classmate

Date

Page

Mathematical Methods MA-203
Assignment - 1

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1) $f(t) = e^t \dfrac{d^n}{dt^n}\left[t^n e^{-t}\right]$

Laplace of $t^n e^{-t}$,

$L[t^n] = \dfrac{\Gamma(n+1)}{s^{n+1}}$

Using the property $L[e^{at} f(t)] = F(s-a)$

We get;

$$L[t^n e^{-t}] = \frac{\Gamma(n+1)}{(s+1)^{n+1}} \quad \text{--- (1)}$$

Next using the derivative property of laplace;

$$L\left[\frac{d^n}{dt^n} g(t)\right] = s^n L[g(t)] - s^{n-1} g(0) - s^{n-2} g'(0) - s^{n-3} g''(0) \cdots - g^{(n-1)}(0) \quad \text{--- (11)}$$

Here $g(t) = t^n e^{-t}$

We can clearly see that all n order derivative of f(t) become 0 at t=0

$L[g(t)] = \dfrac{\Gamma(n+1)}{(s+1)^{n+1}}$ ~~or~~ [From (1)]

Now to calculate $L\left[\dfrac{d^n}{dt^n} g(t)\right]$ we put the values in eq^n (11) to get:

$$L\left[\frac{d^n}{dt^n} t^n e^{-t}\right] = \frac{s^n \Gamma(n+1)}{(s+1)^{n+1}}$$

Vanshika

classmate
Date
Page

Vanshika

Again we use the property $L[e^{at}f(t)] = F(s-a)$

so $$\boxed{L\left[e^{t}\frac{d^n}{dt^n}(t^n e^{-t})\right] = \frac{(s-1)^n \, \boxed{(n+1)}}{s^{n+1}}}$$

2) Given equation is $\frac{d^4u}{dx^4} = A\delta\left(x-\frac{1}{4}\right)$

Taking laplace transform both the sides:

$$L\left[\frac{d^4u}{dx^4}\right] = A\,L\left[\delta\left(x-\frac{1}{4}\right)\right]$$

RHS: We know that Laplace transform of δ (dirac-delta) is equal to 1

using the second shifting theorem of laplace transform which states that:

$$L[G(t)] = e^{-as}f(s) \quad \text{were } G(t) = \begin{cases} F(t-a) & t \geq a \\ 0 & t < a \end{cases}$$

$$L\left[\delta\left(x-\frac{1}{4}\right)\right] = e^{\frac{1}{4}s} \quad [f(s) = 1 \text{ for dirac delta}]$$

$\therefore$ RHS $= A e^{\frac{1s}{4}}$

using the laplace of derivatives for 4^{th} order:

LHS becomes:

$$s^{4}L[u(x)] - s^{4-1}u(0) - s^{4-2}u'(0) - s^{4-3}u''(0) - s\,u'''(0)$$

Putting the given values: $u(0) = 0$ and $u''(0) = 0$

Also let $u'(0) = B$ and $u'''(0) = C$

LHS $= s^4 L[u(x)] - Bs^2 - C$

Vanshika

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The overall equation becomes:

$$s^4 L[u(x)] - Bs^2 - C = Ae^{\frac{ls}{4}}$$

$$\Rightarrow L[u(x)] = \frac{Ae^{\frac{ls}{4}} + Bs^2 + C}{s^4}$$

$$L[u(x)] = \frac{Ae^{\frac{ls}{4}}}{s^4} + \frac{B}{s^2} + \frac{C}{s^4}$$

$$L[u(x)] = \frac{A}{6} e^{\frac{ls}{4}} \frac{3!}{s^4} + B\frac{\sqrt{2}}{s^2} + \frac{C}{6}\frac{3!}{s^4}$$

Taking laplace inverse on both sides:

$$u(x) = \frac{A}{6}\left(x + \frac{l}{4}\right)^3 + Bx + \frac{C}{6}x^3$$

Putting $u(l) = 0$ and $u'(l) = 0$ in the above equation:

$$0 = \frac{A}{6}\left(\frac{5l}{4}\right)^3 + Bl + \frac{C}{6}l^3 \quad \text{--- (i)}$$

$$u'(x) = \frac{A}{2}\left(x + \frac{l}{4}\right)^2 + B + \frac{C}{2}x^2$$

$$\Rightarrow 0 = \frac{A}{2}\left(\frac{5l}{4}\right)^2 + B + \frac{C}{2}l^2 \quad \text{--- (ii)}$$

Deviding (i) by l on both sides:

$$0 = \frac{A}{6}\frac{125l^2}{64} + B + \frac{C}{6}l^2 \quad \text{--- (iii)}$$

Subtracting (iii) from (ii):

$$0 = \frac{A}{2}\cdot\frac{25}{16}l^2 - \frac{A}{6}\frac{125}{64}l^2 + \frac{C}{2}l^2 - \frac{C}{6}l^2$$

$$\Rightarrow -\frac{Cl^2}{3} = Al^2\left[\frac{25}{32} - \frac{125}{6\times 64}\right]$$

Vanshika

classmate

Date
Page

$\Rightarrow$ ~~x~~ Vanshika

$\Rightarrow \frac{-c}{3} = A \cdot \frac{175}{384} \Rightarrow \boxed{c = -\frac{175}{128}}$

Adding (ii) and (iii)

$$0 = A\frac{25l^2}{32} + A\frac{125l^2}{384} + 2B + -\frac{175}{192}l^2$$

$$\Rightarrow \frac{175}{192}l^2 = Al^2\frac{425}{384} + 2B$$

$$\Rightarrow \boxed{B = \frac{175l^2}{384} - \frac{425}{768}Al^2}$$

$$\boxed{u(x) = \frac{A}{6}\left(x + \frac{l}{4}\right)^3 + \frac{175l^2x}{384} - \frac{425Al^2x}{768} - \frac{175}{768}x^3}$$

3) We need to find $F_s\left[\frac{e^{-ax}}{x}\right]$

The fourier sine transform is given by:

$$F_s(s) = \int_0^\infty f(x)\sin sx\,dx = \int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx \quad \text{(1)}$$

Differentiating both sides wrt s:

$$\frac{d}{ds}[F_s(s)] = \int_0^\infty \frac{e^{-ax}}{x} x\cos sx\,dx$$

$$= \int_0^\infty e^{-ax}\cos sx\,dx$$

using formula: $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$;

$$= \frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\Big|_0^\infty$$

Vanshika

classmate

Date
Page

Vanshika

$$= \frac{1}{a^2+s^2}\left[0-1(-a+0)\right]$$

$$\Rightarrow \frac{d}{ds}\left[F_s(s)\right] = \frac{a}{a^2+s^2}$$

$$\Rightarrow F_s(s) = \int \frac{a}{a^2+s^2}\,ds = a\int \frac{ds}{a^2+s^2}$$

$$= \frac{a}{a}\tan^{-1}\frac{s}{a} + c$$

$$\Rightarrow F_s(s) = \tan^{-1}\frac{s}{a} + c$$

To find the integration constant c, we u
the boundary condition values:
From eq^n (1): when $s=0 \Rightarrow F_s(s)=0$.
Putting this in the above equation we
$\boxed{c=0}$ so,

$$\boxed{F_s(s) = \tan^{-1}\frac{s}{a}}$$

Now by inverse fourier sine transform

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,dx$$

$$\Rightarrow \frac{e^{-ax}}{x} = \frac{2}{\pi}\int_0^\infty \tan^{-1}\left(\frac{s}{a}\right)\sin sx\,ds$$

Put $x=1$ on both sides:

$$\frac{e^{-a}}{1} = \frac{2}{\pi}\int_0^\infty \tan^{-1}\frac{s}{a}\sin s\,ds$$

changing variable from s to x

$$\boxed{\int_0^\infty \tan^{-1}\frac{x}{a}\sin x\,dx = \frac{\pi}{2}e^{-a}}$$

Vanshika

classmate

Date
Page

Vanshika

4) Given: $f(x) = e^{-ax}$

To find fourier cosine transform:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

Drawing an analogy with laplace transform and using the second shifting theorem we get

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} \qquad \left[\because L[\cos ax] = \frac{s}{a^2+s^2} \right]$$

Now;

Let $g(x) = \begin{cases} 1 & 0 < x < b \\ 0 & x > b \end{cases}$

$$\Rightarrow F_c[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos sx \, dx \qquad \text{[Definition]}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^b \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^b$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin bx}{s}$$

Now consider $I = \int_0^\infty \frac{(t-3)\sin(t+3) + (t+3)\sin(t-3)}{(t^2+4)(t^2-9)} dt$

$$\Rightarrow I = \int_0^\infty \frac{1}{t^2+4} \left(\frac{\sin(t+3)}{t+3} + \frac{\sin(t-3)}{t-3} \right) dt$$

On replacing t with s in above integral

Vanshika

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$$I = \int_0^\infty \frac{\cancel{s}}{s^2+2^2}\left[\frac{\sin(s+3)}{(s+3)} + \frac{\sin(s-3)}{s-3}\right] ds$$

$$= \int_0^\infty \frac{1}{2}\sqrt{\frac{\pi}{2}}\left[\sqrt{\frac{2}{\pi}}\frac{2}{s^2+2^2}\right]\left[\frac{\sin(s+3)}{s+3} + \frac{\sin(s-3)}{s-3}\right] ds$$

$$= \int_0^\infty \frac{\sqrt{\pi}}{2\sqrt{2}} F_c(f(x)\left[\frac{\sin(s+3)}{s+3} + \frac{\sin(s-3)}{s-3}\right] ds$$

where $f(x) = e^{-ax}$, $a = 2$

$\Rightarrow$ $$I = \frac{\pi}{2}\int_0^\infty F_c(f(x))\frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\frac{\sin(s+3)}{s+3} + \sqrt{\frac{2}{\pi}}\frac{\sin(s-3)}{s-3}\right] ds$$

we know that:

$$F_c\left(g(x)\cos ax\right) = \frac{1}{2}\left[F_c(s-a) + F_c(s+a)\right]$$

$\Rightarrow$ $$I = \frac{\pi}{2}\int_0^\infty F_c[f(x)] \cdot F_c[h(x)]\, ds$$

where $h(x) = g(x)\cos 3x$, $b = 1$

using the Preseval's identity for the fourier cosine transform we have:

$$I = \frac{\pi}{2}\int_0^\infty f(x)\cdot h(x)\, dx$$

$\Rightarrow$ $$I = \frac{\pi}{2}\int_0^\infty e^{-2x} g(x)\cos 3x\, dx$$

$$= \frac{\pi}{2}\int_0^1 e^{-2x}\cos 3x\, dx$$ [definition of $g(x)$ is applied].

using formula: $\int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + c$

$$I = \frac{\pi}{2}\frac{e^{-2x}}{13}\left[-2\cos 3x + 3\sin 3x\right]_0^1$$

$$I = \frac{\pi}{26}\left[-2e^{-2}\cos 3 + 3e^{-2}\sin 3 + 2\right]$$ Ans:

Vanshika

classmate
Date
Page

Vanshika

5) Given equation: $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$

Since the values $u(0,t)$ or $u_x(0,t)$ are not given and $x \in (-\infty, \infty)$ we will use the complex fourier transform to solve the problem.

Taking fourier transform on both sides of equation:

$$\frac{d\bar{u}}{dt} = -\alpha s^2 \bar{u}$$

Here $\bar{u}$ represents the fourier transform of u

$$\int \frac{d\bar{u}}{\bar{u}} = -\int \alpha s^2 \, dt$$

$\Rightarrow \log \frac{\bar{u}}{A} = -\alpha s^2 t$ [A is integration constant]

$\Rightarrow \bar{u} = A e^{-\alpha s^2 t}$ or $\bar{u}(s,t) = A e^{-\alpha s^2 t}$ --- (1)

Putting $t = 0$ on both sides;

$\bar{u}(s,0) = A$

Also it is given that $u(x,0) = e^{-x^2}$

Taking fourier transform on both sides;

$$\bar{u}(s,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 - isx)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{is}{2\alpha}\right)^2} e^{-\frac{s^2}{4a^2}} \, dx$$

Putting $\left(x - \frac{is}{2\alpha}\right) = t$

$dx = \frac{dt}{\alpha}$

Vanshika

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$$= \int_{-\infty}^{\infty} e^{-t^2} e^{-\frac{s^2}{4}} dt$$

$$= e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

Now we know that $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$

$$\Rightarrow \bar{u}(s,0) = \sqrt{\pi}\, e^{-\frac{s^2}{4}}$$

So $A = \sqrt{\pi}\, e^{-s^2/4}$

Putting this value of A in ①:

$$\bar{u}(s,t) = \sqrt{\pi}\, e^{-\frac{s^2}{4}} e^{-\alpha s^2 t}$$

Taking inverse fourier transform on both sides:

$$u(x,t) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \sqrt{\pi}\, e^{-\frac{s^2}{4} - \alpha s^2 t - isx}$$

comparing it with the given integral in the form of ε and K:

$$\boxed{K(x-\varepsilon, t) = \frac{1}{2\sqrt{\pi \alpha t}} e^{-\frac{(x-\varepsilon)^2}{4\alpha t}}}$$

Vanshika