MA-203 Assignment-2 →

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Sol^n 1) Given,

$$\phi(x) = \begin{cases} x & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Taking fourier transform on both sides →

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \phi(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x e^{i\omega x}\,dx \quad — ①$$

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x\{\cos\omega x + i\sin\omega x\}\,dx = \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1} x\cos(\omega x)\,dx + i\int_{-1}^{1} x\sin(\omega x)\,dx\right]$$

Since $x\cos x$ is odd function & $x\sin x$ is even function

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}}\left[0 + 2i\int_{0}^{1} x\sin(\omega x)\,dx\right]$$

$$\hat{\phi}(\omega) = \frac{2i}{\sqrt{2\pi}}\left\{\left[\frac{-x\cos\omega x}{\omega} + \frac{\sin\omega x}{\omega^2}\right]_0^1\right\} = \frac{2i}{\sqrt{2\pi}}\left\{\frac{-\cos\omega}{\omega} + \frac{\sin\omega}{\omega^2}\right\}$$

$$\hat{\phi}(\omega) = i\sqrt{\frac{2}{\pi}}\left\{\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right\} \quad — ②$$

Now from eq^n ①

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i\omega x}\phi(x)\,dx$$

$$\hat{\phi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x\,dx \Rightarrow 0 \quad \text{as } x \text{ is a odd function.}$$

$$\therefore \boxed{\hat{\phi}(0) = 0}$$ Ans of (i) part

Also from eq^n (2)

$$\hat{\phi}(\omega) = i\sqrt{\frac{2}{\pi}}\left\{\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right\}$$

Taking inverse fourier transform on both sides

$$F^{-1}(\hat{\phi}(\omega)) = \phi(x)$$

Since by parseval's identity we have,

$$\int_{-\infty}^{\infty} |\hat{\phi}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |\phi(x)|^2 dx \qquad \text{where } |f(x)| = f(x)\cdot\bar{f}(x)$$

$$\Rightarrow \int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\, i\left\{\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right\}\right)^2 d\omega = \int_{-1}^{1} x^2 dx$$

$$\Rightarrow \int_{-\infty}^{\infty} i^2\left(\frac{2}{\pi}\right)\left\{\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right\}^2 \cdot d\omega = \int_{-1}^{1} x^2 dx$$

$$\Rightarrow \frac{2}{\pi}\left\{\frac{\omega\cos\omega - \sin\omega}{\omega^2}\right\}^2 d\omega = 2\int_{0}^{1} x^2 dx \qquad (\text{as } x^2 \text{ is an even function})$$

$$\Rightarrow \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\omega\cos\omega - \sin\omega}{\omega^2}\right)^2 d\omega = \frac{1}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty}\left(\frac{\omega\cos\omega - \sin\omega}{\omega^2}\right)^2 d\omega = \frac{\pi}{3}$$

Replacing ω with ξ we get.

$$\boxed{\int_{-\infty}^{\infty}\left(\frac{\xi\cos\xi - \sin\xi}{\xi^2}\right)^2 d\xi = \frac{\pi}{3}}$$

Ans of (ii) part.

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Sol^n 2) Yes, Fourier transform of a constant function exists.

Proof :→ Consider a constant function, like →

$$K(x) = C$$

Using Linearity this function's fourier transform can we written as →

$$F(K(x)) = C\,F(1) \quad — \quad (1)$$

Therefore if fourier transform exist for 1 then fourier transform of $C(x)$ also exist.

Now we can write,

$$g(x) = 1 = f(x) + h(x) \quad — (2)$$

where $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$, $h(x) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 1 \end{cases}$

Let $F(\omega)$ & $H(\omega)$ be fourier transform of $f(x)$ & $h(x)$ respectively.

Let's have a function,

$$f_a(x) = \begin{cases} e^{-ax} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

So, $f(x) = \lim_{a \to 0} f_a(x)$

$$F(\omega) = \lim_{a\to 0} F(f_a)(\omega) = \lim_{a \to 0} \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{+i\omega x} \cdot e^{-ax}\, dx$$

$$\Rightarrow \quad F(\omega) = \frac{1}{\sqrt{2\pi}} \lim_{a \to 0} \left(\frac{1}{a - i\omega}\right)$$

by Rationalizing we get

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \lim_{a \to 0} \left(\frac{a}{a^2+\omega^2} + i\frac{\omega}{a^2+\omega^2}\right)$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\left(\pi\,\delta(\omega) + \frac{i}{\omega}\right)$$

where $\delta(\omega) = \lim_{a \to 0} \frac{1}{\pi}\left(\frac{a}{a^2+\omega^2}\right)$

↓

by def. of dirac delta function.

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Similarly for $H(\omega)$,

where, $h_a(x) = \begin{cases} 0 & x > 0 \\ e^{ax} & x \leq 0 \end{cases}$

$$H(\omega) = \lim_{a \to 0} F(h_a)(\omega)$$

~~where~~ $H(\omega) = \frac{1}{\sqrt{2\pi}} \lim_{a \to 0} \int_{-\infty}^{0} e^{i\omega x} \cdot e^{ax} dx$

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \lim_{a \to 0} \left(\frac{1}{a + i\omega} \right) = \frac{1}{\sqrt{2\pi}} \lim_{a \to 0} \left(\frac{a}{a^2 + \omega^2} - i \times \frac{\omega}{a^2 + \omega^2} \right)$$

$$\Rightarrow \quad H(\omega) = \frac{1}{\sqrt{2\pi}} \left(\pi \delta(\omega) - \frac{i}{\omega} \right)$$

Now, from eq^n (2) we have,

$$g(x) = f(x) + h(x)$$

$$F(g(x)) = F(\omega) + H(\omega)$$

$$= \frac{1}{\sqrt{2\pi}} \left(2\pi \delta(\omega) \right)$$

$$F(g(x)) = \sqrt{2\pi}\, \delta(\omega)$$

from eq^n (1)

$$F(K(x)) = c\, F(1) = c\, F(g(x))$$

$$\Rightarrow F(K(x)) = \sqrt{2\pi}\, \delta(\omega) . c$$

$$\Rightarrow \boxed{F(c) = \sqrt{2\pi}\, c\, \delta(\omega)}$$ Ans.

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Sol^n 3)

Given eq^n →

$u_t = \alpha u_{xx} \Rightarrow \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ — (1)

& the boundary condition given is →

$u(x,0) = \phi(x)$ — (2)

Let $\bar{u}(\omega,t)$ be fourier transform of $u(x,t)$ then,

Taking fourier transform of eq^n (1)

$\frac{\partial \bar{u}}{\partial t} = \alpha(-i\omega)^2 \bar{u}$ $\qquad F\left(\frac{d^n y}{dx^n}\right) = (-i\omega)^n F(y)$

$\frac{\partial \bar{u}}{\bar{u}} = -\alpha \omega^2 \partial t \Rightarrow \ln \bar{u}(\omega,t) = \alpha\omega^2 t + K$ — (3)

$\hookrightarrow$ independent of t

apply Fourier transformation on both sides of eq^n (2)

$\bar{u}(\omega,0) = \hat{\phi}(\omega)$

from eq^n (3)

$\ln \bar{u}(\omega,0) = K \Rightarrow \hat{\phi}(\omega) = e^K$

$\Rightarrow \quad \ln \bar{u}(\omega,t) = -\alpha\omega^2 t + \ln \hat{\phi}(\omega)$

$\bar{u}(\omega,t) = e^{\ln \hat{\phi}(\omega)} \cdot e^{-\alpha\omega^2 t}$

$\bar{u}(\omega,t) = \hat{\phi}(\omega) \cdot e^{-\alpha\omega^2 t}$ — (4)

if $f(x)$ & $g(x)$ be inverse fourier transform of $\hat{\phi}(\omega)$ & $e^{-\alpha\omega^2 t}$

by convolution theorem →

$F^{-1}(\hat{\phi}(\omega) \cdot e^{-\alpha\omega^2 t}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) \cdot g(x-\xi)\, d\xi$ — (5)

$f(x) = F^{-1}(\hat{\phi}(\omega)) = \phi(x)$

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$$g(x) = F^{-1}\left(e^{-\alpha\omega^2 t}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \cdot e^{-\alpha\omega^2 t} \, d\omega$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha t \times \left(\omega^2 + \frac{ix\omega}{\alpha t}\right)} \cdot d\omega$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha t\left[\left(\omega + \frac{i\omega}{2\alpha t}\right)^2 - \left(\frac{ix}{2\alpha t}\right)^2\right]} \cdot d\omega$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4\alpha t}} \cdot e^{-\alpha t\left(\omega + \frac{ix}{2\alpha t}\right)^2} \cdot d\omega$$

Let $\sqrt{\alpha t}\left(\omega + \frac{ix}{2\alpha t}\right) = h \Rightarrow d\omega = \frac{dh}{\sqrt{\alpha t}}$

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{4\alpha t}} \cdot \int_{-\infty}^{\infty} \frac{e^{-h^2}}{\sqrt{\alpha t}} dh \Rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{4\alpha t}} \times \frac{\sqrt{\pi}}{\sqrt{\alpha t}}$$

as $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$g(x) = \frac{1}{\sqrt{2\alpha t}} \times e^{-\frac{x^2}{4\alpha t}}$$

from eq^n 5 $\rightarrow$

$$F^{-1}\left(\hat{\phi}(\omega) \cdot e^{-\alpha\omega^2 t}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\varepsilon) \cdot \frac{1}{\sqrt{2\alpha t}} e^{-\frac{(x-\varepsilon)^2}{4\alpha t}} d\varepsilon \quad — \quad (6)$$

from eq^n 4 $\rightarrow$

$$u(x,t) = \frac{1}{2\sqrt{\pi\alpha t}} \int_{-\infty}^{\infty} \phi(\varepsilon) \cdot e^{-\frac{(x-\varepsilon)^2}{4\alpha t}} d\varepsilon \quad — \quad (7)$$

On comparing ~~[illegible]~~ 7 with $\rightarrow$

$$u(x,t) = \int_{-\infty}^{\infty} \phi(\varepsilon) \cdot K(x-\varepsilon, t) \, d\varepsilon$$

we get,

$$\boxed{K(x-\varepsilon, t) = \frac{1}{2\sqrt{\pi\alpha t}} e^{-\frac{(x-\varepsilon)^2}{4\alpha t}}}$$

Ans.

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Sol^n 4)

(i) Given $\phi(x) = \frac{2}{\pi} \tan^{-1}\left(\frac{x}{a}\right)$, $a>0$

Let's consider a function $f(\omega) = \frac{e^{-a\omega}}{\omega}$

Let $F(x)$ be fourier sine inverse transform of $f(\omega)$, then

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\omega) \sin(\omega x)\, dx$$

$$\Rightarrow F(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-a\omega}}{\omega} \sin(\omega x)\, dx \quad — (1)$$

$$\frac{d(F(x))}{dx} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-a\omega}}{\omega} (\omega \cos(\omega x)\, dx$$

$$\frac{d(F(x))}{dx} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a\omega} \cos(\omega x)\, dx = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+\omega^2}$$

by solving above differential eq^n we get,

$$F(x) = \sqrt{\frac{2}{\pi}} \int \frac{a}{a^2+x^2}$$

$$F(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right) + C \quad — (2)$$

for $x=0$, we get $F(0) = 0$

$\therefore$ $C = 0$

$$\Rightarrow \quad F(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right)$$

Now by our assumption,

$\cancel{F_s^{-1}(f(s))}$ $F_s^{-1}(f(s)) = F(x)$ where $F_s^{-1}(f(s))$ is fourier inverse sine transform

So, $F_s(F(x)) = f(s)$

$$F_s\left(\sqrt{\frac{2}{\pi}} \tan^{-1} \frac{x}{a}\right) = \frac{e^{-as}}{s}$$

$F_s(f(x)) \Rightarrow$ fourier sine transform.

$$\Rightarrow F_s\left(\sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right)\right) = \sqrt{\frac{2}{\pi}} \frac{e^{-as}}{s}$$

by linearity property

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(ii) Given function

$$\sigma(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

$$\Rightarrow \quad \sigma(x) = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \infty$$

The R.H.S is series expansion of e^{-x}

$$\Rightarrow \quad \sigma(x) = e^{-x}$$

Fourier transform of $\sigma(x) \Rightarrow$

$$\sigma_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin sx \, dx$$

$$\sigma_s(s) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+s^2} (-s\cos sx - \sin sx) \right]_0^{\infty} \quad \left(\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2} [a\sin bx - b\cos bx] \right)$$

$$\boxed{\sigma_s(s) = \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+1} \right)}$$

Ans

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Sol^n 5) Given $\rightarrow$ $F_c(\phi(x)) = \sum_{n=0}^{\infty} (-1)^n s^{2n}$ $F_c \rightarrow$ Fourier cosine transform

$$\sum_{n=0}^{\infty} (-1)^n s^{2n} \Rightarrow 1 - s^2 + s^4 - s^6 + \ldots \infty$$

it's a G.P. so, it's sum $\Rightarrow \dfrac{1}{1-(-s^2)} \Rightarrow \dfrac{1}{1+s^2}$

$$\therefore \quad F_c(\phi(x)) = \frac{1}{1+s^2} \quad \text{— (1)}$$

Applying fourier inverse cosine transformation $\rightarrow$

$$\phi(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\phi) \cdot \cos sx \cdot ds.$$

$$\phi(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{1+s^2} \cdot \cos(sx) ds \quad \text{— (2)}$$

differentiating w.r.t. x on both sides $\rightarrow$

$$\phi'(x) = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{s \sin(sx) ds}{1+s^2} = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{s^2 \sin(sx) ds}{(1+s^2) s}$$

$$\phi'(x) = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \left[\frac{\sin(sx)(s^2+1-1) ds}{(1+s^2) s} \right] = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \left(\frac{\sin(sx)}{s} - \frac{\sin(sx)}{s(1+s^2)} \right) ds$$

$$\phi'(x) = -\sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2} - \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin(sx) ds}{s(1+s^2)} \right) \quad \text{— (3)}$$

differentiating w.r.t. x on both sides $\rightarrow$

$$\phi''(x) = 0 + \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\cancel{\sin(sx) ds}\, s \cos(sx) ds}{(s^2+1) s} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\cos(sx) ds}{1+s^2}$$

As $\phi''(x) = \phi(x)$, the solution of this diff. eq^n can be given as $\rightarrow$

$$\phi(x) = Ae^{x} + Be^{-x}$$

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So, $\phi(0) = A + B = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos(0)}{1+s^2}$ using ②

$= \sqrt{\frac{2}{\pi}} \left(\tan^{-1}(\infty) - \tan^{-1}(0) \right) \Rightarrow$

$\sqrt{\frac{2}{\pi}} \times \left(\frac{\pi}{2} - 0 \right) \Rightarrow \sqrt{\frac{\pi}{2}} \Rightarrow \phi(0) = \sqrt{\frac{\pi}{2}}$ — ④

$\phi'(0) = A - B = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin(0)\, ds}{s(1+s^2)}$ using ③

$\Rightarrow \sqrt{\frac{2}{\pi}} \times -\frac{\pi}{2} \Rightarrow \phi'(0) = -\sqrt{\frac{\pi}{2}}$ — ⑤

Using ④ & ⑤ we get

$A = 0 \qquad B = \sqrt{\frac{\pi}{2}}$

$\Rightarrow \quad \phi(x) = 0 . e^{-x} + \sqrt{\frac{\pi}{2}} . e^{-x}$

$\Rightarrow \quad \boxed{\phi(x) = \sqrt{\frac{\pi}{2}}\, e^{-x}} \rightarrow$ Ans