Lecture 2: Functions of Random Vectors:II

16 March, 2018

Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

Example 2.1 *Show that the sum of two independent Poisson random variables with parameters μ and λ respectively, is Poisson with parameter $\mu + \lambda$.*

Solution: Let $X \sim Poisson(\mu)$ and $Y \sim Poisson(\lambda)$. Recall that a Poisson random variable takes values $0, 1, 2, \cdots$ and

$$P(X = k) = \frac{\mu^k e^{-\mu}}{k}, \quad P(Y = k) = \frac{\lambda^k e^{-\lambda}}{k}, k = 0, 1, 2, \cdots$$

Then $X + Y$ is a discrete random variable taking values $\{0, 1, 2, \cdots\}$. To see this,

Fix $x = 0$, then $z = 0 + y$. Now run through the all y values, we get $z = 0, 1, 2, 3, \cdots$
Fix $x = 1$, then $z = 1 + y$. Now run through the all y values, we get $z = 1, 2, 3, \cdots$
Fix $x = 2$, then $z = 2 + y$. Now run through the all y values, we get $z = 2, 3, 4, \cdots$

Also note that independence of X and Y, implies that $f(x, y) = f_X(x) f_y(y)$. For $k \in \{0, 1, 2, \cdots\}$

$$\begin{aligned}
P(X + Y = k) &= \sum_{(x,y):x+y=k} f(x, y) = \sum_{i=0}^{k} f_X(i) f_Y(k - i) \\
&= \sum_{i=0}^{k} \frac{e^{-\mu}\mu^i}{i!} \times \frac{e^{-\lambda}\lambda^{k-i}}{k - i!} \\
&= e^{-(\mu+\lambda)} \sum_{i=0}^{k} \frac{\mu^i \lambda^{k-i}}{i!k - i!} \\
&= e^{-(\mu+\lambda)} \times \frac{1}{k!} \sum_{i=0}^{k} k! \frac{\mu^i \lambda^{k-i}}{i!k - i!} \\
&= e^{-(\mu+\lambda)} \times \frac{1}{k!} (\mu + \lambda)^k \quad \text{(binomail formula)}
\end{aligned}$$

■

Functions of Random Vectors with Joint Density:

We have noticed that if X and Y are discrete random variables on the sample space Ω and $g : \mathbb{R}^2 \to \mathbb{R}$ be any function, then $g(X, Y)$ is a discrete random variable.

Now if X and Y have joint pdf and $g : \mathbb{R}^2 \to \mathbb{R}$ is any function. Can we say that the random variable $g(X, Y)$ also has a pdf? It is very natural (generalizing the discrete case) to think that the answer must be yes. Before we answer this question, let us consider the same question for a single random variable.

Recall that a random variable X with the following pdf

$$f(x) = \begin{cases} 0 & \text{if } \ x < 0 \\ \lambda e^{-\lambda x} & \text{if } \ x \geq 0 \end{cases}.$$

is called a exponential random variable with parameter $\lambda > 0$. We write this $X \sim \exp(\lambda)$. Then distribution function of X is given by

$$F_X(x) = \begin{cases} 0 & \text{if } \ x < 0 \\ 1 - e^{-\lambda x} & \text{if } \ x \geq 0 \end{cases}.$$

Example 2.2 *Let* $X \sim \exp(5)$. *Find the pdf of the random variable* $\min\{X, 10\}$ *(if it exists).*

Solution: First we determine the distribution function of random variable $Y := \min\{X, 10\}$.

Let $x \in \mathbb{R}$ be given. Then we want to compute the event $\{Y \leq x\}$ in terms of random variable X and constant 10. Note that $Y(\omega) \leq x \iff$ either $X(\omega) \leq x$ or $x \geq 10$ or both. Therefore we have

$$\{Y \leq x\} = \{\omega : X(\omega) \leq x\} \cup \{\omega : 10 \leq x\}$$

Note that if $10 > x$ then $\{10 \leq x\} = \emptyset$ and if $10 \leq x$ then $\{10 \leq x\} = \Omega$. Hence

$$\{Y \leq x\} = \begin{cases} \{X \leq x\} & \text{if } \ x < 10 \\ \Omega & \text{if } \ x \geq 10 \end{cases}$$

Hence distribution function of Y denoted by F_Y is

$$F_Y(x) = \begin{cases} F_X(x) & \text{if } \ x < 10 \\ 1 & \text{if } \ x \geq 10 \end{cases} = \begin{cases} 0 & \text{if } \ x < 0 \\ 1 - e^{-5x} & \text{if } \ 0 \leq x < 10 \\ 1 & \text{if } \ x \geq 10 \end{cases}$$

Now we differentiate the function F_Y to get the pdf of random variable Y.

$$f_Y(x) = \begin{cases} 0 & \text{if } \ x < 0 \\ 5e^{-5x} & \text{if } \ 0 < x < 10 \\ 0 & \text{if } \ x > 10 \end{cases}$$

We don't bother regarding differentiability of F_Y at points $x = 0, 10$ where functions changes it's definition because if you change the value of a pdf at finitely many points it does not affect the distribution function (and hence we may define it equal to zero at both points.) If f_Y is a probability density function then it must be non-negative on $\mathbb{R}$ and it integrate to 1 on $\mathbb{R}$.

$$\int_{-\infty}^{\infty} f_Y(y)dy = \int_0^{10} 5e^{-5y}dy = -e^{-5y}\Big|_0^{10} = 1 - e^{-50} < 1.$$

What went wrong? If a random variable has pdf then it's distribution function is continuous. Since F_Y is discontinuous at $x = 10$ hence Y can not have pdf. ■

Remark 2.3 *Many students think that if a random variable is not continuous, i.e., which does not have pdf then it is discrete. The random variable Y in Example 2.2, is not continuous. What is range of the Y? X is an exponential random variable so it's range is $[0, \infty)$.*

This tells us that range of Y is $[0, 10]$. Random variable Y takes values over an interval but it is not continuous!!! Of course it is not discrete also.

If X and Y have joint pdf and g is a function such that $Z = g(X, Y)$ has pdf. Then, how to compute the PDF of Z? first determine the CDF of Z using joint density of (X, Y) and then differentiate it.

Example 2.4 *Let X and Y be random variables having joint density f. Find the density of $X + Y$.*

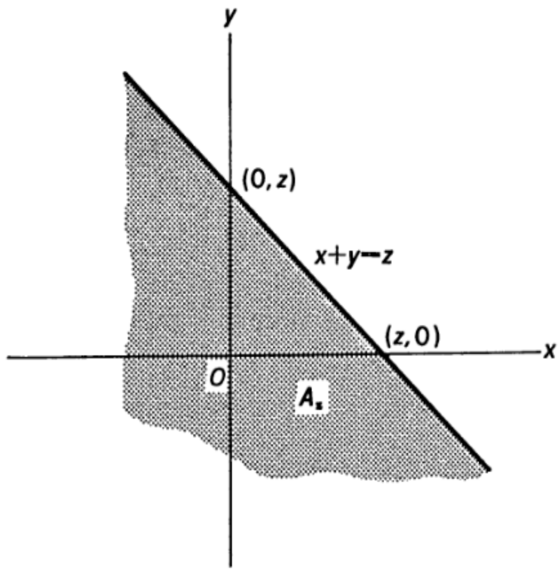
Solution: Define $Z := X + Y$. For fixed $z \in \mathbb{R}$ the event $\{Z \leq z\}$ is equivalent to the event $\{(X, Y) \in A_z\}$, where A_z is the subset of $\mathbb{R}^2$ defined by $A_z = \{(x, y) \in \mathbb{R}^2 | x + y \leq z\}$. Thus

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P((X, Y) \in A_z) \\ &= \iint\limits_{A_z} f(x, y)dxdy \end{aligned}$$

If we can find a nonnegative function g such that

$$\iint\limits_{A_z} f(x, y)dxdy = \int_{-\infty}^{z} g(t)dt, \quad \forall z \in \mathbb{R}$$

then g is necessarily a density of Z. Note that A_z is just the half-plane to the lower left of the line $x + y = z$ as shown in following figure.



Thus

$$F_Z(z) = \iint_{A_z} f(x,y)dxdy = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z-x} f(x,y)dy\right)dx$$

Make the change of variable $y = s - x$ in the inner integral. Then

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z} f(x,s-x)ds\right)dx \\ &= \int_{-\infty}^{z}\left(\int_{-\infty}^{\infty} f(x,s-x)dx\right)ds \end{aligned}$$

where we have interchanged the order of integration in last step. Thus the density of $X+Y$ is

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x,z-x)dx$$

■