# Quantum Computation & Quantum Information

Room No. : 2011

3 quizzes → 25%
Mid-term → 25%
End-term → 50%

{ IBM-Q

On sub-atomic level
+
Newton's law fail.

→ Google
→ IBM
→ AirBus Quantum challenge

Q To what extent Newton's law are right?

~~Could~~

↳ Quantum Mechanics being used even outside atomic level.

Machine Computers → Computers Nowadays
Fundamentally same

→ { DNA computation
Peptide computation }
Fundamentally same

Classical computers (Machine Computers) → Quantum computers
Fundamentally different

{ Quantum computers on cloud? By IBM

~~Rea~~

Realization that physics of universe is not entirely based on Newtonian physics, but quantum mechanics

{ Feyman & Manen }

→ Basis of Quantum computation

1995 → Peter Shor → Algorithm

Q Gilbert asked, "Can All mathematic problems be solved by algorithms?"

↓

Turing in Answer

↳ Turing Machine Model

Computational Complexity

Q Moore's law & its limitations in regard of quantum mechanics?

(Reducing size of device beyond a threshold quantum mechanics has to work)

⇒ Church Turing Thesis → Any problem that can be solved on some computational model it can be solved on Turing Machine Model too

↳ Later it was proved wrong

Machine

↙ Deterministic Machine

↓

output is obtained

↘ Probablistic Machine

Problems related to quantum computation?

(i) The basic unit of quantum info whose state is unknown can not be copied.

Noise → Disturbance needed to run a system

But Noise has to be protected too from the system

} Contradictory statements

Analog Computation?

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_{app}$$

(Electric system)

Or

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = F$$

(Mechanical system)

* Another example

Allow the ball to be steady at equilibrium position

↳ Analog computation examples

* **Classical Computation :-**

based on 'classical mechanics'

Finite state machine (FSM)
PDA ?
Turing Machine (TM)

} Mathematical models, practically not applicable

Turing Machine → has following components :

Software ↳

(i) A finite set $S$ (alphabet)

(ii) An element $\sqcup \in S$ (Blank symbol)

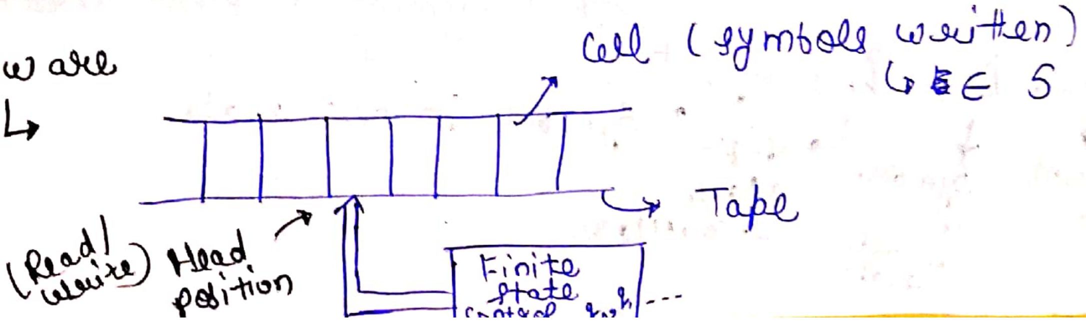
(iii) A set $A \subset S$ (external alphabet) such that $\sqcup \notin A$

(iv) A finite set $Q$ whose elements are the state of a turing machine

(v) An initial state $q_0 \in Q$

(vi) A Transition function $\delta : Q \times S \rightarrow Q \times S \times \{-1, 0, 1\}$

Hardware ↳



Two inputs ⇒ $q \in Q$
$S_p \in S$

$\delta(q, S_p) = (q', S_p', \Delta P)$

Transition function

(For each new turing machine, this function will have to be evaluated differently)

$\Delta P \in \{-1, 0, +1\}$

$-1 \rightarrow$ Head moving left by one unit
$0 \rightarrow$ Head not moving
$+1 \rightarrow$ Head moving right by one unit

Turing machine takes up the inputs. Based on those inputs, searches for transition functions out of the list & computes output.

$$\langle \underbrace{\alpha, \sqcup, \sqcup, \sqcup, \ldots}_{S} ; \underset{\uparrow \text{Head}}{0} ; \underset{\text{Internal state}}{q_0} \rangle \text{ (Initial state, here)}$$

$$\langle S_0, S_1, \ldots S_{p-1}, S_p, S_{p+1} ; P + \Delta P ; q' \rangle$$

Q Suppose we are given a string of 0's & 1's and we want to replace by $1 \sqcup \sqcup \sqcup \ldots$

Ans

$S = \{0, 1, \sqcup, \triangleright\}$

↳ special symbol indicating beginning of string

We also define a special state $q_H$, called halting state
(Machine stops when in state $q_H$)

$\delta(q_i, S_i) = (q_{i+1}, S_i', \Delta P)$

$(q_0, \triangleright, q_1, \triangleright, +1)$

$q_0$ → Current state
$\triangleright$ → Symbol in current state
$q_1$ → Next state encounter
$\triangleright$ → Symbol in next state
$+1$ → Movement in right direction (of head)

$(q_0, \triangleright, q_1, \triangleright, +1)$
$(q_1, 0, q_1, \sqcup, +1)$
$(q_1, 1, q_1, \sqcup, +1)$
$(q_1, \sqcup, q_2, \sqcup, -1)$
$(q_2, \sqcup, q_2, \sqcup, -1)$
$(q_2, \triangleright, q_3, \triangleright, +1)$
$(q_3, \sqcup, q_H, 1, 0)$

Note :- Inputs are not changed simultaneously, but one by one
(if $q_i \to q_{i+1}$, $S_p \to$ same
if $S_p \to$ changes $q_i \to q_i$ )

Q Design a turing machine which will compute the parity of a binary string that is input to the machine. It writes '0' if the parity is even & writes '1' if the parity is odd.

* Computable function :- A function 'f' : $A^* \to A^*$ is called if $\exists$ a turing machine 'M' such that the function '$\phi_M$' that computes is equal to 'f' for every $x \in A^*$

$$\phi_M(x) = f(x) \quad \forall\ x \in A^*$$

$A^* \to$ Set of all strings over A

Church - Turing Thesis :- The class of functions computable by a turing machine corresponds exactly to the class of functions which are regarded as being computable by an algorithm.
(empirical statement → proof can not be found)

## Algorithm $\Leftrightarrow$ Turing Machine

or,

As soon as an algorithm of a problem is found, its turing machine exists too (can be found)

Universal turing machine? → Programmable

↳ $U([M], x)$

↓

'Halting problem' → solved by Turing using Turing Machine model

Computational complexity :- → Intrinsic efficiency of an algorithm

In Turing machine → Movement of head (Physics involved)

Classical mechanics → mass of object, force, initial position

⇓

final position can be computed

↓

Deterministic

Non-classical mechanics → Probabilistic (output in terms of probability)

↓

Definitive probabilistic? (What is that?)

→ To compare complexity → a common 'reference' is must (Time, space occupied..)

↳ Important parameters to decide complexity

Variants of Turing Machine:→

(i) Multi-tape TM
(ii) Probabilistic TM
(iii) Non-deterministic TM

Q Now, which Turing Machine to choose for solving problems?

* Complexity Classes:-

Coarse-based classification

The classification will be based on 'Asymptotic behavior' $(n \to \infty)$

{n = No. of inputs}

e.g: $2n^3 + 4n + 3$

Asymptotic Notation:-

(i) O (big O notation) → worst case scenario (sets upper bound for no. of resources used)

(ii) $\Omega$ (Omega notation) → Best-case scenario (sets lower bound for ~~resource~~ resource requirement)

→ If $f(n)$ & $g(n)$ are two functions, one says $f(n)$ is in class $O(g(n))$ if $\exists$ constant $c$ & $n_0$ such that for all $n > n_0$:

$$f(n) \leq c\, g(n)$$

→ And, $f(n)$ is in class $\Omega(g(n))$ if $\exists$ constants $c$ & $n_0$ such that for all $n > n_0$:

$$f(n) \geq c\, g(n)$$

(iii) $\Theta$ (Theeta) → $f(n)$ is $\Theta(g(n))$ if $f(n)$ is both $\Omega(g(n))$ & $O(g(n))$

i.e. $f(n) \sim c\, g(n)$ for all $n > n_0$ ($c \to$ constant)

e.g. Let $f(n) = n^2$ & $g(n) = \log n$

→ $f(n)$ is $\Omega(g(n))$

Note :- It is clear that if $f(n)$ is $O(g(n))$ then $g(n)$ is $\Omega(f(n))$

e.g. $f(n) = n^3$
$g(n) = 2^n$

→ $f(n)$ is $O(g(n))$

* The class of functions computable in polynomial time by a TM is called $\boxed{\text{class 'P'}}$

→ If we move from one variant to other variant, the class 'P'. & The change is by a polynomial to bring its time complexity to be again a polynomial &

→ The space requirement for problems in class P is obviously a polynomial in the size of input.

A function (decision problem) F is said to be computable / decidable in polynomial space if ∃ a TM that computes / decides F & runs in space $S(n) = poly(n)$

The class of all such problems is called 'PSPACE'

Note :- It is clear that $\boxed{P \sqsubseteq PSPACE}$

Q Prove that $f(n)$ is $O(g(n))$ iff $g(n)$ is ~~[illegible]~~ $\Omega(f(n))$

Q Show that $\log(n)$ is $O(n^k)$ for any $K > 0$.

* Problems for which resource requirement grows faster than any polynomial in 'n' are (wrongly) said to belog to exponential class 'EXP'

$c^n$ → Actual exponential

$c^{\log n}$ → (wrongly termed) exponential

* NP class → decision problems {Yes or No}

Non-deterministic polynomial

⇓

Some problems are not solvable on standard turing machine in polynomial time, but solvable on Non-deterministic turing machine in polynomial time.

e.g. Factoring problem {Finding factors ...} → NP class problem

Non-deterministic machine makes random guess & checkes if this guess is the answer we're looking for.

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$\boxed{P \subseteq NP}$

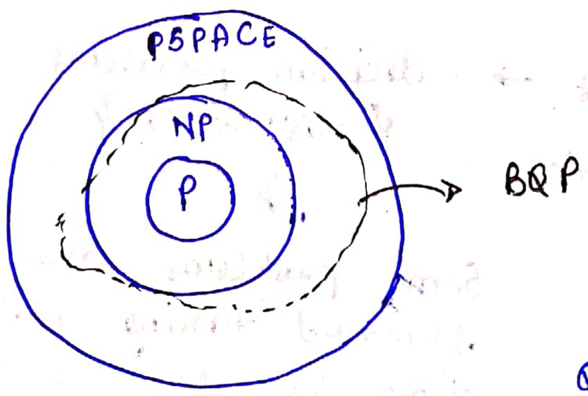
e.g. Deciding whether a number is ~~even~~ prime
Numbers or composite
Number → 'P class'

* BPP class →
Bounded error probabilistic

* BQP class →
Bounded quantum probabilistic

$$\boxed{L \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP}$$

L → Log space



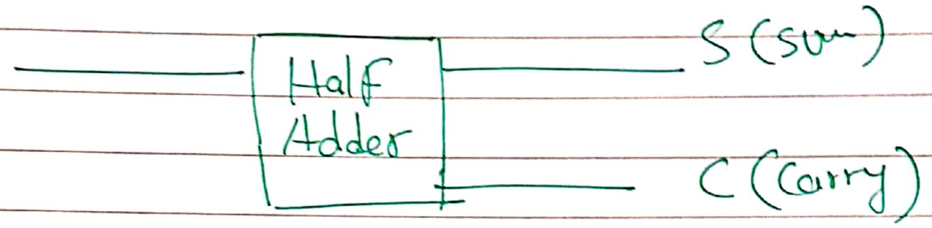

Any problem outside PSPACE Quantum computers can not solve.

Circuit wires & gates

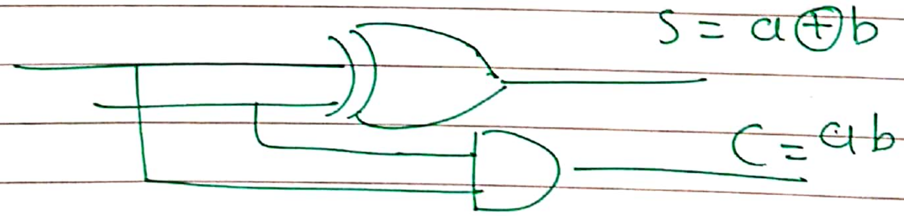
Binary Logic — $B = \{0, 1\}$

Acyclic Circuit i.e. —

e.g. Half adder



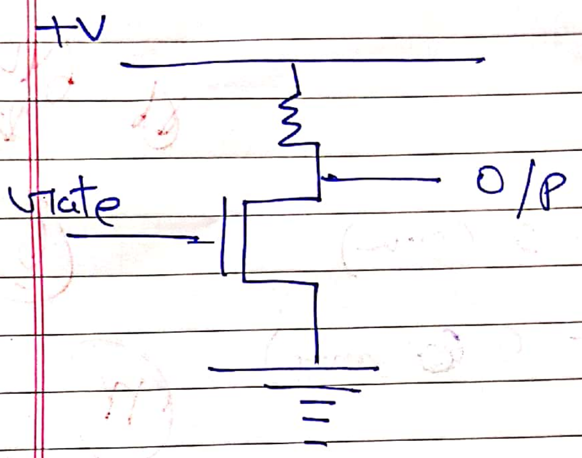

| a | b | S | C |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |



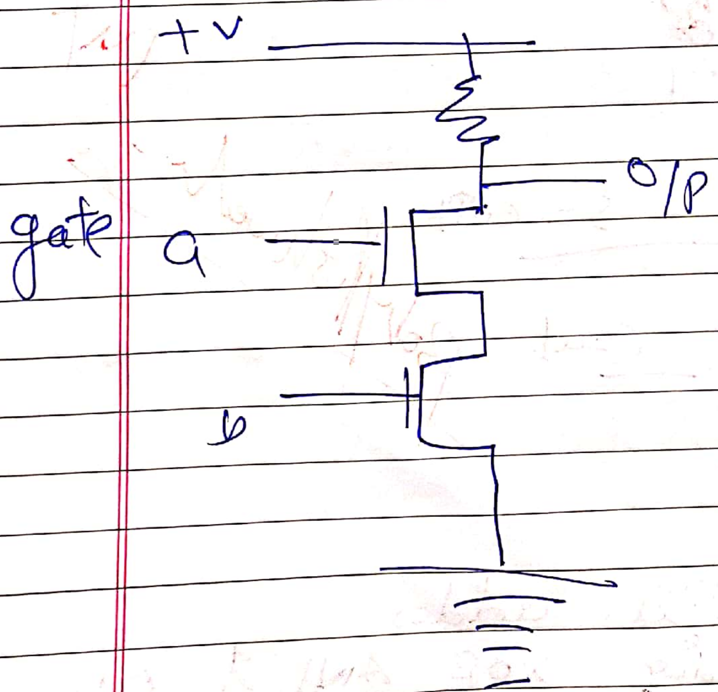

Q- Construct a full adder
(1) directly using XOR, AND & OR
(2) using half adder.

Q- ~~How many half~~

Q. How many half adder are required to add two n-bit binary no. 2 Design the Corresponding Ckt.




Not gate




NAND gate

Ex - Contruct NOR gate

– A general logic gate is a function

$$f : B^n \longrightarrow B^m$$

where $B = \{0, 1\}$

if $m = 1$ i.e. if

$f : B^n \rightarrow B^1$ then $f$ is called a Boolean function.

Completeness / basis

$\underbrace{\text{AND, OR, NOT}}$ form a complete set

$$a + b = \overline{(\bar{a}\,\bar{b})}$$

(De morgan theorem)

Note: NAND is a universal gate.

Q → Proof that AND OR & NOT form a complete set for any boolean function

Note:

Any boolean function can be represented in terms of its truth table.

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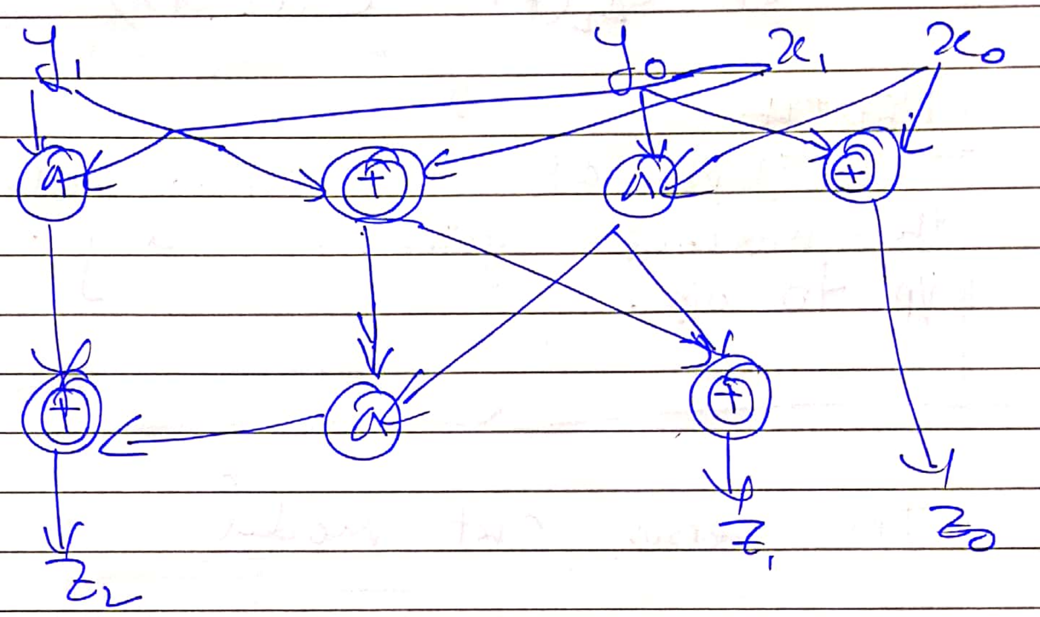
Ex - How many distinct n-variable Boolean function are possible.

Circuits can also be represented, in terms of graph (acyclic)




Check that the circuit represent binary addition of 2-bit binary no.

Size (a measure of complexity of circuit)

The minimal size of a circuit over a basis A (($\wedge$, $\oplus$) in the graph) that computes a given function f is called the circuit complexity of the function (w.r.t to basis (A)) & is denoted by $C_A(f)$

In going from one basis $A_1$ to another basis $A_2$ the circuit complexity changes only by a constant ~~con~~ multiple i.e.

$$C_{A_1}(f) = O(C_{A_2}(f))$$

and ~~ct~~ $C_{A_2}(f) = O(C_{A_1}(f))$

## DEPTH

depth of a Boolean circuit ~~d~~ is the max no. of gates on any path from i/p to o/p.

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## TM versus Ckt model

Circuit model is ~~not~~ non-uniform ~~as~~ opposed to the TM model which is uniform

Circuit family (for a particular computational problem)

Notatn : $\{C_n\}$

i/p size

## Consistent family

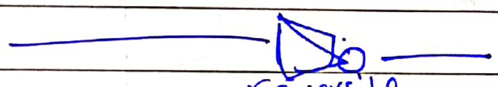
if $m < n$ & if the i/p $x$ is of size is $m$ then the o/p

$$C_m(x) = C_n(x)$$

~~$C(x)$~~

If a TM model o/p a description of $C_n \, \forall \, n \in N$ in polynomial time then the Ckt family is said to be polynomial time uniform

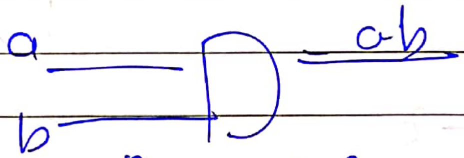
## Reversible Computation



reversible

| $x$ | $\bar{x}$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

Knowing the output you can predict input.



irreversible

information eraser

| a | b | ab |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

Note: Randauer principle ~~the~~ the energy dissipated by a computer in erasing a single bit is

$K_B T \log 2$ ("information is physical")

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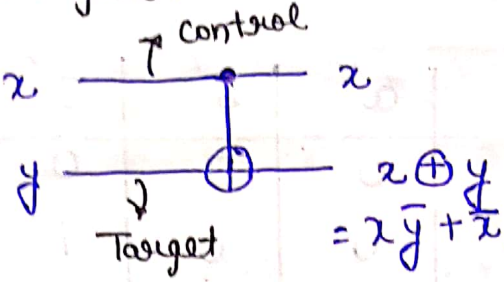
Reversible computation :-

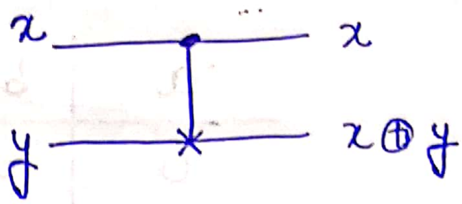
(i) NOT gate

(ii) CNOT gate :-




$= x\bar{y} + \bar{x}y$

Or,



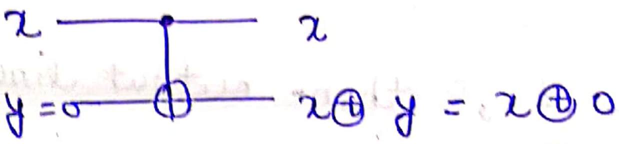

(a) If control = 0 → Target has not meaning (Target line is untouched)

(b) If control = 1, then -

output on the target line is : $1 \oplus y = \bar{y}$

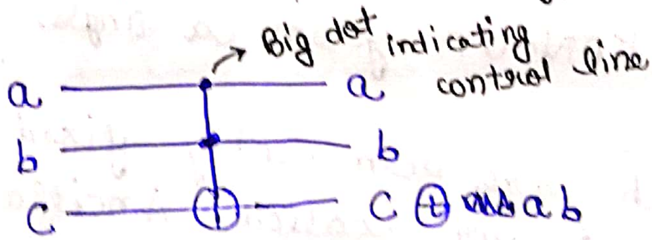
i.e. CNOT as NOT gate

(c) We also see than CNOT can implement FANOUT as well.




$x \oplus y = x \oplus 0$
$= x.$

(iii) CCNOT (Controlled controlled NOT)
{Toffoli gate}




Reversible universal

(a) If c = 0 then output line is ab
i.e. AND operation

(b) Suppose we fix $a = 1$, but leave $b$ & $c$ arbitrary, then output on target line is –

$c \oplus b \Rightarrow$ XOR gate

Truth table ↳

| a | b | c | a' | b' | c' |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 |

(c) If $a = b = 1$, then output line

$\bar{C} \rightarrow$ NOT gate

Q Construct a NAND gate using a single CCNOT gate.

Q Construct FANOUT from a single CCNOT

Note:- Bits which are permanently fixed to some definite value are called Ancilla bit

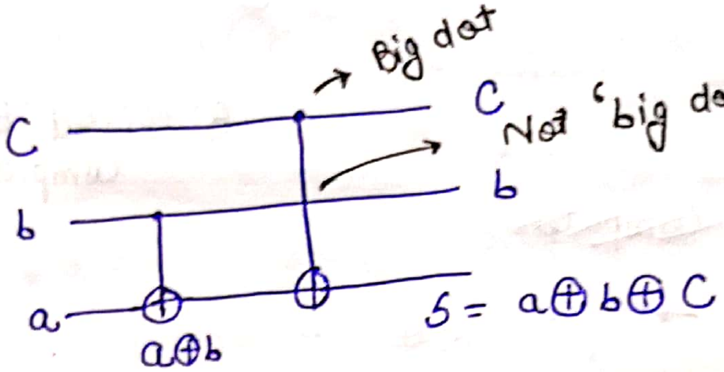
## Binary Addition using CNOT / CCNOT :-

If $a$, $b$ are bits to be summed & $c$ is a ~~pre~~ previous carry, then sum is -

$$S = a \oplus b \oplus c$$

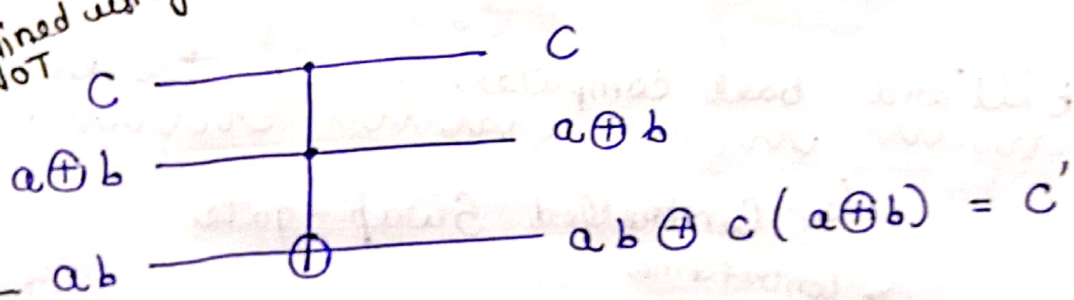
& carry $c' = ab \vee bc \vee ca$

$$= ab \oplus bc \oplus ca$$




$$c' = ab \oplus c(a + b)$$

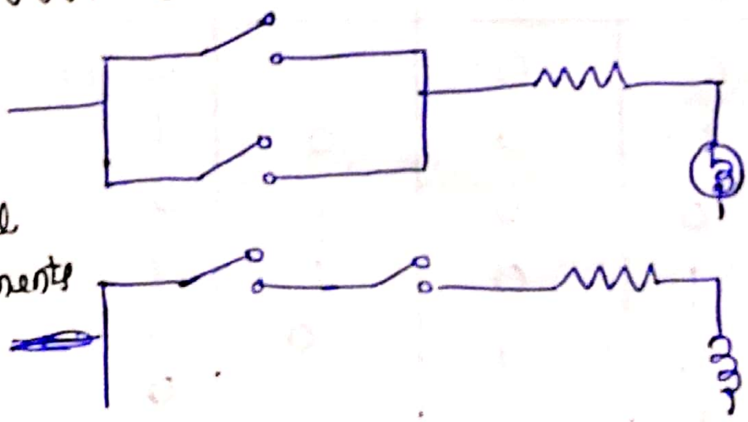
obtained using CNOT




can be obtained using CCNOT

## * Physics & Computation :-

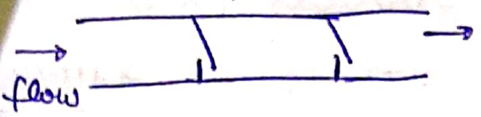
Electrical Arrangements



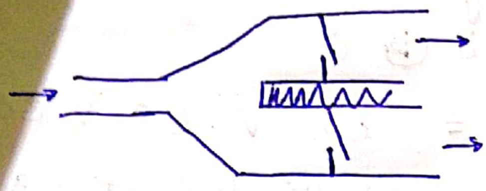
OR gate

NOT gate

Mechanical Arrangements ⇒




AND gate

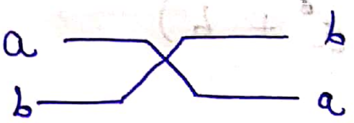


OR gate

Billiard ball Computers?

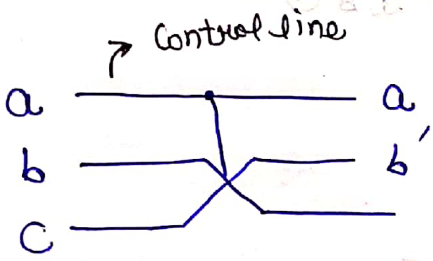
~~* Billiard ball computers~~

* Swap gate →




* Billiard ball computer or the Fredkin gate :-

↳ Controlled Swap gate



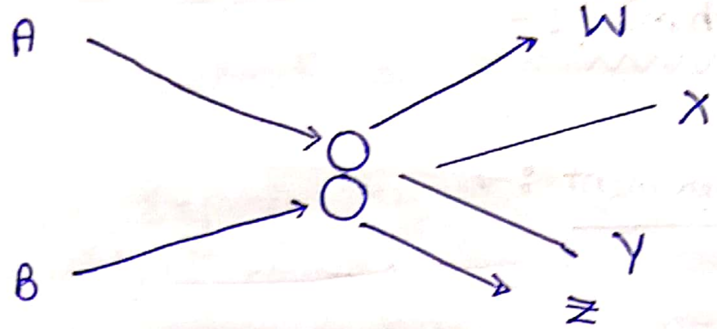

→ Universal

| a | b | c | a' | b' | c' |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

$a = 0 \Rightarrow b' = b, \; c' = c$

$a = 1 \Rightarrow b' = c, \; c' = b$

* Take example of two balls colliding :→




(a) If $A = 1, \; B = 0 \Rightarrow$ No collision

$\Rightarrow Y = 1; \; W = Z = X = 0$ ↳ $Y = A \wedge \bar{B}$

(b) If $A = 0, \; B = 1 \Rightarrow X = 1$

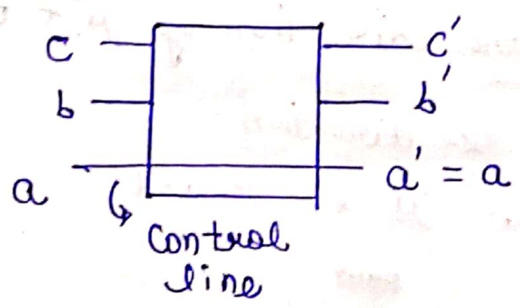
& $W = Y = Z = 0$ ↳ $X = \bar{A} \wedge B$

(c) If $A = B = 1$

$\Rightarrow W = Z = 1 \Rightarrow W = Z = A \wedge B$

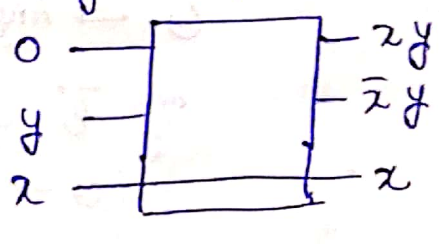
& $X = Y = 0$

Amazingly, this trivial collision is also performing computation

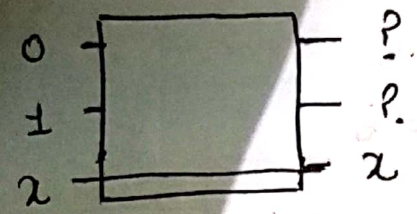



$\Rightarrow$

If we fix one input

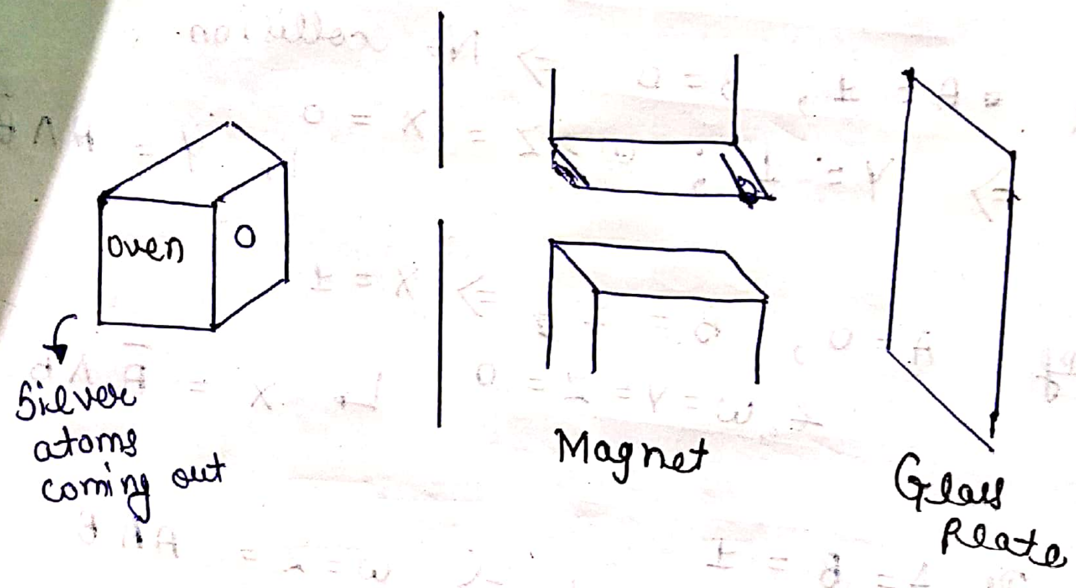


Q




## * Quantum Mechanics :-

### Stern- Gerlach Experiment :→




Every silver atom behaves like a little magnet. If it has magnetic moment $\vec{\mu}$ & the magnetic field $\vec{B}$ then change in energy -

$$\Delta U = -\vec{\mu}\cdot\vec{B}$$
$$= -\mu B \cos\theta$$

$\theta \rightarrow$ Angle b/w the direction of $\vec{\mu}$ & $\vec{B}$

If $\vec{B}$ is along z- direction

$\theta \rightarrow$ Angle b/w $\vec{\mu}$ & z-axis.

And, force $= -\underbrace{\vec{\nabla}}_{\text{gradient}} \underbrace{V}_{\text{Potential}}$

Because of sharp magnet pole the $\vec{B}$ field is not uniform & there will be a force of the Ag atoms

$$F_z = -\frac{\partial(\Delta U)}{\partial z}$$

$$F_z = \mu \frac{dB}{dz} \cos\theta$$

Expected result → a continuous line appearing on glass plate
(As per laws of classical mechanics)

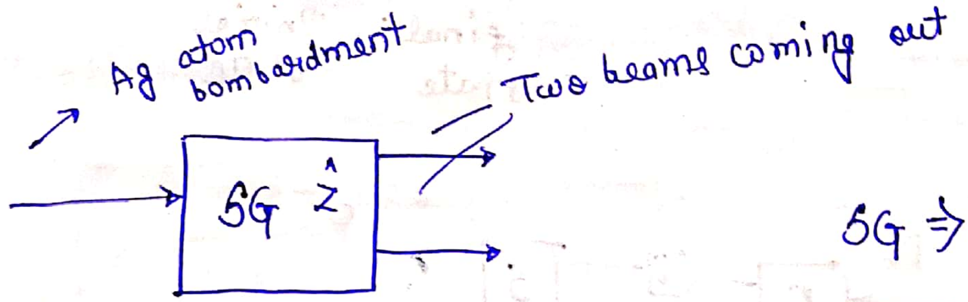
deviates Maxwell's electrodynamics law

Actual result → Two dots on both ~~extre~~ extreme

Ag atoms → One unpaired electron {responsible for this behavior}

Conclusion ↳ We say that magnetic moment & the spin of electron are quantized. And, one may say the electron has 2 possible states.

Note:- If atoms → element are changed, No. of possible states of electron are changed too.

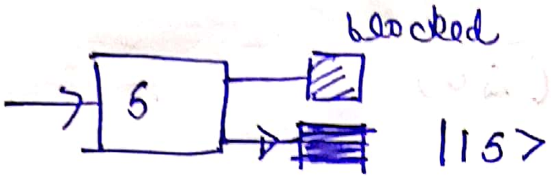


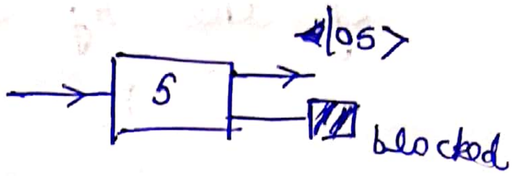

S-type
SG apparatus

SG ⇒ Stern Gerlach

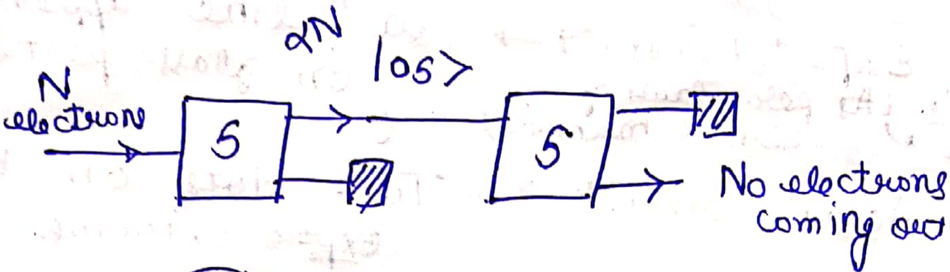
(Unpaired) Electrons coming out on the upper arm of S-type apparatus are said to be in state $|0S\rangle$

And, $e^-$ in lower arm are said to be in state $|1S\rangle$



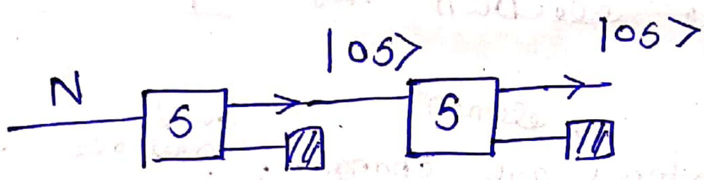

e.g. (i)




{Probability amplitude for $e^-$ in $|0S\rangle$ state to go in $|1S\rangle$ state} Probability of this happening $\langle 1S|0S\rangle = 0$

final state   Initial state

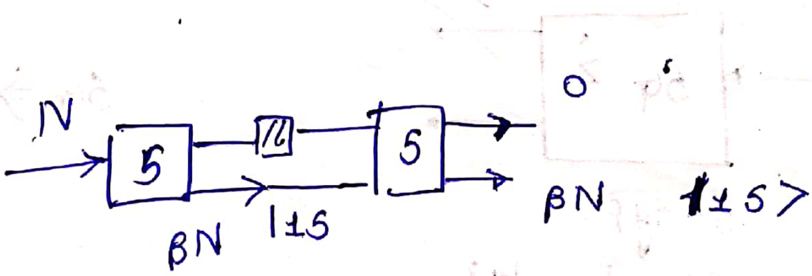
(ii)




$\langle 0S|0S\rangle = 1$

final state   Initial state

(iii)




$\therefore \quad \langle 0S|1S\rangle = 0$

$\langle 1S|1S\rangle = 1$

|  |  | Initial |  |
|---|---|---|---|
|  |  | $\vert 0S\rangle$ | $\vert 1S\rangle$ |
| Final | $\vert 0S\rangle$ | 1 | 0 |
|  | $\vert 1S\rangle$ | 0 | 1 |

In general, the notation $\langle b|a\rangle$ gives $\boxed{\text{"probability amplitude"}}$ for starting state $|a\rangle$ to be found in state $|b\rangle$

→ The probability for the process is given by the absolute square of probability amplitude

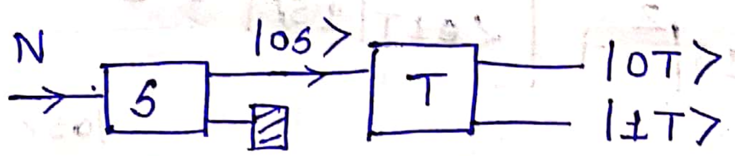
i.e. $P(|a\rangle \rightarrow |b\rangle) = |\langle b|a\rangle|^2$

$= \langle b|a\rangle \langle b|a\rangle^*$

$*$ denotes 'complex conjugate'

S-apparatus → when magnetic field aligned along z-axis

* T-apparatus → Magnetic field randomly aligned (not along z-axis necessarily)




It is found that:

$\langle 0T|0S\rangle \neq 0$

$\langle 1T|0S\rangle \neq 0$

Similarly, $\langle 0T|1S\rangle \neq 0$ & $\langle 1T|1S\rangle \neq 0$

| | | Initial | |
|---|---|---|---|
| | | $\lvert 0S\rangle$ | $\lvert 1S\rangle$ |
| Final | $\lvert 0T\rangle$ | $\langle 0T\lvert 0S\rangle$ | $\langle 0T\lvert 1S\rangle$ |
| | $\lvert 1T\rangle$ | $\langle 1T\lvert 0S\rangle$ | $\langle 1T\lvert 1S\rangle$ |

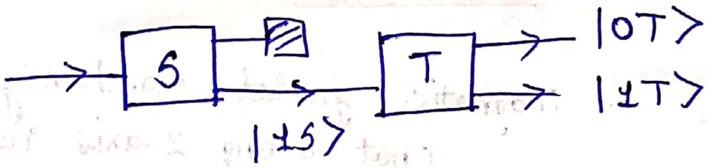
If no $e^-$ are created or absorbed, then the no. of $e^-$ coming out of T-type arrangements with both output channels $|0T\rangle$ & $|1T\rangle$ open is same as no. of $e^-$ entering the T-type apparatus in the $|0S\rangle$ state

$\downarrow$

∴ Conservation of $e^-$ no. or probability conservation

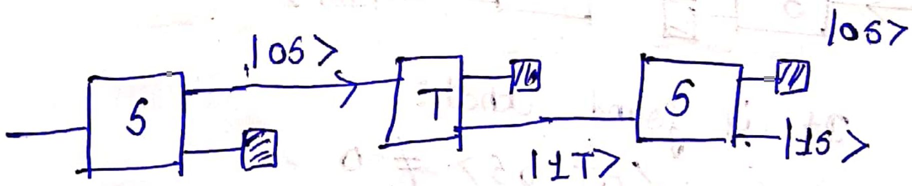
$$|\langle 0T|0S\rangle|^2 + |\langle 1T|0S\rangle|^2 = 1$$

e.g. What is the corresponding equation for the arrangements ?




$$|\langle 0T|1S|^2 + |\langle 1T|1S\rangle|^2 = 1$$

*

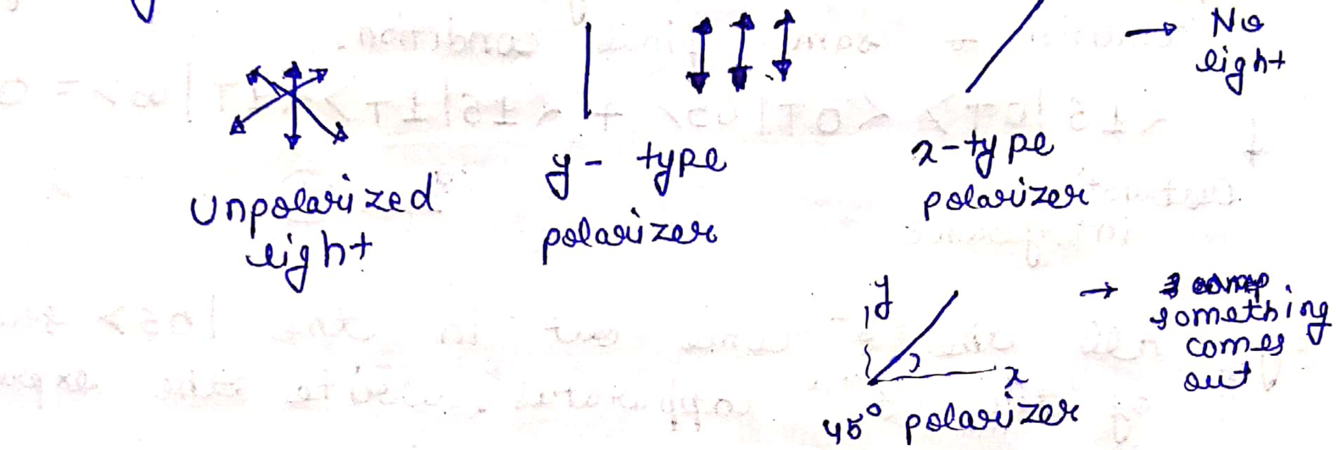



We find that the output from $|1S\rangle$ channel of the 3rd apparatus is not zero.
Likewise, the output from the $|0S\rangle$ arm is, in general, non-zero.

Explanation :→

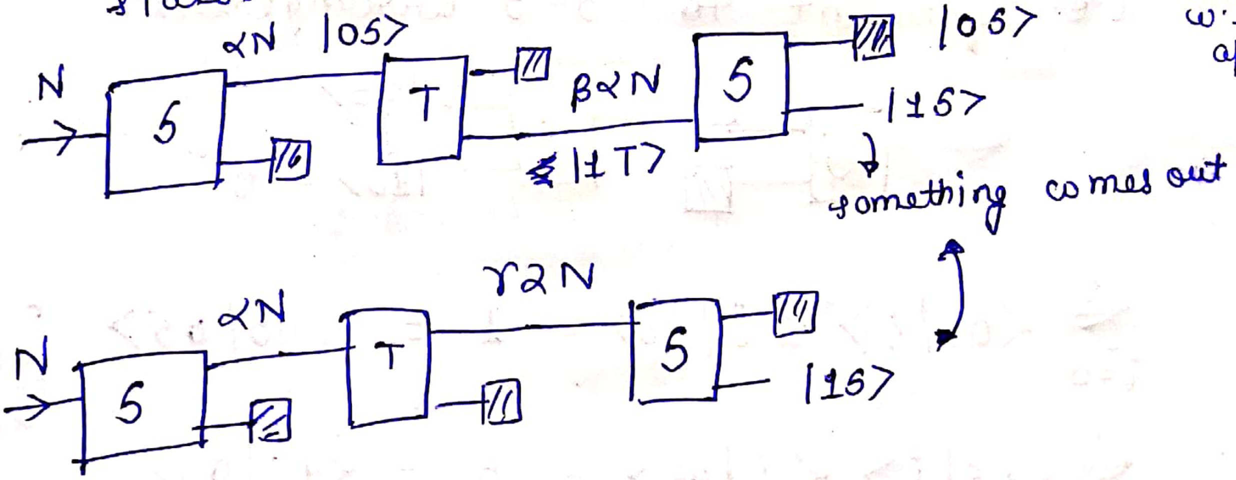
Selection of $|\mathbb{1}T\rangle$ state by 2nd apparatus completely destroys any information about the previous state of the $e^-$ in the previous case {namely $|0S\rangle$}

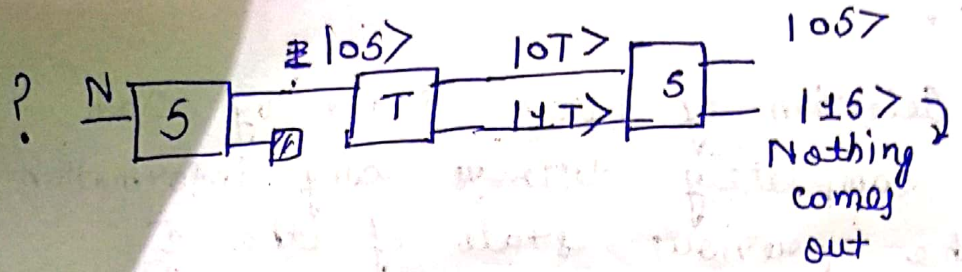
→ Something similar happens in polarization of light.




* **Base states :–** We say that filtering process separates the system in base states (w.r.t. that particular apparatus) states of the system in a of the future behavior of the system depends only on that particular base state state.

→ $|0S\rangle$, $|\mathbb{1}S\rangle$ are base states w.r.t. S apparatus






(why?)

Probability amplitude interfere if alternative paths are available for the same starting condition & same final condition.

$$\langle 1S|0T\rangle\langle 0T|0S\rangle + \langle 1S|1T\rangle\langle 1T|0S\rangle = 0 \quad \text{—} \; (1)$$

Destructive interference

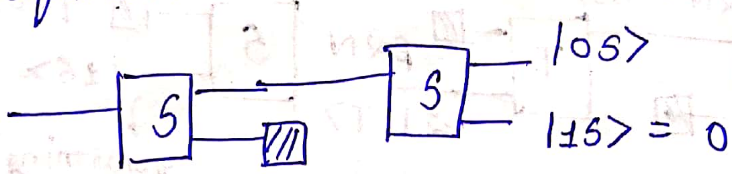
e.g. All the $e^-$ come out in the $|0S\rangle$ state of the 3rd apparatus. Write the expression similar to (1) to describe this.

$$\langle 0S|0T\rangle\langle 0T|0S\rangle + \langle 0S|1T\rangle\langle 1T|0S\rangle = 1 \quad \text{—} \; (2)$$

Constructive interference

Note:- Interestingly, a $\boxed{\text{wide open}}$ T-apparatus is equivalent to the apparatus being present not at all.

i.e. equivalent to S-S arrangement.




$$\sum_{i=0}^{1} \langle 0S|i\rangle\langle i|0S\rangle = 1 = \langle 0S|0S\rangle$$ Initial state & final state same

$$\sum_{i=0}^{1} \langle 1S|i\rangle\langle i|0S\rangle = 0 = \langle 1S|0S\rangle$$ Initial & final state different

Nothing special about intermediate wide open $|0S\rangle$ & $|1S\rangle$ states.

$$\langle x|p\rangle = \sum_{i=0}^{1} \langle x|i\rangle\langle i|p\rangle$$

$\left\{ \text{Jokingly, } \sum_{i=0}^{1} |i\rangle\langle i| = 1 \right\}$

Note :- $\langle j|i\rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ $\quad i, j \rightarrow$ Bases States

Here, $|i\rangle$ & $|j\rangle$ are states w.r.t. same SG apparatus.

$\langle j|i\rangle = \delta_{ji}$ $\left\{ \underline{\text{Kronecker}} \text{ Delta Symbol} \right\}$

So, $\delta_{ji} = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

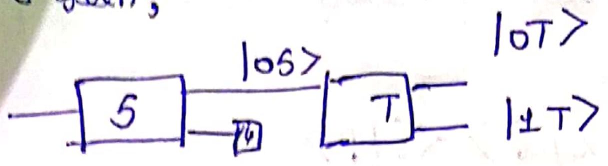
$\delta_{00} = 1 = \delta_{11}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\delta_{01} = 0 = \delta_{10}$

Note:- $e^-$ has momentum & position states as well but in quantum computation magnetic spin state of $e^-$ is basically helpful

Duality of $e^- \rightarrow$ particle & Wave nature

We have seen,




$$\langle 0T|0S\rangle \langle 0T|0S\rangle^* + \langle 1T|0S\rangle \langle 1T|0S\rangle^* = 1 \quad \text{— (3)}$$

Comparing eq$^n$ (2) & (3)

$$\boxed{\begin{aligned}\langle 0T|0S\rangle^* &= \langle 0S|0T\rangle \\ \langle 1T|0S\rangle^* &= \langle 0S|1T\rangle\end{aligned}}$$

Summary :-

(i) $\langle j|i\rangle = \delta_{ij}$

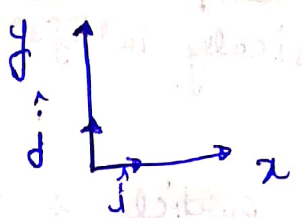
$$\delta_{ij} = \delta_{ji} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

where $|i\rangle$ & $|j\rangle$ are base states w.r.t. a particular S-G apparatus.

(ii) $\sum_{i=0}^{1} \langle x|i\rangle \langle i|\phi\rangle = \langle x|\phi\rangle$

(iii) $\langle x|\phi\rangle = \langle \phi|x\rangle^*$

Consider 2-D Euclidean space $R^2$




$$\hat{i}\cdot\hat{i} = 1 = \hat{j}\cdot\hat{j}$$
$$\hat{i}\cdot\hat{j} = 0 = \hat{j}\cdot\hat{i}$$

→ Compare it summary (i)

It seems that $\hat{i}$ & $\hat{j}$ are in some sense similar to $|\hat{i}\rangle$ & $|\hat{j}\rangle$ {the base states of $e^-$}

$$\vec{v}\cdot\vec{\omega} = v_x\omega_x + v_y\omega_y$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

$$v_x = v\cos\theta_1 = \vec{v}\cdot\hat{i}$$

$$v_y = \vec{v}\cdot\hat{j}$$

$$\omega_x = \vec{\omega}\cdot\hat{i}$$

$$\& \quad \omega_y = \vec{\omega}\cdot\hat{j}$$

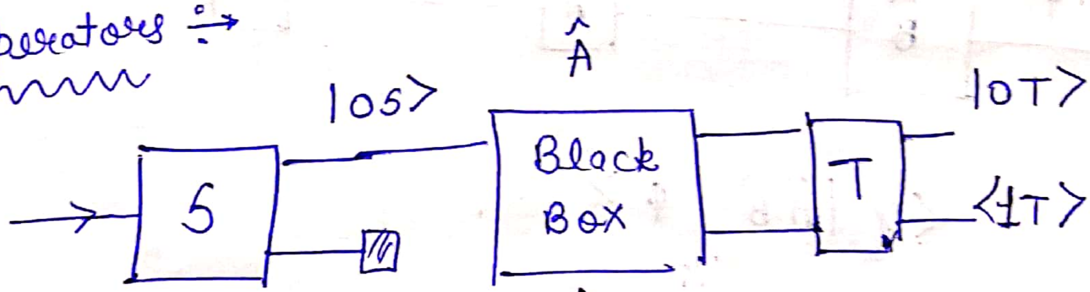
$$\vec{v}\cdot\vec{\omega} = (\vec{v}\cdot\hat{i})(\hat{i}\cdot\vec{\omega}) + (\vec{v}\cdot\hat{j})(\hat{j}\cdot\vec{\omega})$$

Compare it with (ii)

And, $\cancel{\vec{v}\cdot\hat{j} = \hat{i}\cdot\hat{j} = (\hat{i}\cdot\hat{j})}$ $\quad \vec{v}\cdot\hat{i} = \hat{i}\cdot\vec{v} = (\hat{i}\cdot\vec{v})^*$

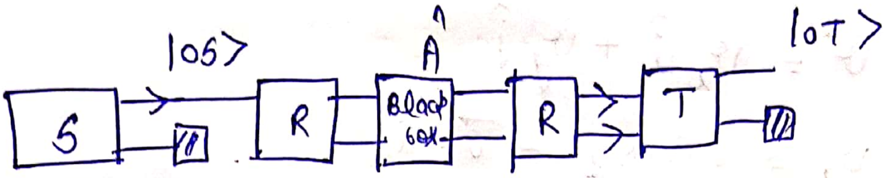
Conjugate of real no. is that real no. itself.

Operators :-




affects the state of $e^-$ in some way

$$\langle OT | \underbrace{\hat{A}}_{\text{operator}} | OS \rangle$$

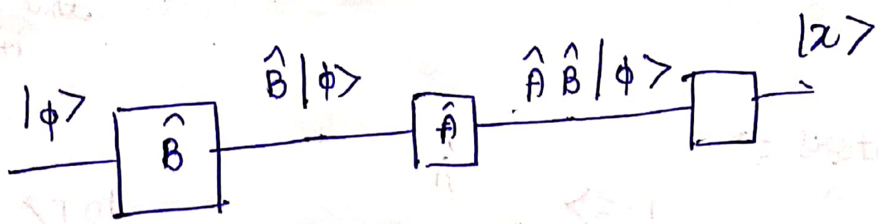



$$\langle \chi | \hat{A} | \phi \rangle = \sum_{i,j} \langle \chi | j \rangle \langle j | \hat{A} | i \rangle \langle i | \phi \rangle$$

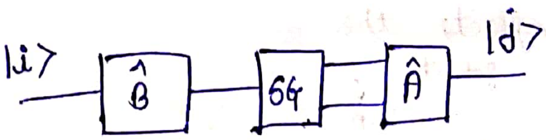
$$\sum |i\rangle\langle i| = 1$$

| | | Initial | |
|---|---|---|---|
| | | $\|0\rangle$ | $\langle 1\rangle$ |
| Final | $\|0\rangle$ | $\langle 0\|\hat{A}\|0\rangle$ | $\langle 0\|\hat{A}\|1\rangle$ |
| | $\|1\rangle$ | $\langle 1\|\hat{A}\|0\rangle$ | $\langle 1\|\hat{A}\|1\rangle$ |

The operator $\hat{A}$ is completely described by giving ~~there~~ these 4 notations.




$$\langle \chi | \hat{A}\hat{B} | \phi \rangle$$




$$\langle \underbrace{j}_{\text{Base State}} | \hat{A}\hat{B} | \underbrace{i}_{\text{Base State}} \rangle = \sum_{K=0,1} \langle j | \hat{A} | K \rangle \langle K | \hat{B} | i \rangle$$

$$= \left( \quad \right) \left( \quad \right)$$

matrix (look at tables initial-final) matrix

$$\langle j S | \phi \rangle = \sum_{i=0}^{1} \langle j S | i T \rangle \langle i T | \phi \rangle$$

S - language

S - T trans-formation (changing)

* Linear Algebra (Vector Spaces) :-

Vector space :- A vector space is a set S whose elements are called vectors {denoted by '$| \rangle$'} with following 2 operations defined :

(i) Vector addition → If $|a\rangle, |b\rangle \in S$ then $|a\rangle + |b\rangle \in S$

(ii) Multiplication by scalar → Multiplication of a vector by a scalar (complex no. for us) if

If $\alpha \in \mathbb{C}$ (complex No.)

& $|a\rangle \in S$, then -

$\Rightarrow \alpha |a\rangle \in S$

Note :- Because the scalars for us belong to $\mathbb{C}$, the vector space is a complex vector space; if scalars $\in \mathbb{R}$ then vector space is called Real vector space.

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In addition,

(i) S consists of a special vector, the zero vector ($0_v$) such that -

$$|a\rangle + 0_v = |a\rangle \quad \forall\ |a\rangle \in S$$

↳ Identity element

(ii) For every $|a\rangle \in S$, there exists a $|a'\rangle \in S$ such that -

$$|a\rangle + |a'\rangle = 0_v$$

↳ Additive element

(iii) Vector addition is commutative

$$|a\rangle + |b\rangle = |b\rangle + |a\rangle$$
$$\forall\ |a\rangle, |b\rangle \in S$$

and Associative

$$|a\rangle + (|b\rangle + |c\rangle) = (|a\rangle + |b\rangle) + |c\rangle$$

(iv) Scalar multiplication follows usual commutative & associative & distributive property

For $\alpha, \beta \in \mathbb{C}$

→ $\alpha(\beta|a\rangle) = (\alpha \cdot \beta)|a\rangle$

→ $(\alpha+\beta)|a\rangle = \alpha|a\rangle + \beta|a\rangle$

→ $\alpha(|a\rangle + |b\rangle) = \alpha|a\rangle + \alpha|b\rangle$

Q. Is the set of all quadratic polynomials over real no. a vector space?

$$P(x) = ax^2 + bx + c$$

* Linear independence of vectors :- A set of vectors $|V_1\rangle, |V_2\rangle, \ldots |V_N\rangle$ is said to be linearly independent if a relation $\sum_{i=1}^{N} \lambda_i |V_i\rangle = 0$ — (L) necessary imply $\lambda_i = 0 \; \forall i$

On the other hand, if (L) holds with at least two $\lambda_i$'s non-zero then the set of vectors is linearly dependent.

e.g. In $\mathbb{R}^2$, $a_x \hat{i} + a_y \hat{j} = 0 \Rightarrow$ imply $a_x = 0 = a_y$

↳ $\hat{i}$ & $\hat{j}$ are linearly independent.

Dimension of a Vector space :→ is max. no. of linearly independent vectors in that space.

Span :- A sub-set of vectors in S with the property that any vector in 'S' can be expressed in terms of vectors in the sub-set is said to ~~span~~ 'Span S'.

e.g. If $|V_1\rangle, |V_2\rangle, \ldots |V_N\rangle \in S$ & satisfy the property than any $|a\rangle \in S$ can be written as $|a\rangle = \sum_{i=1}^{N} \alpha_i |V_i\rangle$ then the set $|V_i\rangle$ span S.

**Basis :-** A set of linearly independent vectors in 'S' which spans S is said to form a basis for S.

* **Postulates of Quantum Mechanics :-**

(i) Any isolated (or closed) physical system has an associated complex vector space also called the state space of the system. The state of the system is described by a vector in the state space (the state vector)

→ Dimension of state space depends on the system under consideration.

$|a\rangle \rightarrow$ $\boxed{\text{Ket Vectors}}$

And, state space is called Ket space.

→ If $|a\rangle$ & $|b\rangle$ are any two ket vectors, then their sum is another ket vector.

$|a\rangle + |b\rangle = |c\rangle$

Similarly, $\alpha|a\rangle \; (\alpha \in \mathbb{C})$ is also ket vector.

**Note :-** In quantum mechanics, $|a\rangle$ & $\alpha|a\rangle$ describe the same state.

(why?) i.e. Scaling a vector does not change its state

✓ $\{\alpha \neq 0\}$

→ If we multiply $|a\rangle$ by a zero number, then we get a zero vector

$$0 \cdot |a\rangle = 0$$

* Linear superposition → A more general linear combination of state vectors will be –

$$\alpha|a\rangle + \beta|b\rangle \ ; \ \alpha, \beta \in \mathbb{C}$$

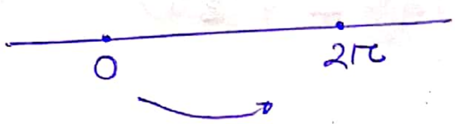
and we say that this represents linear superposition of $|a\rangle$ & $|b\rangle$.

(ii) Observables (for e.g. position, momentum, spin etc.) are represented by linear operators on the vector space

{operators denoted by $\hat{O}$, $\hat{A}$ etc.}

→ Operators act on ket vectors from the left & the result is $\hat{A}(|c\rangle) \equiv \hat{A}|c\rangle$ another ket.

* Distinction b/w vector space & state space

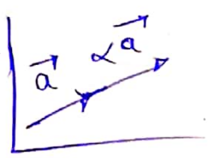



To form a circle

$\pi$, $3\pi$ are distinct vectors in vector space

But $\pi$, $3\pi$ are same vectors in state space

Because they represent same state.




$\vec{a}$ & $\alpha\vec{a}$ are distinct vectors in vector space, but not in state space.

→ Sometimes the action of an operator on a vector gives back same vector multiplied by a number.

$$\hat{A}\,|a\rangle = \alpha\,|a\rangle$$

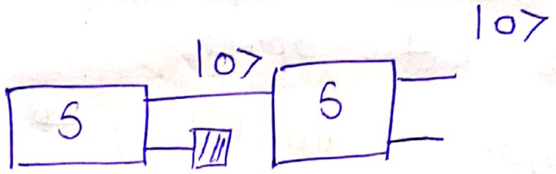
In such a case we say that →

$\hat{A}$ → Eigen Vector

$\alpha$ → Eigen Value

e.g. (i) Suppose you place a vector along z
so now if you rotate (Rotation operator applied) about z-axis, its state the vector won't change.

(ii)




state does not change

$$S_z|0\rangle = \frac{\hbar}{2}|0\rangle$$

* Operators Algebra :-

(1.) If $\hat{X}$ & $\hat{Y}$ are any 2 operators then
$\hat{X} + \hat{Y} = \hat{Y} + \hat{X}$ is also an operator.

$(\hat{X} + \hat{Y})\,|a\rangle = \hat{X}\,|a\rangle + \hat{Y}\,|a\rangle$

(2.) Operators $\hat{A}$ & $\hat{B}$ are said to be equal
$(\hat{A} = \hat{B})$ iff :

$$\hat{A}\,|\alpha\rangle = \hat{B}\,|\alpha\rangle$$

(3.) If $\hat{X}\,|\alpha\rangle = 0,\ \forall\ |\alpha\rangle \in S$, then $\hat{X}$ is called the null operator.

* Linearity of operator →

$$\hat{X}(\alpha|a\rangle + \beta|b\rangle) = \alpha\hat{X}|a\rangle + \beta\hat{X}|b\rangle$$

Quiz : 11/01/2019
[illegible]
12 30PM
Venue - LT-3

(4.) If $\hat{X}$ & $\hat{Y}$ are two operators,

$$\hat{X}\hat{Y}|\alpha\rangle \equiv \hat{X}(\hat{Y}|\alpha\rangle)$$

In general, operators don't commute.

$$\Rightarrow \hat{X}\hat{Y} \neq \hat{Y}\hat{X}$$

$$\Rightarrow \hat{X}\hat{Y} - \hat{Y}\hat{X} \neq 0$$

$$\Rightarrow [\hat{X}\ \hat{Y}] \neq 0$$

denoting Commute — Not commutative

{ If we say $[x, y] = 0$ that means $x$ & $y$ commute. $xy - yx = 0$ }

(5.) Associative law →

$$(\hat{X}\hat{Y})\hat{Z} = (\hat{X})(\hat{Y}\hat{Z})$$

* Inner product :- Additional on the vector space

$$S \times S \to \mathbb{C}$$

Notation → If $|\alpha\rangle$ & $|\beta\rangle$ are two vectors then their inner product is $\langle\alpha|\beta\rangle$

Similarly,
Inner product of $|\beta\rangle$ & $|\alpha\rangle$ is $\langle\beta|\alpha\rangle$

Some requirements for Inner Product →

(i) Inner product has to be linear in 2nd argument i.e.

$$|\beta\rangle = a|\gamma\rangle + b|\delta\rangle$$

then inner product $|\alpha\rangle, |\beta\rangle$ is –

$$\langle\alpha|\beta\rangle = \langle\alpha| \left(a|\gamma\rangle + b|\delta\rangle\right)$$
$$= a\langle\alpha|\gamma\rangle + b\langle\alpha|\delta\rangle$$

(ii) $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$
$$= \overline{(\langle\beta|\alpha\rangle)}$$

$*$ $\dagger$ $-$ $\Rightarrow$ complex conjugate

(iii) $\langle\alpha|\alpha\rangle \geq 0$

$\langle\alpha|\alpha\rangle \rightarrow$ Real no. (why?)
↳ Look at last property

equality holds only when
$|\alpha\rangle = O_V$

**Ortogonal vectors** :– $|a\rangle$ & $|b\rangle$ are orthogonal if:
$$\langle a|b\rangle = \langle b|a\rangle = 0$$

For any vector $|\alpha\rangle$, $\sqrt{|\langle\alpha|\alpha\rangle|}$

↳ Norm (or length) of vector $|\alpha\rangle$

Notation:
$$\| |\alpha\rangle \|$$

So, for a given vector $|\alpha\rangle$

Unit vector $\rightarrow |\tilde{\alpha}\rangle = \dfrac{|\alpha\rangle}{\| |\alpha\rangle \|} = \dfrac{|\alpha\rangle}{\sqrt{|\langle\alpha|\alpha\rangle|}}$

{Normalized vector}

## * Dual Vector (or Bra Vector) :-

Given any $|\alpha\rangle$ in $\mathcal{V}$ we can define a vector dual to $|\alpha\rangle$ denoted by $\langle\alpha|$

Dual vectors can act on Ket vectors to given "scalar". In particular if $|\beta\rangle$ is a ket vector & $\langle\alpha|$ is dual to $|\alpha\rangle$ then -

$$\langle\alpha|\,(|\beta\rangle) = \langle\alpha|\beta\rangle$$

$$\langle\alpha|\cdot|\beta\rangle) \equiv \langle\alpha|\beta\rangle$$

Bra(c)Ket

Q Check that the space of dual vectors is also a vector space. (Use postulates)

Fourier transforms

Dual vector

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

Ket vector

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t}\, dt$$

$$|\alpha\rangle = \sum_{i=1}^{n} c_i |i\rangle$$

(i) $|\alpha\rangle \xrightarrow{\text{Dual}} \langle\alpha|$

(ii) $|\alpha\rangle + |\beta\rangle \xrightarrow{\text{Dual}} \langle\alpha| + \langle\beta|$

(iii) $a|\alpha\rangle + b|\beta\rangle \overset{}{\longleftrightarrow}$ dual ?

let $a|\alpha\rangle + b|\beta\rangle = |\gamma\rangle$ → linearity

↳ dual vector $\langle\gamma|$

So, $\langle\gamma|\delta\rangle = \overline{\langle\delta|\gamma\rangle}$ ↳ $|\gamma\rangle$ is second argument ∴ (i) porcholde of inner product

$= \overline{\langle\delta|\left(a|\alpha\rangle + b|\beta\rangle\right)}$

$= \bar{a}\,\overline{\langle\delta|\alpha\rangle} + \bar{b}\,\overline{\langle\delta|\beta\rangle}$

$= a^* \langle\alpha|\delta\rangle + b^* \langle\beta|\delta\rangle$

$\langle\gamma|\delta\rangle = \left(a^*\langle\alpha| + b^*\langle\beta|\right)|\delta\rangle$

∴ $\langle\gamma| = a^* \langle\alpha| + b^* \langle\beta|$

(iii)

e.g. $|v\rangle = (3+i)|1\rangle + 4i|2\rangle$

$\Rightarrow$ $\langle v| = (3-i)\langle 1| - 4i\langle 2|$

↳ using (iii) property just discussed

Length of $|v\rangle$

if $|1\rangle$ & $|2\rangle$ are unit vectors

$\langle v|v\rangle$ ?

↳ inner product

(iv) $\hat{A}|\alpha\rangle \xrightarrow{\text{dual}}$ ?

$= \langle\alpha|\hat{A}^{+}$

↳ $\hat{A}$ 'dagger'

How?

{adjoint of op. $\hat{A}$}

Let $\hat{A}|\alpha\rangle = |\beta\rangle$ — ①

If $|\alpha\rangle$ is vary other vector

then –
$\langle\alpha|\beta\rangle = \langle\alpha|\hat{A}|\gamma\rangle$ — ②

But
$\langle\alpha|\beta\rangle = \overline{\langle\beta|\alpha\rangle}$

$= \overline{\langle\alpha|\hat{A}^{+}|\alpha\rangle}$

{let $\langle\alpha|\hat{A}^{+} = \langle\beta|$}

$\Rightarrow \overline{\langle\gamma|\hat{A}^{+}|\alpha\rangle} = \langle\alpha|\hat{A}|\gamma\rangle$

$\Rightarrow \langle\gamma|\hat{A}^{+}|\alpha\rangle = \overline{\langle\alpha|\hat{A}|\gamma\rangle}$

* If $\hat{A}^{+} = \hat{A}$ i.e.

$\langle\gamma|\hat{A}^{+}|\alpha\rangle = \langle\gamma|\hat{A}|\alpha\rangle = \overline{\langle\alpha|\hat{A}|\gamma\rangle}$

then $\hat{A}$ is called Hermitian operator (or self adjoint op.)

Q Show that $(\hat{X}\hat{Y})^{+} = \hat{Y}^{+}\hat{X}^{+}$

* Outer Product :–

Op. acting left on ket vector

↳ $\hat{A}|\alpha\rangle$ → ket vector

Op. acting right on Bra vector

↳ $\langle\beta|\hat{A}$ → Bra vector

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Again, $|\alpha\rangle\langle\beta|$ → what does this signify?

↳ ~~$|\alpha\rangle\langle\beta|(|\beta\rangle)$~~

↳ $|\alpha\rangle\langle\beta|\,(|\alpha\rangle)$ → Ket vector

$\Rightarrow \underbrace{|\alpha\rangle}_{\text{Ket vector}} \underbrace{\langle\beta|\alpha\rangle}_{\text{scalar}}$ ⇒ Resultant Ket vector

Similarly,

$\underbrace{(\langle\gamma|)}_{\text{Bra vector}}\,|\alpha\rangle\langle\beta|$

$\Rightarrow \langle\gamma|\alpha\rangle\langle\beta|$ ↳ Bra vector ⇒ Resultant Bra vector

$|\alpha\rangle\langle\beta|$ follows properties of an operator.

$|\alpha\rangle\langle\beta|$ → Outer Product

↳ acting on left from ket vector gives ket vector

& acting on right of bra vector gives bra vector

Q Show that:

$$\underbrace{|\alpha\rangle\langle\beta|}_{\hat{X}} \xrightarrow{\text{dual}} \underbrace{|\beta\rangle\langle\alpha|}_{\hat{X}^{+}}$$

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Statement :- (Theorem) For Hermitian operator i.e. for $\hat{H}$ satisfying $\hat{H}^{+} = \hat{H}$, the eigen values are all real.

$$\hat{H}\,|h_i\rangle = h_i\,|h_i\rangle \quad \text{for } h_i \in R$$

& if $|h_i\rangle$ & $|h_j\rangle$ are eigen vectors belonging to different eigen values :

$$h_i \neq h_j$$

then- $|h_i\rangle$ & $|h_j\rangle$ are orthogonal.

$$\langle h_i | h_j\rangle = \langle h_j | h_i\rangle = 0$$

* $\hat{A}\,|a'\rangle = a'\,|a'\rangle$ → {Eigen equation}

$\hat{A}$ → operator

$a'$ → Corresponding eigen value

$|a'\rangle$ → Eigen ket

Assumption :- It is clear that the eigen kets of an operator (in our case, Hermitian) will generate a vector space.

We assume that these eigenkets span the entire state space of the physical system under consideration.

→ The eigen kets are orthogonal to each other & because we work with unit vectors we have, what is known as "Orthogonal set"

i.e. If $|a'\rangle$ & $|a''\rangle$ are eigen kets of an Hermitian operator, then we have

$$\langle a' | a''\rangle = \delta_{a'a''}$$

Eigen kets span the state space of the system

$\Downarrow$

Any ket (Normalized)
↳ unit vectors
can be expressed as linear combination of eigen kets $|a_i\rangle$

$$|\alpha\rangle = \sum_i C_{a_i} |a_i\rangle \quad \text{--- } (1)$$

Q What are coeff. $C_{a_i}$ ?

Ans Using orthogonality of eigen kets we have -

$$\langle a_j|\alpha\rangle = \sum C_{a_i} \langle a_j|a_i\rangle$$

$$\Rightarrow \langle a_j|\alpha\rangle = \sum C_{a_i} \delta_{ji}$$

{For $i = j$, $\delta_{ji} = 1$}

$$\Rightarrow C_{a_j} = \langle a_j|\alpha\rangle \quad \text{--- } (2)$$

$\therefore$ Again, writing eq$^n$ (1)

$$|\alpha\rangle = \sum_i \langle a_i|\alpha\rangle |a_i\rangle$$

or $$|\alpha\rangle = \sum_i |a_i\rangle\langle a_i|\alpha\rangle$$

$\because |\alpha\rangle$ Arbitrary

$$\Rightarrow \boxed{\sum_i |a_i\rangle\langle a_i| = 1}$$

→ Remember, we have formally used this during S G experiment mathematics

Again,

$$\langle\alpha|\alpha\rangle = 1 = \sum_i \langle\alpha|a_i\rangle\langle a_i|\alpha\rangle$$

$$= \sum_i \langle a_i | \alpha \rangle^* \langle a_i | \alpha \rangle$$

$\langle a_i|\alpha\rangle^* \to C^*_{a_i}$, $\langle a_i|\alpha\rangle \to C_{a_i}$

$$= \sum_i C^*_{a_i} C_{a_i}$$

$$1 = \sum_i |C_{a_i}|^2$$

$$\therefore \boxed{\sum_i |C_{a_i}|^2 = 1}$$

* Projection ~~vectors~~ operator :-

$$P_i = |a_i\rangle\langle a_i|$$

$$\Rightarrow P_i^2 = (|a_i\rangle\langle a_i|)(|a_i\rangle\langle a_i|)$$

$$= |a_i\rangle \underbrace{\langle a_i | a_i\rangle}_{1} \langle a_i|$$

$$= |a_i\rangle\langle a_i|$$

$$= P_i$$

$|b\rangle\langle a| \Rightarrow$ Dagger $|a\rangle\langle b|$

And, $P_i^+ = |a_i\rangle\langle a_i|$

$$= P_i$$

$\left.\begin{array}{l} P_i^2 = P_i \\ \& \ P_i^+ = P_i \end{array}\right\}$ Any vector following these two properties is projection vector

* Unitary operators :- If $\hat{U}$ is an operator such that -

$$UU^+ = U^+U = 1$$

$$\Rightarrow U^+ = U^{-1}$$ → Inverse of U

Unitary operator take orthogonal basis to ortogonal basis. (what does this mean?)

Suppose, $|i\rangle, |j\rangle$ are such that -

$$\langle i|i\rangle = 1 = \langle j|j\rangle$$

$$\& \ \langle i|j\rangle = 0 = \langle j|i\rangle$$

$$|i\rangle \xrightarrow{U} |i'\rangle = U|i\rangle$$

$$|j\rangle \longrightarrow |j'\rangle = U|j\rangle$$

$$\therefore \langle i|j\rangle \xrightarrow{U} \langle i'|j'\rangle$$

Dual vector's operation

$$= \langle i|U^{+} \ U|j\rangle$$

$$= \langle i|j\rangle$$

$$= \delta_{ij}$$

$|i'\rangle = U|i\rangle$

Dual vector $\quad$ dual vector

$$\langle i'| = \langle i|U^{+}$$

Q Show that for any 2 vectors $|\alpha\rangle$, $|\beta\rangle$, then inner product $\langle\alpha|\beta\rangle$ does not change under unitary transformation.

↳ Proved just above

* Matrix Representation :-

$$\sum_i |a_i\rangle\langle a_i| = 1$$

Let $\hat{X}$ be any operator on a n-dimensional space

$$\Rightarrow \hat{X} = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_j\rangle \underbrace{\langle a_j | \hat{x} | a_i\rangle}_{n^2 \text{ no.}} \langle a_i|$$

which can be arranged as $n \times n$ matrix

$$\langle a_j | \hat{x} | a_i \rangle$$

$\langle a_j|$ → Row, $|a_i\rangle$ → Column

$$\hat{X} \equiv \begin{pmatrix} \langle a_1|X|a_1\rangle & \langle a_1|X|a_2\rangle & \cdots & \langle a_1|X|a_n\rangle \\ \langle a_2|X|a_1\rangle & \langle a_2|X|a_2\rangle & \cdots & \langle a_2|X|a_n\rangle \\ \vdots & \vdots & & \\ \langle a_n|X|a_n\rangle & \langle a_n|X|a_2\rangle & \cdots & \langle a_n|X|a_n\rangle \end{pmatrix}$$

$$\hat{X}^{+} = ?$$

* $\langle a_i | X^+ | a_j\rangle = \overline{\langle a_j | X | a_i\rangle}$ → Complex conjugate of each element of Transpose of matrix

(i) If $X$ is Hermitian

$$X^+ = X$$

$$\Rightarrow \langle a_i | X | a_j\rangle = \overline{\langle a_j | X | a_i\rangle}$$

↳ Diagonal elements → Real No.

* Let $|\gamma\rangle = X|\alpha\rangle$

$$\Rightarrow \langle a_i | \gamma\rangle = \langle a_i | X | \alpha\rangle$$

$$= \sum_{j=1}^{n} \langle a_i | X | a_j\rangle \langle a_j | \alpha\rangle$$

$$\cancel{a_j}\,|\alpha\rangle \equiv \begin{pmatrix} \langle a_1|\alpha\rangle \\ \langle a_2|\alpha\rangle \\ \vdots \\ \langle a_n|\alpha\rangle \end{pmatrix}$$

↳ column matrix

Similarly,

$$\therefore\ |\gamma\rangle \equiv \begin{pmatrix} \langle a_1|\gamma\rangle \\ \langle a_2|\gamma\rangle \\ \vdots \\ \langle a_n|\gamma\rangle \end{pmatrix}$$

Q. Show that $\langle\gamma| = \langle\alpha| X$ allows one to represent dual vectors by row vectors.

$$\langle\gamma| = \left( \langle\gamma|a_1\rangle \quad \langle\gamma|a_2\rangle \ ---\ \langle\gamma|a_n\rangle \right)$$

$$= \left( \langle a_1|\gamma\rangle^* \quad \langle a_2|\gamma\rangle^* \ ----\ \langle a_n|\gamma\rangle^* \right)$$

* Under duality

↳ $|\gamma\rangle \leftrightarrow \langle\gamma|$

Note:- Representation of $|\gamma\rangle$ is related to representation of $\langle\gamma|$ by complex conjugate Transpose

{ Just as for $X$ & $X^+$ }

$$\Rightarrow \underbrace{\langle\alpha|\beta\rangle}_{\text{Inner product}} = \underbrace{\left(\langle a_1|\alpha\rangle^* \;\; \langle a_2|\alpha\rangle^* \;\dots\; \langle a_n|\alpha\rangle^*\right)}_{\text{Row }\cancel{\text{vector}}\text{ matrix}} \underbrace{\begin{pmatrix} \langle a_1|\beta\rangle \\ \langle a_2|\beta\rangle \\ \vdots \\ \langle a_n|\beta\rangle \end{pmatrix}}_{\text{Column matrix}}$$

Q $|\alpha\rangle\langle\beta| = ?$

* Representation of $\hat{A}$ is its own eigen basis.

$$\hat{A} \equiv \begin{pmatrix} \langle a_1|A|a_1\rangle & \langle a_1|A|a_2\rangle & \cdots & \\ \langle a_2|A|a_1\rangle & \langle a_2|A|a_2\rangle & \cdots & \\ \vdots & & & \\ \langle a_n|A|a_1\rangle & \langle a_n|A|a_2\rangle & \cdots & \langle a_n|A|a_n\rangle \end{pmatrix}$$

$$\hat{A}\,|a_i\rangle \equiv a_i\,|a_i\rangle$$

$$\Rightarrow \langle a_j|A|a_i\rangle = a_i\,\delta_{ij}$$

$$\Rightarrow \quad \hat{A} = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ & & \ddots \\ 0 & & a_n \end{pmatrix} = \sum a_i\,|a_i\rangle\langle a_i|$$

e.g. $|0\rangle$, $|1\rangle$

$\langle 0|0\rangle = 1 = \langle 1|1\rangle$

$\langle 0|1\rangle = 0 = \langle 1|0\rangle$

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$$\therefore\ |0\rangle \equiv \begin{pmatrix} \langle 0|0\rangle \\ \langle 1|0\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\&\ |1\rangle = \begin{pmatrix} \langle 0|1\rangle \\ \langle 1|1\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let us define an operator $Z$

$Z|0\rangle = |0\rangle$ → operation of $Z$ on $|0\rangle$ gives $|0\rangle$

$\&\ Z|1\rangle = -|1\rangle$

$$\therefore\ Z \equiv \begin{pmatrix} \langle 0|Z|0\rangle & \langle 0|Z|1\rangle \\ \langle 1|Z|0\rangle & \langle 1|Z|1\rangle \end{pmatrix}$$

$$\equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$Z^{\dagger} = Z$ → Hermitian operator

Q Representation of $|0\rangle\langle 1|$ ?

Q Are the states

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\&\ |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$ orthogonal ?

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Q What is the matrix representation of $|0\rangle$ in the $(|+\rangle, |-\rangle)$ basis?

Q Representation of $z$ in $(|+\rangle, |-\rangle)$ basis

* Measurement postulates:-

$$|\alpha\rangle = \sum_{i=1}^{n} C_{a_i} |a_i\rangle$$

(i) If $|\alpha\rangle$ is an vector, then after the measurement it jumps to one of eigen kets.

i.e. $|\alpha\rangle \xrightarrow[\hat{A}]{\text{Measurement}} |a_i\rangle$

But, if $|\alpha\rangle$ happens to be an eigen ket say $|a_j\rangle$ then $|a_j\rangle \xrightarrow[\hat{A}]{\text{Measure}} |a_j\rangle$

(ii) In general, the probability to be found in state $|a_i\rangle$ are measurement is -

$$\cancel{|a_i\rangle\langle\alpha|} \quad |\langle a_i|\alpha\rangle|^2 = |C_{a_i}|^2$$

$$\& \sum |C_{a_i}|^2 = 1 \rightarrow \{\text{Sum of probability}\}$$

* Expectation value of $\hat{A}$ in state $|\alpha\rangle$:-

$$\langle \hat{A} \rangle = \langle \alpha | \hat{A} | \alpha \rangle$$

$$= \sum_{i,j} \langle \alpha | a_j \rangle \langle a_j | \hat{A} | a_i \rangle \langle a_i | \alpha \rangle$$

$$= \sum_{i,j} a_i \langle \alpha | a_j \rangle \langle a_i | \alpha \rangle \delta_{ij}$$

$$= \sum_{i,j} a_i \, |C_{a_i}|^2$$