Lecture 3: Density of Function of two Random Variables

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Example 3.1 *The joint density function of X and Y is given by*

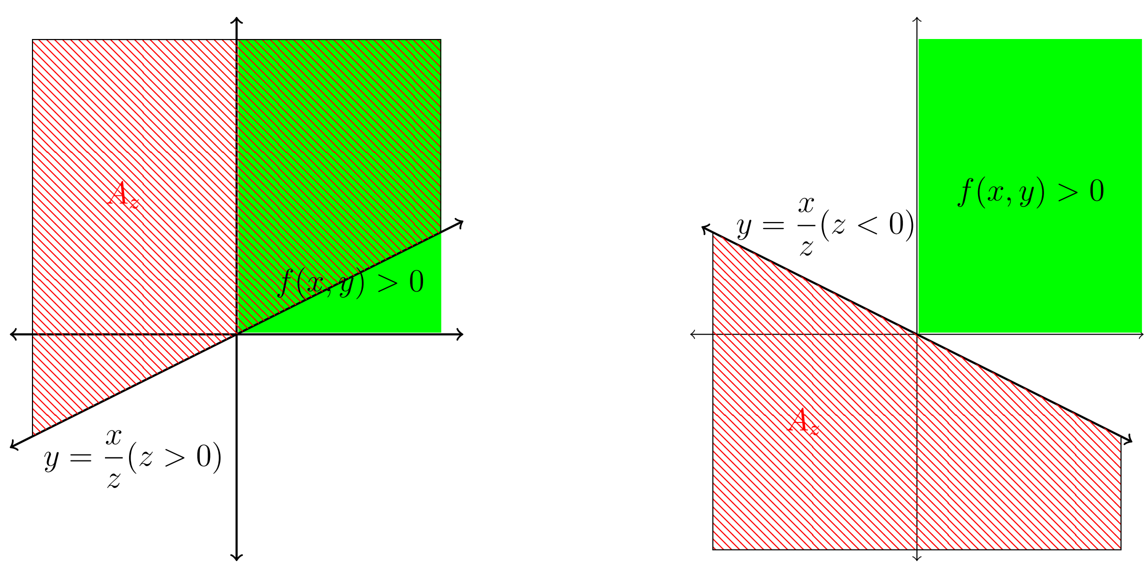
$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{if} \quad 0 < x, y < \infty \\ 0 & \textit{otherwise} \end{cases}.$$

Find the density function of the random variable $\dfrac{X}{Y}$.

Solution: Let $Z := \dfrac{X}{Y}$. Let $z \in \mathbb{R}$ be given. Then

$$\{Z \le z\} = \{\frac{X}{Y} \le z\} = \{(X,Y) \in A_z\},$$

where $A_z = \{(x,y) \in \mathbb{R}^2 : \frac{x}{y} \le z\}$. The following picture depicts the set A_z:



Therefore if $z \geq 0$,

$$F_Z(z) := P\{(X,Y) \in A_z\} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{zy} f(x,y)dx \right) dy = \int_0^{\infty} e^{-y} \left(\int_0^{zy} e^{-x} dx \right) dy$$
$$= \int_0^{\infty} e^{-y} \left(\int_0^{zy} e^{-x} dx \right) dy = \int_0^{\infty} e^{-y} \left[-e^{-x} \right]_0^{zy} dy = \int_0^{\infty} e^{-y} \left[1 - e^{-zy} \right] dy$$
$$= \left[-e^{-y} + \frac{1}{z+1} e^{-(z+1)y} \right]_0^{\infty} = -0 + 0 + 1 - \frac{1}{z+1} = 1 - \frac{1}{z+1}$$

Therefore if $z < 0$,

$$F_Z(z) := P\{(X,Y) \in A_z\} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{zy} f(x,y)dx \right) dy = 0$$

Hence

$$F_Z(z) = \begin{cases} 1 - \dfrac{1}{z+1} & \text{if} \quad z \geq 0 \\ 0 & \text{if} \quad z < 0 \end{cases}.$$

Since F_Z is a continuous function, we may differentiate it to get the density

$$f_Z(z) = \begin{cases} \dfrac{1}{(z+1)^2} & \text{if} \quad z > 0 \\ 0 & \text{if} \quad z \leq 0 \end{cases}.$$

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Example 3.2 *Let X and Y be two independent $U(0,1)$ random variables. Find the pdf of XY (if it exists).*

Solution: Let $Z := XY$. Then Let $z \in \mathbb{R}$ be given. Then

$$\{Z \leq z\} = \{(X,Y) \in A_z\},$$

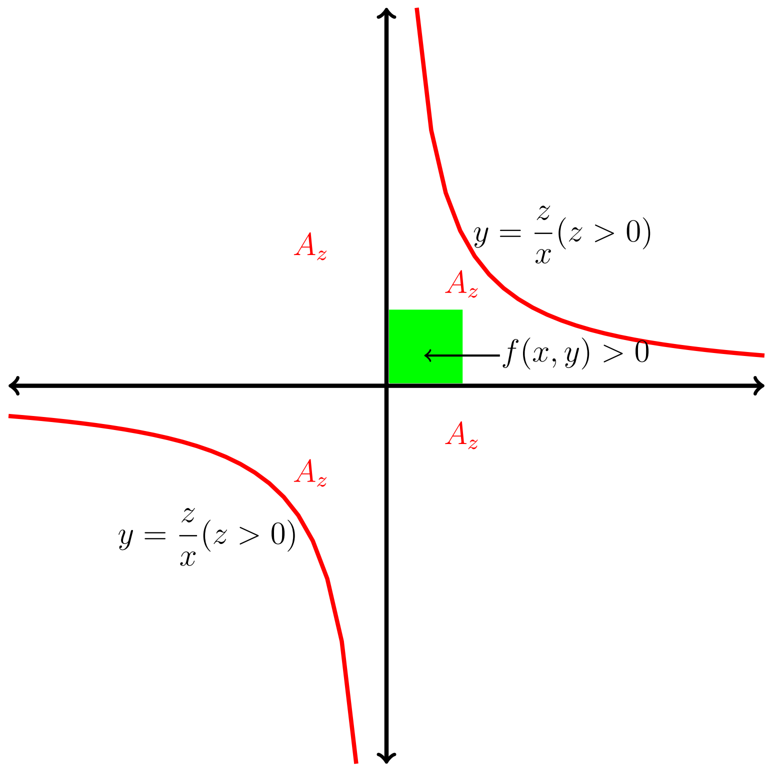
where $A_z = \{(x,y) \in \mathbb{R}^2 : xy \leq z\}$. Since X and Y are independent, the joint pdf of X and Y is $f(x,y) = f_X(x) f_Y(y)$, where

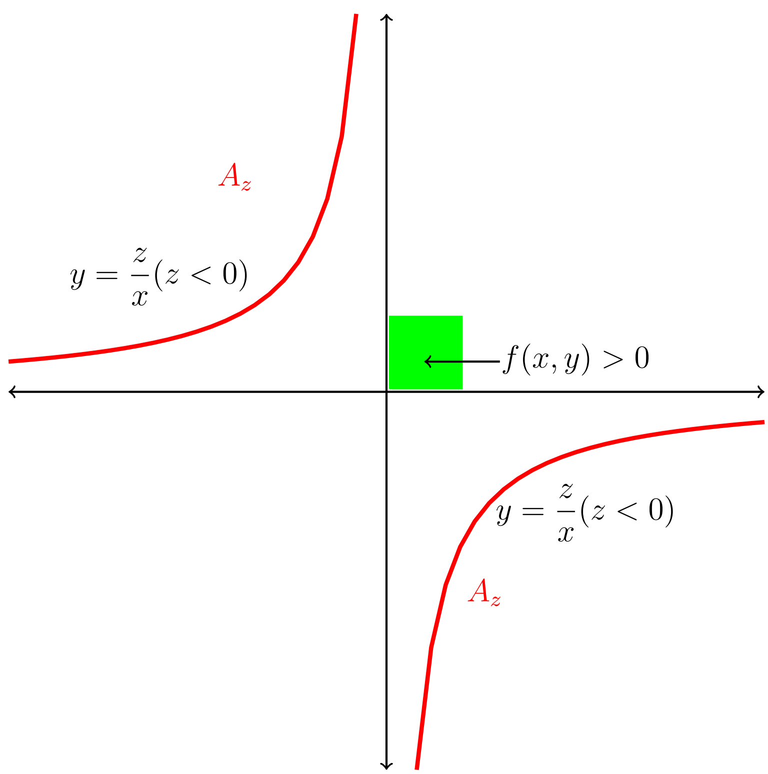
$$f_X(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{otherwsie} \end{cases}, \quad f_Y(y) = \begin{cases} 1 & \text{if} \quad 0 < y < 1 \\ 0 & \text{otherwsie} \end{cases},$$

Hence

$$f(x,y) = \begin{cases} 1 & \text{if} \quad 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwsie} \end{cases}.$$

The following picture depicts the set A_z:

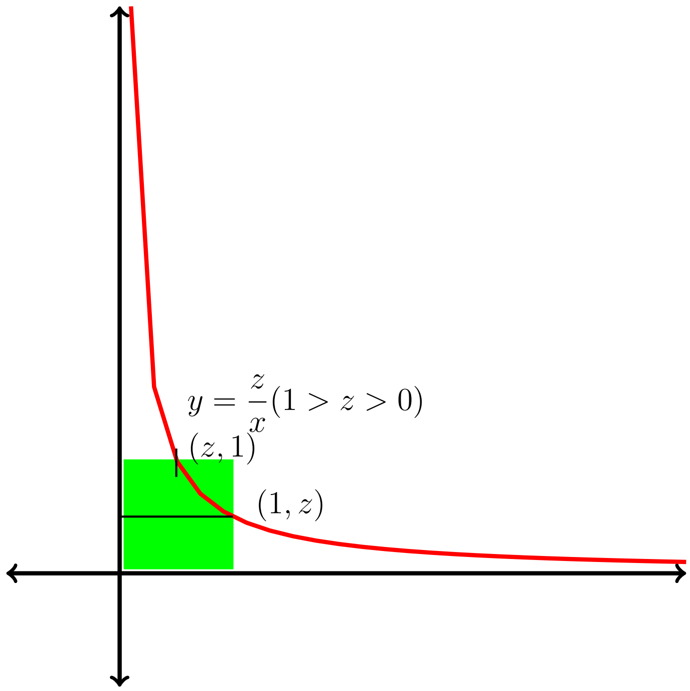




From the equation $y = \frac{z}{x}$ it is clear that if $z \geq 1$ then the curve $xy = z$ will not intersect with the open unit square $(0,1)\times(0,1)$ (because if $x, y \in (0,1)$ then $0 < xy < 1$). Therefore A_z contains the open unit square. Hence for $z \geq 1$,

$$F_Z(z) := P\{(X,Y) \in A_z\} = \int_0^1 \int_0^1 f(x,y)dxdy = 1$$

Now let $0 < z < 1$, then see the picture below.



$$F_Z(z) := P\{(X,Y) \in A_z\} = \int_0^1 \left(\int_0^z f(x,y)dy\right)dx + \int_z^1 \left(\int_0^{z/y} f(x,y)dx\right)dy = z + \int_z^1 \frac{z}{y}dy$$
$$= z + z\left[\ln y\right]_z^1 = z - z\ln z$$

From the figure it is clear that if $z \leq 0$ ($z = 0$ is just both the axis), $A_z \cap \{(x,y) : f(x,y) > 0\} = \emptyset$ therefore

$$F_Z(z) := P\{(X,Y) \in A_z\} = 0$$

Hence

$$F_Z(z) = \begin{cases} 0 & \text{if} \quad z \leq 0 \\ z - z\ln z & \text{if} \quad 0 < z < 1 \\ 1 & \text{if} \quad z \geq 1 \end{cases}.$$

Since F_Z is a continuous function (because $\lim_{z \to 0^+} z \ln z = \lim_{z \to 0^+} \frac{\ln z}{\frac{1}{z}} = \lim_{z \to 0^+} \frac{\frac{1}{z}}{-\frac{1}{z^2}} = \lim_{z \to 0^+} -z = 0$), we may differentiate it to get the density

$$f_Z(z) = \begin{cases} 1 - (\ln z + z\frac{1}{z}) = -\ln z & \text{if} \quad 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Just a check on answer: as $z < 1$, $-\ln z > 0$ so $f_Z \geq 0$. And integration by parts,

$$\int_0^1 -\ln x \; dx = \Big[x - x \ln x \Big]_0^1 = 1$$

$\blacksquare$