MA203: Mathematical Methods Assignment 1 & 2.

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Q1.

Ans-

$$f(t) = e^t \frac{d^n}{dt^n}[t^n e^{-t}].$$

Laplace transform of $t^n e^{-t}$.

As, $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$

using the first shifting property: $L[e^{at} f(t)] = f(s-a)$

$\therefore$

$$L[t^n e^{-t}] = \frac{n!}{(s+1)^{n+1}}$$

Using the Laplace property of derivatives,

$$L\left[\frac{d^n}{dt^n} g(t)\right] = s^n L[g(t)] - s^{n-1} g(0) - s^{n-2} g'(0) - \dots - g^{(n-1)}(0).$$

Here, $g(t) = t^n e^{-t}$.

We can clearly see that all n order derivatives of $g(t)$ will be zero at $t=0$.

i.e $g(0) = g'(0) = g''(0) = \dots \; g^{(n-1)}(0) = 0$.

$$\therefore \cancel{L[g(t)]} = L\left[\frac{d^n}{dt^n} g(t)\right] = \frac{s^n \cdot n!}{(s+1)^{n+1}}$$

Again, using first shifting property, $L[e^{at} f(t)] = f(s-a)$

$\therefore$ Ans -

$$\boxed{L\left\{e^t \frac{d^n}{dt^n}[t^n e^{-t}]\right\} = \frac{(s-1)^n \, n!}{s^{n+1}}}$$

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Q2.

Ans-

$$\frac{d^4 u}{dx^4} = A\,\delta\left(x - \frac{l}{4}\right)$$

$$\Rightarrow L\left[\frac{d^4(u)}{dx^4}\right] = A\,L\left[\delta\left(x - \frac{l}{4}\right)\right]$$ taking laplace on both sides

$$\Rightarrow s^4 L(u(x)) - s^3 u(0) - s^2 u'(0) - s u''(0) - u'''(0)$$

$$= A\, e^{-\left(\frac{l}{4}\right)s} \cdot L(\delta(x))$$

In second shifting property,

if $L^{-1}(f(s)) = F(t)$

then $L^{-1}(e^{-as} f(s)) = F(t-a)\,H(t-a)$

So, $L^{-1}(L(\delta(x))) = \delta(x)$.

then $L^{-1}(e^{-\frac{l}{4}s} L(\delta(x))) = \delta\left(x - \frac{l}{4}\right) H\left(x - \frac{l}{4}\right)$

$$\Rightarrow \begin{cases} \delta(x-a) & x \geq \frac{l}{4} \\ 0 & x < \frac{l}{4}. \end{cases} \qquad \left(\forall\ a = \frac{l}{4}\right)$$

$\Rightarrow \{\delta(x-a)$; because $\delta(x-a) = 0$ at $x < l/4$.

$$\therefore L^{-1}\left(e^{-\frac{l}{4}s} L(\delta(x))\right) = \delta\left(x - \frac{l}{4}\right)$$

$$\Rightarrow L\left(\delta\left(x - \frac{l}{4}\right)\right) = e^{-\frac{l}{4}} L(\delta(x)).$$

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$u(0) = u''(0) = 0$ (As given)

$$s^4 L(u(x)) - s^2 u'(0) - u'''(0) = A e^{-\frac{ls}{4}} \cdot L(\delta(x))$$

$$\Rightarrow L(u(x)) = \frac{A e^{-\frac{ls}{4}}}{s^4} + \frac{u'(0)}{s^2} + \frac{u'''(0)}{s^4} \quad \because L(\delta(x)) = 1$$

$$\Rightarrow u(x) = A L^{-1}\left(\frac{e^{-\frac{ls}{4}}}{s^4}\right) + u'(0) L^{-1}\left(\frac{1}{s^2}\right) + u'''(0) L^{-1}\left(\frac{1}{s^4}\right)$$

$$\Rightarrow u(x) = \frac{A}{3!} H\left(x - \frac{l}{4}\right) \cdot \left(x - \frac{l}{4}\right)^3 + u'(0)x + \frac{u'''(0) x^3}{3!}$$

{ using second shifting and, $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$ — (1)

& $u(l) = 0$ as given

So, $u(l) = \frac{A}{3!} H\left(l - \frac{l}{4}\right)\left(l - \frac{l}{4}\right)^3 + u'(0) l + \frac{u'''(0)(l)^3}{3!}$

&

for $x > l/4$,

$$u(x) = \frac{A}{3!}\left(x - \frac{l}{4}\right)^3 + u'(0)x + \frac{u'''(0)}{3!} x^3$$

So,

$u'(x) \quad \forall \quad x > l/4$

$$u'(x) = \frac{A}{3!}\left(x - \frac{l}{4}\right)^2 + u'(0) + 3 \frac{u'''(0)}{3!} x^2$$

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$\therefore \quad \frac{A}{3!} \cdot 3\left(l-\frac{l}{4}\right)^2 + u'(0) + \frac{3u'''(0)\,l^2}{3!} = 0$ As given,

— (2)

$\therefore$ From results (1) & (2) we have.

$u'(0) = \frac{9}{256} A l^2 \qquad u'''(0) = \frac{-81}{128} A.$

So,

$$\boxed{u(x) = \frac{A}{3!} H\left(x-\frac{l}{4}\right)\left(x-\frac{l}{4}\right)^3 + \frac{9}{256} x A l^2 - \frac{27}{256} A x^3}$$

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Q3.

Ans - $F_s\left[\frac{e^{-ax}}{x}\right]$

The fourier sine transform of $f(x)$ is,

$$F_s(s) = \int_0^\infty f(x)\sin sx \cdot dx = \int_0^\infty \frac{e^{-ax}}{x}\sin sx \cdot dx \quad -(1)$$

Differentiating on both sides wrt s,

$$\frac{d}{ds}[F_s(s)] = \int_0^\infty \frac{e^{-ax}}{x} \cdot x \cdot \cos sx \cdot dx = \int_0^\infty e^{-ax} \cdot \cos sx \cdot dx.$$

$$\left\{ \text{As, } \int e^{-ax} \cdot \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c \right\}$$

$$\frac{d}{ds}[F_s(s)] = \frac{e^{-as}}{a^2+s^2}(-a\cos sx + s\sin sx)\Big|_0^\infty$$

$$= \frac{1}{a^2+s^2}[0 - 1(-a+0)]$$

$$\Rightarrow \frac{d}{ds}[F_s(s)] = \frac{a}{a^2+s^2}$$

$$F_s(s) = \int \frac{a}{a^2+s^2} ds = a\int \frac{1}{a^2+s^2} \cdot ds = \frac{a}{a} \cdot \tan^{-1}\frac{s}{a} + c$$

$$F_s(s) = \tan^{-1}\left(\frac{s}{a}\right) + c$$

To find c (Integration constant) we will use the boundary condition values.

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In equation (1),

when $s=0 \Rightarrow F_s(s)=0$.

$\therefore$ putting this in above eq^n we will get $c=0$.

So,

$$\boxed{F_s(s) = \tan^{-1}\left(\frac{s}{a}\right)}$$

By inverse fourier sine transform, we have,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \cdot dx$$

$$\Rightarrow \frac{e^{-ax}}{x} = \frac{2}{\pi} \int_0^\infty \tan^{-1}\left(\frac{s}{a}\right) \cdot \sin sx \cdot ds.$$

Putting $x=1$ on both sides

$$e^{-a} = \frac{2}{\pi} \int_0^\infty \tan^{-1}\left(\frac{s}{a}\right) \cdot \sin sx \cdot ds.$$

Now,

changing the variable from s to x,

$$\boxed{\int_0^\infty \tan^{-1}\left(\frac{x}{a}\right) \cdot \sin x \cdot dx = \frac{\pi}{2} e^{-a}.}$$

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Q4.

Ans-

$f(x) = e^{-ax}$

Fourier cosine transform of function $f(x)$;

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cdot \cos sx \cdot dx = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

Using the second shifting theorem for fourier transform,

As,

$$L[\cos ax] = \frac{s}{a^2+s^2}$$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

As,

$$g(x) = \begin{cases} 1 & ; \quad 0 < x < b \\ 0 & ; \quad x > b \end{cases}$$

By definition,

$$F_c[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cdot \cos sx \cdot dx = \sqrt{\frac{2}{\pi}} \int_0^b \cos sx \cdot dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^b = \sqrt{\frac{2}{\pi}} \frac{\sin bx}{s}$$

Now, let

$$I = \int_0^\infty \left(\frac{(t-3)\sin(t+3) + (t+3)\sin(t-3)}{(t^2+4)(t^2-9)} \right) dt$$

$$= \int_0^\infty \frac{1}{(t^2+4)} \left[\frac{\sin(t+3)}{(t+3)} + \frac{\sin(t-3)}{(t-3)} \right] dt$$

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$$I = \int_0^\infty \frac{1}{s^2+2^2}\left[\frac{\sin(s+3)}{(s+3)} + \frac{\sin(s-3)}{(s-3)}\right] ds \quad \left\{\text{Replacing variables}\right\}$$

$$= \int_0^\infty \frac{1}{2}\sqrt{\frac{\pi}{2}}\left[\sqrt{\frac{2}{\pi}}\cdot\frac{2}{s^2+2^2}\right]\left[\frac{\sin(s+3)}{(s+3)} + \frac{\sin(s-3)}{(s-3)}\right] ds$$

$$= \int_0^\infty \frac{\sqrt{\pi}}{2\sqrt{2}}\left[F_c\{(f(x))\}\right]\left[\frac{\sin(s+3)}{(s+3)} + \frac{\sin(s-3)}{s-3}\right] ds \quad \left\{f(x) = e^{-ax}, a=2\right\}$$

$$I = \frac{\pi}{2}\int_0^\infty F_c(f(x))\,\frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\,\frac{\sin(s+3)}{s+3} + \sqrt{\frac{2}{\pi}}\,\frac{\sin(s-3)}{s-3}\right] ds.$$

As we know, $F_c(g(x)\cos ax) = \frac{1}{2}\left[F_c(s-a) + F_c(s+a)\right]$

$$I = \frac{\pi}{2}\int_0^\infty F_c[f(x)]\cdot F_c[h(x)]\, ds$$

$\{$ where, $h(x) = g(x)\cos 3x$, $b=1.\}$

Using, perseval's identity for fourier cosine transform:

$$I = \frac{\pi}{2}\int_0^\infty f(x).\, h(x)\, dx = \frac{\pi}{2}\int_0^\infty e^{-2x} g(x).\cos 3x\, dx.$$

$$= \frac{\pi}{2}\int_0^1 e^{-2x}\cos 3x . dx \quad \{\text{Definition of } g(x) \text{ is applied.}\}$$

$$\left\{\text{As, } \int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + c\right\}$$

$$I = \frac{\pi}{2}\,\frac{e^{-2x}}{13}\left[-2\cos 3x + 3\sin 3x\right]_0^1$$

$$\boxed{I = \frac{\pi}{26}\left(\frac{(3\sin(3) - 2\cos(3))}{e^2} + 2\right)}$$

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Q5-

Ans- Given,

$$\frac{du}{dt} = \alpha \frac{\delta^2 u}{dx^2}$$

Since, the values $u(0,t)$ or $u_x(0,t)$ are not given, and $x \in (-\infty, \infty)$ we will use the complex fourier transform to solve the problem.

Taking fourier transform on both sides for $\frac{d\bar{u}}{dt} = -\alpha s^2 \bar{u}$

Here, $\bar{u}$ = fourier transform of u

$$\int \frac{\partial \bar{u}}{\bar{u}} = -\int \alpha s^2 dt$$

$$\Rightarrow \log \frac{\bar{u}}{A} = -\alpha s^2 t \qquad [A = \text{Integration const.}]$$

$$\Rightarrow \bar{u} = A e^{-\alpha s^2 t} \quad \text{or} \quad \bar{u}(s,t) = A e^{-\alpha s^2 t} \qquad -(1)$$

Putting $t=0$ on both sides, $\bar{u}(s,0) = A$.

Also, as given $u(x,0) = e^{-x^2}$

Taking Fourier Transform on both sides,

$$\bar{u}(s,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{isx} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 - isx)} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{is}{2}\right)^2} \cdot e^{-s^2/4} \cdot dx.$$

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$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-s^2/4} \int_{-\infty}^{\infty} e^{-t^2} \cdot dt$$

$\left[\text{Putting } x - \frac{is}{2} = t \Rightarrow dx = dt \right.$

and $\int_{-\infty}^{\infty} e^{-x^2} \cdot dx = \sqrt{\pi}$.

$$\bar{u}(s,0) = \frac{e^{-s^2/4}}{\sqrt{2}}.$$

So, $$A = \frac{(e^{-s^2/4})}{\sqrt{2}}.$$

Putting A in (1) eq^n.

$$\bar{u}(s,t) = \frac{e^{-s^2/4}}{\sqrt{2}} \cdot e^{-\alpha s^2 t}.$$

Taking the Inverse fourier transform on both sides -

$$u(x,t) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} \cdot e^{-s^2/4 - \alpha s^2 t - (sx)}.$$

Comparing with the given integral in the form of ε and K.

$$\boxed{K(x-\varepsilon, t) = \frac{1}{2\sqrt{\pi\alpha t}} \cdot e^{-\frac{(x-\varepsilon)^2}{4\alpha t}}}$$