Lecture 10: Characteristic Function

27 March, 2018

Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

Example 10.1 *Let $X \sim N(0,1)$. Find it's characteristic function.*

Solution:

$$\begin{aligned}
\phi_X(t) &= E[\cos tX + i\sin tX] = E[\cos tX] + iE[\sin tX] \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cos tx \ e^{-\frac{x^2}{2}} dx + i \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \sin tx \ e^{-\frac{x^2}{2}} dx
\end{aligned}$$

Since characteristic function exists for every random variable, therefore both the improper integral exists. So value both improper integrals agrees with their Cauchy principle value. We have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \sin tx \ e^{-\frac{x^2}{2}} dx = \lim_{a\to\infty} \int_{-a}^{a} \frac{1}{\sqrt{2\pi}} \sin tx \ e^{-\frac{x^2}{2}} dx = 0,$$

because sin is an odd function. Also

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cos tx \ e^{-\frac{x^2}{2}} dx = \lim_{a\to\infty} \int_{-a}^{a} \frac{1}{\sqrt{2\pi}} \cos tx \ e^{-\frac{x^2}{2}} dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \cos tx \ e^{-\frac{x^2}{2}} dx = e^{-\frac{t^2}{2}}$$

where the last integral can computed using differentiation under integration. Let $t \in \mathbb{R}$ be given. Define

$$\begin{aligned}
I(t) = \int_0^{\infty} \cos tx \ e^{-\frac{x^2}{2}} dx \implies I'(t) &= -\int_0^{\infty} x \sin tx \ e^{-\frac{x^2}{2}} dx \\
&= -\left[-\sin tx e^{-\frac{x^2}{2}} \Big|_0^{\infty} + \int_0^{\infty} t \cos tx \ e^{-\frac{x^2}{2}} dx \right] \\
&= 0 - tI(t)
\end{aligned}$$

Therefore $\ln I(t) = -\frac{t^2}{2} + C \implies I(t) = Ke^{-\frac{t^2}{2}}$. Also $I(0) = \int_0^{\infty} e^{-\frac{x^2}{2}} dx = \frac{\sqrt{2\pi}}{2}$. So $K = \sqrt{\frac{\pi}{2}}$. ■

Example 10.2 *Let X be a random variable and a and b are real constants, then*

$$\phi_{a+bX}(t) = E\left[e^{it(a+bX)}\right] = E\left[e^{ita}e^{itbX}\right] = e^{ita}E\left[e^{itbX}\right] = e^{ita}\phi_X(bt)$$

Example 10.3 *Let $X \sim N(\mu, \sigma^2)$. Then it is implicit that $\sigma > 0$. Then $Y = \frac{X-\mu}{\sigma}$ has mean zero and variance 1. Also $Y \sim N(0,1)$. To see this,*

$$\begin{aligned}
F_Y(x) &= P(Y \leq x) \\
&= P(\frac{X-\mu}{\sigma} \leq x) \\
&= P(X \leq \sigma x + \mu) \\
&= \int_{-\infty}^{\sigma x+\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt \\
&= \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2}} \sigma du \quad (\text{put } t = \sigma u + \mu\ ) \\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du
\end{aligned}$$

Hence by Example 10.2, $X = \sigma Y + \mu$ has the characteristic function

$$\phi_X(t) = \phi_{\sigma Y+\mu}(t) = e^{it\mu}\phi_Y(\sigma t) = e^{it\mu}e^{-\frac{\sigma^2 t^2}{2}}$$

Example 10.4 *Let X and Y be independent random variables. Show that*

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$$

Solution:

$$\begin{aligned}
\phi_{X+Y}(t) &= E[e^{it(X+Y)}] \\
&= E[e^{itX}e^{itY}] = E[e^{itX}]E[e^{itY}] = \phi_X(t)\phi_Y(t)
\end{aligned}$$

■

More generally, if $X_1, X_2, \cdots, X_n$ are n independent random variables, then

$$\phi_{X_1+X_2+\cdots+X_n}(t) = \phi_{X_1}(t)\phi_{X_2}(t)\cdots\phi_{X_n}(t).$$

Example 10.5 *Compute the characteristic function of a Binomial(n,p) random variables.*

Solution: A Binomial(n,p) random variable is a sum of n independent Bernoulli(p) random variables. Therefore it's characteristic function is

$$\left[e^{it}p + (1-p)\right]^n.$$

■

Theorem 10.6 (Uniqueness Theorem) *Let X_1 and X_2 be two random variables such that $\phi_{X_1} = \phi_{X_2}$. Then X_1 and X_2 have same distribution.*

Example 10.7 *Let $X \sim B(n_1, p)$ and $Y \sim B(n_2, p)$ be two independent Binomial random variables. Show that $X + Y$ is a Binomial$(n_1 + n_2, p)$ random variable.*

Solution: Let $X \sim B(n_1, p)$ and $Y \sim B(n_2, p)$ be two independent random variables. Therefore the characteristic function of $X + Y$ is

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) = \left[e^{it}p + (1-p)\right]^{n_1} \left[e^{it}p + (1-p)\right]^{n_2} = \left[e^{it}p + (1-p)\right]^{n_1+n_2}.$$

RHS is a characteristic function of a Binomial$(n_1 + n_2, p)$ random variable, therefore by uniqueness theorem $X + Y \sim Binomial(n_1 + n_2, p)$. ■

Example 10.8 *Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent normal random variable. Then show that $X + Y$ is a $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.*

Solution: Hence we have

$$\phi_X(t) = e^{it\mu_1} e^{-\frac{\sigma_1^2 t^2}{2}}, \quad \phi_Y(t) = e^{it\mu_2} e^{-\frac{\sigma_2^2 t^2}{2}}.$$

Now

$$\begin{aligned}\phi_{X+Y}(t) &:= E\left[e^{it(X+Y)}\right] = E\left[e^{itX}e^{itY}\right] = E\left[e^{itX}\right]E\left[e^{itY}\right] = \phi_X(t)\phi_Y(t) \\ &= e^{it(\mu_1+\mu_2)} e^{-\frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}\end{aligned}$$

Now right hand side is the characteristic function of a normal random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. Therefore by uniqueness theorem, we conclude $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ ■