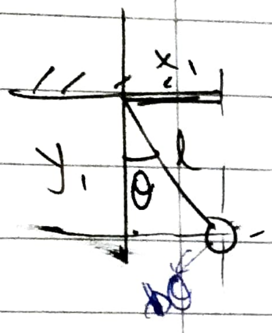
Simple Pendulum Using Lagrange's Eqn.



$x_1 = l \sin\theta.$ $\qquad \dot{x}_1 = l \cos\theta \, \dot{\theta}$

$y_1 = -l \cos\theta$ $\qquad \dot{y}_1 = l \sin\theta \, \dot{\theta}$

$$T = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{y}_1^2 \right)$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2$$

$$L = T - V$$
$$= \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta.$$

$$V = m g y_1$$
$$= -mgl\cos\theta$$

$$\frac{d}{dt}\left(m l^2 \dot{\theta} \right) + mg l \sin\theta = 0 - b\dot{\theta}$$

$$\cancel{m} l^{\cancel{2}} \ddot{\theta} + \cancel{m} g \cancel{l} \sin\theta = 0.$$

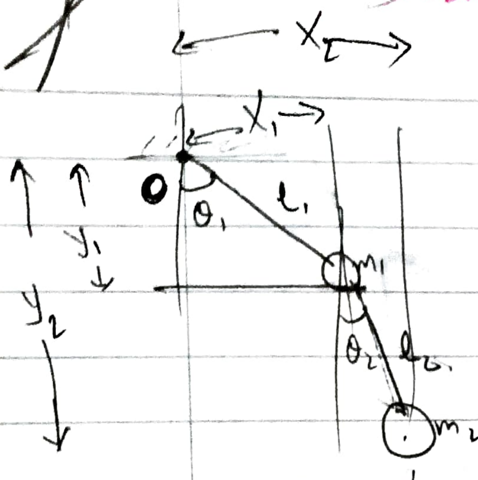
$$l \ddot{\theta} + g \sin\theta = 0.$$
$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0.$$

$$\dot{\theta} = -\frac{g}{l} \sin\theta \, t + c \qquad \cancel{\dot{\theta} = +\frac{g}{L}\cos\theta + c}$$

$$\cancel{\theta = \frac{g}{L}\sin\theta + c\theta + d.}$$

$$\theta = -\frac{g}{l} \sin\theta \frac{t^2}{2} + ct + d$$

Double Pendulum Using Lagrange's Eqn.



$$x_1 = l_1 \sin\theta_1$$
$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

$$y_1 = -l_1 \cos\theta_1$$
$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

$$\dot{x}_1 = l_1 \dot{\theta}_1 \cos\theta_1 \qquad \dot{y}_1 = l_1 \dot{\theta}_1 \sin\theta_1$$
$$\dot{x}_2 = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2 \qquad \dot{y}_2 = l_1 \dot{\theta}_1 \sin\theta_1 + l_2 \dot{\theta}_2 \sin\theta_2$$

$$KE = T = \frac{1}{2} m_1 (v_x^2 + v_y^2) + \frac{1}{2} m_2 (v_x^2 + v_y^2)$$

$$= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m (l_1^2 \dot{\theta}_1^2 (1)) + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 \dot{\theta}_1 l_2 \dot{\theta}_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2))$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2))$$

$$PE = V = m_1 g y_1 + m_2 g y_2$$
$$= -m_1 g l_1 \cos\theta_1 - m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$
$$= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

$$L = T - V$$
$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2$$
$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \cos\theta_1$$
$$+ m_2 g l_2 \cos\theta_2$$

Lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0.$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1+m_2)\, l_1^2\, \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1-\theta_2).$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1-\theta_2)$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - (m_1+m_2)\, g\, l_1 \sin\theta_1$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - m_2 g l_2 \sin\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1+m_2)\, l_1^2\, \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2)$$
$$\cancel{+} - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1-\theta_2)\, \dot{\theta}_1$$
$$+ m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1-\theta_2)\, \dot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) +$$
$$- m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2)$$

①
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0.$$

$$(m_1+m_2)\, l_1^2\, \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) - \cancel{m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2)}$$
$$+ m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) + \cancel{m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2)}$$
$$+ (m_1+m_2)\, g\, l_1 \sin\theta_1 = 0$$

$$(m_1+m_2)\, l_1^2\, \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2)$$
$$+ (m_1+m_2)\, g\, l_1 \sin\theta_1 = 0$$

$$(m_1+m_2)\, l_1\, \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2)$$
$$+ (m_1+m_2)\, g \sin\theta_1 = 0. \quad ①$$

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0.$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0.$$

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$$l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g \sin\theta_2 = 0.$$