Lecture 11: Three Inequalities

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Definition 11.1 *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. We say that*

1. *f is convex on I or concave upward on I if for any $x_1, x_2 \in I$ and any $t \in (0,1)$ we have*
$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2).$$

2. *f is concave on I or concave downward on I if for any $x_1, x_2 \in I$ and any $t \in (0,1)$ we have*
$$f((1-t)x_1 + tx_2) \geq (1-t)f(x_1) + tf(x_2).$$

It follows from the definition that f is convex iff $-f$ is concave.

Theorem 11.2 (Jensen's Inequality) *Let $f : I \to \mathbb{R}$ be a convex function where $I \subset \mathbb{R}$ is an interval and X be a random variable such that X and $f(X)$ has finite mean. Then*

$$f(EX) \leq E[f(X)].$$

If f is a concave function then $-f$ is convex so by Jensen's inequality

$$\begin{aligned} -f(EX) &\leq E(-f(X)) \\ &= -E[f(X)] \quad \text{(By linearity of expectation)} \\ \implies f(EX) &\geq E[f(X)] \end{aligned}$$

Example 11.3 *Note that $f(x) = |x|$ is a convex function hence by Jensen's inequality*

$$EX \leq |EX| \leq E|X|.$$

Definition 11.4 *Let r be a positive real number and X be a random variable. Then $E[X^r]$ is called the r-th moment of X about origin or central moment of X of order r.*

$E|X|^r$ is called the r-th absolute moment of X about origin or central moment of X of order r.

We know from definition of expectation that, $E\left[X^r\right]$ exists and is a finite number if $E\left[|X^r|\right] < \infty$. Therefore from above observation

$$E[X^r] \leq |E[X^r]| \leq E\left[|X|^r\right].$$

Example 11.5 *If the moment of order $q > 0$ exists for a random variable X, then show that moments of order p, where $0 < p < q$ exist.*

Solution: Let $f : (0, \infty) \to \mathbb{R}$ be defined as $f(x) = x^r$, where $r > 1$ is a real number. Then $f'(x) = rx^{r-1}$, $f''(x) = r(r-1)x^{r-2}$. Since $r > 1$, $f''(x) > 0$ on $(0, \infty)$, i.e., f is a convex function on $(0, \infty)$. Hence by Jensen's inequality,

$$[E|X|]^r \leq E\left[|X|^r\right] \implies [E|X|] \leq \left(E\left[|X|^r\right]\right)^{\frac{1}{r}}. \tag{11.1}$$

Let $0 < p < q$. Then we take $r = \frac{q}{p} > 1$ in (11.1) and we get

$$[E|X|] \leq \left(E\left[|X|^{\frac{q}{p}}\right]\right)^{\frac{p}{q}}. \tag{11.2}$$

Now replacing $|X|$ by $|X|^p$ in (11.2), we get

$$[E|X|^p] \leq \left(E\left[|X|^q\right]\right)^{\frac{p}{q}}$$

If $E|X|^q < \infty$ then $(E|X|^q)^{\frac{p}{q}} < \infty$ and therefore $[E|X|^p] < \infty$. ∎

Example 11.6 *Let X be a random variable with $EX = 10$. Show that $E\left[\ln\sqrt{X}\right] \leq \dfrac{1}{2}\ln 10$.*

Solution: Consider $f(x) = \ln\sqrt{x} = \frac{1}{2}\ln x$, for $x \in (0, \infty)$. Then $f'(x) = \dfrac{1}{2x}$ and $f''(x) = -\dfrac{1}{2x^2} < 0$ on $(0, \infty)$. Hence f is a concave function. Therefore by Jensen's inequality

$$\frac{1}{2}\ln 10 = f(EX) \geq E[f(X)] = E\left[\ln\sqrt{X}\right].$$

∎

Now we derive some important inequalities. These inequalities use the mean and possibly the variance of a random variable to draw conclusions on the probabilities of certain events. They are primarily useful in situations where exact values or bounds for the mean and variance of a random variable X are easily computable, but the distribution of X is either unavailable or hard to calculate.

Theorem 11.7 (Markov Inequality) *Let X be a non-negative random variable with finite nth moment. Then we have for each $\epsilon > 0$,*

$$P\{X \geq \epsilon\} \leq \frac{E[X^n]}{\epsilon^n}$$

Loosely speaking, Markov inequality asserts that if a nonnegative random variable has a small nth central moment, then the probability that it takes a large value must also be small.

As a corollary we have the Chebyshev's inequality.

Corollary 11.8 (Chebyshev's inequality) *Let X be a random variable with finite mean μ and finite variance σ^2. Then for every $\epsilon > 0$,*

$$P\{|X - \mu| \geq \epsilon\} \leq \frac{\sigma^2}{\epsilon^2}$$

Proof: The proof of Chebyshev's inequality follows by replacing X by $|X - \mu|$ in the Markov inequality and realizing that $|X - \mu|^2 = [X - \mu]^2$. ■

Remark 11.9 *Loosely speaking, Chebyshev's inequality asserts that if a random variable has small variance, then the probability that it takes a value far from its mean is also small. Note that the Chebyshev inequality does not require the random variable to be nonnegative.*

Example 11.10 *Let $X \sim B(n,p)$. Estimate $P(X \geq \alpha n)$, where $p < \alpha < 1$ using Markov (for first moment) and Chebyshev's inequality. Compare both the estimates for $p = \frac{1}{2}$ and $\alpha = \frac{3}{4}$.*

Solution: Note that X takes values $\{0, 1, \cdots, n\}$, hence is a nonnegative random variable and $EX = np$. Applying Markov's inequality, we obtain

$$P(X \geq \alpha n) \leq \frac{EX}{\alpha n} = \frac{pn}{\alpha n} = \frac{p}{\alpha}$$

Chebyshev's inequality gives estimate for $P(|X - EX| \geq \alpha n)$ so we have rewrite the event $\{X \geq \alpha n\}$ so that we can use the Chebushev's inequality.

$$\begin{aligned} P\{X \geq \alpha n\} &= P\{X - np \geq \alpha n - np\} \\ &\leq P\left(|X - np| \geq \alpha n - np\right) \quad (\because \{|Y| \geq a\} = \{Y \leq -a\} \cup \{Y \geq a\}) \\ &\leq \frac{\text{var(X)}}{(\alpha n - np)^2} = \frac{np(1-p)}{n^2(\alpha - p)^2} = \frac{p(1-p)}{n(\alpha - p)^2} \end{aligned}$$

By Markov inequality for $p = \frac{1}{2}$ and $\alpha = \frac{3}{4}$, we

$$P\left(X \geq \frac{3n}{4}\right) \leq \frac{2}{3}$$

By Chebyshev's inequality for $p = \frac{1}{2}$ and $\alpha = \frac{3}{4}$, we

$$P\left(X \geq \frac{3n}{4}\right) \leq \frac{4}{n}$$

If $n \geq 6$ then estimate given by Chebyshev's are sharper than the estimates provided by Markov inequality. Also as n increases, estimate given by Chebyshev's inequality decreases, i.e., gives much information whereas the estimates provided by Markov inequality remains constant as n varies.

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