Non-homogeneous recurrences

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = f(n)$$

Example

$$t_n - 2t_{n-1} = n \quad \text{--- 1)}$$

replace n by $n-1$

$$t_{n-1} - 2t_{n-2} = n-1 \quad \text{--- 2)}$$

1) − 2)

$$t_n - 3t_{n-1} + 2t_{n-2} = 1 \quad \text{--- 3)}$$

replace n by $n-1$

$$t_{n-1} - 3t_{n-2} + 2t_{n-3} = 1 \quad \text{--- 4)}$$

3) − 4)

Homogeneous $\boxed{t_n - 4t_{n-1} + 5t_{n-2} - 2t_{n-3} = 0}$ — 5)

$$x^3 - 4x^2 + 5x - 2 = 0$$

$\downarrow$ 1, 1, 2

$$t_n = (c_1 + c_2 n)1^n + c_3 2^n$$
$$= c_1 + c_2 n + c_3 2^n$$
$$t_n = O(2^n)$$

Prob. $t_n - 2t_{n-1} = \underbrace{n + 2^n}_{f(n)}$ $\quad n \geq 1, \; t_0 = 0$

for eliminating 2^n — 1)

replace n by $n-1$

2^{n-1} (RHS)

$\downarrow$ multiply by 2

roots: 1, 1, 2, 2

$t_n = (c_1 + c_2 n)1^n + (c_3 + c_4 n) \cdot 2^n$

$= \cancel{c_1}\; O(n2^n)$

$= 2^n$ — 2)

1) − 2)

$= 0$

Change of variable method

$T_n = a T_{\frac{n}{b}} + f(n)$

$T_n = 2 T_{n/2} + cn$

$n/2$

$S(k)\ T_{2^k}$

$n = 2^k$

$T(n) = a T\left(\frac{n}{b}\right) + f(n)$

$S(k) = T(b^k)$ $\quad n = b^k$

$S(k) = T(b^k) = a\, T(b^{k-1}) + f(b^k)$

$S(k) = a S(k-1) + f(b^k)$

$S(k) - a S(k-1) = f(b^k)$

Prob.

$$T_n = \begin{cases} 2 & n = 1 \\ 3T_{n/2} + n \log n & n > 1 \end{cases}$$

$T_n = 3 T_{n/2} + n \log n$, $t_1 = 1$

$a = 3,\ b = 2$

$n = 2^k$

$S(k) = T_{2^k} = 3 T_{2^{k-1}} + k 2^k$

$S(k) = 3 S(k-1) + k 2^k$

$S(k) - 3 S(k-1) = k \cdot 2^k$

↓

Homogeneous RR

$2, 2, 3$

$c_1 = 8$

$c_2 = -6$

~~$c_3 = -2$~~

$S(k) = c_1 3^k + (c_2 + c_3 k) \cdot 2^k$

$n = 2^k$

$k = \log n$

~~$T(n)$~~

$S(k) = 8 \cdot 3^k - 6\, 2^k - 2\, k \cdot 2^k$

↓

$T(2^k) = T(n) = 8 \cdot 3^{\log n} - 6 \cdot 2^{\log n} - 2 \log n \cdot 2^{\log n}$

$= 8 \cdot 3^{\log n} - 6n - 2n \log n$

$T_n = 4T_{n/3} + cn$

$n = 3^k$

$n = 3^k$

Prob: $T_n = \sqrt{n} \cdot T_{\sqrt{n}} + n$

$T_n = aT_{n/b} + f(n)$

$\boxed{n = 2^k} \Rightarrow k = \log_2 n$

$$T(2^k) = 2^{k/2}\, T(2^{k/2}) + 2^k \quad — 1)$$

divide both the sides by 2^k

$$\frac{T(2^k)}{2^k} = \frac{T(2^{k/2})}{2^{k/2}} + 1$$

$T_n = T_{n/2} + 1$
$= O(\log n)$

$$Y(k) = Y(k/2) + 1$$

$$Y(k) = \log k$$

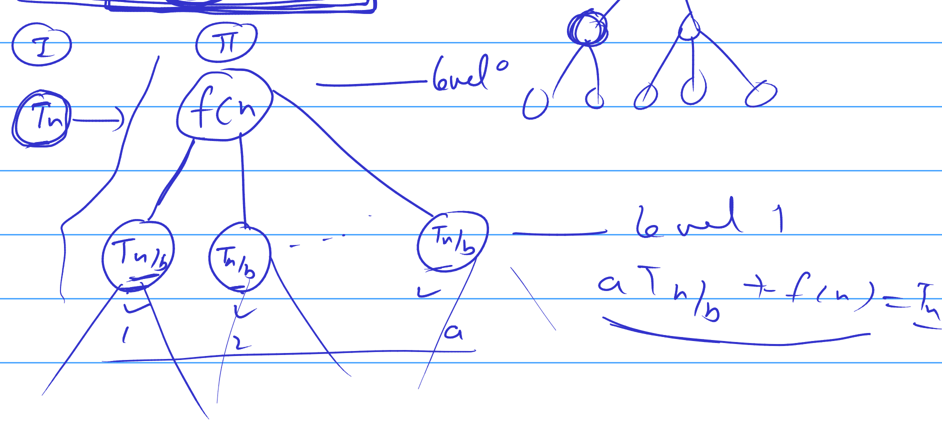
$$\frac{T(2^k)}{2^k} = \log k \Rightarrow T(2^k) = 2^k \log k$$

$$T(n) = 2^{\log n} \log\log n$$

$$\boxed{T_n = n \log\log n}$$

Recursion Tree Method

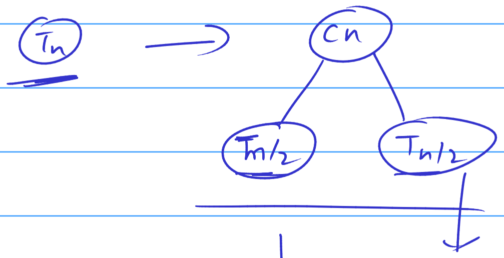
$\boxed{T_n} = a\boxed{T_{n/b}} + f(n)$



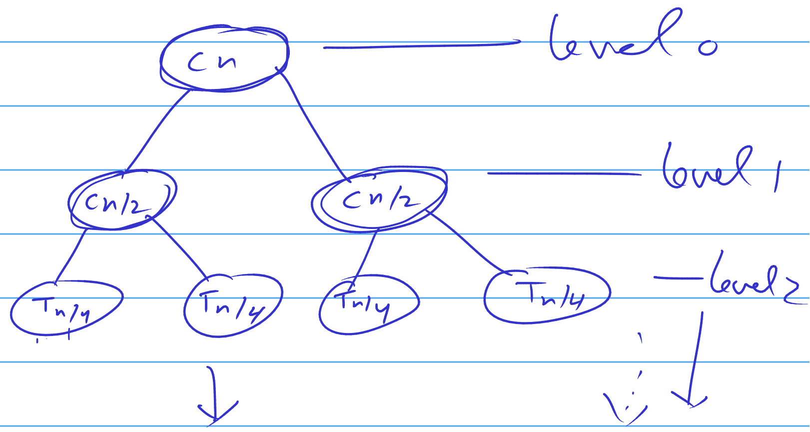
$T_n = a T_{n/b} + f(n)$

$T_n = 2T_{n/2} + cn \qquad n \geq \quad , \ T_1 = c$

$n = 2^k$



$T_{n/2} = 2 T_{n/4} + \frac{cn}{2}$



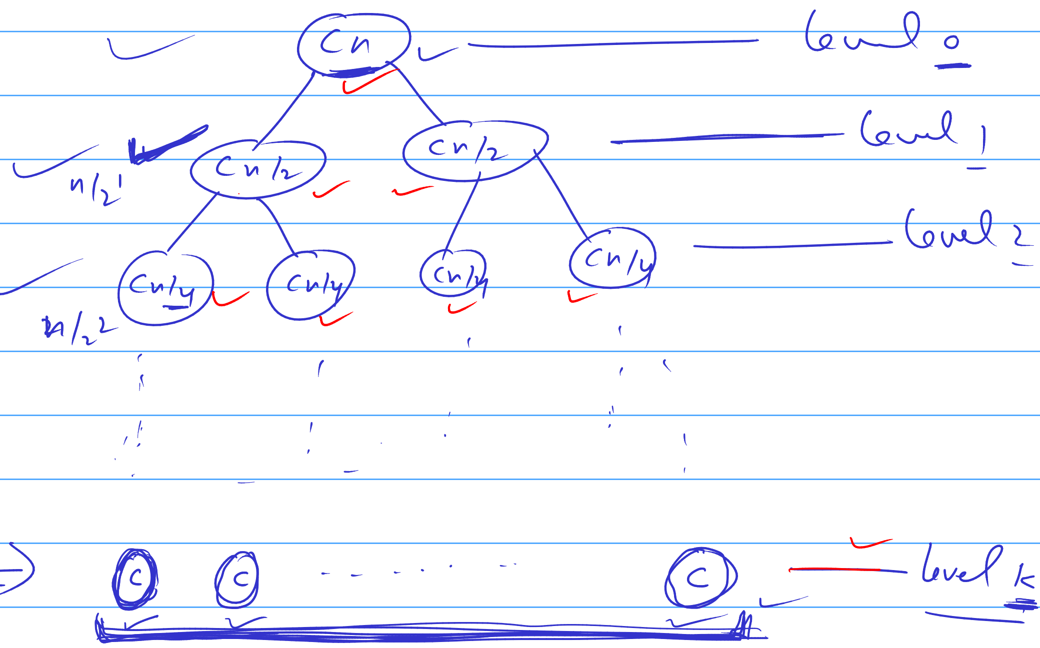
$T_n = 2T_{n/2} + cn$

2^1

$2^2 = 4$

2^3

$T_{n/2^k} \Rightarrow 2^k \Rightarrow T_1 = c$



$\cancel{2^{k+1}}$ $\quad 2^k$

$T_n = cn + cn + cn + \cdots + 2^k \cdot c$

cn

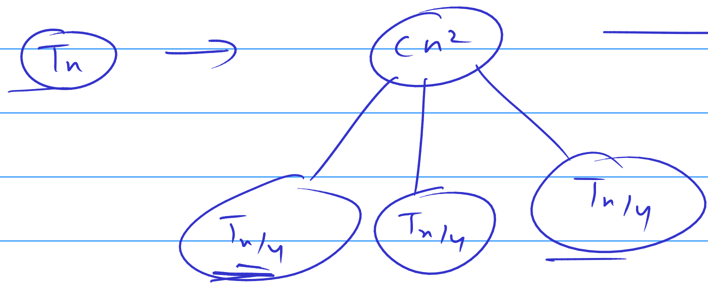
$= cn(k+1)$

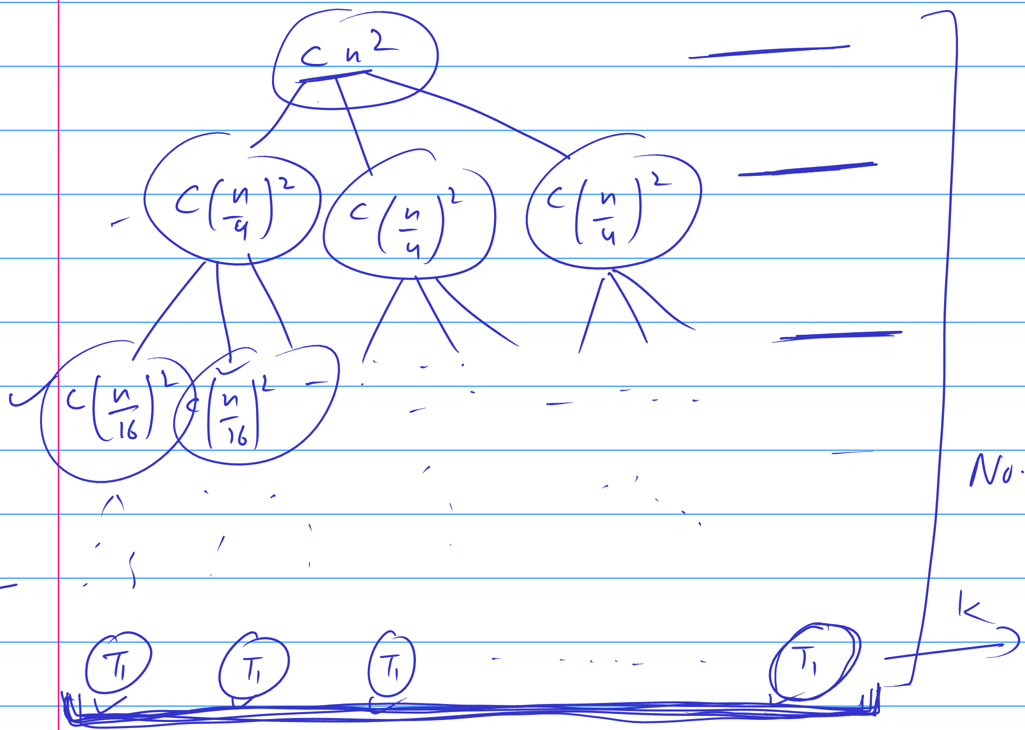
$= \Theta(n \log n) \Leftarrow T_n = 2T_{n/2} + cn$

$T_n = a\,T_{n/b} + f(n)$

$T_n = 3\,T_{n/4} + cn^2$

$n = 4^k$





$n = 4^k$

$k = \log_4 n$

Height ? = k+1

No. of leaves ? 3^k

$3^{\log_4 n} = n^{\log_4 3}$

$$T(n) = \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3})$$

$$\leq \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 +$$

$$T(n) = c_1\, cn^2 + n^{\log_4 3}$$

$$= O(n^2)$$

Prob: $T_n = T_{n/3} + T_{2n/3} + cn$ — left to students