Mathematical Methods MA-203   Mid-sem Exam.

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Q1.)

a)

Ans- $f(t) = H(t-\pi) \cosh(3t) \sin t \cos t.$

Considering that $H(t-\pi)$ is a heavysided unit step function.

Using second shifting Theorem,

If $L\{f(t)\} = f(s)$ and $G(t) = F(t-a) H(t-a) = \begin{cases} F(t-a) & , t \geq a \\ 0 & , t < a \end{cases}$

then $L\{G(t)\} = e^{-as} f(s).$

So, according to the question, $a = \pi$

$F(t-\pi) = \cosh(3t) \sin t \cos t.$

Let, $t-\pi = x \Rightarrow t = x+\pi$

$$F(x) = \cosh(3x+3\pi) \sin x \cdot \cos x = \left(\frac{e^{3\pi} e^{3x} + e^{-3\pi} e^{-3x}}{4} \right) \sin x \cdot \cos x \cdot 2$$

$$F(x) = \left[\frac{e^{3\pi} e^{3x} + e^{-3\pi} e^{-3x}}{4} \right] \sin 2x$$

Taking laplace transform on both sides and using Linearity property:

$$L[F(x)] = \frac{e^{3\pi}}{4} L\left[e^{3x} \sin 2x\right] + \frac{e^{-3\pi}}{4} L\left[e^{-3x} \sin 2x\right]$$

As we know,

$L[\sin at] = \frac{a}{a^2+s^2}$ and $L[e^{at} f(t)] = f(s-a)$ $\left\{\text{First shifting theorem}\right\}$

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$$L[F(x)] = \frac{e^{3\pi}}{4} \cdot \frac{2}{[(s-3)^2+4]} + \frac{e^{-3\pi}}{4} \cdot \frac{2}{(s+3)^2+4}$$

$\therefore$

$$L[G(t)] = \frac{e^{-\pi s}}{2}\left[\frac{e^{3\pi}}{(s-3)^2+4} + \frac{e^{-3\pi}}{(s+3)^2+4}\right]$$

Ans - $\boxed{L[H(t-\pi)\cosh(3t)\sin t\cos t] = \frac{e^{-\pi s}}{2}\left\{\frac{e^{3\pi}}{(s-3)^2+4} + \frac{e^{-3\pi}}{(s+3)^2+4}\right\}}$

(b)

$$f(t) = 1 - \frac{t^2}{4} + \frac{t^4}{64} - \frac{t^6}{2304} + \cdots$$

$$= \frac{1}{1^2} - \frac{t^2}{1.2^2} + \frac{t^4}{1^2\cdot 2^2\cdot 4^2} - \frac{t^6}{1^2\cdot 2^2\cdot 4^2\cdot 6^2} + \frac{t^8}{1^2\cdot 2^2\cdot 4^2\cdot 6^2\cdot 8^2} - \cdots$$

$$= \frac{1}{2^0\cdot 1^2} - \frac{t^2}{2^2\cdot 1^2} + \frac{t^4}{2^4(2\cdot 1)^2} - \frac{t^6}{2^6(3\cdot 2\cdot 1)^2} + \frac{t^8}{2^8\cdot(4\cdot 3\cdot 2\cdot 1)^2} - \cdots$$

From the pattern observed in the above series,

$$T_n = \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} \quad \text{for} \quad n = 0, 1, 2, 3\ldots$$

here, T_n represents the n^{th} term in the series $f(t)$

$$\therefore f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}\cdot(n!)^2} \quad \text{where} \quad n = 0, 1, 2, 3\ldots$$

$$L[t^n] = \frac{n!}{s^{n+1}} \Rightarrow L[t^{2n}] = \frac{(2n)!}{s^{2n+1}} \quad (\because n \text{ is an integer})$$

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So, $L[T_n] = L\left[\frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}\right] = \frac{(-1)^n}{2^{2n}} \cdot \frac{(2n)!}{(n!)^2 s^{2n+1}}$

$\because f(t) = \sum_{n=0}^{\infty} T_n$ (Linearity Property)

$$L[f(t)] = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} \cdot (n!)^2 \cdot s^{2n+1}}$$

$$= \frac{1}{s} - \frac{2!}{2^2 \cdot 1 \cdot s^3} + \frac{4!}{2^4 (2!)^2 \cdot s^5} - \frac{6!}{2^6 \cdot (3!)^2 \cdot s^7} + \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2s^2} + \frac{3}{8s^4} - \frac{5}{16s^6} + \frac{3 \cdot 5}{2^7 s^8} - \cdots\right]$$

$$L[f(t)] = \frac{1}{s}\left[1 - \frac{1}{2^1 s^2} + \frac{3}{2^3 s^4} - \frac{5}{2^4 s^6} + \frac{3 \cdot 5}{2^7 s^8} - \cdots\right] \quad -(1)$$

By Binomial Theorem,

$$(x+1)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

$$\left(1+\frac{1}{s^2}\right)^{-1/2} = 1 - \frac{1}{2s^2} + \frac{1}{2} \times \frac{3}{2} \times \frac{1}{2!} \cdot \frac{1}{s^4} - \frac{1}{2} \times \frac{3}{2} \times \frac{5}{2} \times \frac{1}{3!} \times \frac{1}{s^6} + \cdots$$

$$= 1 - \frac{1}{2s^2} + \frac{3}{8s^4} - \frac{5}{16s^6} + \frac{3 \cdot 5}{2^4 \cdot s^8} + \cdots$$

$$\left(1+\frac{1}{s^2}\right)^{-\frac{1}{2}} = 1 - \frac{1}{2^1 s^2} + \frac{3}{2^3 s^4} - \frac{5}{2^4 s^6} + \frac{3 \cdot 5}{2^7 \cdot s^8} + \cdots \quad -(2)$$

From eqns (1) and (2):

$$L[f(t)] = \frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-1/2}$$

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$$L[f(4t)] = \frac{1}{4} \times \frac{1}{\left(\frac{s}{4}\right)} \left[1 + \frac{1}{\left(\frac{s}{4}\right)^2}\right]^{-\frac{1}{2}}$$

$\left\{ \text{change of scale property: } L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right) \right\}$

$$L[f(4t)] = \frac{1}{s}\left(1 + \frac{16}{s^2}\right)^{-\frac{1}{2}}$$

As we know, $L[t f(t)] = \frac{-d}{ds} L[f(t)]$

Multiplying on both sides by 4;

$$L[4t\, f(4t)] = -4 \frac{d}{ds}\left[\frac{1}{s} \times \left(1 + \frac{16}{s^2}\right)^{-1/2}\right] = -4\left[\frac{-1}{s^2}\left(1 + \frac{16}{s^2}\right)^{-1/2} + \left(\frac{-1}{2s}\right)\left(1 + \frac{16}{s^2}\right)^{-3/2} \times 16(-2) \times \frac{1}{s}\right]$$

$$= -4\left[\frac{-1}{s^2}\left(1 + \frac{16}{s^2}\right)^{-1/2} + \frac{16}{s^4}\left(1 + \frac{16}{s^2}\right)^{-3/2}\right]$$

$$= -4\left[\frac{-1}{s^2}\left(1 + \frac{16}{s^2}\right)^{-3/2} \cdot \left(1 + \frac{16}{s^2}\right) + \frac{16}{s^4}\left(1 + \frac{16}{s^2}\right)^{-3/2}\right]$$

$$L[4t\, f(4t)] = 4\left[\frac{1}{s^2}\left(1 + \frac{16}{s^2}\right)^{-3/2}\right] \quad -(3)$$

Dividing Both sides by 4;

$$L[t f(4t)] = \frac{1}{s^2}\left(1 + \frac{16}{s^2}\right)^{-3/2}$$

By definition of laplace transform:

$$L[t f(4t)] = \int_0^\infty e^{-st}\, t f(4t)\, dt$$

$$= \frac{1}{s^2}\left(1 + \frac{16}{s^2}\right)^{-3/2} \quad [\text{from } (3)]$$

putting $s = 3$.

$$\int_0^\infty e^{-3t}\, t f(4t) \cdot dt = \frac{1}{9}\left(1 + \frac{16}{9}\right)^{-3/2} = \frac{1}{9} \cdot \frac{(25)^{-3/2}}{9^{-3/2}} = \frac{1}{9} \cdot \frac{3^3}{5^3} = \frac{3}{125}$$

Ans - $\boxed{\int_0^\infty e^{-3t}\, t f(4t) \cdot dt = \frac{3}{125}}$

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Q2.)

(a)

Ans- $\varphi(t) = \chi_{[0,1]}$ $\quad\quad$ $\chi_{[0,1]}$ - characteristic function on $[0,1]$

$$\varphi(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$\therefore F\{\varphi(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \cdot \varphi(t) \cdot dt = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} 0 \cdot e^{ist} dt + \int_{0}^{1} e^{ist} \cdot 1 \cdot dt + \int_{1}^{\infty} e^{ist} \cdot 0 \cdot dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + \frac{e^{ist}}{is} \Big|_0^1 + 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(e^{is} - 1)}{is}$$

$$F\{\varphi(2021t)\} = \frac{1}{2021} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{(e^{is/2021} - 1)}{s/2021} \quad \left\{ \text{Change of scale Property} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{(e^{is/2021} - 1)}{is}$$

Since, $h(t) = \varphi(2021t)$

$$\therefore \hat{h}(t) = F\{\varphi(2021t)\} = \frac{1}{\sqrt{2\pi}} \frac{(e^{is/2021} - 1)}{is}$$

$$\therefore \hat{h}(0) = \lim_{s \to 0} \frac{1}{\sqrt{2\pi}} \cdot \frac{(e^{is/2021} - 1)}{is}$$

$$\hat{h}(0) = \lim_{s \to 0} \frac{1}{\sqrt{2\pi}} \cdot \frac{(i/2021) \cdot e^{is/2021}}{i} \quad \text{(L'hostipal's Rule)}$$

Ans $\Rightarrow$ $\boxed{\hat{h}(0) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2021}}$

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(b)

Ans- $\phi(t) = \delta(t)$; $\hat{g}(\omega) = \hat{\phi}(\omega/2022)$

To find $g(t)$

$\hat{g}(\omega) = \hat{\phi}(\omega/2022) = \frac{1}{2022}\ \phi(\omega/2022)\cdot 2022$

$\hat{g}(\omega) = \hat{f}\,(\phi(2022t))\cdot 2022$ {change of scale property of fourier transformation.}

$\Rightarrow f(g(t)) = f(\phi(2022t))\cdot 2022$

Taking Inverse fourier transform on both sides,

$F^{-1}\{f(g(t))\} = F^{-1}\{f(\phi(2022t))\}\cdot 2022$

$g(t) = \phi(2022t)\cdot 2022$

$g(t) = \delta(2022t)\cdot 2022$

Now we can say that,

$$\delta(t) = \begin{cases} \infty & t=0 \\ 0 & \text{otherwise} \end{cases}$$

&

$$\delta(2022t) = \begin{cases} \infty & 2022t = 0 \Rightarrow t=0 \\ 0 & \text{otherwise.} \end{cases}$$

By the property of $\delta(x)$: $\delta(ax) = 1/|a|\cdot\delta(x)$

from this we can conclude,

$\delta(2022t) = \delta(t)$

$g(t) = \frac{1}{2022}\cdot\delta(2022t)\cdot 2022$

Ans $\Rightarrow$ $\boxed{g(t) = \delta(t)}$

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Q3.)

(a)

Ans- To find the inverse fourier sine transform of $F(s) = e^{-s^2/2}$.

By the definition of inverse fourier sine transform,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-s^2/2} \sin(sx)\, ds$$

Differentiating both sides wrt x:

$$\frac{d}{dx} f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d}{dx} e^{-s^2/2} \sin(sx)\, ds.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty s\, e^{-s^2/2} \cdot \cos(sx) \cdot ds \qquad -(1)$$

Now considering $\int_0^\infty s \cdot e^{-s^2/2}\, ds$, Let $\frac{s^2}{2} = a \Rightarrow sds = da$.

The integral changes to $\int_0^\infty e^{-a}\, da = -e^{-a}\Big|_0^\infty = 1 \qquad -(2)$

Applying integration by-parts to (1):

$$\frac{d}{dx} f(x) = \sqrt{\frac{2}{\pi}} \left[\cos sx \int_0^\infty e^{-s^2/2} s \cdot ds \Big|_0^\infty - x \int_0^\infty \sin sx \cdot e^{-s^2/2} \cdot ds \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[(-\cos sx\, e^{-s^2/2} \Big|_0^\infty - x \int_0^\infty \sin sx \cdot e^{-s^2/2} \cdot ds \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[1 - x\sqrt{\frac{\pi}{2}}\, f(x) \right] = \sqrt{\frac{2}{\pi}} - x f(x)$$

Let $y = f(x)$

$$\therefore \frac{dy}{dx} = \sqrt{\frac{2}{\pi}} - xy$$

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$\therefore \frac{dy}{dx} + xy = \sqrt{\frac{2}{\pi}}$

Integrating factor for the above equation is,

$e^{\int x dx} = e^{-x^2/2}.$

solution is $y e^{x^2/2} = \sqrt{\frac{2}{\pi}} \int e^{x^2/2} . dx$

$\Rightarrow y e^{x^2/2} = \sqrt{\frac{2}{\pi}} \left(\sqrt{\frac{\pi}{2}} (-i) \operatorname{erf}\left(\frac{ix}{\sqrt{2}}\right) \right) + c$

$= -i \cdot \operatorname{erf}\left(\frac{ix}{\sqrt{2}}\right) + c_1$ where $c_1 = \sqrt{\frac{2}{\pi}} c.$

substituting $y = f(x)$ back and $f(0) = 0.$

$0 = -i \cdot \operatorname{erf}(0) + c_1$ $\therefore c_1 = 0$ & $c = 0.$

$\therefore$ Ans $\Rightarrow$

$$\boxed{f(x) = -i \operatorname{erf}\left(\frac{ix}{\sqrt{2}}\right) e^{-x^2/2}.}$$

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(b)

Ans- Given eq^n $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

since it is given that $x>0$ and the value of $u_x(0,t)$ is given.
so, we will take fourier cosine transform to solve the eq^n
Taking fourier cosine transform both sides,

$$F_c\left[\frac{\partial u}{\partial t}\right] = F_c\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\frac{\partial \bar{u}_c}{\partial t} = -\sqrt{\frac{2}{\pi}}\, u_x(0,t) - s^2 \bar{u}_c$$ {where $\bar{u}_c$ denotes the fourier cosine transform of $u(x,t)$.}

Putting $u_x(0,t) = \cos t$ we get:

$$\frac{\partial \bar{u}_c}{\partial t} = -\sqrt{\frac{2}{\pi}} \cos t - s^2 \bar{u}_c$$

The equation becomes a linear differential eq^n.
Integrating factor (I.F) $= e^{s^2 t}$

$$\bar{u}_c e^{s^2 t} = -\sqrt{\frac{2}{\pi}} \int \cos t \, e^{s^2 t} \, dt$$

Evaluating the integral in RHS by integration by parts we get:

$$\bar{u}_c e^{s^2 t} = -\sqrt{\frac{2}{\pi}}\, e^{s^2 t} \left(\frac{\sin t + s^2 \cos t}{1+s^4} \right) + c$$

$$\bar{u}_c = -\sqrt{\frac{2}{\pi}} \left(\frac{\sin t + s^2 \cos t}{1+s^4} \right) + c e^{-s^2 t}$$

Putting $t=0$ & $\bar{u}_c(s,0) = -\sqrt{\frac{2}{\pi}} \left(\frac{s^2}{1+s^4} \right) + c$

Also, it's given that $u(x,0) = \frac{1}{1+x^2}$

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Taking fourier cosine transform on both sides,

$$\bar{u}(s,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} \cos sx \, dx = I \quad \text{(let)}$$

i.e $$I = \int_0^\infty \frac{1}{1+x^2} \cos x \cdot dx$$

Differentiating wrt s: $$\frac{dI}{ds} = -\int_0^\infty \frac{x}{1+x^2} \sin(sx) \cdot dx$$

Multiplying and dividing by x:

$$\frac{dI}{ds} = -\int_0^\infty \frac{x^2}{x(1+x^2)} \sin(sx)\, dx = -\int_0^\infty \frac{x^2+1-1}{x(1+x^2)} \sin(sx)\,dx$$

$$= -\int_0^\infty \frac{\cancel{x^2+1}}{x\cancel{(1+x^2)}} \sin(sx)\, dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\, dx$$

As we know $$\int_0^\infty \frac{\sin(sx)}{x}\, dx = \frac{\pi}{2}.$$

so $$\frac{dI}{ds} = \frac{-\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\, dx$$

Again differentiating wrt s:

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x \cos sx}{x(1+x^2)}\, dx = \int_0^\infty \frac{\cos sx}{(1+x^2)}\, dx = I$$

$$\frac{d^2I}{ds^2} - I = 0$$

Solving this second order linear differential eq^n, we get,

$$I = c_1 e^{s} + c_2 e^{-s}$$

$$\frac{dI}{ds} = c_1 e^{s} - c_2 e^{-s}$$

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putting $s=0$ in the both equations:

$$\frac{dI}{ds}\Big|_{s=0} = c_1 + c_2.$$

$$\frac{dI}{ds}\Big|_{s=0} = c_1 - c_2$$

Also, $I\Big|_{s=0} = \int_0^\infty \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big|_0^\infty = \frac{\pi}{2}.$

& $\frac{dI}{ds}\Big|_{s=0} = \frac{-\pi}{2}$

$c_1 + c_2 = \pi/2 \qquad c_1 - c_2 = \pi/2$

$\therefore \quad c_1 = 0$ and $c_2 = \frac{\pi}{2}$

Putting these values in $I = c_1 e^{s} + c_2 e^{-s}$
we get, $\boxed{I = \frac{\pi}{2} e^{-s}}$

Now: $\bar{u}_c(s,0) = \frac{\pi}{2} e^{-s} = -\sqrt{\frac{2}{\pi}} \frac{s^2}{1+s^4} + C$

$$\boxed{C = \frac{\pi}{2} e^{-s} + \sqrt{\frac{2}{\pi}} \frac{s^2}{1+s^4}}$$

Putting this value and taking inverse fourier cosine transform,
we get:

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(-\sqrt{\frac{2}{\pi}} \left(\frac{\sin t + s^2 \cos t}{1+s^4} \right) + \frac{\pi}{2} e^{-s} + \sqrt{\frac{2}{\pi}} \frac{s^2}{(1+s^4)} e^{-s^2 t} \right) \cos sx \, ds$$

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Q4.)

a)

Ans - Given equation is $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$\because$ the value of $u(0,t)$ is known and $x>0$. So we can use fourier sine transform:

Taking fourier sine transform on both sides;

$$\frac{\partial \bar{u}_s}{dt} = \sqrt{\frac{2}{\pi}}\, s\, u(0,t) - s^2 \bar{u}_s$$

Putting $u(0,t)=0$ we get $\frac{d\bar{u}_s}{dt} = -s^2 \bar{u}_s$.

solving this differential equation. $\log \frac{\bar{u}_s}{A} = -s^2 t$.

$\bar{u}_s = A e^{-s^2 t}$ where A is the integration const. putting $t=0$ on both sides,

$\bar{u}_s(s,0) = A$

It is given that $u(x,0) = p(x)$

Taking fourier sine transform:

$$\bar{u}_s(s,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty p(x) \sin sx \cdot dx.$$

so, $A = \sqrt{\frac{2}{\pi}}\, F_s[P(x)]$

$$\bar{u}_s(s,t) = \sqrt{\frac{2}{\pi}}\, F_s[P(x)]\, e^{-s^2 t}$$

Taking fourier sine inverse transform to get,

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s\{p(x)\}\, e^{-\sqrt{\frac{2}{\pi}} s^2 t} \sin(sx) \cdot dx$$

where $F_s\{p(x)\} = A$ denots the fourier sine inverse of $p(x)$.

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(b)

Ans- Given eq^n: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

since it is given that $u(0,t)=0$ and $x>0$ so we can use fourier sine transform.

$$\frac{\partial \bar{u}_s}{\partial t} = \sqrt{\frac{2}{\pi}}\, s\, u(0,t) - s^2 \bar{u}_s$$

Putting $u(0,t)=0$ we get; $\frac{\partial \bar{u}_s}{\partial t} = -s^2 \bar{u}_s$

~~Putting~~ $\bar{u}_s = A e^{-s^2 t}$ where A is integration constant.

Putting $t=0$ on both sides; $\bar{u}_s(s,0) = A$

It is given that $u(x,0) = \begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \end{cases}$

Taking fourier sine transform,

$$\bar{u}_s(s,0) = \sqrt{\frac{2}{\pi}}\left[\int_0^1 x \sin sx\, dx + \int_1^2 (2-x)\sin sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-\cos s}{s} - \frac{\sin s}{s^2} + \frac{2\cos 2s}{s} - \frac{2\cos s}{s} + \frac{2\cos 2s}{s} - \frac{\cos s}{s} + \frac{\sin 2s}{s^2} - \frac{\sin s}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2\cos 2s}{s} - \frac{\cos s}{s} + \frac{\sin 2s - \sin s}{s^2}\right] = A.$$

$$\text{So, } u_s(s,t) = \sqrt{\frac{2}{\pi}}\, e^{-s^2 t}\left[\frac{2\cos 2s - \cos s}{s} + \frac{\sin 2s - \sin s}{s^2}\right]$$

Now,

taking inverse fourier sine transform to get;

$$u(x,t) = \frac{2}{\pi}\int_0^\infty e^{-s^2 t}\sin sx\left[\frac{2\cos 2s - \cos s}{s} + \frac{\sin 2s - \sin s}{s^2}\right] dx$$

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Q5.)

Ans- We have the laplace equation $\frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} = 0$.

where $u(0,y) = u(l,y) = u(x,0)$ and $u(x,a) = \frac{\sin n\pi x}{l}$

Let us assume the solution to be of the from $u(x,y) = X(x) \cdot Y(y)$.

Then we have: $\frac{\partial u}{\partial x} = X'Y$ $\qquad \frac{\partial u}{\partial y} = XY'$

$\frac{\partial^2 u}{\partial x^2} = X''Y$ $\qquad \frac{\partial^2 u}{\partial y^2} = XY''$

The given equation now becomes;

$X''Y + XY'' = 0$

Using the method of separation of variables

$\frac{X''}{X} = \frac{-Y''}{Y} = k$

following three cases arise,

Case- I

$k = -u^2$ where $u \neq 0$ is a real number we will have:

$\frac{d^2X}{dx^2} + u^2 X = 0$ and $\frac{d^2Y}{dy^2} - u^2 Y = 0$

On solving we have,

$X = (c_1 \cos ux + c_2 \sin ux)$ and $Y = c_3 e^{-u y} + c_4 e^{-u y}$

Combining the two trial solution is,

$u(x,y) = (c_1 \cos ux + c_2 \sin ux) \cdot (c_3 e^{uy} + c_4 e^{-uy})$

Case-II

$k = u^2$, $u \neq 0$ is a real number we will have:

$\frac{d^2X}{dx^2} - u^2 X = 0$ and $\frac{\partial^2 Y}{\partial y^2} + u^2 Y = 0$

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On solving we get;

$X = C_5 e^{ux} + C_6 e^{-ux}$ and $Y = C_7 \cos uy + C_8 \sin uy$.

$u(x,y) = (C_5 e^{ux} + C_6 e^{-ux}) \cdot (C_7 \cos uy + C_8 \sin uy)$.

Case III

$k = 0$, we will have:

$\frac{d^2 X}{dx^2} = 0$ and $\frac{d^2 Y}{dy^2} = 0$

On solving the above equations:

$X = C_9 x + C_{10}$ and $Y = C_{11} y + C_{12}$

$u(x,y) = (C_9 x + C_{10})(C_{11} y + C_{12})$.

According to the given conditions:

in case 2 we have:

$u(x,0) = C_7 \cos uy \cdot (C_5 e^{ux} + C_6 e^{-ux})$

$= C_7 = 0$

$u(l,y) = C_5 (e^{ul} - e^{-ul}) \; C_7 \sin ky = 0$

This cannot be true as $(e^{ui} - e^{-ui})$ is a non-zero term and the constants cannot be zero.

Next in case III: $u(0,y) = C_{10}(C_{11} y + C_{12}) \Rightarrow C_{10} = 0$.

$u(x,y) = C_9 x \, (C_{11} y + C_{12})$

$u(x,0) = 0 \Rightarrow C_9 x (C_{12}) \Rightarrow C_{12} = 0$

$u(x,y) = C_3 x \cdot C_{11} y$

$u(l,y) = C_3 l C_{11} y = 0$

But this is contradictory as C_3, C_{11} and l are all non-zero quantities. Hence this solution is also discarded.

Shruti

So, the only acceptable solution to the problem in case I.

$u(x,y) = (c_1 \cos ux + c_2 \sin ux)\cdot(c_3 e^{uy} + c_4 e^{-uy})$

$u(0,y) = c_1 (c_3 e^{uy} + c_4 e^{-uy}) \Rightarrow c_1 = 0$

So, $u(x,y) = c_2 \sin ux \, (c_3 e^{uy} + c_4 e^{-uy})$

$u(l,y) = c_2 \sin ul \cdot c_3 (e^{uy} - e^{-uy}) = 0$

$\Rightarrow \sin ul = 0$ or $ul = n\pi \Rightarrow u = \frac{n\pi}{l}$

$$\Rightarrow \boxed{u(x,y) = c_2 \sin\left(\frac{n\pi}{l} x\right) c_3 (e^{uy} - e^{-uy})}$$