MA 203: Mathematical Methods

End-Semester Exam

Enroll	
Page No:	
Date: / /	

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1.) (a) To prove:

$$L^{-1}\left(\frac{1}{s}\sin\frac{1}{s}\right) = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} + \cdots$$

By the Maclaurin series expansion of sin(s),

$$\sin(s) = s - \frac{s^3}{3!} + \frac{s^5}{5!} + \frac{s^7}{7!} + \cdots$$

$$\therefore \sin\left(\frac{1}{s}\right) = \frac{1}{s} - \frac{1}{3!}\cdot\frac{1}{s^3} + \frac{1}{5!}\cdot\frac{1}{s^5} + \frac{1}{7!}\cdot\frac{1}{s^7} + \cdots$$

$$\Rightarrow \frac{1}{s}\cdot\sin\left(\frac{1}{s}\right) = \frac{1}{s^2} - \frac{1}{3!}\cdot\frac{1}{s^4} + \frac{1}{5!}\cdot\frac{1}{s^6} - \frac{1}{s^8}\cdot\frac{1}{7!}$$

Applying Inverse Laplace Transform on both sides

$$L^{-1}\left\{\frac{1}{s}\cdot\sin\frac{1}{s}\right\} = L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^4}\cdot\frac{1}{3!} + \frac{1}{s^6}\cdot\frac{1}{5!} - \frac{1}{s^8}\cdot\frac{1}{7!} + \cdots\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2}\right\} - L^{-1}\left\{\frac{1}{3!}\cdot\frac{1}{s^4}\right\} + L^{-1}\left\{\frac{1}{5!}\cdot\frac{1}{s^6}\right\} - L^{-1}\left\{\frac{1}{7!}\cdot\frac{1}{s^8}\right\} + \cdots$$

(By Linearity Property of Inverse Laplace transform)

$$= \frac{t}{\Gamma(2)} - \frac{t^3}{\Gamma(4)\cdot 3!} + \frac{t^5}{\Gamma(6)\cdot 5!} - \frac{t^7}{\Gamma(8)\cdot 7!} + \cdots$$

$$\left[\because L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}\right]$$

$$= t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)} - \frac{t^7}{(7!)^2} + \cdots$$

$(\because \Gamma(n+1) = n!)$

$$\therefore L^{-1}\left\{\frac{1}{s}\cdot \sin\frac{1}{s}\right\} = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} + \cdots$$

Hence Proved

1.) (b)

$$\int_0^\infty f(x) \cos\alpha x \cdot dx = e^{-\alpha}$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cdot \cos\alpha x \cdot dx = \sqrt{\frac{2}{\pi}}\cdot e^{-\alpha}$$

$$F_c\{f(x)\}(\alpha) = \sqrt{\frac{2}{\pi}}\cdot e^{-\alpha}$$

$$\left(\because F_c\{f(x)\}(s) = \sqrt{\frac{2}{\pi}}\cdot\int_0^\infty f(x)\cdot \cos sx\cdot dx\right)$$

Taking the Inverse Fourier cosine transform on both sides

$$F_c^{-1}\{F_c\{f(x)\}\} = F_c^{-1}\left\{\sqrt{\frac{2}{\pi}}\cdot e^{-\alpha}\right\}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\cdot e^{-\alpha}\cdot \cos\alpha x\cdot dx$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\alpha}\cdot \cos\alpha x\cdot dx = \frac{2}{\pi}\left[e^{-\alpha}\cdot \frac{(x\cdot \sin\alpha x - \cos\alpha x)}{x^2+1}\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 - \frac{1\cdot(0-1)}{x^2+1}\right]$$

Ans $\Rightarrow$ $\boxed{f(x) = \frac{2}{\pi}\cdot\frac{1}{x^2+1}}$

Page No:

Date: / /

2.)

Given the second order linear differential equation :

$$2x^2y'' - xy' + (x-5)y = 0 \qquad -(1)$$

$$\Rightarrow \quad y'' - \frac{y'}{2x} + \frac{(x-5)}{2x^2}y = 0.$$

Comparing with the standard form,

$$y'' + P(x)y' + Q(x)y = 0$$

So,

$$P(x) = \frac{-1}{2x} \quad \text{and} \quad Q(x) = \frac{(x-5)}{2x^2}$$

As we can see that both $P(x)$ and $Q(x)$ are not analytic at $x=0$.

$\therefore$ $x=0$ is a singular Point.

And,

$x \cdot P(x) = -1/2$ and $x^2 Q(x) = (x-5)/2$ are analytic at $x=0$

$\therefore$ $x=0$ is a Regular singular Point.

$\therefore$ By Frobenius Method, let the trial solution be

$$y = \sum_{m=0}^{\infty} C_m \cdot x^{m+r} \qquad r \in \mathbb{R}, \; C_0 \neq 0$$

$\therefore$ On differentiating wrt x

$$y' = \sum_{m=0}^{\infty} C_m (m+r) \cdot x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} C_m (m+r)(m+r-1) \cdot x^{m+r-2}$$

Substituting values of y, y'' and y' in (1)

Shruti

Enroll

Page No:

Date: / /

$$2x^2\sum_{m=0}^{\infty} C_m(m+r)(m+r-1)\cdot x^{m+r-2} - x\sum_{m=0}^{\infty} C_m(m+r)\cdot x^{m+r-1} + (x-5)\sum_{m=0}^{\infty} C_m\cdot x^{m+r} = 0$$

$$2\sum_{m=0}^{\infty} C_m(m+r)(m+r-1)\, x^{m+r} - \sum_{m=0}^{\infty} C_m(m+r)\, x^{m+r} + (x-5)\sum_{m=0}^{\infty} C_m\, x^{m+r} = 0$$

$$\sum_{m=0}^{\infty} C_m\left(2(m+r)(m+r-1) - (m+r) - 5\right)x^{m+r} + \sum_{m=0}^{\infty} C_m\, x^{m+r+1} = 0 \quad -(2)$$

Equating to zero the coefficient of the smallest power of x in the identity above ie x^r term.

$C_0\,(2r^2 - 3r \cancel{\ } - 5) = 0$, $C_0 \neq 0$

This is the indial equation which gives the values of r -

$2r^2 - 3r - 5 = 0$

$(2r-5)(r+1) = 0$

$\Rightarrow r = -1,\ 5/2$ (real and unequal roots, not differing by an integer.)

Now, in series equation

$$\sum_{m=1}^{\infty} C_m\left(2(m+r)^2 - 3(m+r) - 5\right)x^{m+r} + \sum_{m=1}^{\infty} C_{m-1}\, x^{m+r} = 0$$

Equating to zero the coefficients of x^{m+r},

$$C_m\left(2(m+r)^2 - 3(m+r) - 5\right) + C_{m-1} = 0.$$

$$\Rightarrow C_m = \frac{-C_{m-1}}{2(m+r)^2 - 3(m+r) - 5} \quad \text{or,}$$

$$C_m = \frac{C_{m-1}}{5 - (m+r)(2m+2r-3)} \quad \text{(Recurrence Relation).}$$

Shruti

Enroll

Page No:

Date: / /

For $r=-1$, we get recurrence relation of first solution $y_1(x)$-

$$C_m = \frac{C_{m-1}}{5-(m-1)(2m-5)}, \quad m \geq 1.$$

$$C_1 = \frac{C_0}{5}, \quad C_2 = \frac{C_1}{6} = \frac{C_0}{30}, \quad C_3 = \frac{C_2}{3} = \frac{C_0}{90}, \ldots$$

$$y_1(x) = \sum_{m=0}^{\infty} C_m x^{m-1}$$

$$= x^{-1}\left(C_0 + \frac{C_0}{5}x + \frac{C_0 x^2}{30} + \frac{C_0}{90}x^3 + \cdots\right)$$

$$y_1(x) = C_0 x^{-1}\left(1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \cdots\right)$$

For $r=5/2$, we get recurrence relation of second solution $y_2(x)$.

$$C_m = \frac{C_{m-1}}{5-(2m+5)(m+1)}, \quad m \geq 1.$$

$$C_1 = \frac{C_0}{-9}, \quad C_2 = \frac{C_1}{-22} = \frac{C_0}{198}, \quad C_3 = \frac{C_2}{-39} = \frac{-C_0}{7722}, \ldots$$

$$y_2(x) = \sum_{m=0}^{\infty} C_m x^{m+5/2}$$

$$= x^{5/2}\left(C_0 - \frac{C_0 x}{9} + \frac{C_0 x^2}{198} - \frac{C_0 x^3}{7722} + \cdots\right)$$

$$= C_0 \cdot x^{5/2}\left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \cdots\right)$$

since, $y_1(x)$ and $y_2(x)$ are linearly independent, the general solution of the given equation is

$$y(x) = C_1 y_1(x) + C_2 y_2(x), \quad C_1, C_2 \in \mathbb{R}.$$

$$y_1(x) = C_0 x^{-1}\left(1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \cdots\right)$$

Enroll

Page No:

Date: / /

Shruti

$$y_2(x) = c_0 x^{5/2}\left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \cdots\right)$$

Soln -

$$\boxed{\begin{aligned} y_1(x) &= c_0 x^{-1}\left[1 + \frac{x}{5} + \frac{x^2}{30} + \cdots\right] = x^{-1}\sum_{m=0}^{\infty} c_m x^m \ \& \ c_m = \frac{c_{m-1}}{5-(m-1)(2m-5)} \\ y_2(x) &= c_0 x^{5/2}\left[1 - \frac{x}{9} + \frac{x^2}{128} - \cdots\right] = x^{5/2}\sum_{m=0}^{\infty} c_m x^m \ \& \ c_m = \frac{c_{m-1}}{5+(m+1)(2m+5)} \end{aligned}}$$

Shruti

Page No:

Date: / /

3.)

(a) Given an orthonormal set of functions $\{\phi_0, \phi_1, \phi_2, \dots\}$ in $L^2[-1, 1]$ such that $\phi_n = C_n P_n$, where C_n is a constant and P_n is Legendre's Polynomial of degree n, $n \in \mathbb{N} \cup \{0\}$.

so,

the set of functions must be orthogonal in $L^2[-1,1]$ and each of them has norm 1.

Since Legendre's polynomials are orthogonal,

$$\int_{-1}^{1} P_m(x) P_n(x) \cdot dx = 0 , \quad \text{if } m \neq n.$$

So, $$\int_{-1}^{1} \phi_m(x) \phi_n(x)\, dx = \int_{-1}^{1} C_m P_m(x) C_n P_n(x) \cdot dx$$

$$= \int_{-1}^{1} (C_m C_n) (P_m(x) P_n(x)) \cdot dx$$

$$\Rightarrow \int_{-1}^{1} \phi_m(x) \phi_n(x)\, dx = 0 , \quad m \neq n.$$

The norm of $f(x)$ is $\|f(x)\|$ defined by $\|f(x)\| = \left(\int_a^b f^2(x) dx\right)^{1/2}$ on $a \leq x \leq b$.

So, $$\left(\int_{-1}^{1} \phi_n^2(x)\, dx\right)^{1/2} = \|\phi_n(x)\|$$

Since $\|\phi_n(x)\| = 1$ and $\int_{-1}^{1} P_m(x) P_n(x) dx = \frac{2}{2n+1}$, for $m=n$,

$$\left(\int_{-1}^{1} C_n^2 P_n^2(x) dx\right)^{1/2} = 1$$

$$\Rightarrow C_n \sqrt{\frac{2}{2n+1}} = 1 \Rightarrow C_n = \pm\sqrt{\frac{2n+1}{2}}$$

Enroll

Page No:

Shruti

Date: / /

So, $\int_{-1}^{1} \phi_m(x)\phi_n(x)\,dx = \begin{cases} 0 & , \text{if } m \neq n \\ 1 & , \text{if } m = n \end{cases}$

Now, by generating function of legendre's polynomials,

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad , |x| \leq 1 \quad , |z| \leq 1.$$

Putting $x = 1$

$$(1-2z+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(1)$$

$$\Rightarrow (1-z)^{-1} = \sum_{n=0}^{\infty} z^n P_n(1)$$

Since $|z| < 1$, binomial theorem can be used for expansion of $(1-z)^{-1}$,

$$1 + z + z^2 + \cdots + z^n = \sum_{n=0}^{\infty} P_n(1) z^n.$$

Equating the coefficient of z^n from both sides, we get
$P_n(1) = 1.$

Similarly for $x = -1$, we get $P_n(-1) = (-1)^n$

So,

(i) $\phi_6(1) = c_6 P_6(1)$

$$= \sqrt{\frac{2\times 6+1}{2}} \cdot 1 = \pm\sqrt{\frac{13}{2}}$$

(ii) $\phi_7(-1) = c_7 P_7(-1)$

$$= \sqrt{\frac{2\times 7+1}{2}}\,(-1)^7 = \mp\sqrt{\frac{15}{2}}$$

Shruti

Enroll

Page No:

Date: / /

3) (b)

To check whether $\frac{d}{dx}\left(J_5^2(x)\right) = \frac{x}{10}\left[J_6^2 - J_4^2\right]$ is True or False.

$J_n(x)$ is the Bessel function of the first kind of order n.
We know the recurrence relations of Bessel's function.

$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$ -(1) $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$ -(2)

Using the recurrence relations : $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

$$\frac{d}{dx}\left(J_5^2(x)\right) = 2\cdot J_5(x)\cdot\frac{d}{dx}\left(J_5(x)\right)$$

$$= J_5(x)\cdot 2\cdot J_5'(x)$$

$$= \frac{x}{2\cdot 5}\left[J_4 + J_6\right]\cdot\left[J_4 - J_6\right]$$

$$= \frac{x}{10}\left[J_4^2 - J_6^2\right]$$

$$\therefore \frac{d}{dx}\left(J_5^2(x)\right) = \frac{x}{10}\left[J_4^2 - J_6^2\right]$$

$\Rightarrow$ The given statement is false.

Page No:

Date: / /

Shruti

4.)(a.)

$$H_n\left\{\frac{f(x)}{x}\right\} = \int_0^\infty \frac{f(x)}{x} \cdot x \cdot J_n(sx) \cdot dx$$

$$= \int_0^\infty f(x) \, J_n(sx) \cdot dx$$

{ Using Recurrence Relation of Bessel function

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] \Big\}$$

$$= \int_0^\infty f(x) \cdot \frac{sx}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] dx$$

$$= \frac{s}{2n}\left[\int_0^\infty f(x) \cdot x \cdot J_{n-1}(x) \cdot dx + \int_0^\infty f(x) \cdot x \cdot J_{n+1}(x) dx\right]$$

$$= \frac{s}{2n}\left[H_{n-1}\{f(x)\} + H_{n+1}\{f(x)\}\right]$$

$$\therefore H_n\left\{\frac{f(x)}{x}\right\} = \frac{s}{2n}\left[H_{n-1}\{f(x)\} + H_{n+1}\{f(x)\}\right]$$

Hence, Proved.

$$H_n\left\{\frac{\delta(x-a)}{x \cdot x^{K-1}}\right\} = \int_0^\infty \frac{\delta(x-a)}{x^{K-1}} \cdot J_n(sx) \cdot dx$$

$$= \frac{s}{2n}\left[\int_0^\infty \frac{\delta(x-a)}{x^{K-2}} \cdot J_{n-1}(sx) dx + \int_0^\infty \frac{\delta(x-a)}{x^{K-2}} J_{n+1}(sx) dx\right]$$

Enroll

Shruti

Page No:

Date: / /

As, $\int_{-\infty}^{\infty} \delta(x-a) \cdot f(x) = f(a)$

$$\int_{-\infty}^{a-\epsilon} \delta(x-a) \cdot f(x) + \int_{a-\epsilon}^{a+\epsilon} \delta(x-a) \cdot f(x) + \int_{a+\epsilon}^{\infty} \delta(x-a) f(x) = f(a) \quad \epsilon > 0$$

$$\int_{-\infty}^{a-\epsilon} 0 \cdot f(x) + \int_{a-\epsilon}^{a+\epsilon} \delta(x-a) \cdot f(x) + \int_{a+\epsilon}^{\infty} 0 \cdot f(x) = f(a)$$

$$\int_{a-\epsilon}^{a+\epsilon} \delta(x-a) \cdot f(x) = f(a).$$

$$\therefore \int_{-\infty}^{\infty} \delta(x-a) \cdot f(x) = \int_{a-\epsilon}^{a+\epsilon} \delta(x-a) \cdot f(x) = f(a) \quad \epsilon > 0$$

since $a > 0$ (given in question)

$$\therefore \int_{-\infty}^{\infty} \delta(x-a) f(x) = \int_{0}^{\infty} \delta(x-a) \cdot f(x) = \int_{a-\epsilon}^{a+\epsilon} \delta(x-a) \cdot f(x) = f(a) \quad , \epsilon > 0$$

$$H_n\left\{\frac{\delta(x-a)}{x^k}\right\} = \frac{s}{2n}\left[\int_0^{\infty} \delta(x-a) \cdot \frac{J_{n-1}(sx)\,dx}{x^{k-2}} + \int_0^{\infty} \delta(x-a) \cdot \frac{J_{n+1}(sx)\,dx}{x^{k-2}}\right]$$

$$= \frac{s}{2n}\left[\frac{J_{n-1}(sa)}{a^{k-2}} + \frac{J_{n+1}(sa)}{a^{k-2}}\right]$$

$$= \frac{s}{2n \cdot a^{k-2}}\left[J_{n-1}(sa) + J_{n+1}(sa)\right]$$

$$= \frac{s}{2na^{k-2}} \cdot \frac{2n}{sa} \cdot J_n(sa)$$

$$= \frac{J_n(sa)}{a^{k-1}} \qquad \therefore \boxed{H_n\left\{\frac{\delta(x-a)}{x^k}\right\} = \frac{J_n(sa)}{a^{k-1}}}$$

4.)(b) Given differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r \leq \infty, \quad z \geq 0$$

Subjected to conditions -

$u \to 0$ as $z \to \infty$ and $r \to \infty$

$u(r,0) = f(r)$

$u(r,z)$ is bounded.

Taking Hankel Transform of zero order on both sides of the given equation,

$$\int_0^\infty \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\right) r \cdot J_0(sr) \cdot dr = 0$$

where J_0 is the Bessel's function of first kind of order 0.

$$\Rightarrow \int_0^\infty \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r}\right) r\, J_0(sr) \cdot dr + \int_0^\infty \frac{\partial^2 u}{\partial z^2}\, r\, J_0(sr) \cdot dr = 0$$

We first calculate the first integral.

$$\because \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right)$$

$$\therefore \int_0^\infty \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) J_0(sr) \cdot dr = \left(r \cdot \frac{\partial u}{\partial r}\right) J_0(sr) \Big|_0^\infty - \int_0^\infty s\, J_0'(sr)\, r \cdot \frac{\partial u}{\partial r} \cdot dr$$

Assuming $r\frac{\partial u}{\partial r} \to 0$ as $r \to \infty$,

$$0 - s\int_0^\infty J_0'(sr)\, r\, \frac{\partial u}{\partial r} \cdot dr = -s\left[r J_0'(sr)\, u \Big|_0^\infty - \int_0^\infty \frac{d}{dr}\left(r J_0'(sr)\right) u\, dr\right.$$

Shruti

$$= -s\left[0 - \int_0^\infty \frac{d}{dr}\left(r J_0'(sr)\right) U\,dr\right]$$

since $u \to 0$ as $r \to \infty$ and $U(r,z)$ is bounded.

$$\therefore \int_0^\infty \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) r J_0(sr)\,dr = s\int_0^\infty \frac{d}{dr}\left(r J_0'(sr)\right) U\,dr.$$

Now from Bessel's equation,

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0$$

$$\Rightarrow x J_0''(x) + J_0'(x) + x J_0(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left(x J_0'(x)\right) + x J_0(x) = 0$$

$$\Rightarrow \frac{1}{s}\frac{d}{dx}\left(sx J_0'(sx)\right) + sx J_0(sx) = 0$$

(Putting $x = sx$)

$$\Rightarrow \frac{d}{dx}\left(x J_0'(sx)\right) = -sx J_0(sx)$$

$$\therefore \int_0^\infty s\frac{d}{dr}\left(r J_0'(sr)\right) U\,dr = -s^2\int_0^\infty r J_0(sr)\, U\,dr$$

$$\therefore \int_0^\infty \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) r J_0(sr)\,dr = -s^2\bar{u},$$

where $\bar{u}$ is the Hankel transform of zero order of $v(r,z)$.

Now,

$$-s^2\bar{u}(s,z) + \frac{\partial^2\bar{u}}{\partial z^2}(s,z) = 0$$

or

$$\frac{\partial^2\bar{u}(s,z)}{\partial z^2} - s^2\bar{u}(s,z) = 0$$

which is a second order homogeneous linear differential equation whose general solution is given by,

Shruti

Enroll

Page No:

Date: / /

$\bar{u}(s,z) = c_1 e^{sz} + c_2 e^{-sz}$ where c_1 and c_2 are constants.

from the conditions given,

since $u \to 0$ as $z \to \infty$ and $r \to \infty$, $\bar{u}(s,z)$ should also be finite as $z \to \infty$, which is possible only if $c_1 = 0$.

$$\therefore \bar{u}(s,z) = c_2 e^{-sz}$$

Also, $u(s,0) = f(s)$

Taking Hankel transform of zero order on both sides,

$$\bar{u}(s,0) = \int_0^\infty f(r)\, r\, J_0(sr)\, dr$$

$$\bar{u}(s,0) = \bar{f}(s) \qquad (H_0\{f(r)\} = \bar{f}(s))$$

$$\Rightarrow c_2 = \bar{f}(s)$$

$$\Rightarrow \bar{u}(s,z) = \bar{f}(s) e^{-sz}$$

Taking inverse Hankel transform of zero order on both sides,

$$u(s,z) = \int_0^\infty \bar{f}(s) e^{-sz}\, s\, J_0(sr)\, ds$$

is the required solution.

Shruti

Ans 5.)

$xy'' + y' = 0$; i) $y(x)$ is bounded as $x \to 0$
ii) $y(1) = \alpha y'(1)$, $\alpha \neq 0$

The above equation can be written as -

$$Ly(x) = f(x)$$

where $L = \left[\frac{d}{dx}\left(p(x) \cdot \frac{d}{dx}\right)\right]$ and $f(x) = 0$.

$$p(x) = x$$

$$xy'' + y' = 0$$

$\frac{d}{dx}(xy') = 0$, $xy' = c$; c - constant $(c \in \mathbb{R})$

$$y' = c/x$$

$y(x) = c \ln x + k$; $k \in \mathbb{R}$

For $y(x)$ to be bounded as $x \to 0$, c has to be zero.

$\therefore c = 0$

also, $y(1) = \alpha \cdot y'(1)$

$$k = \alpha \cdot c/1$$

$$k = 0$$

$\therefore y(x) = 0$ Trivial solution is the only solution of $Ly(x) = 0$

Let the Green's function be

$$G(x,t) = \begin{cases} G_1(x,t) = A \ln x + B & , 0 \leq x < t \\ G_2(x,t) = C \ln x + D & , t < x \leq 1 \end{cases}$$

where A, B, C, D are real constants.

Enroll
Page No:
Date: / /

By properties of Green's function,

1) $G_1(x,t)$ is bounded at $x \to 0$

$G_1(x,t) = A\ln x + B$, for $G_1(x,t)$ to be bounded as $x \to 0$,
A should be 0

$\therefore \underline{A = 0}$ - (1)

$G_2(1,t) = \alpha \cdot G_2'(1,t)$

$0 + D = \alpha \left[\frac{C}{x}\right]_{x=1} \Rightarrow \underline{D = \alpha C}$ - (2)

2) $\lim_{x \to t} G_1(x,t) = \lim_{x \to t} G_2(x,t)$

$\lim_{x \to t} [0 + B] = \lim_{x \to t} [C\ln x + D]$ $(G_1(x,t) = A\ln x + B = 0 + B = B)$ from (1)

$B = C\ln t + D$

$B = C\ln t + \alpha C$ from (2)

$\underline{B = C(\alpha + \ln t)}$

3) $\left.\frac{\partial G_2(x,t)}{\partial x}\right|_{x=t} - \left.\frac{\partial G_1(x,t)}{\partial x}\right|_{x=t} = \frac{-1}{P(t)}$

$\left.\frac{\partial}{\partial x}[C\ln x + D]\right|_{x=t} - \left.\frac{\partial}{\partial x}[B]\right|_{x=t} = \frac{-1}{P(t)}$

$\left.\frac{c}{x}\right|_{x=t} - 0 = \frac{-1}{P(t)}$

$\frac{c}{t} = \frac{-1}{t}$ $(\because p(x) = x)$

$(c = -1)$

Enroll

Shruti

Page No:

Date: / /

Now from (1), (2), (3), (4)

$A = 0$ $\qquad$ $C = -1$

$B = -(\alpha + \ln t)$ $\qquad$ $D = -\alpha$

$\therefore$ $G_1(x,t) = A\ln x + B = -(\alpha + \ln t)$

$G_2(x,t) = C\ln x + D = -\ln x - \alpha$

$\therefore$

$$\boxed{G(x,t) = \begin{cases} -(\alpha + \ln t), & 0 \le x < t \\ -(\alpha + \ln x), & t < x \le 1 \end{cases}}$$