POU

Books → The early universe by E W Kolb & M S Turner
Gravitation & Geometry by S Weinberg.

Topics:

→ General introduction
→ Dynamics of space time: General theory of relativity.
→ Standard geometry: space time for cosmology
[described by the robertson walker metric]
→ Time evolution for cosmology → friedman eq^n
→ Expansion age of the universe
→ Dark energy

Homogenous Isotropic

space time → relativity → evolution eq^{n}s
[Friedman eq^n]

3 - Dimensions

$$dl^2 = dx^2 + dy^2 + dz^2$$
$$= [dx \;\; dy \;\; dz] \begin{bmatrix} \eta_{xx} & \eta_{xy} & \eta_{xz} \\ \eta_{yx} & \eta_{yy} & \eta_{yz} \\ \eta_{zx} & \eta_{zy} & \eta_{zz} \end{bmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$$i = 1, 2, 3 \longleftrightarrow x, y, z$$

$$dl^2 \sum_{i,j=1}^{3} dx_i, dx_j \cdot \eta_{x_{ij}}$$

η_{ij} : metric of space.

coordinates $\xrightarrow{\quad}$ distance

4-D

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

metric of space time of
$g_{\mu\nu}$ $\mu, \nu = 0, 1, 2, 3$

$$0 = ds^2 = c^2 dt^2 - dl^2$$

$$c^2 dt^2 = dl^2$$

Age : $\int_{\text{Big bang}}^{\text{Today}} dt$ → Cant be calc.

Contents of universe.
- Light
- Luminous matter
- Dark matter
- ?? (dark energy)

$\omega < 0$

(i) eqn of state is :-

Pressure $P = \omega \rho$ ↔ energy density.

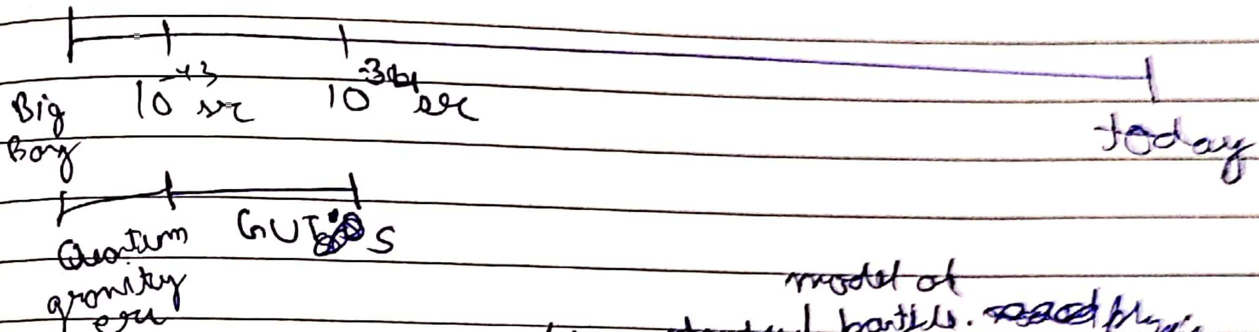
→ Thermodynamics
→ History of universe.
→ structure formation in the universe.

standard model of particle physics.

Interaction
- ~~G~~ Gravity
- EM (magnetism)
- Weak nuclear
- string " —— "

string theory / n theory / theory of everything

grand unified theory (GUTs)



model of
2 comes from standard particle. ~~model~~ physics.

$$\rho = K g_* T^4$$

$\propto P(t)$

multiple degree of freedom g_*

$$P = \rho_{\epsilon} + (g_*, T)$$

★ Structure Formation of Univers.

Homogeneous Isotropic smooth ⟶ Lumpy universe Galaxies, stars.

Universe

Cosmic microwave. Background radiation.

Key observational facts

$\rho \to$ Energy density

$$\Omega = \frac{\rho}{3H^2/8\pi G} = \frac{\rho}{\rho_c}$$

Where $\rho_c = \dfrac{3H^2}{8\pi G}$

ρ_c → critical density.

H = hubble constant

G = Newton's gravitational const.

$\Omega > 1$ closed universe

$\Omega = 1$ Flat universe $R \to \infty$

$\Omega < 1$ Open universe. open

ρ : Luminous → ρ_{Lum} → $\Omega_{Lum} = \dfrac{\rho_{Lum}}{\rho_c}$

ρ : Dark Matter → ρ_{DM} → $\Omega_{DM} = \dfrac{\rho_{DM}}{\rho_c}$

$\Omega \sim 0.01$

$\Omega \sim 0.2 - 0.3$

$\Omega_{DE} \sim 0.7 - 0.8$

On thing else ? dark energy

Hubble space telescope → H

Age of the universe → t_0

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in matter universe $t_0 = \frac{2}{3} H^{-1}$

$\int_0^{t_0} dt$

Dark energy / vacuum energy →

Relationship b/w P & ρ is different
eqn of state = $P = \omega \rho$

Matter $\omega \geq 0$
vacuum energy / dark energy $\omega < 0$

Faster expansion → H↑

Introduction.

- General theory of relativity
  - einstein eqn
  - Dynamics of space time.

linked to contents →
stress energy
Tensor
↓
Matter ← Energy density, pressure
Position

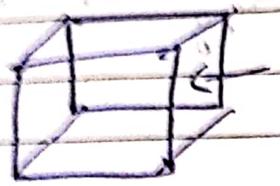
→ Tensor

vector transform in the same way from one reference frame to another.

Tensor transform like the product of vectors

$$T_{ij} = v_i v_j$$

Stress tensor.



$$\underbrace{\sigma_{ij}}_{3D} = \text{stress tensor} \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

in 4D with time

$$S_{\mu\nu} = \begin{pmatrix} S_{01} & S_{02} & S_{03} & S_{04} \\ S_{10} & & & \\ S_{20} & & \sigma_{ij} & \\ \underbrace{S_{30}}_{\text{energy density}} & & & \end{pmatrix}$$

For a perfect fluid

$$S_{\mu\nu} = \begin{pmatrix} \overset{\text{energy density}}{\rho} & & & 0 \\ & -p & & \\ & & -p & \\ 0 & & & p \end{pmatrix}$$

p = pressure.

C_6H_6

$\boxed{EC}$

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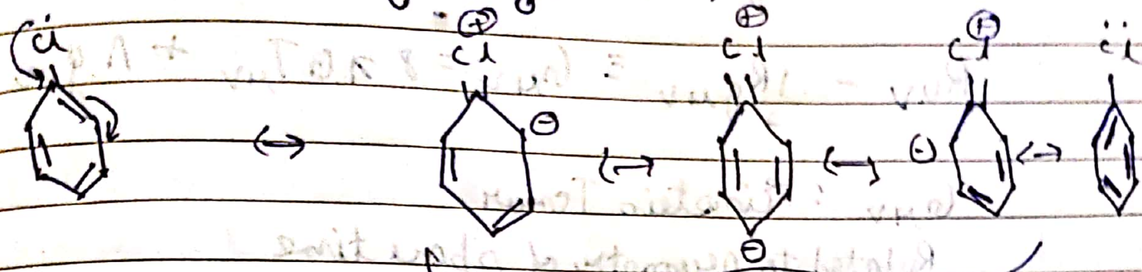
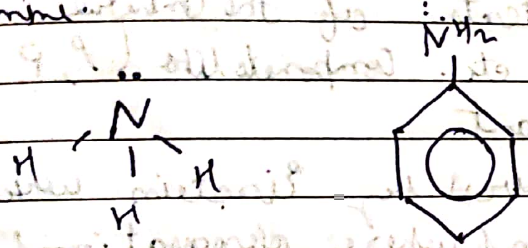
Application of resonance effect.

Low reactivity of aryl & vinyl halides.



double bond so stronger & stable

Compare the basic nature b/w ammonia & aromatic amine.



$NH_4^{\oplus}$

Aniline.

$O_2N-\overset{O}{\overset{\|}{C}}-OH$ $H-\overset{O}{\overset{\|}{C}}-OH$ $H_3C-\overset{O}{\overset{\|}{C}}-OH$

I $H^{\oplus}$ II   III

acidic   weakest

CF_3COOH CHF_2COOH CH_2COOH

Highly electro-ve so EWG

$I > II > III$

General theory of Relativity

- The central eq^n is the einstein eq^n

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} \equiv G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$$

- $G_{\mu\nu}$: Einstein Tensor
  Related to geometry of space time
  as $R_{\mu\nu} \rightarrow$ Curvature tensor
  $R \rightarrow$ Curvature Scalar

  $T_{\mu\nu}$: Stress energy Tensor
  Describes the contents of the universe
  - matter radiation etc. components like (ρ, P etc.)

something to do with vacuum energy ← Λ : cosmological constant.
Free parameter introduced by Einstein which actually has important implication & where observational impact has been observed in the last 20 years.

$g_{\mu\nu}$: Space-time metric which describes the basic properties of space-time

For 4D space-time $\mu\nu = \underbrace{0}_{time}, \underbrace{1, 2, 3}_{space}$

(i) $g_{\mu\nu}$ the metric of space time

Consider first 3D space

$$dl^2 = dx^2 + dy^2 + dz^2$$

$$= \begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ (dx\ \ dy\ \ dz) \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{c} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$dl^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \eta_{ij}\, dx^i dx^j$$

using the einstein summation convention (all repeated indices are summed over all possible values - in this case 1,2,3)

$$dl^2 = \boxed{\eta_{ij}}\, dx^i\, dx^j$$

space (Interval distance squared) metric coordinate differential

- Now for 4-D

In 4-D, the invariant space-time interval ds^2, The metric $g_{\mu\nu}$ and the coordinates are related by:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

Where a sum over repeated indices is implied. To illustrate, let us note that for flat (minkowski) space of special relativity, the metric is more explicitly given by [with $dx^0 = c\,dt$ $dx^1 = dx$ $dx^2 = dy$ $dx^3 = dz$]

$$ds^2 = c^2 (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

$$= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

For most relativity applications (including cosmology) It is standard to use units such that $c = 1$

The metric tensor $g_{\mu\nu}$ can be written in metric format. for case above

$$g_{\mu\nu} = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \mathrm{Diag}(1, -1, -1, -1)$$

It can be written as $$\begin{bmatrix} dx^0 & dx^1 & dx^2 & dx^3 \end{bmatrix} \begin{pmatrix} g_{\mu\nu} \end{pmatrix} \begin{bmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix}$$

For light $ds^2 = 0$

$$c^2 dt^2 = dx^2 + dy^2 + dz^2$$

In another frame

$$c^2 dt'^2 = dx'^2 + dy'^2 + dz'^2$$

→ speed of light same in all frame

$ds'^2 = 0 = ds^2$ independent of frame of reference.

Note that the quantity dx^μ has an index μ which ranges over 4 values (0, 1, 2, 3) and called a 4 vector

4-vectors ~~are~~ come in 2 varities - and denoted by having an index up or down - through they are simply related to each other through the metric tensor

$$dx_\mu = g_{\mu\nu}\, dx^\nu$$

Lower index covariant

(covariant) metric tensor

upper index contravariant

Tensor transform like the product of vector

$g_{\mu\nu}$ rank 2 tensor → has 2 indices & transform like product of 2 vectors.

$T_{\mu\nu k}$: " — " — 3 " — — "

$R_{\mu\nu k\lambda}$: " — " 4 " — " transform like product of 4 vectors.

The contravariant metric tensor $g^{\mu\nu}$ is related to the covariant metric tensor $g_{\mu\nu}$ by the relation

$$g^{\mu\nu} g_{\nu k} = \delta^\mu_k = \mathrm{Diag}(1,1,1,1)$$

Thus given a covariant metric tensor we can determine the contravariant metric tensor.

The contravariant metric tensor can be used to raised indices:

$\boxed{dx^\mu} = g^{\mu\nu} dx_\nu$

$\boxed{\text{Kronecker Delta}}$

→ $\delta^\mu_k = 0 \quad \mu \neq k$

$= 1 \quad \mu \neq k$

Metric Determine Geometry

In particular, it determines the curvature tensor which enters into the einstein's eq^n

Curvature Tensor : $R_{\alpha\beta}$ & $R^{\mu}_{\alpha\nu\beta}$ ↳ Riemann curvature tensor

↓

Ricci curvature tensor

$$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta} = \partial_\mu \Gamma^{\mu}_{\alpha\beta} - \partial_x \Gamma^{\mu}_{\mu\beta} + \Gamma^{\mu}_{\rho\mu}\Gamma^{\rho}_{\alpha\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\alpha\mu}$$

where the connection coefficient (also known as Levi-Civita connection) $\Gamma^{\mu}_{\alpha\beta}$ is:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma} \left(\partial_\alpha g_{\gamma\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta} \right)$$

& $\partial_x = \frac{1}{\partial x^\alpha}$ = α = 0, 1, 2, 3

1. Consider the following matrix:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu =$$

$$= c^2 dt^2 - \frac{a^2(t) dr^2}{1-kr^2} - a^2(t) r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

where t is the time coordinate an r, θ, φ are the spherical coordinates for space.

The indices be denoted choosing units s.t. c = 1 & letting by $\nu = 0$, i

$i = 1, 2, 3$ (r, θ, ϕ)

time component

space component varying from 1 to 3

use

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right)$$

to determine expressions for

1. Γ^{i}_{j00}
2) Γ^{0}_{ij}
3) $\Gamma^{i}_{j\lambda}$
4) ~~$\Gamma^{r}_{j\lambda}$~~ 4) Γ^{r}_{rr}
5) $\Gamma^{r}_{\theta\theta}$
6) $\Gamma^{r}_{\phi\phi}$

II Use the expression:

$$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta} =$$

$$\partial_\mu \Gamma^{\mu}_{\alpha\beta} - \partial_\alpha \Gamma^{\mu}_{\mu\beta} + \Gamma^{\mu}_{\rho\mu}\Gamma^{\rho}_{\alpha\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\alpha\mu}$$

To calculate the following components of the curvature tensor

1) R_{00}
2) R_{ij} (a) R_{rr} (b) $R_{\theta\theta}$ (c) $R_{\phi\phi}$

3) Compute the Ricci Scalar: $R = g^{\mu\nu} R_{\mu\nu} = R_{00} + g^{ij} R_{ij}$

Ex→ Calculate $\Gamma^{\theta}_{\theta r}$

Given $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right)$

$\therefore \Gamma^{\mu}_{0\mu} = \frac{1}{2} g^{0\nu}(\partial_0 g_{\nu r} + \partial_r g_{0r} - \partial_\nu g_{0r})$

Note: $g_{00} = 1 \quad g_{rr} = \frac{-a^2}{1-kr^2}$;

$g_{\theta\theta} = -a^2r^2, \; g_{\phi\phi} = -a^2r^2\sin^2\theta$

$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

$= g_{00}\, dx^0 dx^0 + g_{01}\, dx^0 dx^1 + g_{10}\, dx^1 dx^1 + g_{11}\, dx^1 dx^1$

comparing coefficient with

$$ds^2 = c^2dt^2 - a(t)\frac{dr^2}{1-kr^2} - a^2(t)\, r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

we get g_{00}, g_{rr}, $g_{\phi\phi}$ & $g_{\theta\theta}$ all other are $g_{\mu\nu} = 0$

Thus,

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2r^2 & 0 \\ 0 & 0 & 0 & -a^2r^2\sin^2\theta \end{pmatrix}$$

Also $g^{k\mu} g_{\mu\nu} = \delta^k_\nu$

where $\delta^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{Diag}(1,1,1,1)$

$$\therefore \quad g^{\kappa\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-kr^2}{-a^2} & 0 & 0 \\ 0 & 0 & \frac{-1}{a^2 r^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{-a^2 r^2 \sin^2\theta} \end{pmatrix}$$

$$\therefore \quad g^{\theta\theta} = \frac{-1}{a^2 r^2}$$

$$\& \quad g^{\theta\phi} = 0 = g^{\theta r} \cdot g^{\theta 0} = 0$$

$$\Gamma^{\theta}_{\theta r} = \frac{1}{2} g^{\theta\theta} \left(\partial_\theta g_{\theta r} + \partial_r g_{\theta\theta} - \partial_\theta g_{\theta r} \right)$$
$$+ \frac{1}{2} \cancel{g^{\theta\phi}}^{\to 0} \left(\quad \right)$$
$$+ \frac{1}{2} \cancel{g^{\theta r}}^{\to 0} \left(\quad \right)$$
$$+ \frac{1}{2} \cancel{g^{\theta 0}}^{\to 0} \left(\quad \right)$$

$$\therefore \quad \Gamma^{\theta}_{\theta r} = \frac{1}{2} \left(\frac{-1}{a^2 r^2} \right) \left[\partial_\theta 0 + \partial_r (-a^2 r^2) - \partial_\theta 0 \right]$$

$$= \frac{1}{2} \left(\frac{-1}{a^2 r^2} \right) \frac{\partial(-a^2 r^2)}{\partial r}$$

$$= \frac{1}{2} \left(\frac{1}{a^2 r^2} \right) (+a^2) \frac{\partial r^2}{\partial r} = \frac{1}{\cancel{2}} \frac{1}{\cancel{a^2} r^2} \cancel{a^2}(\cancel{2} r)$$

$$\boxed{\Gamma^{\theta}_{\theta r} = \frac{1}{r}}$$

Note $\left[\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (\partial_{\alpha} g_{\nu\beta} + \partial_{\beta} g_{\alpha\nu} - \partial_{\nu} g_{\alpha\beta})\right.$

$$\Gamma^{\theta}_{\theta r} = \frac{1}{2} g^{\theta\nu} (\partial_{\theta} g_{\nu r} + \partial_{r} g_{\theta\nu} - \partial_{\nu} g_{\theta r})$$

$\alpha \quad \beta$

ν is a repeated indise d so has to be summed over all the possible values 0, 1, 2, 3 or in our case over $0, r, \theta, \phi$

$$\Gamma^{\theta}_{\theta r} = \frac{1}{2} g^{\theta\theta} (\partial_{\theta} g_{\theta r} + \partial_{r} g_{\theta\theta} - \partial_{\theta} g_{\theta r}) + \text{---}$$

HW 1

1) $ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} =$

$$c^2 dt^2 - \frac{a^2(t) dr^2}{1-kr^2} - a^2(t) r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \quad \text{---} (1)$$

using $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (\partial_{\alpha} g_{\nu\beta} + \partial_{\beta} g_{\alpha\nu} - \partial_{\nu} g_{\alpha\beta})$ — (a)

so for $\Gamma^{i}_{j0} = \frac{1}{2} g^{i\nu} (\partial_{j} g_{\nu 0} + \partial_{0} g_{j\nu} - \partial_{\nu} g_{j0})$ — (2)

$\alpha \quad \beta$

$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$

$\mu, \nu = 0, i, j, k$

$= g_{00} dx^0 dx^0 + g_{01} dx^0 dx^1 + g_{10} dx^1 dx^0 + g_{11} dx^1 dx^1$

comparing coefficients with ①

$g_{00} = 1$ (time component)

$g_{01} = \dfrac{-a^2}{1-kx^2}$ (x component)

$g_{10} = -a^2 x^2$ (θ comp.)

$g_{11} = -a^2 x^2 \sin^2\theta$ (ϕ comp.)

$$
\begin{array}{cc}
 & \begin{array}{cccc} 00 & 0i & 0j & 0k \end{array} \\
0,i,j,k & \\
g_{\mu\nu} = & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \left(\dfrac{-a^2}{1-kx^2}\right) & 0 & 0 \\ 0 & 0 & -a^2x^2 & 0 \\ 0 & 0 & 0 & -a^2x^2\sin^2\theta \end{bmatrix}
\end{array}
$$

$$g^{\kappa\mu} g_{\mu\nu} = \delta^{\kappa}_{\nu} = \text{Diag}$$

$$g^{\kappa\mu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1-kx^2}{-a^2} & 0 & 0 \\ 0 & 0 & \dfrac{-1}{a^2x^2} & 0 \\ 0 & 0 & 0 & \dfrac{-1}{a^2x^2\sin^2\theta} \end{bmatrix}$$

① from eq ②

$\nu = 0, i, j, k$

$$\Gamma^{i}_{j\theta} = \cancel{\frac{1}{2} g^{i0}}\!\rightarrow 0 \left(\qquad \right)$$

$$+\frac{1}{2} g^{ii} \left(\partial_\alpha \cancel{g_{i0}}^{\,\rightarrow 0} + \partial_\beta \cancel{g_{ji}}^{\,\rightarrow 0} + \partial_\nu \cancel{g_{j0}}^{\,\rightarrow 0} \right)$$

$$+\cancel{\frac{1}{2} g^{ij}}\!\rightarrow 0 \left(\qquad \right)$$

$$+\cancel{\frac{1}{2} g^{ik}}\!\rightarrow 0 \left(\qquad \right)$$

$$\frac{1}{2}\left(\frac{1-kx^2}{-a^2}\right)\left(\partial_t \right) = 0$$

Answers for $\Gamma^{\mu}_{\alpha\beta}$

Recall that $\mu, \alpha, \beta = 0, i$
$i, j \rightarrow r, \theta, \phi$

The non vanishing components are

$$\Gamma^{i}_{j0} = \frac{1}{c}\frac{\dot{a}}{a}\delta^{i}_{j} \quad \text{where } \dot{a} = \frac{da}{dt}$$

$$\Gamma^{0}_{ij} = -\frac{1}{c}\frac{\dot{a}}{a} g_{ij}$$

~~recall~~ $\Big[g_{00} = 1 \quad g_{rr} = \frac{-a^2}{1-kr^2} \quad g_{\theta\theta} = -a^2 r^2$

$g_{\phi\phi} = -a^2 r^2 \sin^2\theta \Big]$

The non vanishing components are

$$\Gamma^{r}_{rr} = \frac{kr}{1-kr^2}$$

$$\Gamma^{r}_{\theta\theta} = -r(1-kr^2)$$

$$\Gamma^{r}_{\phi\phi} = -r\sin^2\theta(1-kr^2)$$

$$\Gamma^{\theta}_{\theta r} = \frac{1}{r}$$

$$\Gamma^{\phi}_{\phi r} = \frac{1}{r} \qquad \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}_{\phi\theta} = \cot\theta.$$

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Greek letters $\alpha, \beta, \gamma = 0, 1, 2, 3$ $\quad i, j, k, l = 1, 2, 3$

⊙ Solved example for ricci curvature tensor component
~~Component~~

Use
$$R_{\alpha\beta} = R^{\mu}_{\ \alpha\mu\beta} = \partial_\mu \Gamma^{\mu}_{\alpha\beta} - \partial_\alpha \Gamma^{\mu}_{\mu\beta} + \Gamma^{\mu}_{\rho\mu}\Gamma^{\rho}_{\alpha\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\alpha\mu}$$

Complete R_{00}:
$$R_{00} = \partial_\mu \Gamma^{\mu}_{00} - \partial_0 \Gamma^{\mu}_{\mu 0} + \Gamma^{\mu}_{\rho\mu}\Gamma^{\rho}_{00} - \Gamma^{\mu}_{\rho 0}\Gamma^{\rho}_{0\mu}$$

using ~~techni~~ technique directly already for computing $\Gamma^{\alpha}_{\beta\gamma}$ can show that $\Gamma^{\mu}_{00} = 0$

$$R_{00} = -\partial_0 \Gamma^{\mu}_{\mu 0} - \Gamma^{\mu}_{\rho 0}\Gamma^{\rho}_{0\mu}$$

Further the only ~~non~~ non vanishing components of the form $\Gamma^{\mu}_{\mu 0}$ & $\Gamma^{\mu}_{\rho 0}$ etc. are

$$\Gamma^{i}_{j0} = \frac{1}{c}\frac{\dot{a}}{a}\delta^{i}_{j} \qquad i, j = 1, 2, 3$$

$$\therefore R_{00} = -\partial\left(\frac{1}{c}\frac{\dot{a}}{a}\delta^{i}_{i}\right) - \Gamma^{i}_{j0}\Gamma^{j}_{0i}$$

$$= -\partial_0\left(\frac{1}{c}\frac{\dot{a}}{a}\delta^{i}_{j}\right) - \left(\frac{1}{c}\frac{\dot{a}}{a}\right)^2 \delta^{i}_{j}\delta^{j}_{i}$$

Question what's a) $\delta^{i}_{j}\delta^{j}_{i}$?

b) δ^{i}_{i}.

$$\delta^i_j = 1 \quad \text{if } i = j$$
$$= 0 \quad \text{if } i \neq j$$

$$(I)^i_j \cdot (I)^j_i = (I)^i_i = \delta^i_i$$

$$\delta^i_j \cdot \delta^j_i = \delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 = 3$$

$$\delta^\mu_\mu = \delta^1_1 + \delta^2_2 + \delta^3_3 + \delta^4_4 = 4$$

4 possible vals.

$$R_{00} = -3 \frac{1}{c} \frac{\partial}{\partial t}\left(\frac{\dot{a}}{ca}\right) - 3\left(\frac{1}{c^2} \frac{\dot{a}^2}{a^2}\right)$$

$$= -\frac{3}{c} \left[\frac{c\, a\ddot{a} - \dot{a}c\dot{a}}{c^2 a^2}\right] - 3 \frac{1}{c^2} \frac{\dot{a}^2}{a^2}$$

$$= -\frac{3}{c^2} \frac{\ddot{a}}{a}$$

$$\boxed{R_{00} = -\frac{3}{c} \frac{\ddot{a}}{a}} \qquad \ddot{a} = \frac{d^2 a}{dt^2}$$

Going back of einstien eq^n

The geometry of the universe is captured in terms of the einstein curvature tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Ricci curvature tensor

metric tensor

curvature scalar $R = g^{\mu\nu} R_{\mu\nu}$

$G_{\mu\nu}$ is related to contents of the universe as described by the stress energy tensor $T_{\mu\nu}$ by the einstein eqn:

$$\left[G_{\mu\nu} - \underbrace{g_{\mu\nu}\frac{\Lambda}{c^2}}_{\text{Cosmological constant}}\right] = \underbrace{\frac{8\pi G_N}{c^4}}_{\text{constants of nature}} T_{\mu\nu}$$

$T_{\mu\nu}$ = Stress energy tensor

For an ideal fluid $T_{\mu\nu}$ is

$T_{\mu\nu}$ = stress energy tensor

For An ideal fluid $T_{\mu\nu}$ is

$T_{00} = \rho$ $\quad T_{ij} = -P g_{ij}$ all other components $= 0$

energy density

Pressure

Stress ~~energy~~ energy tensor $T_{\mu\nu}$ related to simple measurable things like $\underbrace{\text{energy density on pressure}}_{\text{studied in thermodynamics.}}$

The metric describing the spacetime for our universe is called the ~~expansion~~ Roberison Walker metric and is given by:

$$ds^2 = dt^2 - R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right\}$$

$$= g_{\mu\nu} dx^{\mu} dx^{\nu}$$

from which we can extract the various components of $g_{\mu\nu}$. [Note: We have set $c = 1$ in Array metric]

The space time described by this robertson walker metric can be interpreted as a 3-D sphere embeded by a 4 D space. This can be arrived at mathematically

Consider first the 2-D curved space in a 3 D euclidian space with coordinates X_1, X_2, X_3

The eqn of the 2 sphere of radius R is

$$x_1^2 + x_2^2 + x_3^2 = R^2$$

The element of length in 3D space

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2$$

Since we are only interested in the 2-D surface described by $x_1^2 + x_2^2 + x_3^2 = R^2$ (The surface of the sphere of radius R) we actually only need 2 independent coordinates. If we choose x_1 & x_2 As the 2 independent coordinates then x_3 can be taken as the "fictious" third spatial coordinate and eliminated from dl^2 by using ~~Eqn (2.02)~~ eqn (2.2) in the form below

$$2x_1 dx_1 + 2x_2 dx_2 + 2x_3 d_3 = 0$$

$$dx_3 = -\frac{(2x_1 dx_1 + 2x_2 dx_2)}{}$$

2-sphere 2 independent coordinates x_1, x_2

$$\boxed{dx_3 = \frac{-(x_1 dx_1 + x_2 dx_2)}{\sqrt{R^2 - x_1^2 - x_2^2}}} \quad (2.4)$$

substituting (2.4) in (2.3) and we get

$$dl^2 = dx_1^2 + dx_2^2 + \frac{(x_1 dx_1 + x_2 dx_2)^2}{R^2 - x_1^2 - x_2^2} \quad (2.5)$$

~~Standard Cosmology~~

↓

Space-time metric for homogeneous isotropic universe

↓

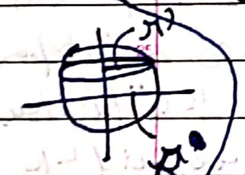
Geometry 3D sphere embeded in 4-D space

looking first at 2-D sphere in 3D space.

eq^n for 2-sphere in 3D ($\to S^2$)

$$dl^2 = dx_1^2 + dx_2^2 + \frac{(x_1 dx_1 + x_2 dx_2)^2}{R^2 - x_1^2 - x_2^2}$$

Now introduce the coordinates r' & θ in terms of x_1 & x_2 as



$$x_1 = r' \cos\theta, \quad x_2 = r' \sin\theta \quad (2.6)$$

$0 \le \frac{r'}{R} = r \le 1$

Physically, r' & θ are polar coordinates in the x_3 plane ($x_3^2 = R^2 - r'^2$). In terms of the new coordinates r' & θ become

[using $dx_1 = -r' \sin\theta \, d\theta + dr' \cos\theta \quad (2.7)$

$dx_2 = r' \cos\theta \, d\theta + \sin\theta \, dr' \quad (2.8)$

$$dl^2 = \cos^2\theta\, dr'^2 + r'^2 \sin^2\theta\, d\theta^2 - \cancel{2r' \sin\theta \cos\theta\, dr' d\theta} + r'^2 \cos^2\theta\, d\theta + \sin^2\theta\, dr'^2 + \cancel{2r' \cos\theta \sin\theta\, dr' d\theta} + \frac{\left(r' \cos^2\theta\, dr' - \cancel{r' \cos\theta \sin\theta\, d\theta} + \cancel{c' \sin\theta \cos\theta\, d\theta} + r' \sin^2\theta\, dr'\right)^2}{R^2 - r'^2}$$

using trigonometric identities & simplifying we get:

$$dl^2 = dr'^2 + r'^2 d\theta^2 + \frac{r'^2 dr'^2}{R^2 - r'^2}$$

$$= dr'^2 \left[1 + \frac{r'^2}{R^2 - r'^2}\right] + r'^2 d\theta^2$$

$$\boxed{dl^2 = \frac{R^2 dr'^2}{R^2 - r'^2} + r'^2 d\theta^2} \qquad 2.9$$

we can further define the coordinate

$$r \equiv \frac{r'}{R} \quad \text{such that} \quad 0 \leq r < 1$$

then $r' = Rr$, where recall that R is a constant

so $dr' = R\, dr$

Thus

$$dl^2 = \frac{R^2 dr^2}{1 - r^2} + R^2 r^2 d\theta^2$$

$$dl^2 = R^2 \left\{ \frac{dr^2}{1 - r^2} + r^2 d\theta^2 \right\} \qquad 2.10$$

Note similarity of metric to the robertson walker metric. The similarity will become exact when we go to 3-D sphere embeded in 4D space.

Now consider the 3-D space embeded in 4D space given by

$$R^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \quad (2.11)$$

[where we have introduce the 'fictitious' 4-D coordinate x_4]

consider now:

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

then the fictitious coordinate (x_4) can be removed to get:

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 d_2 + x_3 dx_3)}{R^2 - x_1^2 - x_2^2 - x_3^2} \quad (2.12)$$

In terms of the polar coordinates

$$x_1 = r' \sin\theta \cos\phi \quad (2.13)$$

$$x_2 = r' \sin\theta \sin\phi \quad (2.14)$$

$$x_3 = r' \cos\theta$$

(and $r = r'/R$) the metric is then given by

$$ds^2 = dt^2 - R^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right. \quad 2.16$$

with $k = +1$

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Thus we see that the robertson walker metric for $k = +1$ can be interpreted as that for a (closed) sphere with radius R ~~...~~ the $k = -1$ case can be reconciled by making the $R \to iR$ (This is called hyper sphere & has an imaginary radius) It is corresponds to open universe. The limiting case is $k = 0$ which is the boundary limit b/w the open & close universe. It is called the flat universe.

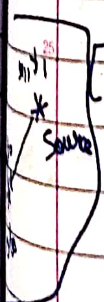
Kinematics of the robertson walker (RW) metric.

→ Red shifting of light

Consider a light emitted from a distant source at coordinate $r = r_1$ at time t_1 which arrives at a detector at time t_0 at coordinate $r=0$ light travel along the shortest path (called a "geodesic") which is given by $ds^2 = 0$ with $d\theta = 0 = d\phi$

Thus, the coordinate distance & time will be related by:

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_{0}^{r_1} \frac{dr}{(1-kr^2)^{1/2}} = f(r_1) \qquad (2.17)$$



[recall that RW metric in general is given by)

$$ds^2 = dt^2 - R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right\}$$

setting $ds^2 = 0$ & $d\theta = 0 = d\phi$ gives (2.17)]

The wave crest emitted at a time $t_1 + \delta t_1$ will

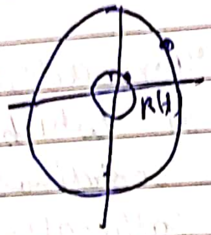
δt is time b/w ...

$\delta t \approx 10^{-14}$

arrive at the detector. ~~[illegible]~~ at the time $t_0 + \delta t_0$. The equation of motion will be the same as (2.17), with $t_1 \to t_1 + \delta t_1$ & $t_0 \to t_0 + \delta t_0$ since $r(t_1)$ is constant [The source is fixed in the comoving coordinate system] ⇒

$$\boxed{\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{R(t)}} \qquad (2.18)$$

⊙ Co-moving coordinate system.



r, θ, ϕ

same or fixed

$R(t)$ changes with time.

↓

physical coordinate system.

Let $\displaystyle\int \frac{dt}{R(t)} = I$.

Then (2.18) ⇒

$$I(t_0) - I(t_1) = I(t_0 + \delta t_0) - I(t_1 + \delta t_1)$$

$$I(t_1 + \delta t_1) - I(t_1) = I(t_0 + \delta t_0) - I(t_0)$$

$$\int_{t_1}^{t_1+\delta t_1} \frac{dt}{R(t)} = \int_{t_0}^{t_0+\delta t_0} \frac{dt}{R(t)}$$

$\lambda = c\,\delta t$

For δt sufficiently small corresponding to wavelength

$$\lambda = \left(\overset{\nearrow 1}{c}\right) \cdot \delta t \ll c\,t,\ R(t)$$

can be taken to be a constant over the integration time & we get $\delta t \approx 10^{-14}$ sec 10^{14} Hz is freq of light wave.

$$\boxed{\frac{\delta t_1}{R(t_1)} = \frac{\delta t_0}{R(t_0)}} \quad (2.19)$$

For $\delta t_1 \rightarrow$ time b/w succesive crests of emitted light
($\delta t_1 \rightarrow$ Wavelength of the emitted light $= \lambda_1$
& $\delta t_0 \rightarrow$ time b/w succesive wave crest of detector light.

& $c\,(\delta t_0) \rightarrow$ wavelength of the detected light $= \lambda_0$

Thus, $$\frac{\lambda_1}{\lambda_0} = \frac{R(t_1)}{R(t_0)} \quad \frac{\overset{\nearrow 1}{c}(\delta t_1)}{c(\delta t_0)}$$

Astronomers define the "red shift" of an object z, in terms of the ratio of the detected wavelength to the emitted wavelength

$$\boxed{1 + z = \frac{\lambda_0}{\lambda_1} = \frac{R(t_0)}{R(t_1)}} \quad 2.21$$

Any increase in $R(t)$ leads to a red shift (increase in wavelength) of light from distant source. to be
Today astronomers observe ~~on~~ distant galaxies the the universe is ~~expa~~ red shifted & we conclude expanding.

Dynamics time evolution

We start with einstein eq^n put in the metric for the homogenous isotropic universe → Roberton walker metric.
→ Arrive at the eq^n describing time evolution of universe → Fridmann eq^n.

Recall that the einstein eq^n is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\nu}$$

where $G_{\mu\nu}$: Einstein tensor
$R_{\mu\nu}$: curvature tensor
R : ricci scalar
$T_{\mu\nu}$: ~~stress~~ stress energy tensor
Λ : cosmological const.
$g_{\mu\nu}$: space-time metric

The space time metric is given by

$$ds^2 = dt^2 - R^2(t) \left\{ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \right\}$$

$$= g_{\mu\nu} dx^\mu dx^\nu$$

This is the RW metric already discussed also note this metric is very similar to metric used in HW 1 & 2. In fact if we set $c=1$ & use mapping $a(t) \to R(t)$ you get metric above.

Thus the non vanishing elements of the metric tensor are
$g_{00} = 1$, $g_{rr} = \frac{-R^2}{1-kr^2}$, $g_{\theta\theta} = -R^2 r^2$, $g_{\phi\phi} = -R^2 r^2 \sin^2\theta$

now using

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\beta} - \partial_\nu g_{\alpha\beta})$$

And, $R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta} = \partial_\mu \Gamma^{\mu}_{\alpha\beta} - \partial_\alpha \Gamma^{\mu}_{\mu\beta} + \Gamma^{\mu}_{\rho\mu}\Gamma^{\rho}_{\alpha\beta}$
$- \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\alpha\mu}$

$$R = g^{\mu\nu} R_{\mu\nu}$$

Can determine all quantities on LHS of einstein eq^n.
Further using $T_{00} = \rho$ & $T_{ij} = -P g_{ij}$ for $T_{\mu\nu}$
can determine RHS of the einstein eq^n.

Thus we get

$$R_{00} = -3\frac{\ddot{R}}{R}$$

$$R_{ij} = -\left[\frac{\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2} + \frac{2k}{R^2}\right] g_{ij}$$

(where i, j go over the spatial coordinates & the 0 component is the time component)

Further, $\dot{R} = \frac{dR}{dt}$

$\ddot{R} = \frac{d^2R}{dt^2}$

The ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ is:

$$R = -6\left[\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right]$$

scale factor

Recall that the einstein eq^n is:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$$

The 0-0 component of the einstein eq^n gives:

$$-3\frac{\ddot{R}}{R} + \frac{1}{2} 6\left[\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right] = 8\pi G \rho + \Lambda$$

$$\boxed{3\left[\frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right] = 8\pi G\rho + \Lambda} \quad (2.22)$$

The i-i component of the einstein eq^n gives

$$-\left[\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} + 2\frac{k}{R^2}\right] + \frac{1}{2}\cdot 6\left[\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right]$$

$$= 8\pi G(-\rho) + \Lambda$$

$$\Rightarrow \boxed{\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = 8\pi G(-\rho) + \Lambda} \quad (2.23)$$

where $\ddot{R} = \frac{d^2 R}{dt^2}$ & $\dot{R} = \frac{dR}{dt}$

photon $\to e^{-}$

can simplify the eq^n by introducing

$$\rho_{vacuum} = \frac{\Lambda}{8\pi G} \quad (2.24a)$$

& $$P_{vacuum} = \frac{-\Lambda}{8\pi G} \quad (2.24b)$$

Then the earlier eq^n (2.22) & (2.23) take the form

$$\boxed{3\left[\frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right] = 8\pi G\left[\rho + \rho_{vacuum}\right]} \quad (2.25)$$

& $$\boxed{\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi G\left(P + P_{vacuum}\right)} \quad (2.26)$$

Further defining

$$\rho_{total} = \rho + \rho_{vacuum}$$

$$P_{total} = P + P_{vacuum}$$

we get more compact eq^n.

Thus:

$$\boxed{3\left[\frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right] = 8\pi G \rho_{total}} \quad 2.27$$

& $$\boxed{\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi G P_{total}} \quad 2.28$$

The eqn (2.27) & (2.28) which follow from the einstein eqn can be used to describe the time evolution of the universe.

Consider now how the energy changes in a comoving volume as given by the expression:

$$\frac{d(\rho R^3)}{dt} = \frac{3}{8\pi G}\frac{d}{dt}\left[\dot{R}^2 R + kR\right] \quad (2.29)$$

where we have used eqn (2.2.7) $\times \frac{R^3}{8\pi G}$ & applied the time derivative to both sides.

From eqn (2.29) we can expand to get

$$\frac{d(\rho R^3)}{dt} = \frac{3}{8\pi G}\left[2\dot{R}\ddot{R}R + \dot{R}^2\dot{R} + k\dot{R}\right]$$

$$= \frac{3\dot{R}}{8\pi G}\left[2\ddot{R}R + \dot{R}^2 + k\right]$$

Thus

$$\frac{d(\rho R^3)}{dt} = \frac{3\dot{R}R^2}{8\pi G}\left[\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right]$$

$$= \frac{3\dot{R}R^2}{8\pi G}\left[-8\pi G P\right]$$ from eq (2.28)

$$= [-P]\left\{\frac{dR^3}{dt} \to 3R^2\frac{dR}{dt}\right\}$$

change in energy inside a comoving body

change in volume

$$\boxed{\frac{d}{dt}\left(\rho R^3\right) = -P\,\frac{d(R^3)}{dt}} \qquad (2.30)$$

The physical content of the above eq^n is the first law of thermodynamics in the expanding universe.

Let us now use the eq^n of state which relates pressure P & energy density ρ as:

$$\boxed{P = \omega\rho} \qquad (2.31)$$

Then using eq^n (2.30) & (2.31) we get:

$$d(\rho R^3) = -P\,d(R^3)$$

$$d(\rho R^3) = -\omega\rho\, dR^3$$

$$R^3\, d\rho + \rho\, 3R^2 dR + \omega\rho\, 3R^2 dR = 0$$

$$R^3 \frac{d\rho}{dR} + 3R^2\rho + 3\omega R^2\rho = 0$$

Thus,

$$\boxed{\frac{d\rho}{dR} = -3(1+\omega)\frac{\rho}{R}} \qquad (2.32)$$

From (2.32) it follows that

$$\boxed{\rho = C R^{-3(1+\omega)}} \qquad (2.33)$$

where C is a constant

Value of w is different for diff quantities
for vacuum energy $w = -1$

This can be verified by differentiating or by integrating

$$\int \frac{d\rho}{\rho} = -\int 3(1+w)\frac{dR}{R}$$

Thus in general

$$\rho = R^{-3(1+w)}$$

where w enters in the eq^n of state $p = w\rho$

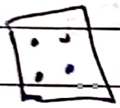
For radiation

$$p = \frac{1}{3}\rho \Rightarrow w = \frac{1}{3}$$

thus $\rho \propto R^{-3\left(1+\frac{1}{3}\right)}$

$\boxed{\rho \propto R^{-4}}$ For radiation.

Note that the number density of photons $\sim \frac{1}{R^3}$


4 no. in volume.

universe is expanding as $(R(t))^3$ and photons are constant number,
so density $= \frac{1}{R^3}$ for photons

Energy density = Number density $\times$ energy of each particle.

$$= \frac{1}{R^3} \times \frac{1}{R}$$

energy of photon $= \frac{hc}{\lambda}$

$$= \frac{1}{R^4}$$

$\lambda \uparrow$ with expanding universe.

$P = f(\rho)$ EQN of state $\rightarrow$ Friedmann Eq^{n}s
First law of thermodynamics

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(i) Next, comment (Non-Relativistic matter) $v \ll c$

For Matter

$$P = \rho_m R T = \rho_m V^2_{thermal}$$

ρ_m → Massdensity, where $V_{Thermal} = \sqrt{RT}$ → thermal speed of molecules.

And energy density ρ is:

$$\rho = \rho_m c^2$$

c → speed of light ($E = mc^2$, $\frac{E}{V} = \frac{m}{V}c^2$)

$$w = \frac{P}{\rho} = \frac{\rho_m v^2}{\rho_m c^2} = \left(\frac{v}{c}\right)^2 \approx 0$$

Thus for matter we have $w = 0$

$$\boxed{\rho \propto R^{-3}}$$

For vacuum energy recall eq^n (2.24)

$$\rho_{vacuum} = \frac{\Lambda}{8\pi G}$$

& $$P_{vacuum} = \frac{-\Lambda}{8\pi G}$$

$$\therefore P_{vacuum} = -\rho_{vacuum}$$

Thus

$$\rho \propto R^0 \quad \text{energy const}$$

... is a constant

(a) Hubble constant changes in million years not 100s of years.

(0-0) $ij - \dot{j} - p - k$

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	Radiation	Matter	Vacuum
summarizing	$\rho \propto R^{-4}$	$\rho \propto R^{-3}$	$\rho \propto R^{0}$

Implication of Friedman eq^n

Open / Closed / Flat universe condition.

Recall the 0-0 componet of Einstein Eq^n gave us

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \quad (2.34)$$

The quantity $\frac{\dot{R}}{R} = H$ is called the hubble parameter. It is a measure of how fast the universe is expanding.

H is a function of time on the cosmological scale. The hubble time H^{-1} sets the time scale for expansion.

The hubble constant H_0 is the present observed value of the hubble parameter.

Eq^n (2.34) can be re-written as

$$H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho$$

$$1 + \frac{k}{R^2 H^2} = \frac{8\pi G\rho}{3H^2}$$

$$\boxed{\frac{k}{R^2H^2} = \frac{\rho}{3H^2/8\pi G} - 1} \quad (2.35)$$

H
$G \Rightarrow$ things collapse fast — some are later open / close — universe.

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Defining $\rho_c = \dfrac{3H^2}{8\pi G}$ → critical density

& $\Omega = \dfrac{\rho}{\rho_c}$

Eqn (2.35) can be written completely as:

$$\boxed{\frac{k}{R^2H^2} = \Omega - 1} \quad (2.36)$$

Recall from our discussion of the Robertson walker metric that depending on the 3 values of k the universe was either closed, open or Flat.

$$\begin{bmatrix} k = +1 & \Rightarrow \text{closed} & \Leftrightarrow & \Omega > 1 \\ k = 0 & \Rightarrow \text{Flat} & \Leftrightarrow & \Omega = 1 \\ k = -1 & \Rightarrow \text{open} & \Leftrightarrow & \Omega < 1 \end{bmatrix}$$

Thus the geometry of the universe (k) is tied to the contents of the universe (ρ/Ω) & the critical ~~dividin~~ dividing line is $\rho_c = \dfrac{3H^2}{8\pi G}$ This follo- -ws from Einstein eqn / Fridmann eqn

(i) Now computing energy density in radiation & matter

The observed ~~xxx~~ value of the Hubble constant H_0 is in the range of 70 – 80 $km\,s^{-1}\,Mpc^{-1}$ & it is usually expressed as $H_0 = 100\,h\ km\,s^{-1}\,Mpc^{-1}$ with an ~~xx~~ dimensionless parameter ~~x~~ h of order unity which is in the range of 0.7 to 0.8.

the ~~xxx~~ uncertainity captured in the

$H \equiv \frac{\dot{R}}{R} = \frac{dR}{dt}$ $\boxed{\text{km } T^{-1}}$ $\rho h^2 = \frac{3H^2}{8\pi G}$ $\rho \times h^2$ km s^{-1} Mpc^{-1} parsec distance only.

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The matter density measured in the universe today is

$$\boxed{\Omega_m h^2 = \Omega_{DM} h^2 + \Omega_{luminous} h^2 \simeq 0.1} \quad 2.37$$

dark matter

[corresponding to $\Omega_m = 0.25$]

The radiation density (in the cosmic microwave background photons):

$$\boxed{\Omega_\gamma h^2 \approx 4.3 \times 10^{-5}} \quad 2.38$$

corresponding to a number density of photons

$$n_\gamma = 410 \text{ cm}^{-3}.$$

Today there is more matter than (CMBR) radiation but it was not so in the past.

Recall from our earlier discussion,

$$\rho_{radiation} \propto R^{-4} \quad \& \quad \rho_{matter} \propto R^{-3}$$

Thus $$\frac{\rho_{radiation}}{\rho_{matter}} \propto \frac{R^{-4}}{R^{-3}} = \frac{1}{R}$$

Thus, at earlier times when universe was smaller (i.e. R was smaller) then the relative value of $\rho_{radiation}$ was higher. In fact,

$$\frac{(\rho_{radiation} / \rho_{matter})_{earlier}}{(\rho_{rad} / \rho_{mat})_{today}} = \frac{R_{today}}{R}$$

$$= \frac{R_{today}}{R_{earlier}} = (1+z)$$

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where z is the redshift & we had earlier derived the relationship $\frac{R(t_0)}{R(t_1)} = (1+z) = \frac{\lambda_0}{\lambda_1}$

Thus,

$$\boxed{\left(\frac{\rho_{radiation}}{\rho_{matter}}\right)_{earlier} = (1+z)\left(\frac{\rho_{rad}}{\rho_{matter}}\right)_{today}}$$

An important epoch in the universe's history is the epoch of Matter - Radiation equality. At matter radiation equality

$$\left(\frac{\rho_{rad}}{\rho_{mat}}\right)_{equality} = 1$$

Eq 2.39 then ⇒

$$1 = (1 + z_{EQ})\left(\frac{\rho_{radiation}}{\rho_{matter}}\right)_{today}$$

where z_{EQ} is the redshift of which Matter radiation equality happens.

From our earlier discussion W.K.T.

$$\left(\frac{\rho_{rad}}{\rho_{matt}}\right)_{today} = \frac{\Omega_{r} h^2}{\Omega_m h^2} = \frac{4.3 \times 10^{-5}}{0.12}$$

$$(1+z_{EQ}) = \frac{0.12}{4.3 \times 10^{-12}}$$

$$\boxed{z_{EQ} \simeq 3000} \quad (2.40)$$

1) z_{EQ} redshift at which $\rho_{rad} = \rho_{matter}$ $z_{EQ} \simeq 3000$
2) $H^2 \propto \rho$
3) $\Omega \simeq 1$ ($\Omega \simeq 1$) an [illegible]

This is the redshift at which matter & radiation had equal density.

$z = 3000$

$z < 3000$ ρ_m | $z > 3000$ ρ_r

⊙ Computing value of Ω

Recall that the friedmann Eq^n: (0-0 component of einstein eq^n)

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G \rho}{3}$$

can be written as:

$$\frac{k}{H^2 R^2} = \Omega - 1 \quad (2.41)$$

by defining $\Omega = \frac{\rho}{\rho_c}$ and $\rho_c = \frac{3H^2}{8\pi G}$

Note that $\Omega - 1$ is not a constant but in general evolves with time.

In fact at early times the energy density is high & the hubble parameter H is also high and in comparison the ~~[illegible]~~ k/R^2 term is small & can be neglected in the evolution eq^{n}s thus we get

$$H^2 \propto \rho$$

$H^2 \propto R^{-3}$ for MD (matter dominated)

$H^2 \propto R^{-4}$ for RD (Radiation dominated)

$R \simeq 1$ $\qquad \frac{R}{R_0} = (1+z)^{-1}$

$\frac{R_{Eq}}{R_0}\left(\frac{R}{R_{Eq}}\right)^2 = 10^{-4}(1+z)^{-2}$

Thus

$$H^2 R^2 \propto R^{-1} \quad \text{for MD} \quad (2.42)$$

$$\& \quad H^2 R^2 \propto R^{-2} \quad \text{for RD} \quad (2.43)$$

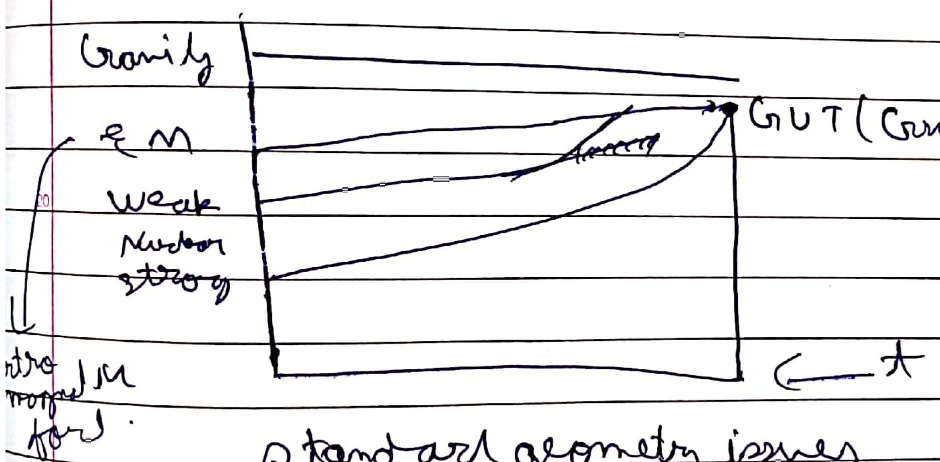
Since $|\Omega - 1|$ is at most of order 1 today using (2.41), (2.42), (2.43) that

$$10^{-3} \leftarrow |\Omega - 1| \simeq \frac{R}{R_0} = (1+z)^{-1} \quad (MD)$$

(approx) at $z_{Eq} = 3000$

$$\left(\frac{R_{Eq}}{R_0}\right)\left(\frac{R}{R_{Eq}}\right)^2 = 10^{-4}(1+z)^{-2} \quad (RD)$$

Today the value of Ω is very close to 1. Thus we see that at early times Ω had to be even closer to 1.



standard geometry issues
1) Relic problem (monopole)
2) $\boxed{\text{Horizon}}$

$e^{Ht} \rightarrow$ cosmological Inflation

$\Downarrow$

$\Omega \rightarrow 1$

MIDSEM