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Verification using a 6-DOF Mars Descent

Szmuk, Utku, and Açıkmeşe 2017 have used successive convexification for the non-convex minimum-fuel OCP for 6 DOF rocket powered Mars landing Scenario. We use this case to verify our SCvx algorithm. It is a fixed final time problem and in the inertial and body reference frames. The vehicle has a single engine and is capable of commanding various thrust magnitudes at different engine gimbal angles. The gravity, COM and inertia are assumed to remain constant. The state vector and its differential is given by:

$$\boldsymbol{x}(t) = \begin{pmatrix} m(t) \\ \boldsymbol{r}^{\mathtt{I}}(t) \\ \boldsymbol{v}^{\mathtt{I}}(t) \\ \boldsymbol{q}_{\mathtt{B/I}}(t) \\ \boldsymbol{\omega}^{\mathtt{B}}_{\mathtt{B/I}}(t) \\ \boldsymbol{T}^{\mathtt{B}}(t) \\ \dot{\boldsymbol{T}}^{\mathtt{B}}(t) \end{pmatrix} \quad \dot{\boldsymbol{x}}(t) = \begin{pmatrix} -\alpha\|\boldsymbol{T}^{\mathtt{B}}(t)\| \\ \boldsymbol{v}^{\mathtt{I}}(t) \\ \boldsymbol{a}^{\mathtt{I}}(t) \\ \dot{\boldsymbol{q}}_{\mathtt{B/I}}(t) \\ \dot{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}}(t) \\ \dot{\boldsymbol{T}}^{\mathtt{B}}(t) \\ \boldsymbol{u}(t) \end{pmatrix} = \begin{pmatrix} -\alpha\|\boldsymbol{T}^{\mathtt{B}}(t)\| \\ \boldsymbol{v}^{\mathtt{I}}(t) \\ \frac{1}{m(t)}\boldsymbol{C}_{\mathtt{B/I}}\boldsymbol{T}^{\mathtt{B}}(t) + \boldsymbol{g}^{\mathtt{I}} \\ \frac{1}{2}\boldsymbol{\omega}^{\mathtt{B}}_{\mathtt{B/I}}(t) \otimes \boldsymbol{q}_{\mathtt{B/I}}(t) \\ J^{-1}[\boldsymbol{r}^{\mathtt{B}}_{e}\times]\boldsymbol{T}^{\mathtt{B}}(t) - [\boldsymbol{\omega}^{\mathtt{B}}_{\mathtt{B/I}}(t)\times]J\boldsymbol{\omega}^{\mathtt{B}}_{\mathtt{B/I}}(t)) \\ \dot{\boldsymbol{T}}^{\mathtt{B}}(t) \\ \boldsymbol{u}(t) \end{pmatrix} \tag{C.1}$$

The non-linear dynamics are linearised by Taylor series to get the form as discussed in Eq. 7.42 to 7.46. The linearised equations are given below:

$$\begin{aligned} f_{\boldsymbol{T}^{\mathtt{B}},t} &= \left(\|\bar{\boldsymbol{T}}^{\mathtt{B}}(t)\| + \frac{\bar{\boldsymbol{T}}^{\mathtt{B}}(t)}{\|\bar{\boldsymbol{T}}^{\mathtt{B}}(t)\|}(\boldsymbol{T}^{\mathtt{B}}(t) - \bar{\boldsymbol{T}}^{\mathtt{B}}(t)\right) \\ f_{\boldsymbol{a}^{\mathtt{I}},t} &= \bar{\boldsymbol{a}}^{\mathtt{I}}(t) + \frac{\partial \boldsymbol{a}^{\mathtt{I}}}{\partial \boldsymbol{x}}\Big|_{\bar{\boldsymbol{x}}(t)}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t)) \\ f_{\dot{\boldsymbol{q}}_{\mathtt{B/I}},t} &= \dot{\bar{\boldsymbol{q}}}_{\mathtt{B/I}}(t) + \frac{\partial \dot{\boldsymbol{q}}_{\mathtt{B/I}}}{\partial \boldsymbol{x}}\Big|_{\bar{\boldsymbol{x}}(t)}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t)) \\ f_{\dot{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}},t} &= \dot{\bar{\boldsymbol{\omega}}}^{\mathtt{B}}_{\mathtt{B/I}}(t) + \frac{\partial \dot{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}}}{\partial \boldsymbol{x}}\Big|_{\bar{\boldsymbol{x}}(t)}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t)) \end{aligned} \tag{C.2}$$

Now we will derive the linearisation matrices for continuous time, for which we first derive the partial differentials with respect to state and controls.

Translational Acceleration:

$$\begin{aligned} \boldsymbol{a}^{\mathtt{I}}(t) &= \frac{1}{m(t)}\boldsymbol{C}_{\mathtt{B/I}}(t)\boldsymbol{T}^{\mathtt{B}}(t) + \boldsymbol{g}^{\mathtt{I}} \\ &= \frac{1}{m(t)}\left[\boldsymbol{q}^{*}_{\mathtt{B/I}}(t) \otimes \boldsymbol{T}^{\mathtt{B}}(t) \otimes\right]\boldsymbol{q}_{\mathtt{B/I}}(t) + \boldsymbol{g}^{\mathtt{I}} \\ &= \frac{1}{m(t)}\left[\boldsymbol{T}^{\mathtt{B}}(t) \otimes \boldsymbol{q}_{\mathtt{B/I}}(t) \odot\right]\boldsymbol{q}^{*}_{\mathtt{B/I}}(t) + \boldsymbol{g}^{\mathtt{I}} \end{aligned} \tag{C.3}$$

We find the partial differentials below:

$$\begin{aligned}
\Psi_m &= \left.\frac{\partial \boldsymbol{a}^{\mathrm{I}}}{\partial m}\right|_{\bar{\boldsymbol{x}}} = \frac{1}{\bar{m}^2(t)}\bar{\boldsymbol{C}}_{\mathrm{B/I}}(t)\bar{\boldsymbol{T}}^{\mathrm{B}} \\
\Psi_q &= \left.\frac{\partial \boldsymbol{a}^{\mathrm{I}}}{\partial \boldsymbol{q}_{\mathrm{B/I}}}\right|_{\bar{\boldsymbol{x}}} = \frac{1}{\bar{m}(t)}\left[\bar{\boldsymbol{q}}^*_{\mathrm{B/I}}(t)\otimes\bar{\boldsymbol{T}}^{\mathrm{B}}(t)\otimes\right] + \frac{1}{m(t)}\left[\bar{\boldsymbol{T}}^{\mathrm{B}}(t)\otimes\bar{\boldsymbol{q}}_{\mathrm{B/I}}(t)\odot\right](-1\ -1\ -1\ 1)^{\mathrm{T}} \\
\Psi_T &= \left.\frac{\partial \boldsymbol{a}^{\mathrm{I}}}{\partial \boldsymbol{T}^{\mathrm{B}}}\right|_{\bar{\boldsymbol{x}}} = \frac{1}{\bar{m}(t)}\bar{\boldsymbol{C}}_{\mathrm{B/I}}(t)
\end{aligned} \tag{C.4}$$

Attitude Kinematics:

$$\begin{aligned}
\dot{\boldsymbol{q}}_{\mathrm{B/I}}(t) &= \frac{1}{2}\left[\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}}(t)\otimes\right]\boldsymbol{q}_{\mathrm{B/I}}(t) \\
&= \frac{1}{2}\left[\boldsymbol{q}_{\mathrm{B/I}}(t)\odot\right]\begin{pmatrix}\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}}(t)\\ 0\end{pmatrix}
\end{aligned} \tag{C.5}$$

We find the partial differentials below:

$$\begin{aligned}
\Upsilon_q &= \left.\frac{\partial \dot{\boldsymbol{q}}_{\mathrm{B/I}}}{\partial \boldsymbol{q}_{\mathrm{B/I}}}\right|_{\bar{\boldsymbol{x}}} = \frac{1}{2}\left[\bar{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/I}}(t)\otimes\right] \\
\Upsilon_\omega &= \left.\frac{\partial \dot{\boldsymbol{q}}_{\mathrm{B/I}}}{\partial \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}}}\right|_{\bar{\boldsymbol{x}}} = \frac{1}{2}\left[\bar{\boldsymbol{q}}_{\mathrm{B/I}}(t)\odot\right]
\end{aligned} \tag{C.6}$$

Attitude Dynamics:

$$\begin{aligned}
\dot{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/I}}(t) &= J^{-1}\left([\boldsymbol{r}^{\mathrm{B}}_e\times]\boldsymbol{T}^{\mathrm{B}}(t) - \left[\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}}(t)\times\right]J\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}}(t)\right) \\
&= J^{-1}\left([\boldsymbol{r}^{\mathrm{B}}_e\times]\boldsymbol{T}^{\mathrm{B}}(t) + \left[J\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}}(t)\times\right]\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}}(t)\right)
\end{aligned} \tag{C.7}$$

We find the partial differentials below:

$$\begin{aligned}
\Theta_\omega &= \left.\frac{\partial \dot{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/I}}}{\partial \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}}}\right|_{\bar{\boldsymbol{x}}} = J^{-1}\left(-\left[\bar{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/I}}(t)\times\right]J + \left[J\bar{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/I}}(t))\times\right]\right) \\
\Theta_T &= \left.\frac{\partial \dot{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/I}}}{\partial \boldsymbol{T}^{\mathrm{B}}}\right|_{\bar{\boldsymbol{x}}} = J^{-1}[\boldsymbol{r}^{\mathrm{B}}_e\times]
\end{aligned} \tag{C.8}$$

The compiled $\boldsymbol{A}(t)$ matrix is given on the next page along with the control matrix. Since the state differential has controls as the last vector, it can be directly computed as given in $\boldsymbol{B}(t)$. And for $\boldsymbol{z}(t)$, the continuous time matrix is as follows:

$$\boldsymbol{z}(t) = \begin{pmatrix} -\alpha\|\bar{\boldsymbol{T}}^{\mathrm{B}}(t)\| \\ \bar{\boldsymbol{v}}^{\mathrm{I}}(t) \\ \frac{1}{\bar{m}(t)}\bar{\boldsymbol{C}}_{\mathrm{B/I}}\bar{\boldsymbol{T}}^{\mathrm{B}}(t) + \boldsymbol{g}^{\mathrm{I}} \\ \frac{1}{2}\bar{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/I}}(t)\otimes\bar{\boldsymbol{q}}_{\mathrm{B/I}}(t) \\ J^{-1}[\boldsymbol{r}^{\mathrm{B}}_e\times]\bar{\boldsymbol{T}}^{\mathrm{B}}(t) - [\bar{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/I}}(t)\times]J\bar{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/I}}(t)) \\ \dot{\bar{\boldsymbol{T}}}^{\mathrm{B}}(t) \\ \bar{\boldsymbol{u}}(t) \end{pmatrix} - \boldsymbol{A}(t)\begin{pmatrix} \bar{m}(t) \\ \bar{\boldsymbol{r}}^{\mathrm{I}}(t) \\ \bar{\boldsymbol{v}}^{\mathrm{I}}(t) \\ \bar{\boldsymbol{q}}_{\mathrm{B/I}}(t) \\ \bar{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/I}}(t) \\ \bar{\boldsymbol{T}}^{\mathrm{B}}(t) \\ \dot{\bar{\boldsymbol{T}}}^{\mathrm{B}}(t) \end{pmatrix} - \boldsymbol{B}(t)\bar{\boldsymbol{u}}(t) \tag{C.9}$$

From TPBVP problem formulation, we have the initial and final required state values which give us the $\boldsymbol{x}_0$ and $\boldsymbol{x}_f$ for a closed time, t_f problem. The in between states and state differential solutions can be found by discretisation. For discretisation the total time is divided into K nodes and the solutions at these points is calculated as per the Eq.7. to 7. The discretisation is given in page 8.

$$
\boldsymbol{A}(t)=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha(\frac{\bar{\boldsymbol{T}}^{\mathtt{B}}(t)_1}{\|\bar{\boldsymbol{T}}^{\mathtt{B}}(t)\|}) & -\alpha(\frac{\bar{\boldsymbol{T}}^{\mathtt{B}}(t)_2}{\|\bar{\boldsymbol{T}}^{\mathtt{B}}(t)\|}) & -\alpha(\frac{\bar{\boldsymbol{T}}^{\mathtt{B}}(t)_3}{\|\bar{\boldsymbol{T}}^{\mathtt{B}}(t)\|}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\bar{m}^2}(\bar{\boldsymbol{C}}_{\mathtt{B/I}}(t)\bar{\boldsymbol{T}}^{\mathtt{B}})_1 & 0 & 0 & 0 & 0 & 0 & 0 & \Psi_{q_{11}} & \Psi_{q_{12}} & \Psi_{q_{13}} & \Psi_{q_{14}} & 0 & 0 & 0 & \frac{1}{\bar{m}}\bar{\boldsymbol{C}}_{\mathtt{B/I}_{11}} & \frac{1}{\bar{m}}\bar{\boldsymbol{C}}_{\mathtt{B/I}_{12}} & \frac{1}{\bar{m}}\bar{\boldsymbol{C}}_{\mathtt{B/I}_{13}} & 0 & 0 & 0 \\
\frac{1}{\bar{m}^2}(\bar{\boldsymbol{C}}_{\mathtt{B/I}}(t)\bar{\boldsymbol{T}}^{\mathtt{B}})_2 & 0 & 0 & 0 & 0 & 0 & 0 & \Psi_{q_{21}} & \Psi_{q_{22}} & \Psi_{q_{23}} & \Psi_{q_{24}} & 0 & 0 & 0 & \frac{1}{\bar{m}}\bar{\boldsymbol{C}}_{\mathtt{B/I}_{21}} & \frac{1}{\bar{m}}\bar{\boldsymbol{C}}_{\mathtt{B/I}_{22}} & \frac{1}{\bar{m}}\bar{\boldsymbol{C}}_{\mathtt{B/I}_{23}} & 0 & 0 & 0 \\
\frac{1}{\bar{m}^2}(\bar{\boldsymbol{C}}_{\mathtt{B/I}}(t)\bar{\boldsymbol{T}}^{\mathtt{B}})_3 & 0 & 0 & 0 & 0 & 0 & 0 & \Psi_{q_{31}} & \Psi_{q_{32}} & \Psi_{q_{33}} & \Psi_{q_{34}} & 0 & 0 & 0 & \frac{1}{\bar{m}}\bar{\boldsymbol{C}}_{\mathtt{B/I}_{31}} & \frac{1}{\bar{m}}\bar{\boldsymbol{C}}_{\mathtt{B/I}_{32}} & \frac{1}{\bar{m}}\bar{\boldsymbol{C}}_{\mathtt{B/I}_{33}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}_3} & -\bar{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}_2} & \bar{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}_1} & \bar{\boldsymbol{q}}_{\mathtt{B/I}_4} & -\bar{\boldsymbol{q}}_{\mathtt{B/I}_3} & \bar{\boldsymbol{q}}_{\mathtt{B/I}_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}_3} & 0 & \bar{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}_1} & \bar{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}_2} & \bar{\boldsymbol{q}}_{\mathtt{B/I}_3} & \bar{\boldsymbol{q}}_{\mathtt{B/I}_4} & -\bar{\boldsymbol{q}}_{\mathtt{B/I}_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}_2} & -\bar{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}_1} & 0 & \bar{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}_3} & -\bar{\boldsymbol{q}}_{\mathtt{B/I}_2} & \bar{\boldsymbol{q}}_{\mathtt{B/I}_1} & \bar{\boldsymbol{q}}_{\mathtt{B/I}_4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}_1} & -\bar{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}_2} & -\bar{\boldsymbol{\omega}}^{\mathtt{B}}_{\mathtt{B/I}_3} & 0 & -\bar{\boldsymbol{q}}_{\mathtt{B/I}_1} & -\bar{\boldsymbol{q}}_{\mathtt{B/I}_2} & -\bar{\boldsymbol{q}}_{\mathtt{B/I}_3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Theta_{\omega_{11}} & \Theta_{\omega_{12}} & \Theta_{\omega_{13}} & \Theta_{T_{11}} & \Theta_{T_{12}} & \Theta_{T_{13}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Theta_{\omega_{21}} & \Theta_{\omega_{22}} & \Theta_{\omega_{23}} & \Theta_{T_{21}} & \Theta_{T_{22}} & \Theta_{T_{23}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Theta_{\omega_{31}} & \Theta_{\omega_{32}} & \Theta_{\omega_{33}} & \Theta_{T_{31}} & \Theta_{T_{32}} & \Theta_{T_{33}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad
\boldsymbol{B}(t)=\begin{bmatrix}
0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1
\end{bmatrix} \tag{C.10}
$$