Arnav Harshad Chavan
Reg No: 211060007 S.Y. B.Tech (Electronics)
VJTI

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Maths Assignment 1 (R4MA2003S)

1) Show that the foll. functions are analytic and find their derivatives

i) $f(z) = (z+1)e^z$

$\rightarrow f(z) = ze^z + e^z$

$= (x+iy)e^{x+iy} + e^{x+iy}$

$= e^x(x+iy)[\cos y + i\sin y] + e^x[\cos y + i\sin y]$

$= (e^x\cos y + ie^x\sin y)[x+iy+1]$

$= (x+1)(e^x\cos y + ie^x\sin y) + iy(e^x\cos y + ie^x\sin y)$

$= xe^x\cos y + ixe^x\sin y + e^x\cos y + ie^x\sin y + iye^x\cos y - ye^x\sin y$

$u+iv = (xe^x\cos y + e^x\cos y - ye^x\sin y) + i(xe^x\sin y + e^x\sin y + ye^x\cos y)$

On comparing, we get

$u = xe^x\cos y + e^x\cos y - ye^x\sin y$; $v = xe^x\sin y + e^x\sin y + ye^x\cos y$

$u_x = e^x\cos y + xe^x\cos y + e^x\cos y - ye^x\sin y$

$u_y = -xe^x\sin y - e^x\sin y - e^x\sin y - ye^x\cos y$

$v_x = e^x\sin y + xe^x\sin y + e^x\sin y + ye^x\cos y$

$v_y = xe^x\cos y + e^x\cos y - ye^x\sin y$

The C-R eqns are: ① $u_x = v_y$ & ② $u_y = -v_x$

Since the C-R eqns are satisfied and the partial derivatives are unique and continuous, hence the function $f(z) = (z+1)e^z$ is analytic

$f'(z) = u_x + iv_x$

$$\boxed{f'(z) = (e^x\cos y + xe^x\cos y + e^x\cos y - ye^x\sin y) + i(e^x\sin y + xe^x\sin y + e^x\sin y + ye^x\cos y)}$$

ii) $f(z) = (z+i)^3$

$\rightarrow f(z) = z^3 + i^3 + 3z^2 i + 3z(i)^2$

$= z^3 - i + i3z^2 - 3z$

$= (x+iy)^3 - i + i\times 3\times(x+iy)^2 - 3(x+iy)$

$= x^3 - iy^3 + 3x^2 iy - 3xy^2 - 3x - 3iy + i3x^2 - i3y^2 - 6xy$

$u+iv = (x^3 - 3xy^2 - 3x - 6xy) + i(-y^3 + 3x^2y - 3y + 3x^2 - 3y^2)$

On comparing, we get

$u = x^3 - 3xy^2 - 3x - 6xy$; $v = -y^3 + 3x^2y - 3y + 3x^2 - 3y^2$

$U_x = 3x^2 - 3y^2 - 3 - 6y$ $\quad V_x = 6xy + 6x$

$U_y = -6xy - 6x$ $\quad V_y = -3y^2 + 3x^2 - 3 - 6y$

The C-R eq^ns are: ① $U_x = V_y$ & ② $U_y = -V_x$

Since the C-R eq^ns are satisfied and the partial derivatives are unique and continuous, hence the function $f(z) = (z+1)^3$ is analytic

$f'(z) = U_x + iV_x$

$$\boxed{f'(z) = (3x^2 - 3y^2 - 6y - 3) + i(6xy + 6x)}$$

2) Find analytic function $f(z) = u + iv$, where

$u(x,y) = \frac{x}{2}\log(x^2+y^2) - y\tan^{-1}\frac{y}{x} + \sin x \cosh y$

→ let $u(x,y) = \phi(x,y) = \frac{x}{2}\log(x^2+y^2) - y\tan^{-1}\frac{y}{x} + \sin x\cosh y$ — ①

Partially diff. eq^n ① w.r.t 'x' & 'y', we get

$$\phi_x(x,y) = \frac{x}{2} \times \frac{2x}{(x^2+y^2)} - y \frac{\times x^2}{(x^2+y^2)} \times \left(\frac{-1}{x^2}\right) + \cos x\cosh y + \frac{1}{2}\log(x^2+y^2)$$

$$\therefore \phi_x(x,y) = \frac{x^2}{x^2+y^2} + \frac{y}{(x^2+y^2)} + \cos x\cosh y \quad — ②$$

Partially diff. eq^n ① w.r.t 'y', we get

$$\phi_y(x,y) = \frac{x}{2(x^2+y^2)} \times 2y - \left(\tan^{-1}\frac{y}{x} + y \cdot \frac{x^2}{(x^2+y^2)} \times \frac{1}{x}\right) + \sin x \sinh y$$

$$\phi_y(x,y) = \frac{xy}{(x^2+y^2)} - \tan^{-1}\frac{y}{x} - \frac{xy}{(x^2+y^2)} + \sin x\sinh y \quad — ③$$

$$\phi_x(z,0) = \frac{z^2}{z^2+0^2} + 0 + \cos z(1) = 1 + \cos z + \log z \quad — ④$$

$$\phi_y(z,0) = 0 + 0 = 0 \quad — ⑤$$

By Milne-Thompson Method,

$$f(z) = \int [\phi_x(z,0) - i\phi_y(z,0)]\,dz + c$$

$$f(z) = \int [(1 + \log z + \cos z) - i(0)]\,dz + c$$

$$f(z) = z + z(\log z - 1) + \sin z + c$$

$f(z) = z\log z + \sin z + c$

$\therefore f(z) = (x+iy)\log(x+iy) + \sin(x+iy) + c$

$= (x+iy)\left[\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}\right] + \sin x\cos iy + \cos x\sin iy + c$

$= \frac{x}{2}\log(x^2+y^2) + ix\tan^{-1}\frac{y}{x} + \frac{iy}{2}\log(x^2+y^2) - y\tan^{-1}\frac{y}{x} + \sin x\cosh y$

$+ i\cos x\sinh y + c$

$$\boxed{f(z) = \left(\frac{x}{2}\log(x^2+y^2) - y\tan^{-1}\frac{y}{x} + \sin x\cosh y\right) + i\left(x\tan^{-1}\frac{y}{x} + \frac{y}{2}\log(x^2+y^2) + \cos x\sinh y\right) + c}$$

3) Find analytic function $f(z) = u+iv$, where $v = \frac{x}{x^2+y^2} + \cosh x\cos y$

$\rightarrow$ let $v(x,y) = \varphi(x,y) = \frac{x}{x^2+y^2} + \cosh x\cos y$ — ①

Partially diff eqn ① w.r.t 'x' & 'y', we get

$\varphi_x(x,y) = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} + \sinh x\cos y = \frac{y^2-x^2}{(x^2+y^2)^2} + \sinh x\cos y$ — ②

$\varphi_y(x,y) = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} - \cosh x\sin y = \frac{-2xy}{(x^2+y^2)^2} - \cosh x\sin y$ — ③

$\varphi_x(z,0) = \frac{0-z^2}{(z^2+0)^2} + \sinh z = \sinh z - \frac{1}{z^2}$ — ④

$\varphi_y(z,0) = 0 - 0 = 0$ — ⑤

By Milne-Thompson Method

$f(z) = \int[\varphi_y(z,0) + i\varphi_x(z,0)]\,dz + c$

$f(z) = \int\left[0 + i\left(\sinh z - \frac{1}{z^2}\right)\right]dz + c$

$\therefore f(z) = i\left(\cosh z + \frac{1}{z}\right) + c$

$\therefore f(z) = i\left[\frac{1}{x+iy} + \cosh(x+iy)\right]$

$$f(z) = i\left[\frac{x-iy}{x^2+y^2} + \cosh x \cosh iy - \sinh x \sinh iy\right] + c$$

$$f(z) = i\left[\frac{x-iy}{x^2+y^2} + \cosh x \cos y - i \sinh x \sin y\right] + c$$

$$f(z) = \frac{ix}{x^2+y^2} + \frac{y}{x^2+y^2} + i\cosh x \cos y + \sinh x \sin y + c$$

$$\boxed{f(z) = u + iv = \left(\frac{y}{x^2+y^2} + \sinh x \sin y\right) + i\left(\frac{x}{x^2+y^2} + \cosh x \cos y\right) + c}$$

4) Find analytic function $f(z) = u+iv$, where $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$

$\rightarrow$ $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ — ①

Partially diff. eq^n ① w.r.t 'x', we get

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{(\cosh y - \cos x)(0 + \sin x + \cos x) - (e^y - \cos x + \sin x)(\sin x)}{(\cosh y - \cos x)^2}$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{\sin x \cosh y + \cos x \cosh y - \cancel{\sin x \cos x} - \cos^2 x - e^y \sin x + \cancel{\sin x \cos x} - \sin^2 x}{(\cosh y - \cos x)^2}$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{\sin x \cosh y + \cos x \cosh y - e^y \sin x - 1}{(\cosh y - \cos x)^2} \quad \text{— ②}$$

Partially diff eq^n ① w.r.t 'y', we get

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{(\cosh y - \cos x)(e^y) - (e^y - \cos x + \sin x)(\sinh y)}{(\cosh y - \cos x)^2}$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{e^y \cosh y - e^y \cos x - e^y \sinh y + \cos x \sinh y - \sin x \sinh y}{(\cosh y - \cos x)^2} \quad \text{— ③}$$

$\because$ $f(z)$ is an Analytic function the C-R eq^{ns} are satisfied

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\sin x \cosh y + \cos x \cosh y - e^y \sin x - 1}{(\cosh y - \cos x)^2} \quad \text{— ④}$$

$$-\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\frac{e^y\cosh y-e^y\cos x-e^y\sinh y+\cos x\sinh y-\sin x\sinh y}{(\cosh y-\cos x)^2} \quad -⑤$$

Adding eq^{ns} ④ & ⑤, we get

$$2\frac{\partial u}{\partial y}=\frac{\sin x\cosh y+\cos x\cosh y-e^y\sin x-1+e^y\cosh y-e^y\cos x-e^y\sinh y+\cos x\sinh y-\sin x\sinh y}{(\cosh y-\cos x)^2}$$

Put $x=z$ & $y=0$, we get

$$\left.\frac{\partial u}{\partial y}\right|_{(z,0)}=\frac{\sin z\cosh(0)+\cos z\cosh(0)-e^0\sin z-1+e^0\cos(h0)-e^0\cos z-e^0(0)+\cos z(0)+0}{2(\cosh(0)-\cos z)^2}$$

$$\phi_y(z,0)=\frac{\cancel{\sin z}+\cancel{\cos z}-\cancel{\sin z}-\cancel{1}+\cancel{1}-\cancel{\cos z}-0-0+0}{2(1-\cos z)^2}=0$$

$$\therefore \phi_y(z,0)=0 \quad -⑥$$

Subtracting eq^n ⑤ from eq^n ④, we get

$$2\frac{\partial u}{\partial x}=\frac{\sin x\cosh y+\cos x\cosh y-e^y\sin x-1-e^y\cosh y+e^y\cos x+e^y\sinh y-\cos x\sinh y+\sin x\sinh y}{(\cosh y-\cos x)^2}$$

Put $x=z$ & $y=0$, we get

$$\left.\frac{\partial u}{\partial x}\right|_{(z,0)}=\frac{\sin z\cosh(0)+\cos z\cosh(0)-e^0\sin z-1-e^0\cosh(0)+e^0\cos z+e^0(0)-0+0}{2(\cosh(0)-\cos z)^2}$$

$$\phi_x(z,0)=\frac{\cancel{\sin z}+\cos z-\cancel{\sin z}-1-1+\cos z}{2(1-\cos z)^2}=\frac{-\cancel{2}}{\cancel{2}(1-\cos z)^2}$$

$$\phi_x(z,0)=\frac{-1}{2}\operatorname{cosec}^2\frac{z}{2} \quad -⑦$$

By Milne-Thompson Method,

$$f(z)=\int[\phi_x(z,0)-i\phi_y(z,0)]\,dz+c$$

$$=\int\left[\frac{-1}{2}\operatorname{cosec}^2\frac{z}{2}-i(0)\right]dz+c$$

$$f(z)=\frac{-1}{\cancel{2}}\times -\cot\frac{z}{2}\times\cancel{2}+c$$

$$\therefore \boxed{f(z)=\cot(z/2)+c}$$

5) Show that the function $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is Harmonic Find analytic function $f(z) = u + iv$ and the Harmonic conjugate v.

→ $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$ — ①

$$\frac{\partial u}{\partial x} = 6xy + 4x \qquad \frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$$

$$\frac{\partial^2 u}{\partial x^2} = 6y + 4 \qquad \frac{\partial^2 u}{\partial y^2} = -6y - 4$$

Now, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Hence, the Laplaces Equation is satisfied.

$\therefore$ $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is a Harmonic function

Harmonic conjugate of $u = v$

$\therefore$ By Cauchy-Riemann Eqns method

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy$$

since $f(z)$ is analytic, C-R eqns are satisfied.

$$\therefore dv = \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) \cdot dy$$

$$dv = (-3x^2 + 3y^2 + 4y)dx + (6xy + 4x)dy$$

Integrating both sides, we get

$$v = -\int 3x^2 dx + \int 3y^2 dx + \int 4y dx + \int 6xy dy + \int 4x dy + c$$

$$v = -x^3 + \int d(3xy^2) + \int d(4xy) + c$$

$$\therefore v = -x^3 + 3xy^2 + 4xy + c$$

$$\therefore \boxed{v = 3xy^2 + 4xy - x^3 + c}$$

Now, $f(z) = u + iv$

$$\boxed{f(z) = (3x^2y + 2x^2 - y^3 - 2y^2) + i(3xy^2 + 4xy - x^3 + c)}$$

6) Find orthogonal trajectories of the family of curves $x^4 - 6x^2y^2 + y^4 = c$

→ Let $u(x,y) = x^4 - 6x^2y^2 + y^4$

$$u_x = 4x^3 - 12xy^2 \qquad u_y = -12x^2y + 4y^3$$

$f'(z) = u_x + iv_x = u_x - iu_y$

$\therefore f'(z) = (4x^3 - 12xy^2) - i(4y^3 - 12x^2y)$

Put $x = z$ & $y = 0$

$\therefore f'(z) = [4z^3 - 12z(0)] - i[4(0)^3 - 12z^2(0)]$

$\therefore f'(z) = 4z^3 - i(0) = 4z^3$

Integrating w.r.t 'z', we get

$\int f'(z) = \int 4z^3 dz + c$

$\therefore f(z) = z^4 + c$

Put $z = x + iy$

$f(z) = (x+iy)^4 + c$

$= x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 + c$

$f(z) = x^4 + 4x^3yi - 6x^2y^2 - 4xy^3i + y^4 + c$

$\therefore f(z) = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3) + c$

Imaginary part $(v) = 4x^3y - 4xy^3$

$\therefore$ $\boxed{\text{Required orthogonal trajectory is: } 4x^3y - 4xy^3 = c_1}$

7) Find Bilinear Transformation which maps the points $z = -2, i, 2$ onto the points $w = 0, i, -i$ respectively. Also find fixed points if any

→ $z_1 = -2,\ z_2 = i,\ z_3 = 2$; $w_1 = 0,\ w_2 = i,\ w_3 = -i$

Cross-Ratio Formula

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\therefore \frac{(w)(2i)}{(w+i)(i)} = \frac{(z+2)(i-2)}{(z-2)(i+2)}$$

$$\frac{2w}{w+i} = (-1)\frac{(z+2)}{(z-2)}\left(\frac{3-4i}{5}\right)$$

$5(z-2)(2w) = (-1)(z+2)(3-4i)(w+i)$

$(10-5z)(2w) = (z+2)(3-4i)(w+i)$

$20w - 10zw = (3z - 4zi + 6 - 8i)(w+i)$

$20w - 10zw = 3zw - 4zwi + 6w - 8wi + 3iz + 4z + 6i + 8$

$20w - 10zw - 3zw + 4zwi - 6w + 8wi = 3iz + 4z + 6i + 8$

$14w - 13zw + 4zwi + 8wi = 3iz + 4z + 6i + 8$

$w[14 - 13z + 4zi + 8i] = z(3i+4) + 2(3i+4)$

$\therefore w[z(-13+4i) + 14 + 8i] = (z+2)(3i+4)$

$\therefore \boxed{w = \left[\dfrac{(3i+4)(z+2)}{(-13+4i)z + 2(7+4i)}\right]}$

8) Find Bilinear Transformation which maps the points $z = -1, 1, \infty$ onto the points $w = -i, -1, i$ respectively using Cross-Ratio property

$\rightarrow$ $z_1 = -1,\ z_2 = 1,\ z_3 = \infty$; $w_1 = -i,\ w_2 = -1,\ w_3 = i$

Cross-Ratio Property:

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\therefore \frac{(w+i)(-1-i)}{(w-i)(-1+i)} = \frac{(z+1)}{2}$$

$$\therefore (-1)\frac{(w+i)(i+1)}{(w-i)(i-1)} = \frac{(z+1)}{2}$$

$$\therefore \frac{(w+i)}{(w-i)} \times i = \frac{(z+1)}{2}$$

$$\therefore 2i(w+i) = (w-i)(z+1)$$

$$\therefore 2iw - 2 = wz + w - iz - i$$

$$\therefore 2iw - wz - w = 2 - iz - i$$

$$\therefore w(2i - z - 1) = -iz - (i-2)$$

$$\therefore \boxed{w = \left[\frac{-iz - (i-2)}{-z - (1-2i)}\right]}$$

9) Find the analytic function whose real part is

$$u = (x-1)^3 - 3xy^2 + 3y^2$$

$\rightarrow$ $u = (x-1)^3 - 3xy^2 + 3y^2$

$u = x^3 - 3x^2(1) + 3x(1)^2 - 1^3 = x^3 - 3x^2 + 3x - 1$ — ①

Let $u(x,y) = \phi(x,y) = x^3 - 3x^2 + 3x - 1 - 3xy^2 + 3y^2$ — ①

Partially diff eqⁿ ① w.r.t 'x' & 'y', we get

$\phi_x(x,y) = 3x^2 - 6x + 3 - 3y^2$ — ②

$\phi_y(x,y) = -6xy + 6y$ — ③

$\phi_x(z,0) = 3z^2 - 6z + 3$ — ④

$\phi_y(z,0) = -6z(0) + 6(0) = 0$ — ⑤

By Milne-Thompson Method

$f(z) = \int [\phi_x(z,0) - i\phi_y(z,0)]\,dz + c$

$= \int (3z^2 - 6z + 3)\,dz + c$

$\boxed{f(z) = z^3 - 3z^2 + 3z + c}$

Put $z = x + iy$

$\therefore f(z) = (x+iy)^3 - 3(x+iy)^2 + 3(x+iy) + c$

$= x^3 + (iy)^3 + 3x^2(iy) + 3x(iy)^2 - 3(x^2 - y^2 + 2ixy) + 3x + 3iy + c$

$= x^3 - iy^3 + 3x^2yi - 3xy^2 - 3x^2 + 3y^2 - 6xyi + 3x + 3yi + c$

$\therefore \boxed{u + iv = (x^3 - 3xy^2 - 3x^2 + 3y^2 + 3x) + i(-y^3 + 3x^2y - 6xy + 3y) + c}$

which is the reqd Analytic function

10) Find the Analytic function whose imaginary part is

$$v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

→ Let $v = \psi(x,y) = \dfrac{\sinh 2y}{\cos 2x + \cosh 2y}$ — ①

Partially diff eqⁿ ① w.r.t 'x' & 'y', we get

$\psi_x(x,y) = \dfrac{(\cos 2x + \cosh 2y)(0) - \sinh 2y(-2\sin 2x)}{(\cos 2x + \cosh 2y)^2} = \dfrac{2\sin 2x \sinh 2y}{(\cos 2x + \cosh 2y)^2}$ — ②

$\psi_y(x,y) = \dfrac{(\cos 2x + \cosh 2y)(2\cosh 2y) - \sinh 2y(2\sinh 2y)}{(\cos 2x + \cosh 2y)^2}$ — ③

$\psi_x(z,0) = 0$ — ④

$\psi_y(z,0) = \dfrac{(\cos 2z + 1)\times 2 - 0}{(\cos 2z + 1)^2} = \dfrac{2}{(1 + \cos 2z)} = \dfrac{2}{2\cos^2 z} = \sec^2 z$ — ⑤

By Milne-Thompson Method

$f(z) = \int [\varphi_y(z,0) + i\,\varphi_x(z,0)]\,dz + c$

$f(z) = \int (\sec^2 z + i0)\,dz + c$

$\boxed{f(z) = \tan z + c}$

which is the required analytic function

11) Find the analytic function $f(z)$ whose real part is $r^2\cos 2\theta - r\sin\theta$

→ let $u(r,\theta) = r^2\cos 2\theta - r\sin\theta$ — ①

Partially diff eqn ① w.r.t 'r' & 'θ', we get

$\frac{\partial u}{\partial r} = 2r\cos 2\theta - \sin\theta$ — ② $\quad \frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta - r\cos\theta$ — ③

By using Cauchy-Riemann Equation Method

$dv = \frac{\partial v}{\partial r}.dr + \frac{\partial v}{\partial \theta}.d\theta$ — ④

since $f(z)$ is analytic, hence C-R eqns are satisfied

$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ & $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

$\therefore dv = \left(\frac{-1}{r}\frac{\partial u}{\partial \theta}\right).dr + \left(r\frac{\partial u}{\partial r}\right).d\theta$

$\therefore dv = (2r\sin 2\theta + \cos\theta)dr + (2r^2\cos 2\theta - r\sin\theta)d\theta$ — ⑤

Integrating both sides, we get

$v = \int 2r\sin 2\theta\, d\theta + \int \cos\theta\, dr + \int 2r^2\cos 2\theta\, d\theta - \int r\sin\theta\, d\theta + c$

$v = \int d(r^2\sin 2\theta) + \int d(r\cos\theta) + c$

$\boxed{v = r^2\sin 2\theta + r\cos\theta + c}$

Now, $f(z) = u + iv = (r^2\cos 2\theta - r\sin\theta) + i(r^2\sin 2\theta + r\cos\theta) + c$

$= r^2\cos 2\theta + ir^2\sin 2\theta - r\sin\theta + ir\cos\theta + c$

$= [r^2(\cos 2\theta + i\sin 2\theta)] + r[i^2\sin\theta + i\cos\theta] + c$

$= (re^{i\theta})^2 + i(re^{i\theta}) + c$

$\boxed{f(z) = (z)^2 + iz + c}$

This is the required Analytic function

12) Determine 'c' such that $u = \sin x . \cosh cy$ is harmonic & find the harmonic conjugate

→ $u = \sin x . \cosh cy$ — ①

$\frac{\partial u}{\partial x} = \cos x . \cosh cy$ $\qquad$ $\frac{\partial u}{\partial y} = c \sin x \sinh cy$

$\frac{\partial^2 u}{\partial x^2} = -\sin x . \cosh cy$ $\qquad$ $\frac{\partial^2 u}{\partial y^2} = c^2 \sin x \cosh cy$

∵ $u(x,y)$ is harmonic, hence it satisfies the Laplace Equation

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore -\sin x . \cosh cy + c^2 \sin x . \cosh cy = 0$

$\therefore \quad c^2 = 1$

$\therefore \quad \boxed{c = \pm 1}$

$\therefore u = \sin x \cosh y$

Harmonic conjugate of $u = v$

∴ By C-R eq^ns method / Exact differentiation method

$dv = \frac{\partial v}{\partial x} . dx + \frac{\partial v}{\partial y} . dy$

$\therefore dv = \left(-\frac{\partial u}{\partial y}\right) . dx + \left(\frac{\partial u}{\partial x}\right) . dy$

$\therefore dv = (-\sin x \sinh y) dx + (\cos x \cosh y) . dy$

Integrating both sides, we get

$v = \int -\sin x \sinh y \, dx + \int \cos x \cosh y . dy + c$

$\therefore v = \int d(\cos x . \sinh y) + c$

$\therefore \boxed{v = \cos x . \sinh y + c}$

13) An electrostatic field in the xy-plane is given by the potential function $\phi = x^2 - y^2 - 2xy - 2x + 3y$. Find the stream function. Find the complex potential function

→ Let potential function $= u(x,y) = \phi(x,y) = x^2 - y^2 - 2xy - 2x + 3y$ — ①

Partially diff eq^n ① w.r.t 'x' & 'y', we get

$\phi_x(x,y) = 2x-2y-2$ — ②

$\phi_y(x,y) = -2y-2x+3$ — ③

Put $x=z$ & $y=0$

$\phi_x(z,0) = 2z-2$ — ④

$\phi_y(z,0) = -2z+3$ — ⑤

By Milne-Thompson method,

$f(z) = \int [\phi_x(z,0) - i\phi_y(z,0)]\,dz + c$

$\therefore f(z) = \int [(2z-2) - i(-2z+3)]\,dz + c$

$\therefore f(z) = \left[\frac{\cancel{2}z^2}{\cancel{2}} - 2z + i\left(\frac{\cancel{2}z^2}{\cancel{2}} - 3z\right)\right] + c$

$\therefore \boxed{f(z) = (z^2-2z) + i(z^2-3z) + c}$

This is the required Analytic function / Complex Potential Function

Put $z = x+iy$

$f(z) = [(x+iy)^2 - 2(x+iy)] + i[(x+iy)^2 - 3(x+iy)] + c$

$= (x^2-y^2+2xyi-2x-2yi) + i(x^2-y^2+2xyi-3x-3iy) + c$

$\therefore f(z) = (x^2-y^2-2x-2xy+3y) + i(x^2-y^2+2xy-2y-3x) + c$

Now, $f(z) = \phi(x,y) + i\psi(x,y)$

Comparing both sides, we get

$\boxed{\psi(x,y) = (x^2-y^2+2xy-2y-3x) + c}$

This is the required stream function.

14) Find the LFT which maps the points 0, i, -1 of z-plane onto i, 1, 0 respectively of w-plane

→ $z_1 = 0,\ z_2 = i,\ z_3 = -1$; $w_1 = i,\ w_2 = 1,\ w_3 = 0$

Cross-Ratio Property

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\therefore \frac{(w-i)(1)}{w(i-1)} = \frac{z(i+1)}{(z+1)i}$$

$\therefore (iz+i)(w-i) = wz(i^2-1)$

$\therefore izw - i^2z + iw - i^2 = -2wz$

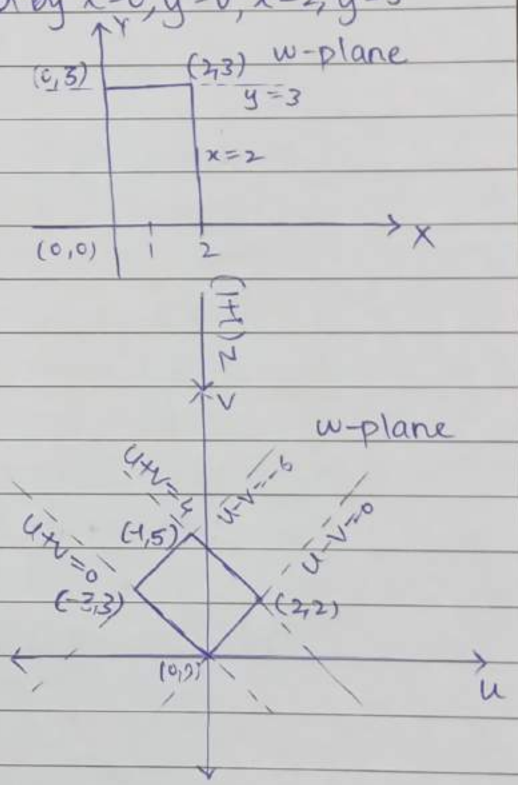
$\therefore izw + z + iw + i = -2wz$

$\therefore izw + 2wz + iw = -(z+1)$

$\therefore w(iz+2z+i) = -(z+1)$

$\therefore \boxed{w = \left[\dfrac{-(z+1)}{z(2+1)+i}\right]}$

15) Find the image of the rectangle bounded by $x=0, y=0, x=2, y=3$ under the mapping $w=(1+i)(z)$



→ $w = (1+i)z = (i+1)(x+iy)$

$\therefore w = x+iy+ix-y = (x-y)+i(x+y)$

$\therefore u+iv = (x-y)+i(x+y)$

$\therefore u = (x-y)$ — ① & $v = (x+y)$ — ②

Adding ① & ②, we get

$u+v = 2x$ — ③

Subtracting ② from ①, we get

$u-v = -2y$ — ④

For $x=0$ & $y=0$, eq^{ns} ③ & ④ become

$u+v=0$ & $u-v=0$

$\therefore v=-u$ & $v=u$

For $x=2$ & $y=3$, eq^{ns} ③ & ④ become

$u+v=4$ & $u-v=-6$

16) Find the inverse $z=f^{-1}(w)$ for $w=f(z)=\dfrac{4z+i}{-3iz+1}$

→ $w = \dfrac{4z+i}{-3iz+1}$

$\therefore w(-3iz+1) = 4z+i$

$\therefore -3wzi+w = 4z+i$

$\therefore\ 4z + 3wzi = w - i$

$\therefore\ z(4+3wi) = (w-i)$

$$\therefore\quad \boxed{z = \frac{(w-i)}{(4+3wi)}}$$

17) Find the image of the circle $x^2+y^2=1$ under $w = \dfrac{5-4z}{4z-2}$

→ $w = \dfrac{5-4z}{4z-2}$

$\therefore\ 4wz - 2w = 5 - 4z$

$\therefore\ 4wz + 4z = 5 + 2w$

$$\therefore\quad z = \frac{(5+2w)}{(4w+4)} \quad \text{— } ①$$

Now, eq^n of circle in z-plane is $x^2+y^2=1 \Rightarrow |z|=1$

$\therefore\ \left|\dfrac{5+2w}{4w+4}\right| = 1$

$\therefore\ |5+2u+i2v| = |4u+i4v+4|$

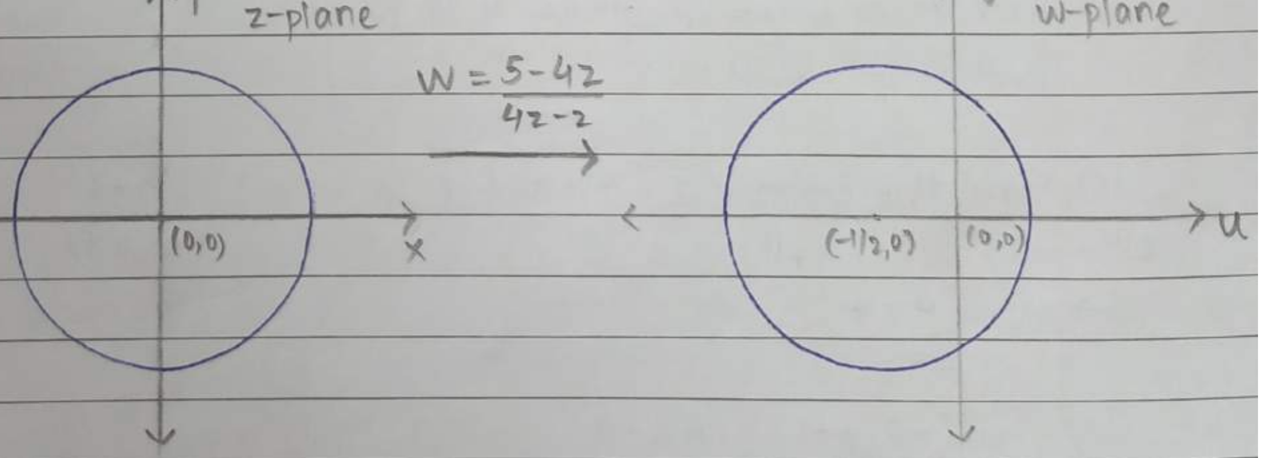
$\therefore\ (5+2u)^2 + (2v)^2 = (4u+4)^2 + (4v)^2$

$\therefore\ 25 + 4u^2 + 20u + 4v^2 = 16u^2 + 16 + 32u + 16v^2$

$\therefore\ 12u^2 + 12v^2 + 12u - 9 = 0$

$\therefore\ u^2 + v^2 + u - \dfrac{3}{4} = 0 \Rightarrow \boxed{\left(u+\dfrac{1}{2}\right)^2 + (v-0)^2 = (1)^2}$

This is a circle in w-plane with centre $(-1/2, 0)$ & radius $|r| = 1$



Yandwu
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