ASSIGNMENT - 1

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1. Verify CHT for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ and hence find A^{-1} & A^4

$\rightarrow$ Char eq^n of A is $\lambda^3 - a_1\lambda^2 + a_0\lambda - |A| = 0$

$$a_1 = \text{sum of diagonals}$$
$$= 1 - 1 - 1 = -1$$

$$a_0 = \text{sum of minors of diagonal elements}$$
$$= \begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$
$$= -3 - 10 - 5$$
$$= -18$$

$$|A| = 1(-3) - 2(-14) + 3(5) = 40$$

$\therefore$ Char eq^n: $\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$

By Cayley - Hamilton theorem

$$A^3 + A^2 - 18A - 40 = 0$$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

$$A^3 + A^2 - 18A - 40I = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

$$- 18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

now to find A^{-1} multiply by A^{-1}

$$40A^{-1} = A^2 + A - 18I$$

$$A^{-1} = \frac{1}{40} \begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

To find A^4 multiply eq^n ① by A

$$A^4 + A^3 - 18A^2 - 40A = 0$$

$$A^4 = 40A + 18A^2 - A^3$$

$$A^4 = \begin{bmatrix} 40 & 80 & 120 \\ 80 & -40 & 160 \\ 120 & 10 & -40 \end{bmatrix} + \begin{bmatrix} 252 & 54 & 144 \\ 216 & 162 & -36 \\ 36 & 72 & 252 \end{bmatrix}$$

$$- \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$$

2. Verify CHT for A, and find the matrix $2A^5 - 3A^4 + A^2 - 4I$

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

→ char eq^n : $\lambda^2 - 5\lambda + 7 = 0$

By CHT : $A^2 - 5A + 7I$

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$A^2 - 5A + 7I$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\therefore \quad A^2 - 5A + 7I = 0$$

For finding the matrix $2A^5 - 3A^4 + A^2 - 4I$ we have to divide it by $A^2 - 5A + 7I = 0$

$$\begin{array}{r|l}
 & 2\lambda^3 + 7\lambda^2 + 21\lambda + 57 \\
\hline
\lambda^2 - 5\lambda + 7 & 2\lambda^5 - 3\lambda^4 + \lambda^2 - 4 \\
 & -\,2\lambda^5 - 10\lambda^4 + 14\lambda^3 \\
 & \quad\;\; + \qquad\; - \\
\hline
 & 7\lambda^4 - 14\lambda^3 + \lambda^2 - 4 \\
 & -\,7\lambda^4 - 35\lambda^3 + 49\lambda^2 \\
 & \quad\;\; + \qquad\; - \\
\hline
 & 21\lambda^3 - 48\lambda^2 - 4 \\
 & 21\lambda^3 - 105\lambda^2 + 147\lambda \\
 & - \qquad + \qquad - \\
\hline
 & 57\lambda^2 - 147\lambda - 4 \\
 & 57\lambda^2 - 285\lambda + 3 \\
 & - \qquad + \qquad - \\
\hline
 & 138\lambda - 403 \\
\end{array}$$

By Cayley's Hamilton theorem

$$2A^5 - 3A^4 + A^2 - 4I = 138A - 403I$$

$$= 138 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 403 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 138 \\ -138 & -127 \end{bmatrix}$$

3. (ii) $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ Determine whether matrix is diagonalisable or not.

$\rightarrow$ Char eq^n of A is given by : $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

here, $S_1 = 3 - 3 + 7 = 7$

$S_2 = (-21 + 20) + (21 - 15) + (-9 + 20)$

$= 16$

$S_3 = 3(-1) - 10(-14 + 12) + 5(-10 + 9) = 12$

$\therefore$ Char eq^n : $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

$$\begin{array}{c|cccc} 2 & 1 & -7 & 16 & -12 \\ & \downarrow & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\therefore (\lambda - 2)(\lambda^2 - 5\lambda + 6) = 0$

either $\lambda = 2$ or $\lambda^2 - 2\lambda - 3\lambda + 6 = 0$

$\lambda(\lambda - 2) - 3(\lambda - 2) = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\lambda = 2, \lambda = 3$

$\therefore \lambda_1 = \lambda_2 = 2$, $\lambda_3 = 2$ are eigen values of A

for $\lambda_1 = \lambda_2 = 2$, here AM = 2

corresponding eigen vector is given by

$(A - \lambda I) X = 0$

$(A - 2I) X = 0$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & -25 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2/3$, $R_3 \to R_3/5$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduced form of eq^n is

$x + 10y + 5z = 0 \quad — (1)$

$\therefore\ 5y + 2z = 0$

$2z = -5y$

$z = -\frac{5}{2}y$

let $y = 2k \quad \ldots \quad k \neq 0$

$z = -\frac{5}{2} \times 2k = -5k$

from ①, $\therefore\ x = -10y - 5z$

$= -20k - 5(-5k)$

$= -20k + 25k = 5k$

$$\therefore\ X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$$

here $GM = 1$

$\because\ AM \neq GM$ — ①

But for $\lambda_3 = 3$, $AM = GM = 1$ — ②

($\because\ AM = 1$ & $AM \geq GM$

$\therefore\ GM = 1$)

from ① & ②, Matrix A is not diagonalizable

iii) $A = \begin{bmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & 2 \end{bmatrix}$

→ Char eq^n: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

here, $S_1 = -17 + 19 + 2 = 4$

$S_2 = (38 + 54) + (-34 - 54) + (-323 + 324) = 5$

$S_3 = -17(38 + 54) - 18(-36 - 54) - 6(-162 + 171)$

$S_3 = 2$

$\therefore\ \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -4 & 5 & -2 \\ & \downarrow & 1 & -3 & 2 \\ \hline & 1 & -3 & 2 & \boxed{0} \end{array}$$

$\therefore (\lambda-1)(\lambda^2-3\lambda+2) = 0$

$\lambda = 1$ or $\lambda^2 - 3\lambda + 2 = 0$

$\lambda(\lambda-2)-(\lambda-2) = 0$

$\lambda = 2, 1$

$\lambda_1 = \lambda_2 = 1, \ \lambda_3 = 2$ are eigen values of A

for $\lambda_1 = \lambda_2 = 1$, AM = 2, corresponding eigen vector is given by,

$(A - \lambda I)X = 0$

$(A - I)X = 0$

$$\begin{bmatrix} -18 & 18 & -6 \\ -18 & 18 & -6 \\ -9 & 9 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} -18 & 18 & -6 \\ 0 & 0 & 0 \\ -9 & 9 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 2R_3 - R_1$

$$\begin{bmatrix} -18 & 18 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} -18 & 18 & -6 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduced form of eq^n is:

$-18x + 18y - 6z = 0 \quad — ①$

$8z = 0, \quad \therefore z = 0$

from ①, $18x = 18y$

$x = y$

let $y = k$. .. $k \neq 0$

$x = k$

$\therefore X_1 = k\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, GM = 1

here AM $\neq$ GM — ①

But for $\lambda_3 = 2$, $AM = GM = 1$ —② ($AM = 1$ & $AM \geq GM$ $\therefore$ GM should be 1)

from ① & ②
A is not diagonalisable

4.(i) If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, find A^{100} using CHT.

→ Char eqn : $\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = 3 - 1 = 2$

$S_2 = -3 + 4 = 1$

$\lambda^2 - 2\lambda + 1 = 0$, $\lambda_1 = \lambda_2 = 1$

let, $f(A) = A^{100}$

By CHT

$A^{100} = a_0 A + a_1 I$ —①

By CHT, let $f_{100}(\lambda_1) = \lambda_1^{100}$

$\lambda_1^{100} = a_0\lambda_1 + a_1$ —②

$1 = a_0 + a_1$ —③

diff ② wrt λ_1

$100\lambda_1^{99} = a_0$

$a_0 = 100$ —④

from ③ & ④, $100 + a_1 = 1$

$a_1 = -99$

put value of a_0 & a_1 in ①

$A^{100} = 100A - 99I$

$= \begin{bmatrix} 300 & -400 \\ 100 & -100 \end{bmatrix} - \begin{bmatrix} 99 & 0 \\ 0 & 99 \end{bmatrix}$

$\therefore A^{100} = \begin{bmatrix} 201 & -400 \\ 100 & -199 \end{bmatrix}$

4. (ii) If $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, find A^n & e^A

$\rightarrow$ Char eq^n : $\lambda^2 - S_1\lambda + S_2 = 0$

$$\lambda^2 - 8\lambda + 12 = 0$$
$$\lambda^2 - 6\lambda - 2\lambda + 12 = 0$$
$$\lambda(\lambda-6) - 2(\lambda-6) = 0$$
$$(\lambda-6)(\lambda-2) = 0$$
$$\lambda_1 = 6, \lambda_2 = 2$$

let $f(A) = A^n$ By CHT

$$A^n = a_0 A + a_1 I \quad \text{— (1)}$$

for, $\lambda_1 = 6$, let $f(\lambda_1) = \lambda_1^n$, By CHT

$$\lambda_1^n = a_0\lambda_1 + a_1$$

$$\therefore 6^n = 6a_0 + a_1 \quad \text{— (2)}$$

for $\lambda_2 = 2$, let $f(\lambda_2) = \lambda_2^n$, By CHT

$$\lambda_2^n = a_0\lambda_2 + a_1$$
$$2^n = 2a_0 + a_1 \quad \text{— (3)}$$
$$a_1 = 2^n - 2a_0 \quad \text{— (4)}$$

from (2) & (4)

$$6^n = 6a_0 + 2^n - 2a_0$$
$$6^n - 2^n = 4a_0$$
$$a_0 = \frac{6^n - 2^n}{4} = \frac{2^n(3^n-1)}{4}$$
$$a_0 = 2^{n-2}(3^n-1)$$

put a_1 & a_0 values in (1)

$$A^n = 2^{n-2}(3^n-1)\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix} + [2^n - 2^{n-1}(3^n-1)]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5/4 \cdot 2^n(3^n-1) & 3/4 \cdot 2^n(3^n-1) \\ 1/4 \cdot 2^n(3^n-1) & 3/4 \cdot 2^n(3^n-1) \end{bmatrix} +$$

$$+\begin{bmatrix} 2^n - \frac{2^n}{2}(3^n-1) & 0 \\ 0 & 2^n - \frac{2^n}{2}(3^n-1) \end{bmatrix}$$

Let $2^n(3^n-1) = x$

$$= \begin{bmatrix} 5x/4 & 3x/4 \\ x/4 & 3x/4 \end{bmatrix} + \begin{bmatrix} 2^n - x/2 & 0 \\ 0 & 2^n - x/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2^n + 3/4\, x & 3x/4 \\ x/4 & 2^n + x/4 \end{bmatrix}$$

$$\therefore A^n = \begin{bmatrix} 2^n + 3/4 \cdot 2^n(3^n-1) & 3/4 \cdot 2^n(3^n-1) \\ 1/4 \cdot 2^n(3^n-1) & 2^n + 1/4 \cdot 2^n(3^n-1) \end{bmatrix}$$

Now for e^A,

let $f(A) = e^A$, By CHT

$$e^A = a_0 A + a_1 I \quad — ⑤$$

$f(\lambda_1) = e^{\lambda_1}$, By CHT

$$e^{\lambda_1} = a_0\lambda_1 + a_1$$

$$e^6 = 6a_0 + a_1 \quad — ⑥$$

$f(\lambda_2) = e^{\lambda_2}$, By CHT

$$e^{\lambda_2} = a_0\lambda_2 + a_1$$

$$e^2 = 2a_0 + a_1$$

$$a_1 = e^2 - 2a_0 \quad — ⑦$$

put ⑦ in ⑥

$$e^6 = 6a_0 + e^2 - 2a_0$$

$$e^6 - e^2 = 4a_0$$

$$a_0 = \frac{e^2}{4}(e^4-1)$$

put a_0 in ⑦, $a_1 = e^2 - 2\frac{e^2}{4}(e^4-1)$

$$= e^2 - \frac{e^2}{2}(e^4-1)$$

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Put a_1 & a_0 values in ⑤

$$e^A = \frac{e^2}{4}(e^4-1)\begin{bmatrix}5 & 3\\ 1 & 3\end{bmatrix} + \left[e^2 - \frac{e^2}{2}(e^4-1)\right]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}5/4\, e^2(e^4-1) & 3/4\, e^2(e^4-1)\\ 1/4 \cdot e^2(e^4-1) & 3/4\, e^2(e^4-1)\end{bmatrix} + \begin{bmatrix}e^2 - e^2/2(e^4-1) & 0\\ 0 & e^2 - \frac{e^2}{2}(e^4-1)\end{bmatrix}$$

let $e^2(e^4-1) = y$

$$= \begin{bmatrix}5y/4 & 3y/4\\ y/4 & 3y/4\end{bmatrix} + \begin{bmatrix}e^2 - y/2 & 0\\ 0 & e^2 - y/2\end{bmatrix}$$

$$= \begin{bmatrix}e^2 + 5y/4 & 3y/4\\ y/4 & e^2 - y/2\end{bmatrix}$$

$$\therefore\ e^A = \begin{bmatrix}e^2 + 3/4\, e^2(e^4-1) & 3/4\, e^2(e^4-1)\\ 1/4\, e^2(e^4-1) & e^2 + 1/4\, e^2(e^4-1)\end{bmatrix}$$

iii) If $A = \begin{bmatrix}-1 & 4\\ 2 & 1\end{bmatrix}$, prove that $3\tan A = A \tan 3$

→ Char eqn: $\lambda^2 - S_1\lambda + S_2 = 0$

$$\lambda^2 - 9 = 0$$
$$(\lambda+3)(\lambda-3) = 0$$
$$\lambda_1 = 3 \text{ or } \lambda_2 = -3$$

let $f(\lambda_1) = \tan\lambda_1$

By CHT, $\tan\lambda_1 = a_0\lambda_1 + a_1$

$\tan A = A a_0 + a_1$ — ①

$f(\lambda_1) = \tan\lambda_1$

$\therefore \tan\lambda_1 = a_0\lambda_1 + a_1$

$\tan 3 = 3a_0 + a_1$ — ②

$f(\lambda_2) = \tan\lambda_2$

$\tan\lambda_2 = a_0\lambda_2 + a_1$

$-\tan 3 = -3a_0 + a_1$ — ③

add ② & ③

$a_1 = 0$

from ② put $a_1 = 0$

$a_0 = \frac{\tan 3}{3}$

put values of a_1 & a_0 in ①

$\tan A = \frac{\tan 3}{3} \cdot A + 0$

$3 \tan A = A \tan 3$

iv) If $A = \begin{bmatrix} 1 & 20 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, find $\tan A$

→ Char eqⁿ: $\lambda^3 - S_1\lambda^2 + S_2\lambda \cancel{+} - S_3 = 0$

$S_1 = 0$

$S_2 = 0 + 0 + (-1) = -1$

$S_3 = 1(0) - 20(0) + 0 = 0$

∴ Char eqⁿ: $\lambda^3 - \lambda = 0$

$\lambda(\lambda^2 - 1) = 0$

$\lambda = 0, \quad \lambda^2 - 1 = 0$

$(\lambda - 1)(\lambda + 1) = 0$

$\lambda = 1, \lambda = -1$

let $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -1$

let $f(A) = \tan A$

By CHT, $\tan A = a_0 A^2 + a_1 A + a_2 I$ — ①

$f(\lambda_1) = \tan \lambda_1$

By CHT, $\tan \lambda_1 = a_0 \lambda_1^2 + a_1 \lambda_1 + a_2$

$a_2 = 0$ — ②

$f(\lambda_2) = \tan \lambda_2$

By CHT, $\tan \lambda_2 = a_0 \lambda_2^2 + a_1 \lambda_2 + a_2$

$\tan 1 = a_0 + a_1 + a_2$ — ③

$f(\lambda_3) = \tan \lambda_3$

By CHT, $\tan \lambda_3 = a_0 \lambda_3^2 + a_1 \lambda_3 + a_2$

$-\tan(1) = a_0 - a_1 + a_2$ — ④

from ② & ③

$a_0 + a_1 = \tan(1)$ — ⑤

from ② & ④

$a_0 - a_1 = -\tan(1)$ — ⑥

Add ⑤ & ⑥,

$$\begin{aligned} a_0 + a_1 &= \tan(1) \\ a_0 - a_1 &= -\tan(1) \\ \hline 2a_0 &= 0 \\ a_0 &= 0 \end{aligned}$$

from ③, $a_1 = \tan(1)$

put a_1, a_2, a_0 values in ①

$\tan A = \tan(1)\, A$

$$\tan A = \tan(1) \begin{bmatrix} 1 & 20 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Determine whether matrices are derogatory or non-derogatory & find its minimal polynomial.

1) $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

→ Char eq^n: $4\lambda^2 - \lambda^3 + \lambda - 4 = 0$

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$(\lambda - 1)(\lambda + 1)(\lambda - 4) = 0$

$\lambda = 1, -1, 4$ are eigen values

Minimal polynomial $\Rightarrow (\lambda - 1)(\lambda + 1)(\lambda - 4) = 0$

$$= (A+I)(A-I)(A-4I)$$

$$= \begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 1 & -4 & -4 \end{bmatrix} \begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 18 & 18 \\ 5 & 6 & 6 \\ -5 & 6 & -6 \end{bmatrix} \begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix A is non-derogatory

2) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

→ Char eq^n: $\lambda^3 + \lambda^2 - 2\lambda - 45 = 0$

$(\lambda+3)(\lambda+3)(\lambda-5) = 0$

$\lambda_1 = -3, \quad \lambda_2 = -3, \quad \lambda_3 = 5$

Minimal polynomial $(\lambda+3)(\lambda-5)$

$$(A+3I)(A-5I) = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ Matrix A is derogatory

3) $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

→ Char eq^n: $-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$

$(\lambda-1)(\lambda-2)(\lambda-2) = 0$

$\lambda = 1, 2, 2$ are eigen values

Minimal poly^n: $(\lambda-1)(\lambda-2) = 0$ or

$(\lambda-1)(\lambda-2)^2 = 0$

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$$(A-I)(A-2I) = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & 5 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -6 & 6 \\ 2 & -2 & 2 \\ -4 & 4 & -4 \end{bmatrix}$$

$$(A-I)(A-2I)^2 = \begin{bmatrix} 6 & -6 & 6 \\ 2 & -2 & 2 \\ -4 & 4 & -4 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix A is non-derogatory.

6. Find the orthogonal matrix P which will diagonalise the matrix A, where matrix A is

i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Char eq^n: $\lambda^3 - 14\lambda^2 = 0$

$\lambda = 14, 0, 0$

When $\lambda = 14$

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 - 10x_2 + 6x_3 = 0$$
$$3x_1 + 6x_2 - 5x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -10 & 6 \\ 6 & -5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 6 \\ 3 & -5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & -10 \\ 3 & 6 \end{vmatrix}}$$

$$\frac{x_1}{14} = \frac{x_2}{28} = \frac{x_3}{\cancel{3}\,42}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$$

$$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

When $\lambda = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$\cancel{R_2 \to}$ $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 3x_3 = 0$$

$$x_1 = -2x_2 - 3x_3$$

$$x_3 = -t, \quad x_2 = -s$$

$$x_1 = 2s + 3t$$

$$X = \begin{bmatrix} 2s + 3t \\ -s + 0t \\ -t + 0s \end{bmatrix} = s \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad X_3 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore X_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix} \qquad X_3 = \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ -1/\sqrt{10} \end{bmatrix}$$

$\therefore$ Diagonal matrix $= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 14 \end{bmatrix}$

Orthogonal modal matrix, $P = \begin{bmatrix} 2/\sqrt{5} & 3/\sqrt{10} & 1/\sqrt{14} \\ 1/\sqrt{5} & 0 & 2/\sqrt{14} \\ 0 & -1/\sqrt{10} & 3/\sqrt{14} \end{bmatrix}$

ii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$\rightarrow$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} = 0$$

$$\lambda^3 - 18\lambda + 45\lambda = 0$$

$$\lambda(\lambda-3)(\lambda-15) = 0$$

$$\lambda = 0, 3, 15$$

* For $\lambda = 0$, $AX = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{5} = \frac{-x_2}{-10} = \frac{x_3}{10}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

* For $\lambda = 3$, $[A - \lambda_2 I] X = 0$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 4 & -4 \\ -4 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -6 & -4 \\ 2 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -6 & 4 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16}$$

$$X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

For $\lambda = 15$

$$[A - \lambda_3 I] X = 0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} -1 & 2 & 6 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} X = 0$$

$R_2 \to R_2 - 6R_1$, $R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} -1 & 2 & 6 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix} X = 0$$

$$-x_1 + 2x_2 + 6x_3 = 0$$

$$-20x_2 - 40x_3 = 0$$

$$x_3 = t, \quad x_2 = -2x_3$$

$$x_3 = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Now $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

Normalized matrix

$\therefore$ Orthogonal matrix is

$$P = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

7. # Cayley Hamilton Theorem

Every square matrix satisfies its own characteristic eqⁿ

Proof: Let $(-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots a_n) = 0$ as a char eqⁿ of A, we know

$$A\, adj(A) = |A|$$

$$\Rightarrow (A - \lambda I)\, adj(A - \lambda I) = |A - \lambda I|$$

$$\Rightarrow (A - \lambda I)\, adj(A - \lambda I) = (-1)^n(\lambda^n + a_1\lambda^{n-1} + \dots a_n) \quad \text{--- (1)}$$

$$(A - \lambda I) = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$$

Let $adj(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \dots B_{n-1}$

Now (1) becomes, $(A - \lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \dots B_{n-1})$

$$= (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots a_n)$$

Equating co-efficients of alike power of λ, we get

$$\left.\begin{array}{rcll} -B_0 &=& (-1)^n & \text{———} \times A^n \\ AB_0 - B_1 &=& (-1)^n a_1 & \text{———} \times A^{n-1} \\ AB_1 - B_2 &=& (-1)^n a_2 & \text{———} \times A^{n-2} \\ \vdots & & & \vdots \\ AB_{n-2} - B_{n-1} &=& (-1)^n a_{n-1} & \text{———} \times A \\ AB_{n-1} &=& (-1)^n a_n & \text{———} \times I \end{array}\right\} \text{multiplying both sides}$$

$$\Rightarrow \quad \begin{array}{rcl} -A^n B_0 &=& (-1)^n A^n \\ A^n B_0 - A^{n-1} B_1 &=& (-1)^n a_1 A^{n-1} \\ A^{n-1} B_1 - A^{n-2} B_2 &=& (-1)^n a_2 A^{n-2} \\ \vdots & & \\ A^2 B_{n-2} - AB_{n-1} &=& (-1)^n a_{n-1} A \\ AB_{n-1} &=& (-1)^n a_n I \end{array}$$

Adding all above eq^{n}s :

$$0 = (-1)^n \left(A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots a_n I\right)$$

• Th :- Prove that the e value of a real-symmetric or Hermitian matrix are always real.

Proof : Let A is a real symmetric matrix

$\Rightarrow A = A^T$ also $A = A^\theta$ — ①

θ. Transjugate (Transpose & Conjugate)

Again let λ is an e value of A

$\Rightarrow$ There exists $X \neq 0$ such that $AX = \lambda X$ — ②

Taking tranjugate of ② on both side we get

$X^\theta A^\theta = \bar{\lambda} X^\theta$ — ③

Multiplying eq^n ③ by X on both side, we get

$$X^\theta A X = \bar{\lambda} X^\theta X$$

From ② $X^\theta \lambda X = \bar{\lambda} X^\theta X$

$$\Rightarrow (\lambda - \bar{\lambda}) X^\theta X = 0$$

Since $X^\theta X \neq 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$$\Rightarrow \lambda = \bar{\lambda}$$

It is possible if λ is real

Th :- Prove that the e values of a real skew symmetric Skew - Hermitian matrix are either zero or purely imaginary.

→ Let A is a real skew symm matrix

$$A = - A^\theta$$

$$A X = \lambda X$$

$$X^\theta A^\theta = \bar{\lambda} X^\theta$$

from ① we get $-X^\theta A = \bar{\lambda} X^\theta$

multiplying by X, $-X^\theta A X = \bar{\lambda} X^\theta X$

$$(\lambda + \bar{\lambda}) X^\theta X = 0$$

Since $X^\theta X \neq 0 \Rightarrow \lambda + \bar{\lambda} = 0$

$$\lambda = -\bar{\lambda}$$

It is possible if λ is zero or purely imaginary

Th :- Prove that the e values of a unitary matrix/ orthogonal matrix are of unit modulus.

→ Let A is an unitary matrix

$$\Rightarrow A A^\theta = I \quad \text{— ①}$$

Let λ is an e value of A

$$\Rightarrow \exists\ X = 0 \text{ such that } A X = \lambda X \quad \text{— ②}$$

Taking tranjugate on both sides of ② we get

$$X^\theta A^\theta = \bar{\lambda} X^\theta \quad \text{— ③}$$

Post multiplying ③ by 'AX' we get

$$X^\theta A^\theta A X = \bar{\lambda} X^\theta A X$$

From ① & ②, we get

$$X^{\theta} I X = \bar{\lambda} X^{\theta} \lambda X$$

$$\Rightarrow X^{\theta} X = \bar{\lambda} \lambda X^{\theta} X$$

$$(\bar{\lambda}\lambda - 1) X^{\theta} X = 0$$

Since $X^{\theta} X \neq 0 \Rightarrow \bar{\lambda}\lambda = 1$

$$\Rightarrow |\lambda| = 1$$

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Th : Prove that eigen value of involuntary matrix are either +1 & -1.

→ Let A be an involuntary matrix

$$A^2 = I \quad - ①$$

Let λ is an eigen value

$\exists\, x \neq 0$ such that $AX = \lambda X$ — ②

Pre-multiplying eq^n ② by A, we get

$$A^2 X = \lambda A X$$

$$I X = \lambda \lambda X$$

$$X = \lambda^2 X$$

Since $X \neq 0$, $\lambda^2 = 1$

$$\lambda = \pm 1$$

Th: Prove that the e ~~val~~ vectors corresponding to distinct e value of a real-symmetric matrix are always orthogonal.

→ $X_1 = (x_1, y_1, z_1)$

$X_2 = (x_2, y_2, z_2)$

$$X_1 X_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$\underline{X_1 X_2^T = 0}$ → with this we prove

Let λ_1 & λ_2 are distinct e values of A

$\exists\, x_1$ & $x_2 \neq 0$ st. $AX_1 = \lambda_1 X_1$ — ①

& $AX_2 = \lambda_2 X_2$ — ②

Given $A = A^T$ — ③

Taking transpose of ① we get

$$X_1^T A^T = \lambda_1 X_1^T \Rightarrow X_1^T A = \lambda_1 X_1^T \quad -④$$

Post multiplying ④ by X_2

$$X_1^T A X_2 = \lambda_1 X_1^T X_2$$

By ② we get,

$$X_1^T \lambda_2 X_2 = \lambda_1 X_1^T X_2$$

$$\Rightarrow (\lambda_1 - \lambda_2) X_1^T X_2 = 0$$

Since λ_1 & λ_2 are distinct, i.e

$$\lambda_1 \neq \lambda_2 \Rightarrow X_1^T X_2 = 0$$

$\Rightarrow X_1$ & X_2 are orthogonal

9. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix}$ are non-diagonalizable but AB is diagonalizable.

→ * $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Char eq^n of A : $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda-1)^2 = 0$, $\lambda = 1, 1$

for $\lambda = 1$ $[A - \lambda_1 I]X = 0$

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Rank of matrix is 1 & no. of variable is 2 so no. of soln is 1

Geometric multiplicity is 1 but algebric multiplicity is 2, hence it is non-diagonalizable

* $B = \begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix}$

Char eq^n of B : $\lambda^2 - 4\lambda + 4 = 0$

$(\lambda-2)^2 = 0$, $\lambda = 2, 2$

for $\lambda = 2$, $[B - \lambda_2 I]X = 0$

$$\begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Rank is 1 & no. of variables are 2 so geometrical multiplicity is 2-1 = 1 & algebric multiplicity is 2 as AM ≠ GM it is non-diagonalisable.

* $C = AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1/2 & 2 \end{bmatrix}$

Char eq^n of C : $\lambda^2 - 5\lambda + 4 = 0$

$\lambda = 1, 4$

for $\lambda = 1$, $[C - \lambda_3 I]X = 0$

$$\begin{bmatrix} 2 & 4 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{4} R_1$

$$\therefore \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Rank is 1, also AM & GM of eigen value 1 is equal

for $\lambda = 4$, $[A - 4I] X = 0$

$$\begin{bmatrix} -2 & 4 \\ 1/2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + \frac{1}{2} R_1$

$$\begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Rank of matrix is 1, $-x_1 + 4x_2 = 0$

There are only one soln, hence AM & GM of eigen value 4 is equal, so C is diagonalisable.

10. If $A = \begin{bmatrix} x & 4x \\ 2 & y \end{bmatrix}$ have Eigen values 5 & -1, find values of x & y

→ Char eq^n: $\lambda^2 - (x+y)\lambda + xy - 8x = 0$

Sum of diagonal elements = sum of eigen values

$$x + y = 4$$

$|A|$ = products of eigen value

$$xy - 8x = -5$$

$$y = 4 - x$$

$$x(4-x) - 8x = -5$$

$$4x - x^2 - 8x = -5$$

$$-4x = x^2 + 5 = 0$$

$$x^2 + 4x - 5 = 0$$

$$x = 1, \quad x = -5$$

When $x = 1$, $y = 3$

$x = -5$, $y = 9$

11. Using Cayley-Hamilton theorem find A^{100} where

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$\rightarrow$ char eq^n of A: $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$$\lambda^2(\lambda - 2) - (\lambda - 2) = 0$$

$$(\lambda - 2)(\lambda^2 - 1) = 0$$

$$\lambda = 2, 1, -1$$

According to CHT

$$A^3 - 2A^2 - A + 2I = 0$$

$$\phi(A) = A^{100} = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$$

$$\lambda^{100} = \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 \quad \text{—} \ 1$$

put $\lambda = 1, -1$

$$\therefore \ 1 = \alpha_2 + \alpha_1 + \alpha_0 \quad \text{—} \ 2$$

$$4 = \alpha_2 - \alpha_1 + \alpha_0 \quad \text{—} \ 3$$

diff 1 wrt λ

$$100\lambda^{99} = 2\alpha_2 \lambda + \alpha_1$$

$$\lambda = 1, \quad 100 = 2\alpha_2 + \alpha_1 \quad \text{—} \ 4$$

After solving 2, 3, 4

$$\alpha_2 = 50, \quad \alpha_1 = 0, \quad \alpha_0 = 49$$

$$A^{100} = 50A^2 - 49I$$

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} 50 & 300 & 100 \\ 0 & 200 & 50 \\ 0 & 0 & 50 \end{bmatrix} + \begin{bmatrix} -49 & 0 & 0 \\ 0 & -49 & 0 \\ 0 & 0 & -49 \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} 1 & 300 & 100 \\ 0 & 151 & 50 \\ 0 & 0 & 1 \end{bmatrix}$$

13) Reduce the following quadratic forms by orthogonal transformation. Write the corresponding linear transformation.

i) $4xy + 2yz - 4xz - 6x^2 - 3y^2 - 3z^2$

$\rightarrow$

$$[x \quad y \quad z]\begin{bmatrix} -6 & 2 & -2 \\ 2 & -3 & 1 \\ -2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Char eq^n of A: $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\lambda_1 = 2, \ \lambda_2 = 2, \ \lambda_3 = 8$

The e vectors to $\lambda_1 = \lambda_2 = 2$ given by

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_1 \rightarrow R_1/2 \qquad R_2 \rightarrow R_2 + R_1$

$R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 - x_2 + x_3 = 0$

$x_2 = 2x_1 + x_3$

Putting $x_1 = 0$ & $x_3 = 1$ we get $X_1 \ (0, 1, 1)$

$x_3 = 0$ & $x_1 = 1$, $\quad X_2 \ (1, 2, 0)$

For $\lambda_1 = 8$, e vector is given by

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 + R_1$

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$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_1 \to R_1/2$, $R_2 \to -R_2/3$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 - x_3 = 0$$
$$x_2 + x_3 = 0$$

putting $x_3 = 1$, $X_3 = (2, -1, 1)$

Normalised e vectors are

$$\frac{x_1}{|x_1|} = \frac{(0,1,1)}{\sqrt{2}} = (0, 1/\sqrt{2}, 1/\sqrt{2}) = y_1$$

$$\frac{x_2}{|x_2|} = \frac{(1,2,0)}{\sqrt{5}} = (1/\sqrt{5}, 2/\sqrt{5}, 0) = y_2$$

$$\frac{x_3}{|x_3|} = \frac{(2,-1,1)}{\sqrt{6}} = (2/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6}) = Y_3$$

Orthogonal model matrix $P = [y_1^T, y_2^T, y_3^T]$

$$P = \begin{bmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{6} \\ 1/\sqrt{2} & 2/\sqrt{5} & -1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 1/\sqrt{6} \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = P^TAP$$

Canonical form of QF is Z^TDZ

when $Z = [z_1, z_2, z_3]$ when LT is $X = PZ$

Canonical form $\to 2z_1^2 + 2z_2^2 + 8z_3^2$

& LT or orthogonal transformation

$$X = \begin{bmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{6} \\ 1/\sqrt{2} & 2/\sqrt{5} & -1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$x = \frac{z_2}{\sqrt{5}} + 2z_3$$

$$y = \frac{z_1}{\sqrt{2}} + \frac{2z_2}{\sqrt{5}} - \frac{z_3}{\sqrt{6}}$$

$$z = \frac{z_1}{\sqrt{2}} + \frac{z_3}{\sqrt{6}}$$

Value class

(r) rank = 3 (non-zero entities)

(p) index = 3 (+ve entries)

Signature $= 2p - r$

$= 2(3) - 3 = 3$

Nature since $r = n$ & $p = n$ $\Rightarrow$ +ve

ii) $10x^2 + 2y^2 + 5z^2 + 6yz - 10xz - 4xy$

$\rightarrow$ Matrix of quadratic form: $A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$

Char eq^n: $\lambda^3 - 17\lambda^2 + 42\lambda - 0 = 0$

$\lambda_1 = 14, \quad \lambda_2 = 3, \quad \lambda_3 = 0$

for $\lambda = 14$, $[A - 14\lambda]X = 0$

$$\begin{bmatrix} -4 & -2 & -5 \\ -2 & -12 & 3 \\ -5 & 3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4x_1 - 2x_2 - 5x_3 = 0$$
$$-2x_1 - 12x_2 - 3x_3 = 0$$
$$-5x_1 + 3x_2 \cancel{+} 9x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -12 & 3 \\ 3 & -9 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 3 \\ -5 & 9 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -12 \\ -5 & 3 \end{vmatrix}}$$

$$\frac{-x_1}{99} = \frac{x_2}{33} = \frac{x_3}{66}$$

$$X_1 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

for $\lambda = 3$, $[A - 3I]X = 0$

$$\begin{bmatrix} 7 & -2 & -5 \\ -2 & -1 & 3 \\ -5 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 3 \\ -5 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ -5 & 3 \end{vmatrix}}$$

$$\frac{x_1}{-11} = \frac{-x_2}{11} = \frac{x_3}{-11}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

for $\lambda = 0$ $[A]X = 0$

$$\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 3 \\ -5 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 2 \\ -5 & 3 \end{vmatrix}}$$

$$\frac{x_1}{1} = \frac{-x_2}{5} = \frac{x_3}{4}$$

$$X_3 = \begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}$$

Normalised e vectors, $\dfrac{x_1}{|x_1|} = \left(-1/\sqrt{42}, 5/\sqrt{42}, -4/\sqrt{42}\right)$

$$\frac{x_2}{|x_2|} = \left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$$

$$\frac{x_3}{|x_3|} = \left(-3/\sqrt{14}, 1/\sqrt{14}, 2/\sqrt{14}\right)$$

Orthogonal modal matrix, $P = \begin{bmatrix} -1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ 5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ -4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{bmatrix}$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

Canonical form $Z^T D Z$

$$z = [z_1 \quad z_2 \quad z_3]$$

$\therefore$ Canonical form $\cancel{Z^T D Z} = 3z_2^2 + 14z_3^2$

linear transformation, $X = PZ$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ 5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ -4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$x = \frac{-1}{\sqrt{42}} z_1 + \frac{1}{\sqrt{3}} z_2 - \frac{3}{\sqrt{14}} z_3$$

$$y = \frac{5}{\sqrt{42}} z_1 + \frac{1}{\sqrt{3}} z_2 + \frac{1}{\sqrt{14}} z_3$$

$$z = \frac{-4}{\sqrt{42}} z_1 + \frac{1}{\sqrt{3}} z_2 + \frac{2}{\sqrt{14}} z_3$$