MA 203: Assignment-3

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Q1.)

a)

Ans- Given the second order linear differential equation:

$$y'' - xy = 0 \quad -(1)$$

comparing with the standard form,

$$y'' + P(x)y' + Q(x)y = 0$$

So,

$$P(x) = 0 \quad \text{and} \quad Q(x) = -x.$$

And both of them are analytic at $x=0$.

$\therefore$ $x=0$ is an Ordinary point of the above differential eq^n.

So, let the trial solution be -

$$y = \sum_{m=0}^{\infty} C_m \cdot x^m \quad -(1)$$

On differentiating w.r.t x,

$$y' = \sum_{m=1}^{\infty} C_m \cdot m \cdot x^{m-1} \quad \text{and} \quad y'' = \sum_{m=2}^{\infty} C_m \cdot m \cdot (m-1) \cdot x^{m-2} \quad -(2)$$

Substituting (1) & (2) in the given differential equation,

$$\sum_{m=2}^{\infty} C_m \cdot m(m-1)x^{m-2} - x \cdot \sum_{m=0}^{\infty} C_m \cdot x^m = 0$$

$$C_2 \cdot 2 \cdot (1) \cdot x^0 + \sum_{m=3}^{\infty} C_m \cdot m \cdot (m-1) \cdot x^{m-2} - \sum_{m=0}^{\infty} C_m x^{m+1} = 0$$

$$2C_2 + \sum_{m=3}^{\infty} C_m \cdot m \cdot (m-1) \cdot x^{m-2} - \sum_{m=3}^{\infty} C_{m-3} \cdot x^{m-2} = 0$$

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Comparing the coefficients of same powers of x.

$2C_2 = 0 \Rightarrow C_2 = 0$

$C_m \cdot m(m-1) - C_{m-3} = 0 \Rightarrow C_m = \frac{C_{m-3}}{m(m-1)}$

$\therefore$

$C_2 = 0$ and $C_m = \frac{C_{m-3}}{m(m-1)}$ {It's a Recurrence Relation}

$\Rightarrow C_3 = \frac{C_0}{6}, \quad C_4 = \frac{C_1}{12}, \quad C_5 = \frac{C_2}{20} = 0, \quad C_6 = \frac{C_3}{30} = \frac{C_0}{180}$

$C_7 = \frac{C_4}{42} = \frac{C_1}{504}, \quad C_8 = \frac{C_5}{56} = 0$

$\therefore y = \sum_{m=0}^{\infty} C_m \cdot x^m = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + \ldots$

$y = C_0 + C_1 x + 0 + \frac{C_0}{6} x^3 + \frac{C_1}{12} x^4 + 0 + \frac{C_0}{180} x^6 + \ldots$

$y = C_0 \left[1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960} + \ldots \right]$ where $C_0, C_1 \in R$

$+ C_1 \left[x + \frac{x^4}{12} + \frac{x^7}{504} + \ldots \right]$

$$\boxed{\begin{aligned} &\therefore \; y = C_0\, y_1(x) + C_1\, y_2(x) \\ &\text{where, } \; y_1(x) = 1 + \frac{x^3}{6} + \frac{x^6}{180} + \ldots \\ &\phantom{\text{where, }} \; y_2(x) = x\left[1 + \frac{x^3}{12} + \frac{x^6}{504} + \ldots\right] \end{aligned}}$$

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Q1.) (b)

Ans- We need to check if the given eqn

$$\int_0^x x^{-n} J_{n+1}(x) \cdot dx = \frac{1}{2^n \cdot \Gamma(n+1)} - x^{-n} \cdot J_n(x) \text{ is}$$

True or false.

So, from Recurrence relations properties of bessel function.

$$\int_0^x x^{-n} \cdot J_{n+1}(x) \cdot dx = \int_0^x \frac{d}{dx}\left(-x^{-n} \cdot J_n(x)\right) \cdot dx$$

$$\frac{d}{dx}\left[x^{-n} \cdot J_n(x)\right] = \frac{d}{dx}\left[x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{2r+n}\right]$$

$$= \frac{d}{dx}\left[\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{2r} \cdot \frac{1}{2^n}\right]$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot 2r \cdot \frac{x^{2r-1}}{2^{2r}} \cdot \frac{1}{2^n}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{2r-1} \cdot \frac{1}{2^n}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!\,\Gamma(n+k+2)} \cdot \left(\frac{x}{2}\right)^{2k+1} \cdot \frac{1}{2^n}$$

$$= (-1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma((n+1)+k+1)} \cdot \left(\frac{x}{2}\right)^{2k+n+1} \cdot x^{-n}$$

$$= (-1)\, J_{n+1}(x) \cdot x^{-n}$$

$$= -x^{-n} \cdot J_{n+1}(x)$$

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$$\int_0^x x^{-n} \cdot J_{n+1}(x) \cdot dx = \int_0^x \frac{d}{dx}\left(-x^{-n} \cdot J_n(x)\right) dx$$

$$= \left[-x^{-n} \cdot J_n(x)\right]_0^x$$

$$= -x^{-n} \cdot J_n(x) + \lim_{x \to 0} \left[x^{-n} \cdot J_n(x)\right]$$

Now, $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$

$$x^{-n} \cdot J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{2r} \cdot \frac{1}{2^n}$$

$$\lim_{x \to 0} \left[x^{-n} \cdot J_n(x)\right] = \frac{(-1)^0}{0!\,\Gamma(n+1)} \cdot \frac{1}{2^n} + 0 + 0 + \ldots$$

$$= \frac{1}{2^n \cdot \Gamma(n+1)}.$$

$$\therefore \boxed{\int_0^x x^{-n} \cdot J_{n+1}(x) \cdot dx = \frac{1}{2^n\,\Gamma(n+1)} - x^{-n} \cdot J_n(x)}$$

Hence, the given statement is True.

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Q2.) (a)

Ans -

$$\int_{-1}^{1} x^2 P_n^2(x) \cdot dx$$

From the Recurrence Relations of Legendre's polynomials,

$$x \cdot P_n = \frac{(n+1)}{(2n+1)} P_{n+1} + \frac{n}{(2n+1)} P_{n-1}$$

$$x^2 \cdot P_n = x \cdot \frac{(n+1)}{(2n+1)} \cdot P_{n+1} + x \cdot \frac{n}{(2n+1)} \cdot P_{n-1}$$

$$x^2 \cdot P_n^2 = x \cdot \frac{(n+1)}{(2n+1)} \cdot P_{(n+1)} \cdot P_n + x \cdot \frac{n}{(2n+1)} \cdot P_{n-1} \cdot P_n$$

$$\therefore \int_{-1}^{1} x^2 \cdot P_n^2(x) \cdot dx = \int_{-1}^{1} \frac{n+1}{2n+1} \cdot x \cdot P_n \cdot P_{n+1} \cdot dx + \int_{-1}^{1} \frac{n}{2n+1} \cdot x \cdot P_n \cdot P_{n-1} \cdot dx.$$

$$= \frac{n+1}{(2n+1)} \cdot \frac{2(n+1)}{4(n+1)^2 - 1} + \frac{n}{(2n+1)} \cdot \frac{2n}{4n^2 - 1}$$

(using property, $\int_{-1}^{1} x \cdot P_n(x) \cdot P_{n-1}(x)\, dx = \frac{2n}{4n^2-1}$)

$$= \frac{1}{(2n+1)} \cdot \left[\frac{2(n+1)^2}{4(n+1)^2 - 1} + \frac{2n^2}{4n^2 - 1} \right]$$

$\therefore$ Hence, Shown ✓

Q2) (b)

Ans - Bessel equation of order 0 is given as,

$$x^2 y'' + xy' + x^2 y = 0$$

(The Bessel eq^n of order n is - $x^2y'' + xy' + (x^2-n^2)y = 0$)

The solution of the above eq^n is given by Bessel function of order 0, i.e. $y(x) = J_0(x)$

If we take $y(x) = \dfrac{z(x)}{\sqrt{x}}$, $x > 0$

i.e $z(x) = \sqrt{x} \cdot y(x) = \sqrt{x} \cdot J_0(x)$

& for $x>0$, $J_0(x)$ and $z(x) = \sqrt{x} \cdot J_0(x)$ have some positive zeroes.

Putting it in the above Bessel feq^n, we get

$$x^2 \frac{d^2}{dx^2}\left[\frac{z(x)}{\sqrt{x}}\right] + x \cdot \frac{d}{dx}\left[\frac{z(x)}{\sqrt{x}}\right] + x^2 \frac{z(x)}{\sqrt{x}} = 0$$

$$\frac{d}{dx}\left(\frac{z(x)}{\sqrt{x}}\right) = \frac{z'(x)}{\sqrt{x}} - \frac{z(x)}{2x\sqrt{x}}$$

$$\frac{d^2}{dx^2}\left(\frac{z(x)}{\sqrt{x}}\right) = \frac{z''(x)}{\sqrt{x}} - \frac{z'(x)}{x\sqrt{x}} + \frac{3z(x)}{4x^2\sqrt{x}}$$

$$x^2\left[\frac{z''(x)}{\sqrt{x}} - \frac{z'(x)}{x\sqrt{x}} + \frac{3z(x)}{4x^2\sqrt{x}}\right] + x\left[\frac{z'(x)}{\sqrt{x}} - \frac{z(x)}{2x\sqrt{x}}\right] + \frac{x^2 z(x)}{\sqrt{x}} = 0$$

$$x\sqrt{x} \cdot z''(x) + \left[\frac{3}{4} - \frac{1}{2}\right] \cdot \frac{z(x)}{\sqrt{x}} + x\sqrt{x} \cdot z(x) = 0$$

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$$z''(x) + \frac{z(x)}{4x^2} + z(x) = 0 \quad , \quad x>0.$$

$$z''(x) + \left(\frac{1}{4x^2} + 1\right) . z(x) = 0$$

Let there be another differential equation

$$z''(x) + z(x) = 0$$

one possible solution of above equation can be,

$$z(x) = \sin(x).$$

$\therefore$ $z(x) = \sin(x)$ and $z(x) = \sqrt{x}\cdot J_0(x)$ are non-trivial solutions of $z'' + z = 0$ respectively for $x>0$

$$z'' + \left[\frac{1}{4x^2} + 1\right] z = 0.$$

also, $1 < 1 + \dfrac{1}{4x^2}$ $\left\{ g_1 \le g_2 \text{ in } z'' + g_1(x) z = 0 \quad g_1(x) = 1. \quad z'' + g_2(x) z = 0 \quad g_2(x) = \dfrac{1}{4x^2} + 1 \right\}$

$\therefore$ By strum comparison Theorem,
Between any two consecutive zeroes of $\sin x$ (say θ_1 and θ_2) there is atleast one zero of $\sqrt{x}\cdot J_0(x)$.

And, since on $(0,\infty)$, $\sin(x)$ has infinite zeroes
$\therefore$ $\sqrt{x}\cdot J_0(x)$ also has infinite zeroes.
$\Rightarrow$ $J_0(x)$ has infinite zeroes.

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Proof of Strum Comparison Theorem

In the interval (a, b), let y_1 and y_2 be non-trivial solution of the differential equations,

$$y'' + g_1(x)y = 0 \quad \& \quad y'' + g_2(x)\, y = 0$$

respectively, where g_1 & g_2 are continuous real valued functions in (a, b) such that $g_1 \leq g_2$

Let x_k and x_{k+1} be the consecutive roots of y_1 on (a, b). without loss of generality, we can assume that $y_1(x) > 0$ on (x_k, x_{k+1}) which implies that $y_1'(x_k) \geq 0$ and $y_1'(x_{k+1}) \leq 0$.

Let's assume that y_2 has no zeroes in (x_k, x_{k+1}) - (1)

If $y_2(x) > 0$ on (x_k, x_{k+1}), then

$$y_1'' + g_1(x) \cdot y_1 = 0 \qquad y_2'' + g_2(x) \cdot y_2 = 0$$

$$y_2\left(y_1'' + g_1(x) \cdot y_1\right) = 0 \qquad y_1\left(y_2'' + g_2(x) \cdot y_2\right) = 0$$

Subtracting the two equations,

$$y_1'' \cdot y_2 - y_2'' \cdot y_1 = \left[g_2(x) - g_1(x)\right] y_1 \cdot y_2$$

Integrating both sides on $[x_k, x_{k+1}]$

$$\int_{x_k}^{x_{k+1}} [g_2(x) - g_1(x)]\, y_1(x) \cdot y_2(x) \cdot dx = \int_{x_k}^{x_{k+1}} [y_1'' \cdot y_2(x) - y_2''(x) \cdot y_1(x)] \cdot dx.$$

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$$= \left[y_1'(x) \cdot y_2(x) \right]_{x_k}^{x_{k+1}} - \int_{x_k}^{x_{k+1}} y_1'(x) \cdot y_2'(x) \cdot dx$$

$$- \left[y_2'(x) \cdot y_1(x) \right]_{x_k}^{x_{k+1}} + \int_{x_k}^{x_{k+1}} y_1'(x) \cdot y_2'(x) \cdot dx.$$

$$= y_1'(x_{k+1}) \cdot y_2(x_{k+1}) - y_1'(x_k) \cdot y_2(x_k)$$

The L.H.S. $\int_{x_k}^{x_{k+1}} [g_2(x) - g_1(x)] y_1(x) \cdot y_2(x) \cdot dx$ is positive (LHS>0)

since by assumption,
$g_1(x) \leq g_2(x)$, $y_1(x) > 0$, $y_2(x) > 0$ on (x_k, x_{k+1})

but the RHS, $y_1'(x_{k+1}) \cdot y_2(x_{k+1}) - y_1'(x_k) \cdot y_2(x_k)$ is a non-positive, since $y_1'(x_k) \geq 0$, $y_1'(x_{k+1}) \leq 0$, $y_2(x_k) \geq 0$, $y_2(x_{k+1}) \geq 0$

which leads to a contradiction,
(If we take $y_2(x) < 0$, then again we arrive at similar contradiction.)
$\Rightarrow$ The assumption in (1) is false.

i.e y_2 has atleast one zero in (x_k, x_{k+1}).

Q3.)

Ans- $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = \lambda y$, $0 < x < \pi$; $y(0) = y(\pi) = 0$

$y'' - 3y' - \lambda y = 0$

Solving the above second order homogeneous diff. eq^n.

→ auxilliary eq^n : $r^2 - 3r - \lambda = 0$

$D = 9 + 4\lambda$

1) If $\lambda > \frac{-9}{4}$ ($D > 0$, real and distinct roots).

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ (r_1 and r_2 are roots of auxiliary eq^n)

satisfying the given boundary conditions,

$y(0) = c_1 \cdot 1 + c_2 \cdot 1$

$0 = c_1 + c_2$

$\underline{c_1 = -c_2}$

$y(\pi) = c_1 e^{r_1 \pi} + c_2 e^{r_2 \pi}$

$0 = c_1 \left[e^{r_1 \pi} - e^{r_2 \pi} \right]$ $(c_1 = -c_2)$

$0 = e^{r_1 \pi} - e^{r_2 \pi}$

$\Rightarrow r_1 = r_2$

But $D > 0$ and roots are real and distinct

∴ No possible sol^n for $\lambda > -9/4$.

2) If $\lambda = -9/4$ ($D = 0$, real and identical roots)

$y = (c_1 + c_2 x) e^{r_1 x}$ (r_1 is root of auxiliary eq^n).

Satisfying the given boundary conditions,

$y(0) = (c_1 + 0) \cdot 1 \Rightarrow 0 = c_1$

$y(\pi) = (c_1 + c_2 \pi) e^{r_1 \pi} = 0 = c_2 \pi \cdot e^{r_1 \pi} \Rightarrow \underline{c_2 = 0}$

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$\therefore y(x) = (0 + 0 \cdot x) e^{r_1 x} = 0$

which is a trivial solution.

$\therefore$ No non-trivial solution for $\lambda = -9/4$.

3) If $\lambda < -9/4$ ($D < 0$, complex roots)

$y = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$ { $\alpha \pm \beta i$ are roots of auxiliary eq^n $\alpha = 3/2$, $\beta = \sqrt{-(9+4\lambda)}/2$. }

satisfying the given boundary conditions,

$y(0) = 1 \cdot [c_1 + 0]$

$0 = c_1$

$y(\pi) = e^{\alpha\pi}[0 + c_2 \sin(\beta\pi)] = 0$

$\Rightarrow \beta\pi = n\pi, \quad n = 1, 2, 3, \ldots$

$\beta = n$

$\frac{\sqrt{-(9+4\lambda)}}{2} = n$

$-\frac{(9+4\lambda)}{4} = n^2$

$\lambda = \frac{-9-4n^2}{4}$

$\lambda = \frac{-9}{4} - n^2, \quad n = 1, 2, 3, \ldots$

(since for $n=0$, $\lambda = \frac{-9}{4}$ gave trivial solution.)

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$\therefore\ y = e^{\alpha x} \cdot c_2 \cdot \sin(\beta x)$

$$\boxed{\begin{array}{ll} y(x) = c_2 \cdot e^{3x/2} \cdot \sin\left[\sqrt{\frac{-(9+4x)}{\cancel{4}2}} \cdot x \right] & : \text{Eigen function} \\ \text{where,} \quad \lambda = -\frac{9}{4} - n^2 , \quad n \in \mathbb{N} & : \text{Eigenvalue.} \end{array}}$$

OR

$$\boxed{\begin{array}{l} y_n(x) = c\, e^{3x/2} \sin(nx) \\ \lambda_n = -\frac{9}{4} - n^2 , \quad n \in \mathbb{N} \end{array}}$$