Analytical tractability and computability of the phase diagram of ionic compounds in strong electric fields

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Abstract

1 Introduction

2 Gibbs ensemble of ions in different phases

In this article, one explains the statistical mechanics of the ions in the isothermal-isobaric ensemble, also known as the Gibbs ensemble, well described in [4, 5]. This ensemble of particles is in a container which is maintained at constant temperature T and pressure P, and has fixed number of particles N. The partition function for the Gibbs ensemble can be computed from that of the canonical ensemble as follows [4, 5],

$$Z(N, P, T) = \int_{V=0}^{V=\infty} \frac{PdV}{k_B T} e^{-\frac{PV}{k_B T}} Z_c(N, V, T) \tag{1}$$

where k_B is the Boltzmann constant and Z_c the partition function for the canonical ensemble, which in turn can be computed from the Hamiltonian $H(\mathbf{r}, \mathbf{p})$ in conservative systems,

$$Z_c(N, V, T) = \frac{1}{h^{3N} C} \int d^{3N} \mathbf{r} \int d^{3N} \mathbf{p} \quad exp\left(-\frac{H(\mathbf{r}, \mathbf{p})}{k_B T} \right) \tag{2}$$

where h is the Planck's constant introduced to ensure non-dimensionalisation of the partition function and C is the over-counting factor. Note that the integration of the position variables happen over the 'available volume' to the particles from the total volume V of the system. After having computed the partition

function, one can extract the complete thermodynamics from the Gibbs' free energy,

$$G(N, P, T) = -k_B T log(Z(N, P, T)) \tag{3}$$

and the Helmholtz free energy $F = G - PV$. In the subsequent sections, we will consider the different assumptions for the Hamiltonian of the ionic compound in its different phases in the presence of strong background electric field and compute the Gibbs partition function from this to prove its analytical tractability, i.e. the existence of a closed-form expression in terms of all the variables in our parameter space which eradicates the necessity to compute the partition function by numerical integration techniques such as Monte Carlo stochastic sampling.

2.1 Solid state thermodynamics

In this subsection and the next, we assume for simplicity an electric field E_0 in the x-direction of the presumed anisotropy (isotropy for the fluid phases) of the arrangement of ions (prove the choice of the electric field here). About the arrangement of ions, the case of a solid salt, with simple cubic unit cell propagating throughout its crystallography, is considered, for example caesium chloride (CsCl). The radius of caesium ions is 174 pm and those of the chloride ions is 181 pm. One can easily show that the lattice constant computed from the radii and the geometry is $a = 419$ pm. For generalisation to similar binary ionic compounds with the anions and cations of same magnitude of charge, I would assume the *charge* to be q (for instance, in the case of CsCl, $q = 1.6 \times 10^{-19}$ C).

Before proceeding further, it is important to enunciate on the assumption of the *strong electric field*. Throughout this article, until and unless stated, the underlying assumption is that the electric field E_0 is way larger than the characteristic electrostatic forces of the ions, i.e.

$$E_0 >> \frac{q}{4\pi\epsilon_0 a^2} \tag{4}$$

where $\epsilon_0 = 8.85 \times 10^{-12}$ Fm^{-1} is the vacuum permittivity. Plugging in the numbers one obtains $\frac{q}{4\pi\epsilon_0 a^2} = 8.31 \times 10^8$ NC^{-1}. The software which was written in [2] to simulate and calculate the models discussed now, or otherwise, considered electric fields of the order of 10^{15}-10^{20} NC^{-1}. In this regime, we will *not be considering the influence of Coulombic forces* in the Hamiltonian in this case (or even in the fluid case). Also, I am ignoring the oscillatory modes of the crystal lattice therewith.

The Hamiltonian of this system is fairly simple; $H_- = \frac{-qE_0na}{k_BT}$ for the negative ions and $H_+ = \frac{qE_0na}{k_BT}$ for the positive ions, $0 \leq n \leq \lfloor N^{\frac{1}{3}} \rfloor - 1$. Thus the

canonical partition function (including the correction factors)

$$Z_c(N, V, T) = \frac{N^{\frac{2}{3}}}{\Gamma(\frac{N}{2} + 1)^2} \sum_{n=0}^{\lfloor N^{\frac{1}{3}} \rfloor - 1} e^{-\frac{qE_0 na}{k_B T}} + e^{\frac{qE_0 na}{k_B T}} \tag{5}$$

$$\implies Z_c(N, V, T) = \frac{2N^{\frac{2}{3}}}{\Gamma(\frac{N}{2} + 1)^2} \sum_{n=0}^{\lfloor N^{\frac{1}{3}} \rfloor - 1} sinh\left(\frac{qE_0 na}{k_B T}\right)$$

$$\implies Z_c(N, V, T) = \frac{N^{\frac{2}{3}}}{\Gamma(\frac{N}{2} + 1)^2} \left(\frac{cosh(\frac{qE_0 a}{2k_B T}(2N^{\frac{1}{3}} - 1)) - cosh(\frac{qE_0 a}{2k_B T})}{sinh(\frac{qE_0 a}{2k_B T})} \right)$$

As such, there is no explicit dependence of the canonical partition function on the volume; the implicit relationship between V and N is dictated by the symmetric geometry, $V = Na^3$. This simplifies the calculation of the Gibbs partition function,

$$Z(N, P, T) = \int_{V=0}^{V=\infty} \frac{PdV}{k_B T} e^{-\frac{PV}{k_B T}} Z_c(N, V, T)$$

$$\rightarrow Z(N, P, T) = \int_{V=0}^{V=\infty} \frac{PdV}{k_B T} e^{-\frac{PV}{k_B T}} \frac{N^{\frac{2}{3}}}{\Gamma(\frac{N}{2} + 1)^2} \left(\frac{cosh(\frac{qE_0 a}{2k_B T}(2N^{\frac{1}{3}} - 1)) - cosh(\frac{qE_0 a}{2k_B T})}{sinh(\frac{qE_0 a}{2k_B T})} \right)$$

$$\rightarrow Z(N, P, T) = \frac{N^{\frac{2}{3}}}{\Gamma(\frac{N}{2} + 1)^2} \left(\frac{cosh(\frac{qE_0 a}{2k_B T}(2N^{\frac{1}{3}} - 1)) - cosh(\frac{qE_0 a}{2k_B T})}{sinh(\frac{qE_0 a}{2k_B T})} \right)$$

; The independence of the Gibbs' partition function of the pressure is reminiscent of the solid phase is resistant to external pressure. The usual values of this function has enormous exponents so the log-log plots of the same have been plotted (an example being the figure 1) and explored in [2] and is highly recommended for the reader to explore the symbolic computation. As one can see, the computation of the partition function is fairly trivial in the solid phase, which becomes a bit more involved in the case of fluid phases as is described in the next subsection.

2.2 Partition function for the molten salts and salt vapours

As mentioned above, the Hamiltonian of the system will not be including the electrostatic forces between the ions in the strong electric field regime. The energy for the system (i.e. the Hamiltonian) of the molten salts will be described with the following physical assumption; since there is a translational degree of freedom with a restriction of viscous forces against the ions, the anions rearrange
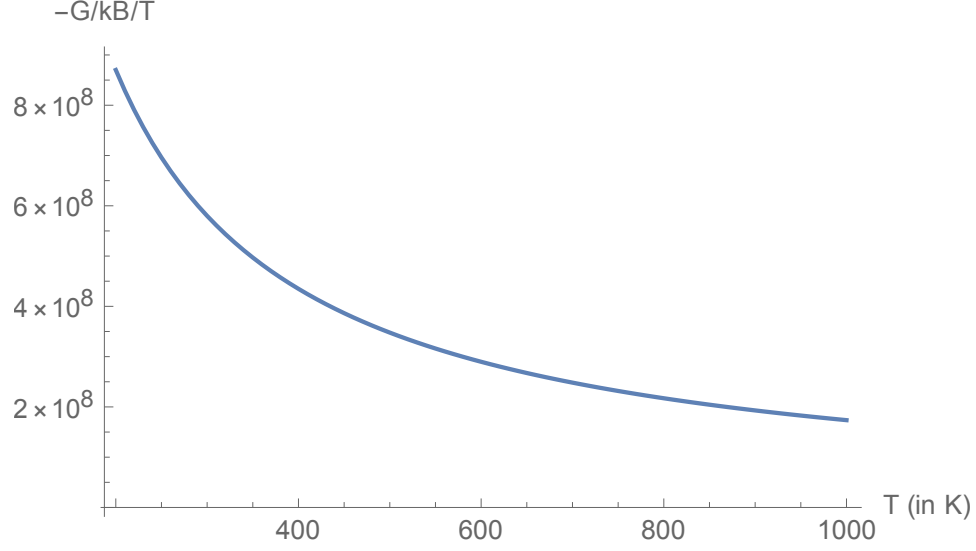
Figure 1: The plot of the non-dimensionalized Gibbs' free energy $\frac{-G}{k_B T}$ against the temperature T. Here $N = 5 \times 10^4$, $q = 1.6 \times 10^{-19}$ C, $E_0 = 10^{15}$ NC^{-1} and $a = 419$ pm

themselves away from the electric field E_0, i.e. in the regime $0 \leq x \leq \frac{L}{2}$ and the cations rearrange themselves from the electric field, i.e. in the regime $\frac{L}{2} \leq x \leq L$. The Hamiltonian for the molten salts is given as follows:

$$H = \frac{\mathbf{p} \cdot \mathbf{p}}{2m} - qE_0 x, \forall 0 \leq x \leq \frac{L}{2} \tag{6}$$

and

$$H = \frac{\mathbf{p} \cdot \mathbf{p}}{2m} + qE_0 x, \forall \frac{L}{2} \leq x \leq L \tag{7}$$

where $L = V^{\frac{1}{3}}$. The canonical partition function $Z_{c,-}$ for one single anion,

$$Z_{c,-} = \frac{1}{h^3} \int_{p_x=-\infty}^{\infty} \int_{p_y=-\infty}^{\infty} \int_{p_z=-\infty}^{\infty} \int_V dp_x dp_y dp_z d^3\mathbf{r} \quad e^{-\frac{H(\mathbf{r},\mathbf{p})}{k_B T}}$$

$$= \frac{1}{h^3} \left(\int_{p=-\infty}^{p=\infty} dp \quad e^{-\frac{p^2}{2mk_B T}} \right)^3 L^2 \int_{x=0}^{x=\frac{L}{2}} dx \quad e^{\frac{qE_0 x}{k_B T}}$$

$$\rightarrow Z_{c,-} = \frac{k_B T L^2}{qE_0} \left(\frac{2\pi m k_B T}{h^2} \right)^{\frac{3}{2}} \left(e^{\frac{qE_0 L}{2k_B T}} - 1 \right)$$

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Similarly for the canonical partition function $Z_{c,+}$ for one single cation,

$$Z_{c,+} = \frac{1}{h^3} \int_{p_x=-\infty}^{\infty} \int_{p_y=-\infty}^{\infty} \int_{p_z=-\infty}^{\infty} \int_V dp_x dp_y dp_z d^3\mathbf{r} \quad e^{-\frac{H(\mathbf{r},\mathbf{p})}{k_B T}}$$

$$= \frac{1}{h^3} \left(\int_{p=-\infty}^{p=\infty} dp \quad e^{-\frac{p^2}{2mk_B T}} \right)^3 L^2 \int_{x=\frac{L}{2}}^{x=L} dx \quad e^{\frac{-qE_0 x}{k_B T}}$$

$$\rightarrow Z_{c,+} = \frac{k_B T L^2}{qE_0} \left(\frac{2\pi m k_B T}{h^2} \right)^{\frac{3}{2}} \left(e^{\frac{-qE_0 L}{2k_B T}} - e^{\frac{-qE_0 L}{k_B T}} \right)$$

From the above simple canonical partition functions, one can extend these to compute the complete canonical partition function.

$$Z_c(N,V,T) = (Z_{c,+})^{\frac{N}{2}} + (Z_{c,-})^{\frac{N}{2}}$$

$$= \left(\frac{k_B T L^2}{qE_0} \right)^{\frac{N}{2}} \left(\frac{2\pi m k_B T}{h^2} \right)^{\frac{3N}{2}} \left(\left(e^{\frac{-qE_0 L}{2k_B T}} - e^{\frac{-qE_0 L}{k_B T}} \right)^{\frac{N}{2}} + \left(e^{\frac{qE_0 L}{2k_B T}} - 1 \right)^{\frac{N}{2}} \right)$$

Finally I will take into account the over-counting factor $\left(\frac{N}{2} \right)! \left(\frac{N}{2} \right)! = \Gamma(\frac{N}{2} + 1)^2$.

$$Z_c(N,V,T) = \frac{V^N}{\Gamma(\frac{N}{2}+1)^2} \left(\frac{k_B T}{qE_0} \right)^{\frac{N}{2}} \left(\frac{2\pi m k_B T}{h^2} \right)^{\frac{3N}{2}} \left(\left(e^{\frac{-qE_0 L}{2k_B T}} - e^{\frac{-qE_0 L}{k_B T}} \right)^{\frac{N}{2}} + \left(e^{\frac{qE_0 L}{2k_B T}} - 1 \right)^{\frac{N}{2}} \right)$$

$$Z_c(N,L,T) = \frac{L^{3N}}{\Gamma(\frac{N}{2}+1)^2} \left(\frac{k_B T}{qE_0} \right)^{\frac{N}{2}} \left(\frac{2\pi m k_B T}{h^2} \right)^{\frac{3N}{2}} \left(\left(e^{\frac{-qE_0 L}{2k_B T}} - e^{\frac{-qE_0 L}{k_B T}} \right)^{\frac{N}{2}} + \left(e^{\frac{qE_0 L}{2k_B T}} - 1 \right)^{\frac{N}{2}} \right)$$

We come back to the definition of the Gibbs' partition function $Z(N,P,T) = \int dV \quad Z_c(N,V,T)e^{-\frac{PV}{k_B T}}$. I think it is convenient to resort to the reminiscent algebraic transformation $V = L^3 \rightarrow dV = 3L^2 dL$. Therefore the Gibbs' partition function for the molten salt can be calculated as,

$$Z(N,P,T) = \int_{L=0}^{L=\infty} dL \quad \frac{3L^{3N+2}}{\Gamma(\frac{N}{2}+1)^2} \left(\frac{k_B T}{qE_0} \right)^{\frac{N}{2}}$$

$$\left(\frac{2\pi m k_B T}{h^2} \right)^{\frac{3N}{2}} \left(\left(e^{\frac{-qE_0 L}{2k_B T}} - e^{\frac{-qE_0 L}{k_B T}} \right)^{\frac{N}{2}} + \left(e^{\frac{qE_0 L}{2k_B T}} - 1 \right)^{\frac{N}{2}} \right)$$

The partition function linearly depends on the integrals,

$$I_1 = \int_{L=0}^{L=\infty} dL \quad L^{3N+2}\left(e^{\frac{-qE_0L}{2k_BT}} - e^{\frac{-qE_0L}{k_BT}}\right)^{\frac{N}{2}} = \int_{L=0}^{L=\infty} dL \quad L^{3N+2}e^{\frac{-NqE_0L}{k_BT}}\left(1 - e^{\frac{-qE_0L}{k_BT}}\right)^{\frac{N}{2}}$$

$$I_2 = \int_{L=0}^{L=\infty} dL \quad L^{3N+2}\left(e^{\frac{qE_0L}{2k_BT}} - 1\right)^{\frac{N}{2}} = \int_{L=0}^{L=\infty} dL \quad L^{3N+2}e^{\frac{NqE_0L}{k_BT}}\left(1 - e^{\frac{-qE_0L}{2k_BT}}\right)^{\frac{N}{2}}$$

The two integrals look similar and can be computed by similar methods (as is explained a little further) but cannot be directly related by trivial transformation of the variables. Both of them are otherwise intractable at this stage, which could be proven by Liouville's theorem [2, 6, 7]. But one can simplify one step further from here using binomial theorem.

$$I_1 = \int_{L=0}^{L=\infty} dL \quad L^{3N+2}\left(e^{\frac{-qE_0L}{2k_BT}} - e^{\frac{-qE_0L}{k_BT}}\right)^{\frac{N}{2}} = \int_{L=0}^{L=\infty} dL \quad L^{3N+2}e^{\frac{-NqE_0L}{k_BT}}\left(1 - e^{\frac{-qE_0L}{k_BT}}\right)^{\frac{N}{2}}$$

$$= \int_{L=0}^{L=\infty} dL \quad L^{3N+2}e^{\frac{-NqE_0L}{k_BT}}\sum_{k=0}^{\frac{N}{2}}\binom{\frac{N}{2}}{k}(-1)^k e^{\frac{-kqE_0L}{k_BT}}$$

$$\rightarrow I_1 = \sum_{k=0}^{\frac{N}{2}}\binom{\frac{N}{2}}{k}(-1)^k \int_{L=0}^{L=\infty} dL \quad L^{3N+2}e^{\frac{-(N+k)qE_0L}{k_BT}}$$

Now the simplification of the integral [2] leads to,

$$\int_{L=0}^{L=\infty} dL \quad L^{3N+2}e^{\frac{-(N+k)qE_0L}{k_BT}} = \left(\frac{(N+k)qE_0}{k_BT}\right)^{-3(N+1)}\Gamma(3N+3)$$

Thus,

$$I_1 = \sum_{k=0}^{\frac{N}{2}}\binom{\frac{N}{2}}{k}(-1)^k\left(\frac{(N+k)qE_0}{k_BT}\right)^{-3(N+1)}\Gamma(3N+3)$$

Similarly, I can compute the other integral to be the following.

$$I_2 = \int_{L=0}^{L=\infty} dL \quad L^{3N+2}\left(e^{\frac{qE_0L}{2k_BT}} - 1\right)^{\frac{N}{2}} = \int_{L=0}^{L=\infty} dL \quad L^{3N+2}e^{\frac{NqE_0L}{k_BT}}\left(1 - e^{\frac{-qE_0L}{2k_BT}}\right)^{\frac{N}{2}}$$

$$= \int_{L=0}^{L=\infty} dL \quad L^{3N+2}e^{\frac{NqE_0L}{k_BT}}\sum_{k=0}^{\frac{N}{2}}\binom{\frac{N}{2}}{k}(-1)^k e^{\frac{-kqE_0L}{2k_BT}}$$

$$\rightarrow I_2 = \sum_{k=0}^{\frac{N}{2}} \binom{\frac{N}{2}}{k} (-1)^k \int_{L=0}^{L=\infty} dL \ L^{3N+2} e^{\frac{(2N-k)qE_0L}{2k_BT}}$$

Now the above integral diverges, as $0 \leq k \leq \frac{N}{2}$, so we place a cutoff volume $L_c = V_c^{\frac{1}{3}}$,

$$I_2 = \sum_{k=0}^{\frac{N}{2}} \binom{\frac{N}{2}}{k} (-1)^k \int_{L=0}^{L=L_c} dL \ L^{3N+2} e^{\frac{(2N-k)qE_0L}{2k_BT}}$$

$$\rightarrow I_2 = \sum_{k=0}^{\frac{N}{2}} \binom{\frac{N}{2}}{k} (-1)^{n+k} \left(\frac{qE_0(2n-k)}{2k_BT} \right)^{-3(n+1)} \left(\Gamma\left(3n+3, \frac{-qE_0L_c(2n-k)}{2k_BT}\right) - \Gamma(3n+3) \right)$$

Thus the liquid state Gibbs' partition function is given by,

$$Z(N,P,T) = \frac{3}{\Gamma(\frac{N}{2}+1)^2} \left(\frac{k_BT}{qE_0} \right)^N \left(\frac{2\pi m k_B T}{h^2} \right)^{\frac{3N}{2}} \sum_{k=0}^{\frac{N}{2}-1} \binom{\frac{N}{2}}{k} \left[(-1)^k \left(\frac{(N+k)qE_0}{k_BT} \right)^{-3(N+1)} \right.$$

$$\left. \Gamma(3N+3) + (-1)^{k+n} \left(\frac{(2N-k)qE_0}{k_BT} \right)^{-3(N+1)} \left\{ \Gamma\left(3N+3, \frac{-qE_0L_c(2N-k)}{2k_BT}\right) - \Gamma(3N+3) \right\} \right]$$

3   Monte Carlo methods to compute the partition function and phase transitions

4   Conclusion

References

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