Lecture 12: Law of Large Numbers

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Definition 12.1 *We say that two random variables X and Y are independent if events $\{X \in A\}$ and $\{Y \in B\}$ are independent for every Borel subsets A and B of $\mathbb{R}$.*

Recall the notion of (mutual or total) independence for multiple events

Definition 12.2 *We say that the events $A_1, A_2, \cdots, A_n$ are independent if for any distinct $A_{i_1}, A_{i_2}, \cdots, A_{i_m}, (2 \leq m \leq n)$ from $\{A_1, A_2, \cdots, A_n\}$*

$$P\left(A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m}\right) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_m}).$$

Definition 12.3 *We say $X_1, X_2, \cdots, X_n$ are independent if events $\{X_1 \in A_1\}, \{X_2 \in A_2\}, \cdots, \{X_n \in A_n\}$ are independent for all $A_1, A_2, \cdots, A_n$ Borel subsets of $\mathbb{R}$.*

When we study law of large numbers and central limit theorem we encounter with the sequence of independent random variables $X_1, X_2, \cdots$. So we need to understand the meaning of independence for countably infinite collection of random variables. The following definition applies to any arbitrary (even uncountable!!) collection.

Definition 12.4 *Let T be an index set. We say that the family or class of random variables $\{X_t | t \in T\}$ is independent if every finite sub-collection $\{X_{t_1}, X_{t_2}, \cdots, X_{t_n}\}$ is independent, where each $t_i \in T$.*

Theorem 12.5 (Weak Law of Large numbers) *Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables, each having finite mean μ. Then for every $\delta > 0$,*

$$\lim_{n\to\infty} P\left\{\left|\frac{S_n}{n} - \mu\right| \geq \delta\right\} = 0, \quad \textit{or equivalently} \quad \lim_{n\to\infty} P\left\{\left|\frac{S_n}{n} - \mu\right| < \delta\right\} = 1 \tag{12.1}$$

where $S_n = X_1 + X_2 + \cdots + X_n$.

Remark 12.6 *1. The weak law of large numbers states that for large n, the bulk of the distribution of $\frac{S_n}{n}$ is concentrated near μ . That is, if we consider a positive length interval $[\mu-\delta, \mu+\delta]$ around μ, then there is high probability that S_n/n will fall in that interval; as $n \to \infty$, this probability converges to 1 . Of course, if δ is very small, we may have to wait longer (i.e. , need a larger value of n) before we can assert that S_n/n is highly likely to fall in that interval.*

2. *To understand the convergence in weak law, think in terms of PMF (if X_i are discrete random variables) or PDF (if X_i's have the pdf then we know that S_n will posses a pdf) of random variable S_n/n. Weak law states that "almost all" of the PMF or PDF of S_n/n is concentrated within δ neighborhood of μ for large values of n.*

3. *The limit in (12.1) means: $\forall\ \delta, \epsilon > 0$, there exists $n_0(\epsilon, \delta)$ such that for all $n \geq n_0(\epsilon, \delta)$ we have*

$$P\left\{\omega : \left|\frac{S_n}{n} - \mu\right| < \delta\right\} > 1 - \epsilon.$$

If we refer to δ as the accuracy level and ϵ as the confidence level, the weak law takes the following intuitive form: for any given level of accuracy and confidence, S_n/n will be equal to μ, within these levels of accuracy and confidence, provided n is large enough.

Example 12.7 *Let $X_1, X_2, \cdots$ be independent and identically distributed random variables with $E[X_i] = 0$ and $Var(X_i) = 1$ for all i. Let $S_n = X_1 + X_2 + \cdots + X_n$. Then, for any $x > 0$, compute $\lim_{n\to\infty} P\left(-nx < S_n < nx\right)$.*

Solution: For any $x > 0$, we have

$$\begin{aligned} P\left(-nx < S_n < nx\right) &= P\left(-x < \frac{S_n}{n} < x\right) = P\left(\left|\frac{S_n}{n} - 0\right| < x\right) \\ &= 1 - P\left(\left|\frac{S_n}{n} - 0\right| \geq x\right) \end{aligned}$$

By weak law of large numbers, we have

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - 0\right| \geq x\right) = 0$$

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Random Sampling

Let $X_1, \cdots, X_n$ be n independent random variables having the same distribution. These random variables may be thought of as n independent measurements of some quantity that is distributed according to their common distribution (e.g., height of students in LNMIIT campus). In this sense we sometimes speak of the random variables $X_1, \cdots, X_n$ as constituting a random sample of size n from this distribution.

Suppose that the common distribution of these random variables has finite mean μ. Then for n sufficiently large we would expect that the sample mean $\frac{S_n}{n} = (X_1 + \cdots + X_n)/n$ should be close to ture mean μ.

The weak law of large numbers asserts that the sample mean of a large number of independent identically distributed random variables is very close to the true mean, with high probability.