Lecture 1: Functions of Random Vectors

14 March, 2018

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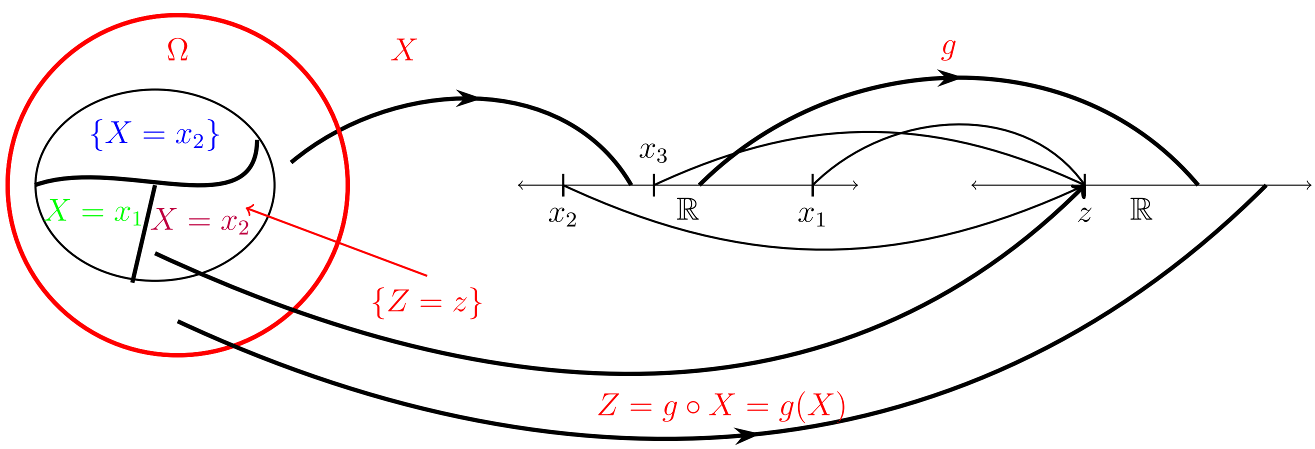
When there are multiple random variables of interest, it is possible to generate new random variables by considering functions involving several of these random variables . In particular, suppose we have two random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$, and $g : \mathbb{R}^2 \to \mathbb{R}$ be a function. Then $Z := g(X, Y) : \Omega \to \mathbb{R}$ defines another random variable. If both X, Y are discrete (A random variable that can take on at most a countable number of possible values is said to be discrete) then Z is also discrete. Now next natural question is what is the PMF of random variable Z?

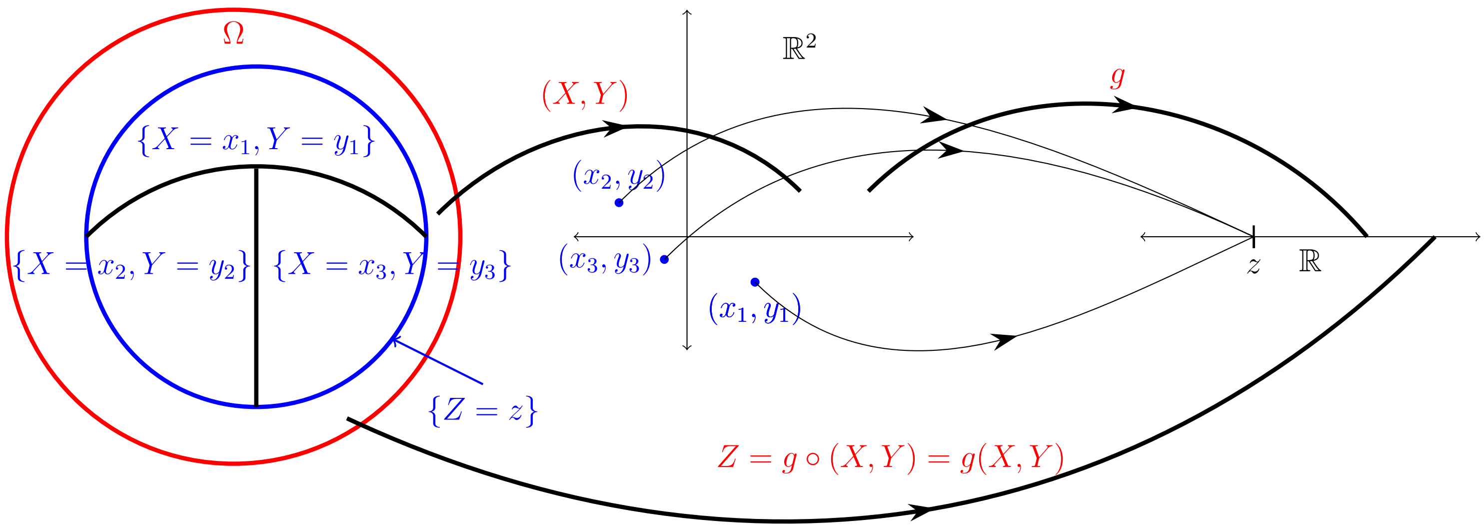
Before we derive the formula for the PMF of real-valued function of random vector (X, Y), it will be helpful to recall the derivation of PMF of real-valued function of a single random variable. Let $X : \Omega \to \mathbb{R}$ be a discrete random variable with PMF f_X and $g : \mathbb{R} \to \mathbb{R}$ be a function. Then the pmf of $Z = g(X)$ can be obtained as follows:

$$
\begin{aligned}
f_Z(z) &:= P\{Z = z\} = P\{\omega \in \Omega : Z(\omega) = z\} = P\{\omega \in \Omega : g(X(\omega)) = z\} \\
&= P\left\{ \bigcup_{x:g(x)=z} \{\omega \in \Omega : X(\omega) = x\} \right\} \quad \text{(See the figure below)} \\
&= \sum_{x:g(x)=z} P\{X = x\} \quad \text{(By additivity of probability measure)}
\end{aligned}
$$



Now we can employ the same idea to derive PMF of $Z = g(X, Y)$ from the joint PMF $f(x, y)$ according to

$$\begin{aligned} f_Z(z) &= P\{Z = z\} = P\{g(X, Y) = z\} \\ &= P\left\{ \bigcup_{(x,y):g(x,y)=z} \{X = x, Y = y\} \right\} \quad \text{(See figure below)} \\ &= \sum_{(x,y):g(x,y)=z} f(x, y). \end{aligned}$$



Example 1.1 *Let X and Y be random variables with the joint pmf given by the following table.*

X \ Y	*-1*	*0*	*2*	*6*
-2	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{1}{9}$
1	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$
3	0	0	$\frac{1}{9}$	$\frac{4}{27}$

Find the PMF of $|Y - X|$.

Solution: So here $g(x, y) = |y - x|$. First we compute the range of the random variable $Z = g(X, Y) = |Y - X|$.

Fix $x = -2$ and run through the all y values, we get $z = |y - (-2)| = |y + 2|, 1, 2, 4, 8$

Fix $x = 1$ and run through the all y values, we get $z = |y - 1|, 2, 1, 1, 5$

Fix $x = 3$ and run through the all y values, we get $z = |y - 3|, 4, 3, 1, 3$

So range of random variable Z is $\{1, 2, 3, 4, 5, 8\}$. Now

$$P(Z=1) = \sum_{(x,y):|y-x|=1} f(x,y) = f(-2,-1) + f(1,0) + f(1,2) + f(3,2) = \frac{1}{9} + 0 + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}.$$

$$P(Z=2) = \sum_{(x,y):|y-x|=2} f(x,y) = f(-2,0) + f(1,-1) = \frac{1}{27} + \frac{2}{9} = \frac{7}{27}.$$

$$P(Z=3) = \sum_{(x,y):|y-x|=3} f(x,y) = f(3,0) + f(3,6) = 0 + \frac{4}{27} = \frac{4}{27}.$$

$$P(Z=4) = \sum_{(x,y):|y-x|=4} f(x,y) = f(-2,2) + f(3,-1) = \frac{1}{27} + 0 = \frac{1}{27}.$$

$$P(Z=5) = \sum_{(x,y):|y-x|=5} f(x,y) = f(1,6) = \frac{1}{9}.$$

$$P(Z=8) = \sum_{(x,y):|y-x|=8} f(x,y) = f(-2,6) = \frac{1}{9}.$$

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✠ **Tips For Exam** One check regarding the calculations of pmf in Example 1.1 is, $\sum_{z \in R_Z} f_Z(z) = 1$, where R_Z denotes the range of the random variable Z.

Example 1.2 *Let X and Y be random variables with the joint pmf $P\{X = i, Y = j\} = \frac{1}{N^2}, i, j = 1, 2, \cdots, N$. Find pmf of random variable* $\min\{X, Y\}$.

Solution: Set $Z := \min\{X, Y\}$. Since X and Y are discrete random variable, hence Z will also be a discrete random variable. Also it is clear that range of random variable Z would be $\{1, 2, \cdots, N\}$. Now we find its pmf,

$$P\{Z = i\} = \sum_{(x,y):\min\{x,y\}=i} f(x,y)$$

Now both x and y ranges over the set $\{1, 2, \cdots, N\}$.

X \ Y	1	2	$\cdots$	i	$\cdots$	N
1	$\frac{1}{N^2}$	$\frac{1}{N^2}$	$\cdots$	$\frac{1}{N^2}$	$\cdots$	$\frac{1}{N^2}$
2	$\frac{1}{N^2}$	$\frac{1}{N^2}$	$\cdots$	$\frac{1}{N^2}$	$\cdots$	$\frac{1}{N^2}$
$\vdots$	$\frac{1}{N^2}$	$\frac{1}{N^2}$	$\cdots$	$\frac{1}{N^2}$	$\cdots$	$\frac{1}{N^2}$
i	$\frac{1}{N^2}$	$\frac{1}{N^2}$	$\cdots$	$\frac{1}{N^2}$	$\cdots$	$\frac{1}{N^2}$
$\vdots$	$\frac{1}{N^2}$	$\frac{1}{N^2}$	$\cdots$	$\frac{1}{N^2}$	$\cdots$	$\frac{1}{N^2}$
N	$\frac{1}{N^2}$	$\frac{1}{N^2}$	$\cdots$	$\frac{1}{N^2}$	$\cdots$	$\frac{1}{N^2}$

Hence for given $i \in \{1, 2, \cdots, N\}$,

$$
\begin{aligned}
\sum_{(x,y):\min\{x,y\}=i} f(x,y) &= \sum_{y=i}^{N} f(i,y) + \sum_{x=i+1}^{N} f(x,i) \\
&= \frac{(N-(i-1))}{N^2} + \frac{N-i}{N^2} \\
&= \frac{2N-2i+1}{N^2}
\end{aligned}
$$

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