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CSO 322 Theory of computation
Assignment 1

1)
a) Let the finite alphabet be $\Sigma = \{a_1 \cdots a_l\}$ and let \$ be some character not in Σ.
Let $L = \{w_1 \cdots w_n\}$ be a finite language over Σ. We can consider the strings $\$w_1 \$w_2 \$w_3 \cdots \$w_{3n}$ to be a number in base $|\Sigma|+1$ by associating the symbols $a_1 \cdots a_l$, \$ with the base - $(|\Sigma|+1)$ digits $0, \cdots l-1, l$ respectively.

This gives a map from the set of finite languages over Σ to a subset of the integers, so that set of languages is countable.

b)
Given
$$M = \{Q, \Sigma, \delta, q_0, F\}, q$$

K must accept $\{w \in \Sigma^*$: when M is run on input w, the state q is never visited

For this any string that visits q will never be allowed to have it and all other states would be accepting states (as any string that doesn't visit q must be accepted).

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$$M' = (Q, \Sigma, \delta', q_0, F')$$

where

$$F' = Q / \{q\}$$

$$\delta'(p, a) = \delta(p, a) \; \forall \; p \in Q/\{q\}, a \in \Sigma$$

$$\delta'(p, a) = q \; \forall \; a \in \Sigma$$

c) $\{a^{n^2} \mid n \geq 0\}$ or $\{a^{2n} \mid n \geq 0\}$ be the two non-regular language.

Since, it is non regular, its wards can be arbitrary lang.

This means that its prefix language contains all words $e, a, a^2, ----$

That is, its prefix language is regular.

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a) →

Given

$A \subseteq (\Sigma \cup \Gamma)^*$ is a regular language.

To prove

The language $B = \{\pi(w) : w \in A\}$ is regular.

Proof →

By Klein's Theorem (part - I), we know that any regular language can be accepted by a finite language.

So, there exists a DFA M, for which language A is accepted.

Since DFA M will be defined over alphabet set $(\Sigma \cup \Gamma)$, we can define

$$M = (Q, \Sigma \cup \Gamma, \delta, q_0, F) \quad [M \text{ accepts } A]$$

$$\delta : Q \times (\Sigma \cup \Gamma) \rightarrow Q$$

To prove that language B is regular, we need to show that there exists a DFA, NFA or ϵ-NFA (finite automaton) which accepts B.

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We try to find an ϵ-NFA that accepts B

So, we define an ϵ-NFA M' over Σ, such that

$$M' = (Q, \Sigma, \delta', q_0, F) \quad , \quad P(Q) = \text{powerset of } Q$$

$\delta' : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow P(Q)$ define as-

$$\delta'(q, m) = \{\delta(q, m)\}, \; \forall m \in \Sigma, q \in Q$$

$$\delta'(q, \epsilon) = \{\delta(q, b) ; b \in \Gamma\}, \; \forall q \in Q$$

Intutively, we see that we constructed M' from M, by replacing all occurences of b (where $b \in \Gamma$) by equivalent ϵ-transition i.e.

For every $q \in Q$, we ~~replaced~~ required the requirement of a character b ($b \in \Gamma$) to transition from q to $\delta(q, b)$, by providing an epsillon transition as a substitute.

Hence, for example, if our string in A was "abbaib", then we constructed M' such that it accepts "$ab\epsilon a\epsilon b$" which is equivalent to "abab" $= \pi(w)$

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Hence ϵ-NFA M', accepts the language

$$B = \{\pi(w) : w \in A\}$$

So, B is a regular language.

b) →

Given

$C \subseteq \Sigma^*$ is a regular language

To prove

The language $D = \{w \in (\Sigma \cup \Gamma)^* : \pi(w) \in C\}$

is regular.

Proof →

let M be a DFA accepting given regular language C (By Kleen's theorem, Part-I) such a finite automaton must exist, and we know that any NFA or ENFA can be converted into a DFA. So let M be such a DFA.

Suppose

$$M = (Q, \Sigma, \delta, q_0, F)$$

Now we try to define a DFA M', which accept language D from M

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since ~~the~~ strings in D will be of the form

$w_0 u_1 w_1 u_2 w_2$ ---- ,

where

$u_1, u_2 \in C$

$w_1, w_2, w_3 \in \Gamma^*$

Hence, if we define identity transitions for all $q \in Q$ with input symbol $\in \Gamma$, we will have successfully constructed M' from M.

So defining DFA M':

$M' = (Q, \Sigma \cup \Gamma, \delta', q_0, F)$

$\delta' : Q \times (\Sigma \cup \Gamma) \to Q$ as:

$\delta'(q, m) = \delta(q, m)$ for every $q \in Q, m \in \Sigma$

and

$\delta'(q, b) = q$ for every $q \in Q, b \in \Gamma$

clearly, M' also accepts language C.
since, C is a regular language, M' is a DFA.

Since, M' is a DFA, any language accepted by it is a regular language (Kleen's theorem)

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But M' accepts language D

Hence, D is a regular language.

Hence proved

(M' is similar to M, with only difference being that we have accomodated elements of K in M', by mapping them to identify transitions.

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3) →

Given →

$A \subseteq \Sigma^*$ is a regular language

To prove →

$B = \{ uv^R : u, v \in \Sigma^* \text{ and } uv \in A \}$

Proof →

Define an NFA

$N = (R, \Sigma, \eta, q_0, G)$

as follows

The state set of N is defined as

$R = Q \cup (Q \times Q)$

and the accept states of N are

$G = \{ (P, P) : P \in Q \}$

It remains to describe the transition function η of N, which will be as follows:

1) $\eta(p, a) = \{ \delta(p, a) \}$ for all $p \in Q$ and $a \in \Sigma$

2) $\eta(p, \varepsilon) = \{ (p, q) : q \in F \}$

3) $\eta((p, q), a) = \{ (p, r) : \delta(r, a) = q \}$ for all $p, q \in Q$ and $a \in \Sigma$

4) $\eta((p, q), \varepsilon) = \phi$ for every $p, q \in F$

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The idea behind this NFA is that its computation has two phases, which corresponds naturally to the strings u and v, in the definition of B.

The first phase of the computation of N indicated by its state being contained in Q, while the second phase is indicated by its state being contained in Q X Q.

Here, N begins in the state q_0 and therefore in the first phase. In first phase, N simply mimics M directly, which is the point of the transitions listed in item 1) of the list above.

The start of the second phase happens when N follows one of the ϵ-transitions listed in item 2) of the list.
such a transition has 2 effects:
It records the state that N was in immediately prior to the phase change, and It guesses the final state that M would be in if it were given the input uv.

The second phase of N is effectively to run M backwards, as the rules in item 3) allow while all along remembering the state that N was in immidiately prior to the

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phase change.

Acceptance occur when running M backwa
in this way reaches this remembered
state.

Since, $L(N) = B$.

Hence

B is regular.

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4)

Given:

Language $A \subseteq \{0\}^*$

$A = \{0^{rs} : r, s \in N, r, s \geq 3\}$ —— (1)

To prove

A is non-regular.

Proof →

We can use the pumping lemma to establish that a given language is not regular, by supplying a decomposition of w into xyz and finally show that xy^iz is ~~reg~~ not in the regular language.

Now we prove that A is non-regular by using method of contradiction.

Let A be a regular language accepted by finite automaton M.

Then by pumping lemma, we have -

If A is a regular language, then there exists a constant $n \geq 1$ (which depends on A) ∃ for every string $w \in L$ such that $|w| \geq n$, we can break w into 3 substrings, $w = xyz$ such that

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i) $y \neq \varepsilon$ — ②

ii) $|xy| \leq n$ — ③

iii) the string $xy^k z \in A$, $\forall k \geq 0$ — ④

Now from ①

$r \geq 3$, $s \geq 3 \Rightarrow rs \geq 9$

Let w be a string of length $n+1$

$w =$ '000000' $(n+1)$ times

$= xyz$ (say)

Let $|x| = x_1$, $|y| = y_1$, $|z| = z_1$

using ② and ③ we can say

$y_1 \neq 0$ — ⑤ $(\because y \neq \varepsilon)$

$|xy| = |x_1 + y_1| \leq n+1$ — ⑥

$x_1 + y_1 + z_1 = n+1$ — ⑦

Let us choose

$x = \varepsilon$ $(\Rightarrow x_1 = 0)$

$y = 000 \cdots 0$ (n times $y_1 = n \neq 0$)

$z = 0$ $(z_1 = 1)$ $(n \neq 0, \because n+1 \geq 9 \Rightarrow n \geq 8)$

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∴ ⑤ ⑥ are satisfied ($y \neq \varepsilon$ and $|xy| \leq n+1$)
where n is our length of string w

• Now as per our assumption, A is regular
so ④ should also be satisfied
$\forall k \geq 0$, the string $xy^kz \in A$

$$xy^kz = \underbrace{\varepsilon}_{x} \underbrace{000\cdots\cdots0}_{\substack{y^k \\ nk \text{ times '0'}}} \underbrace{0}_{z} = \underset{(nk+1)\text{ times}}{00\cdots0}$$

To show $xy^kz \in A$.
By ① $A = \{0^{rs} ; r, s \in \mathbb{N} ; r, s \geq 3\}$.

Now, we know that
there exist infinitely many prime numbers
so let us choose $n = 11, k = 6$ $(n \geq 0, k \geq 0)$
$nk+1 = (11)(6)+1 = 67$ which is a prime number

But if our assumption was true
$\Rightarrow xy^kz \in A$
$0^{nk+1} \in A$
$0^{nk+1} = 0^{rs}$
$nk+1 = rs$
but $nk+1 = 67 = 1 \times 67 \neq$ product of 2 nubers each ≥ 3
$(\because 1 \nleq 3)$

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$\therefore xy^k z \notin A$

$\therefore$ our assumption that A is regular language is wrong

$\therefore$ A is non-regular

Hence proved

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5) $\rightarrow$ Alone