14/10/19 (After Mid Term)

* Age of the Universe (with vaccum energy) ⇒

$$t_0 = \int dt = H_0^{-1} \Omega_v^{-1/2} \int \frac{1}{R} \frac{dR}{\left[1 + \frac{1}{R^2}\frac{\Omega_M}{\Omega_v}\right]^{1/2}}$$

Recall $\Omega_M + \Omega_v = 1$ ⇒ $\Omega_M = 1 - \Omega_v$

⇒ Integral $= \frac{1}{3} \ln \left(\frac{\frac{1}{\Omega_v^{1/2}} + 1}{\frac{1}{\Omega_v^{1/2}} - 1} \right)$

where Integral $= \int_0^1 \frac{1}{R} \frac{dR}{\left[1 + \frac{1}{R^2}\frac{\Omega_M}{\Omega_v}\right]^{1/2}}$

Thus,

$$t_0 = H_0^{-1} \Omega_v^{-1/2} \frac{1}{3} \ln \left(\frac{\frac{1}{\Omega_v^{1/2}} + 1}{\frac{1}{\Omega_v^{1/2}} - 1} \right)$$

$$= H_0^{-1} \Omega_v^{-1/2} \frac{1}{3} \ln \left(\frac{1 + \Omega_v^{1/2}}{1 - \Omega_v^{1/2}} \right)$$

$$= H_0^{-1} \Omega_v^{-1/2} \frac{1}{3} \ln \left(\frac{(1 + \Omega_v^{1/2})^2}{(1 - \Omega_v)} \right)$$

$$= H_0^{-1} \Omega_v^{-1/2} \frac{1}{3} \ln \left[\left[\frac{(1 + \Omega_v^{1/2})}{(1 - \Omega_v)^{1/2}} \right]^2 \right]$$

$$t_0 = H_0^{-1} \Omega_v^{-1/2} \frac{2}{3} \ln \left(\frac{(1 + \Omega_v^{1/2})}{(1 - \Omega_v)^{1/2}} \right)$$

Note → as $\Omega_v \longrightarrow 1$ $t_0 \longrightarrow \infty$

Thus large age is not problem when vaccum energy is present.

• Recall that:

$$H_0 = 100\,h \text{ kms}^{-1} \text{ Mpc}^{-1}$$

$$\Rightarrow H_0^{-1} = h^{-1} * 9.78 * 10^9 \text{ years.}$$

$$\Rightarrow H_0^{-1} = 13.6 * 10^9 \text{ years.}$$

$$\Rightarrow t_0 = \frac{2}{3} \times 13.6 \times 10^9 \text{ years} \times \Omega_v^{-1/2} \ln\left[\frac{1+\Omega_v^{1/2}}{(1-\Omega_v)^{1/2}}\right]$$

From observations we know that:

$$t_0 = 13.6 \times 10^9 \text{ years.}$$

Thus, we get

$$\boxed{13.6 \times 10^9 = 9 \times 10^9 \times \Omega_v^{-1/2} \ln\left[\frac{1+\Omega_v^{1/2}}{(1-\Omega_v)^{1/2}}\right]} \quad (3.11)$$

⇒

We can solve 3.11 & get:

$$\boxed{\Omega_v \approx 0.75}$$

This is discovered in 1995.

Thus abt $\frac{3}{4}$th of the energy density of the Universe is in the form of vaccum energy or dark energy.

• Recall that the dark energy has the eqn of state which is of the form $P = \omega \rho$ with $\omega < 0$.

* Other Implications of Friedmann's Eq^n ⇒

• Scale factor as a f^n of time ⇒.

Consider the observationally & theoretically supported case of the flat $k = 0$, $\Omega = 1$ universe.

In this case the Friedmann's Eq^n is:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho \quad \text{—— (3.12)}$$

Recall that we have the 1st law of thermodynamics

$$\boxed{d(\rho R^3) = -P\, d(R^3)}$$

together with the eq^n of the state: $P = \omega \rho$ —— (3.13)

which implies ⟶ $\rho \propto R^{-3(1+\omega)}$ —— (3.14)

or,

$$\boxed{\frac{\rho}{\rho_0} = \left(\frac{R}{R_0}\right)^{-3(1+\omega)}} \quad \text{—— (3.15)}$$

Putting 3.15 in 3.12 ⇒

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G \rho_0}{3}\left(\frac{R}{R_0}\right)^{-3(1+\omega)}$$

$$\Rightarrow \dot{R}^2 = \frac{8\pi G \rho_0}{3} \frac{R^{-3-3\omega+2}}{R_0^{-3(1+\omega)}}$$

Choosing $R_0 = 1$.

$$\Rightarrow \dot{R} = \sqrt{\frac{8\pi G \rho_0}{3}}\, R^{-1/2(1+3\omega)}$$

$$\Rightarrow \frac{dR}{dt} = \sqrt{\frac{8\pi G \rho_0}{3}}\, R^{-1/2(1+3\omega)}$$

$$\text{or, } \int \frac{dR}{R^{-1/2(1+3\omega)}} = \int \sqrt{\frac{8\pi G \rho_0}{3}}\, dt$$

$$\Rightarrow \int R^{1/2(1+3\omega)}\, dR = \sqrt{\frac{8\pi G \rho_0}{3}} \int dt$$

$$\text{or, } R^{3/2(1+\omega)} = \frac{3}{2}(1+\omega)^2 \sqrt{\frac{8\pi G \rho_0}{3}}\, t$$

$$\Rightarrow \left[R^{3/2(1+\omega)}\right]^{\frac{2}{3(1+\omega)}} = \left(4(1+\omega)\pi G \rho_0\right)^{\frac{2}{3(1+\omega)}} t^{\frac{2}{3(1+\omega)}}$$

$$\Rightarrow \boxed{R = k\, t^{\frac{2}{3(1+\omega)}}} \qquad (3.16)$$

- For Radiation Dominated Universe (RD) $\rightarrow$ $\omega = 1/3$.

$$\Rightarrow \frac{2}{3(1+\omega)} = \frac{2}{3\left(1+\frac{1}{3}\right)} = \frac{1}{2}$$

$$\Rightarrow \boxed{R = k_1 t^{1/2}} \quad \text{for RD Universe.} \qquad (3.17)$$

For $\omega = 0$ (MD universe)

$$\frac{2}{3(1+\omega)} = \frac{2}{3}$$

Thus, $\boxed{R = k_2 t^{2/3}}$ for MD Universe

$$(3.18)$$

For $\omega = -1$ (Vaccum Dominated Universe)

$$\frac{2}{3(1-1)} \longrightarrow \infty$$

So, expansion is actually faster than any polynomial in t.
(faster than any t^n with fixed n).

* Now, we will have a deeper look on the 3^{rd} case →

Consider the Friedmann Eq^n for the vaccum energy case more carefully:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho \qquad (3.19)$$

for vaccum energy $\omega = -1$, we know that $\rho = \rho_0 = \text{const}$.

$$(3.20)$$

Putting 3.20 in 3.19 ⇒

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_0 \qquad (3.21)$$

Thus we get

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G \rho_0}{3} = \text{const} = H_0$$

$$\Rightarrow \frac{\dot{R}}{R} = H_0$$

$$\Rightarrow \frac{1}{R}\frac{dR}{dt} = H_0 \Rightarrow \int \frac{1}{R} dR = H_0 \int dt$$

$$\Rightarrow \ln R = H_0 t \Rightarrow \boxed{R(t) = \exp(H_0 t)}$$

(3.23)

Summary →

① For RD universe ⇒ $\omega = 1/3$

$R = k_1 t^{1/2}$

② For MD universe ⇒ $\omega = 0$

$R = k_2 t^{2/3}$

③ For VD universe ⇒ $\omega = -1$

$\boxed{R(t) = \exp(H_0 t)}$

So, From Big Bang expansion is 1st RD then MD & today it is VD in nature.

* Thermodynamics ⇒

Quantity of physical interest that are thermodynamic in nature

$n \longrightarrow$ No. density of particles.

$\rho \longrightarrow$ Energy

$P \longrightarrow$ Pressure due to gas of particles.

These can be given in terms of phase space distribution (sometimes called the occupancy) function $f(\vec{p})$ where $\vec{p}$ is

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p})\, d^3p \quad \text{——— (5.1)}$$

$$\rho = \frac{g}{(2\pi)^3} \int E(\vec{p})\, f(\vec{p})\, d^3p \quad \text{——— (5.2)}$$

$$P = \frac{g}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E}\, f(\vec{p})\, d^3p \quad \text{——— (5.3)}$$

where g = # of internal degrees of freedom.

(eg. $g = 2$ for a spin $1/2$ particle such as an e^-),

&

$$E^2 = |\vec{p}|^2 + m^2 \quad \text{——— (5.4)}$$

i.e. E is the energy.

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• n, ρ & P all depends upon the distribution f^n

f

Value of this depends on thermal eqb. & many other things.

$f \longrightarrow$ Defined & fixed in Thermal Eqb.
$\rightarrow$ Depends on Spin of particles.

$S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \longleftarrow$ Half Integer

Anti Symmetric under exchange of particles.

Wavefunction: $\psi \longrightarrow -\psi$

Fermions obey Fermi-Dirac Statistics

Integer $\longrightarrow S = 0, 1, 2$

Symmetric under exchange of particles.

Wave f^n: $\psi \longrightarrow \psi$

Bosons obey Bose-Einstein Statistics

- Particles with Integer Spin $(s = 0, 1, 2)$ are Bosons & their phase space distribution is given by the Bose-Einstein Distribution:

$$f(\vec{p}) = \frac{1}{\exp((E-\mu)/T) - 1} \qquad (5.5)$$

Eg $\rightarrow$ photon

- Particles with half integer spin $(s = 1/2, 3/2, 5/2)$ are Fermions & their Phase Space Distribution is:

$$f(\vec{p}) = \frac{1}{\exp[(E-\mu)/T] + 1} \qquad (5.6)$$

Eg. $\rightarrow$ Electron

μ is chemical potential & describes chemical eqb.

Note: $E^2 = p^2 + m^2$

Units:

$\boxed{k_B = 1} \longrightarrow$ We have taken this as unit.

$e^{-\Delta E / k_B T}$

For eg. → for species "i" interacting with species "j, k & l" by the reaction $i + j \rightleftharpoons k + l$

the chemical potentials obeys the relationship:

$$\mu_i + \mu_j \rightleftharpoons \mu_k + \mu_l.$$

Thus μ captures the information abt the chemical eqb. Frequently, we combine the description of Fermi-Dirac species & Bose einstein species & write the expression as:

$$f(\vec{p}) = \frac{1}{\exp[(E-\mu)/T] \pm 1} \quad (5.7)$$

$+ \rightarrow$ Fermi-Dirac Species.

$- \rightarrow$ Bose-Einstein ".

* Taking the distribution fⁿ f given by (5.7) & inserting into the general expression for n, ρ & P given by (5.1), (5.2) & (5.3) respectively, we get:

$$n = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp[(E-\mu)/T] \pm 1} E\, dE$$

$$\rho = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp[(E-\mu)/T] \pm 1} E^2 dE$$

&,

$$P = \frac{g}{6\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{3/2}}{\exp[(E-\mu)/T] \pm 1} dE.$$

where we have changed the variables of integration from p to E & used the relationship (5.4) i.e.

$$E^2 = |\vec{p}|^2 + m^2.$$

Derivation →

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p})\, d^3p$$

$$= \frac{g}{8\pi^3} \int f(\vec{p})\, 4\pi p^2 dp$$

$$= \frac{g\, 4\pi}{8\pi^3} \int f(\vec{p})\, p^2 dp.$$

$$= \frac{g}{2\pi^2} \int f(\vec{p})\, p^2 dp.$$

$E^2 = |\vec{p}|^2 + m^2$

$p = 0$
$\Rightarrow E = m$

$2E\,dE = 2p\,dp.$

$\Rightarrow p^2 dp = p\,E\,dE$

$= (E^2 - m^2)^{1/2} E\,dE$

$$\Rightarrow \frac{g}{2\pi^2} \int_m^\infty f(\vec{p})\, (E^2 - m^2)^{1/2} E\,dE$$

$$n = \frac{g}{2\pi^2} \int_m^\infty \frac{1}{\exp[(E-\mu)/T] \pm 1} (E^2 - m^2)^{1/2} E\,dE$$

[Note → $d^3p = dp_x\, dp_y\, dp_z$ in cartesian coor.

$d^3p = p^2 dp\, d(\sin\theta)\, d\phi$ in spherical coo.

→ We will use this.

$$\underbrace{\int d\sin\theta}_{2} \underbrace{\int d\phi}_{2\pi} \Rightarrow 4\pi$$

So, $d^3p = 4\pi p^2 dp$].

* For ρ ⇒

$$\rho = \frac{g}{(2\pi)^3} \int E(\vec{p}) f(\vec{p})\, d^3p = \frac{g}{2\pi^2} \int_m^\infty p f(\vec{p}) E^2 dE$$

[∵ $d^3p = 4\pi p^2 dp$].

$$= \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2} E^2 dE}{\exp[(E-\mu)/T] \pm 1}$$

* For P ⇒

$$P = \frac{g}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E} f(\vec{p})\, d^3p$$

$$= \frac{g}{2\pi^2}\,\frac{1}{3}\int_m^\infty \frac{|\vec{p}|^2 f(\vec{p})\, p\, \cancel{E}\, dE}{\cancel{E}}$$

$$= \frac{g}{6\pi^2}\int_m^\infty |\vec{p}|^3 f(\vec{p})\, dE$$

$$\boxed{P = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2 - m^2)^{3/2}\, dE}{\exp[(E-\mu)/T] \pm 1}}$$

* <u>Summarize</u> $\Rightarrow$ the expression of n, ρ & P

- $$\boxed{n = \frac{g}{2\pi^2}\int_m^\infty \frac{1}{\exp[(E-\mu)/T] \pm 1}\,(E^2 - m^2)^{1/2}\, E\, dE} \quad \text{(5.8)}$$

- $$\boxed{\rho = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2 - m^2)^{1/2} E^2\, dE}{\exp[(E-\mu)/T] \pm 1}} \quad \text{——— (5.9)}$$

- $$\boxed{P = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2 - m^2)^{3/2}}{\exp[(E-\mu)/T] \pm 1}} \quad \text{——— (5.10)}$$

Q. How to use these expressions ??

$n(T, \mu, m) \longrightarrow$ Number density

$\rho(T, \mu, m) \longrightarrow$ Energy density

$P(T, \mu, m) \longrightarrow$ Pressure.

$\rightarrow$ High T limit

$\rightarrow$ Low T limit

eg - [for Electron]

$\rho = \sum_i \rho_i$

$\rho_{electron}$.

$\langle E_e \rangle \sim k_B T$

$\langle p \rangle \approx k_B T$ $\boxed{E^2 = p^2 + m^2}$.

$T >> m(c^2)$

High T $\longrightarrow$ Relativistic limit. $T >> \mu$.

Heaviest particle in Std. Model of Particle Physics.

$m_T \sim 100$ GeV

$T >> 100$ GeV.

$T \sim 1$ eV

I For high T limit (Applicable to the Early universe).

$T >> m$ $T >> \mu$.

① $$\rho = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2} E^2 dE}{\exp\left[\left(\frac{E-\mu}{T}\right)\right] \pm 1}.$$

1st let us use $\longrightarrow$ $\mu << T \Rightarrow e^{-\mu/T} = 1$.

$$\Rightarrow \rho = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2} E^2 dE}{\exp[E/T] \pm 1}.$$

Now use $k^2 = E^2 - m^2$ $\Rightarrow 2k\,dk = 2E\,dE$

Thus,

$$\rho = \frac{g}{2\pi^2}\int_0^\infty \frac{k(k^2+m^2)^{1/2}\, k\, dk}{\exp\left[\frac{(k^2+m^2)^{1/2}}{T}\right] \pm 1}.$$

Now use $m << T$ & set $k' = k/T$ $\Rightarrow dk = T\,dk'$

$$\Rightarrow \rho = \frac{g}{2\pi^2}\int_0^\infty \frac{(k'T)^3\, T\, dk'}{\exp[k'] \pm 1}.$$

$$\Rightarrow \boxed{\rho = \frac{gT^4}{2\pi^2}\int_0^\infty \frac{k'^3\, dk'}{\exp[k'] \pm 1}.} \quad (5.11)$$

- **For Bosons** $\Rightarrow$ $-$ sign $\boxed{\rho = \frac{\pi^2}{30} g T^4}$ (5.12)

- **For Fermion** $\Rightarrow$ $+$ sign $\boxed{\rho = \frac{7}{8}\left(\frac{\pi^2}{30}\right) g T^4}$ (5.13)

$\boxed{g \longrightarrow \text{no. of integral degree of freedoms}}$

21/10/13 ρ, P, n

High T limit evaluated.

(2) Consider n in high T limit:

$$n = \frac{g}{2\pi^2}\int_m^E \frac{(E^2 - m^2)^{1/2}\, E\, dE}{\exp[(E/\mu)/T] \pm 1}$$

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$T >> \mu$

$$\Rightarrow n = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2} E\,dE}{\exp[E/T] \pm 1}$$

$E^2 = k^2 + m^2$ & $2E\,dE = 2k\,dk$ & $m << T \Rightarrow$

$$n = \frac{g}{2\pi^2} \int_0^\infty \frac{k^2 dk}{\exp[E/T] \pm 1}$$

Let $k' = k/T \Rightarrow dk = T\,dk'$

$$\Rightarrow n = \frac{g}{2\pi^2} \int_0^\infty \frac{T^3 k'^2 dk'}{\exp[k'] \pm 1}$$

$$\Rightarrow n = g T^3 \frac{1}{2\pi^2} \underbrace{\int_0^\infty \frac{k'^2 dk'}{\exp[k'] \pm 1}}_{\text{Pure no.}}$$

• Thus <u>For Bosons</u> $\Rightarrow$

Riemann - Zeta f^n for 3

$$n = g T^3 \left(\frac{\zeta(3)}{\pi^2} \right) \quad \text{—— (5.14)}$$

; where $\zeta(3) = \underline{\underline{1.202}}$

• <u>For Fermion</u> (+ sign in demand)

$$\boxed{n = g T^3 \left(\frac{3}{4}\right) \left[\frac{\zeta(3)}{\pi^2} \right]} \quad \text{—— (5.15)}$$

(3) <u>For Pressure P</u> →

$$P = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2-m^2)^{3/2}\,dE}{\exp[(E-\mu)/T]\pm 1}$$

Again using $\mu << T$ & $k^2+m^2 = E^2$ & $kdk = EdE$ we get:

$$P = \frac{g}{6\pi^2}\int_0^\infty \frac{k^3\,dk}{\exp[k/T]\pm 1}$$

Let $k' = k/T$ & $dk = Tdk'$

$$\Rightarrow P = \frac{g}{6\pi^2}\int_0^\infty \frac{T^4\,k'^3\,dk}{\exp[k']\pm 1}$$

Thus,

$$P = \frac{gT^4}{6\pi^2}\int_0^\infty \frac{k'^3\,dk'}{\exp[k']\pm 1} \quad \text{——— (5.16)}$$

Comparing (5.16) with (5.11) for Energy density (ρ) we note:

$$\boxed{P = \frac{1}{3}\rho} \quad \text{——— (5.17)}$$

Note that (5.17) is the eq^n of <u>State for Radiation & Relativistic Matter</u> which we have <u>now derived from fundamental Thermodynamics</u>.

~~For Bosons →~~

$$\cancel{n = g T^3 \left(\frac{\zeta(3)}{\pi^2} \right)}$$ 5.

II For Low T limit

Now, consider the non-relativistic case $m >> T$.

In this case the no. density is given by :

① g° $n = \frac{g}{(2\pi)^3} \int f(\vec{p}) \, 4\pi p^2 dp$

$$= \frac{g}{2\pi^2} \int f(\vec{p}) \, p^2 dp$$

$$f(p) = \frac{1}{\exp[(E-\mu)/T] \pm 1}.$$

Binomial approx.

$$E = (p^2 + m^2)^{1/2} = m\left(1 + \frac{p^2}{m^2}\right)^{1/2} \approx m\left(1 + \frac{p^2}{2m^2}\right)$$

$$f = \frac{1}{\exp[(E-\mu)/T] \pm 1} = \frac{1}{\exp\left[\frac{m}{T}\left(1 + \frac{p^2}{2m^2}\right) - \frac{\mu}{T}\right] \pm 1}$$

It will repeat in f & p.

$\frac{m}{\cancel{p}} >> 1$ $\Rightarrow$ $\exp\left[\frac{m}{T}\left(1 + \frac{p^2}{2m^2}\right) - \frac{\mu}{T}\right] >> 1.$
T

$$\Rightarrow f = \frac{1}{\exp\left[\frac{m}{T}\left(1 + \frac{p^2}{2m^2}\right) - \frac{\mu}{T}\right]} = \exp\left[-\frac{(m-\mu)}{T}\right] \cdot \exp\left[\frac{-p^2}{2mT}\right]$$

$$\left[\; n = \frac{g}{2\pi^2} \exp\left[-\frac{(m-\mu)}{T}\right] \cdot \int \exp\left[\frac{-p^2}{2mT}\right] \cdot p^2 dp \right.$$

Let $p'^2 = \dfrac{p^2}{2mT} \Rightarrow p = p'(2mT)^{1/2}$

or, $dp = (2mT)^{1/2} dp'$

$$\Rightarrow n = \frac{g}{2\pi^2} \exp\left[\frac{-(m-\mu)}{T}\right] (2mT)^{3/2} \underbrace{\int_0^\infty \exp(-p'^2)\, p'^2 dp'}_{\text{Pure Number} = \frac{\sqrt{\pi}}{4}}$$

$$\Rightarrow \boxed{n = g \left(\frac{mT}{2\pi}\right)^{3/2} \exp\left[\frac{-(m-\mu)}{T}\right]} \quad \text{——} (5.24)$$

(c) Similarly for the pressure P in $m \gg T$ limit :

$$P = \frac{g}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E} f(\vec{p})\, d^3p$$

$\Downarrow$ (final expression).

$$\boxed{P = g \exp\left[\frac{-(m-\mu)}{T}\right] \left(\frac{mT}{2\pi}\right)^{3/2} T} \quad \text{——} (5.25)$$

Comparing (5.25) & (5.24) [expression for n], we get :

$$\boxed{P = nT} \quad \text{———} (5.26).$$

where P is Pressure, n is number density & T is Temperature.

Finally, consider ρ :

(3) $$\rho = \frac{g}{(2\pi)^3} \int E(p) f(p) \, d^3p.$$

Making the same approximations & similar steps, we get :

$$\boxed{\rho = g \left(\frac{mT}{2\pi} \right)^{3/2} \exp\left[\frac{-(m-\mu)}{T} \right] m} \quad \text{———} (5.27)$$

From (5.27) & (5.24), we get : $\boxed{\rho = mn}$ ——— (5.28)

Further we note from (5.26) that $P = nT$ ——— (5.26).
& (5.28) tells us $\underline{\rho = mn}$

Dividing (5.26) by (5.28) ⇒.

$$\frac{P}{\rho} = \frac{\cancel{n} T}{m \cancel{n}} = \frac{T}{m}$$

Thus, $$P = \left(\frac{T}{m} \right) \rho \quad \text{———} (5.29)$$

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In the Non-Relativistic limit $\frac{T}{m} \ll 1$.

$$\Rightarrow P \ll \rho \quad \text{——— (5.30)}$$

or,

$P = \omega\rho$, with $\omega \to 0$ or the $\omega = 0$ will be eq^n of state for non-relativistic matter.

* Things to Note & Remember $\Rightarrow$

① In the Relativistic limit :

$$T \gg m \quad \&$$

$$\rho \sim T^4$$

$$n \sim T^3$$

$$P \sim T^4.$$

② In the non-relativistic limit :

$$m \gg T \quad \text{and}.$$

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left[\frac{-(m-\mu)}{T}\right].$$

$$\rho = mn$$

$$P = nT \ll \rho.$$

→ In non-relativistic limit all quantities are exponentially suppressed by the $\exp\left[\frac{-(m-\mu)}{T}\right]$ factor & small compared to the similar quantities in relativistic limit

In general, the total energy density & Pressure for all spheres in equilibrium can be expressed in terms of the photon temperature T as:

$$\rho = T^4 \sum_{i = \text{all species}} \left(\frac{T_i}{T}\right)^4 \frac{g_i}{2\pi^2} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2} u^2 du}{\exp[u - y_i] \pm 1} \quad (5.31)$$

$$P = T^4 \sum_{i = \text{all species}} \left(\frac{T_i}{T}\right)^4 \frac{g_i}{6\pi^2} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{3/2} du}{\exp[u - y_i] \pm 1} \quad (5.32)$$

where $x_i = \frac{m_i}{T}$, $y_i = \frac{\mu_i}{T}$

By introducing the quantity T_i we allow that species i may have a thermal distribution but with a temp. different from than that of photons.

One thing to note is that ρ & P of non-relativistic species (with $m \gg T$) are exponentially smaller than that of relativistic species (with $m \ll T$).

Thus it is a very good approximation to keep only the relativistic species in the sums for ρ & P given in (5.31) & (5.32). Thus,

$$\boxed{\rho = \frac{\pi^2}{30} g_* T^4} \quad (5.33)$$

& $$\boxed{P = \frac{\rho}{3} = \frac{\pi^2}{90} g_* T^4}$$ ——— (5.34)

$(m \ll T)$

where g_* counts the no. of effectively massless degree of freedom.

Further g_* can be expressed as :

$$g_* = \sum_{i = \text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i = \text{fermions}} g_i \left(\frac{T_i}{T}\right)^4$$

——— (5.35).

The factor $\frac{7}{8}$ is due to the different distribution functions (Fermi-dirac & Bose Einstein) for Fermions & Bosons.

- g_* of course depends on T.

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post diwali missing one class. 6-11-17

History of the universe
$\Downarrow$
Determined by contents of the universe
$\downarrow$
Std Model
$\downarrow$
$g_*(T)$

Connect to discussion on cosmology earlier. Recall Friedmann's equation relates expansion of universe to ρ & P.
The total energy density ρ is just the sum of the energy density over all the particles species

$$\rho = \sum_i \rho_i(\overset{\text{temperature}}{T})$$

where i runs over all particle species.
Also note that ρ for non-relativistic species ~~suppress~~ suppressed by exponential Boltzman factor.
Thus sum over all species to a very good approximate ~~Rel~~ includes only relativistic species. Thus ρ is

$$\rho(T) = \frac{\pi^2}{30} g_*(T)\, T^4 \qquad (6.1)$$

where g_* counts the total no. of effective degrees of freedom.

$$g_*(T) = g_b(T) + \frac{7}{8} g_f(T)$$

& $g_b = \sum_i g_i$ over relativistic bosons

$g_f = \sum_i g_i$ " " fermions

Pressure P is $P = \frac{1}{3}\rho(T)$

Reasons for g_* to change as T changes:

1) Species changing from relativistic species to non-relativistic species. This is determined by the mass of the species as compared to the temperature T. As Universe evolves & T drops, more & more species become non-relativistic.

2) Particle-Antiparticle annihilations also decrease the effective number of degrees of freedom g_*. Thus for example, e^- & e^+ annihilate with each other through the reaction:

$$e^+ + e^- \rightarrow \gamma + \gamma$$

and this leads to a decrease in g_*.

Let's pick up History of Universe where we had last left it - after the end of Quantum Gravity era and the end of grand unification which happens at $T \sim 10^{14} - 10^{16}$ GeV. There is no dramatic change between 10^{14} GeV and 10^{3} GeV [Meaning $g_* =$ Constant during this period & $\rho \propto T^4$ & $\rho \propto R^{-4}$ & $R \propto T^{-1}$ as discussed earlier]

strongly interactive particle

For $T > m_t =$ mass of the Top quark $\simeq \overset{175}{\cancel{176}}$ GeV
(Heaviest particle in Standard Model)
all known particles are relativistic.

We added up their degrees of freedom to obtain:

$$g_b = 28$$
$$g_f = 90$$
$$\Rightarrow g_* = g_b + \frac{7}{8} g_f = 106.75$$

By $T \sim 100$ GeV, top quark annihilation happens. This means the degree of freedom corresponding to t & $\bar{t}$ disappear from the sum for g_*. The number of degrees of freedom that disappear at $T \sim 100$ GeV is:

$$g_* = \frac{7}{8} \times 2 \times 2 \times 3 = 10.5$$

3 colours

fermions

2 species

top & anti-top

spin $\frac{1}{2}$

up & down

Thus, number of degrees of freedom decreases by 10.5 after top quark and top (t & $\bar{t}$) annihilate. Since we started out with 106.75 degrees of freedom. After t-$\bar{t}$ annihilations ($T \sim 100$ GeV), the effective number of degrees of freedom is:

$$g_* = 106.75 - 10.5 = 96.25$$

The next set of particles to annihilate are the ~~Higgs~~ Higgs Boson and the Electroweak Gauge Bosons $W^{\pm}, Z^0$

The corresponding degrees of freedom which disappear after annihilation are:

$W^{\pm}, Z^0$: $3 \times 3 = 9$

No. of particles

spin = 1

Higgs Boson

H^0 : $1 \times 1 = 1$

# of particles

spin = 0

All Bosons : Total of 10 degrees of freedom disappear after Higgs Boson & Electroweak Gauge Bosons annihilate.

$W^{\pm}, Z^{0}$ annihilate at $T \sim 80$ GeV. After $T < 80$ GeV, we have $g_* = 96.25 - 10 = 86.25$ degrees of freedom left. Next event is the annihilation of the bottom & anti-bottom (b & $\bar{b}$) quark annihilations. This happens at ~~$T \sim 4.6$~~ $T \sim 4$ GeV.

Number of degree of freedom g_* that disappear:

$$\cancel{g_* = \frac{7}{8} * \frac{2}{7}} \qquad g_* = \frac{7}{8} \times 2 \times 2 \times 3 = 10.5$$

Fermions, 2 species, Spin 1/2, colours

Thus, after b-$\bar{b}$ annihilations for $T < 4$ GeV we have

~~g_*~~ $g_* = 86.25 - 10.5 = 75.75$

At $T \sim 3$ GeV both the charm-anti-charm (c & $\bar{c}$) quark annihilations as well as the τ^--τ^+ lepton annihilations occur: The number of degrees of freedom that disappear are:

c-$\bar{c}$: $g_* = \frac{7}{8} \times 2 \times 2 \times 3 = 10.5$

Fermion, species, spin, colours

τ^--τ^+: $g_* = \frac{7}{8} \times 2 \times 2 = 3.5$

total $= 14.0$

↳ don't have colour degrees of freedom

After c-$\bar{c}$ & ~~Φ~~ $\tau^- \tau^+$ annihilations for $T < 1$ GeV we have

$$g_* = 75.75 - 14.0 = 61.75$$

Next event: Quark Hadron Phase Transition

This is a major transition

Quark Hadron Phase Transition

Quarks & gluons → Baryons : n, p
($q, \bar{q}, \ldots$)
&
Mesons : $\pi^{+}, \pi^{-}, \pi^{0}$

In the quark hadron phase transition, the quarks get bound together to form the hadrons. Thus, for example, the (uud) quarks get bound together to form the proton (p) & (udd) quarks get bound together to form ~~that~~ neutron (n) : Similarly, u & d quarks get bound together to form the $\pi^{\pm}, \pi^{0}$ mesons. Before Phase transition you have the ~~quark~~ Quark gluon Plasma. After Phase Transition, you've Hadrons, in the form of a Hadron gas. The lightest Hadrons are the proton (p), neutron (n) & the pions ($\pi^{\pm}, \pi^{0}$). Of all the Hadrons, the only ones that are relativistic after the Quark Hadrons Phase Transition are the pions ($\pi^{\pm}, \pi^{0}$).

After the Quark Hadron Phase transition, the only particle ~~space~~ ~~spieces~~ species left in larger numbers are the pions, muons, electrons, neutrinos & the photons. We will thus count the no. of degrees of freedom corresponding to these particles to determine g_* after the Quark gluon Phase Transition