Department of Mathematical Sciences, IIT (B.H.U)

Mathematical Methods: MA-203

Even Semester 2021-22

Tutorial Sheet- 5 & 6

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1. Find the Fourier transforms of each the following functions:

$$(a)\ f(x) = x\exp(-a|x|)\,, a > 0, \qquad (b)\ f(t) = x\exp(-ax^2)\,, a > 0,$$
$$(c)\ f(x) = u(x), the\ unit\ step\ function \quad (d) f(t) = K, a\ constant.$$

2. Find the Fourier transform of $f(x) = e^{-|x|}$ and hence, evaluate $\int_0^\infty \frac{x sinmx}{1+x^2}dx$.

3. Find Fourier Cosine transform of $\frac{1}{1+x^2}$ and also find Fourier Sine transform of $\frac{x}{1+x^2}$.

4. Find f(x), if its Fourier Sine transform is $\frac{e^{-as}}{s}$. Hence, deduce $F_s^{-1}\left(\frac{1}{s}\right)$.

5. If Fourier Cosine transform of f(x) is $\frac{1}{2}tan^{-1}(\frac{2}{s^2})$, then find f(x).

6. Find Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$.

   Hence, find $\int_0^\infty \frac{sin^4 t}{t^4}dt$.

7. Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ for $0 \le x < \infty,\ t > 0$ given the conditions

$$(i)\ u(x,0) = 0 \text{ for } x \ge 0 \qquad (ii)\ \frac{\partial u}{\partial x}(0,t) = -a\ (constant) \qquad (iii)\ u(x,t)\ is\ bounded.$$

8. Using suitable Fourier transformation solve

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$$

   If $u(0,t) = 0,\ u(x,0) = e^{-x}\,,\ u(x,t)\ is\ bounded.$

9. Apply appropriate Fourier transform to solve the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2};\ x > 0, \qquad t > 0$$

   subjected to the conditions

$$(i)\ u_x(0,t) = 0 \qquad (ii)\ u(x,0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases} \qquad (iii)\ u(x,t)\ is\ bounded.$$

END

TUTORIAL SHEET - 5&6

Solution 1.

(a) $f(x) = x\exp(-a|x|),\ a>0$

Since we know that if $F\{f(x)\} = F(s)$

Then $F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} F(s)$ —— ①

So, first we will find Fourier transform of $e^{-a|x|},\ a>0$

we know that $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} e^{-a|x|}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{isx} e^{ax}\,dx + \int_{0}^{\infty} e^{isx} e^{-ax}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(a+is)x}\,dx + \int_{0}^{\infty} e^{-(a-is)x}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{\frac{e^{(a+is)x}}{a+is}\right\}_{-\infty}^{0} + \left\{\frac{e^{-(a-is)x}}{-(a-is)}\right\}_{0}^{\infty}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a+is} + \frac{1}{a-is}\right] = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+s^2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

Now, from equ^n ①, we have

$$F\{xf(x)\} = -i\frac{d}{ds} F(s)$$

$$\Rightarrow F\{xe^{-a|x|}\} = -i\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}\right]$$

$$= -ia\sqrt{\frac{2}{\pi}} \left[\frac{-2s}{(a^2+s^2)^2}\right]$$

$$= 2\sqrt{\frac{2}{\pi}} \frac{ias}{(a^2+s^2)^2}$$

(b) $f(x) = xe^{-ax^2}$, $a>0$

First we will find Fourier transform of e^{-ax^2}.

Using the identity $x^2 - isx = x^2 - isx + \frac{s^2}{4} - \frac{s^2}{4}$

$$= \left(x - \frac{is}{2}\right)^2 + \frac{s^2}{4}$$

We see that $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} e^{-\cancel{a}x^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 - isx)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left\{\left(x - \frac{is}{2}\right)^2 + \frac{s^2}{4}\right\}}\, dx$$

$$= \frac{e^{-s^2/4}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{is}{2}\right)^2}\, dx$$ Let $x - \frac{is}{2} = y$

$$= \frac{e^{-s^2/4}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2}\, dy$$ $\left[\because \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\right.$

$$= \frac{e^{-s^2/4}}{\sqrt{2\pi}} \cdot \sqrt{\pi} = \frac{e^{-s^2/4}}{\sqrt{2}}$$

So, $F\{e^{-x^2}\} = \frac{e^{-s^2/4}}{\sqrt{2}}$

$[\because$ change of scale property: if $F\{f(x)\} = F(s)$, then $F\{f(ax)\} = \frac{1}{|a|} F(s/a)$

Therefore, we have

$$F\{e^{-ax^2}\} = F\{e^{-(\sqrt{a}x)^2}\} = \frac{1}{\sqrt{2a}} e^{-\frac{1}{4}\left(\frac{s}{\sqrt{a}}\right)^2} = \frac{1}{\sqrt{2a}} e^{-\frac{s^2}{4a}}$$

$\left[\because \text{~~since~~} \quad F\{xf(x)\} = -i\frac{d}{ds} F(s)\right]$

Then $F\{xe^{-ax^2}\} = -i\frac{d}{ds}\left[\frac{1}{\sqrt{2a}} e^{-\frac{s^2}{4a}}\right] = \frac{is}{(2a)^{3/2}} e^{-s^2/4a}$

(c) $f(x) = u(x)$, the unit step function; $u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

let $u_\alpha(x) = \begin{cases} e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$$u(x) = \lim_{\alpha \to 0} u_\alpha(x)$$

$$F(u(x)) = \lim_{\alpha \to 0} F(u_\alpha)(s) = \lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha x} e^{isx} \, dx$$

$$= \lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha - is}$$

$$= \lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi}} \left[\frac{\alpha}{\alpha^2 + s^2} + \frac{is}{\alpha^2 + s^2} \right] \quad \text{— (1)}$$

$\because$ we know that if

$$R_\varepsilon(x) = \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2} \quad \text{for } \varepsilon > 0$$

Then $\delta(x) = \lim_{\varepsilon \to 0} R_\varepsilon(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{otherwise} \end{cases}$ — (2)

where $\delta(x)$ is dirac delta function

From (1) & (2), we have

$$F(u(x)) = \frac{1}{\sqrt{2\pi}} \left[\pi \delta(s) + \frac{i}{s} \right]$$

(d) $f(x) = K$, a constant

Let $u'(x) = \begin{cases} 0 & x > 0 \\ 1 & x \le 0 \end{cases}$

Similarly as part (c), we can show that

$$F(u'(x)) = \frac{1}{\sqrt{2\pi}} \left[\pi \delta(s) - \frac{i}{s} \right] \quad \text{— (1)}$$

$$f(x) = K\left[u(x) + u'(x)\right]$$

where $u(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases}$

and $F(u(x)) = \frac{1}{\sqrt{2\pi}} \left[\pi \delta(s) + \frac{i}{s} \right]$ — (2)

using equ's (1) & (2), we obtain

$$F(f(x)) = K\left[F(u(x) + u'(x))\right]$$

$$= \frac{K}{\sqrt{2\pi}} \left[\pi \delta(s) + \pi \delta(s) + \cancel{\frac{i}{s}} - \cancel{\frac{i}{s}} \right]$$

$$= \sqrt{2\pi}\, K\, \delta(s)$$

Solution 2: First we will find Fourier sine transform of $e^{-|x|}$. Then we will evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx$

The Fourier sine transform of $f(x)$ is given by

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx \, f(x)\, dx$$

$\because$ x is +ve in the interval $[0, \infty]$.

i.e. $e^{-|x|} = e^{-x}$

$$\Rightarrow \quad F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \, dx \qquad \Bigg| \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$$

$$\Rightarrow \quad F_s(s) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right]_0^\infty$$

$$\Rightarrow \quad F_s(s) = \sqrt{\frac{2}{\pi}} \left(\frac{s}{1+s^2} \right)$$

Now, by inverse Fourier sine transform,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds \qquad \because f(x) = e^{-x} \text{ in the interval } (0, \infty)$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sx \, ds$$

$$\Rightarrow \quad \int_0^\infty \frac{s}{1+s^2} \sin sx \, ds = \frac{\pi}{2} e^{-x}$$

changing x to m,

$$\int_0^\infty \frac{s}{1+s^2} \sin ms \, ds = \frac{\pi}{2} e^{-m}$$

or $$\int_0^\infty \frac{x}{1+x^2} \sin mx \, dx = \frac{\pi}{2} e^{-m}$$

Solution 3. First we will find Fourier cosine transform of $\frac{1}{1+x^2}$.

By definition, we have

$$F_c\left\{\frac{1}{1+x^2}; s\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos sx}{1+x^2}\,dx = I, \text{ (say)} \quad \text{—} (1)$$

Differentiating (1) w.r. to s, we get

$$\frac{dI}{ds} = \frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos sx}{1+x^2}\,dx = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin sx}{1+x^2}\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x^2\sin sx}{x(1+x^2)}\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(x^2+1-1)\sin sx}{x(1+x^2)}\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x}\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx$$

$$= -\sqrt{\frac{\pi}{2}} + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx \qquad \because \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

So

$$\frac{dI}{ds} = -\sqrt{\frac{\pi}{2}} + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx \quad \text{—} (2)$$

Differentiating (2) w.r. to s, we obtain

$$\frac{d^2I}{ds^2} = \frac{d}{ds}\left[-\sqrt{\frac{\pi}{2}} + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx\right]$$

$$= 0 + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos sx}{1+x^2}\,dx = I$$

Therefore

$$\frac{d^2I}{ds^2} - I = 0 \quad \text{or} \quad (D^2-1)I = 0 \quad \text{—} (3)$$

whose solution is

$$I = Ae^{s} + Be^{-s} \quad \text{—} (4)$$

and

$$\frac{dI}{ds} = Ae^{s} - Be^{-s}$$

when $s=0$, we find from (1) & (2)

$$I = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos sx}{1+x^2}\,dx$$

at $s=0$

$$I = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}\,dx = \sqrt{\frac{2}{\pi}}\tan^{-1}x\Big|_0^\infty = \sqrt{\frac{\pi}{2}}$$

and $\frac{dI}{ds} = -\sqrt{\frac{\pi}{2}}$

Again putting $s=0$ in (3) & (4) . and using above results, we get

$$A+B = \sqrt{\frac{\pi}{2}} \quad \text{and} \quad A-B = -\sqrt{\frac{\pi}{2}}$$

Solving these , we get $A=0$ and $B = \sqrt{\frac{\pi}{2}}$

Hence, from (1) and (3), we have

$$F_c\left\{\frac{1}{1+x^2}\right\} = \sqrt{\frac{\pi}{2}}\, e^{-s} \Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos sx}{1+x^2}\,dx = \sqrt{\frac{\pi}{2}}\, e^{-s}$$

Now, differentiating the above equation w.r.to s, we obtain

$$-\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin sx}{1+x^2}\,dx = -\sqrt{\frac{\pi}{2}}\, e^{-s}$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin sx}{1+x^2}\,dx = \sqrt{\frac{\pi}{2}}\, e^{-s}$$

$$\Rightarrow F_s\left\{\frac{x}{1+x^2}\right\} = \sqrt{\frac{\pi}{2}}\, e^{-s}$$

Solution 4: Given that $F_s\{f(x)\} = \frac{e^{-as}}{s}$

By the inversion formula for Fourier sine transform, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx\,ds \qquad \text{— (1)}$$

Differentiating this w.r to x by the Leibnitz's rule, we have

$$\frac{df}{dx} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\cos sx\,ds$$

$$\frac{df}{dx} = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+x^2}, \quad a>0$$

On integrating it, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right) + A \quad \text{— (2)}$$

Now when $x = 0$, Then from eqution (1) $f(0) = 0$

So, from equ^n (2), we have

$$f(0) = A = 0$$

Thus $f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right)$

Putting $a = 0$ in this result

$F_s^{-1}\left\{\frac{1}{s}\right\} = \sqrt{\frac{2}{\pi}} \tan^{-1}(\infty) = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$

Solution 5: $\because$ $F_c\{f(x)\} = \frac{1}{2} \tan^{-1}\left(\frac{2}{s^2}\right)$

By Fourier cosine inversion formula, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds = \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty \tan^{-1}\left(\frac{2}{s^2}\right) \cos sx \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \tan^{-1}\left(\frac{2}{s^2}\right) \cos sx \, ds$$

Now $\tan^{-1}\left(\frac{2}{s^2}\right) = \tan^{-1} \frac{2}{(s^2-1)+1} = \tan^{-1} \frac{2}{(s-1)(s+1)+1}$

$$= \tan^{-1}\left(\frac{\frac{2}{(s-1)(s+1)}}{1 + \frac{1}{(s-1)(s+1)}}\right) = \tan^{-1}\left(\frac{\frac{1}{s-1} - \frac{1}{s+1}}{1 + \frac{1}{(s-1)(s+1)}}\right)$$

$$= \tan^{-1}\frac{1}{s-1} - \tan^{-1}\frac{1}{s+1}$$

Therefore, $$f(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \left(\tan^{-1}\frac{1}{s-1} - \tan^{-1}\frac{1}{s+1}\right) \cos sx \, ds \quad \text{— (1)}$$

Now, we have

$$\int_0^\infty \tan^{-1}\left(\frac{1}{s-1}\right) \cos sx \, dx = \tan^{-1}\frac{1}{s-1} \frac{\sin sx}{x} \Big|_0^\infty - \int_0^\infty \frac{-1/(s-1)^2}{1+\left(\frac{1}{s-1}\right)^2} \frac{\sin sx}{x} \, dx$$

$$= \frac{1}{x} \int_0^\infty \frac{\sin sx}{(s-1)^2+1} \, dx$$

Similarly, we have

$$\int_0^\infty \tan^{-1}\left(\frac{1}{s+1}\right) \cos sx \, ds = \frac{1}{x} \int_0^\infty \frac{\sin sx}{(s+1)^2+1} \, dx$$

substituting these in equ^n ①, we find that

$$f(x) = \frac{1}{x\sqrt{2\pi}} \int_0^\infty \left\{ \frac{\sin sx}{(s-1)^2+1} - \frac{\sin sx}{(s+1)^2+1} \right\} ds$$

$$= \frac{1}{2x\sqrt{2\pi}} \int_{-\infty}^\infty \left\{ \frac{\sin sx}{(s-1)^2+1} - \frac{\sin sx}{(s+1)^2+1} \right\} ds$$

$$\left\{ \because \int_{-\infty}^\infty \frac{\sin sx}{(x-b)^2+a^2} dx = \frac{\pi}{a} e^{-as} \sin bs \ , (s>0) \right\}$$

using this formula, we have

$$f(x) = \frac{1}{2x\sqrt{2\pi}} \left\{ \frac{\pi}{1} e^{-x} \sin sx - \frac{\pi}{1} e^{-x} \sin(-x) \right\}$$

$$f(x) = \sqrt{\frac{\pi}{2}} \left(\frac{e^{-x}\sin x}{x} \right)$$

solution 8: First we will find Fourier transform of

$$f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0 & |x|>1 \end{cases}$$

By the definition of Fourier transform

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x)\, e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (1-|x|)\,(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (1-|x|) \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^1 (1-|x|) \sin sx\, dx$$

(Even function) (odd function)

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x) \cos sx\, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ (1-x) \frac{\sin sx}{s} \right\}_0^1 + \int_0^1 \frac{\sin sx}{s} dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[0 + \left\{ -\frac{\cos sx}{s^2} \right\}_0^1 \right] = \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos sx}{s^2} \right)$$

using Parseval's identity, we get

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

$$\Rightarrow \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4} ds = \int_{-1}^{1} (1-|x|)^2 dx$$

$$\Rightarrow \frac{4}{\pi}\int_{0}^{\infty} \frac{(1-1+2\sin^2\frac{s}{2})^2}{s^4} ds = \int_0^1 (1-x)^2 dx + \int_{-1}^{0} (1+x)^2 dx = \frac{2}{3}$$

$$\Rightarrow \frac{16}{\pi}\int_0^{\infty} \frac{\sin^4\frac{s}{2}}{s^4} ds = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

Putting $\frac{s}{2} = x$, we get $\boxed{\int_0^{\infty} \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}}$

Solution 7:

Note: If $u(x,t)$ is given at $x=0$, then take Fourier sine transform and if $\frac{\partial u}{\partial x}$ at $x=0$ is given, then use Fourier Cosine transform.

Given that $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$ for $0 \le x < \infty$, $t \ge 0$ — (1)

(i) $u(x,0) = 0$ for $x \ge 0$

(ii) $\frac{\partial u}{\partial x}(0,t) = -a$ (const.) (iii) $u(x,t)$ is bounded

Since $\frac{\partial u}{\partial x}(0,t)$ is given, therefore taking Fourier Cosine transform of (1) on both sides, we get

$$F_c\left(\frac{\partial u}{\partial t}\right) = F_c\left(K\frac{\partial^2 u}{\partial x^2}\right) \qquad \left[\because F_c\{f''(x)\} = -s^2F_c(s) - \sqrt{\frac{2}{\pi}}f'(0)\right]$$

$$\Rightarrow \frac{d\bar{u}}{dt} = K\left\{-s^2\bar{u} - \sqrt{\frac{2}{\pi}}\cdot\frac{\partial u}{\partial x}(0,t)\right\} \quad \{\text{using (ii)}\}$$

$$= -Ks^2\bar{u} + \sqrt{\frac{2}{\pi}}Ka$$

$$\Rightarrow \frac{d\bar{u}}{dt} + Ks^2\bar{u} = \sqrt{\frac{2}{\pi}}Ka$$

This is linear in $\bar{u}$, therefore I.F. is $e^{\int Ks^2 dt} = e^{Ks^2t}$

$\cancel{\bar{u}(s,t) = \sqrt{\frac{2}{\pi}}}$. Thus general solution of this first order linear D.E. is

$$e^{ks^2t}\,\bar{u}(s,t) = \sqrt{\frac{2}{\pi}}\,ka\int e^{ks^2t}\,dt = \sqrt{\frac{2}{\pi}}\,ka\,\frac{e^{ks^2t}}{ks^2} + C_1$$

$$\Rightarrow \bar{u}(s,t) = \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2} + C_1 e^{-ks^2t} \qquad \text{— (1)}$$

Since $u(x,0) = 0$, $x \geq 0 \Rightarrow \bar{u}(s,0) = 0$

Hence $\bar{u}(s,0) = \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2} + C_1$

$\Rightarrow 0 = \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2} + C_1 \Rightarrow C_1 = -\sqrt{\frac{2}{\pi}}\,\frac{a}{s^2}$

Therefore, from eqn (1), we obtain

$\cancel{\bar{u}(s,t) = \sqrt{\frac{2}{\pi}}\frac{a}{s^2}(1 - e^{-1}}$

$$\bar{u}(s,t) = \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2}\left(1 - e^{-ks^2t}\right)$$

Taking the inverse Fourier cosine transform, we get

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{a}{s^2}\left(1 - e^{-ks^2t}\right)\cos sx\,ds$$

Solution 8: Given that $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ — (1)

and $u(0,t) = 0$, $u(x,0) = e^{-x}$, $u(x,t)$ is bounded

Taking Fourier sine transform of eqn (1), we get

$$\frac{d\bar{u}}{dt} = 2\left(-s^2\bar{u} + \sqrt{\frac{2}{\pi}}\,s\,u(0)\right) \qquad \because u(0,t) = 0$$

$$= -2s^2\bar{u} + 0$$

$$\Rightarrow \frac{d\bar{u}}{dt} = -2s^2\bar{u}$$

$$\Rightarrow \bar{u}(s,t) = c\,e^{-2s^2t} \qquad \text{— (2)}$$

Since $u(x,0) = e^{-x}$

$\Rightarrow \bar{u}(s,0) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+1}\right)$ (By Fourier sine transform

From ②, we get

$$\bar{u}(s,t) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+1}\right)e^{-2s^2t}$$

By inversion formula, we get

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1}e^{-2s^2t}\sin sx\,ds$$

Solution 9: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$; $x>0$, $t>0$ — ①

and (i) $u_x(0,t)=0$ (ii) $u(x,0)=\begin{cases} x & 0\le x\le 1 \\ 0 & x>1\end{cases}$ and $u(x,t)$ is bounded

Applying Fourier cosine transform on equ^n ①, we get

$$\frac{d\bar{u}}{dt} = -s^2\bar{u} - \sqrt{\frac{2}{\pi}}\frac{\partial u}{\partial x}(0,t) \qquad \because u_x(0,t)=0$$

$$\Rightarrow \frac{d\bar{u}}{dt} = -s^2\bar{u}$$

$$\Rightarrow \bar{u}(s,t) = Ae^{-s^2t} \quad \text{— ②}$$

Since $u(x,0) = \begin{cases} x & 0\le x\le 1 \\ 0 & x>1\end{cases}$

$$\Rightarrow \bar{u}(s,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^1 x\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{x\sin sx}{s}\Big|_0^1 - \int_0^1 \frac{\sin sx}{s}dx\right]$$

$$\bar{u}(s,0) = \sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s} + \frac{\cos s-1}{s^2}\right] \quad \text{— ③}$$

Using ② and ③, we get

$$A = \sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s} + \frac{\cos s-1}{s^2}\right]$$

$$\Rightarrow \bar{u}(s,t) = \sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s} + \frac{\cos s-1}{s^2}\right]e^{-s^2t}$$

Taking the inverse Fourier cosine transform, we get

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin s}{s} + \frac{\cos s-1}{s^2}\right)e^{-s^2t}\cos sx\,ds$$