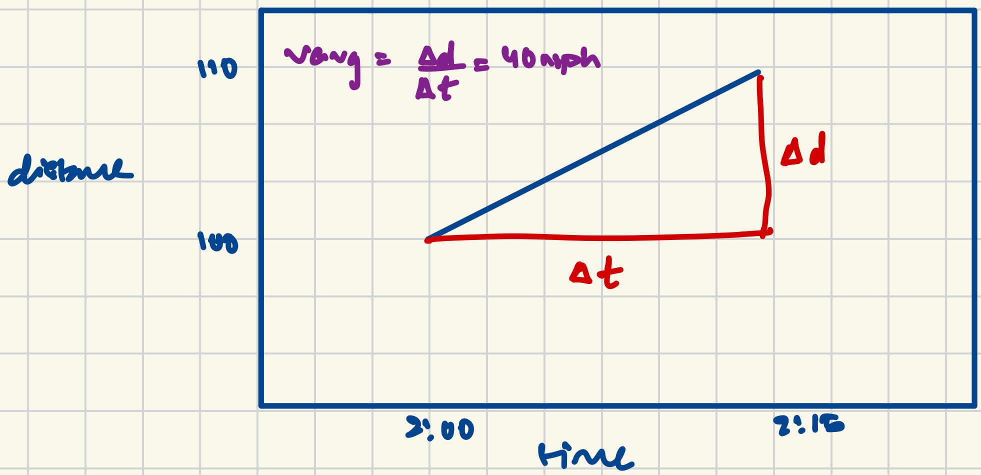
The Velocity Problem

Note:

$v_{avg} = \frac{\text{Change in Distance}}{\text{Change in time}}$



- The secant line would be the avg velocity.

- Average Velocity = $\frac{\text{Distance Travelled}}{\text{Time Elapsed}}$, which is represented by the slope of the secant line.

- Instantaneous Velocity = velocity at a given instant of time, which is represented by the slope of the tangent line.

Example:

If a rock is thrown upward on the planet mars with the velocity of 10 m/s, it's height in meters t seconds later is given by $y = 10t - 1.86t^2$

a) Find the average velocity over the given time intervals

i) $[1, 2] \leftarrow t$

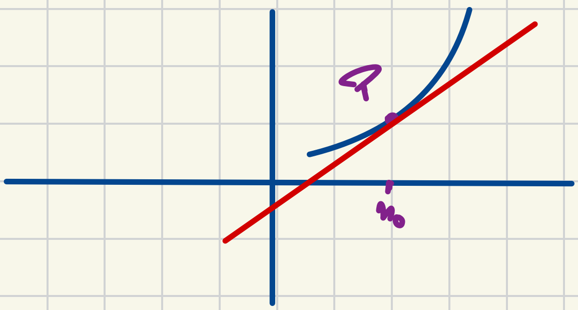
$$\text{average velocity} = \frac{\text{Change in Distance}}{\text{Change in time}}$$

$$y(1) = 10(1) - 1.86(1)^2 = 8.14$$
$$y(2) = 10(2) - 1.86(2)^2 = 12.56$$

$$\therefore \text{avg velocity} = \frac{12.56 - 8.14}{2 - 1} = \boxed{4.42\text{m/s}}$$

Geometric Interpretation of a Derivative

Find the tangent line to $y = f(x)$, at $P(x_0, y_0)$



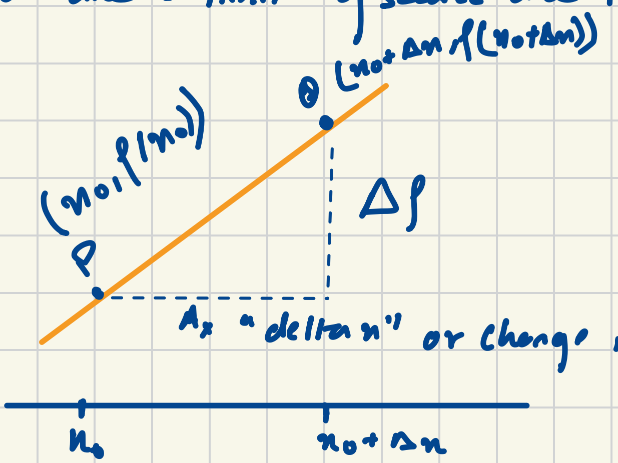
Tangent line: $y - y_0 = m(x - x_0)$

$$y_0 = f(x_0)$$

$$\text{slope} = m = f'(x_0)$$

Def'n: $f'(x_0)$, the derivative of f at x_0, is the slope of the tangent line to $y = f(x)$ at P.

Tangent line = limit of secant line PQ as $Q \to P$, where P is fixed.



$$\underbrace{m}_{\text{slope of tangent line}} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} : \text{slope of Secant PQ}$$

$$\boxed{\begin{matrix} f'(x_0) \\ (x) \end{matrix} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}}$$

Ex:

$$f(x) = \frac{1}{x}$$

$$\frac{\Delta f}{\Delta x} = \frac{\frac{1}{x_0+\Delta x} - \frac{1}{x_0}}{\Delta x}$$

$$\frac{1}{\Delta x}\left(\frac{x_0 - (x_0+\Delta x)}{x_0(x_0+\Delta x)}\right)$$

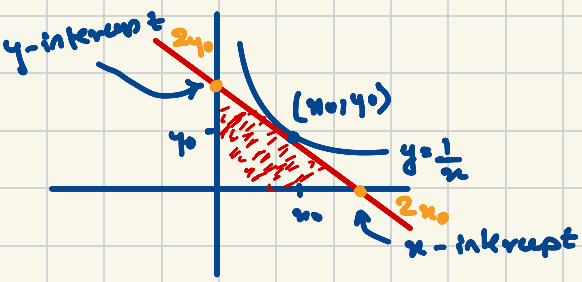
$$\frac{1}{\cancel{\Delta x}}\left(\frac{-\cancel{\Delta x}}{x_0(x_0+\Delta x)}\right)$$

$$f'(x_0)\ \frac{-1}{x_0(x_0+\Delta x)} \xrightarrow[\Delta x \to 0]{} \boxed{\frac{-1}{x_0^2}}$$

< 0, negative slope

$x_0 \to \infty$ less steep

Find the area of triangle enclosed by the axis and the tangent to $y = 1/x$.



$$y - y_0 = \underbrace{\frac{-1}{x_0^2}}_{m}(x - x_0)$$

when $y = 0$

$$0 - \frac{1}{x_0} = -\frac{1}{x_0^2}(x - x_0)$$

$$-\frac{1}{x_0} = \frac{-x}{x_0^2} + \frac{1}{x_0}$$

$$\cancel{-}\frac{2}{\cancel{x_0}} = \cancel{-}\frac{x}{x_0^{\cancel{2}}},\ x = 2x_0$$

when $x = 0$

$y = 2y_0$ {Using Symmetry}

Area of triangle $= \frac{1}{2}(2x_0)(2y_0)$

$= 2x_0 y_0$

More Notations

$y = f(x)$, $\Delta y = \Delta f$

$$\underbrace{f'}_{\text{Newton}} = \underbrace{\frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx} f = \frac{d}{dx} y}_{\text{Leibniz}}$$

(Omits x_0)

Note: $y = c$, c = constant horizontal line

Note: $x = c$, c = constant vertical line

Ex:

$4x + 2y - 8 = 0$

$2y = 8 - 4x$

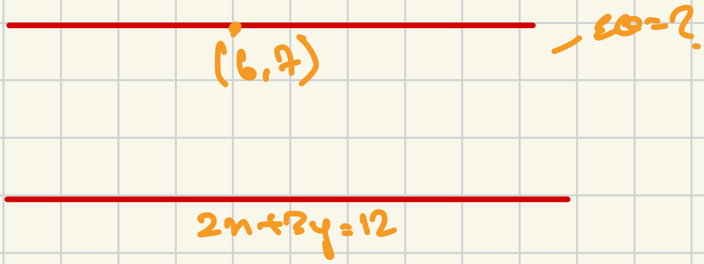
$y = \frac{1}{2}(8 - 4x)$

$y = -2x + \frac{8}{2}$

Parallel line: lines that have same slope.

Perpendicular line: lines that have negative reciprocal slope.

Ex: Find EQ thru $(6,7)$ AND parallel to
$2x + 3y = 12$



Now,

$2x+3y=12$

$3y = 12-2x$

$y = \frac{12}{3} - \frac{2x}{3}$

$y = 4 - \frac{2}{3}x$

$m = -\frac{2}{3}$ (Since both lines are parallel)

$y - y_1 = m(x - x_1)$

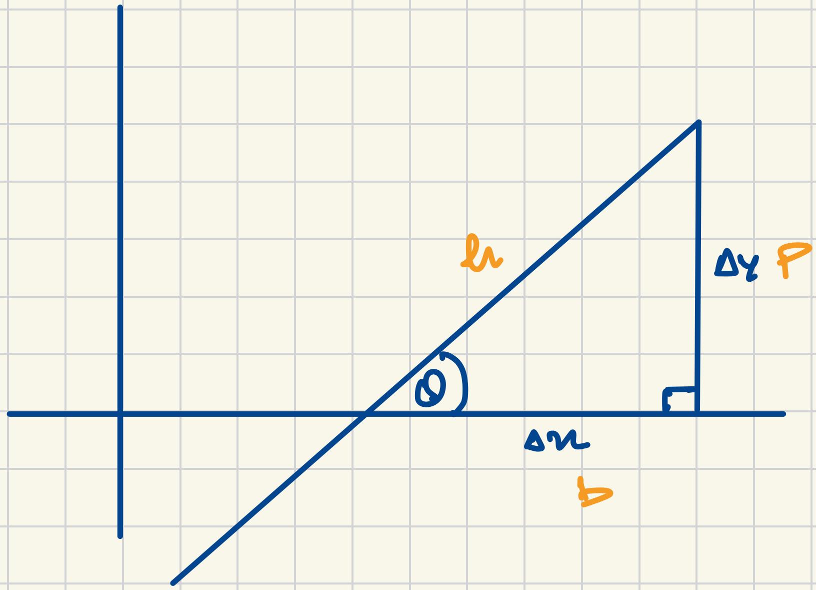
$y - 7 = -\frac{2}{3}(x-6)$

$3y - 21 = -2x + 12$

$3y = -2x + 33$

$$\boxed{y = -\frac{2x}{3} + 11}$$

Angle of Inclination



Here,

$$\tan\theta = \frac{P}{b}$$

or,

$$\tan\theta = \frac{\Delta y}{\Delta x} = m$$

Here,

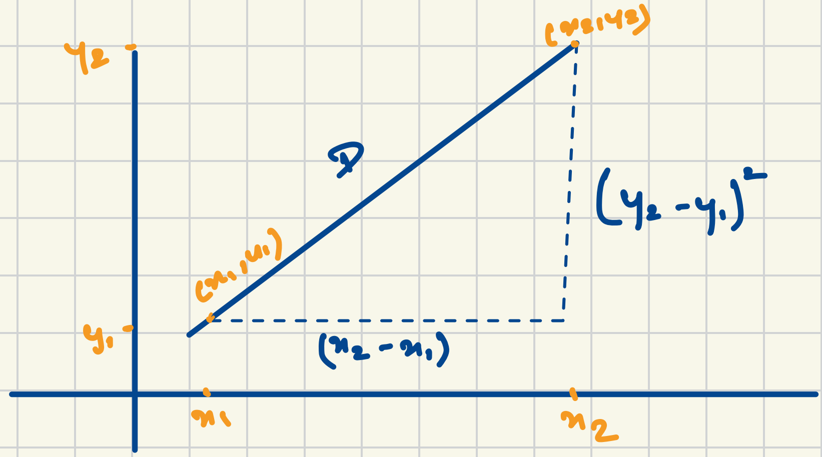
$\angle$n $\theta = 30^\circ$

$$\tan 30^\circ = m$$

$$m = \frac{1}{\sqrt{3}}$$

$$\boxed{\therefore m = \frac{1}{\sqrt{3}}}$$

Distance Formula



$$D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Functions

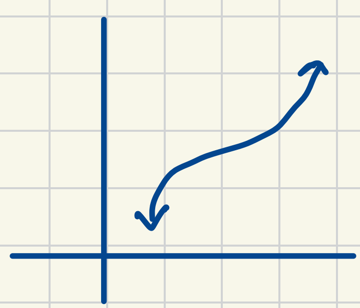
Some Expression where each input (x) determines one specific (y) output → Meaning doesn't happen again.

Ex

Fish	lbs
1	3.2
2	1.4
3	2.8
4	7.3
5	3.2

✓

$A = \pi r^2$



$x^2 + y^2 = 25$ (To check if this is a function)

$y^2 = 25 - x^2$

$y = \pm\sqrt{25 - x^2}$ (Not a function)

BUT....

$f(x) = +\sqrt{25 - x^2}$ ✓ function

$g(x) = -\sqrt{25 - x^2}$ ✓ function

PIECE-WISE Function

The function changes depending on the value of (x)

ABSOLUTE-VALUE (gives you distance from x)

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

* You can graph piece wise function by graphing each piece INDIVIDUALLY, For The given domain.

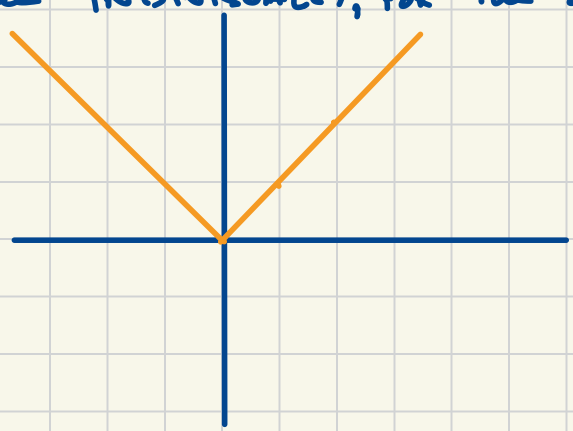


fig: $f(x) = |x|$

$$g(x) = \begin{cases} 0, & x \leq -1 \\ \sqrt{1-x^2}, & -1 < x < 1 \\ x, & x \geq 1 \end{cases}$$

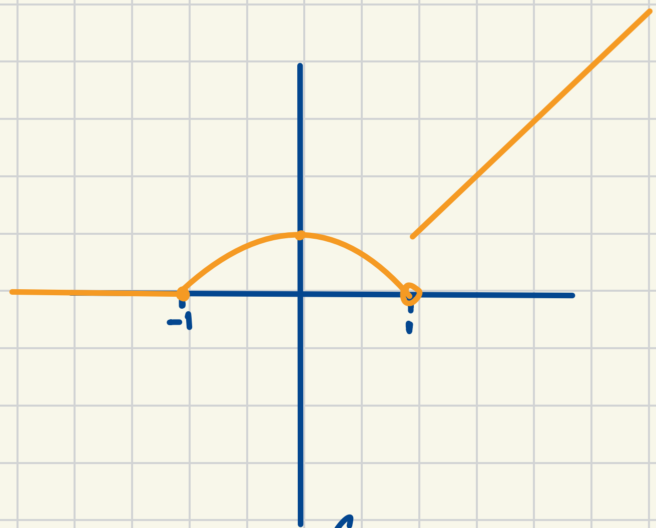


fig: $$g(x) = \begin{cases} 0, & x \leq -1 \\ \sqrt{1-x^2}, & -1 < x < 1 \\ x, & x \geq 1 \end{cases}$$

Domain and Range

DOMAIN : All INPUT values (x-value)

Range : All output values (y-values)

Ex:

AREA OF SQUARE

$A = s^2, \quad s \geq 0$

$y = \frac{1}{x}, \quad x \neq 0$

$f(x) = \sqrt{x}, \quad x \geq 0$

Check Denominators/Roots that might give you undefined / imaginary number

Natural Domain: Everything / inputs that work in the formula.

Ex

$f(x) = x^3$ D: All real numbers, $x \in \mathbb{R}$

Ex $g(x) = \dfrac{1}{(x-1)(x-3)}$ D: $x \in \mathbb{R}$ except $x \neq 1, x \neq 3$

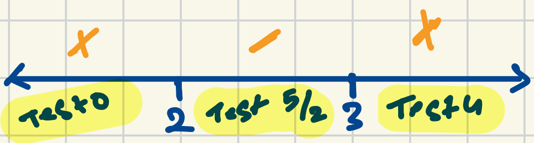
Ex $h(x) = \tan x$ D: $x \in \mathbb{R}$ except
or, $h(x) = \dfrac{\sin x}{\cos x}$ $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\ldots$

Ex $f(x) = \sqrt{x^2 - 5x + 6}$: $D = (-\infty, 2] \cup [3, \infty)$

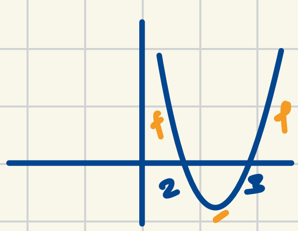
Here,

$x^2 - 5x + 6 = 0$

$x^2 - (2+3)x + 6 = 0$
$x^2 - 2x - 3x + 6 = 0$
$x(x-2) - 3(x-2) = 0$
$(x-3)(x-2) = 0$



Sign Analysis Test

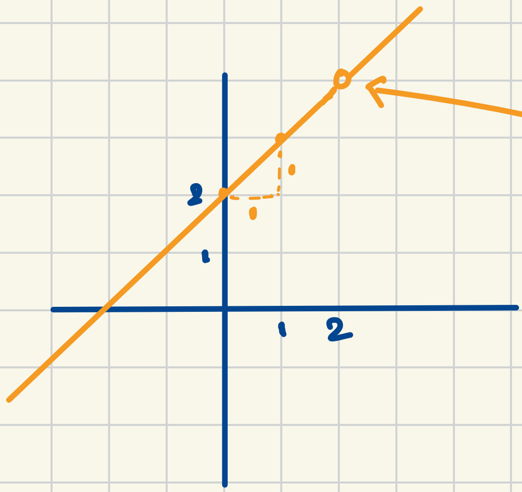


Ex $f(x) = \dfrac{x^2-4}{x-2}$ D: $x \neq 2$

$f(x) = \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$

You still need to keep original domain even if you simplify!

$f(x) = (x+2)$ D: All $x \in \mathbb{R}$? X , D: $x \neq 2$



If you can cancel your Domain Problem, it's a hole $0/0$

if you cannot cancel your Domain Problem, it's an asymtote. $\dfrac{\text{Something}}{0}$

Ex $g(x) = \dfrac{3x}{x-4}$ $x \neq 4$

Since, we cannot factor the domain problem, it's a "vertical ASYMTOTE"

Ex $f(x) = 2 + \sqrt{x-1}$ $x - 1 \geq 0$

D: $[1, \infty)$

R: $[2, \infty)$

$f(1) = 2/0$

Ex $y = \dfrac{x+1}{x-1}$ D: $x \in \mathbb{R}$ except $x \neq 1$

Range: $y \in \mathbb{R}$ except $y \neq 1$

[Asymtote]

Even and ODD function

$f(-x) = f(x)$ **Even**

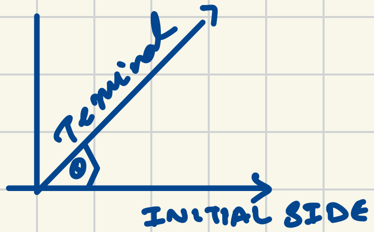
Symmetric about Y Axis

$f(-x) = -f(x)$ **ODD**

SYMMETRIC about the origin.

TRIG FUNCTIONS

ANGLES



COUNTER-CLOCKWISE GIVES POSITIVE ANGLES

CLOCKWISE GIVES Negative Angles

RADIANS Vs Degrees

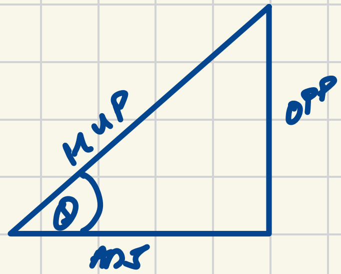
$2\pi = 360$ DEGREES

NOTE! CONVERT

Degrees $\rightarrow$ RADIANS : $* \frac{\pi}{180}$

Radians $\rightarrow$ Degrees : $* \frac{180}{\pi}$

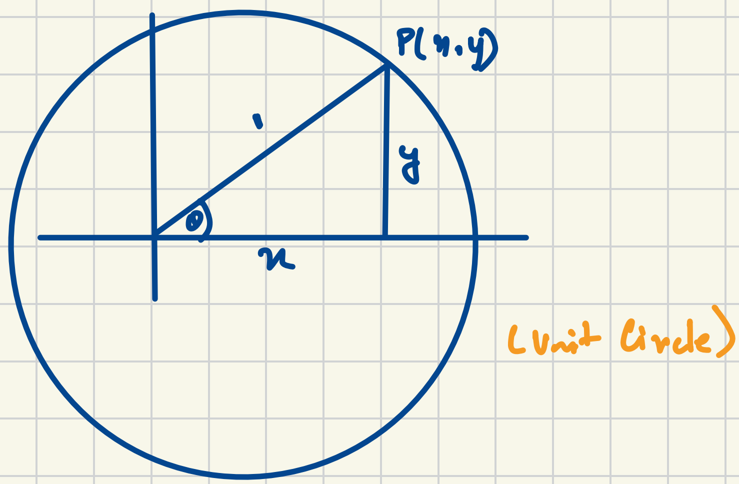
TRIG FUNCTIONS



$\sin\theta : \dfrac{OPP}{HYP} \xrightarrow{\text{Reciprocal}} \csc\theta : \dfrac{HYP}{OPP}$

$\cos\theta : \dfrac{ADJ}{HYP} \xrightarrow{\text{Reciprocal}} \sec\theta : \dfrac{HYP}{ADJ}$

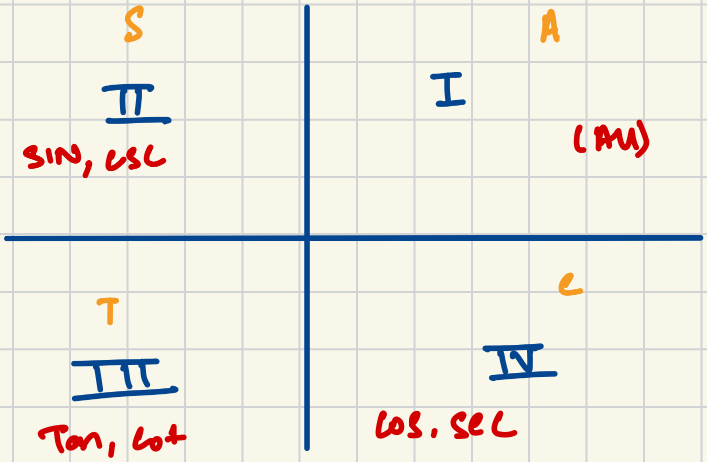
$\tan\theta : \dfrac{OPP}{ADJ} \xrightarrow{\text{Reciprocal}} \cot\theta : \dfrac{ADJ}{OPP}$



* $\sin\theta = \dfrac{y}{1} = y$ * $\csc\theta = \dfrac{1}{y}$

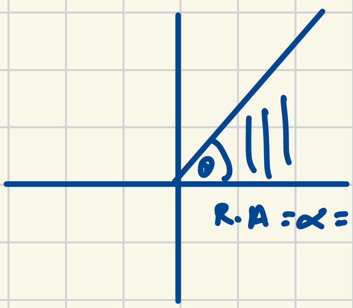
* $\cos\theta = \dfrac{x}{1} = x$ * $\sec\theta = \dfrac{1}{x}$

* $\tan\theta = \dfrac{y}{x}$ * $\cot\theta = x/y$

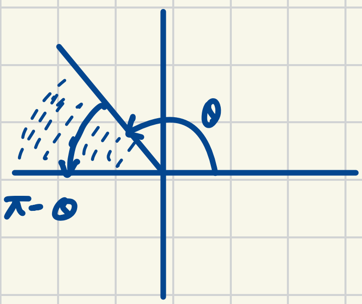


REFERENCE ANGLE (α)

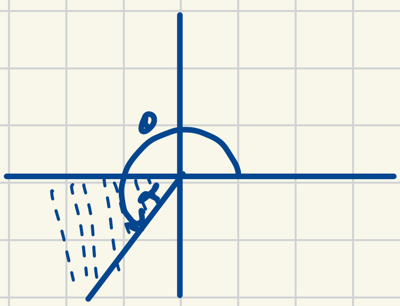
IDEA: Make an Acute Angle w/ x-axis, Then use ASTC



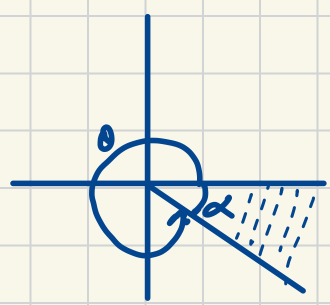
R.A $= \alpha = \theta$



$\alpha = \pi - \theta$



$\alpha = \theta - \pi$



$\alpha = 2\pi - \theta$

GRAPH

$y = A\cos(Bx)$

$y = \sin(x)$

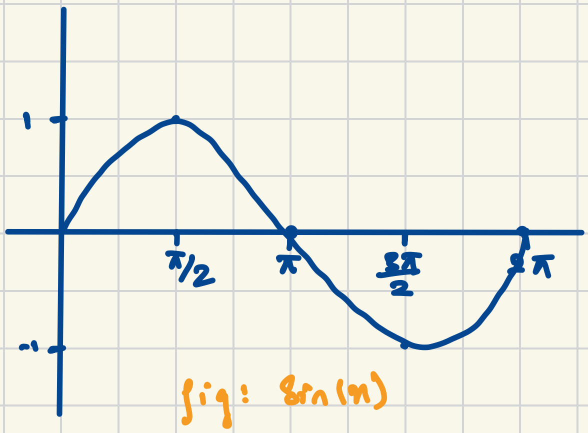
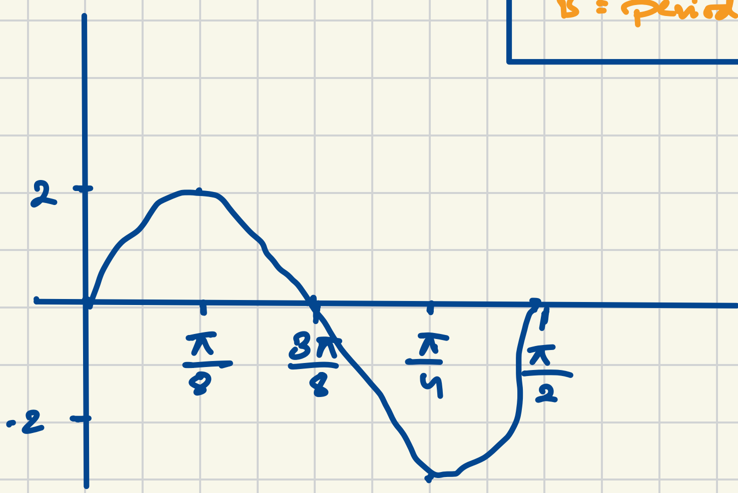


fig: $\sin(x)$

$y = 2\sin(4x)$

$y = A\sin(Bx)$

$|A|$ = Amplitude
B = period $= \frac{2\pi}{B}$



Here,
Amplitude $= |2| = 2$
Period $= \frac{2\pi}{4} = \frac{\pi}{2}$

360°
- (180°)
(270°)

	0°	30°	45°	60°	90°
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Ex

$y = -3\cos\left(\frac{1}{2}x\right)$

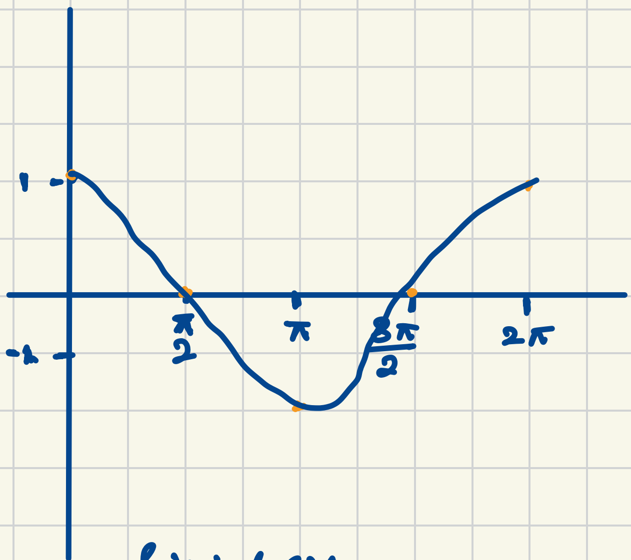
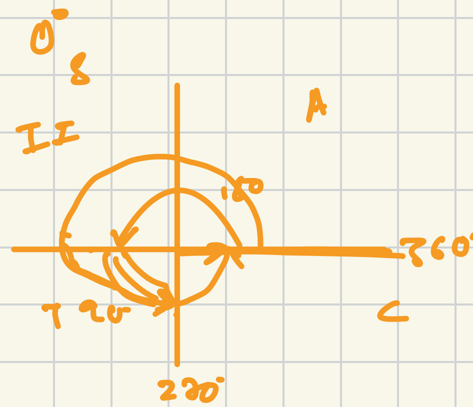


fig: $\cos x$



$A = |-3| = 3$

$\text{Period} = \frac{2\pi}{1/2} = 4\pi$

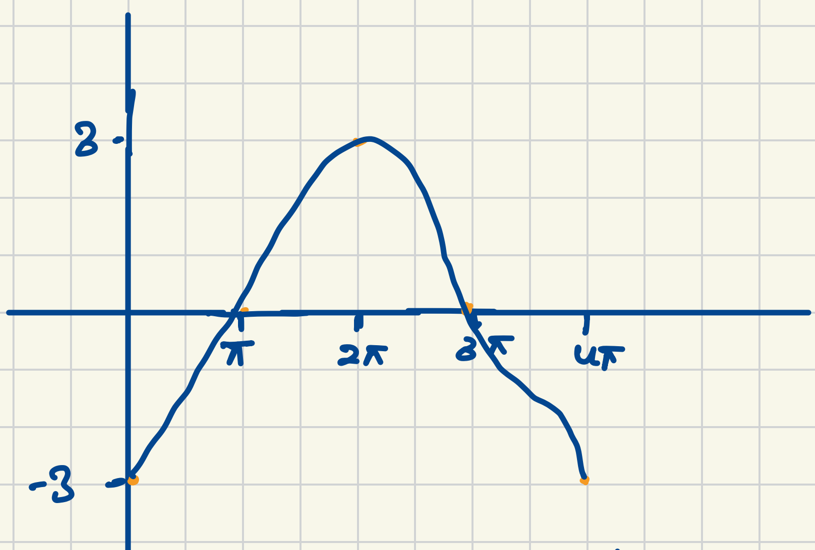


fig: $-3\cos\left(\frac{1}{2}x\right)$

Note: -ve sign, will flip the graph vertically through the x-axis

$y = A\sin(Bx - C)$

or,

$y = A\sin\left[B\left(x - \frac{C}{B}\right)\right]$

$y = A\cos(Bx - C)$

or,

$y = A\cos\left[B\left(x - \frac{C}{B}\right)\right]$

The $\frac{C}{B}$ is a "translation"
(A shift Along the x-axis)

If $-\frac{C}{B}$, shift Right of x-axis

If $+\frac{C}{B}$, shift left of x-axis

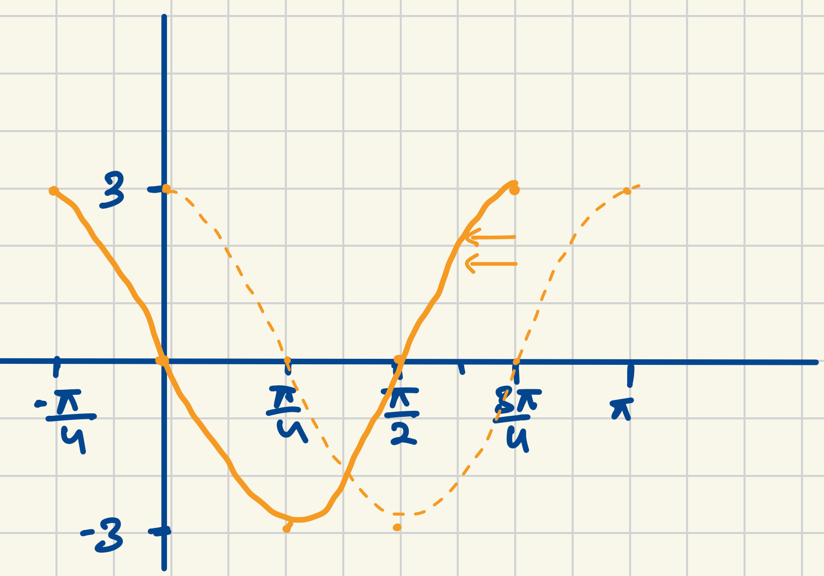
ex $y = 3\cos\left(2x + \frac{\pi}{2}\right)$

$\therefore y = 3\cos\left[2\left(x + \frac{\pi}{4}\right)\right]$

$A = |3| = 3$

Period $= \frac{\cancel{2}\pi}{\cancel{2}} = \pi$

TRANSLATION $= \frac{\pi}{4}$ (left shift)



COMBINING FUNCTIONS

$f(x) = 1 + \sqrt{x-2}$ $\qquad$ $g(x) = x - 3$

* $(f+g)(x) = f(x) + g(x)$
  $= (1+\sqrt{x-2}) + (x-3)$

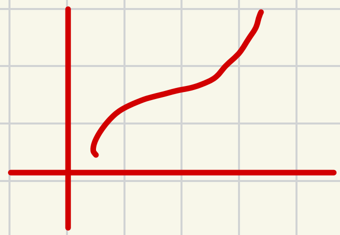
* $(f-g)(x) = f(x) - g(x)$
  $= (1+\sqrt{x-2}) - (x-3)$

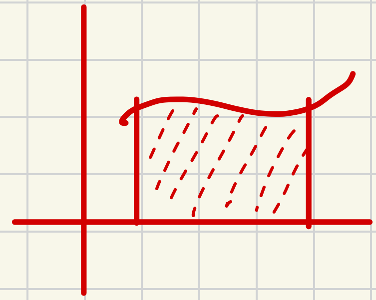
* $(f \cdot g)(x) = f(x) \cdot g(x)$
  $= (1+\sqrt{x-2}) \cdot (x-3)$

* $(f/g)(x) = \dfrac{f(x)}{g(x)}$
  $= \dfrac{1+\sqrt{x-2}}{x-3}$ $\qquad$ D: $x \geq 2$, except $x \neq 3$

LIMITS

Two Goals For Calculus

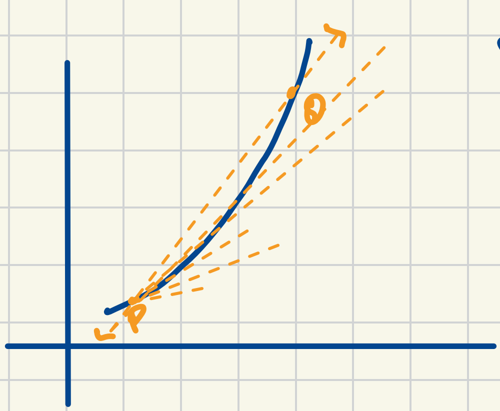




- Find the slope/ TANGENT to the curve at any point.
- Find the area under the curve b/w two points

Tangent Problem:

Find Tangent at Point P

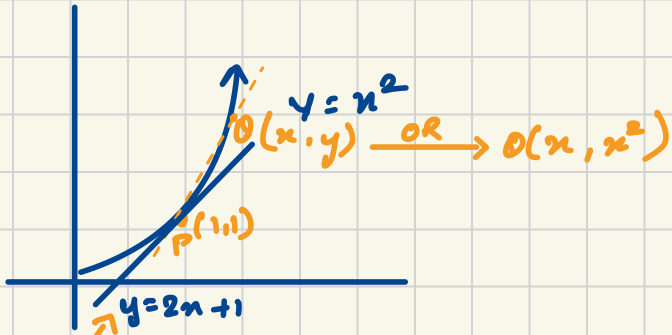


$\vec{PQ}$ is on secant line (has two points)

Keep moving Q really really close to P (which is fixed btw). so we get an approximate for the tangent line at P.

Note: we can't let Q = P, because you need two points to have a line, but we can let it get really really close, the secant line would be identical to the tangent line.

Ex



Find the eqn of the tangent line @ (1,1)

EQN of tangent line

$$y - y_1 = m_{TAN}(x - x_1)$$

$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{x^2-1}{x-1}$$

* AS $Q \to P$

$m_{sec} \to m_{TAN}$

* $x \neq 1$ {Because it gives undefined at $x=1$, as $Q \neq P$}

SO

$$m_{sec} = \frac{x^2-1}{x-1}$$

$$m_{sec} = \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}, \quad D: x \neq 1$$

$m_{sec} = (x+1)$, so what happens when $x \to 1$

$m_{sec} \to 2$

Hence,

$m_{TAN} = 2$

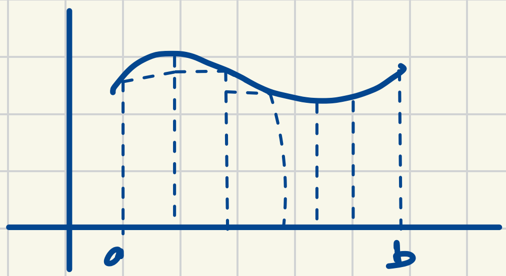
Hence,

EQN of tangent

$$y - 1 = 2(x-1)$$

$$\therefore y = 2x - 1$$

AREA PROBLEM



Make infinite number of rectangles under the curve and add the area of all the Rectangles.

Limits

"what does the function do as the variable approaches A given value?"

$f(x) = x^2$, what happens as $x \to 2$?

x	1.5	1.9	1.99	2	2.001	2.1	2.5
	2.25	3.61	3.9601		4.004	4.41	6.25

$\rightarrow 4 \leftarrow$

The function <u>must</u> APPROACH the <u>SAME</u> value from the left and the right for it to exist.

$$\lim_{x \to 2} x^2 = 4$$

In General

$$\lim_{x \to a} f(x) = L$$

"x never gets to a"

Ex

$$\lim_{x \to 1} \frac{x-1}{x^2-1} \qquad D : x \neq 1$$

0.5	.99	.999	1	1.001	1.01	1.5
.68	.503	.5002		.499	.498	.4

$\longrightarrow \frac{1}{2} \longleftarrow$

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

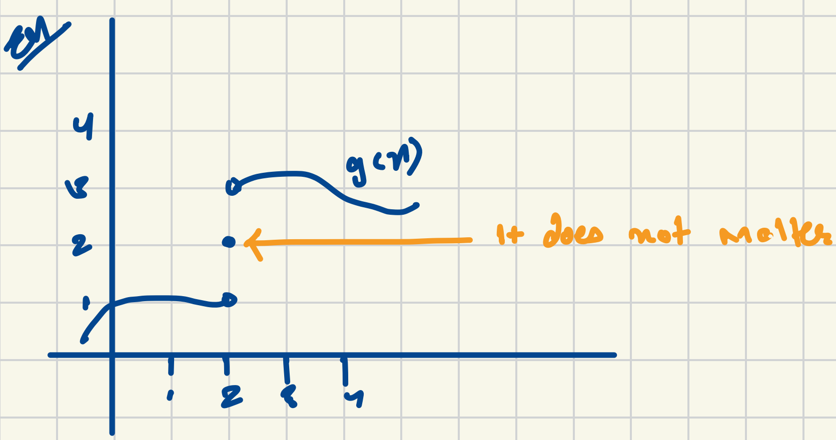
NOTE:

For a limit to exist at a point "a"

$$\lim_{x \to a} f(x) = L$$

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Ex



$$\lim_{x \to 2} g(x) = \text{D.N.E as } \boxed{\lim_{x \to 2^-} = 1 \neq \lim_{x \to 2^+} = 3}$$

Ex

$\lim_{x \to 0} f(x) = \frac{1}{x}$

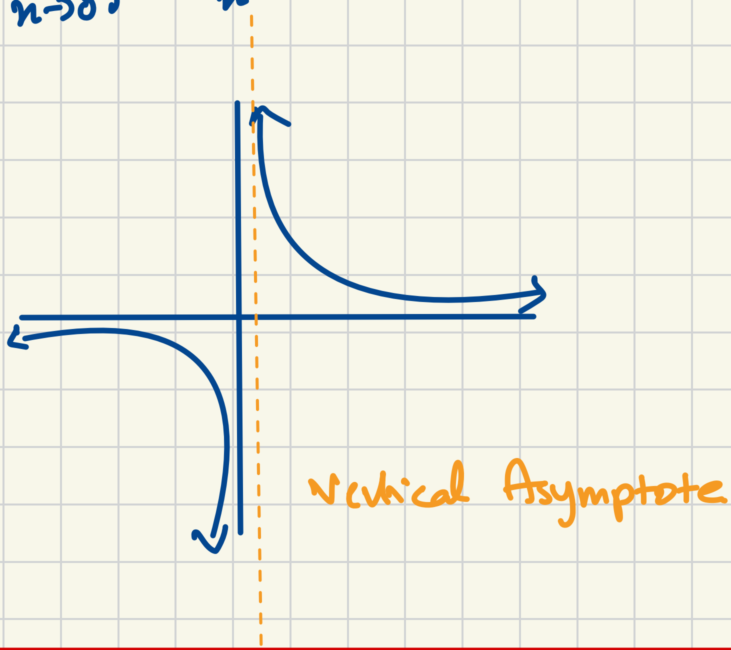
Here,

$\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0^+} f(x) = \infty$

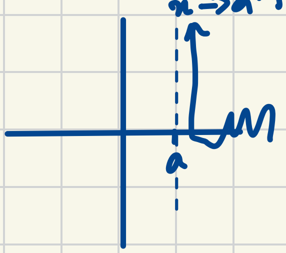
Hence,

$\lim_{x \to 0} f(x) = \frac{1}{x} = \text{D.N.E}$

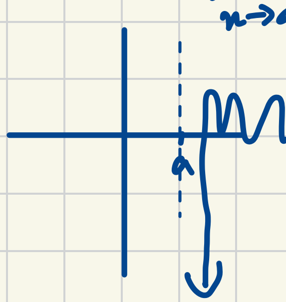


ASYMPTOTES

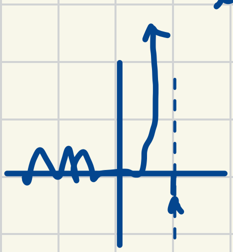
$\lim_{x \to a^+} f(x) = +\infty$



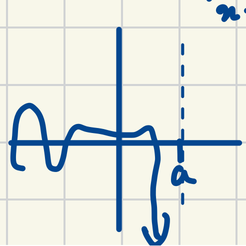
$\lim_{x \to a^+} f(x) = -\infty$



$\lim_{x \to a^-} f(x) = +\infty$



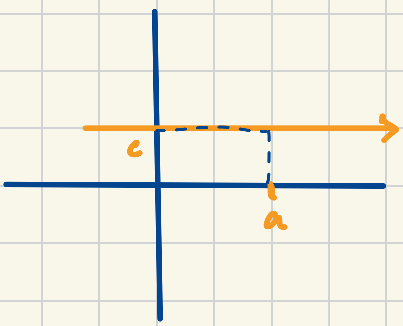
$\lim_{x \to a^-} f(x) = -\infty$



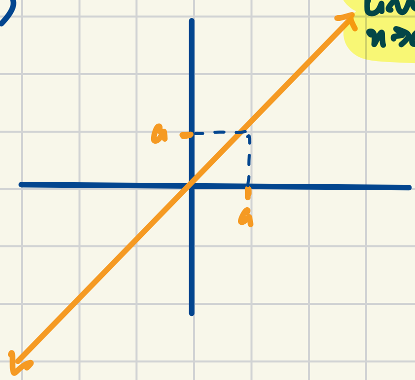
Computing Limits

Note

$\lim_{x \to} c = c$ where c: constant



2) $\lim_{x \to a} x = a$



3) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

4) $\lim_{x \to 0^+} \frac{1}{x} = \infty$

5) $\lim_{x \to 0} \frac{1}{x} =$ D.N.E

PROPERTIES

we have,

$$\lim_{x \to a} f(x) = L_1 \quad , \quad \lim_{x \to a} g(x) = L_2$$

we can,

1) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

2) $\lim_{x \to a} [f(x) . g(x)] = \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$

3) $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, $\lim_{x \to a} g(x) \neq 0$

4) $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x) \right]^n \longrightarrow \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

radical

Ex

$$\lim_{x \to 2} (x^3 - 2x + 7)$$

$$= \lim_{x \to 2} x^3 - \lim_{x \to 2} 2x + \lim_{x \to 2} 7$$

$$= \left[\lim_{x \to 2} x \right]^3 - \left[\lim_{x \to 2} 2 . \lim_{x \to 2} x \right] + \lim_{x \to 2} 7$$

$$= 2^3 - [2 . 2] + 7$$

$$= 8 - (4) + 7$$

$$\therefore \lim_{x \to 2} (x^3 - 2x + 7) = 11$$

FOR ANY POLYNOMIAL, JUST Plugging in the value in the function WORKS.

Polynomial function.

$$\lim_{x \to a} P(x) = P(a)$$

Ex $\lim_{x \to 2} (x^5 - 3x + 4)^3$

$\lim_{x \to 2} = (2^5 - 3(2) + 4)^3$

$\lim_{x \to 2} = 27000$

Ex $\lim_{x \to 2} \frac{4x^2+1}{x-3}$, $x \neq 3$

$$\frac{\lim_{x \to 2} 4x^2+1}{\lim_{x \to 2} x-3}$$

we can do this as long as the denominator does not equal to zero

$\lim_{x \to 2} \frac{4x^2+1}{x-3} = -17/1$

Ex $\lim_{x \to 1} \sqrt[3]{\frac{5x+2}{x^2+1}}$

Not a polynomial

$\lim_{x \to 1} \sqrt[3]{\frac{5(3)+7}{(1)^2+1}}$

$= \sqrt[3]{\frac{15+7}{2}}$

$= \sqrt[3]{6}$

$$\lim_{x \to 2} \frac{x^2-4}{x-2} \quad , x \neq 2$$

$$\lim_{x \to 2} \frac{(\cancel{x-2})(x+2)}{(\cancel{x-2})} \quad , \text{ since, } x \text{ never equal } 2$$

$$\lim_{x \to 2} (x+2)$$

Ex $\lim_{x \to -4} \frac{2x+8}{x^2+x-12} \qquad x \neq -3$

when,

$f(-4) = 0/0$ (hole)

now,

$$\lim_{x \to -4} = \frac{2(x+4)}{x^2-3x+4x-12}$$

$$\lim_{x \to -4} = \frac{2(x+4)}{x(x-3)+4(x-3)}$$

$$\lim_{x \to -4} = \frac{2\cancel{(x+4)}}{\cancel{(x+4)}(x-3)}$$

$$\lim_{x \to -4} = \frac{2}{(x-3)}$$

$$\lim_{x \to -4} = \frac{2}{-4-3}$$

$$\therefore \lim_{x \to -4} = \frac{2}{-7}$$

D: $x \neq 5, x \in \mathbb{R}$

ex

$$\lim_{x \to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} \rightarrow \frac{x^2 - (5-2)x - 10}{x^2 - (5+5)x + 25} \rightarrow \frac{x^2 - 5x + 2x - 10}{x^2 - 5x - 5x + 25}$$

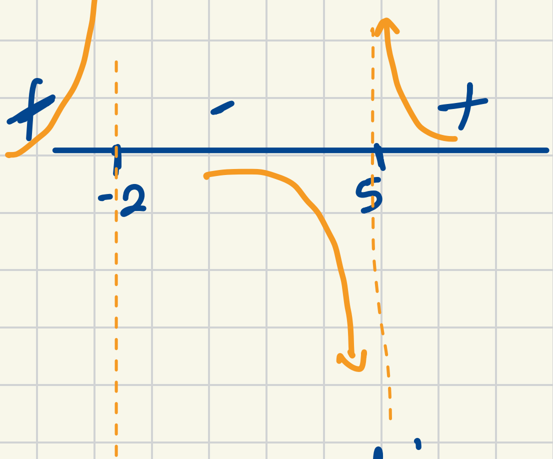
$$\lim_{x \to 5} \frac{x(x-5) + 2(x-5)}{x(x-5) - 5(x-5)}$$

$$\lim_{x \to 5} = \frac{(x+2)\cancel{(x-5)}}{(x-5)\cancel{(x-5)}}$$

Since, I cannot get rid of this, this function is an asymptote.

$$\lim_{x \to 5} = \frac{x+2}{x-5}$$

} Since, we cannot factor/simplify any more, we will be using sign analysis test.



$\lim_{x \to 5}$. D.N.E as $\lim_{x \to 5^-}$ Doesnot approach $\lim_{x \to 5^+}$

Note:

① If $p(x) = \frac{0}{0}$, factor and simplify.

② If you canot cancel / simplify the problem, that is an Asymptote. Do A SIGN ANALYSIS TEST

Eg $\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1}$ $\quad D: x \neq 1$

$\Rightarrow \lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1}$

$\Rightarrow \lim_{x \to 1} \frac{(\sqrt{x}+1)(x-1)}{(\sqrt{x})^2 - 1^2}$

$\Rightarrow \lim_{x \to 1} \frac{(\sqrt{x}+1)\cancel{(x-1)}}{\cancel{(x-1)}}$

$\Rightarrow \lim_{x \to 1} (\sqrt{x}+1)$

$\Rightarrow \boxed{\lim_{x \to 1} (\sqrt{x}+1) = 2}$

Eg $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$ $\quad D: x \neq 0$

$\Rightarrow \lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$\Rightarrow \lim_{x \to 0} \frac{(\sqrt{1+x})^2 - 1^2}{x(\sqrt{1+x}+1)}$

$\Rightarrow \lim_{x \to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)}$

$\Rightarrow \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{1+x}+1)}$

$\Rightarrow \lim_{x \to 0} \frac{1}{1+\sqrt{1+x}}$

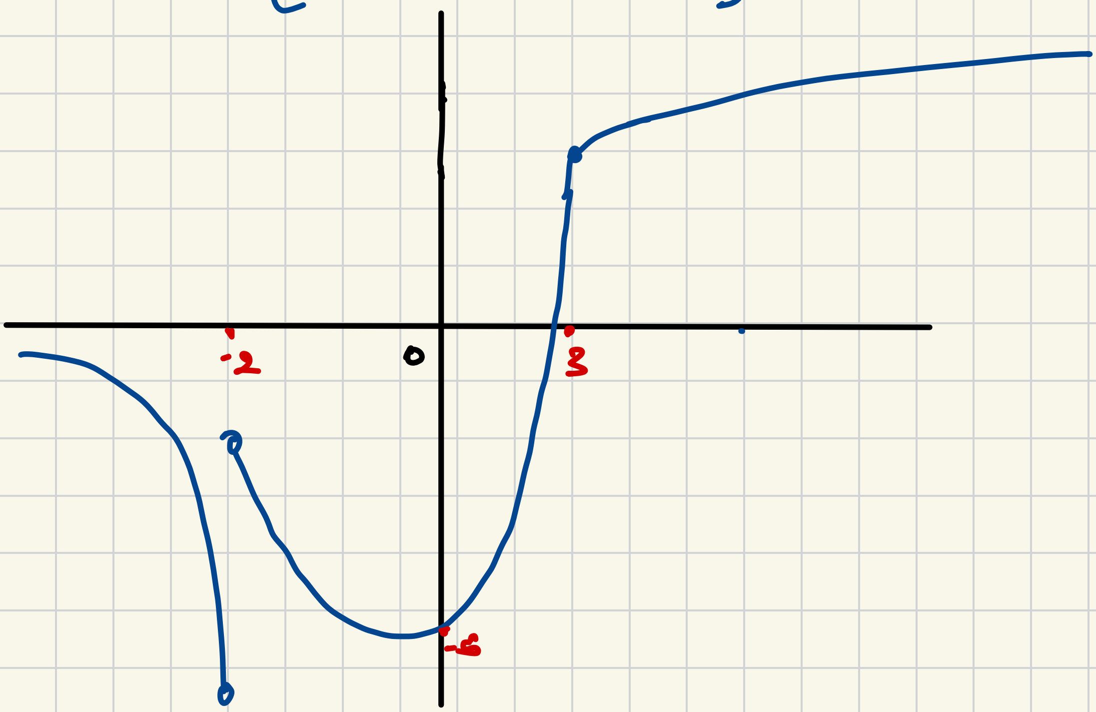
$\Rightarrow \lim_{x \to 0} = \frac{1}{2}$

Piece-wise Limits:

Find 1 sided limits and see if they are equal.

Ex

$$f(x) = \begin{cases} \dfrac{1}{x+2} & , x < -2 \\ x^2 - 5 & , -2 < x \leq 3 \\ \sqrt{x+12} & , x > 3 \end{cases}$$



$$\lim_{x \to 2^-} \frac{1}{x+2} = +\infty$$

$$\lim_{x \to 2^+} x^2 - 5 = -1$$

$$\lim_{x \to 2} f(x) = \text{D.N.E}$$

$$\lim_{x \to 3^-} x^2 - 5 = 4$$

$$\lim_{x \to 3^+} \sqrt{x+12} = 4$$

$$\lim_{x \to 3} f(x) = 4$$

Limits of Trig Functions

Sin(x) and cos(x) are continuous everywhere

$\lim_{x\to a} \sin(x) = \sin(a)$ & $\lim_{x\to a} \cos(x) = \cos(a)$

What about Tan(x)?

$\lim_{x\to a} \tan(x) = \lim_{x\to a} \frac{\sin(x)}{\cos(x)} \longrightarrow \frac{\lim_{x\to a} \sin(x)}{\lim_{x\to a} \cos(x)}$

$\longrightarrow \frac{\sin(a)}{\cos(a)} = \tan(a)$ $\quad \cos(a) \neq 0$

$\lim_{x\to a} \tan(x) = \tan(a)$, $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ ....

Ex $\lim_{x\to 1} \cos\left(\frac{x^2-1}{x-1}\right)$

$\therefore \lim_{x\to 1} \cos\left(\frac{x^2-1}{x-1}\right)$ cos is continuous, so by composition

$\cos\left[\lim_{x\to 1} \frac{x^2-1}{x-1}\right] \longrightarrow \cos\left[\lim_{x\to 1} x+1,\ x \neq 1\right]$

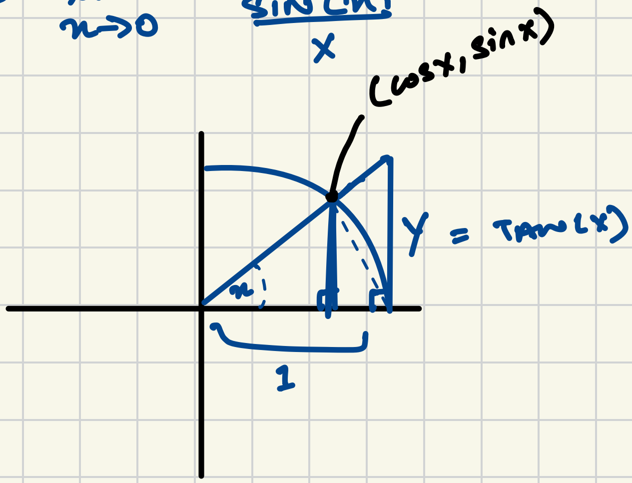
$\longrightarrow \cos\left[\lim_{x\to 1} x+1\right] = \cos(2)$ //

Ex $\lim_{x\to \frac{\pi}{2}} \left[3x^2 + \cos x\right]$

$\longrightarrow 3\left(\frac{\pi}{2}\right)^2 + \cos\left(\frac{\pi}{2}\right) \longrightarrow \frac{3\pi^2}{4} + 0$

SQUEEZE THEOREM

ex $\lim_{x \to 0} \frac{\sin(x)}{x}$



x = ANGLE

$\tan x = \frac{y}{1}$

$y = \tan(x)$

AREAS

Big Δ $\frac{1 \cdot \tan(x)}{2}$

Sector Δ $\frac{1^2 \cdot x}{2}$

Small Δ $\frac{1 \cdot \sin(x)}{2}$

$$\frac{1 \cdot \sin(x)}{2} < \frac{1 \cdot x}{2} < \frac{1 \cdot \tan x}{2}$$

OR

$$\sin x < x < \tan x$$

Divide all by $\sin(x)$

$$\frac{\sin(x)}{\sin(x)} < \frac{x}{\sin(x)} \leq \frac{\tan x}{\sin(x)}$$

$$1 \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)}$$

$$1 \geq \frac{\sin x}{x} \geq \cos(x)$$

According to squeeze theorem

$\lim_{x \to 0} 1 \to 1$ $\lim_{x \to 0} \frac{\sin x}{x}$ $\lim_{x \to 0} \cos(x) \to 1$

$\downarrow$

1

$$\lim_{x \to 0} \frac{1-\cos(x)}{x}$$

$$\Rightarrow \lim_{x \to 0} \frac{1-\cos(x)}{x} \cdot \frac{1+\cos(x)}{1+\cos(x)}$$

$$\Rightarrow \lim_{x \to 0} \frac{1^2-\cos^2 x}{x(1+\cos(x))} \rightarrow \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos(x))}$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{(1+\cos x)}$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{1+\cos x}$$

$$\Rightarrow 1 \cdot \left[\lim_{x \to 0} \frac{\sin x}{1+\cos x} \right]$$

$$\Rightarrow 1 \cdot \frac{0}{2}$$

$$\Rightarrow 0$$

$$\therefore \lim_{x \to 0} \frac{1-\cos x}{x} = 0$$

$$\lim_{x\to 0} : \frac{\tan x}{x} \to \lim_{x\to 0} \frac{\frac{\sin x}{\cos x}}{\frac{x}{1}}$$

$$\lim_{x\to 0} = \frac{\sin x}{\cos x} \times \frac{1}{x}$$

$$\lim_{x\to 0} = \frac{\sin x}{x} \cdot \lim_{x\to 0} = \frac{1}{\cos x}$$

$$1 \cdot 1$$

$$\therefore \lim_{x\to 0} \frac{\tan x}{x} = 1$$

Ex/ $\lim_{x\to 0} \frac{\sin(2x)}{x}$

$$\lim_{x\to 0} \frac{\sin(2x)}{x} \cdot \frac{2}{2}$$

$$\lim_{x\to 0} \frac{2\sin(2x)}{2x}$$

$$2 \cdot \lim_{x\to 0} \frac{\sin(2x)}{2x}$$

$$2 \cdot 1$$

$$\Rightarrow 2$$

Ex

$$\lim_{x\to 0} \frac{\sin(5x)}{\sin(6x)}$$

$$\lim_{x\to 0} \frac{\sin(5x)}{\sin(6x)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$\lim_{x\to 0} \frac{\frac{\sin(5x)}{x}}{\frac{\sin(6x)}{x}}$$

$$\lim_{x\to 0} \frac{\sin(5x)}{x} \cdot \frac{5}{5}$$

$$\lim_{x\to 0} \frac{5(\sin 5x)}{5x}$$

$$5 \cdot \lim_{x\to 0} \frac{\sin 5x}{5x} \rightarrow 5 \cdot 1 = 5$$

By squeeze theorem

$$\lim_{x\to 0} \frac{\sin 6x}{x} \cdot \frac{6}{6}$$

$$6 \cdot \lim_{x\to 0} \frac{\sin 6x}{6x}$$

$$6 \cdot 1 = 6$$

By squeeze theorem

$$\therefore \lim_{x\to 0} \frac{\sin(5x)}{\sin(6x)} = \frac{5}{6}$$

Ex $\lim_{x\to 0} \frac{\sin(x^2)}{x}$

$$\lim_{x\to 0} \frac{\sin(x^2)}{x} \cdot \frac{x}{x}$$

$$\lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{\sin(x^2)}{x^2}$$

$$0 \cdot 1$$

$$\therefore \lim_{x\to 0} \frac{\sin(x^2)}{x} = 0$$

Ex $\lim_{x\to 0} \frac{\sin^2(x)}{x}$

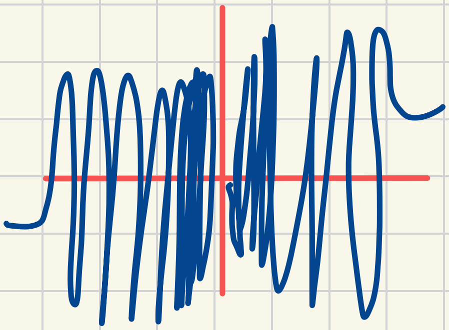
$$\lim_{x\to 0} \frac{\sin x}{x} \cdot \sin x$$

$$1 \cdot \lim_{x\to 0} \sin x$$

$$1 \cdot 0$$

$$\therefore \lim_{x\to 0} \frac{\sin^2(x)}{x} = 0$$

Ex $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = \text{DNE}$



Ex $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$\rightarrow -|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$

we know

$\lim_{x \to 0} -|x| = 0$ and $\lim_{x \to 0} |x| = 0$

By squeeze theorem,

$x \cdot \sin\left(\frac{1}{x}\right) = 0$

Ex $\lim_{x \to 0} \frac{2 - \cos(3x) - \cos(4x)}{x}$

$\lim_{x \to 0} \frac{1 + 1 - \cos 3x - \cos 4x}{x}$

$\lim_{x \to 0} \frac{1 - \cos 3x + 1 - \cos 4x}{x}$

$\lim_{x \to 0} \frac{1 - \cos 3x}{x} + \lim_{x \to 0} \frac{1 - \cos 4x}{x}$

$\lim_{x \to 0} \frac{3}{3} \cdot \frac{1 - \cos 3x}{x} + \lim_{x \to 0} \frac{4}{4} \cdot \frac{1 - \cos 4x}{x}$

$$3.\lim_{x\to 0} \frac{1-\cos 3x}{3} + 4.\lim_{x\to 0} \frac{1-\cos 4x}{4}$$

$$3.0 + 4.0$$

$$\therefore \lim_{x\to 0} \frac{2\cdot\cos(3x) - \cos 4x}{x} = 0$$

Ex

$$\lim_{x\to 0} \frac{x^2 \cdot 3\sin(x)}{x}$$

$$\lim_{x\to 0} \frac{x^{\cancel{2}}}{\cancel{x}} - \frac{3\sin(x)}{x}$$

$$\lim_{x\to 0} x - \lim_{x\to 0} 3\frac{\sin(x)}{x}$$

$$0 - 3.1 = -3$$

$$\therefore \lim_{x\to 0} \frac{x^2 - 3\sin(x)}{x} = -3$$

Ex $$\lim_{T\to 0} \frac{T^2}{1-\cos^2 T}$$

$$\lim_{T\to 0} \frac{T^2}{\sin^2 T} \rightarrow \lim_{x\to 0} \frac{T}{\sin T} \cdot \lim_{x\to 0} \frac{T}{\sin T}$$

$$\lim_{T\to 0} \lim_{x\to 0}\left[\frac{\sin T}{T}\right]^{-1} \cdot \lim_{x\to 0}\left[\frac{\sin T}{T}\right]^{-1}$$

$$\lim_{T\to 0} 1^{-1} \cdot 1^{-1}$$

$$\Rightarrow \therefore \lim_{T\to 0} \frac{T^2}{1-\cos^2 T} = 1$$

Ex $\lim_{x \to 0} \dfrac{x}{\cos(\frac{1}{2}\pi - x)}$ → Half Angle Identity

$\lim_{x \to 0} \dfrac{x}{\sin(x)} = 1$

$\therefore \lim_{x \to 0} \dfrac{x}{\sin(x)} = 1$

Ex $\lim_{\theta \to 0} \dfrac{\theta^2}{1-\cos\theta} \cdot \dfrac{1+\cos\theta}{1+\cos\theta}$

$\lim_{\theta \to 0} \dfrac{\theta^2(1+\cos\theta)}{1-\cos^2\theta}$

$\theta^2 \lim_{\theta \to 0} \dfrac{1+\cos\theta}{\sin^2\theta}$

$\lim_{\theta \to 0} \left(\dfrac{\theta}{\sin\theta}\right)^2 \cdot (1+\cos\theta)$

$\lim_{\theta \to 0} 1^2 \cdot (1+1)$

$\lim_{\theta \to 0} \dfrac{\theta^2}{1-\cos\theta} = 2$

Continuity

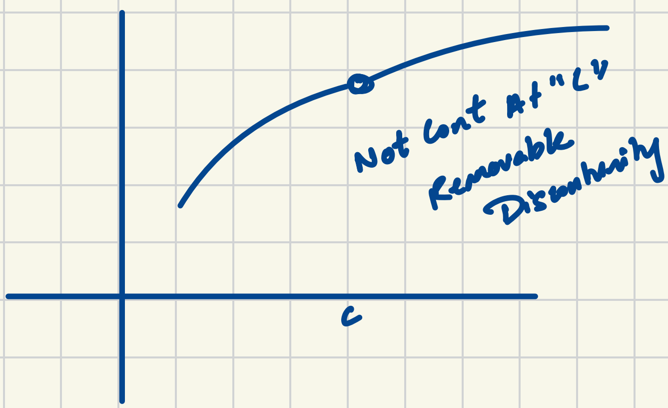
A curve is CONTINUOUS if the function does not have any "holes" AND "Breaks" OR "ASYMPTOTES."

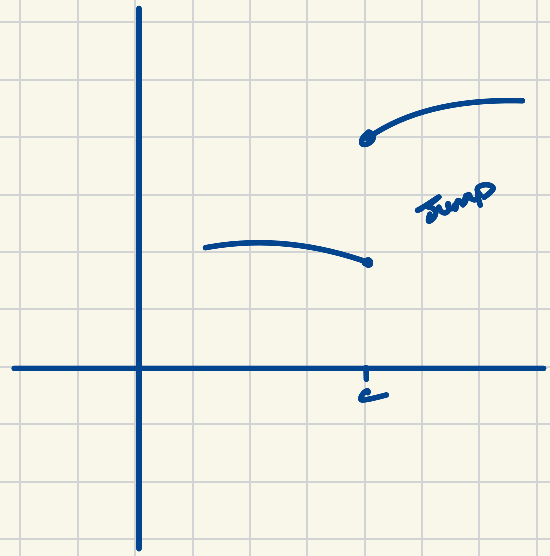
CONTINUOUS at a point 'C' if :

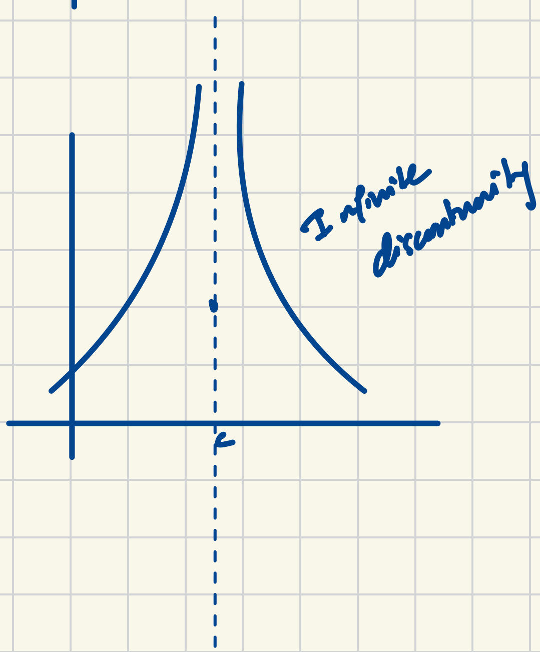
1. $f(c)$ IS defined.

2. $\lim_{x \to c} f(x)$ exists.

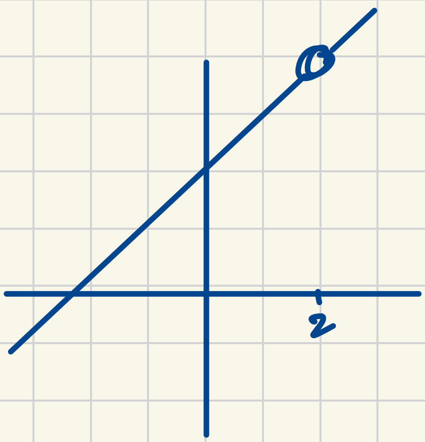
3. $\lim_{x \to c} f(x) = f(c)$

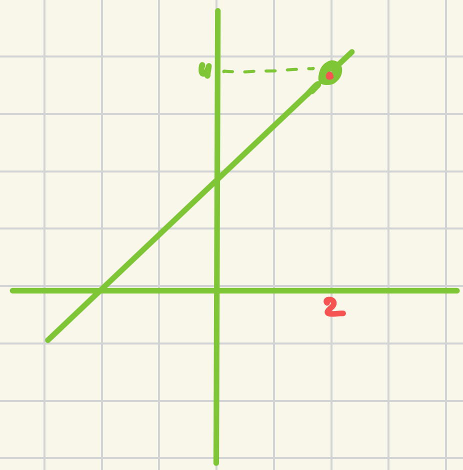






$f(x) = \frac{x^2-4}{x-4}$; $g(x) = \left\{ \begin{array}{ll} \frac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{array} \right\}$; $h(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$





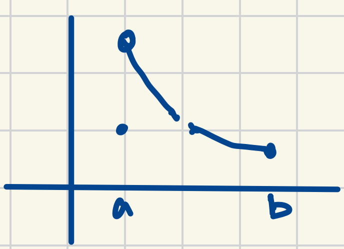
This function is continuous at every point between a & b, Then f is continuous on (a,b)

$f(x) = \frac{x^2-4}{x-2}$

$f(x) = \frac{(\cancel{x-2})(x+2)}{\cancel{x-2}}$

$f(x) = x+2$

What about Endpoints Continuity.



Continuous on [a,b]

In General:

- Continuity from the left at Point c:
  $\lim_{x \to c^-} f(x) = f(c)$ (Like Point 'B')

- Continuity from the Right at Point c
  $\lim_{x \to c^+} f(x) = f(c)$

SQUEEZE THEOREM
(Used mostly on Trig Functions)

Q) $8-x^3 \leq h(x) \leq x^2+8$

$\lim_{x \to 0} h(x)$

$\lim_{x \to 0} 8-x^3 = 8$ $\qquad\qquad$ $\lim_{x \to 0} x^2+8 = 8$

$\therefore \lim_{x \to 0} h(x) = 8$

Q) Find the limit $\lim_{x \to 0} (x\cos(x))$ using the

using the squeeze theorem

$g(x) \leq x\cos(x) \leq f(x)$

we know that

$-1 \leq \cos(x) \leq 1$

$-x \leq x\cos(x) \leq x$

$\lim_{x \to 0} -x \leq \lim_{x \to 0} x\cos(x) \leq \lim_{x \to 0} x$

So, when x approaches zero:

$0 \leq \lim_{x \to 0} x\cos x \leq 0$

$\boxed{\therefore \lim_{x \to 0} x\cos x = 0}$

Checking Continuity on End-Points

Ex Prove $f(x) = \sqrt{16-x^2}$ is continuous on $[-4,4]$

1. Check $(-4,4)$

$$\lim_{x\to c} \sqrt{16-x^2} = \sqrt{16-c^2} = f(c)$$

2. Check $\lim_{x\to -4^+} f(x)$

$$\lim_{x\to -4^+} \sqrt{16-(-4)^2} = 0 = f(-4)$$

3. Check $\lim_{x\to 4^-} f(x)$

$$\lim_{x\to 4^-} \sqrt{16-(-4)^2} = 0 = f(4)$$

Properties of limits

f \$ g are continuous at Point 'c'

$$\left.\begin{array}{l} 1.\ f+g \\ 2.\ f-g \\ 3\ f\cdot g \end{array}\right\} \text{ARE ALL Continuous at Point } c$$

4 f/g is continuous at 'c' unless $g(c)=0$

* if $g(c)=0$ There is,
A discontinuity at 'c' for f/g

ex $f(x) = \dfrac{x^2-4}{x^2+x-6}$

Find x where discontinuous

x^2+x-6

$x^2+(3-2)x-6$

$x^2+3x-2x-6$

$x(x-2)+3(x-2)$

$(x+3)\ (x-2)$

$\therefore f(x)$ is discontinuous at $x=-3$, $x=2$

when $x=2$

$f(2) = 0/0$, here it is a hole at $x=2$

PROVE $f(x) = |x|$ is continuous Everywhere

$$f(x) = |x| = \begin{cases} x & , x>0 \\ 0 & , x=0 \\ -x & , x<0 \end{cases}$$

Polynomial here Continuous

To check it's Continuity

$\lim_{x\to0^+} x = 0 \qquad \lim_{x\to0^-} -x = 0$

$\therefore$ since, $f(0)$ exists $\lim_{x\to0} f(x)$ exists

and

$f(0) = \lim_{x\to0} f(x)$

hence $f(x)$ is continuous everywhere.

Compositions w Limits

If $\lim_{x \to c} g(x) = L$ AND f is continuous at L

Then
$$\lim_{x \to c} f[g(x)] = f(L) = f\left(\lim_{x \to c} g(x)\right)$$

* we can seperate
Limits BY Composition

Ex $\lim_{x \to 4} |10 - 3x^2| \rightarrow$

$\hookrightarrow \left|\lim_{x \to 4} 10 - 3x^2\right|$

$\hookrightarrow |-38|$

$\hookrightarrow 38$ //

* If 2 Function are continuous Everywhere, Their compositions are continuous Everywhere.

* If f is continuous on it's domain, $f^{-1}(x)$ will also be continuous on it's Respective domain. (The Range of f)

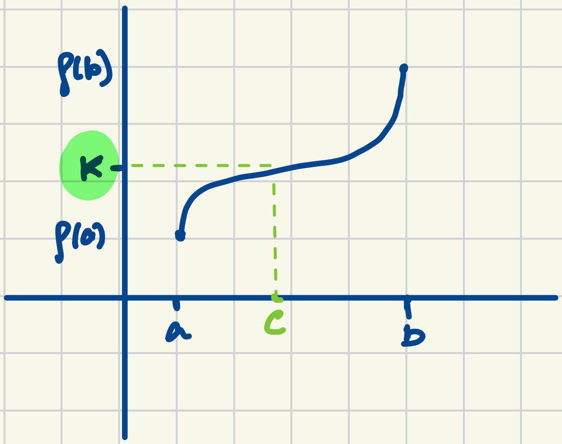
Ex

$f(x) = x^3$
Poly / cont from $(-\infty, \infty)$
RANGE $(-\infty, \infty)$

Therefore

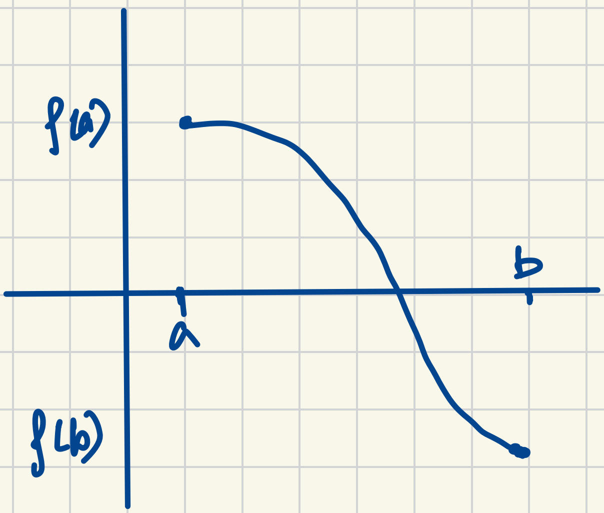
$f^{-1}(x) = \sqrt[3]{x}$
cont $(-\infty, \infty)$

INTERMEDIATE VALUE THEOREM



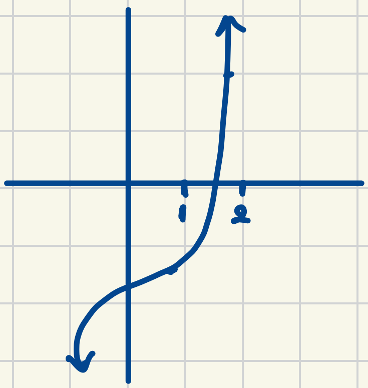
f is CONT ON [a,b]

SAY K is b/w f(a) & f(b)
THEN, there is at-least one x value 'c', where 'c' is between a & b &, such that f(c) = K



If f(a) & f(b) have DIFFERENT SIGNS, there MUST BE K, at-least 1 Root on [a,b]

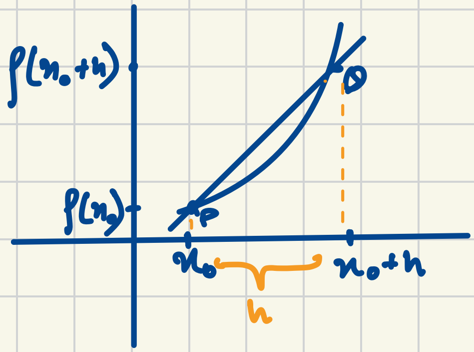
Ex



← There is a point c that gives f(c)=0, in between a & b

TANGENT LINE & Rate of change

using limits to find slope of the tangent line to the curve



$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{\cancel{x_0} + h - \cancel{x_0}} \quad \therefore \quad m = \frac{f(x+h) - f(x)}{h}$$

This is the curve of the above secant line.

To get the slope of the tangent.

$Q \to P$. How?

Let $h \to 0$ on $m_{TAN}\ \dfrac{f(x+h) - f(x)}{h}$

$$\boxed{\text{Here, } m_{TAN} \lim_{x \to 0} \frac{f(x_0+h) - f(x)}{h}}$$

Ex Find the EQ of the tangent line to $y = x^2$ at the point $(1,1)$

Here,

$$f(x) = x^2$$

$$m_{TAN} = \lim_{h \to 0} \frac{f(x_0+h) - f(x)}{h}$$

$$x_0 = 1$$

$$m_{TAN} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$m_{TAN} = \lim_{h \to 0} \frac{(1+h)^2 - (1)^2}{h}$$

$$m_{TAN} = \lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h}$$

$$M_{TAN} = \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}}$$

$$M_{TAN} = \lim_{h \to 0} 2+h$$

$\therefore$ $M_{TAN} = 2$ //

Now to find the equation of the tangent line at $(1,1)$

$m_{tan} = 2$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x-1)$$

$$y - 1 = 2x - 2$$

$$\boxed{\therefore\ y = 2x - 1}$$

Ex Find the Eq of the tangent line $y = 3/x$ at $(3,1)$

$$m_{tan} = \boxed{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$ ← "Instantaneous slope"

$$m_{tan} = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$

$$m_{tan} = \lim_{h\to 0} \frac{\frac{3}{3+h} - 1}{h}$$

$$m_{tan} = \lim_{h\to 0} \frac{\frac{3-3-h}{3+h}}{h}$$

$$m_{tan} = \lim_{h\to 0} \frac{-\cancel{h}}{3+h} \times \frac{1}{\cancel{h}}$$

$$m_{tan} = \lim_{h\to 0} -\frac{1}{3+h}$$

$$m_{tan} \lim_{h\to 0} = -\frac{1}{3+h}$$

$$\boxed{m_{tan} = -\frac{1}{3}}$$

slope of tangent at this point

Eqn of the tangent:

$$y - y_1 = m(x - x_1$$

$$y - 1 = -\frac{1}{3}(x-3)$$

$$y - 1 = -\frac{x}{3} + 1$$

$$\boxed{y = -\frac{x}{3} + 2}$$

ex/ Find slope of TANGENT Lines at $y = \sqrt{x}$ at ANY POINT

Here,

$$f(x+h) = \sqrt{x+h}$$

$$f(x) = \sqrt{x}$$

Then

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$m = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$m = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$m = \lim_{h \to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})}$$

$$m = \lim_{h \to 0} \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$m = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$\boxed{\therefore\ m = \frac{1}{2\sqrt{x}}}$$

Equation to slope at any point x on the curve

Velocity
AND
The Rate of Change

average velocity

$$V_{AVE} : \frac{\text{DISTANCE}}{\text{TIME}} = \frac{f(T_0+h) + f(T_0)}{h}$$

Elapsed time

Starting time

ex. Find average velocity of $s(t) = 1+3t-2t^2$
on $[1,3]$ $T_0=1$, $h=2$

$$V_{AVE} = \frac{S(T+h) - S(T)}{h}$$

$$V_{AVE} : \frac{S(1+2) - S(1)}{2}$$

$$= \frac{-8-2}{2} = \boxed{-5\,m/s}$$

Instantaneous Velocity

$$m = \lim_{h \to 0} \frac{f(T_0+h) - f(T)}{T}$$

$$V_{Instantaneous} \lim_{h \to 0} \frac{f(T_0+h) - f(T_0)}{h}$$

ex. $S(T) = 500 - 16T^2$, Find Instantaneous Velocity at 5 sec.

$$V_{INST} \lim_{h \to 0} \frac{f(T_0+h) - f(T)}{h}$$

$$\lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$$

$$\lim_{h \to 0} \frac{(500 - 16(5+h)^2) - (500 - 16(5)^2)}{h}$$

$$\lim_{h \to 0} \frac{500 - 16(5+h)^2 - 100}{h}$$

$$\lim_{h \to 0} \frac{400 - 16(5+h)^2}{h}$$

$$\lim_{h \to 0} \frac{400 - 16(25 + 10h + h^2)}{h}$$

$$\lim_{h \to 0} \frac{\cancel{400} - \cancel{400} - 160h - 16h^2}{h}$$

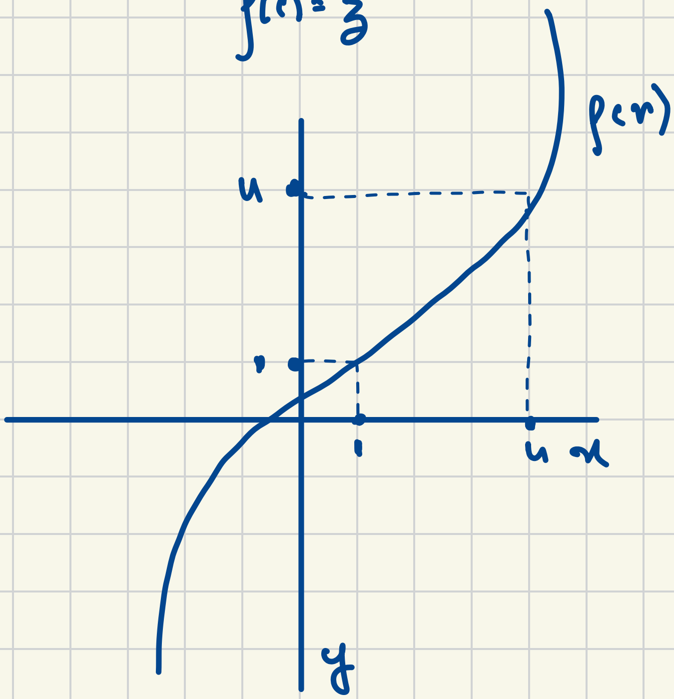
$$\lim_{h \to 0} \frac{-16\cancel{h}(10+h)}{\cancel{h}}$$

$$-16(10+0)$$

$$\boxed{V_{INS} : -160\,m/sec}$$

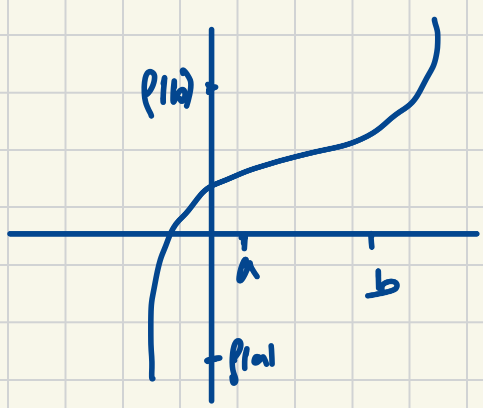
Intermediate Value Theorem

If f is continuous on a closed Interval $[a,b]$ and z is any number b/w $f(a)$ and $f(b)$, there exists at least one $c \in (a,b)$ such that $f(c) = z$

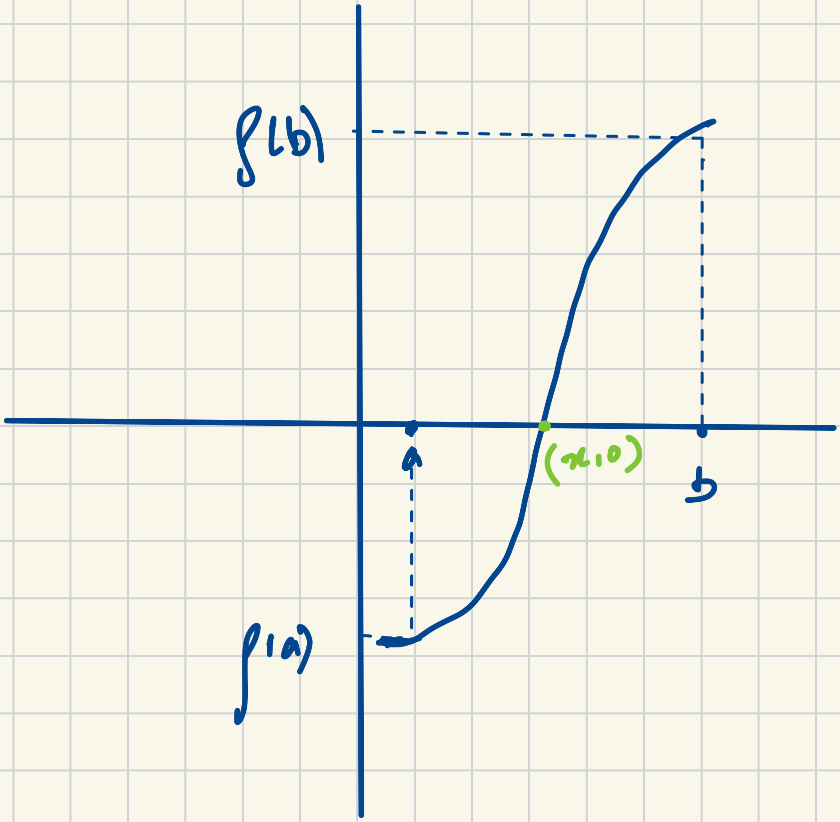


1) $f(x)$ is continuous.

2) There is at least one x, that equal to y b/w $[1,4]$

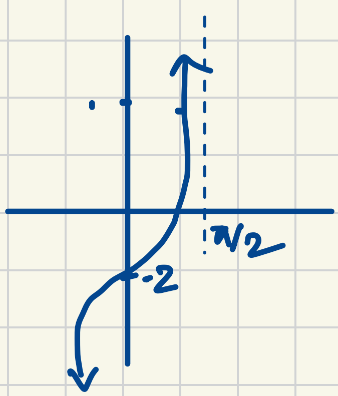


Since, $f(a)$, & $f(b)$ are on opposite ends of the origin on y-axis we can guarantee that a root/solution exists.



a & b are on closed interval on a continuous function, therefore, there must exist an x where cuts through the x axis. hence giving a solution.

Show $\tan x = 2$ has a solution in $[0, \pi/2)$



here, they have two opposite sign y values b/w $[0, \pi/2]$, the root exists

Show $f(x) = x - \cos x$ has one zero.

when $f(0) = 0 - \cos(0)$
$= 0 - 1$
$f(0) = -1$

$f(1) = 1 - \cos(1)$
$= 1 - 0.50.$
$= 0.5\ldots$

Here, since they have two opposite y values b/w $f(0)$ & $f(1)$ the root exists

8) Tangent line to $y = \frac{3}{x}$ @ $(3,1)$

$f'(3+h) = \frac{3}{3+h}$

$f(3) = 1$

here,

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\lim_{h \to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h}$

$\lim_{h \to 0} \frac{\frac{\cancel{3x} - \cancel{3x} - 3h}{x(x+h)}}{h}$

$\lim_{h \to 0} \frac{-3\cancel{h}}{x(x+h)\cancel{h}}$

$\lim_{h \to 0} -\frac{3}{(x^2+xh)}$

$\frac{dy}{dx} = -\frac{3}{x^2}$

$\therefore m_{tan}$ at $(3,1) = -1/3$

$\therefore$ Eq of the tangent line =

$y - 1 = -1/3 (x - 3)$

$3y - 3 = -x + 3$

$3y = -x + 6$

Eqn of the tangent line $\therefore y = -x/3 + 2$

Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ * Slope of a curve at a point

This is an Derivative

Ex Find the Derivative of $f(x) = 2x^2 - 3$
Then find the Eqn of Tangent line at $f(x)$ $(2, 5)$

Here,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 - 3 - (2x^2 - 3)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) - \cancel{3} - 2x^2 + \cancel{3}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{h}(4x + 2h)}{\cancel{h}}$$

$$f'(x) = 4x$$

$\therefore$ $m_{tangent} = f'(2) = 8$

Eqn of tangent line at $(2, 5)$

$$y - 5 = 8(x - 2)$$
$$y - 5 = 8x - 16$$
$$\boxed{y = 8x + 11}$$ ← Eqn of tangent line at $(2, 5)$

Find

$f'(x)$ for $f(x) = 2x^3 - x$

Here,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^3 - (x+h) - (2x^3 - x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2(x^3 + 3x^2 h + 3xh^2 + h^3) - 2x^3 + x - x - h}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{2x^3} + 6x^2 h + 6xh^2 + 2h^3 - \cancel{2x^3} + x - x - h}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{6x^2 h + 6xh^2 + 2h^3 + x - x - h}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{h}(6x^2 + 6xh + 2h^2 + \cancel{x} - \cancel{x} - 1)}{\cancel{h}}$$

$$\therefore \boxed{f'(x) = 6x^2 - 1}$$

Ex $f(x) = \sqrt{x}$, find $f'(x)$ & EQ. of tangent line at $x = 4$

Here,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^{1/2} - (x)^{1/2}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$\therefore f'(x) = \frac{1}{2\sqrt{x}}$$

$\therefore$ at $x = 4$

$$y - 2 = \tfrac{1}{4}(x - 4)$$

$$y - 2 = \tfrac{x}{4} - 1$$

$$\boxed{y = \tfrac{x}{4} + 1}$$

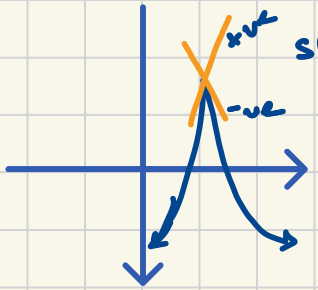
Differentiability

In order for a Derivative to exist at a point

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This must exist.

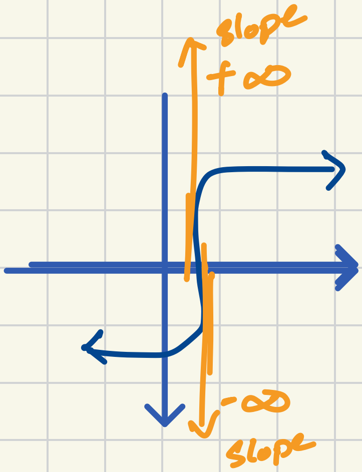
Implications



$\text{slope}_{\text{left}} \neq \text{Slope}_{\text{Right}}$

Cannot take Derivate at a Sharp point.

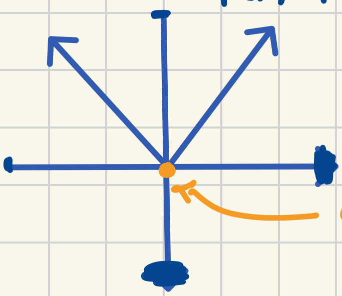
because, the slope from the right does not equal to the slope coming from the right.



OR.. when slope is vertical

Ex Is $|x|$ Differentiable everywhere?

$f(x) = |x|$



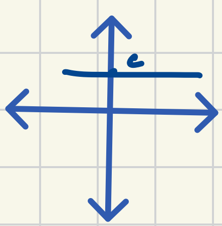
Cannot be differentiated at this point.

Note: If A Function is NOT continuous, it is not continuous

* If A function is Differentiable, it is automatically continuous.

techniques of differentiation

Slope of a constant



$x = 0$

$$\frac{d}{dx}[c] = 0$$

Similarly,

In General:

$$\frac{d}{dx}[x^n] = n x^{(n-1)}$$

$$* \quad \boxed{\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]}$$

$$* \quad \boxed{\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]}$$

<u>Ex</u>

$$\frac{d}{dx}\left[3x^9 - x^{-3}\right] = \frac{d}{dx}\left[3x^9\right] - \frac{d}{dx}\left[x^{-3}\right]$$

$$\therefore \quad = 27x^8 + 3x^{-4}$$

<u>Ex</u> At what points does $y = x^3 - 3x + 4$ have horizontal tangent lines.

we know that,
horizontal tangent line has slope $= 0$.

Here,

$f'(x) = 3x^2 - 3$
here, to find x-values where you have a horizontal slope.

$$0 = 3x^2 - 3$$

$$\frac{3}{3} = x^2$$

$$x^2 = 1$$

$x = \pm\sqrt{1}$ or, at $x=1$ & $x=-1$

$\therefore \ (1, 2) \ (-1, 6)$

$$\frac{dy}{dx}\left[\frac{x^5-2x-3}{3\sqrt{x}}\right]$$

Simplyfing:

$$\frac{x^5}{3x^{1/2}} - \frac{2x}{3x^{1/2}} - \frac{3}{3x^{1/2}}$$

$$\frac{x^{5-1/2}}{3} - \frac{2x^{1-1/2}}{3} - x^{-1/2}$$

$$\frac{x^{9/2}}{3} - \frac{2x^{1/2}}{3} - x^{-1/2}$$

Taking the Derivative:

$$\frac{3\cancel{9}}{2}\cdot\frac{x^{7/2}}{\cancel{3}} - \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}x^{-1/2} + \frac{1}{2}x^{-3/2}$$

$$\boxed{\therefore \frac{dy}{dx} \quad \frac{3}{2}x^{7/2} - \frac{1}{3}x^{-1/2} + \frac{1}{2}x^{-3/2}}$$

Product & Quotient Rules

Ex $f(x) = x^2$ $\qquad g(x) = x^3$

$$\frac{d}{dx}\left[x^2 \cdot x^3\right]$$

Here,

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] \neq \frac{d}{dx}\left[f(x)\right] \cdot \frac{d}{dx}\left[g(x)\right]$$

Correct way: Distribution

$$\frac{d}{dx}\left[x^2 \cdot x^3\right] = \frac{d}{dx}\left[x^5\right] = 5x^4$$

OR USE

Product Rule

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$\frac{d}{dx}\left[x^2 \cdot x^3\right] = \frac{d}{dx}\left[x^2\right] \cdot x^3 + \frac{d}{dx}\left[x^3\right] \cdot x^2$$

$$\frac{dy}{dx} = 2x \cdot x^3 + 3x^2 \cdot x^2$$

$$= 2x^4 + 3x^4$$

$$\boxed{\frac{dy}{dx} = 5x^4}$$

Q) $y = (x^2-1)(3x^4+2x)$

$$\frac{dy}{dx} = 2x(3x^4+2x) + (x^2-1)(12x^3+2)$$

$$= 6x^5 + 4x^2 + 12x^5 + 2x^2 - 12x^3 - 2$$

$$= 18x^5 + 6x^2 - 12x^3 - 2$$

$$\boxed{\therefore f'(x) = 18x^5 - 12x^3 + 6x^2 - 2}$$

Q) $f(x) = (1+x^2).\sqrt{x}$

$$\frac{dy}{dx} = (1+x^2).\tfrac{1}{2}x^{-1/2} + x^{1/2}(2x)$$

$$= \frac{1}{2\sqrt{x}} + \frac{x^2}{2\sqrt{x}} + 2x\sqrt{x}$$

$$\boxed{\therefore f'(x) = \frac{x^2+1}{2\sqrt{x}} + 2x\sqrt{x}}$$

Q) $g(x) = (x^2+1)f(x)$

$f(2) = 3$
$f'(2) - 1$

Find $g'(2) = ?$

$$g'(x) = (x^2+1).f'(x) + 2x.f(x)$$

when $x = 2$

$$g'(2) = (2^2+1).f'(2) + 2(2).f(2)$$

$$g'(2) = 5.(-1) + 4.(3)$$

$$\boxed{\therefore g'(2) = 7}$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - g'(x)\cdot f'(x)}{[g(x)]^2}$$

OR

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot \frac{d}{dx}[f(x)] - f(x)\cdot \frac{d}{d}[g(x)]}{[g(x)]^2}$$

$$y = \frac{x^3 - 3x^2 - 5}{2x+5}$$

$$\frac{dy}{dx} = \frac{(2x+5)\cdot(3x^2-6x) - (2)\cdot(x^3-3x^2-5)}{(2x+5)^2}$$

$$\frac{dy}{dx} = \frac{6x^3 - 12x^2 + 15x^2 - 30x - (2x^3 - 6x^2 - 10)}{(2x+5)^2}$$

$$\frac{dy}{dx} = \frac{6x^3 - 12x^2 + 15x^2 - 30x - 2x^3 + 6x^2 + 10}{(2x+5)^2}$$

$$\boxed{y' = \frac{4x^3 + 9x^2 - 30x + 10}{(2x+5)^2}}$$

$$f(x) = \frac{(3x-1)(x^2-4)}{x^2+2}$$

$$\frac{dy}{dx} = \frac{(x^2+2)\frac{d}{dx}\left[(3x-1)(x^2-4)\right] - \left((3x-1)(x^2-4)\right) \cdot \frac{d}{dx}\left[x^2+2\right])}{(x^2+2)^2}$$

$$= \frac{x^2+2\left[(3x-1)(2x)+(3)(x^2-4))\right] - \left[(3x-1)(x^2-4) \quad 2x\right]}{(x^2+2)^2}$$

$$= \frac{x^2+2\left[6x - 2x + 3x^2 - 12\right] - 2x\left[3x^3 - 12x - x^2 + 4\right]}{(x^2+2)^2}$$

$$= \frac{x^2+2\left[8x^2+4x-12\right] - \left[6x^4 - 24x^2 - 2x^3 + 8x\right]}{(x^2+2)^2}$$

$$= \frac{3x^4 + 4x^3 - 12x^2 + 6x^2 + 8x - 24 - 6x^4 + 24x^2 + 2x^3 - 8x}{(x^2+2)^2}$$

$$= \frac{3x^4 - 6x^4 + 4x^3 + 2x^3 - 12x^2 + 6x^2 + 24x^2 + \cancel{8x} - \cancel{8x} - 24}{(x^2+2)^2}$$

$$\boxed{\therefore f'(x) = \frac{-3x^4 + 6x^3 + 18x^4 - 24}{(x^2+2)^2}}$$

Applications

DVD: $S(T) = \frac{7T}{T^2+1}$, T = Years , $T \geq 0$

① Rate of change for SALES:

$$S(T) = \frac{(T^2+1)\cdot\frac{d}{dt}[7T] - \left(7T\cdot\frac{d}{dt}[T^2+1]\right)}{(T^2+1)^2}$$

$$\boxed{S'(T) = \frac{7-7t^2}{(T^2+1)^2}}$$

② When will Sales Peak?

at Peak $S'(T) = 0$

Here,

$$\frac{7-7t^2}{(T^2+1)^2} = 0$$

$7 - 7t^2 = 0$

$7 = 7t^2$

$t^2 = 1$

$\boxed{t = 1}$

∴ Sale will peak at the 1st year.

③ How FAST will Sales Increase when Movie is Realeased?

↳ This happens when T=0

$$S'(0) = \frac{7-0^2}{(0^2+1)^2}$$

$\boxed{S'(0) = 7 \text{ million DVD/per Year}}$

2.7 Implicit Differentiation

$\boxed{\text{Ex: } y + xy = x}$ → Implicit

Sometimes, we can turn implicit to explicit

For eg:

$y + xy = x$

$\boxed{y = \frac{x}{1+x}}$ → Explicit Form

* Implicit Function can define more than 1 function of x.

Sometimes, you can't turn Implicit into Explicit.

Ex: $2x^3 + 3y^3 = 9xy$

Here,

you must treat y as a Function of x.

Derivative In Respect to x.

$$\frac{d}{dx}\left[2x^3 + 3y^3\right] = \frac{d}{dx}\left[9xy\right]$$

$$6x^2 + 9y^2 \cdot \frac{d}{dx}[y] = 9\left[x\frac{dy}{dx} + y\right]$$

Ex: $3y^2 + \sin(y) = 4x^5$

$$\frac{d}{dx}\left[3y^2 + \sin(y)\right] = \frac{d}{dx}\left[4x^5\right]$$

$$6y \cdot \frac{dy}{dx} + \cos(y) \cdot \frac{dy}{dx} = 20x^4$$

$$\frac{dy}{dx}\left(6y + \cos y\right) = 20x^4$$

$$\boxed{\frac{dy}{dx} = \frac{20x^4}{6y + \cos y}}$$

Ex: $xy = 1$

$$\frac{d}{dx}[x \cdot y] = \frac{d}{dx}[1]$$

$$x \cdot \frac{dy}{dx} + 1 \cdot y = 0$$

$$\frac{dy}{dx}x + y = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-y}{x}}$$

Ex: $3x^2 - y^2 = 16$ Find $\frac{d^2y}{dx^2}$

$$= \frac{d}{dx}[3x^2 - y^2] = \frac{d}{dx}[16]$$

$$= 6x - 2y \cdot \frac{dy}{dx} = 0$$

$$\frac{\cancel{6}^{3}x}{\cancel{2}y} = \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{3x}{y}$$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[3x \cdot y^{-1}\right]$$

$$\frac{d^2y}{dx^2} = 3x \cdot -1y^{-2} \cdot \frac{dy}{dx} + 3 \cdot y^{-1}$$

$$\frac{d^2y}{dx^2} = -\frac{3x}{y^2} \cdot \frac{dy}{dx} + \frac{3}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{3x}{y^2} \cdot \frac{3x}{y} + \frac{3}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{9x^2}{y^3} + \frac{3}{y}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{3y^2 - 9x^2}{y^3}}$$

$y^2 - x + 1 = 0$, Find Slopes at $(2,-1)$ & $(2,1)$

$$\frac{d}{dx}\left[y^2 - x + 1\right] = \frac{d}{dx}[0]$$

$$2y\frac{dy}{dx} - 1 = 0$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

↑ general Slope formula.

Here,

at $(2,-1) \to \frac{1}{2(-1)} = -1/2$

at $(2,1) \to \frac{1}{2(1)} = 1/2$

Two Different Slopes at one point

Ex: $4x^4 + 8x^2y^2 - 25x^2y + 4y^4 = 0$

Find EQ of the tangent line at $(2,1)$

$$\frac{d}{dx}\left[4x^4 + 8x^2y^2 - 25x^2y + 4y^4\right] = \frac{d}{dx}[0]$$

$$16x^3 + 16x^2y\frac{dy}{dx} + 16xy^2 - 25\left(2x \cdot y + x^2 \cdot \frac{dy}{dx}\right) + 16y^3 \cdot \frac{dy}{dx} = 0$$

$$16x^3 + 16x^2y\frac{dy}{dx} + 16xy^2 - 50xy - 25x^2\frac{dy}{dx} + 16y^3 \cdot \frac{dy}{dx} = 0$$

$$16x^3 - 50xy + 16xy^2 + 16x^2y\frac{dy}{dx} - 25x^2\frac{dy}{dx} + 16y^3 \cdot \frac{dy}{dx} = 0$$

$$16x^3 - 50xy + 16xy^2 + \frac{dy}{dx}\left(16x^2y - 25x^2 + 16y^3\right) = 0$$

$$\boxed{\frac{dy}{dx} = \frac{50xy - 16x^3 - 16xy^2}{16x^2y - 25x^2 + 16y^3}}$$

Derivatives of Exponential & logarithmic Functions.

* $\frac{d}{dx}\left[e^{2x}\right] = e^{2x}(2)$
$= 2e^{2x}$

Here,

$\frac{d}{dx}\left[e^{u}\right] = e^{u} \cdot u' \cdot \underbrace{\ln(e)}_{=1}$

$\therefore \frac{d}{dx}\left[e^{u}\right] = u' \cdot e^{u}$

By the same logic

$\frac{d}{dx}\left[e^{x}\right] = e^{x} \cdot (1) \cdot \ln(e)$

$$\boxed{\frac{d}{dx}\left[e^{x}\right] = e^{x}}$$

However,

$$\boxed{\frac{d}{dx}\left[x^{e}\right] = ex^{(e-1)}}$$

likewise,

$\frac{d}{dx}\left[e^{x^2}\right] = e^{x^2}(2x) = 2xe^{x^2}$

Similarly,

constant

$\frac{d}{dx}\left[e^{2}\right] = 0$

* $\frac{d}{dx}\left[e^{\sin x}\right] = e^{\sin(x)} \cdot \cos(x) \cdot (1)$ Derivative

$\therefore = \cos(x) \cdot e^{\sin(x)}$

* $\frac{d}{dx}\left[e^{\cos(2x)}\right] = e^{\cos(2x)} \cdot -\sin(2x)(2)$

$\therefore = -2\sin(2x)\cos(2x)$

likewise,

Derivative of an exp Function

$\frac{d}{dx}\left[a^{u}\right] = a^{u} \cdot u' \cdot \ln(a)$

* $\frac{d}{dx}\left[2^{x}\right] = 2^{x} \cdot (1) \cdot \ln(2)$

$\therefore = 2^{x}\ln(2)$

* $\frac{d}{dx}\left[4^{x^2}\right] = 4^{x^2} \cdot (2x) \cdot (1) \cdot \ln(4)$

$\therefore 2x \cdot 4^{x^2} \cdot \ln(4)$

* $\frac{d}{dx}\left[7^{4x-x^2}\right] = 7^{4x-x^2} \cdot (4 - 2x(1)) \cdot \ln(7)$

$\therefore = 7^{4x-x^2}(4-2x) \cdot \ln(7)$

Natural logs (Derivative)

Note:

$$\frac{d}{dx}\left[\ln(u)\right] = \frac{u'}{u}$$ → Rule for deriving natural log.

* $\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$

* $\frac{d}{dx}\left[\ln(x^2)\right] = \frac{2\cancel{x}}{x^{\cancel{2}}} = \frac{2}{x}$

* $\frac{d}{dx}\left[\ln(2x)\right] = \frac{\cancel{2}}{\cancel{2}x} = \frac{1}{x}$

Note: any real number

$$\frac{d}{dx}\left[\ln(ax)\right] = \frac{1}{x}$$

* $\frac{d}{dx}\left[x^3\right] = \frac{3\cancel{x^2}}{x^{\cancel{3}}} = \frac{3}{x}$

* $\frac{d}{dx}\left[5x^4\right] = \frac{\overset{4}{\cancel{20}}\cancel{x^3}}{\cancel{5}x^{\cancel{4}}} = \frac{4}{x}$

Here,

Note:

$$\frac{d}{dx}\left[x^a\right] = \frac{a}{x}$$

* $\frac{d}{dx}\left[\ln(x+1)\right] = \frac{1}{x+1}$

* $\frac{d}{dx}\left[\ln(x^2+1)\right] = \frac{2x}{x^2+1}$

* $\frac{d}{dx}\left[\ln(\sin x)\right] = \frac{\cos x}{\sin x} = \cot x$

* $\frac{d}{dx}\left[\ln(\cos x)\right] = \frac{-\sin(x)}{\cos x} = -\tan(x)$

Deriving Log Functions

Note!

$\frac{d}{dx}\left[\log_a[u]\right] = \frac{u'}{u \cdot \ln(a)}$ ← Rule for deriving log functions

* $\frac{d}{dx}\left[\log_3[2x]\right] = \frac{\cancel{2}}{\cancel{2}x \cdot \ln(3)} = \frac{1}{x \cdot \ln(3)}$

* $\frac{d}{dx}\left[\log_5(x^2)\right] = \frac{2x}{x^2 \cdot \ln(5)}$

* $\frac{d}{dx}\left[\log_7(1-x)\right] = \frac{-1}{(1-x)\log_e(7)}$

* $\frac{d}{dx}\left[x \cdot \ln(x)\right] = \cancel{x} \cdot \frac{1}{\cancel{x}} + 1 \cdot \ln(x)$

$\therefore = 1 + \ln(x)$

* $\frac{d}{dx}\left[\ln[\ln(\ln(x))]\right] = \frac{\frac{1}{x \ln(x)}}{\ln(\ln(x))}$

$= \frac{1}{x \ln(x) \ln[\ln(x)]}$

* $\frac{d}{dx}\left[\ln(\ln(x))\right] = \dfrac{\frac{1}{x}}{\frac{\ln(x)}{1}} \cdot (1)$

$\therefore \quad = \frac{1}{x\ln(x)}$

Inverse Trig Derivative

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\arccos u = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\arctan u = \frac{u'}{1+u^2} \qquad \frac{d}{dx}\operatorname{arccot} u = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}\operatorname{arcsec} u = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}\operatorname{arccsc} u = \frac{-u'}{|u|\sqrt{u^2-1}}$$

① $y = \frac{1}{\tan^{-1}(x)}$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\tan(x)\right]$

$\frac{dy}{dx} = \frac{1}{1+x^2} \cdot (1)$

Smooth Function:

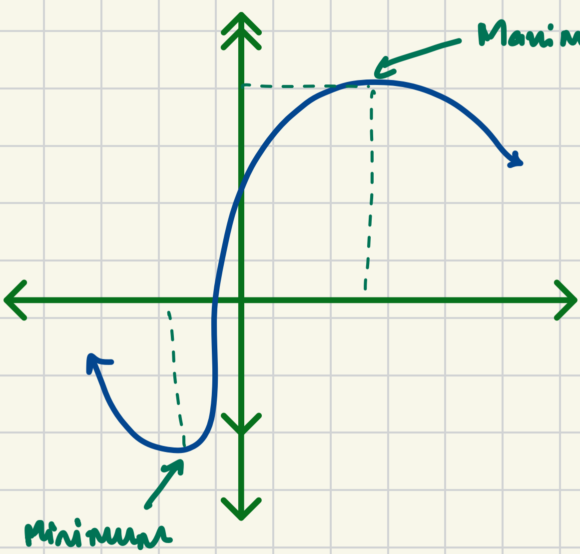
Functions we can derive forever, are called smooth functions

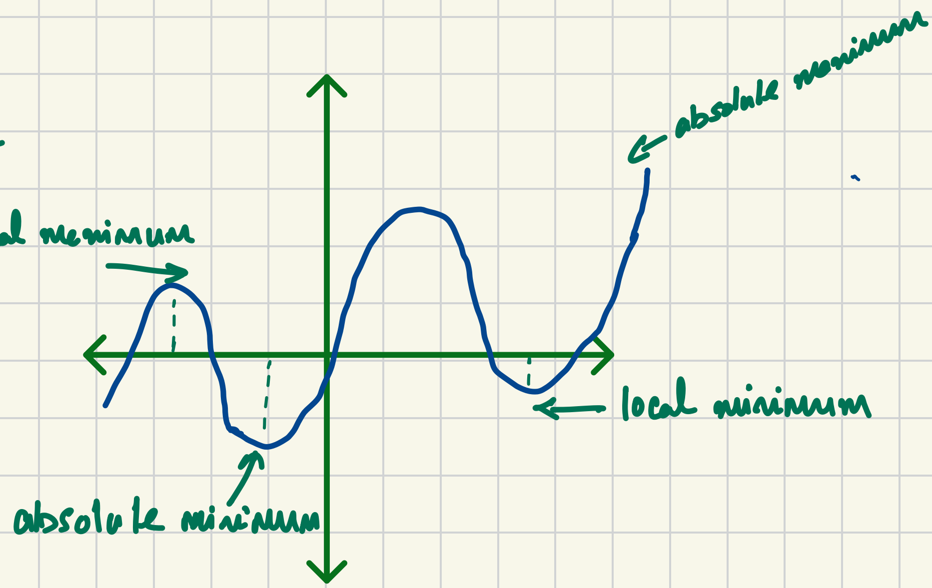
① $\frac{1}{x}$ ② $\sin(x)$, $\cos(x)$

③ polynomials ④ e^x

⑤ $\ln(x)$

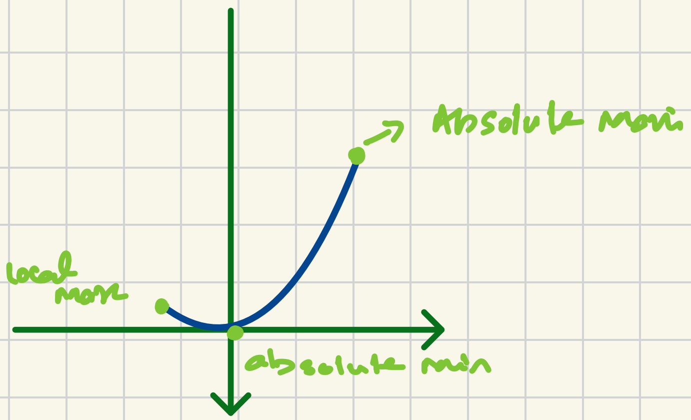
Maximum & Minimum Value



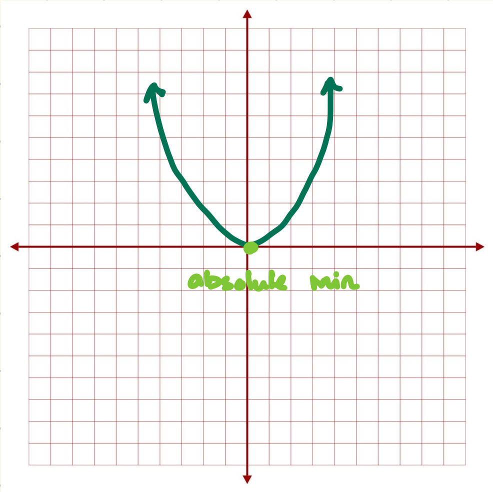


- There can also be an Absolute Maximum (highest extreme value in the function) & Absolute Minimum (lowest extreme value in the function)

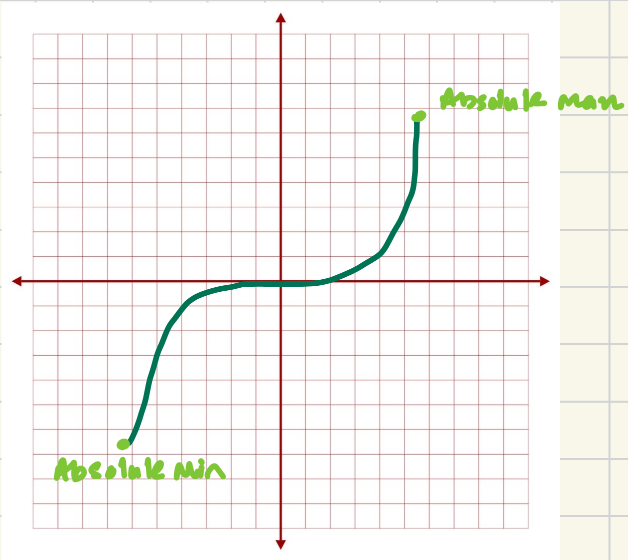
- There can also be local maximum & local minimum value in the function (These points are not the highest or lowest)



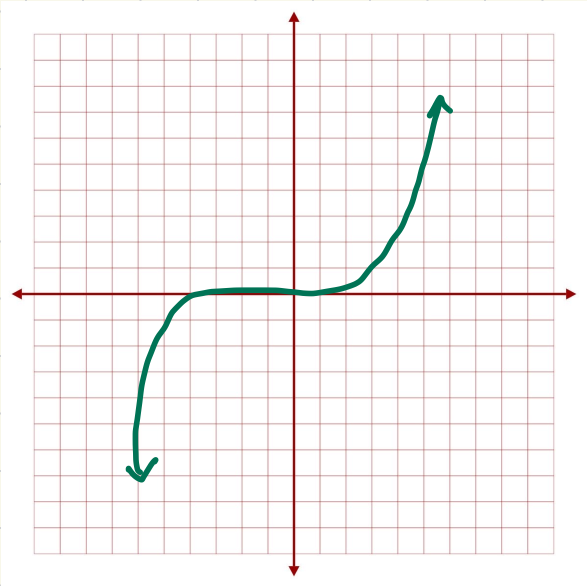
closed function



open function



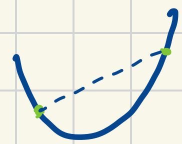
closed Interval function.

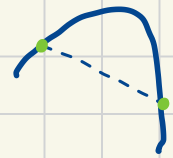


No minimum/maximum

Second Derivative for Maximum | Minimum

Here,

$f''(x) > 0$, Concave up 

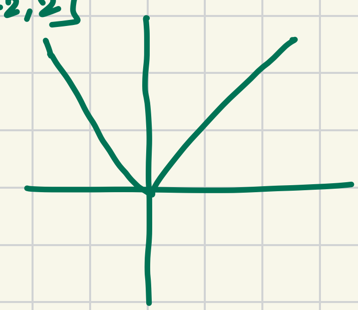
$f''(x) < 0$, Concave down 

Here,

- if $f'(x_0) = 0$ & $f''(x_0) > 0$, the function has local minimum.
- if $f'(x_0) = 0$ & $f''(x_0) < 0$, the function has local maximum.

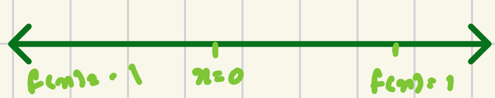
Critical points for absolute values on closed function

$f(x) = |x|$ on $[-2,2]$



critical point at 0, as at $x=0$, there is a global minimum.

First Derivative Test



By the EVT, we have to check $x=-2$, $x=2$

x	$f(x) = \|x\|$
2	$\|2\| = 2$
0	$\|0\| = 0$
-2	$\|-2\| = 2$

Conclusion : $f(x)$ has a global max of 2 at $x=2$ & $x=-2$. Also it has a global minimum of zero at $x=0$

Finding Critical values on an Open function

$f(x) = -(x+4)^3$, $f'(x) = -3(x+4)^3$

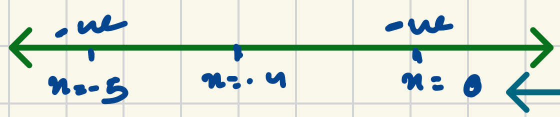
$0 = -3(x+4)^2$

$x = -4$

Hence,

Using Sign Analysis Test

Plug the x values for sign analysis to the derived function.

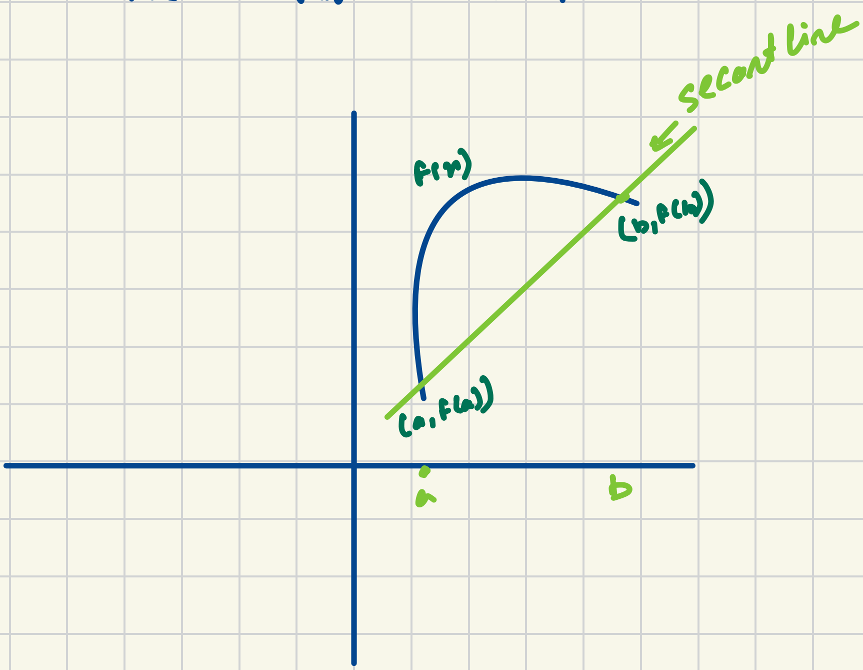


Conclusion, By the First Derivative Test, since $f'(x)$ did not change sign around $x=-4$, this critical point is not an extreme point

ROLL'E'S THEOREM

if f(x) is differentiable on (a, b),
$f'(x) = 0$ for some "c" on (a, b)

Mean - Value THEOREM



If f(x) is Differentiable on (a, b),

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

For some "c" on (a, b)

3.3 First Derivative Test

Recap

$f'(x) > 0 \Rightarrow$ Increasing slope

$f'(x) < 0 \Rightarrow$ Decreasing slope

$f'(x) = 0 \Rightarrow$ CONSTANT / critical Number

How to find Relative Extrema:

Derivative Test

(i) take first Derivative

(ii) Set it equal to 0 for Critical #'s.

(iii) $f'(x) \Rightarrow$ C.N | C.N | C.N |

(iv)

$f'(x) \Rightarrow$ C.N | C.N |

- Find the sign of each interval by PLUGGING into the value into $f'(x)$

(This gives you Increasing / Decreasing)

Ex $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$

$3x^2 - 3 = 0$

$x = 1, -1$

		R.Max		R.Min	
$f'(x)$		$x=-1$		$x=1$	
	+ve	|	-ve	|	+ve
x values:	-2		0		2

R. Max $(-1, 3)$

R Min $(1, -1)$

$f(x) = 8x^{5/3} - 15x^{2/3}$

$f'(x) = 5x^{2/3} - 10x^{-1/3} = 0$

$5x^{-1/3}(x - 2) = 0$

$\frac{5(x-2)}{\sqrt[3]{x}} = 0$ $\quad x - 2 = 0, \quad \sqrt[3]{x} = 0$

$x = 2 \qquad x = 0$

$f'(-1)$ $\quad x=0$ $\quad f'(1)$ $\quad x=2$ $\quad f'(3)$

$f'(x) \rightarrow$ +ve | -ve | +ve

Second Derivative Test

Recall: $f''(x) > 0 \Rightarrow$ Concave up

: $f''(x) < 0 \Rightarrow$ Concave Down

: $f''(x) = 0 \Rightarrow$ Possible Inflection Point (P.I.P)

How to find Inflection Point

2nd Derivative Test

- Find 2nd Derivative
- set $= 0$ for P.I.P
- Make Table

Find sign of each interval by using $f''(x)$

$f''(x)$ —|—|—|—

PIP PIP PIP

$f(x) = x^4 - 4x^3 + 12$

Find Inflection Points

$$\frac{dy}{dx}\left[x^4 - 4x^3 + 12\right]$$

$$\frac{dy}{dx} = 4x^3 - 12x^2$$

$$\frac{dy}{dx}\left[\frac{dy}{dx}\right] = 12x^2 - 24x$$

$$\frac{d^2y}{dx^2} = 12x^2 - 24x = 0$$

$\hookrightarrow$ $12x(x-2) = 0$

$x = 0$, $x = 2$

	+ve		-ve		+ve
$f''(x) \Rightarrow$	$f''(-1)$	0	$f''(1)$	2	$f''(3)$
	C. UP		C Down		C. UP

If the concave changes between the P.I.P, that is an Inflection Point.

Here, the inflection point is at:

$(0, 12)$ $(2, -4)$

Ex find Inflection points:

$f(x) = (x-1)^{1/3}$

$$f'(x) = \frac{1}{3}(x-1)^{-2/3}$$

$$f''(x) = \frac{-2}{9}(x-1)^{-5/3}$$

solving for 0, to get P.I.P

$$0 = -2/9\,(x-1)^{-5/3}$$

$0 = -2/9\,(x-1)^{-5/3}$

$-(x-1)^{-5/3} = 0$

$-\dfrac{1}{\sqrt[3]{(x-1)^5}} = 0$

How, when at $x=1$, an asymptote occurs/undefined derivative

+ve	-ve
$f''(0)$	$f''(2)$

$x = 1$

Concave UP | Concave Down

Since the concave changes, there is an inflection point at $x=1$

$g(1) = (1-1)^{1/3} = 0$

$\therefore = (1,0)$

---

6) $f(x) = x^3 - 3x^2 - 24x + 32$

First Derivative Test

$f'(x) = 3x^2 - 6x - 24$

$0 = 3x^2 - 6x - 24$

$3(x^2 - 2x - 8) = 0$

$3(x^2 - 4x + 2x - 8)$

$3(x(x+2) - 4(x+2))$

$3((x-4)(x+2))$

Rel. Max → $x=-2$

$x=4$ ← Relative Min

$f'(x) \to$ INC + | $x=-2$ | DEC -ve | $x=4$ | INC +ve

Second Derivative Test

$f''(x)\quad 6x - 6 = 0$

$6(x-1) = 0$

$x = 1 = \text{P.I.P}$

$f''(0)$ -ve | $x=1$ | $f''(2)$ +ve

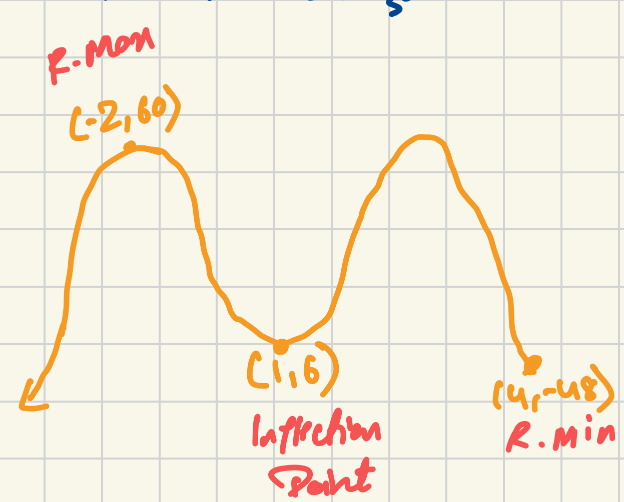
C.D | | C.U

R.Max = $(-2, 60)$

R.Min = $(4, -48)$

I.P = $(1, 6)$

APPROX Visualization



Q) $h(x) = 2+3x-x^3$

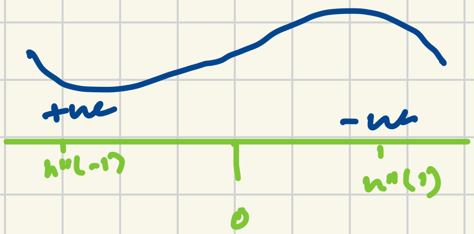
$h'(x) = 3-3x^2$

$h''(x) = -6x$

Inflection point when $h''(x)=0$

$-6x=0$, $x=0$

$\Rightarrow$ Now, we check the sign around $x=0$



Therefore the I.P = $(0,2)$

Now,

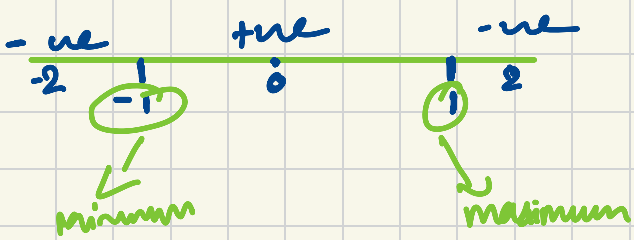
to find minimum/maximum

$h'(x)=0$

$0 = 3-3x^2$

$3x^2 = 3$

$x = \pm 1$



$h(-1) = 2-3+1 = 0$ $\therefore$ MINIMUM at $(-1,0)$

$h(1) = 2+3-1 = 4$ $\therefore$ Maximum at $(1,4)$

Ex $f(x) = e^x$

To prove that the e^x is always concave up with no max or min

$f'(x) = e^x \Rightarrow 0 \Rightarrow$ no x exists

$f''(x) = e^x = 0 \Rightarrow$ no x exists

Thus no critical points or inflection points to check concavity

To check concavity

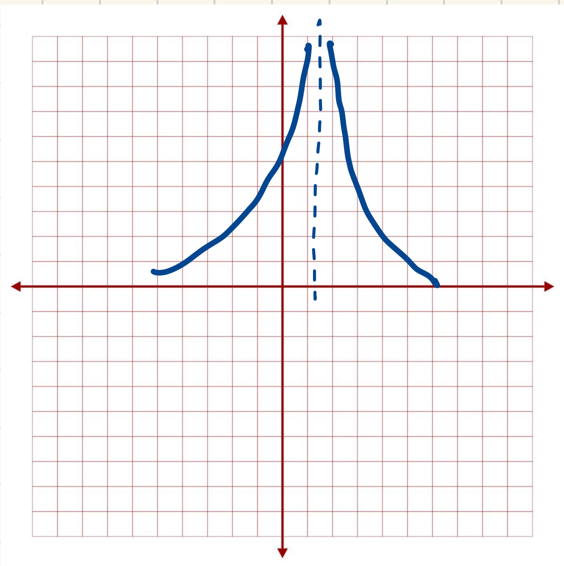
$f''(e^x)$ is always positive for all values of x.

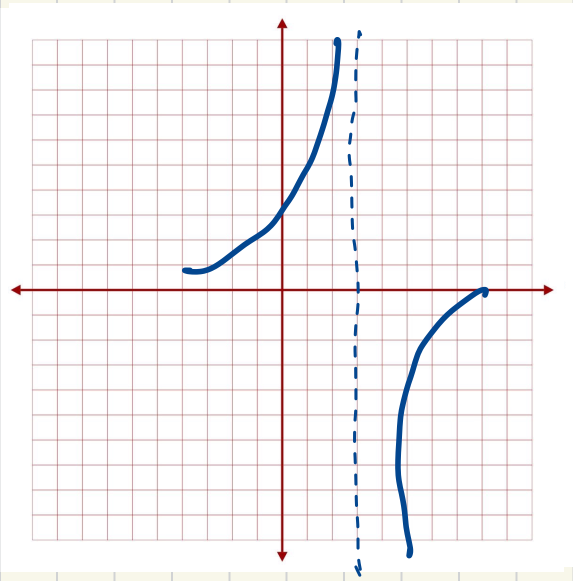
Thus concave up.

Limits @ Infinity

If $\lim_{x \to a} f(x) = \pm\infty$

Gives an horizontal asymptote





$$f(x) = \frac{x}{(x+3)(x-1)}$$

$\therefore$ Discontinuity at $x = -3$ and $x = 1$

as $\frac{\text{something}}{0}$ = Asymptote/Discontinuity

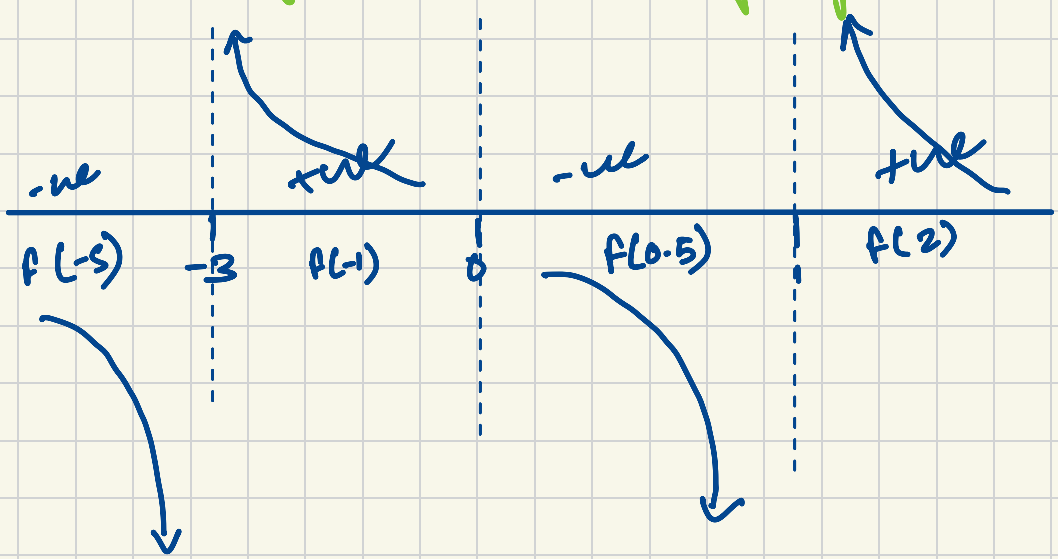
Discontinuity when Denominator $= 0$

CASE 1: "Hole" → A Removable Discontinuity
when NUMERATOR $= 0$ AND DENOMINATOR $= 0$

CASE 2: "VERTICAL ASYMPTOTE" when you cannot "cancel-out"
The Discontinuity

Sign Analysis Test

(to find out what kind of Asymptote it is)



Ex

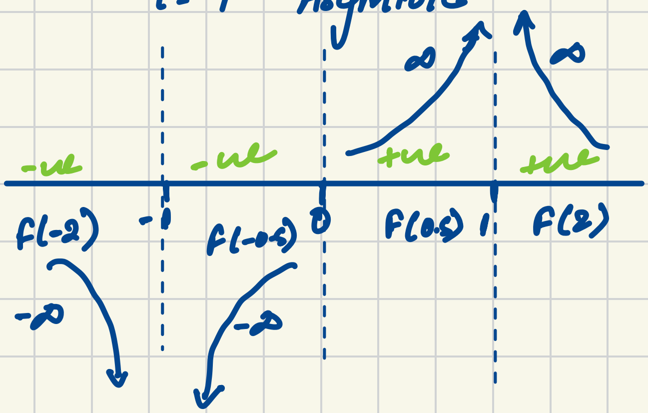
$$\frac{T^3}{(T^2-1)^2}$$

Discontinuity :

$T^2 - 1 = 0$

$T = +1$ Asymptote

$T = -1$ Asymptote



$$\lim_{T\to 1} \frac{T^3}{(T^2-1)^2} = +\infty$$

$$\lim_{T\to -1} \frac{T^3}{(T^2-1)^2} = -\infty$$

WHAT happens as $x \to +\infty$ OR $x \to -\infty$

$$\lim_{x\to +\infty} \frac{1}{x} = 0 \qquad \lim_{x\to -\infty} \frac{1}{x} = 0$$

x	1	10	100	10000
$f(x)$	1	0.1	0.01	0.0001

* If f(x) gets really close to a certain number as $x \to \pm\infty$, the limit exist.

* These are horizontal asymptotes

Ex: $\lim_{x\to\pm\infty} \frac{1}{x^n} \to \left[\lim_{x\to\pm\infty} \frac{1}{x}\right]^n = 0^n = 0$

horizontal asymptote at 0

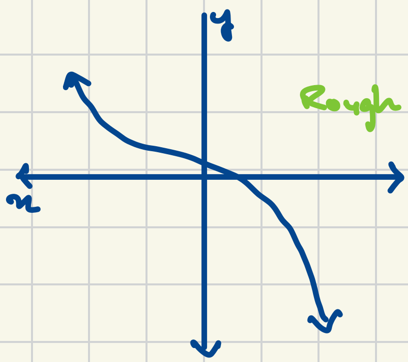
Note: The limit of A Polynomial As $x \to \pm\infty$ goes to $\pm\infty$

Ex $\lim_{x\to+\infty} x^3 = +\infty$ $\quad\quad$ $\lim_{x\to-\infty} x^3 = -\infty$

$\lim_{x\to+\infty} x^2 = \infty$ $\quad\quad$ $\lim_{x\to-\infty} x^2 = \infty$

Also, limits of a Polynomial will follow THE BEHAVIOUR of the Term w/ the highest Power

Ex: $\lim_{x\to-\infty} -8x^3 - 2x^2 - x + 9 = \lim_{x\to-\infty} = +\infty$



Rough sketch of Ans

Ex $\lim_{x \to +\infty} \frac{5x-2}{3x+9} \to \lim_{x \to \infty} \frac{\frac{5\cancel{x}}{\cancel{x}} - \frac{2}{x}}{\frac{3\cancel{x}}{\cancel{x}} + \frac{9}{x}}$

$$\lim_{x \to \infty} \frac{5 - \frac{2}{x}}{3 + \frac{9}{x}} = \frac{5-0}{3+0} = \frac{5}{3}$$

Quick cheat:

Numerator > Denominator = ∞

Denominator > Numerator = 0

Numerator = Denominator = $\frac{\text{coeff of Num}}{\text{coeff of Den}}$

Divide every term by the largest power of x in the Denominator

Ex $\lim_{x \to -\infty} \frac{5x^2 \cdot 4x}{15x^3 \cdot 3} \to \lim_{x \to -\infty} \frac{\frac{5x^2}{x^2} - \frac{4x}{x^2}}{\frac{15x^2}{x^2} - \frac{3}{x^3}}$

$$\to \lim_{x \to -\infty} \frac{\frac{5}{x} + \frac{4}{x^2}}{15 - \frac{3}{x^3}} = \frac{0+0}{15-0} = 0$$

highest power goes to $+\infty$

Ex $\lim_{x \to -\infty} \frac{7x^3 - 2x^2 + 1}{9 - 2x} \to \lim_{x \to -\infty} \frac{7x^2 - 2x + \frac{1}{x}}{\underbrace{\frac{9}{x} - 2}_{\to (0-2)}} = \frac{+\infty}{-2} = -\infty$

Ex $\lim_{x \to +\infty} \sqrt[3]{\frac{2x^2-3}{3x^2-5}} \to \sqrt[3]{\lim_{x \to +\infty} \frac{2x^2-3}{3x^2-5}}$

$$= \sqrt[3]{\frac{2}{3}}$$

Ex $\lim_{x\to+\infty} \frac{\sqrt{x^2-2}}{3x-6} \to \lim_{x\to\infty} \frac{\frac{\sqrt{x^2-2}}{|x|}}{3-\frac{6}{|x|}}$

$\to \lim_{x\to\infty} \frac{\sqrt{\frac{x^2}{x^2}-\frac{2}{x^2}}}{3-\frac{6}{|x|}} \to \lim_{x\to\infty} \frac{\sqrt{1-\frac{2}{x^2}}}{3-\frac{6}{|x|}}$

Hence, $= \frac{\sqrt{1-0}}{3-0}$

$$\boxed{\therefore \lim_{x\to\infty} \frac{\sqrt{x^2-2}}{3x-6} = \frac{1}{3}}$$

Ex $\lim_{x\to+\infty} \sqrt{x^4+2} - x^2$

$\lim_{x\to+\infty} \frac{\sqrt{x^4+2}-x^2}{1} \cdot \frac{\sqrt{x^4+2}+x^2}{\sqrt{x^4+2}+x^2}$

$\lim_{x\to+\infty} \frac{x^4+2-(x^2)^2}{\sqrt{x^4+2}+x^2}$

$\lim_{x\to+\infty} \frac{x^4+2-x^4}{\sqrt{x^4+2}+x^2} \to \lim_{x\to\infty} \frac{2}{\sqrt{x^4+2}+x^2}$

$\lim_{x\to+\infty} \frac{\frac{2}{x^2}}{\frac{\sqrt{x^4+2}}{x^2}+\frac{x^2}{x^2}} \to \frac{\frac{2}{x^2}}{\sqrt{\frac{x^4}{x^4}+\frac{2}{x^4}}+1} \to \frac{\frac{2}{x^2}}{\sqrt{1+\frac{2}{x^4}}+1}$

$= \frac{0}{\sqrt{1+0}+1}$

$= \frac{0}{2} = 0$

Curve Sketching

Ex $y = (x+2)(x-1)^2$

1) Find Intercepts

x: Set $=0$ & solve for 'x' (If Easy)

y: Plug-in 0 & solve for 'y'

$x=-2, \quad x=1$ ← x-intercepts

$y = (0+2)(0-1)^2$

$y = 2(-1)^2 = 2$ ← y-intercept

2) (only for Rational functions)

Find ALL the asymptotes

Vertical: Denominator $=0$
Non-Removable

Hole: Denominator $=0$
(Removable)

Horizontal: Limit as $\pm\infty$

3) 1^{st} Derivative Test:

- CRITICAL Number
- Increase/Decrease

4) 2^{nd} Derivative Test

- Concavity
- INFLECTION Pts

5. Make Table

6. Find ALL Possible **Points** From original functions

7. GRAPH

Continued working on the same problem

$y = (x+2)(x-1)^2 = 0$

Distribute

① $x^3 - 3x + 2$

x intercept = $x = -2$, $x = 1$

$y = 2$

③ $y' = 3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = \pm\sqrt{1}$

$\therefore x = 1, -1$ critical #s

④ $y'' = 6x = 0$

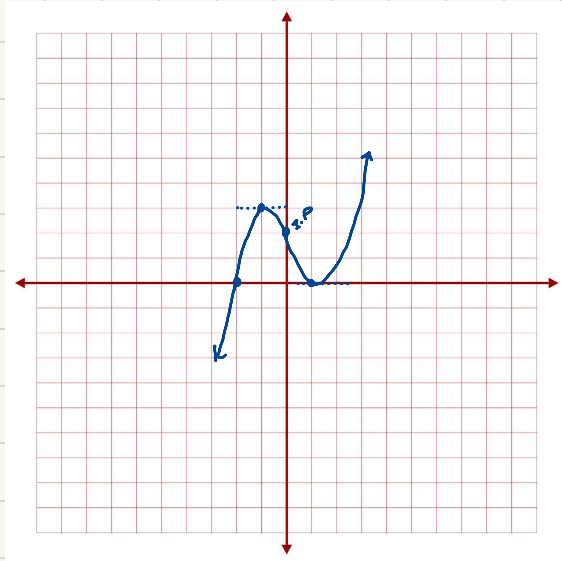
$x = 0$ P.I.P

⑤

	$x=-1$		$x=1$	
$f'(x)$	+ INC	− DEC		+ INC
$f''(x)$	− C. DOWN	$x=0$	+ C. UP	

Points (1,0) (-2,0) x INT
(0,2) y INT

R Max (-1,4) I.P: (0,2)
R Min (1,0)



Ex $y = \dfrac{x^2-1}{x^3}$

$\dfrac{3-1}{\cancel{(\sqrt{3})^3}}$ $\dfrac{2}{.3}$

① x-int $= 0 = x^2 - 1$

$x^2 = 1$

$x = 1, -1$

y-intercept = D.N.E

② Vertical Asymptote occurs where $x^3 = 0 \rightarrow x = 0$

Horizontal Asymptote

$\lim_{x \to \infty} \dfrac{x^2-1}{x^3}$

$\lim_{x \to -\infty} \dfrac{x^2-1}{x^3}$

$\rightarrow \dfrac{\frac{1}{x^3} \cdot \frac{1}{x^3}}{\frac{x^3}{x^3}} = \dfrac{0}{1} = 0$

③ $f'(x) = \dfrac{2x(x^3) - (x^2-1)(3x^2)}{x^6}$

$f'(x) = \dfrac{2x^4 - 3x^4 + 3x^2}{x^6}$

$f'(x) = \dfrac{3x^2 - x^4}{x^6}$

$f''(x) = \dfrac{(6x - 4x^3)(x^6) - (3x^2 - x^4)(6x^5)}{x^{12}}$

$f''(x) = \dfrac{6x^7 - 4x^9 - 18x^7 + 6x^9}{x^{12}}$

$f''(x) = \dfrac{2x^9 - 12x^7}{x^{12}}$

critical numbers

$f'(x) = \dfrac{3x^2 - x^4}{x^6} = 0 \quad = 3x^2 = x^4$

$x = \pm\sqrt{3}, 0$

④ $\dfrac{2x^9 - 12x^7}{x^{12}} = 0$

$x = 0$

$2x^2 = 12$

$x^2 = 6$

$x = \pm\sqrt{6}$

$\therefore$ PIP $= +\sqrt{6}, -\sqrt{6}, 0$

f'(-3) f(-1) 0 f(1) f'(2)

L.Min $-\sqrt{3}$ $\sqrt{3}$ L.Max

− DEC | + INC | + INC | − DEC

$-\sqrt{6}$ − C.DOWN | + C.UP | − C.DOWN | + C.UP $\sqrt{6}$

C.

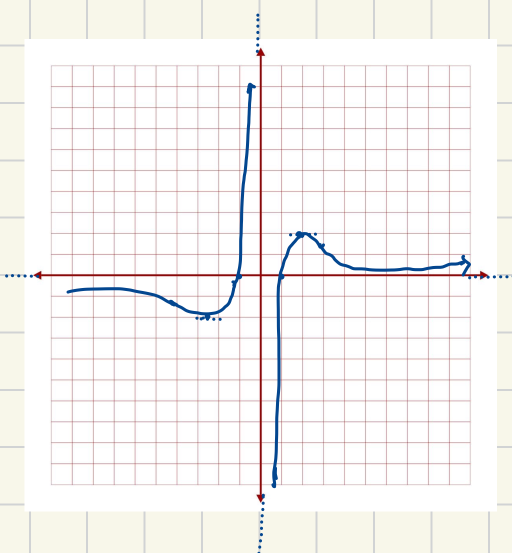
f''(-3) f''(-1) f''(1) f''(3)

0

L Min $(-\sqrt{3}, -0.38)$

I.P $(-\sqrt{6}, 0.34)$

L Max $(\sqrt{3}, .38)$ $(\sqrt{6}, -0.34)$



$f(x) = \frac{2x^2-8}{x^2-16}$

x INT $= 0 = 2x^2-8$

$u = x^2$

$x = \pm 2$

y INT $= \frac{2(0)-8}{0-16}$

y INT $\frac{\cancel{-8}\,1}{\cancel{-16}\,2} = \frac{1}{2}$

Asymptotes

Vertical Asymptote

$x^2 = 16$ or, $x = 4$ or $x = -4$

Horizontal Asymptote

$\lim_{x \to +\infty} \frac{2x^2-8}{x^2-16} = \frac{2}{1} = 2$

$\therefore$ Horizontal Asymptote at $y = 2$

$\lim_{x \to -\infty} \frac{2x^2-8}{x^2-16} \to \frac{2}{1} = 2$

$f'(x) = \frac{d}{dx}\left[\frac{2x^2-8}{x^2-16}\right] = \frac{(4x)(x^2-16) - (2x^2-8)(2x)}{(x^2-16)^2}$

$f'(x) \quad = \frac{4x^3 - 64x - 4x^3 + 16x}{(x^2-16)^2} = \frac{-48x}{(x^2-16)^2}$

Critical points:

$x = 0$

$x = 4$

$x = -4$

$$f''(x) = \frac{d}{dx}\left[= \frac{-48x}{(x^2-16)^2} \right]$$

$$f''(x) = \frac{-48(x^2-16)^2 + 192x^2(x^2-16)}{(x^2-16)^4}$$

$$f''(x) = \frac{48\cancel{(x^2-16)}\left[-(x^2-16) + 4x^2\right]}{(x^2-16)^3}$$

$$f''(x) = \frac{48(3x^2+16)}{(x^2-16)^3}$$

Possible Inflection Points

$48(3x^2+16) = 0$

$x^2 = \frac{-16}{3}$ ← No PIP

Undefined P.I.P

$x^2 - 16 = 0$

$x^2 = 16$

P.IP $x = +4, -4$

Points:

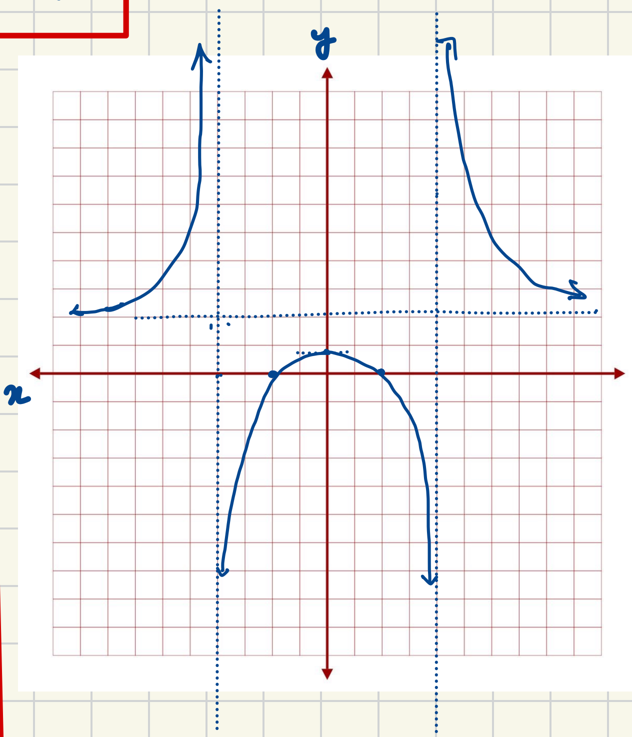
X INT: $(2,0)$ $(-2,0)$

y INT: $(0, \frac{1}{2})$

L. Max: $(0, \frac{1}{2})$

V. Asymptote: $4, -4$

H. Asymptote: 2



L. Max

	-4		0		4	
$f'(x)$	$f'(-5)$ + INC		$f(-1)$ + INC		$f(1)$ − DEC	$f(5)$ − DEC
$f''(x)$	C UP +ve $f''(-5)$		C DOWN −ve		C DOWN −ve	C. UP +ve

OPTIMIZATION

3.7 Applied Max/Min

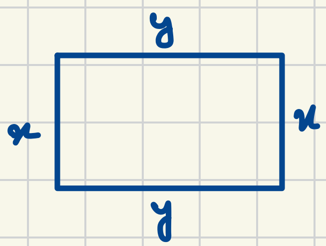
* Maximizing OR Minimizing

Continuous Functions:

1) CLOSED INTERVALS, [,] will HAVE SOLUTIONS
2) OPEN, INFINITE INTERVAL, (,) Might not have SOLUTIONS

Ex

Fence A Rectangular Region and Maximize the Area. $\underbrace{\text{You have 100ft of fence.}}_{\text{CONSTRAINT}}$



⇒ FORMULA

$A = x \cdot y$

substituting from ①

$A = x(50 - x)$

$A = 50x - x^2$

$A' = 50 - 2x$

Here, when $A' = 0$, that would give a Critical point which could be a maximum / Minimum point

$A' = 50 - 2x = 0$

$50 = 2x$

$x = 25$

CONSTRAINT

Perimeter $= 2(x+y) = 100$

$x + y = 50$

$y = 50 - x$ — ①

* Solve the constraint for one variable and sub into the formula.

CHECK

$x = 25 \rightarrow$ Area $\rightarrow 625$ sq ft^2

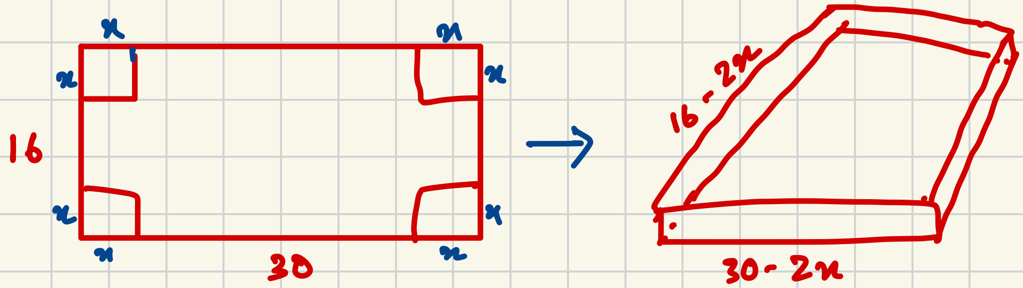
$x = 50$ [Maximum x can be] $\rightarrow$ Area $= 0$ sq ft

$x = 0$ [Minimum x can be] $\rightarrow$ Area $= 0$ sq ft

ENDPOINTS

$x \in [0, 50]$

$\therefore$ THE Maximum area is 625 sq ft^2, when $x = 25$ & $y = 25$



maximize value :

Formula: $(30-2x)(16-2x) \cdot x$

CONSTRAINT: $0 \leq x \leq 8$

$V = 480x - 60x^2 - 32x^2 + 4x^3$

$V = 4x^3 - 92x^2 + 480x$

$V' = 12x^2 - 184x + 480$

Find critical values

$V' = 0$

$12x^2 - 184x + 480 = 0$

$4(3x^2 - 46x + 120) = 0$

$3x^2 - 46x + 120 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{46 \pm \sqrt{2116 - 1440}}{6}$$

$$x = \frac{46 \pm \sqrt{676}}{6}$$

$$x = \frac{46 \pm 26}{6}$$

$\therefore x = \frac{10}{3}, 12$

checking end points

$x = 0$: volume $= 0$

$x = 8$: Volume $= 0$

$x = 10/3$: volume $= 725.926$ inches^3

$f(x) = (x^2)^{1/2}(x^2+25)^{1/2}$

$(x^4+25x^2)^{1/2}$

$\frac{1}{2}(x^4+25x^2)^{-1/2} \cdot (4x^3+50x)$

$f'(x) \; \frac{4x^3+50x}{2\sqrt{x^4+25x^2}}$

$f'(x) \; \frac{2x^3+25x}{\sqrt{x^4+25x^2}}$

$f'(x) \; \frac{2x^2+25}{\sqrt{x^2+25}}$

$f''(x) = \frac{50x}{(x^2+25)^{3/2}}$

3

$f''(x) = 50x$
$x = 0$

$(x^2+25)^{3/2} = 0$

$x^2 = -25$ ← NO PIP

$f'(x) = 0$
$2x^2+25 = 0$

$x^2 = -25/2$

$x^2+25 = 0$
$x^2 = -2$

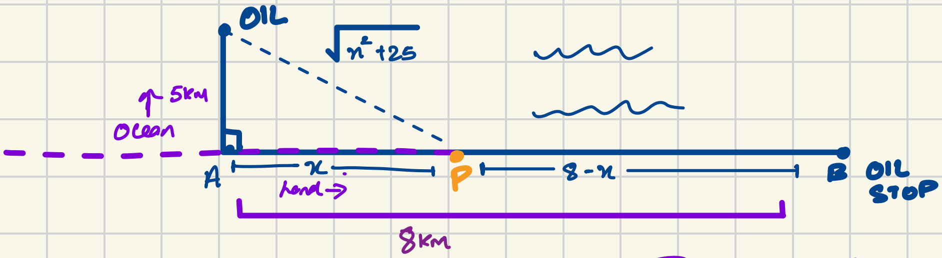
DEC DEC INC IN

-ve -ve +ve +ve

-7 -1 1 5

-ve C.Down | -ve C.Dwn | +ve C.UP | + C.UP

0 ← I.P



Pipe Cost
\$/per km in Ocean
\$ 1/2 per km on land

If we want to minimize cost, how far should 'P' be from point A.

FORMULA

$$\text{Cost} = 1.\$\left(\sqrt{x^2+25}\right) + \$\tfrac{1}{2}(8-x)$$

$$c = \left(\text{Pipe @ Sea}\right) + \left(\text{Pipe @ land}\right)$$

Constraint

$$8 \geq x \geq 0$$

$$c = \sqrt{x^2+25} + 4 - \tfrac{1}{2}x$$

$$c = (x^2+25)^{1/2} + 4 - \tfrac{1}{2}x$$

$$\frac{d}{dx}[c] = \frac{1}{\cancel{2}}(x^2+25)^{-1/2}\cdot\cancel{2}x + 0 - \frac{1}{2}$$

$$c'(x) = \frac{x}{(x^2+25)^{1/2}} - \frac{1}{2}$$

Here, to find critical values

$$\frac{1}{2} = \frac{x}{(x^2+25)^{1/2}}$$

$$(x^2+25)^{1/2} = 2x$$

$$x^2+25 = 4x^2$$

$$25 = 3x^2$$

$$x^2 = \frac{25}{3}$$

$$x = \pm\sqrt{\frac{25}{3}}$$

[we ignore $-\sqrt{\frac{25}{3}}$ as x has to be >0]

$$\therefore x = \sqrt{\frac{25}{3}}$$

$$c''(x) = \frac{d}{dx}\left[\frac{x}{(x^2+25)^{1/2}} - \frac{1}{2}\right]$$

$$\therefore c''(x) = \frac{\left[x \cdot \frac{1}{\cancel{2}}(x^2+25)^{-1/2}\cdot\cancel{2}x\right] + \left[(x^2+25)^{1/2}\right]}{x^2+25}$$

$$\therefore \quad = \frac{\frac{x^2}{(x^2+25)^{1/2}} + (x^2+25)^{1/2}}{x^2+25}$$

$$0 = \frac{x^2}{(x^2+25)^{1/2}} + (x^2+25)^{1/2}$$

$$-(x^2+25) = x^2$$

$$-x^2+25 = x^2$$

$$25 = 2x^2$$

$$x = \pm\sqrt{\frac{25}{2}}$$

[ignored $-\sqrt{\frac{25}{2}}$ as $x>0$]

inflection point

$x = \sqrt{\frac{25}{2}}$

critical point

$x = \sqrt{\frac{25}{3}}$

$x = 0, 8$

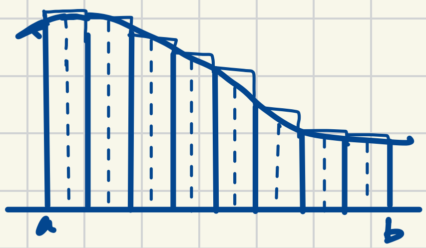
x	y
0	9
$\sqrt{\frac{25}{3}}$	8.33
8	9.423

$\therefore x = \sqrt{\frac{25}{3}}$, $\therefore$ the x value for minimizing the cost of making the pipeline is $\sqrt{\frac{25}{3}}$.

Area Problem (Indefinite Integrals)

Rectangle Method

Divide [a,b] into 'n' equal sub-section & create a Rectangle in each one



For a better approximation, make the space b/w each interval (The width of the Rectangle) approach zero

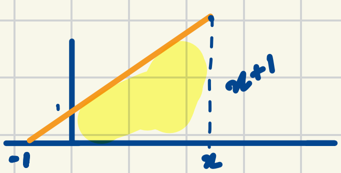
Anti-Derivative Method

If $A(x)$ is the area function. For some $f(x)$, on [a,b]

Then; $A'(x) = f(x)$

→ This implies to find the area under $f(x)$, we must undo $A'(x)$ to find $A(x)$

Ex $f(x) = x+1$ $[-1, x]$



$A(x) = \frac{1}{2} . b . h$

$A(x) = \frac{1}{2}(x+1)(x+1)$

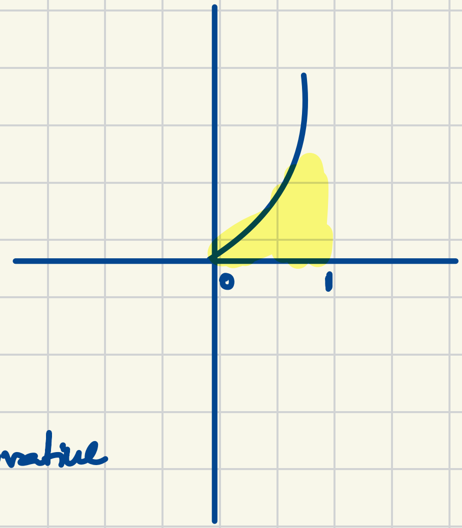
$A(x) = \frac{(x+1)^2}{2}$

OR, $A(x) = \frac{x^2}{2} + x + \frac{1}{2}$

$A'(x) = \frac{2x}{2} + 1$

$A'(x) = x+1 \rightarrow f(x)$

Ex Find Area under $f(x) = x^2$ on $[0,1]$



IDEA : IF $f(x) = x^2$ AND $f(x) = A'(x)$
Then $A'(x) = x^2$

Can you find function $A(x)$ whose Derivative is x^2?

$A(x) = \frac{x^3}{3} \longrightarrow A'(x) = \frac{\cancel{3}x^2}{\cancel{3}} = x^2 + c$

So $A(x) = \frac{x^3}{3} + c$

c is a constant & y-int of $A(x)$ & can be found by $x = 0$

$A(0) = x^3 + c$

$A(0) = c$

Interval is $[0,x]$
when you set $x = 0$, The interval is $[0,0]$
That is the area under a single point $\Rightarrow 0$

Here,
$A(0) = 0$ & $A(0) = c$
$\therefore c = 0$

$A(x) = \frac{x^3}{3}$ on the interval of $[0,1]$, the area

is
$A(1) = \frac{1}{3}$

Indefinite Integral

given a function, f, on some interval, I, F is called an $A(x)$ Anti-Derivative If,

$$F'(x) = f(x)$$

Ex $f(x) = x^2$

$F(x) = \frac{x^3}{3}$ one Anti-D

wny $\frac{d}{dx}\left[\frac{x^3}{3}\right] = x^2 = f(x)$

$F(x) = \frac{x^3}{3} + C$ → This represents the family of all possible Anti-D of x^2.

This Process is also called Integration.

$\int f(x)\,dx = F(x) + C$ ← constant

The Indefinite Integral

Ex Initial Value Problem

$$\frac{dy}{dx} = \frac{1}{(2x)^3}, \quad y(1) = 0$$

$$y = \int \frac{1}{(2x)^3}\,dx \to \int \frac{1}{8} x^{-3}\,dx$$

$$y = \frac{1}{8}\frac{x^{-2}}{-2} = y = -\frac{1}{16x^2} +$$

$$0 = -\frac{1}{16(1)^2} + C$$

$$0 = -\frac{1}{16} + C$$

$$\therefore C = 1/16.$$

Catapult: Accidentaly Shoots
Straight up
w/ An initial velocity of 128 ft/s, from A Height of 16 ft.

- Find the position function For the Height.
- Max Height
- When will it hit ground?

Initial Height: $S(0) = 16$ ft

Velocity: $S'(0) = 128$ ft/sec

Acceleration: $S''(T) = -32$ ft/sec^2

$$S'(T) = \int S''(T)\,dt \rightarrow S'(T) = \int -32$$

$$S'(t) = -32t + c$$

$$S'(0) = 128$$
$$128 = -32(0) + c$$
$$\therefore c = 128$$
$$S'(t) = -32t + 128$$
$$S(t) = \int S'(t)$$

$$S(t) = -\frac{32t^2}{2} + 128t + c$$

Here, we also Know
$$S(0) = 16$$
$$16 = -\frac{32(0)^2}{2} + 128(0) + c$$

$$\therefore c = 16$$
$$\therefore S(t) = -16t^2 + 128 + 16$$

② Max Height

Here,

$s'(t) = -32t + 128$

at $s'(t) = 0$

$0 = -32t + 128$

$32t = 128$

$t = \frac{128}{32}$

$t = 4$

Here,

when $t = 4$

$s(4) = -16(4)^2 + 128(4) + 16 = 272$

$\therefore$ Max height = 272 ft.

$0 = -16t^2 + 128t + 12$

$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\frac{-128 \pm \sqrt{16384 + 768}}{-32}$

$\frac{-128 \pm 130.96}{-32} = -0.0925$ (Ignored)

$t = 8.0925$

Here, when $t = 8.092$ s, catapult hits the ground.

Integration by Substitution.

ex $\int (x^2+1)^{50} \cdot 2x \, dx$

$u = x^2+1$, $du = 2x\,dx$

$dx = \frac{du}{2x}$ — ①

$\int u^{50} \cdot 2x \, dx$ — ①

$\int u^{50} \cdot \cancel{2x} \cdot \frac{du}{\cancel{2x}}$

$\int u^{50} \cdot du$

$\Rightarrow \frac{u^{51}}{51} + C$

or,

$\frac{(x^2+1)^{51}}{51} + C$

---

ex $\int x^2 (x^3-4)^5 dx$

$u = x^3-4$, $du = 3x^2 \cdot dx$

$\int \cancel{x^2} \, u^5 \cdot \frac{du}{3\cancel{x^2}}$

$\int \frac{u^5}{3} du \Rightarrow \frac{u^6}{18} \Rightarrow \frac{(x^3-4)^6}{18} + C$

Ex $\int 2x^3 \sin(x^4)\,dx$

$u = x^4$

$du = 4x^3\,dx$

$dx = \dfrac{du}{4x^3}$

$\int \cancel{2x^3} \sin(u) \cdot \dfrac{du}{\cancel{4x^3}_{2}}$

$\Rightarrow \int \dfrac{\sin(u)}{2}\,du$

$\Rightarrow \dfrac{1}{2} \times -\cos(x^4)$

$\Rightarrow \dfrac{-\cos(x^4)}{2} + C$

Ex $\int \cos(5x)\,dx$

$u = 5x$

$du = 5.dx$

$dx = \dfrac{du}{5}$

$\Rightarrow \int \cos(u) \dfrac{du}{5}$

$\Rightarrow \dfrac{\sin(u)}{5} \Rightarrow \dfrac{\sin(5x)}{5}$

$$\Rightarrow \int \sin^2(x)\cos(x)\,dx$$

$u \to \sin x$

$du \to \cos x\,dx$

$dx = \dfrac{du}{\cos(x)}$ — ①

Here,

substituting — ①

$$\Rightarrow \int u^2 \cancel{\cos(x)} \frac{du}{\cancel{\cos(x)}}$$

$$\Rightarrow \int u^2\,du$$

$$\Rightarrow \frac{u^3}{3} \text{ or, } \frac{\sin^3(x)}{3} + C$$

$$\int \frac{2-x}{\sqrt{2x^2-8x+1}}\,dx$$

$$u = 2x^2 - 8x + 1$$

$$du = (4x-8)\,dx$$

$$\int \frac{\cancel{2-x}}{\sqrt{u}} \cdot \frac{du}{-4\cancel{(2-x)}}$$

$$du = -4(2-x)\,dx$$

$$dx = \frac{du}{-4(2-x)}$$

$$\int \frac{1}{-4\sqrt{u}}\,du$$

$$\frac{-1}{4}\int \frac{1}{\sqrt{u}}\,du \Rightarrow \int u^{-1/2}\,du$$

$$\Rightarrow \frac{-1}{4}\left(\frac{u^{1/2}}{1/2}\right) \Rightarrow \frac{-1}{4}\left(2u^{1/2}\right)$$

$$-\frac{1}{2}\left(\sqrt{2x^2-8x+1}\right)+C$$

Area As An limit

2. $\sum_{k=1}^{n}\left[f(k) \pm g(k)\right] = \sum_{k=1}^{n} f(k) \pm \sum_{k=1}^{n} g(k)$

Sigma Notation

$$\sum_{k=0}^{5} k^3$$

$\Rightarrow 0^3 + 1^3 + 2^3 + 3^3 + 4^3 + 5^3$

Formula

1. $\sum_{k=1}^{n} k = 1+2+3+4+\ldots+n \longrightarrow \frac{n(n+1)}{2}$

2. $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\ldots+n^2 \longrightarrow \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\ldots+n^3 \longrightarrow \left[\frac{n(n+1)}{2}\right]^2$

4. $\sum_{k=1}^{n} = n$

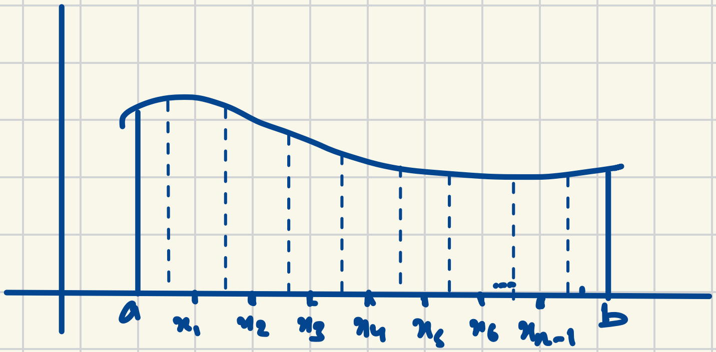
Ex

$$\sum_{k=1}^{10} k(k+1) = \sum_{k=1}^{10} k^2+k = \sum_{k=1}^{10} k^2 + \sum_{k=1}^{10} k$$

$$= \frac{10(10+1)(20+1)}{6} + \frac{10(10-1)}{2}$$

$$= \frac{110 \times 21}{6} + \frac{90}{2} = 440$$

AREA:

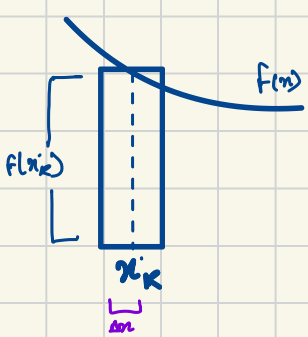


- Cut into "n" Equal sub-sections
- The width of Each Interval

$$\Delta x = \frac{b-a}{n}$$

- Pick an arbitrary Point for each sub-interval. & make a Rectangle.

Height of Each Rectangle:



Height of Each Rectangle:

$f(x_1^i), f(x_2^i) \dots f(x_n^i)$

Area of Rectangle:

b.h

$f(x_1^i)\,\Delta x,\ f(x_2^i)\,\Delta x, \dots f(x_n^i).\Delta x$

1st Rectangle · 2nd Rectangle · 3rd Rectangle

So, $AREA = f(x_1^i)\Delta x + f(x_2^i)\Delta x + \dots + f(x_n^i)\Delta x$

OR

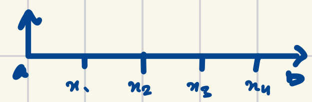
$$AREA = \sum_{x^i=1}^{n} f(x_n^i)\,\Delta x$$

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^{*}) \Delta x$$

This Is an Integral

* we choose x_k^{*} to be the left-End point, Right End-point, or Midpoint of the Sub-Interval.

Ex



LEFT ENDPOINTS

$$x_k^{*} = a + \Delta x (k-1)$$

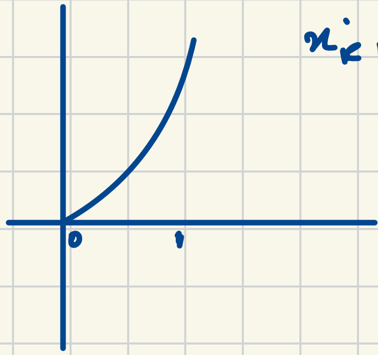
Right EndPoints:

$$x_k^{*} = a + k\Delta x$$

MIDPOINTS:

$$x_k^{*} = a + \Delta x \left(k - \tfrac{1}{2}\right)$$

Ex. Find the Area under $f(x) = x^2$ on $[0,1]$



x_k^* are Right Endpoints

① $\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

② $x_k^* = a + k\Delta x = x_k^* = 0 + k \cdot \frac{1}{n} = \frac{k}{n}$

③ $f(x_k^*) \to f\left(\frac{k}{n}\right) = \left(\frac{k}{n}\right)^2$

④ $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(x_k^*)\Delta x \to \sum\limits_{k=1}^{n} \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} \to$

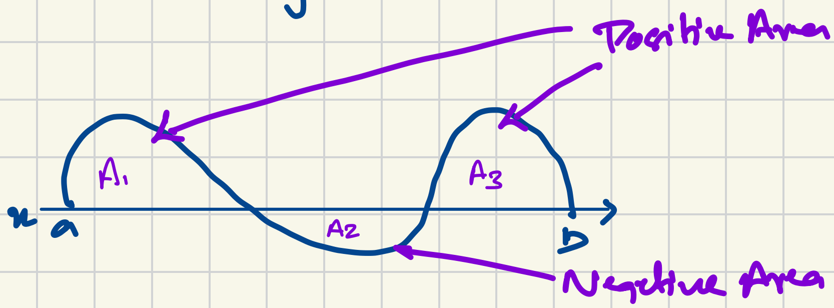
$* \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{k^2}{n^2} \cdot \frac{1}{n} \to \lim\limits_{n\to\infty} \frac{1}{n^3} \cdot \sum\limits_{k=1}^{n} k^2 \to \frac{1}{n^{\cancel{3}2}} \frac{\cancel{n}(n+1)(2n+1)}{6}$

$\to \lim\limits_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \to \lim\limits_{n\to\infty} \frac{2n^2+3n+1}{6n^2}$

$\to \lim\limits_{n\to\infty} \frac{\frac{2\cancel{n^2}}{\cancel{n^2}} + \frac{3\cancel{n}}{n\cancel{^2}} + \frac{1}{n^2}}{\frac{6\cancel{n^2}}{\cancel{n^2}}}$

$\Rightarrow \lim\limits_{n\to\infty} \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} = \frac{2+0+0}{6} = \frac{1}{3}$

Net Signed AREA



$A = (A_1 + A_3) - A_2$

$A = \lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} f(c_k)\, \Delta x$

Ex $f(x) = x - 1$ on $[0,2]$ w/ left endpoints

(i) $\Delta x \to \Delta x = \frac{2-0}{n} = \frac{2}{n}$

(ii) $c_k \to a + (k-1)\Delta x = 0 + (k-1) \cdot \frac{2}{n}$

$c_k = \frac{2(k-1)}{n}$

(iii) $\lim\limits_{n \to \infty} f(c_k) = \lim\limits_{n \to \infty} f\left(\frac{2(k-1)}{n}\right) = \frac{2(k-1)}{n} - 1$

$\to \lim\limits_{n \to \infty} \frac{2(k-1) - n}{n}$

(iv) $\sum\limits_{k=1}^{n} f(c_k)\Delta x \Rightarrow \sum\limits_{k=1}^{n} \left(\frac{2(k-1)}{n} - 1\right) \cdot \frac{2}{n} \Rightarrow \sum\limits_{k=1}^{n} \frac{4(k-1)}{n^2} - \frac{2}{n}$

$\sum\limits_{k=1}^{n} \frac{4(k-1)}{n^2} - \sum\limits_{k=1}^{n} \frac{2}{n} \to \frac{4}{n^2}\left[\sum\limits_{k=1}^{n} (k-1)\right] - \frac{2}{n} \sum\limits_{k=1}^{n} 1$

$\to \frac{4}{n^2}\left[\sum\limits_{k=1}^{n} k - \sum\limits_{k=1}^{n} 1\right] - \frac{2}{\cancel{n}} \cdot \cancel{n}$

$$\rightarrow \frac{4}{n^2}\left[\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1\right] - \frac{2}{\cancel{n}} \cdot \cancel{n}$$

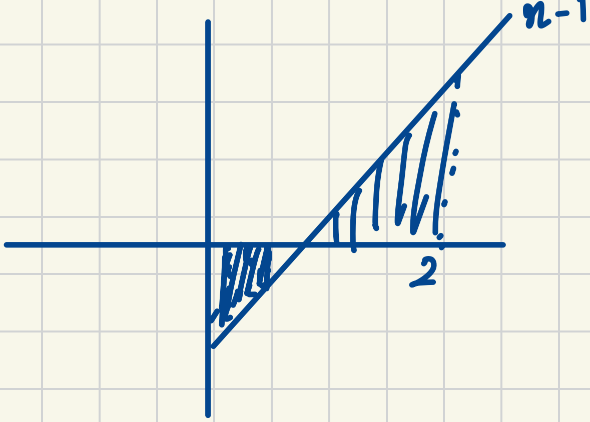
$$\rightarrow \frac{4}{n^2}\left[\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1\right] - 2$$

$$\rightarrow \frac{4n(n+1)}{2n^2} - \frac{4n}{n^2} - 2$$

$$\rightarrow \frac{2(n+1)}{n} - \frac{4}{n} - 2$$

$$\rightarrow \cancel{2} + \frac{2}{n} - \frac{4}{n} - \cancel{2}$$

$$\rightarrow \lim_{n\to\infty} -\frac{2}{n} = 0$$ (Net Area)



Ex $2x - x^3$ $[0,1]$ w/ Right E.P

(i) $\Delta x = \frac{1-0}{n} = \frac{1}{n}$

$C_k = a + k \cdot \Delta x$

(ii) $c_k = 0 + k \Delta x \Rightarrow \frac{1}{n} \cdot k$

(iii) $f(c_k) = 2\frac{k}{n} - \left(\frac{k}{n}\right)^3$

(iv) $\sum_{k=1}^{n} f(c_k)\Delta x \rightarrow \sum_{k=1}^{n}\left[2\left(\frac{k}{n}\right) - \left(\frac{k}{n}\right)^3\right]\frac{1}{n}$

$\Rightarrow \frac{1}{n}\left[\frac{2}{n}\sum_{k=1}^{n} k - \frac{1}{n^3}\sum_{k=1}^{n} k^3\right]$

$\Rightarrow \frac{\cancel{2}}{n\cancel{2}}\left(\frac{\cancel{n}(n+1)}{\cancel{2}} - \frac{1}{n^4}\left(\left[\frac{n(n+1)}{2}\right]^2\right)\right)$

$\Rightarrow \frac{n+1}{n} - \frac{1}{n^{\cancel{4}2}}\left(\frac{\cancel{n^2}(n+1)^2}{4}\right)$

$\Rightarrow \frac{n+1}{n} - \frac{(n+1)^2}{4n^2}$

$\Rightarrow 1 + \frac{1}{n} - \frac{\cancel{n^2}}{4\cancel{n^2}} - \frac{2n}{4n^2} - \frac{1}{4n^2}$

$\Rightarrow 1 + \frac{1}{n} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2}$

$$\Rightarrow \quad 1 + \frac{1}{n} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2}$$

$$\lim_{n \to \infty} f(c(x))\,\Delta x = \lim_{n \to \infty} 1 + \frac{1}{n} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2}$$

$$= 1 + 0 - \frac{1}{4} - 0 - 0$$

$$A \quad = \frac{3}{4}$$

Definite Integral

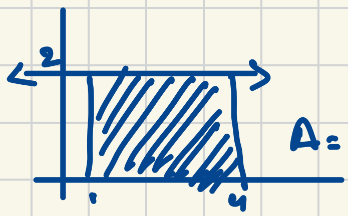
$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$ on $[a,b]$

$A = \int_a^b f(x)\, dx$ (net signed Area)

$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$

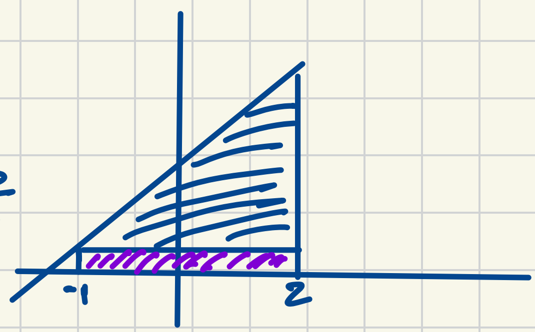
Ex

$\int_1^4 2\, dx = 6$



$A = 3.2 = 6$

Ex $\int_{-1}^{2} x+2\, dx = \frac{15}{2}$

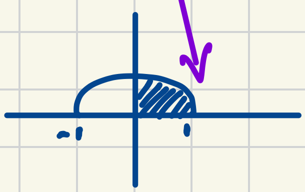


$A = 3.1 = 3$

$A = \frac{3.3}{2}$

$A + A = 3 + \frac{9}{2} = \frac{15}{2}$

$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4}$$



$$A = \frac{\pi r^2}{4} = \frac{\pi}{4}$$

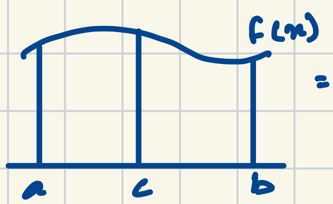
PROPERTIES

1. $\int_a^a f(x)\, dx = 0$ AREA UNDER A Single Point

2. $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$

3. $\int_a^b c \cdot f(x)\, dx = c \cdot \int_a^b f(x) \cdot dx$

4. $\int_a^b f(x) \pm g(x)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

5. $\int_a^b f(x)\, dx =$  $= \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

6) If $f(x) \geq 0$ for all $x \in [a,b]$

THEN $\int_a^b f(x)\,dx \geq 0$

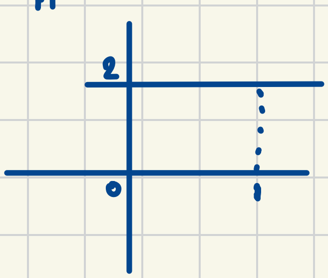
If $f(x) \leq 0$ FOR ALL $x \in [a,b]$

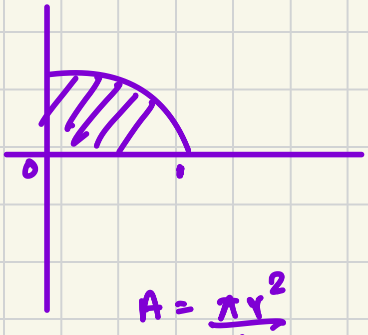
THEN $\int_a^b f(x)\,dx \leq 0$

Ex $\int_0^1 2\left(2-\sqrt{1-x^2}\right) dx$

$$2\left[\int_0^1 2\,dx - \int_0^1 \sqrt{1-x^2}\,dx\right]$$

A





$A = 1 \times 2 = 2$

$A = \frac{\pi r^2}{4}$

$A = \frac{\pi}{4}$

$$2\left[\int_0^1 2\,dx - \int_0^1 \sqrt{1-x^2}\,dx\right]$$

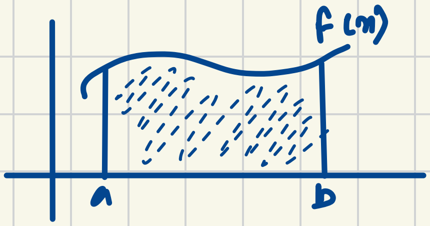
$$2\left[2 - \frac{\pi}{4}\right]$$

$$A = \frac{8-\pi}{2}$$ //

FUNDAMENTAL THEOREM OF CALCULUS

$$A = \int_a^b f(x)\,dx = F(x) + c$$

↑ Area function



we want the area from a to b.

Area = [Area up "b" - Area up to "b"]

So,

$$A = \int_a^b f(x)\,dx = F(b) - F(a)$$

Area function

Ex

$$\int_1^5 x\,dx = \frac{x^2}{2}\Big]_1^5$$

$$A = \frac{5^2}{2} - \frac{1^2}{2} = \frac{25}{2} - \frac{1}{2} = \frac{24}{2} = 12$$

what happened to c?

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = [F(b) + c] - [F(a) + c]$$

$$F(b) + \cancel{c} - F(a) - \cancel{c}$$

$$= F(b) - F(a)$$

1)

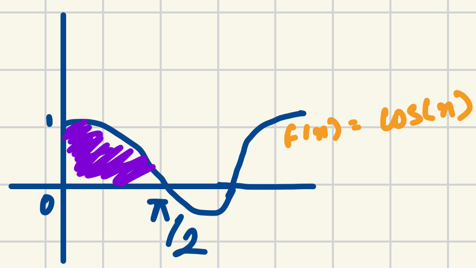
$$\int_1^5 x \, dx$$

$$\Rightarrow \left[\frac{x^2}{2}\right]_1^5$$

$$\Rightarrow \frac{5^2}{2} - \frac{1^2}{2}$$

$$\Rightarrow \frac{25-1}{2} = 12$$

$$\int_0^{\pi/2} \cos(x) \, dx = \sin x \Big]_0^{\pi/2}$$



$$\Rightarrow \sin(\pi/2) - \sin(0)$$

$$\Rightarrow 1 - 0$$

$$\Rightarrow 1$$

---

ex $$\int_4^9 x^2\sqrt{x} \, dx \Rightarrow \int_4^9 x^{5/2} \, dx$$

$$\Rightarrow \int_4^9 x^{5/2} \, dx = \frac{2}{7} x^{7/2}\Big]_4^9$$

$$\Rightarrow \left(\frac{2}{7} \times 9^{7/2}\right) - \left(\frac{2}{7} 4^{7/2}\right)$$

$$\Rightarrow 588.286$$

Ex $\displaystyle\int_0^{\pi/3} \sec^2(x)\,dx \Rightarrow \tan x\Big]_0^{\pi/3}$

$\Rightarrow \tan(\pi/3) - \tan(0)$

$\Rightarrow \sqrt{3} - 0$

$\therefore \sqrt{3}$

Ex $\displaystyle\int_4^0 x^3\,dx \Rightarrow \frac{x^4}{4}\Big]_4^0$

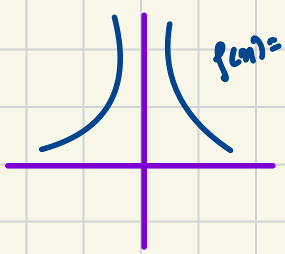
$\Rightarrow \dfrac{0^4}{4} - \dfrac{4^4}{4} = -64$

Ex $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx \Rightarrow -x^{-1}\Big]_{-1}^{1}$ → not a continuous function

$\Rightarrow -(1)^{-1} - (-(-1)^{-1})$

Domain issue at $x=0$
vertical Asymptote

$\Rightarrow -1 - 1$



$f(x) = \frac{1}{x^2}$

$\Rightarrow -2$

Ex $\displaystyle\int_0^6 f(x)\,dx$, $f(x) = \begin{cases} x^3, & x<2 \\ 5x-1, & x\geq 2 \end{cases}$

$\displaystyle\int_0^2 x^3\,dx + \int_2^6 5x-1\,dx$

$$\int_0^2 x^3\,dx + \int_2^6 5x-1\,dx$$

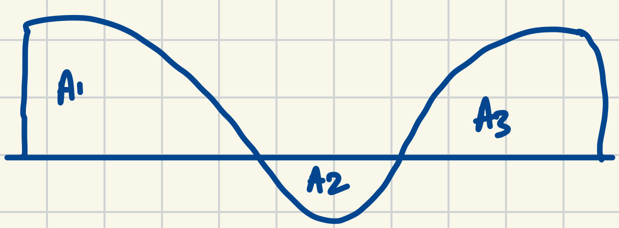
$$\Rightarrow \left[\frac{x^4}{4}\right]_0^2 + \left[\frac{5}{2}x^2 - x\right]_2^6$$

$$\Rightarrow \left[\frac{(2^4)}{4} - 0\right] + \left[\left(\frac{5(6)^2}{2} - 6\right) - \left(\frac{5(\cancel{2})^{\cancel{2}}}{\cancel{2}} - 2\right)\right]$$

$$\Rightarrow \frac{\cancel{16}^{4}}{\cancel{4}} + \frac{\cancel{180}^{90}}{\cancel{2}} - 6 - 8$$

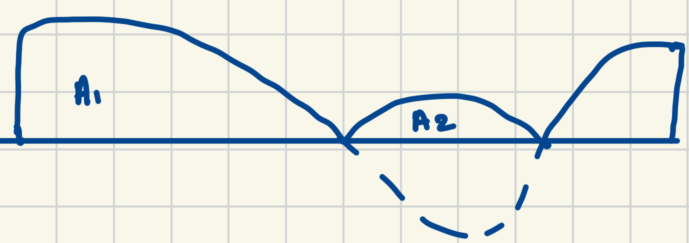
$$\Rightarrow 80$$

Total AREA



Net Signed AREA $\rightarrow \int_a^b f(x)\,dx$

$A = A_1 + A_3 - A_2$



For Total AREA, $\rightarrow \int_a^b |f(x)|\,dx,\ |f(x)| \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$

$A = A_1 + A_2 + A_3$

Change Sign of Any 'Negative AREA'
(AREA Below x-Axis)

Ex. Find Total AREA Between $1-x^2$ & x-Axis on $[0,2]$

$$T.A = \int_0^2 |1-x^2|\, dx$$

Before we find the area:

$$1-x^2=0$$
$$x^2=1$$
$$x=\pm 1$$

0 +ve +1 -ve 2

$$\therefore f(x) \begin{cases} f(x), & x \le 1 \\ -f(x), & x > 1 \end{cases}$$

Total Area $= \int_0^1 1-x^2\,dx + \int_1^2 -(1-x^2)\,dx$

$$\Rightarrow \left[x-\frac{x^3}{3}\right]_0^1 + \left[-x+\frac{x^3}{3}\right]_1^2$$

$$\Rightarrow \left[\left(1-\frac{1}{3}\right)-0\right] + \left[\left(-2+\frac{8}{3}\right)-\left(-1+\frac{1}{3}\right)\right]$$

$$\Rightarrow \frac{2}{3} + \left(\frac{2}{3}+\frac{2}{3}\right) \Rightarrow \frac{6}{3} = 2$$

Part I : If $y = f(x)$ is continuous over $[a, x]$, Then Area is $A(x)$ such that $A'(x) = f(x)$

$$A(x) = \int_a^x f(x)\,dx$$

$$\frac{d}{dx}\left[A(x)\right] = f(x) \to \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

Ex $\frac{d}{dx}\left[\int_1^x T^4\,dT\right] = x^4$

Definite Integrals w/ Substitution

Method 1: Don't change Bounds, Sub Back in for u before Evaluation

Ex $\int_0^2 4x(x^2-1)^3\,dx$ $u = x^2 - 1$

$du = 2x \cdot dx$

$dx = \frac{du}{2x}$

$\cancel{4x}^{2}(x^2-1) \cdot \frac{du}{\cancel{2x}}$

$$\int_0^2 2u^3 \Rightarrow \frac{u^4}{2}$$

OR

$$\left[\frac{(x^2-1)^4}{2}\right]_0^2 \Rightarrow \frac{(2^2-1)^4}{2} - \frac{(0-1)^4}{2}$$

$$\Rightarrow \frac{81}{2} - \frac{1}{2}$$

$$\Rightarrow \frac{80}{2} = 40$$

Method 2: Change Bounds During your first Substitution.
But Do not Sub Back IN For "u"
Before Substitution

Changing the Bounds

$x: 2 \quad : u = 2^2 - 1 = 3$

$x: 0 \quad : u = 0^2 - 1 = -1$

ex $\displaystyle\int_0^2 4x(x^2-1)^3\,dx$

$u = x^2 - 1$
$du = 2x\,dx$
$dx = \frac{du}{2x}$

$\Rightarrow \displaystyle\int_{-1}^{3} 4x u^3\,dx$

$\Rightarrow \displaystyle\int_{-1}^{3} 2u^3 \cdot du = \left.\frac{u^4}{2}\right]_{-1}^{3}$

$\Rightarrow \frac{3^4}{2} - \frac{(-1)^4}{2}$

$\Rightarrow \frac{81}{2} - \frac{1}{2} = 40$

---

ex $\displaystyle\int_0^{\pi/8} \sin^5(2x)\cos(2x)\,dx$

$u = \sin(2x)$
$\frac{du}{dx} = 2\cos 2x.$
$dx = \frac{du}{2\cos(2x)}$

$u = \sin\left(2\frac{\pi}{8}\right) = \frac{1}{\sqrt{2}}$

$u = \sin(0) = 0$

$\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \cancel{\cos(2x)} \cdot \left(\frac{du}{2\cancel{\cos(2x)}}\right) \cdot u^5$

$\frac{1}{2}\displaystyle\int_0^{\frac{1}{\sqrt{2}}} u^5 du \Rightarrow \left.\frac{u^6}{6}\right]_0^{\frac{1}{\sqrt{2}}} \quad \Rightarrow \frac{1}{2} \frac{\left(\frac{1}{\sqrt{2}}\right)^6}{6} - 0$

$\therefore \frac{1}{8} \cdot \frac{1}{12} = \frac{1}{96}$

Ex $\int_{1}^{3} \frac{\cos(\pi/x)}{x^2}\,dx$

$u = \pi/x$

$u = \pi \cdot x^{-1}$

$du = -\pi x^{-2}\,dx$

$du = -\frac{\pi}{x^2}\,dx$

$dx = -\frac{x^2\,du}{\pi}$

when $x=1$

$u = \pi \cdot 1^{-1}$

$u = \pi$

when $x=3$

$u = \pi \cdot 3^{-1}$

$u = \frac{\pi}{3}$

$$\Rightarrow \int_{\pi}^{\pi/3} -\frac{\cos(u)}{\cancel{x^2}} \cdot \frac{\cancel{x^2}\,du}{\pi}$$

$$\Rightarrow \int_{\pi}^{\frac{\pi}{3}} -\frac{\cos(u)}{\pi}\,du$$

$$\Rightarrow -\frac{1}{\pi}\int_{\pi}^{\frac{\pi}{3}} \cos(u)\,du$$

$$\Rightarrow -\frac{1}{\pi}\left[\frac{\sin(u)}{(1)}\right]_{\pi}^{\pi/3}$$

$$\Rightarrow -\frac{1}{\pi}\left\{\left(\sin\left(\frac{\pi}{3}\right) - \sin(\pi)\right)\right\}$$

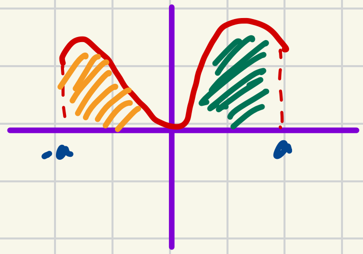
$$\Rightarrow -\frac{1}{\pi}\left\{\frac{\sqrt{3}}{2} - 0\right\}$$

$$-\frac{\sqrt{3}}{2\pi}$$

Even & Odd Functions In Integrals

EVEN: $f(-x) = f(x)$ Symmetric at Y-axis

If $f(x)$ is Even



$$\int_{-a}^{a} f(x)\,dx$$

$$\Rightarrow 2 \cdot \int_{0}^{a} f(x)\,dx$$

Ex

$$\int_{-3}^{3} x^2 + 4\,dx$$

To prove the given function is Even:

$$F(-x) = (-x)^2 + 4 = x^2 + 4 = f(x)$$

Therefore,

$$\Rightarrow 2 \cdot \int_{0}^{3} x^2 + 4 \Rightarrow 2 \cdot \left[\frac{x^3}{3} + 4x \right]_0^3$$

$$\Rightarrow 2 \cdot \left\{ \frac{\cancel{3^3}^{\,2}}{\cancel{3}} + 4(3) - 0 \right\}$$

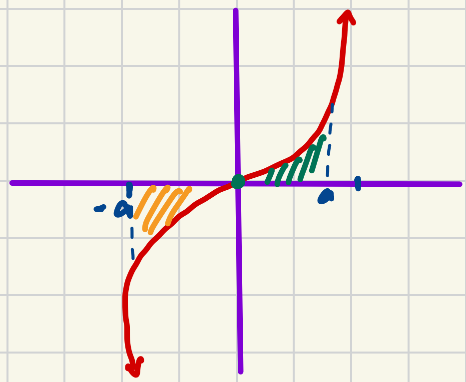
$$\Rightarrow 2 \cdot (9 + 12)$$

$$\Rightarrow 42$$

ODD:

$$f(-x) = -f(x)$$

symmetric to the Origin



If $f(x)$ is <u>odd</u>:

$$\int_{-a}^{a} f(x)\,dx = 0$$

Total AREA:

$$\int_{-a}^{a} |f(x)|\,dx$$

<u>Ex</u> $\displaystyle\int_{-3}^{3} \frac{\sin(x)}{\sqrt{1+x^2}}$

Thought Process:

(i) Cannot Substitute

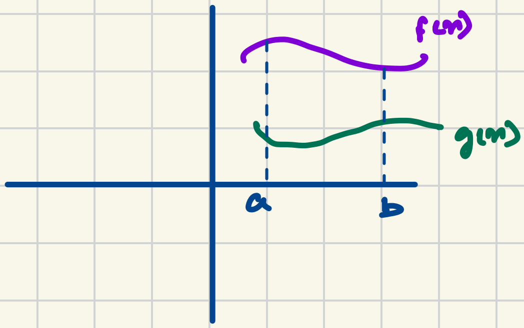
(ii) Check if ODD/Even.

$$f(-x) = \frac{\sin(-x)}{\sqrt{1+(-x)^2}} = \frac{\sin(-x)}{\sqrt{1+x^2}} \rightarrow \frac{-\sin(x)}{\sqrt{1+x^2}} \rightarrow -f(x)$$

$\therefore f(-x) = -f(x) \rightarrow$ ODD Function

$$\hookrightarrow \int_{-3}^{3} \frac{\sin(x)}{\sqrt{1+x^2}} = 0$$

5.1 Area Between Two Curves

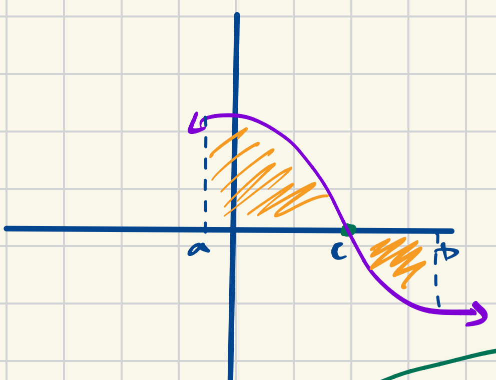


Area between:
$f(x)$ & $g(x)$

$$A = \int_a^b f(x) - \int_a^b g(x)$$

$$A = \int_a^b f(x) - g(x)$$

Note: $f(x) \geq g(x)$
For all $x : [a,b]$
($f(x)$ is greater than $g(x)$)



Equation of x-axis

$$\int_a^c f(x) - 0 + \int_c^b 0 - f(x)$$

$$\int_a^c f(x) - \int_c^b f(x)$$

Ex

Find the area Bounded above $y = 2x+5$ & Bounded Below $y = x^3$ on $[0,2]$

$$A = \int_0^2 (2x+5) - (x)^3$$

$$A = \int_0^2 2x+5-x^3 \Rightarrow \left[\frac{2x^2}{2} + 5x - \frac{x^4}{4}\right]_0^2$$

$$\Rightarrow \left[x^2 + 5x - \frac{x^4}{4}\right]_0^2$$

$$\Rightarrow \left[\left(2^2 + 5(2) - \frac{2^4}{4}\right) - (0)\right]$$

$$\Rightarrow \cancel{4} + 10 - \frac{\cancel{2\times2\times2\times2}}{\cancel{4}}$$

$\therefore$ Area $= 10$

Ex Find Area b/w $y = x^2$ & $y = x+6$

$x^2 = x+6$

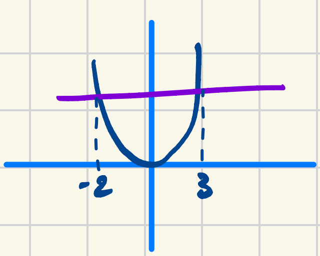
$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3,\ x = -2$

$$A = \int_{-2}^{3} (x+6) - (x)^2$$

$$A = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$



$$A = \left(\frac{3^2}{2} + 6(3) - \frac{3^3}{3}\right) - \left(\frac{4}{2} + 6(-2) - \frac{(-2)^3}{3}\right)$$

$$A = \left(\frac{9}{2} + 18 - 9\right) - \left(2 - 12 + \frac{8}{3}\right) \Rightarrow \left(\frac{9}{2} - 9\right) - \left(-14 + \frac{8}{3}\right) \Rightarrow \frac{125}{6}$$

Steps:

1. Find the x coordinates of the Intersections of the curves (set $f(x) = g(x)$)
2. which Point is on top? (Pick one Point on Interval)
3. Set-up & Solve

Ex Find the area between the x-axis and $y = x^2 - 2x$

$y = 0$

Here,

$0 = x^2 - 2x$

$x(x-2) = 0$

$x = 0$ (a)

$x = 2$ (b)

$y = 0 > y = x^2 - 2x$

Here,

$$\Rightarrow \int_0^2 0 - (x^2 - 2x)$$

$$\Rightarrow \int_0^2 -x^2 + 2x \Rightarrow \left[\frac{-x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}} \right]_0^2$$

$$\left(\frac{-2^3}{3} + 2^2 \right) - \left(0 \right)$$

$$= \frac{-8}{3} + 4$$

$$= \frac{-8+12}{3} \Rightarrow 4/3$$

Ex Find the area bound by $y=x^3$ & $y=x$

$x^3=x$

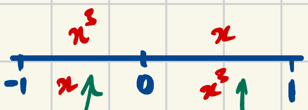
$x^3-x=0$

$x(x^2-1)=0$

$x=0$

$(x+1)(x-1)=0$

$\therefore x=0, x=-1, x=1$



$-1/2$:
$y=x^3 = -\frac{1}{8}$
$y=x = -1/2$

$1/2$: $y=x^3=\frac{1}{8}$
$y=x=\frac{1}{2}$

$$\Rightarrow \int_{-1}^{0} x^3 - x \, dx + \int_{0}^{1} x - x^3$$

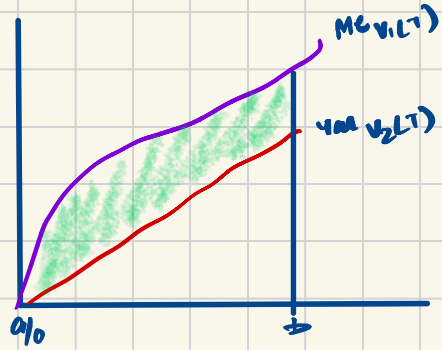
$$\Rightarrow \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1}$$

$$\left[0 - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right)\right] + \left[\frac{1^2}{2} - \frac{1^4}{4} - 0\right]$$

$$-\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$-\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - 1/4$$

$$\frac{2}{2} - \frac{2}{4} \Rightarrow 1 - \frac{1}{2} = 1/2$$

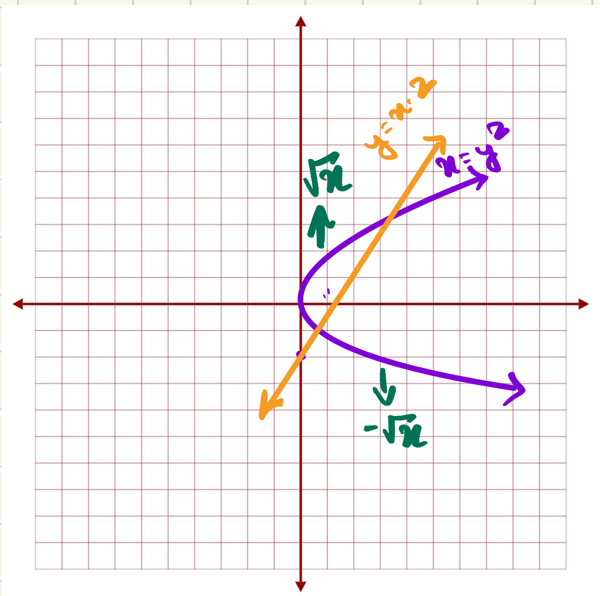


$D = v \cdot T$

In this Case, The Area Represents The Distance CAR "Me" is Ahead of Car "You"

$$A = \int_0^b v_1(T) - v_2(T)\, dT$$

Ex: Find the area Bound By $x = y^2$ AND $y = x - 2$



$\Rightarrow \quad x = y^2 \qquad x = y + 2$

$\Rightarrow \quad y^2 = y + 2$

$\Rightarrow \quad y^2 - y - 2 = 0$

$\Rightarrow \quad y^2 - (2y - y) - 2 = 0$

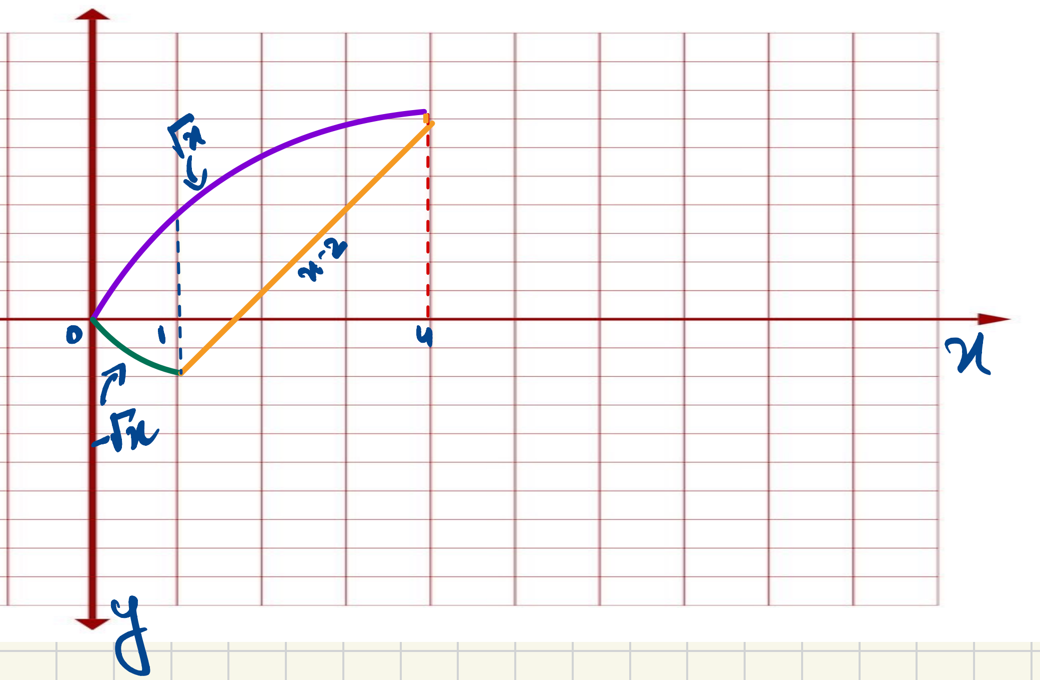
$\quad y^2 - 2y + y - 2 = 0$

$\Rightarrow \quad y(y-2) + 1(y-2) = 0$

$\Rightarrow \quad (y+1)(y-2) = 0$

$\therefore y = -1, \; y = 2$

$x = 1 \qquad x = 4$



$$A = \int_0^1 \sqrt{x} - (-\sqrt{x})\, dx + \int_1^4 \sqrt{x} - (x-2)\, dx$$

$$A = \int_0^1 2\sqrt{x}\, dx + \int_1^4 2 - x + \sqrt{x}\, dx$$

$$A = \left[2x^{3/2} \cdot \frac{2}{3}\right]_0^1 + \left[2x - \frac{x^2}{2} + x^{3/2} \cdot \frac{2}{3}\right]_1^4$$

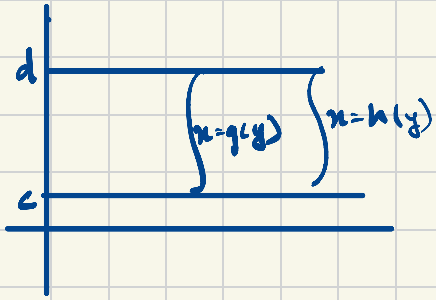
$$A = \left[\frac{4x^{3/2}}{3}\right]_0^1 + \left[2x - \frac{x^2}{2} + \frac{2x^{3/2}}{3}\right]_1^4$$

$$A = \left(\frac{4}{3}\right) + \left[\cancel{8} - \cancel{8} + \frac{2 \cdot 4^{3/2}}{3} - \left(2 - \frac{1}{2} + \frac{2}{3}\right)\right]$$

$$A = \frac{4}{3} + \left[\frac{2 \cdot 4^{3/2}}{3} - \left(\frac{4-1}{2} + \frac{2}{3}\right)\right] \Rightarrow \frac{4}{3} + \left[\frac{2 \cdot 4^{3/2}}{3} - \left(\frac{3}{2} + \frac{2}{3}\right)\right]$$

$$\Rightarrow \frac{4}{3} + \left[\frac{2 \cdot 4^{3/2}}{3} - \left(\frac{9+4}{6}\right)\right] \Rightarrow 4/3 + \left[\frac{2 \cdot 4^{3/2}}{3} - \frac{13}{6}\right] \Rightarrow \frac{27}{6} = 9/2$$

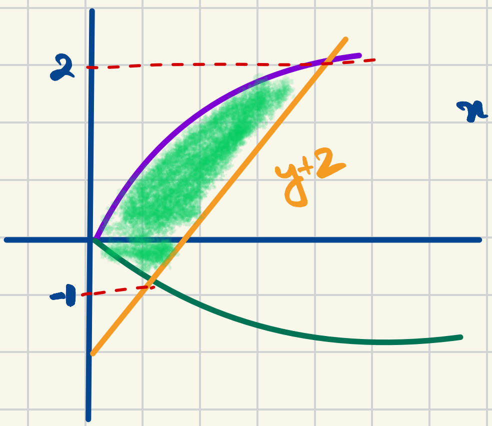
You can do the Same Integration in Respect to y.



$$A = \int_c^d h(y) - g(y)\, dy$$

If $h(y) > g(y)$ For $[c,d]$

FOR EXAMPLE:



$x = y^2 \qquad y = x - 2$

$\therefore\ x = y^2,\ x = y+2$

Both of these Function meet at $y = 2, -1$

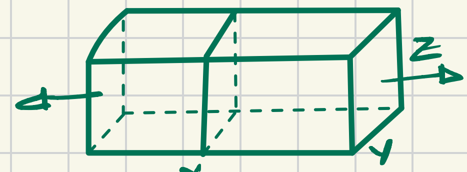
$$A = \int_{-1}^{2} (y+2) - y^2\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$$

$$= \left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right]$$

$$= \left[\left(6 - 8/3\right) - \frac{1}{2} + 2 - \frac{1}{3}\ \right)\Big]$$

$$= \left[6 - \frac{8}{3} - \frac{5}{6} + 2\ \right] \Rightarrow \frac{9}{2}$$

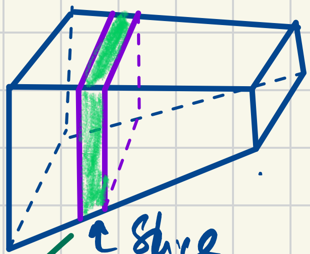
Volume: Disks/Washers



Volume = $(y \cdot z) \cdot x$

Surface Area of the Cross-Section

Volume of Solids By Slicing:

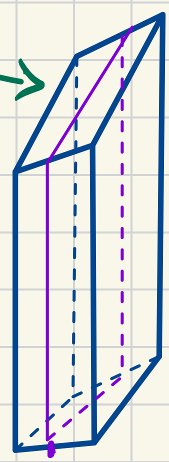


Slice

Cut into thin slabs. Then use Summations to set up an Integral.

Δx

To Do This, Find the Area of Cross-Section.



x_k^*

Δx

Pick Arbitrary Point, x_k^* on Each Sub-Interval

$$V_k = \underbrace{A(x_k^*)}_{\text{Cross-Sectional Area}} \cdot \underbrace{\Delta x}_{\text{Length}}$$

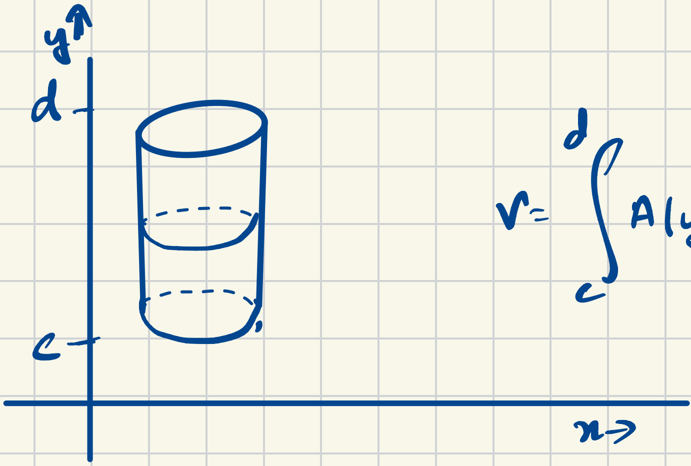
Therefore,

$$V = \sum_{k=1}^{n} A(x_k^*) \cdot \Delta x$$ → (Approximate Volume)

OR

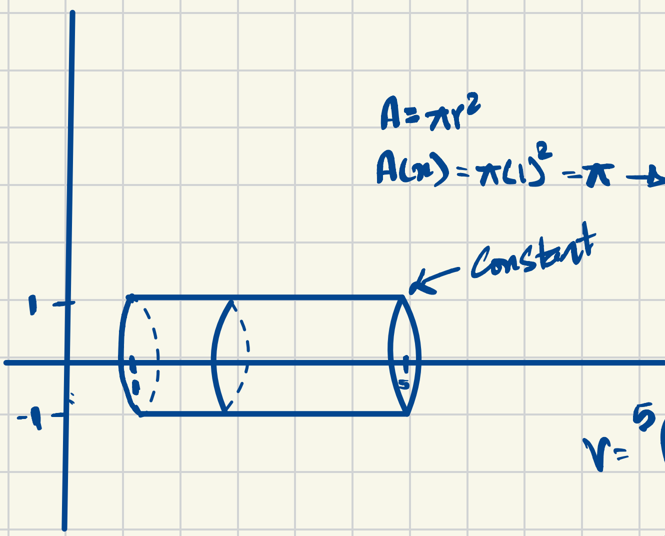
$$V = \lim_{n \to \infty} \sum_{k=1}^{n} A(x_k^*) \cdot \Delta x$$

$$V = \int_a^b A(x)\,dx$$, $A(x)$ = Cross-sectional Area over $[a,b]$



$$V = \int_c^d A(y)\,dy$$

Ex



$A = \pi r^2$

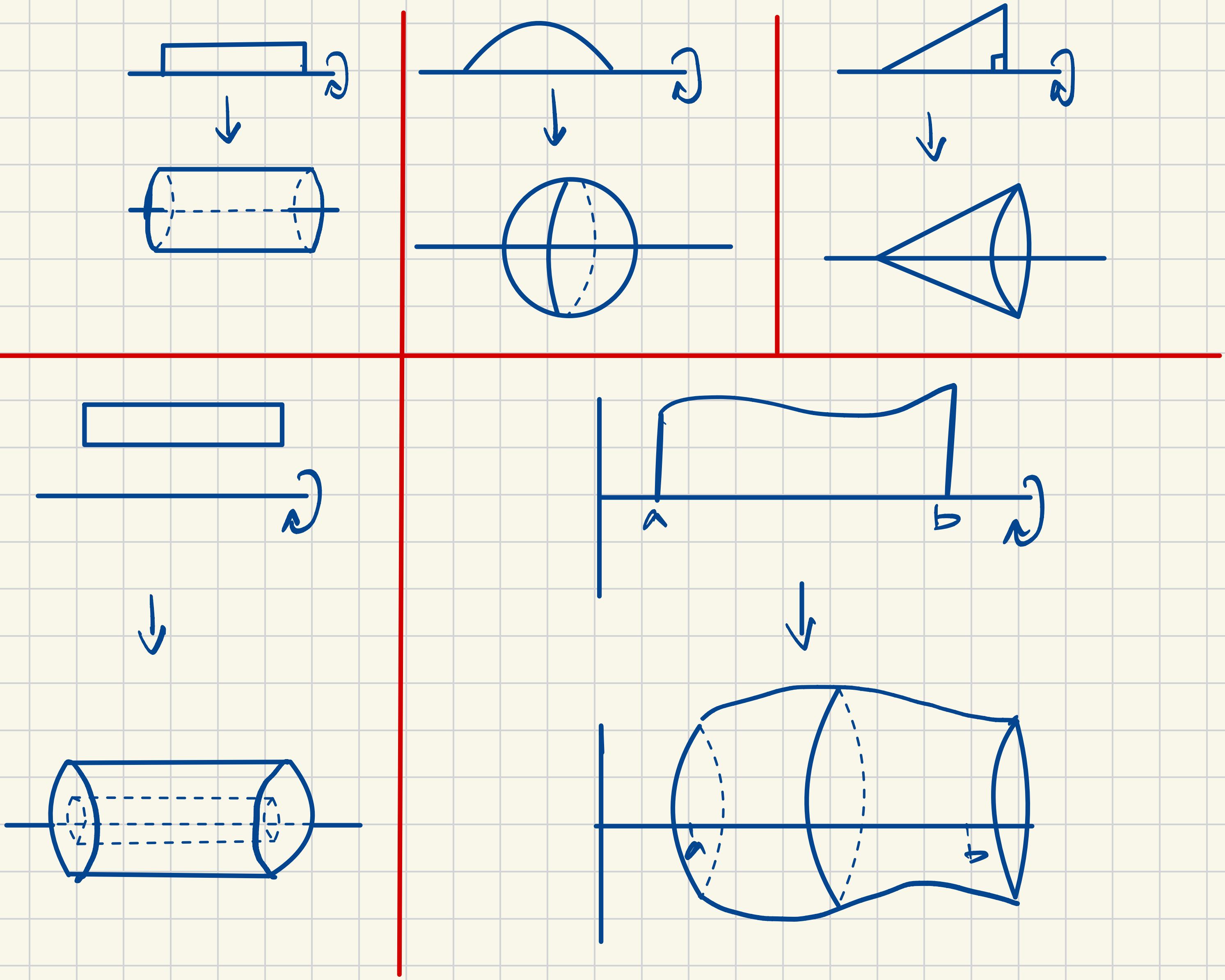
$A(x) = \pi(1)^2 = \pi \rightarrow$ Cross-Sectional Area

$$V = \int_1^5 \pi \cdot dx$$

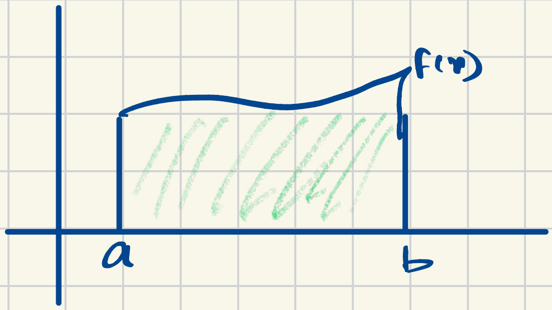
$$V = \Big[\pi x\Big]_1^5$$

$$\therefore V = \Big[5\pi - \pi\Big] = 4\pi$$

Solid Of Revolution

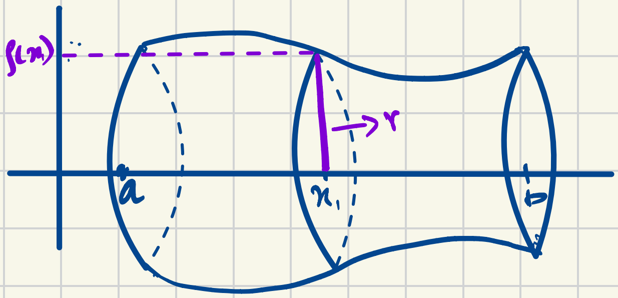


Volume of Solids of Revolution:



f(x) is:

- Continuous and bound by 'a' & 'b' $x=a$, $x=b$
- Rotate about x-Axis
- Sides are $\perp$ to x-Axis.
- When you revolve cross section area is always a circle.



Find Volume by Slicing:

$$V = \int_a^b A(x)\,dx$$

$A(x)$ = Surface Area of Cross-Section

$A(x) = \pi r^2$, Circle, $r = f(x)$

$$A(x) = \pi\left(f(x)\right)^2$$

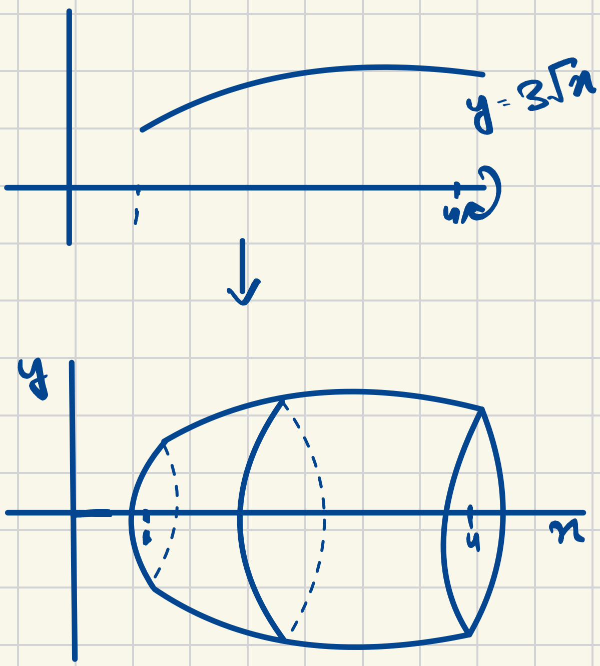
$$\therefore V = \int_a^b \pi\left[f(x)\right]^2 dx$$

For Revolved Solids

Ex

Find the volume of the solid of revolution where $y = 3\sqrt{x}$ on $[1,4]$ is revolved around x-axis.

$\Rightarrow$

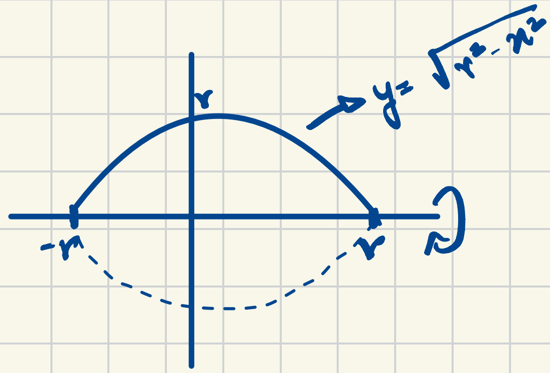


$\Rightarrow \quad V = \int_1^4 \pi \left(f(x)\right)^2 dx \rightarrow \pi \int_1^4 \left(3\sqrt{x}\right)^2 dx$

$\Rightarrow \quad V = \pi \int_1^4 9x \, dx \rightarrow \pi \left[\frac{9x^2}{2}\right]_1^4 \rightarrow \pi \left[\frac{9 \cdot \overset{8}{\cancel{16}}}{\cancel{2}} - \frac{9 \cdot 1}{2}\right]$

$\Rightarrow \quad V = 67.5\pi$

DERIVE VOLUME OF SPHERE



$x^2 + y^2 = r^2$

$y = \pm\sqrt{r^2 - x^2}$

$$V = \int_{-r}^{r} \pi [f(x)]^2 \, dx \rightarrow \pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$\Rightarrow \pi \int_{-r}^{r} r^2 - x^2 \, dx \quad \Rightarrow \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$$

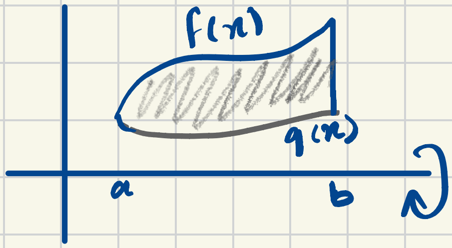
$$\Rightarrow \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right] \Rightarrow \pi \left[r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3} \right]$$

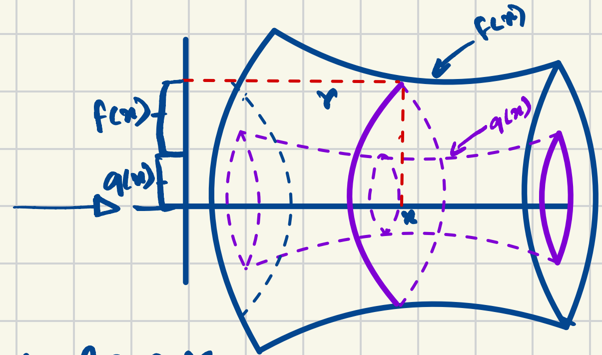
$$\pi \left[2r^3 - \frac{2r^3}{3} \right] \Rightarrow \pi \left[\frac{4r^3}{3} \right]$$

$$\therefore V = \frac{4\pi r^3}{3}$$

Volume By Washers

$A = \pi r^2$





$$V = \int_a^b A(x)\,dx, \quad A(x) = \text{Area of Cross-Section}$$

$$A(x) = A[f(x)] - A[g(x)]$$

$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 \Rightarrow \pi\left[[f(x)]^2 - [g(x)]^2\right]$$

$$\therefore \text{Volume By Washers} = \int_a^b \pi\left[[f(x)]^2 - [g(x)]^2\right] dx$$

Ex Find the volume of the solid created when the area b/w $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ on $[0,2]$ is rotated about x-axis.

$$V = \pi\int_0^2 \left[\tfrac{1}{2}+x^2\right]^2 - [x]^2\,dx \Rightarrow \pi\int_0^2 \tfrac{1}{4} + \cancel{x^2} + x^4 - \cancel{x^2}$$

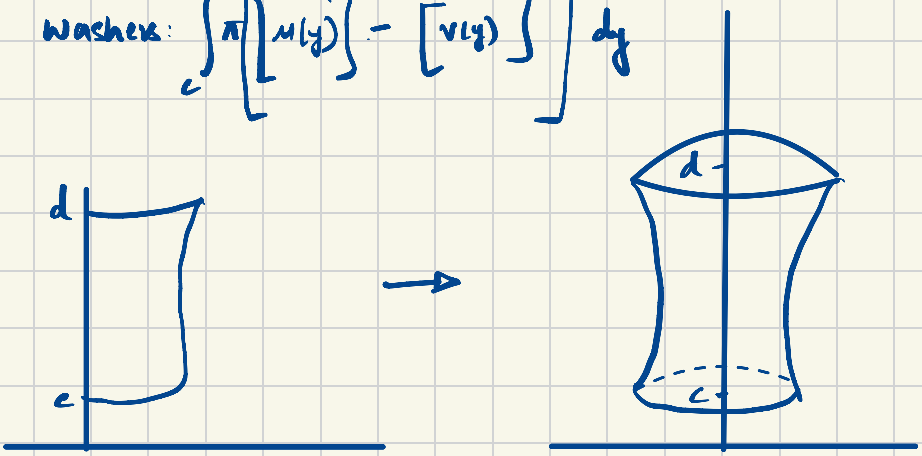
$$\Rightarrow \pi\int_0^2 \tfrac{1}{4} + x^4 \Rightarrow \pi\left[\tfrac{1}{4}x + \tfrac{x^5}{5}\right]_0^2 \Rightarrow \pi\tfrac{1}{2} + \tfrac{2^5}{5}\pi$$

$$V \Rightarrow \frac{69\pi}{10}$$

Volumes where Cross Section is ⊥ to Y-Axis

Disks: $\int_c^d \pi [u(y)]^2 \, dy$

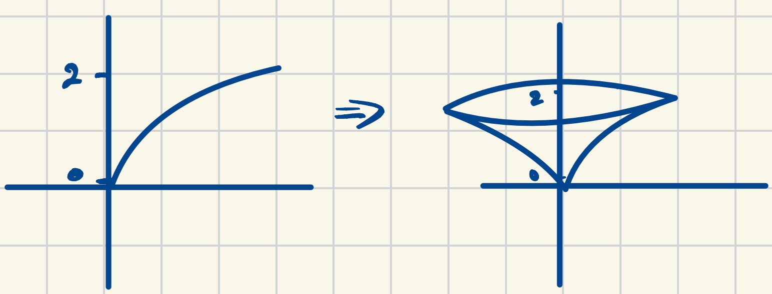
Washers: $\int_c^d \pi \left[[u(y)]^2 - [v(y)]^2 \right] dy$



Ex

$y = \sqrt{x}$ is Revolved around y-axis & Bound by $y=2$ & $y=0$

Find volume.



Note:

* when Revolving about x-axis Put Functions IN Terms of 'x' (Solve For Y)
* when Revolving around y-axis Put Function IN Terms of 'y' (Solve For x)

Eqn in terms of y:

$y = \sqrt{x}$

$y^2 = x$

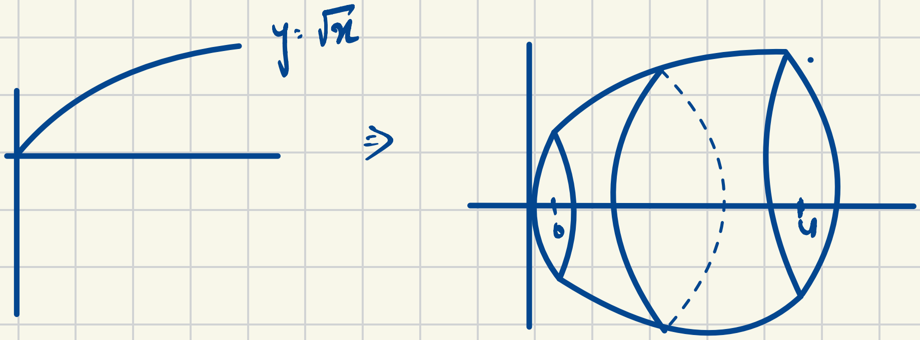
$\therefore x = y^2$

$$V = \int_0^2 \pi [x]^2 \, dy \Rightarrow \pi \int_0^2 [y^2]^2 \, dy \Rightarrow \pi \left[\frac{y^5}{5} \right]_0^2 \Rightarrow \pi \left[\frac{32}{5} \right]$$

$\therefore V = \dfrac{32\pi}{5}$

Here,

For Revolving on X-Axis



Here,

when $y=0$, when $y=2$

$x=0$ $\quad$ $x=4$

$$\int_0^4 \pi (\sqrt{x})^2 \, dx \Rightarrow \pi \int_0^4 x \, dx \Rightarrow \pi \left[\frac{x^2}{2}\right]_0^4$$

$$\Rightarrow \pi 8 \Rightarrow 8\pi$$

FINDING Volume b/w two functions without the given Bounds

Ex: Find the Volume of the solid when the AREA Contained By $y=x^2$ and $y=x^3$ Is Revolved Around x-Axis.

$$V = \int_a^b \pi \left[[f(x)]^2 - [g(x)]^2 \right] dx$$

WHERE Do They Intersect?

$x^3 = x^2 \Rightarrow x^3 - x^2 = 0$

$x^2(x-1) = 0$

$\therefore x=0, x=1$

$$V = \int_0^1 \pi \left[[x^2]^2 - [x^3]^2 \right] dx$$

Which Function is on Top?

$x^2 > x^3$ b/w $[0,1]$

$$V = \pi \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_0^1 \Rightarrow \pi \left[\frac{1}{5} - \frac{1}{7} \right]$$

$$\therefore V \Rightarrow \frac{2}{35}\pi$$

.... $y = x^2$ AND $x = y^2$ AROUND Y AXIS

Solving for x,

$x = y^2$ ✓

$y = x^2 \Rightarrow x = \sqrt{y}$

Here,

Now finding the point both the functions Intersect.

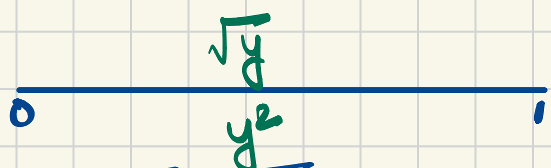
$y^2 = \sqrt{y}$

$y^4 - y = 0$

$y(y^3 - y) = 0$

$y = 0$ $\therefore y = 0, 1$

$y^{\cancel{3}\,2} = \cancel{y}\,1 \Rightarrow y = 1$



$y = 1/2 : \sqrt{y} = \sqrt{1/2}$

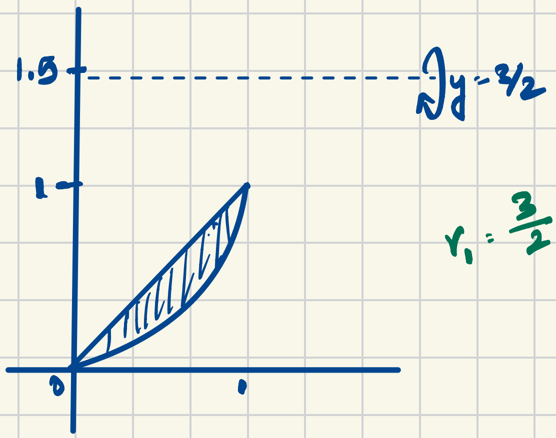
$y = 1/2 : y^2 = \frac{1}{4}$

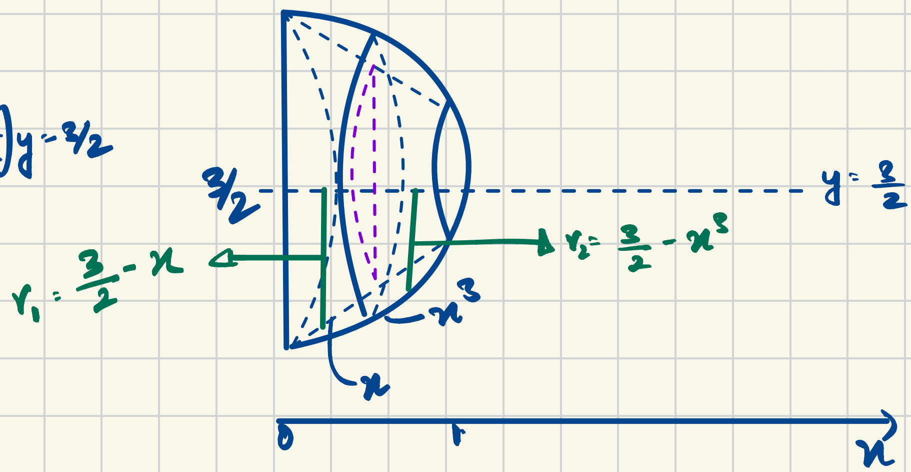
$$V = \int_0^1 \pi\left[[\sqrt{y}]^2 - [y^2]^2\right] dy$$

$$V = \pi\int_0^1 y - y^4 \, dy \Rightarrow \pi\left[\frac{y^2}{2} - \frac{y^5}{5}\right]_0^1$$

$$V = \left[\frac{1}{2} - \frac{1}{5}\right]\pi = \frac{3}{10}\pi$$

Ex: Find the Volume of the solid when the AREA Contained By $y = x$ and $y = x^3$ Is Revolved Around $y = 3/2$





$$V = \int_a^b \pi \left[[f(x)]^2 - [g(x)]^2 \right] dx$$

OR

$$V = \int_a^b \pi [f(x)]^2 - \pi [g(x)]^2 \, dx$$

Radius

Intersect:

$x = x^3$

$x(1 - x^2) = 0$

$x = 0, \; x = 1, -1$

So For Washers like This

outside

Inside

$$V = \int_a^b \pi \left[c - f(x) \right]^2 - \pi \left[c - g(x) \right]^2 dx$$

$$V = \int_0^1 \pi \left(\frac{3}{2} - x^3 \right)^2 - \pi \left(\frac{3}{2} - x \right)^2 dx$$

$$V = \int_0^1 \pi\left(\frac{3}{2} - x^3\right)^2 - \pi\left(\frac{3}{2} - x\right)^2 dx$$

$$V = \pi \int_0^1 \left(\left(\frac{9}{4} - 3x^3 + x^6\right) - \left(\frac{9}{4} - 3x + x^2\right)\right) dx$$

$$V = \pi \left[\cancel{\frac{9}{4}x} - \frac{3x^4}{4} + \frac{x^7}{7} - \cancel{\frac{9}{4}x} + \frac{3x^2}{2} - \frac{x^3}{3}\right]_0^1$$

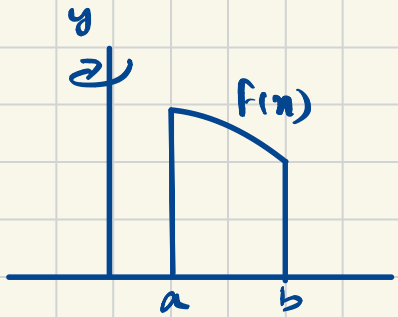
$$V = \pi \left[\frac{3x^2}{2} - \frac{3x^4}{4} + \frac{x^7}{7} - \frac{x^3}{3}\right]_0^1$$

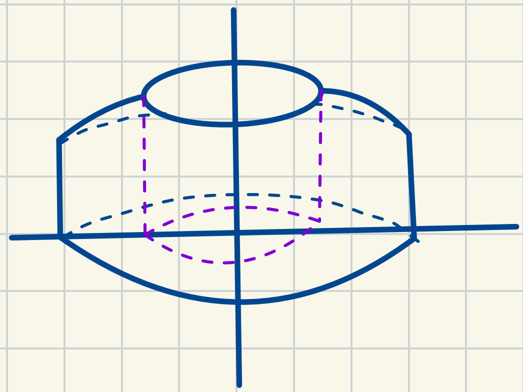
$$V = \pi \left[\left(\frac{3}{2} - \frac{3}{4} + \frac{1}{7} - \frac{1}{3}\right) - 0\right]$$

$$V = 0.56\pi \;//$$

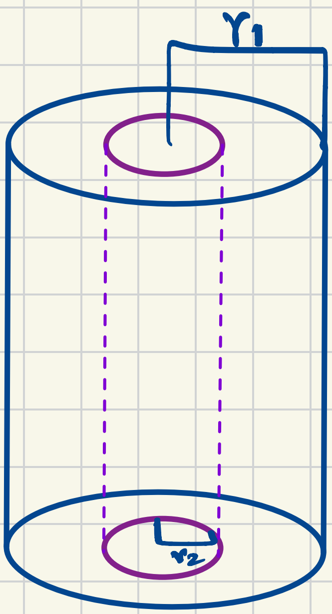
Cylindrical Shells For Volumes

[Future Note: written in terms of x for y-axis & vice-versa.]





For Each Region:



V = AREA of Cross-Section × Height

$$V = \left[\pi r_1^2 - \pi r_2^2\right] \cdot h$$

$$\Rightarrow \pi\left[r_1^2 - r_2^2\right] \cdot h$$

$$\Rightarrow \pi(r_1 + r_2)(r_1 - r_2) \cdot h$$

OR

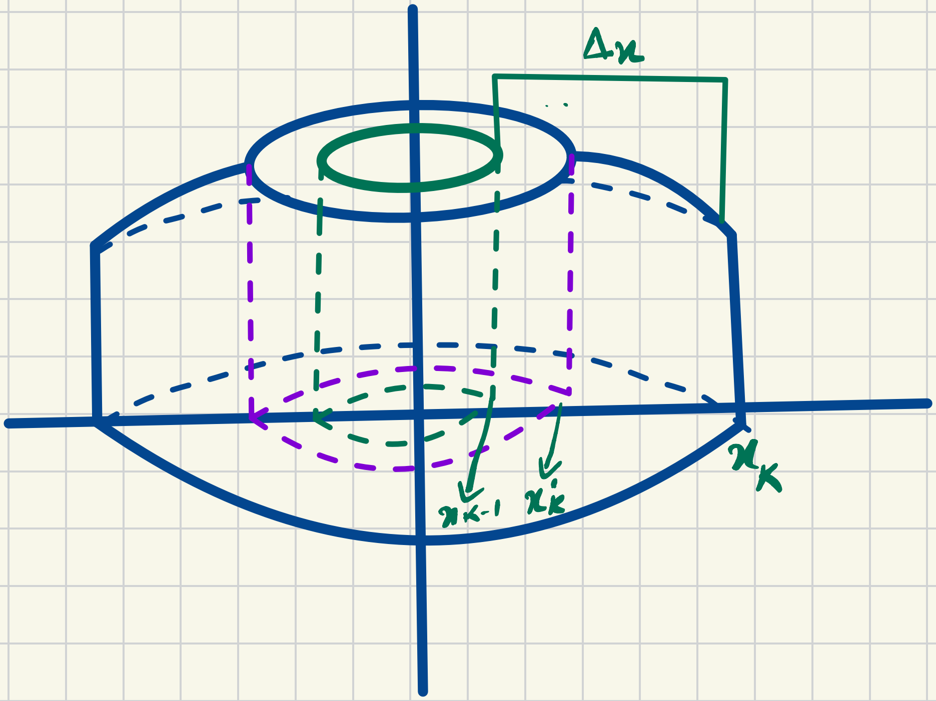
$$\Rightarrow 2\pi \tfrac{1}{2}(r_1 + r_2)(r_1 - r_2)\, h$$

$$V \Rightarrow 2\pi \left[\tfrac{1}{2}(r_1 + r_2)\right] \cdot h \cdot (r_1 - r_2)$$

Avg of radii

Thickness of the Shell

V = 2π * Average Radii * Height * Thickness Radius



- Make a cut at x_k^*, The MIDPOINT OF The Section (Average)

- Thickness is Δx

- Height is $f(x_k^*)$

$\therefore V = 2\pi \cdot$ Average Radii $*$ Height $*$ Thickness

$\Rightarrow 2\pi \cdot x_k^* \cdot f(x_k) * \Delta x$

$\Rightarrow V = \sum_{k=1}^{n} 2\pi (x_k^*) \cdot f(x_k^*) \cdot \Delta x$

OR

$\Rightarrow \lim_{n \to \infty} \sum_{k=1}^{n} 2\pi (x_k) \cdot f(x_k^*) \cdot \Delta x$

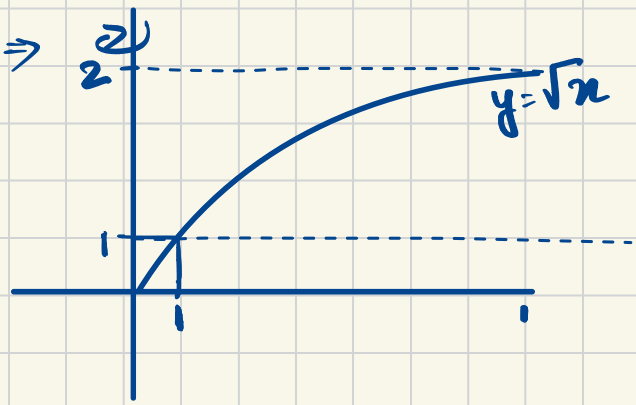
$\therefore V = \int_a^b 2\pi x\, f(x) \cdot dx$ —▷ AROUND Y-AXIS

OR

$V = \int_c^d 2\pi y \cdot g(y) \cdot dy$ —▷ AROUND X-AXIS

CYLINDRICAL SHELLS VOLUME → If Revolve around y, in terms of x [Future note]

Ex: Find The volume of the Region Enclosed by $y = \sqrt{x}$, $x=1$ AND $x=4$ when Revolved Around about y-Axis

⇒ 

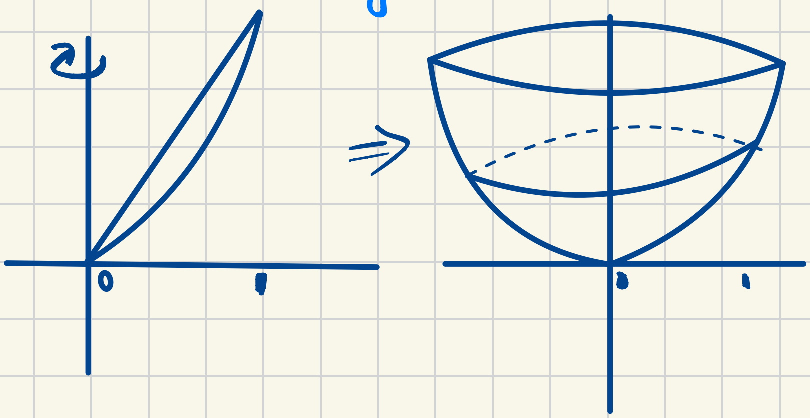
$$V = \int_1^4 2\pi x\sqrt{x} \cdot dx \Rightarrow 2\pi\int_1^4 x^{3/2}\, dx \Rightarrow 2\pi\left[x^{5/2} \cdot \frac{2}{5}\right]_1^4$$

$$\frac{4\pi}{5}\left[x^{5/2}\right]_1^4 \Rightarrow \frac{4\pi}{5}\left[4^{5/2} - 1\right]$$

$$V \Rightarrow \quad \Rightarrow \frac{4\pi}{5} \times 31$$

$$\text{Volume} \Rightarrow \frac{124\pi}{5}$$

Q) Revolve The Region Bound By $y=x$ & $y=x^2$
AROUND The y-axis. Find the The volume



Bounds they Intersect

$x = x^2$

$x - x^2 = 0$

$x(1-x) = 0$

$\boxed{x=0 \quad x=1}$

$x > x^2$ on $[0,1]$

$$V = \int_a^b 2\pi x f(x)\,dx$$

$$V = \int_0^1 2\pi x\left[x - x^2\right] dx$$

$$\Rightarrow 2\pi \int_0^1 x^2 - x^3\,dx$$

$$\Rightarrow 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$\Rightarrow 2\pi \left[\left(\frac{1}{3} - \frac{1}{4}\right) - 0\right]$$

$$\Rightarrow 2\pi \left[\frac{1}{12}\right]$$

$$V \Rightarrow \frac{\pi}{6}$$

Q) Revolve The Region Bound By $x = -y^2 + 6y$ AND $x = 0$ AROUND The x-axis. Find the The volume

$$V = \int_c^d 2\pi y \cdot g(y) \cdot dy$$

$$\Rightarrow \int_0^6 2\pi y \cdot \left[-y^2 + 6y - 0\right] dy$$

$$\Rightarrow 2\pi \int_0^6 -y^3 + 6y^2 \, dy$$

$$\Rightarrow 2\pi \left[-\frac{y^4}{4} + \frac{\cancel{6}^{2} y^3}{\cancel{3}}\right]_0^6$$

$$\Rightarrow 2\pi \left[2y^3 - \frac{y^4}{4}\right]_0^6$$

$$\Rightarrow 2\pi \left(2(6)^3 - \frac{6^4}{4}\right)$$

$\therefore V \Rightarrow 216\pi$

where they Intersect:

$$-y^2 + 6y = 0$$

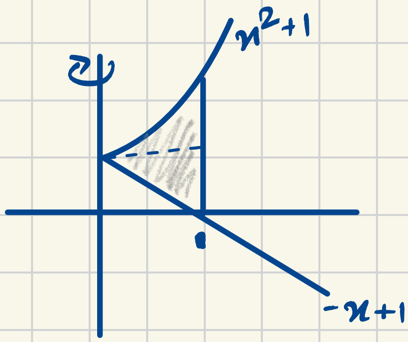
$$y(y-6) = 0$$

$$y = 0, 6$$

$y^2 + 6y$

0

Q) Revolve the Region Bound by $y=x^2+1$, $y=-x+1$ AND $x=1$ AROUND y-Axis



Here,

If you decide to solve this Question by the WASHERS METHOD:

$\Rightarrow$ You would have to Solve three 3 Different Integral with DIFFERENT BOUNDS.

However,

By Doing it through Cylindrical Shells Method:

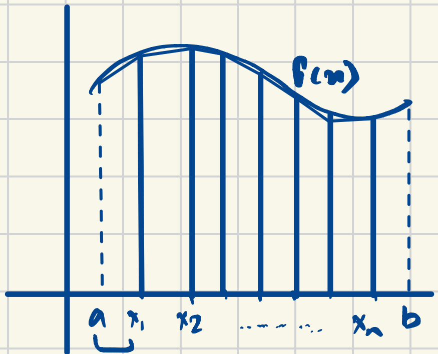
$$V = \int_0^1 2\pi x\left[x^2+1-(-x+1)\right]dx$$

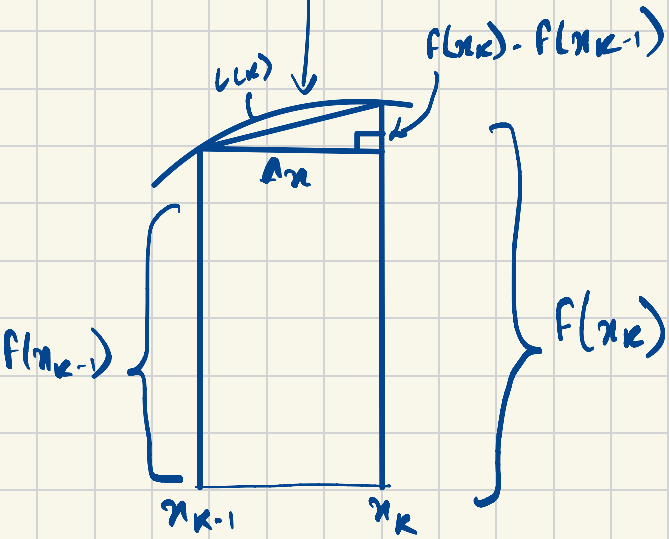
$$V \Rightarrow 2\pi\int_0^1 x\left[x^2+\cancel{1}+x-\cancel{1}\right]dx$$

$$V = 2\pi\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1$$

$$V = 2\pi\left[\frac{1}{4}+\frac{1}{3}\right] \Rightarrow \frac{7\pi}{6}$$

Length of the Curve





$$L_K^2 = \Delta x^2 + \left[f(x_K) - f(x_{K-1})\right]^2 \Rightarrow L_K = \sqrt{\Delta x^2 + [f(x_K) - f(x_{K-1})]^2} \quad \text{— (i)}$$

$$\frac{f(x_K) - f(x_{K-1})}{x_K - x_{K-1}} = f'(x_K^*)$$

ARBITARY Point b/w x_K to x_{K-1}
BY MEAN-VALUE THEOREM

OR

$$\frac{f(x_K) - f(x_{K-1})}{\Delta x} = f'(x_K^*)$$

$$f(x_K) - f(x_{K-1}) = f'(x_K^*)\,\Delta x \quad \text{— (ii)}$$

substituting — (ii) to (i)

$$L_K = \sqrt{\Delta x^2 + [f'(x_K^*) \cdot \Delta x]^2}$$

$$L_K = \sqrt{\Delta x^2 [1 + [f'(x_K)]^2]}$$

$$L_K = \sqrt{1 + f'(x_K)^2} \cdot \Delta x$$

$$\Rightarrow \sum_{K=1}^{n} \sqrt{1 + f'(x_K)^2} \cdot \Delta x$$

$$\Rightarrow \lim_{n \to \infty} \sum_{K=1}^{n} \sqrt{1 + f'(x_K)^2} \cdot \Delta x$$

$$\text{Distance} \Rightarrow \int_a^b \sqrt{1 + [f'(x)]^2} \cdot dx$$

Length of a curve along x-axis

Ex $f(x) = \frac{1}{3}x^3 + \frac{1}{4x}$ on $[1, 3]$

$4x^{-1}$

$-x x^{-1/2}$

Here, $$\int_1^3 \sqrt{1 + [x^2 - 4x^{-2}]^2}\, dx$$

$$\Rightarrow \int_1^3 \sqrt{1 + x^4 - \cancel{2x^2} \cdot \frac{1}{\cancel{4x^2}_{2}} + \left(\frac{-4}{x^2}\right)^2}\, dx$$

$$\Rightarrow \int_1^3 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}}\, dx$$

$$\Rightarrow \int_1^3 \sqrt{\frac{1}{2} + x^4 + \frac{1}{16x^4}}\, dx$$

$$\Rightarrow \int_1^3 \sqrt{\frac{8x^4}{16x^4} + \frac{16x^8}{16x^4} + \frac{1}{16x^4}}\, dx$$

$$\Rightarrow \int_1^3 \sqrt{\frac{16x^8+8x^4+1}{16x^4}}\,dx$$

$$\Rightarrow \int_1^3 \frac{\sqrt{16x^8+8x^4+1}}{\sqrt{16x^4}}\,dx$$

$$\Rightarrow \int_1^3 \frac{\sqrt{16x^8+4x^4+4x^4+1}}{4x^2}$$

$$\int_1^3 \frac{\sqrt{4x^4(4x^4+1)+1(4x^4+1)}}{4x^2}\,dx$$

$$\Rightarrow \int_1^3 \frac{\cancel{\sqrt{(4x^4+1)^2}}}{4x^2}\,dx$$

$$\Rightarrow \int_1^3 x^2+\frac{1}{4x^2}\,dx$$

$$\Rightarrow \left[\frac{x^3}{3}-\frac{1}{4x}\right]_1^3$$

$$\Rightarrow \left(\overset{9}{\cancel{\frac{27}{3}}}-\frac{1}{12}\right)-\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$\Rightarrow \left(9-\frac{1}{12}-\frac{1}{3}+\frac{1}{4}\right)$$

$$\Rightarrow \frac{53}{6}$$

Ex $f(x) = x^{3/2}$ From $(1,1)$ To $(2, 2\sqrt{2})$, Find the length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \cdot dx$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2} \cdot dx$$

$$L = \int_1^2 \sqrt{1 + \frac{9}{4}x} \cdot dx$$

$$L = \int_1^2 \sqrt{\frac{4+9x}{4}} \cdot dx$$

$$L = \frac{1}{2} \int_1^2 \sqrt{4+9x} \cdot dx$$

$$\boxed{\begin{array}{c|l} u = 4+9x & u = 4+9(1) \\ du = 9dx & u_{(1)} = 13 \\ dx = \frac{du}{9} & u(2) = 4+9(2) \\ & u(2) = 22 \end{array}}$$

$$\alpha \Rightarrow \frac{1}{2} \int_1^2 \sqrt{u} \cdot \frac{du}{9}$$

$$\alpha \Rightarrow \frac{1}{18} \int_1^2 u^{1/2} \Rightarrow \frac{1}{18} \left[\frac{2u^{3/2}}{3} \right]_{13}^{22}$$

$$\alpha \Rightarrow 2.09$$

Ex: $y = x^{3/2}$ From $(1,1)$ to $(2, 2\sqrt{2})$, Find length func in y terms

$x = y^{2/3}$, $x' = \frac{2}{3} y^{-\frac{1}{3}}$

$$L = \int_1^{2\sqrt{2}} \sqrt{1 + [g'(x)]^2}\, dy$$

$$L = \int_1^{2\sqrt{2}} \sqrt{1 + \left[\frac{2}{3} y^{-\frac{1}{3}}\right]^2} \cdot dy$$

$$L = \int_1^{2\sqrt{2}} \sqrt{1 + \frac{4}{9} y^{-\frac{2}{3}}} \cdot dy$$

$$L = \int_1^{2\sqrt{2}} \sqrt{\frac{9 + 4y^{-\frac{2}{3}}}{9}} \cdot dy$$

$$L = \frac{1}{3} \int_1^{2\sqrt{2}} \sqrt{\frac{9y^{2/3} + 4}{y^{2/3}}} \cdot dy$$

$$\Rightarrow \frac{1}{3} \int_1^{2\sqrt{2}} \frac{1}{y^{1/3}} \sqrt{9y^{2/3} + 4} \cdot dy$$

$$\Rightarrow \frac{1}{3} \int_{13}^{22} \frac{1}{y^{1/3}} \sqrt{u} \cdot \frac{du}{6y^{-\frac{1}{3}}}$$

$$\frac{1}{3} \int_{13}^{22} \cancel{\frac{1}{y^{1/3}}} \sqrt{u} \cdot \frac{du}{6} \cdot \cancel{y^{1/3}}$$

$u = 9y^{2/3} + 4$

$du = \frac{2}{\cancel{3}} \cdot \cancel{9}^{3} y^{-\frac{1}{3}} + 0\, dy$

$du = 6y^{-\frac{1}{3}}\, dy$

$dy = \frac{du}{6y^{-\frac{1}{3}}}$

when $y = 2\sqrt{2}$

$u \Rightarrow 9\left(2\sqrt{2}\right)^{2/3} + 4 = 22$

when $y = 1$

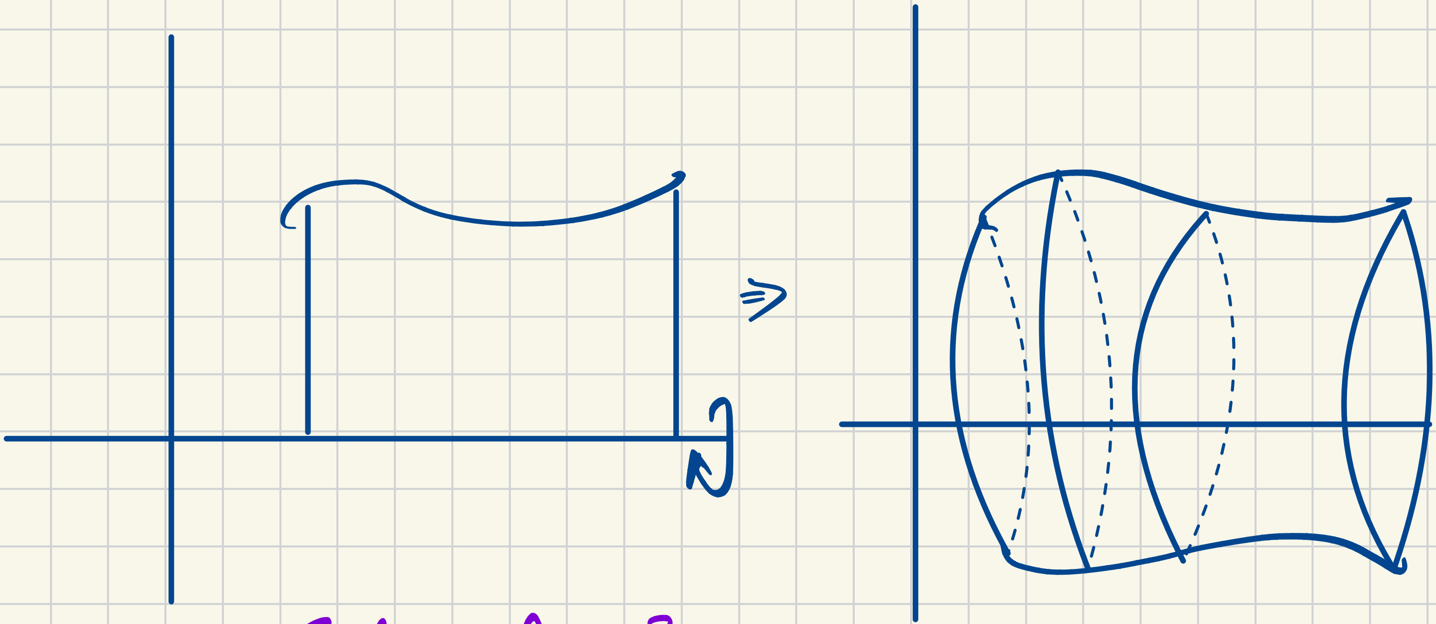
$\Rightarrow 9(1)^{2/3} + 4 = 13$

$$\Rightarrow \frac{1}{18}\int_{13}^{22} \sqrt{u}\, du$$

$$\Rightarrow \frac{1}{18}\left[u^{3/2}\cdot\frac{2}{3}\right]_{13}^{22}$$

$$\Rightarrow \frac{1}{18}\left[(22)^{3/2}\cdot\frac{2}{3} - (13)^{3/2}\cdot\frac{2}{3}\right]$$

$$\approx 2.094$$

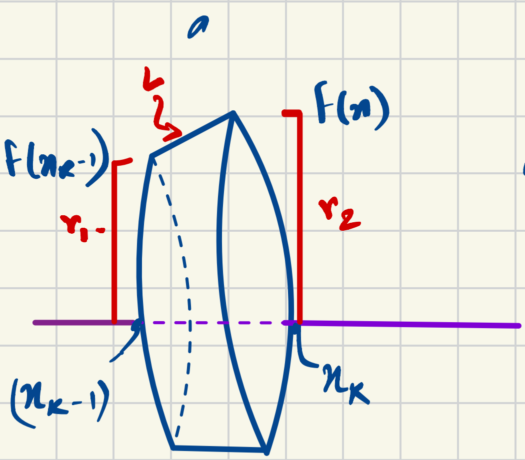


Surface Area = ?

CIRCUMFERENCE OF CIRCLE AT EACH ARBITARY (x_k^i) * LENGTH

OR

$$2\pi \left(f(x_k^i)\right) * \sqrt{1 + f'(x_k^i)}$$



Surface Area of only outward part of the Disk $\Rightarrow \pi(r_{1k} + r_{2k}) \cdot L_k$

$$L_k = \sqrt{\Delta_k^2 + \sqrt{f(x_k) - f(x_{k-1})}}$$

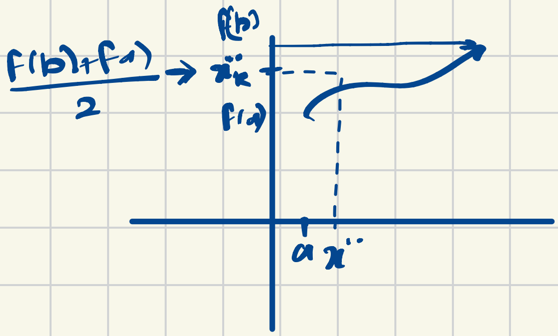
$$SA_k = \pi\left[f(x_k) - f(x_{k-1})\right] \cdot \sqrt{\Delta_k^2 + \sqrt{f(x_k) - f(x_{k-1})}}$$

$$S.A = \pi\left[f(x)+f(x_{k-1})\right] \cdot \sqrt{\Delta_k^2+\left[f'(x_k)\right]^2 \cdot \Delta_x^2}$$

$$= \pi\left[f(x)+f(x_{k-1})\right] \cdot \sqrt{\left[1+\left[f'(x_k)\right]^2\right] \cdot \Delta_k^2}$$

$$= \pi\underbrace{\left[f(x)+f(x_{k-1})\right]}_{2\left[\frac{1}{2}\left[f(x_{k-1})+f(x_k)\right]\right]} \cdot \sqrt{1+\left[f'(x_k)\right]^2} \cdot \Delta x$$

There is some point x_k^{**} in this interval $[x_{k-1}, x_k]$. For which $f(x_k^{**}) = \frac{1}{2}\left[f(x_k)+f(x_k)\right]$



Intermediate Value Theorem

$$\Rightarrow \quad 2\pi f(x_k^{**}) \cdot \sqrt{1+\left[f(x_k^{*})\right]^2} \cdot \Delta x$$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi f(x_k^{**}) \cdot \sqrt{1+\left[f(x_k^{*})\right]^2} \cdot \Delta x$$

OR

$$S.A = \int_a^b 2\pi f(x) \cdot \sqrt{1+\left[f'(x)\right]^2} \cdot \Delta x$$

About x Axis

SURFACE AREA FORMULA

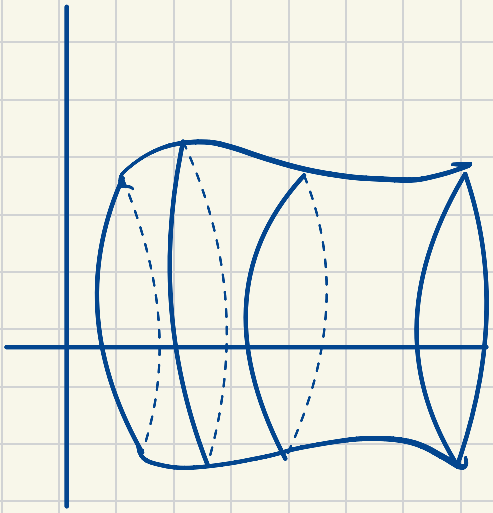
To Sum The Surface Area Proof:

In Short

S.A = Circumference * Length

$\Rightarrow 2\pi r * \text{Length}$

$\Rightarrow 2\pi f(x) * \sqrt{1+[f'(x)]^2} \cdot \Delta x$



OR

$$SA = \int_a^b 2\pi f(x) * \sqrt{1+[f'(x)]^2} \cdot dx$$ → Around x-axis

$$SA = \int_c^d 2\pi g(y) * \sqrt{1+[g'(y)]^2} \cdot dy$$ → Around y-axis

Ex: Find S.A of Revolution of $y=x^3$ Between $x=0$ & $x=1$ when Revolved around x-Axis.

$y = x^2$

$y' = 3x^2$

$$\text{Surface Area} = \int_0^1 2\pi x^3 \cdot \sqrt{1+[3x^2]^2} \cdot dx$$

$$\Rightarrow 2\pi \int_0^1 x^3 \cdot \sqrt{1+9x^4} \cdot dx$$

$$\Rightarrow 2\pi \int_1^{10} \frac{\cancel{x^3} \sqrt{u}}{36\cancel{x^3}}\, du$$

$$\Rightarrow \frac{\cancel{2}^{1}}{3} \cdot \frac{2\pi}{\cancel{36}_{\cancel{18}_{9}}} \Big[u^{3/2} \Big]_1^{10}$$

$$\Rightarrow \frac{\pi}{27} \left[10^{3/2} - 1^{3/2} \right]$$

$$\Rightarrow \frac{\pi}{27} \left[10\sqrt{10} - 1 \right]$$

$$\therefore SA \Rightarrow \frac{10\pi\sqrt{10} - \pi}{27}$$

$u = 1 + 9x^4$,

$du = 36x^3\, dx$

$dx = \dfrac{du}{36x^3}$

when $x=0$, $u=1$

when $x=1$, $u=10$

SA of Revolution Between $x=1$ & $x=2$ Around Y-Axis, $y=x^2$

↳ $x=\sqrt{y}$

$x' = \frac{1}{2}y^{-1/2}$

↳

$$SA \Rightarrow \int_c^d 2\pi g(x) * \sqrt{[1+[g'(y)]^2]}\, dy$$

$$\Rightarrow \int_1^4 2\pi(\sqrt{y}) * \sqrt{[1+(\frac{1}{2y^{1/2}})^2}\, dy$$

$$\Rightarrow 2\pi \int_1^4 \sqrt{y} \cdot \sqrt{1+\left(\frac{1}{4y}\right)}\, dy$$

$$\Rightarrow \pi \int_1^4 \sqrt{4y} \cdot \sqrt{1+\frac{1}{4y}}\, dy$$

$$\Rightarrow \pi \int_1^4 \sqrt{4y+1}\, dy$$

$u = 4y+1$

$du = 4\,dy$

$dy = \frac{du}{4}$

when, $y=1$
$u=5$
when $y=4$
$u=17$

$$\Rightarrow \pi \int_5^{17} \sqrt{u} \cdot \frac{du}{4}$$

$$\Rightarrow \frac{\cancel{2}}{3} \cdot \frac{\pi}{\cancel{4}_2} \left[u^{3/2}\right]_5^{17}$$

$$\Rightarrow \frac{\pi}{6}\left[17^{3/2} - 5^{3/2}\right]$$

$$\Rightarrow \frac{\pi}{6}\left[\sqrt{17^3} - \sqrt{5^3}\right]$$

$$\Rightarrow \frac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right]$$

$$\Rightarrow \quad \frac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right]$$

$$\Rightarrow \quad \frac{17\pi\sqrt{17} - 5\pi\sqrt{5}}{6}$$