Lecture 20: Maximum Likelihood Estimator

13 April, 2018

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Example 20.1 *Let $X_1, X_2, \cdots, X_n$ be a random sample from a geometric(p) distribution. Find the MLE of p.*

Solution: Note that a geometric(p) random variable has the pmf

$$f(x) = p(1-p)^{x-1}, \quad x = 1, 2, \cdots$$

The likelihood function of the given random sample is

$$L(p|x_1, x_2, \cdots, x_n) = [p(1-p)^{x_1-1}][p(1-p)^{x_2-1}]\cdots[p(1-p)^{x_n-1}] = p^n(1-p)^{\sum_{i=1}^n x_i - n}, \text{ for } p \in (0,1).$$

Note that we have parameter range $(0,1)$ rather than $[0,1]$ (if $p = 0$ or $p = 1$ then we pmf becomes identically zero, which is absurd). Set $y = \displaystyle\sum_{i=1}^n x_i$, then $y \geq n$ and taking logarithm of likelihood function we obtain

$$f(p) = \log L(p) = n \log p + (y-n)\log(1-p).$$

$$f'(p) = \frac{n}{p} - \frac{y-n}{1-p} = 0 \implies \frac{n}{p} = \frac{y-n}{1-p} \implies \frac{1-p}{p} = \frac{y-n}{n} \implies \frac{1-p+p}{p} = \frac{y-n+n}{n}$$
$$\implies \frac{1}{p} = \frac{y}{n} \implies p = \frac{n}{y}$$

Note that if $p < \frac{n}{y}$ then

$$\frac{n}{p} > y. \text{ And also } -p > -\frac{n}{y} \implies 1-p > 1-\frac{n}{y} \implies 1-p > \frac{y-n}{y} \implies y > \frac{y-n}{1-p}.$$

Therefore $f'(p) > 0$ on the interval $(0, \frac{n}{y})$ and by reversing the inequality we obtain $f'(p) < 0$ on the interval $(\frac{n}{y}, 1)$. Therefore f is strictly increasing on $(0, \frac{n}{y})$ and strictly decreasing on $(\frac{n}{y}, 1)$.

Alternatively, it is easy to see that $\lim_{p\to 0^+} f(p) = -\infty = \lim_{p \to 1^-} f(p)$. So $p = \frac{n}{y}$ is the point of global maximum. Hence $\hat{p} = \dfrac{n}{\sum_{i=1}^n X_i}$ the MLE of p for the given sample. ■