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CSE (IDD)

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1.

Ans. $f(x) = (x^2-1)^n$

~~we need to~~

So, $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n} (x^2-1)^n$

By the definition of legendre polynomial,

$$P_n(x) = \sum_{r=0}^{M} (-1)^r \frac{(2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!} \times x^{n-2r}$$

where M

$$M = \lfloor n/2 \rfloor = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

By binomial Thm.

$$(x^2-1)^n = \sum_{r=0}^{n} {}^nC_r (x^2)^{n-r} (-1)^r = \sum_{r=0}^{n} {}^nC_r (-1)^r x^{2n-2r}.$$

$$\therefore \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n = \frac{1}{2^n \times n!} \sum_{r=0}^{n} {}^nC_r (-1)^r \frac{d^n}{dx^n} x^{2n-2r}.$$

But,

$$\frac{d^n}{dx^n} x^m = 0 \text{ if } m<n \text{ and } \frac{d^n}{dx^n} x^m = \frac{m!}{(m-n)!} x^{m-n}, \text{ if } m \geq n.$$

$$\therefore \frac{d^n}{dx^n} x^{2n-2r} = 0 \text{ if } 2n-2r<n \text{ ie } r > \frac{n}{2}.$$

we will have to replace $\sum_{r=0}^{n}$ by $\sum_{r=0}^{n/2}$ if n is even &

by $\sum_{r=0}^{(n-1)/2}$ if n is odd.

$$\therefore \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n n!}\sum_{r=0}^{[n/2]} {}^nC_r (-1)^r \frac{d^n}{dx^n} x^{2n-2r}.$$

$$= \frac{1}{2^n n!}\sum_{r=0}^{[n/2]} {}^nC_r (-1)^r \frac{(2n-2r)!}{(2n-2r-n)!} x^{2n-2r-n}$$

$$= \sum_{r=0}^{[n/2]} \frac{n!\,(-1)^r}{r!\,(n-r)!} \cdot \frac{(2n-2r)!}{(n-2r)!} x^{n-2r}.$$

$$= \underline{P_n(x)}$$

Hence, Proved that nth order derivative of $f^n f(x) = (x^2-1)^n$ satisfies legendre's diff. eq$^n$.

Also,

$$f^{(n)}(x) = C\,P_n(x).$$

$$(x^2-1)^n = \sum_{r=0}^{n} {}^nC_r (x^2)^{n-r} (-1)^r.$$

$$= \sum_{r=0}^{n} {}^nC_r (-1)^r x^{2n-2r}$$

$$\frac{d^n}{dx^n}(x^2-1)^n = \sum_{r=0}^{n} {}^nC_r (-1)^r \frac{d^n}{dx^n} x^{2n-2r}.$$

But, $\frac{d^n}{dx^n} x^m = 0$ if $m<n$ & $\frac{d^n}{dx^n} x^m = \frac{m!}{(m-n)!} x^{m-n}$, if $m \geq n$.

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$$\frac{d^n}{dx^n} x^{2n-2r} = 0 \quad \text{if} \quad 2n-r < n \quad \text{ie} \quad r > \frac{n}{2}.$$

we replace $\sum_r$

so, $$\frac{d^n}{dx^n}(x^2-1)^n = \sum_{r=0}^{n} \frac{{}^nC_r(-1)^r (2n-2r)!\, x^{n-2r}}{(n-2r)!}$$

$$= \sum_{r=0}^{[n/2]} \frac{n(-1)^r (2n-2r)!\, x^{n-2r}}{r!\,(n-r)!\,(n-2r)!}$$

$$\frac{d^n}{dx^n}(x^2-1)^n = n!\,2^n P_n(x)$$

$$f^n(x) = 2^n n!\, P_n(x)$$

So, the $n^{th}$ derivative of $f(x) = (x^2-1)^n$ satisfies legendr's diff. eq$^n$ because $P_n(x)$ is legendre's polynomial.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

$$\frac{1}{2^n n!}\left[(1-x^2)(f^n(x))'' - 2x(f^n(x))' + n(n+1) f^n(x)\right] = 0$$

& $$f^n(x) = C\, P_n(x)$$

$\therefore$ $\boxed{C = 2^n n!}$

Rodrigue's formula is - $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

$$2^n n!\, P_n(x) = f^n(x)$$

HENCE PROVED.