}\right][\rho_i] = \left[\frac{V_i}{M}\tilde{C}^i_{lmn}\right]^{\mathrm{T}} P_{\tilde{C}}^{-1}[\tilde{C}_{lmn}]. \tag{14}$$

Define the normal matrix Λ to be

$$\Lambda = \left[\frac{V_i}{M}\tilde{C}^i_{lmn}\right]^{\mathrm{T}} P_{\tilde{C}}^{-1}\left[\frac{V_i}{M}\tilde{C}^i_{lmn}\right] \tag{15}$$

and the data vector z to be

$$z = \left[\frac{V_i}{M}\tilde{C}^i_{lmn}\right]^{\mathrm{T}} P_{\tilde{C}}^{-1}[\tilde{C}_{lmn}] \tag{16}$$

to define our system in a compact notation

$$\Lambda\rho = z. \tag{17}$$

If the number of polyhedra is less than or equal to $(N+1)^2$ then, in general, the normal matrix will be invertible. Note that this will not be the case for certain classes of shape discretization, such as concentric spherical layers, but we will not encounter such degenerate cases in our situations. If the number of polyhedra is greater than $(N+1)^2$ then the system is under-determined and cannot be solved by normal means. In order to accomodate both cases we will solve the equation using singular-value decomposition (SVD). Applying the SVD algorithm to Λ generates

$$\Lambda = VSV^{\mathrm{T}}, \tag{18}$$

where V is an orthonormal matrix and S is a diagonal matrix. We get this particular form of the SVD since Λ is, by definition, a symmetric matrix. The solution is then generated as

$$\rho = V\tilde{S}^{-1}V^{\mathrm{T}}z, \tag{19}$$

where $\tilde{S}^{-1}$ is a diagonal matrix containing the reciprocal of the diagonal entries of S, except when they are smaller than some defined limit, when a zero is instead used in that entry for $\tilde{S}^{-1}$. It has been shown that this generates the so-called minimum length solution for the density vector ρ in the under-determined case (Press et al., 1993).

4. Results

The following results are tests of our basic approach using a 508 tetrahedra model of Eros based on radar data (Mitchell et al., 1998), shown in Fig. 4. In these tests we assemble the tetrahedra into a number of polyhedra and assign densities to each polyhedron to compute the "true" gravity field potential. Then we use our algorithm on the same shape discretization in order to verify that we recover the proper densities for each body. We also investigate the limitations of this method when the number of polyhedra is greater than the total number of gravity coefficients, and verify that the performance of the algorithm degrades once the problem is under-determined. This indicates the importance of estimating asteroid gravity fields to as high a degree and order as possible, as it allows the use of more polyhedra to investigate its internal density structure.

4.1. Under-determined case

We first test the sensitivity of our approach when we have an under-determined case. We assume that 507 of the tetrahedra have a density of 1 g/cm^3 and that one of the tetrahedra has a density that varies from 0 to 5 g/cm^3. The true gravity field is specified up to degree and order 8. We initially assume that the densities of all 508 tetrahedra equal 1 g/cm^3 and compute the correction to this initial assumption. The computed density of the one tetrahedron that has a varying density as a function of the actual density of that tetrahedron is shown in Fig. 1. The minimum, maximum, mean and standard deviation of the computed densities of the other 507 tetrahedra as a function of the density of the single tetrahedron is shown in Fig. 2. If the algorithm works "perfectly" all of these values would be equal to 1. A histogram of the 507 tetrahedra

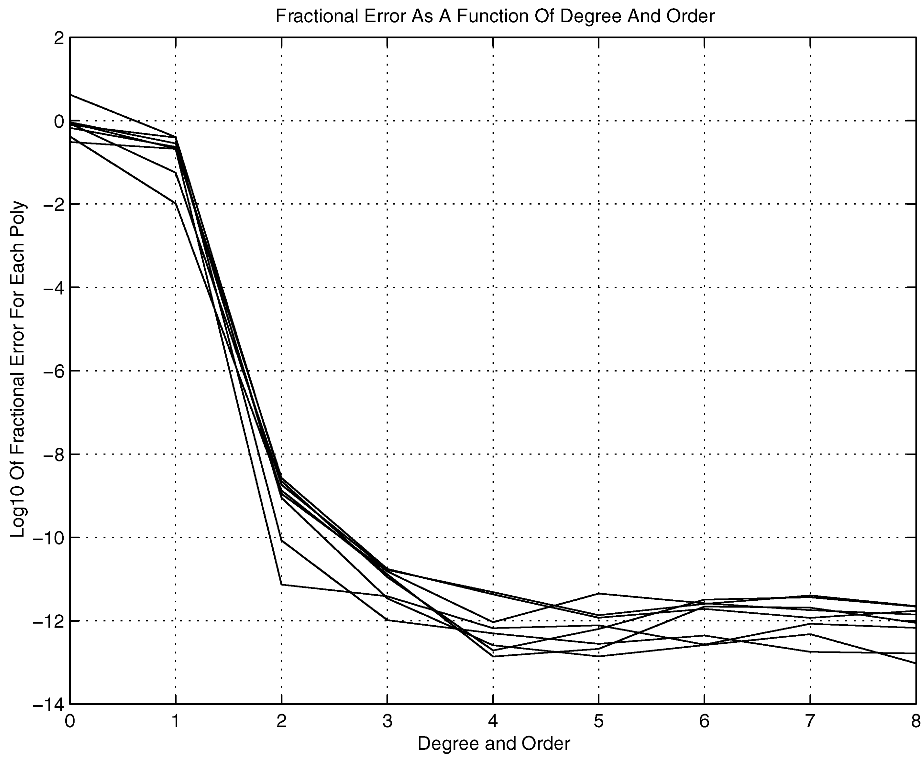

Fig. 5. Fractional error of the computed solution as a function of the degree and order of the measured gravity field, computed for an 8-polyhedron asteroid shape.

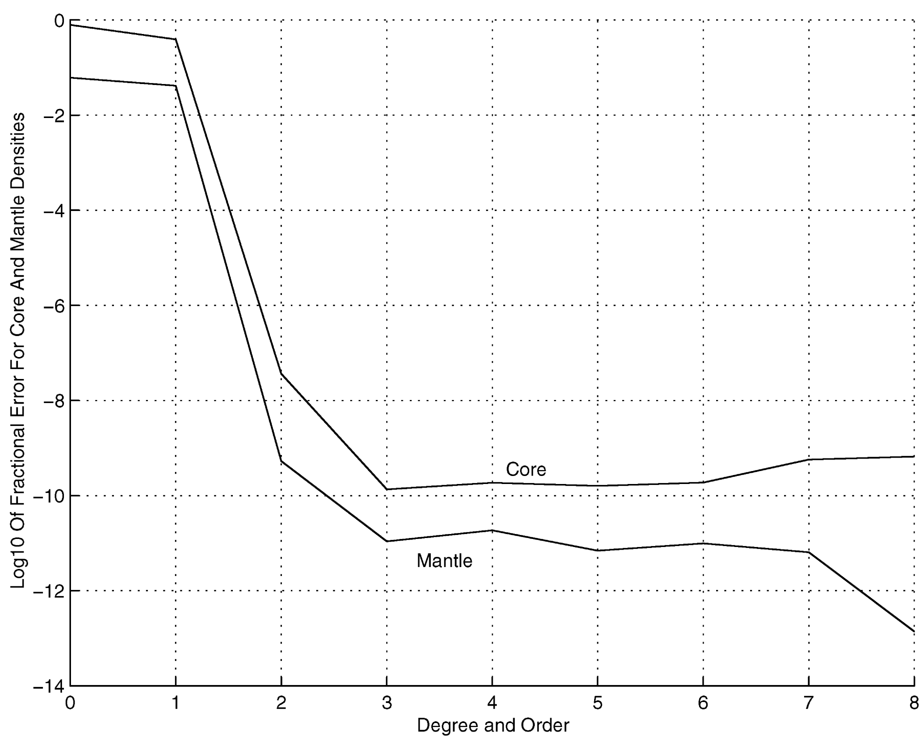

Fig. 6. Fractional error of the computed solution as a function of degree and order of the measured gravity field, computed for an asteroid shape split into an outer mantle polyhedron and an inner core polyhedron.

distributed according to their computed density when the variable tetrahedron has a value of 5 g/cm^3 is shown in Fig. 3. From this we note that the under-determined case is sensitive to the change in one tetrahedron and is able to discriminate that one tetrahedron from the others, even though we do not recover the precise actual values. From Fig. 4 we note that the single density distortion affects our model locally and there is no noteworthy global distortion of the density field. This indicates that our method will be useful in detecting local distortions in the density distribution, allowing for refinement of the initial shape discretization.

4.2. Over-determined case

To investigate the over-determined case we group all 508 tetrahedra into eight separate polyhedra corresponding to eight quadrants in space, grouped according to where the vertices of their base triangles lie (i.e., $\pm x, \pm y, \pm z$). We assign each of the eight quadrants a constant density of $1, 2, \ldots, 8$ g/cm^3, respectively and compute the gravity coefficients for this field up to degree and order 8. Then we apply our method to compute the densities of each polyhedron with no initial estimate. Plotted in Fig. 5 is the fractional error as a function of the degree and order used for the "real" gravity field. For the overdetermined case ($N \geqslant 2$) the fractional error is very small, but increases drastically once the problem becomes under-determined ($N \leqslant 1$). The fractional error is defined as

$$f_{\mathrm{e}} = \frac{\text{computed density} - \text{actual density}}{\text{actual density}}. \tag{20}$$

These results show that our algorithm can properly recover a non-homogeneous density distribution if the problem is over-determined.

4.3. Core–mantle discrimination

We repeat the calculation, now dividing the asteroid shape into two polyhedra, an outer mantle with density 2 g/cm^3 and an inner core with density 5 g/cm^3. The core is formed by splitting each tetrahedron radially, creating an interior tetrahedon of lower radius and an exterior, truncated tetrahedron. The interior and exterior tetrahedra are then identified with each other, respectively, creating a two-component shape model of the asteroid. The results, shown in Fig. 6, indicate that it should be possible to discern such a radial density distribution for all but the lowest order gravity fields. The same technique can be used to create multiple radial layers of differing density.

5. Conclusions

This paper presents a mathematical framework for estimating the internal density distribution of an arbitrarily shaped body, assuming that a measured gravity field and shape for that body exist. The algorithm is explicitly developed and tested with a 508 tetrahedra shape model based on radar measurements of the asteroid 433 Eros (Mitchell et al., 1998). The technique will be especially useful for any and all missions to asteroids and comets, where the determination of internal density distribution will be of paramount interest. By constructing a model of the interior density distribution it also becomes possible to specify a more accurate gravity field model that will enable a spacecraft to accurately predict its trajectory close to the surface of the body. The first planned, practical application of this method will be for the Near Earth Asteroid Rendezvous Mission (Farquhar, 1995), estimating the internal density distribution of asteroid 433 Eros.

Acknowledgements

This work was supported by a Grant from NASA's Planetary Geology and Geophysics Program.

References

Farquhar, R.W., 1995. Introduction: the near-Earth asteroid rendezvous (NEAR) mission. J. Astron. Sci. 43, 349–352.
Kaula, W.M., 1966. Theory of Satellite Geodesy. Blaisdell, New York.
Mitchell, D., Hudson, R.S., Ostro, S.J., Rosema, K., 1998. Shape of asteroid 433 Eros from inversion of goldstone radar doppler spectra. Icarus 131, 4–14.
Press, W.H., Flannery, B.P., Teukolsky, S.A., Vetterling, W.T., 1993. Numerical Recipes in FORTRAN. Cambridge University Press, Cambridge.
Sanso, F., Rummel, R. (Eds.), 1989. Theory of Satellite Geodesy and Gravity Field Determination. Springer, Berlin.
Veverka, J., et al., 1997. An overview of the NEAR multispectral imager (MSI)-near-infrared spectrometer (NIS) investigation. J. Geophys. Res. 102, 23709.
Werner, R.A., 1997. Spherical harmonic coefficients for the potential of a constant-density polyhedron. Comput. & Geosci. 23, 1071.
Werner, R.A., Scheeres, D.J., 1997. Exterior gravitation of a polyhedron derived and compared with harmonic and mascon gravitation representations of asteroid 4769 castalia. Celestial Mech. 65, 313–344.
Yeomans, D.K., et al., 1997. The NEAR radio science investigations. J. Geophys. Res. 102, 23775.
Zuber, M.T., et al., 1997. The NEAR laser ranging experiment. J. Geophys. Res. 102, 23761.