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Question:-1:-> Given:->
$$f(t) = e^t \frac{d^n}{dt^n}(t^n e^{-t})$$

Let's
$$G(t) = \frac{d^n}{dt^n}(t^n e^{-t})$$

$$P(t) = t^n e^{-t}$$

$$L(G(t)) = g(s) = s^n L(P(t)) - s^{n-1}(P(0)) - s^{n-2}P'(0) - s^{n-3}P''(0) - \cdots P^{n-1}(0)$$

Here $P(t) = t^n e^{-t}$

$P'(t) = n t^{n-1} e^{-t} + t^n e^{-t} \Rightarrow P'(0) = 0$

$P^{n-1}(t) = (n!)\, t^{n-(n-1)} e^{-t} + 0 + 0 \cdots$ because each of term of this expression contains t, where $t=0$, that term will be 0

Lowest term of that also contains t,

so,

$P^{n-1}(t) = n!\, t e^{-t} + \cdots$ at $t=0$

$P^{n-1}(0) = 0$, $P^{n-2}(0) = 0$ -- $P(0) = 0$

$$L(g(t)) = g(s) = s^n L(P(t)) \quad \text{-- (1)}$$

$p(t) = t^n e^{-t}$

Let, $m(t) = t^n$

$\cancel{L(p(t)) = P(s)}$

$$L(p(t)) = L(m(t))(s+1)$$

$$L(m(t))(s) = L(t^n) = \frac{\Gamma n+1}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$$L(m(t))(s+1) = \frac{n!}{(s+1)^{n+1}}$$

$$L(p(t)) = \frac{n!}{(s+1)^{n+1}}$$

$$L(g(t))(s) = \frac{s^n \, n!}{(s+1)^{n+1}} \quad \text{using eq - (i)}$$

$$L(f(t)) = L(g(t))(s-1)$$

$$= \frac{(s-1)^n}{(s-1+1)^{n+1}} \times n!$$

$$= \frac{s^n}{(s+1)^{n+1}} \times n! \quad \text{Ans}$$

Sivel

Question: → 2/ ~~Sol~~

$$A\delta\left(x-\frac{l}{4}\right) = \frac{d^4 u}{dx^4}$$

Take laplace both side

~~$L(f(x)\delta(x-a)) = f(a)$~~

We know

$$\int_0^\infty f(x)\,\delta(x-a) = f(a) \quad \text{--- (i)}$$

$$L\left(A\delta\left(x-\frac{l}{4}\right)\right) = A\int_0^\infty e^{-sx}\,\delta(x-l/4)\,dx$$

$$= Ae^{-sl/4} \quad \text{using eq (i)}$$

$$L\left(\frac{d^4 u}{dx^4}\right) = s^4 L(u) - s^3(u(0)) - s^2(u'(0)) - s(u''(0) - u'''(0)$$

$$Ae^{-sl/4} = s^4 L(u(x)) - s^2 u'(0) - u'''(0)$$

$$L(u(x)) = \frac{\Gamma 4\, Ae^{-sl/4}}{6\times s^4} + \frac{u'(0)\Gamma 2}{s^2} + \frac{u'''(0)\,\Gamma 4}{6\times s^4}$$

Laplace inverse both side

$$L^{-1}(L(u(t))) = \frac{A}{6}(t-l/4)^3 \cdot H(t-1/4) + u'(0)t + \frac{u'''(0)\,t^3}{6}$$

By second shifting inverse property

$$U(t) = U'(0)t + \frac{U'''(0)}{6}t^3 \qquad t < 1/4$$

$$U(t) = \frac{A}{6}\left(t - \frac{1}{4}\right)^3 + U'(0)t + \frac{U'''(0)}{6}t^3 \qquad t > 1/4$$

$$U(l) = \frac{9Al^3}{128} + U'(0)l + \frac{U'''(0)}{6}l^3$$

$$0 = \frac{9Al^3}{128} + U'(0)l + \frac{U'''(0)}{6}l^3 \quad \text{--- (ii)}$$

$$U'(t) = \frac{A}{2}\left(t - \frac{1}{4}\right)^2 + U'(0) + \frac{U'''(0)}{2}t^2$$

$$0 = \frac{9Al^2}{32} + \frac{16U'''(0)l^2}{32} + U'(0) \quad \text{--- (iii)}$$

By solving (iii) and (ii)

$$U'(0) = \frac{9Al^2}{256}$$

$$U'''(0) = \frac{-81A}{128}$$

Ans :→ $$U(x) = \begin{cases} \dfrac{9Al^2 x}{256} - \dfrac{27Ax^3}{256} & x < 1/4 \\ \dfrac{A}{6}\left(x - \dfrac{1}{4}\right)^3 + \dfrac{9Al^2 x}{256} - \dfrac{27Ax^3}{256} & x > 1/4 \end{cases}$$

Question: 3

Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$$

~~Lt~~ ~~After~~ $\mathcal{L}(g(x)) \, f(a) = \int_0^\infty e^{-ax} \frac{\sin sx}{x} dx$

where $g(x) = \frac{\sin sx}{x}$, $F_s(f(x)) = \sqrt{\frac{2}{\pi}} \mathcal{L}(g(x)) a \quad \cdots \text{(i)}$

$$\mathcal{L}\left(\frac{1}{x}(\sin ax)\right)(a) = \int_a^{\infty} L(\sin(at))\, da$$

$$= \int_a^{\infty} \frac{s}{s^2+a^2}\, da$$

$$= s \times \left(\frac{1}{s} \tan^{-1}\frac{a}{s}\right)_a^{\infty}$$

$$= \pi/_2 - \tan^{-1}\left(\frac{a}{s}\right)$$

$$= \cot^{-1}\left(\frac{a}{s}\right) = \tan^{-1}\left(\frac{s}{a}\right)$$

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\mathcal{L}(g(x))(a) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) \quad \text{-- by eq. (i)}$$

Now, by inverse fourier sin transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(f(x)) \sin sx\, ds$$

$$\frac{e^{-ax}}{x} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) \sin sx\, ds \quad \text{--- (ii)}$$

Put $x=1$ and $s = x$ in eq -- (ii)

$$\sqrt{\frac{\pi}{2}}\sqrt{\frac{\pi}{2}}\, e^{-a} = \int_0^{\infty} \tan^{-1}\left(\frac{x}{a}\right) \sin x\, dx$$

Ans

$$\Rightarrow \frac{\pi}{2} e^{-a} \text{ Ans}$$

Q8(ii)

Given,

$$f(x) = e^{-ax}$$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos(sx)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2+s^2} \right)$$

$$\because \mathcal{L}(\cos ax) = \frac{s}{a^2+s^2}$$

$$g(x) = \begin{cases} 1, & 0 < x < b \\ 0, & x > b \end{cases}$$

$$F_c(g(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(x) \cdot \cos(sx)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{b} \cos(sx)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin bs}{s}$$

Now

$$I, = \int_0^{\infty} \frac{(t-3)\sin(t+3) + (t+3)\sin(t-3)\, dt}{(t^2+4)\;(t^2-9)}$$

Ans

$$I = \int_0^\infty \frac{1}{t^2+4}\left(\frac{\sin(t+3)}{t+3} + \frac{\sin(t-3)}{t-3}\right) dt$$

$$I = \int_0^\infty \left(\frac{1}{s^2+4}\right)\left(\frac{\sin(s+3)}{s+3} + \frac{\sin(s-3)}{s-3}\right) ds$$

$$I = \int_0^\infty \frac{1}{2}\sqrt{\frac{\pi}{2}}\left[\sqrt{\frac{2}{\pi}} \cdot \frac{2}{s^2+2^2}\right]\left(\frac{\sin(s+3)}{s+3} + \frac{\sin(s-3)}{s-3}\right) ds$$

$$I = \int_0^\infty \frac{\sqrt{\pi}}{2\sqrt{2}} F_c(f(x))\left(\frac{\sin(s+3)}{s+3} + \frac{\sin(s-3)}{s-3}\right) ds \quad \text{--- (ii)}$$

Here, $f(x) = e^{-ax}$ where $a=2$

$f(x) = e^{-2x}$

$$F_c(u(x)\cos ax) = \int_0^\infty u(x)\cos(sx)\cos(ax)\,dx$$

$$= \frac{1}{2}\int_0^\infty u(x)\left(\cos(s+a)x + \cos(s-a)x\right) dx$$

$$F_c[u(x)\cos ax] = \frac{1}{2}\left[F_c(s+a) + F_c(s-a)\right]$$

$$F_c[u(x)\cos 3x] = \frac{1}{2}\left[F_c(s+3) + F_c(s-3)\right]$$

where $F_c(s) \Rightarrow$ Fourier transform of $u(x)$

If $F_c(s) = \sqrt{\frac{2}{\pi}}\frac{\sin s}{s}$ $\quad u(x) = g(x)$ where $b=1$

$$F_c[g(x)\cos 3x] = \cancel{\frac{1}{2}}\sqrt{\cancel{\pi}}\left(\frac{1}{2}\right)\left[\sqrt{\frac{2}{\pi}}\frac{\sin(s+3)}{(s+3)} + \sqrt{\frac{2}{\pi}}\frac{\sin(s-3)}{s-3}\right]$$

$$\sqrt{2\pi}\, F_c(g(x)\cos 3x) = \left[\frac{\sin(s+3)}{s+3} + \frac{\sin(s-3)}{s-3}\right]$$

Now manipulate - eq (ii)

$$= \frac{\pi}{2}\int_0^\infty F_c(g(x)\cos 3x)\, F_c(f(x))$$

Now, from Parseval's identity for fourier

$$I = \frac{\pi}{2}\int_0^\infty e^{-2x} \cdot g(x)\cos 3x\, dx$$

$$= \pi/2 \int_0^1 e^{-2x}\cos 3x\, dx$$

$$= \cancel{\pi/2}$$

$$\because \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + c$$

$$I = \pi/2 \cdot \frac{e^{-2x}}{13}\left[-2\cos 3x + 3\sin 3x\right]\Big|_0^1$$

$$= \frac{\pi}{2\times 13} e^{-2}\left[-2\cos 3 + 3\sin 3 + 2\right]$$

$$= \frac{\pi}{26} e^{-2}\left[-2\cos 3 + 3\sin 3 + 2\right]$$

Question 5 ⇒ Given equation

$$\frac{\partial u}{\partial t} = \frac{\alpha \partial^2 u}{\partial x^2}$$

since the values $u(0,t)$ or $u_x(0,t)$ are not given and $x \in (-\infty, \infty)$, we will use the complex fourier transform to solve the problem

Taking fourier transform both side

$$\frac{\partial \bar{u}}{\partial t} = -\alpha s^2 \bar{u}$$

$\bar{u}$ fourier transform

$$\int \frac{\partial \bar{u}}{\bar{u}} = -\alpha s^2 \int \partial t$$

$$\log \frac{\bar{u}}{A} = -\alpha s^2 t$$

A - constant

$$\bar{u} = A e^{-\alpha s^2 t}$$

$$\bar{u}(s,t) = A(s) e^{-\alpha s^2 t} \quad \text{--- (1)}$$

Putting $t=0$

$$\bar{u}(s,0) = A$$

Also, $u(x,0) = e^{-x^2}$

Take fourier transform both side

$$\bar{u}(s,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x - is)x} \, dx$$

$$\bar{U}(s,0) = \frac{1}{\sqrt{2}} e^{-s^2/4} = A(s)$$

$$\bar{U}(s,t) = \frac{1}{\sqrt{2}} e^{-s^2/4} e^{-\alpha s^2 t}$$

$$= \frac{1}{\sqrt{2}} e^{-s^2 (\alpha t + 1/4)}$$

$$U(x,t) = \frac{\frac{1}{2} e^{-x^2/4\alpha(1/4 + \alpha t)}}{\sqrt{\frac{1}{4} + \alpha t}}$$

$$U(x,t) = \frac{1}{\sqrt{1 + 4\alpha t}} \alpha \, e^{-x^2/(1 + 4\alpha t)}$$

$$U(x,0) = e^{-x^2}$$

$$e^{-x^2} = \int_{-\infty}^{\infty} K(x - \xi, 0) \, e^{-\xi^2} \, d\xi$$

$$\int_{-\infty}^{\infty} f(x - \xi) \, f(\xi) \, d\xi = f(x)$$

At $t = 0$. $K(x - \xi, 0) = f(x, \xi)$

PA :

$K(x-\xi, \xi) = f(x-\xi) \times g(x-\xi), \; t$ — ①

And

$U(x,t) = \int_{-\infty}^{\infty} \cancel{f(x-\xi)\, U(\xi, t)\, \partial\xi}$ — ③

from ①

$$U(x,t) = \int_{-\infty}^{\infty} K(x-\xi, t)\, e^{-\xi^2}\, \partial\xi$$

From ①, ②, ③

$$K(x-\xi, t) = f(x-\xi)\; U(\xi, t)\, e^{\xi^2}$$

Ans

$$U(x,t) = \frac{1}{\sqrt{1+4\alpha t}}\, e^{-x^2/(1+4\alpha t)}$$

$$K(x-\xi, t) = \delta(x-\xi)\; U(\xi, t)\, e^{\xi^2}$$