Assignment-1

1) Simplify the following logical expression using logical equivalences:

$$(P \land Q) \lor (\sim P \land Q)$$

→ Step-by-Step Simplification :-

1 Factor out Q (Distributive Property):

$$(P \land Q) \lor (\sim P \land Q)$$
$$= Q \land (P \lor \sim P)$$

2 Simplify $P \lor \sim P$ (Law of excluded middle):

$$P \lor \sim P = T$$

3. Substitute T into the expression:

$$Q \land T$$

4 Simplify $Q \land T$ (Identity Law):

$$Q \land T = Q$$

Final simplified Expression :-

$$Q$$

2) Show that $(P \to Q) \leftrightarrow (\sim P \lor Q)$ is a tautology using a truth table.

→

P	Q	$\sim P$	$P \to Q$	$\sim P \lor Q$	$(P \to Q) \leftrightarrow (\sim P \lor Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

→ The expression $(P \to Q) \leftrightarrow (\sim P \lor Q)$ is a tautology because it is true for all combinations of P and Q

3) Prove or disprove : $(P \vee Q) \wedge (\sim P \vee \sim Q)$ is a contradiction

→

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim P \vee \sim Q$	$(P \vee Q) \wedge (\sim P \vee \sim Q)$
T	T	F	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	T	F

Hence, from the above table, the expression $(P \vee Q) \wedge (\sim P \vee \sim Q)$ is not a contradiction because it is true for some values of P and Q.

4) Define a relation R on the set $A = \{1, 2, 3\}$ as $R = \{(1,1), (2,2), (3,3), (1,2)\}$

- Is R reflexive? why or why not?
- Is R symmetric? why or why not?
- Is R transitive? why or why not?

→ set $A = \{1, 2, 3\}$ and relation $R = \{(1,1), (2,2), (3,3), (1,2)\}$.

A relation R on a set A is reflexive, if for every $a \in A$, $(a, a) \in R$.
Here, $A = \{1, 2, 3\}$
The pairs $(1,1)$, $(2,2)$ and $(3,3)$ are all in R.
$\therefore (a, a) \in R$, the relation is reflexive

A relation R on a set A is symmetric, if $(a, b) \in R$ $(b, a) \in R$.
Here, $(1,2) \in R$, but $(2,1) \notin R$
$\therefore$ Relation R is not symmetric

A relation R on a set A is transitive, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$

Here :

- $(1,2) \in R$, but there is no $(2,c) \in R$ for any c. So, no violation of transitivity occurs
- For all other pairs, there are no combinations that violate the transitive property.

$\therefore$ R is transitive

=) Here the R is reflexive, transitive but not symmetric
$\therefore$ It is not equivalence relation.

5) Let $A = \{1,2,3,4\}$ and define $R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1)\}$. Is R an equivalence relation? why or why not?

-> To determine if R is an equivalence relation, we check if it satisfies the properties of reflexivity, symmetry, and transitivity.

set $A = \{1,2,3,4\}$, $R = \{(1,1)(2,2),(3,3),(4,4),(1,2)(2,1)\}$

A relation R on a set A is reflexive if, for every $a \in A$, $(a,a) \in R$

Here, $A = \{1,2,3,4\}$.

The pairs $(1,1),(2,2),(3,3),(4,4)$ are all in R

$\therefore$ R is reflexive

A relation R on a set A is symmetric, if $(a,b) \in R$ $(b,a) \in R$

Here : $(1,2) \in R$ and $(2,1) \in R$

$\therefore$ R is symmetric.

A relation R on a set A is transitive if, $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$

Here $(1,2) \in R$, $(2,1) \in R$

$\therefore (1,1) \in R$ is true.

$\therefore$ R is transitive

$\Rightarrow$ Here, R is reflexive, symmetric and transitive

$\therefore$ Relation R is an equivalence relation.

6). Prove by induction that $2^n > n$ for all integers $n \geq 1$

$\rightarrow$ Let $P(n) : 2^n > n$ for all integers $n \geq 1$

– For $n = 1$

$L.H.S = 2^n = 2^1 = 1$

$R.H.S = n = 1$

– Since $2 > 1$

$L.H.S. > R.H.S$

clearly, $2^1 > 1$

$\therefore P(n)$ is true for $n \geq 1$

7) Prove that $3^n + 2^n$ is divisible by 5 for all $n \geq 1$.

$\rightarrow$ Step 1: Base Case

Test if $3^n + 2^n$ is divisible by 5 for $n = 1$

$3^1 + 2^1 = 3 + 2 = 5$

clearly, 5 is divisible by 5

Thus, the base case holds

Step 2: Inductive Hypothesis

Assume that $3^k + 2^k$ is divisible by 5 for $n = k$,

i.e., $3^k + 2^k = 5m$ for some integer m.

Step 3 : Prove for $n = k+1$

we need to show that $3^{k+1} + 2^{k+1}$ is divisible by 5.

$$3^{k+1} + 2^{k+1} = 3.3^k + 2.2^k$$

From the inductive hypothesis, $3^k + 2^k = 5m$.

Let's simplify $3^{k+1} + 2^{k+1}$ % 5 :

1. Modulo Properties :
   - $3 = -2 \pmod 5$
   - 2.2^k remains $2^k \pmod 5$
2. Substituting values :

$$3^{k+1} + 2^{k+1} = (3.3^k) + (2.2^k)$$

Modulo 5, since $3^k + 2^k = 0 \pmod 5$;

we conclude :

$$3^{k+1} + 2^{k+1} = 0 \pmod 5$$

Thus, $3^{k+1} + 2^{k+1}$ is divisible by 5.

→ By induction, we proved "$3^n + 2^n$ is divisible by 5 for all $n \geq 1$"

8) Prove that $n(n^2+5)$ is divisible by 6 for all integers n using PMI.

→ $P(n) = n(n^2+5)$ is divisible by 6.

So, substituting different values for n, we get,

$P(0) = 0(0^2+5) = 0$ which is divisible by 6.

$P(1) = 1(1^2+5) = 6$ which is divisible by 6

$P(2) = 2(2^2+5) = 18$ which is divisible by 6.

$P(3) = 3(3^2+5) = 42$ which is divisible by 6.

Let $P(k) = k(k^2+5)$ be divisible by 6.

So, we get,

$\Rightarrow k(k^2+5) = 6x$

Now, we also get that,

$\Rightarrow P(k+1) = (k+1)((k+1)^2+5) = (k+1)(k^2+2k+6)$

$= k^3 + 3k^2 + 8k + 6$

$= 6x + 3k^2 + 3k + 6$

$= 6x + 3k(k+1) + 6$ [$n(n+1)$ is always even and divisible by 2]

$= 6x + 3 \times 2y + 6$ which is divisible by 6

$\Rightarrow P(k+1)$ is true when $P(k)$ is true

→ Therefore, by mathematical Induction,

" $P(n) = n(n^2+5)$ is divisible by 6, for each natural number n "

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