Lecture 4: Expectation of function of two random variables

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Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

Let us first recall the notion of expectation of single random variable.

Definition 4.1 *Let X be a discrete random variable with the pmf $f_X(x)$. Then expectation of X denoted by $E(X)$ is defined as*

$$E(X) = \sum_{x \in R_X} x f_X(x)$$

provided the right hand side series converges absolutely, i.e., $\sum_{x} |x| f_X(x) < \infty$.

The absolute convergence implies that the infinite sum $\sum_{x} x f_X(x)$ converges to a finite value that is independent of the order in which the various terms are summed.

Remark 4.2

1. *If X has finite range then the sum $\sum_{x \in R_X} x f_X(x)$ is a finite sum and there is no need to check the condition $\sum_{x \in R_X} |x| f_X(x) < \infty$ separately (as finite sum of real numbers is a real number).*
2. *If range of discrete random variables X is infinite then the sum $\sum_{x \in R_X} x f_X(x)$ is an infinite sum and the condition $\sum_{x \in R_X} |x| f_X(x) < \infty$ implies that the infinite sum $\sum_{x \in R_X} x f_X(x)$ converges to a finite value that is independent of the order in which the various terms are summed (see Example 4.3). Note that the series $\sum_{x \in R_X} x f_X(x)$ may converge but the series $\sum_{x \in R_X} |x| f_X(x)$ may not. In that case we say that $E[X]$ does not exist (see Example 4.6 below).*
3. *If $R_X \subset [0, \infty)$ then convergence of the infinite series $\sum_{x \in R_X} x f_X(x)$ is equivalent to absolute convergence.*

Example 4.3 *Consider the convergent series* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ *and one of its rearrangements*

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots \tag{4.1}$$

in which two positive terms are always followed by one negative. If s is the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, *then one can show that the rearrangement series (4.1) converges to a number strictly bigger then s.*

If we rearrange the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ *in such a way that two negative terms follow a positive term:*

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots \tag{4.2}$$

then one can show that the rearranged series (4.2) converges to $\frac{s}{2}$.

In fact, the theorem below says something very dramatic and startling.

Theorem 4.4 (Riemann) *A conditionally convergent series can be made to converge to any arbitrary real number or even made to diverge by a suitable rearrangement of its terms.*

Theorem 4.5 *If the series* $\sum_n a_n$ *converges absolutely, then every rearrangement of the series* $\sum_n a_n$ *converges, and they all converge to the same sum.*

Example 4.6 *Let X have the PMF given by*

$$P\left(X = \frac{(-1)^{n+1}3^n}{n}\right) = \frac{2}{3^n}, \quad n = 1, 2, \cdots$$

Then

$$\sum_{x \in R_X} |x| P(X = x) = \sum_{n=1}^{\infty} \frac{3^n}{n} \times \frac{2}{3^n} = \sum_{n=1}^{\infty} \frac{2}{n} = \infty.$$

Therefore $E[X]$ does not exist, although the series

$$\sum_{x \in R_X} x P(X = x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^n}{n} \times \frac{2}{3^n} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} < \infty.$$

Definition 4.7 *Let X be a random variable with pdf f. Then expectation of X denoted by $E(X)$ is defined as*

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

provided the improper integral in the right hand side converges absolutely, i.e., $\int_{-\infty}^{\infty} |x| f(x) dx < \infty$.

Remark 4.8 *We emphasize that the condition* $\int_{-\infty}^{\infty} |x| f_X(x) dx < \infty$ *must be checked before it can be concluded that* $E[X]$ *exists and equals* $\int_{-\infty}^{\infty} x f_X(x) dx$.

Theorem 4.9

1. *Let X be a discrete random variable with pmf f_X, and let $g : \mathbb{R} \to \mathbb{R}$ be any function. Then,*

$$E[g(X)] = \sum_{x \in R_X} g(x) f_X(x) \tag{4.3}$$

provided $\sum_{x \in R_X} |g(x)| f_X(x) < \infty$.

2. *Let X be a random variable with pdf f and $g : \mathbb{R} \to \mathbb{R}$ be a Borel function (e.g., piecewise continuous, continuous) such that the integral* $\int_{-\infty}^{\infty} |g(x)| f(x) dx < \infty$. *Then,*

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx \tag{4.4}$$

Remark 4.10

1. *If X is discrete, for any function g, random variable $Z := g(X)$ admits pmf. So one can use the formula $E[Z] = \sum_{z \in R_Z} z f_Z(z)$ to compute expectation of Z. But if one is just interested in $E[g(X)]$, then formula (4.3) paves the way without calculating pmf of $g(X)$.*

2. *If X has density then for some functions g it is possible that $g(X)$ does not have pdf (see Example 4.11), but the formula (4.4) paves the way to define the expectation of $g(X)$.*

   Suppose $g(X)$ has pdf then the formula (4.4) tells us how to compute expectation, without calculating the pdf of $g(X)$.

Example 4.11 *Recall Example 2.2, where $X \sim \exp(5)$ and $g(x) = \min\{x, 10\}$. We have seen that $g(X)$ does not have pdf. But we can compute the expectation of $g(X)$ using formula*

(4.4) as follows:

$$
\begin{aligned}
E[\min\{X,10\}] &= \int_{-\infty}^{\infty} \min\{x,10\} f_X(x)dx = 5\int_{0}^{\infty} \min\{x,10\} e^{-5x}dx \\
&= 5\left[\int_0^{10} \min\{x,10\}e^{-5x}dx + \int_{10}^{\infty} \min\{x,10\}e^{-5x}dx\right] \\
&= 5\left[\int_0^{10} xe^{-5x}dx + \int_{10}^{\infty} 10e^{-5x}dx\right] \\
&= 5\left[\left(\frac{e^{-5x}}{25}(-5x-1)\right)_0^{10} - 2\left[e^{-5x}\right]_{10}^{\infty}\right] = \frac{1}{5}\left(-51e^{-50}+1\right) + 10e^{-50} = \frac{1-e^{-50}}{5}
\end{aligned}
$$

There is no need to check the absolute convergence of the improper integral because $g(x)$ is non-negative on $[0,\infty)$.

Theorem 4.12 *1. Let X, Y be two discrete random variable with joint pmf $f(x,y)$. Let $g:\mathbb{R}^2 \to \mathbb{R}$ be a function, then*

$$E[g(X,Y)] = \sum_{x\in R_X}\sum_{y\in R_Y} g(x,y)f(x,y), \tag{4.5}$$

provided $\sum_{x\in R_X}\sum_{y\in R_Y} |g(x,y)|f(x,y) < \infty.$

2. Let X, Y be two absolutely continuous random variable with joint pdf $f(x,y)$. Let $g:\mathbb{R}^2 \to \mathbb{R}$ be a Borel measurable function (e.g., continuous, indicators of reasonable sets (Borel subsets of $\mathbb{R}^2$), and functions that are continuous except across some smooth boundaries) , then

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy \tag{4.6}$$

provided $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(x,y)|f(x,y)dxdy < \infty.$

Example 4.13 *Recall Example 1.1, X and Y were random variables with the joint pmf given by the following table.*

X \ Y	-1	0	2	6
-2	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{1}{9}$
1	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$
3	0	0	$\frac{1}{9}$	$\frac{4}{27}$

Then PMF of $Z := |Y - X|$, *is*

$$P(Z=1)=\frac{1}{3},\quad P(Z=2)=\frac{7}{27},\quad P(Z=3)=\frac{4}{27},\quad P(Z=4)=\frac{1}{27},\quad P(Z=5)=\frac{1}{9},\quad P(Z=8)=\frac{1}{9}$$

So

$$E[Z]=\sum_{z\in R_Z} zP(Z=z)=1\times\frac{1}{3}+2\times\frac{7}{27}+3\times\frac{4}{27}+4\times\frac{1}{27}+5\times\frac{1}{9}+8\times\frac{1}{9}=\frac{26}{9}$$

Suppose we are interested only in $E|Y-X|$ *then by formula (4.5), we have*

$$\begin{aligned}E|Y-X| &= \sum_y\sum_x |y-x|f(x,y)\\ &= \sum_y |y-(-2)|f(-2,y)+\sum_y|y-1|f(1,y)+\sum_y|y-3|f(3,y)\end{aligned}$$

$$\sum_y |y-(-2)|f(-2,y)=1\times\frac{1}{9}+2\times\frac{1}{27}+4\times\frac{1}{27}+8\times\frac{1}{9}=\frac{11}{9}$$

$$\sum_y |y-1|f(1,y)=2\times\frac{2}{9}+1\times 0+1\times\frac{1}{9}+5\times\frac{1}{9}=\frac{10}{9}$$

$$\sum_y |y-3|f(3,y)=4\times 0+3\times 0+1\times\frac{1}{9}+3\times\frac{4}{27}=\frac{5}{9}$$

Example 4.14 *Let X and Y be independent and identically distributed exponential random variables with parameter λ. Find the mean of* $\max\{X,Y\}$.

Solution:

Since

$$f_X(x)=\begin{cases}0 & \text{if} \quad x<0\\ \lambda e^{-\lambda x} & \text{if} \quad x\geq 0\end{cases},\ f_Y(y)=\begin{cases}0 & \text{if} \quad y<0\\ \lambda e^{-\lambda y} & \text{if} \quad y\geq 0\end{cases},$$

Therefore

$$f_X(x)f_Y(y)=\begin{cases}\lambda^2 e^{-\lambda(x+y)} & \text{if} \quad x\geq 0, y\geq 0\\ 0 & ; \quad \text{otherwise}\end{cases},$$

$$
\begin{aligned}
E[\max\{X,Y\}] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max\{x,y\} f(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max\{x,y\} f_X(x)f_Y(y)dxdy \quad (\because X, Y \text{ are independent}) \\
&= \iint\limits_{\{x\geq 0, y\geq 0\}} \max\{x,y\}\lambda^2 e^{-\lambda(x+y)}dxdy \\
&= \iint\limits_{\{x\geq 0, y\geq 0\}\cap\{y\geq x\}} \max\{x,y\}\lambda^2 e^{-\lambda(x+y)}dxdy \\
&\quad + \iint\limits_{\{x\geq 0, y\geq 0\}\cap\{y< x\}} \max\{x,y\}\lambda^2 e^{-\lambda(x+y)}dxdy \\
&= \iint\limits_{\{0\leq x\leq y\}} y\lambda^2 e^{-\lambda(x+y)}dxdy + \iint\limits_{\{0\leq y< x\}} x\lambda^2 e^{-\lambda(x+y)}dxdy \\
&= \lambda^2\int_0^{\infty} ye^{-\lambda y}\left(\int_0^y e^{-\lambda x}dx\right)dy + \lambda^2\int_0^{\infty} xe^{-\lambda x}\left(\int_0^x e^{-\lambda y}dy\right)dx = \frac{3}{2\lambda}
\end{aligned}
$$

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\begin{aligned}
\lambda^2\int_0^{\infty} ye^{-\lambda y}\left(\int_0^y e^{-\lambda x}dx\right)dy &= \lambda^2\int_0^{\infty} ye^{-\lambda y}\left(\frac{1-e^{-\lambda y}}{\lambda}\right)dy \\
&= \lambda\int_0^{\infty}(ye^{-\lambda y} - ye^{-2\lambda y})dy \\
&= \lambda\left(y\frac{1}{-\lambda}e^{-\lambda y}\Big|_0^{\infty} + \int_0^{\infty}\frac{1}{\lambda}e^{-\lambda y}dy\right. \\
&\quad \left. - \left[y\frac{1}{-2\lambda}e^{-2\lambda y}\Big|_0^{\infty} + \int_0^{\infty}\frac{1}{2\lambda}e^{-2\lambda y}dy\right]\right) \\
&= \lambda\left(\frac{1}{-\lambda^2}e^{-\lambda y}\Big|_0^{\infty} - \frac{1}{-4\lambda^2}e^{-2\lambda y}\Big|_0^{\infty}\right) \\
&= \frac{1}{\lambda} - \frac{1}{4\lambda} = \frac{3}{4\lambda} \\
\lambda^2\int_0^{\infty} xe^{-\lambda x}\left(\int_0^x e^{-\lambda y}dy\right)dx &= \frac{3}{4\lambda}
\end{aligned}
$$

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Solution:[Alternate] For $z \in \mathbb{R}$,

$$\{Z = \max\{X,Y\} \leq z\} = \{X \leq z\} \cap \{Y \leq z\}$$

Hence

$$P\{Z \leq z\} = \begin{cases} P(\emptyset) = 0 & \text{if } \quad z < 0 \\ P(X \leq z, Y \leq z) = P(X \leq z)P(Y \leq z) & \text{if } \quad z \geq 0 \end{cases}$$

Hence distribution function of $Z = \max\{X,Y\}$ is denoted by F_Z

$$F_Z(z) = \begin{cases} 0 & \text{if} \quad z < 0 \\ (1 - e^{-\lambda z})^2 & \text{if} \quad z \geq 0 \end{cases}$$

Since F_Z is continuous everywhere, Z has the following pdf

$$f_Z(z) = \begin{cases} 0 & \text{if} \quad z \leq 0 \\ 2\lambda e^{-\lambda z}(1 - e^{-\lambda z}) & \text{if} \quad z > 0 \end{cases}$$

Therefore

$$\begin{aligned} E[Z] &= \int_0^\infty 2z(1 - e^{-\lambda z})(\lambda e^{-\lambda z}) dz = 2\lambda \int_0^\infty z e^{-\lambda z}(1 - e^{-\lambda z}) dz \\ &= 2\lambda \left[\int_0^\infty z e^{-\lambda z} dz - \int_0^\infty z e^{-2\lambda z} dz \right] = \frac{3}{2\lambda} \end{aligned}$$

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