Recurrence relations

$a_n \Rightarrow$ in terms of the previous terms in the sequence

$a_n \to a_0, a_1, a_2, \quad a_{\frac{n}{2}}, \quad \dots, a_{n-2}, a_{n-1}$

$a_n = 2a_{n-1} + 5 + a_{n-2} + a_{n-3}$

$\boxed{a_0, a_1, a_2, \_\_\_\_\_\_, a_n}$

$\to 1, 3, 5, 7, 9, \dots$

$a_1 \quad a_2 \quad a_3$

$\boxed{a_n = a_{n-1} + 2} \quad a_1 = 1, \; a_2 = 3$

$F_n = F_{n-1} + F_{n-2} \longrightarrow 1)$

$F_n = 2F_{n-2} + F_{n-3} \longrightarrow 2)$

Compound Interest

$$\boxed{A = P\left(1 + \frac{r}{100}\right)^n}$$

$\to P_0$

$\to P_n$

$P_n =$

$P_n = P_0\left(1 + \frac{r}{100}\right)^n$

P_0

$P_1 =$

$P_2 =$

$P_3 =$

$\vdots$

$P_{n-1} =$ ____

$\boxed{P_n = P_{n-1} + P_{n-1} \cdot \frac{r}{100}}$ — implicit formula

$T_n = 2T_{n/2} + cn$

$T_n = T_{n/3} + O(1)$

$T_n = T_{n/k} + O(1) = O(\log_k n)$

0 2^3
1

$\boxed{2^5} - 13 = ?$ 19

41

Prob: How many bit strings of length 5 are there that do not have two consecutive 0's?

Sol^n

$a_0, a_1, a_2, \ldots \boxed{a_n}$ length: n $\rightarrow$

$$\boxed{a_n = a_{n-1} + a_{n-2}}$$
$a_1 = 2$
$a_2 = 3$

$a_3 = 5$, $a_4 = 8$, $a_5 = 13$

0 00
1 01
10
11

Any bit string of length $n-1$ with no 2 con. 0's | 1 $\rightarrow a_{n-1}$

Any bit string of length $n-2$ with no 2 con. 0's | 10 $\rightarrow a_{n-2}$

$$\boxed{a_n = a_{n-1} + a_{n-2}} \quad a_0 = 0, a_1 = 2, a_3 = 3$$

0, 2, 3, 5, 8, 13, ...

$$\boxed{F_n = F_{n-1} + F_{n-2} \quad F_0 = 0, F_1 = 1, F_2 = 1}, F_3 = 2, F_4 = 3$$

0, 1, 1, 2, 3, 5, ...

Prob 2:

b_n: no. of bit strings of length n that do have two con. 0's

b_0, b_1, b_2

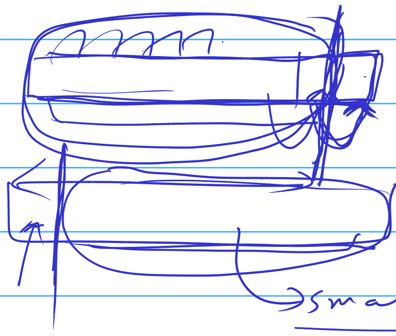
$$b_n = 2^n - a_n$$
$$= 2^n - (a_{n-1} + a_{n-2})$$
$$= 2^n - (2^{n-1} - b_{n-1} + 2^{n-2} - b_{n-2})$$

$$b_n = 2^n - 2^{n-1} + b_{n-1} - 2^{n-2} + b_{n-2}$$

Sol^n of some common recurrence relation

1) $\boxed{t_n = t_{n-1} + n}$ $\quad n \geq 2, \quad t_1 = 1$

Bubble sort, selection sort



$t_n = O(n) + t_{n-1}$

$t_n = n + t_{n-1}$

smallest

substitution method

$\boxed{t_n = t_{n-1} + n}$ — 1) $\quad t_1 = 1$

$= t_{n-2} + (n-1) + (n)$

$= t_{n-3} + (n-2) + (n-1) + (n)$

$= \quad \vdots$

$= t_1 + 2 + 3 + \cdot \quad \cdot \quad + n$

$\boxed{t_n = n(n+1)/2} = O(n^2)$

2) $\boxed{t_n = t_{n/2} + 1}$ $\quad n \geq 2 \qquad t_1 = 1$

D&C

$= t_{n/4} + 1 + 1$

$= t_{n/8} + 1 + 1 + 1$

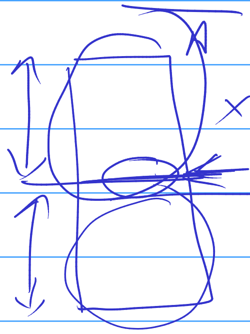
$\vdots$

$= t_{n/2^k} + \underbrace{1 + \cdots + 1}_{k}$

treinty

$n = 2^k$

$k = \log_2 n$



mid

$t_n = 1 + t_{n/2}$

$= t_1 + k = 1 + k = 1 + \log_2 n$

$t_n = O(\log_2 n)$

3) $t_n = t_{n/2} + n$ $\qquad n \geq 2, \quad t_1 = 1 \qquad n = 2^k$

$= t_{n/4} + \frac{n}{2} + n = t_{n/2^3} + \frac{n}{2^2} + \frac{n}{2^1} + \frac{n}{2^0}$

$= 1 + 2 + \cdot \quad \cdot \quad \cdot + 2^{k-1} = O(n)$

4) $\boxed{t_n = 2t_{n/2} + n}$

left to students

$t_n = O(n \log n)$

merge sort

$t_{n/2}$ $t_{n/2}$

merge: $O(n)$

$t_n = t_{n/2} + t_{n/2} + n$

$\boxed{t_n = 2t_{n/2} + n}$

quick sort

$\rightarrow O(n^2)$ worst case

$O(n \log n)$ Best

$O(n \log n)$ Avg

partition: $O(n)$

n-1

1	2	3	4	5	...	n

Worst case:

→ sorted list

→ reverse sorted list

$$\boxed{t_n = O(n) + t_{n-1}}$$
$$= O(n^2)$$

Homogeneous Recurrences:

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = 0$$

$t_n = x^n$

Character eq^n: $a_0 x^n + a_1 x^{n-1} + \cdots = 0$

⇓

roots, $r_1, r_2, \cdots r_k$

$$\boxed{t_n = \sum_{i=1}^{k} c_i r_i^n}$$

c_i's are determined using initial values

$\boxed{F_n - F_{n-1} - F_{n-2} = 0}$

$F_n = F_{n-1} + F_{n-2}$

Example: $t_n - 3t_{n-1} - 4t_{n-2} = 0$, $t_0 = 0$, $t_1 = 1$

⇓

$x^2 - 3x - 4 = 0 \implies r_1 = 4$, $r_2 = -1$

$t_0 = 0 = c_1 + c_2$

$t_1 = 1 = 4c_1 - c_2$

c_1, c_2

$\rightarrow t_n = \boxed{c_1 4^n + c_2 (-1)^n} = O(4^n)$

$$t_n = t_{n-1} + t_{n-2} \qquad \underline{\underline{t_0 = 0,\ t_1 = 1}}$$

$$t_n - t_{n-1} - t_{n-2} = 0$$

$$x^2 - x - 1 = 0$$

$$r_1 = \left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$r_2 = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$t_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$= \underline{\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n} - \underline{\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n}$$

$$\underline{\simeq O(n^{1.6})}$$