Lecture 6: Conditional Distributions & Total Probability

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Example 6.1 *Let X and Y be two random variables having the joint probability density function*

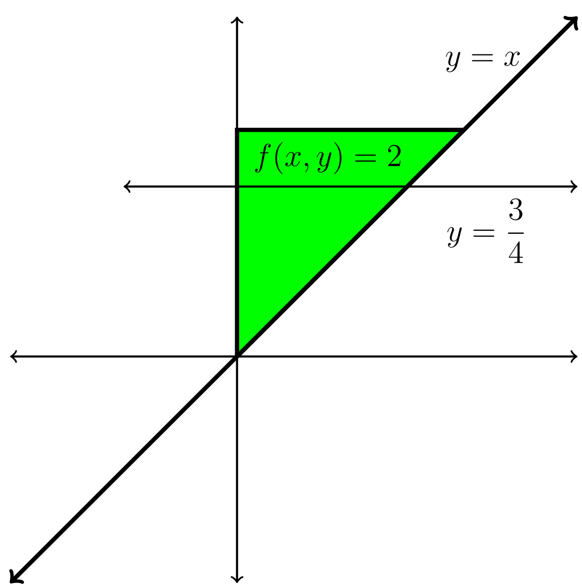
$$f(x,y) = \begin{cases} 2 & \text{if } \; 0 < x < y < 1 \\ 0 & \textit{elsewhere} \end{cases}$$

Then find the conditional probability $P\left(X \leq \frac{2}{3} \middle| Y = \frac{3}{4}\right)$.

Solution: We are suppose to use the following definition

$$P(X \in B | Y = y) = \int_B f_{X|Y}(x|y)dx,$$

i.e., we need to compute the conditional density $f_{X|Y}\left(x \middle| \frac{3}{4}\right)$ and for this we need to compute $f_Y\left(\frac{3}{4}\right)$.



$$f_Y\left(\frac{3}{4}\right) = \int_{-\infty}^{\infty} f\left(x, \frac{3}{4}\right) dx = \int_0^{\frac{3}{4}} f\left(x, \frac{3}{4}\right) dx = \int_0^{\frac{3}{4}} 2dx = 2 \times \frac{3}{4} = \frac{3}{2}.$$

Since $f_Y\left(\frac{3}{4}\right) > 0$, therefore

$$f_{X|Y}\left(x \middle| \frac{3}{4}\right) = \begin{cases} \dfrac{f\left(x, \frac{3}{4}\right)}{f_Y\left(\frac{3}{4}\right)} = \dfrac{2}{\frac{3}{2}} = \dfrac{4}{3} & \text{if} \quad 0 < x < \dfrac{3}{4} \\ 0 & \text{elsewhere} \end{cases}$$

Hence

$$\begin{aligned} P\left(X \leq \frac{2}{3} \middle| Y = \frac{3}{4}\right) &= \int_{-\infty}^{\frac{2}{3}} f_{X|Y}\left(x \middle| \frac{3}{4}\right) dx \\ &= \int_0^{\frac{2}{3}} \frac{4}{3}\, dx \\ &= \frac{8}{9} \end{aligned}$$

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Example 6.2 *Let X and Y be independent continuous random variables with pdf f_X and f_Y respectively. Let $Z = X + Y$. Determine conditional density of Z given X.*

Solution: Basically we first determine the conditional distribution function of Z given X, i.e., $P(Z \leq z|X = x)$. Then we have the relation

$$P(Z \leq z|X = x) = \int_{-\infty}^{z} f_{Z|X}(t|x)dt$$

Now

$$\begin{aligned} P(Z \leq z|X = x) &= P(X + Y \leq z|X = x) \\ &= P(x + Y \leq z|X = x) \\ &= P(x + Y \leq z) \quad (\because X, Y \text{ are indepedent}) \\ &= P(Y \leq z - x) \\ &= \int_{-\infty}^{z-x} f_Y(y)dy \\ &= \int_{-\infty}^{z} f_Y(t - x)dt \qquad (\text{put } y = t - x) \end{aligned}$$

Hence $f_{Z|X}(z|x) = f_Y(z - x)$. ■

Remark 6.3 *In Example 6.2, if we try to compute conditional density of $X+Y$ given X by definition then we require to compute the joint density of $X+Y$ and X. This type of problem we have not studied.*

Rather than going by definition, we adopt the technique of finding pdf of a real-valued function of two random variables. We first compute the conditional distribution function and differentiate it to obtain the conditional density.

Example 6.4 *Suppose that X and Y are independent, identically distributed, geometric random variables with parameter p. Find the conditional pmf of Y given $X+Y=n$ where $n \geq 2$.*

Solution: Since range of X and Y is $\mathbb{N}$, hence the range of the random variable $Z := X+Y$ is $\{2, 3, \cdots\}$. Let $n \geq 2$ be given. So if $X+Y=n$ then Y can only assume values in $\{1, 2, \cdots, n-1\}$. Therefore

$$P(Y=y|Z=n) = 0, \quad \text{for } y = n, n+1, n+2, \cdots$$

For $y \in \{1, 2, \cdots, n-1\}$

$$\begin{aligned}
P(Y=y|Z=n) &= \frac{P(Y=y, X+Y=n)}{P(X+Y=n)} \\
&= \frac{P(Y=y, X=n-y)}{\sum_{k=1}^{n-1} P(X=k, Y=n-k)} \quad \text{(By Total Probability theorem)} \\
&= \frac{P(Y=y)P(X=n-y)}{\sum_{k=1}^{n-1} P(X=k)P(Y=n-k)} \quad (\because X, Y \text{ are independent }) \\
&= \frac{p(1-p)^{y-1}p(1-p)^{n-y-1}}{\sum_{k=1}^{n-1} p(1-p)^{k-1}p(1-p)^{n-k-1}} \\
&= \frac{p^2(1-p)^{n-2}}{\sum_{k=1}^{n-1} p^2(1-p)^{n-2}} \\
&= \frac{1}{n-1}
\end{aligned}$$

This shows that

$$f_{Y|X+Y}(y|n) = \begin{cases} \dfrac{1}{n-1} & \text{if} \quad y = 1, \cdots, n-1, \\ 0 & \text{if} \quad y \geq n \end{cases}$$

Hence Y is geometrically distributed in the original universe, but in the new universe determined by the event $X + Y = n$, Y is uniformly (discrete) distributed over the set $\{1, 2, \cdots, n-1\}$. ∎

Remark 6.5 *Again in Example 6.4, if we go by pmf definitions $\frac{f(z,y)}{f_Z(z)}$ where f the joint pmf of $Z = X + Y$ and Y and f_Z is the pmf of Z, then we need to compute both the quantities. Where as if choose the conditional probability definition, then our job is much easier.*

Law of Total Probability

Recall

Theorem 6.6 (Total Probability Theorem) *Let (Ω, P) be a probability space and $\{A_1, A_2, \cdots, A_N\}$ be a at most countable partition (either $N \in \mathbb{N}$ or $N = \infty$) of Ω such that $P(A_i) > 0$ for all i. Then for any event B,*

$$P(B) = \sum_{i=1}^{N} P(B|A_i)P(A_i).$$

Proposition 6.7 (Law of total probability) *Let Y be a discrete random variable on the sample space Ω. Then for any event B,*

$$P(B) = \sum_{y \in R_Y} P(B|Y = y) f_Y(y), \tag{6.1}$$

where f_Y is the pmf of Y.

Proof: If Y is a discrete random variable with range $R_Y \subset \mathbb{R}$, then the collection of events $\{\{Y = y\}\}_{y \in R_Y}$ form a partition of the sample space Ω. Thus, we can use the total probability theorem.

$$P(B) = \sum_{y \in R_Y} P(B|Y = y)P(Y = y) = \sum_{y \in R_Y} P(B|Y = y) f_Y(y).$$

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