Age of the Universe

$$\dot{R} \equiv \frac{dR}{dt}$$

$$\frac{dR}{dt} = f(R)$$

$$\int \frac{dR}{f(R)} = \int dt \longrightarrow \text{Age}$$

Age can be obtained by integrating Friedmann Eqn Energy density scales as:

$$\frac{\rho}{\rho_0} = \left(\frac{R}{R_0}\right)^{-3} \text{ for matter dominated (MD) Universe} \quad \text{— (1)}$$

$$\frac{\rho}{\rho_0} = \left(\frac{R}{R_0}\right)^{-4} \text{ for radiation dominated (RD) Universe} \quad \text{— (2)}$$

The Friedmann Eqn becomes

$$\left(\frac{\dot{R}}{R_0}\right)^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3}\rho_0 \frac{R_0}{R} \text{ for MD} \quad \text{— (3)}$$

and $$\left(\frac{\dot{R}}{R_0}\right)^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3}\rho_0 \left(\frac{R_0}{R}\right)^2 \text{ for RD} \quad \text{— (4)}$$

Note that eqns (3) and (4) are obtained by taking the Friedmann Eqn, $\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho$

Using (1) and (2) and multiplying by $\left(\frac{R}{R_0}\right)^2$ on both sides.

~~Eqs~~ Eqns (3) and (4) can be integrated with some manipulation, but consider first the simplest case $k=0$. This coresspponds to the observationally and theoretically favoured case of the flat universe.

for flat (K=0) MD universe

$$\left(\frac{\dot{R}_0}{R_0}\right)^2 = \frac{8\pi G}{3} \rho_0 \left(\frac{R_0}{R}\right)$$

$$\dot{R} = \sqrt{\frac{8\pi G}{3} \rho_0 \frac{R_0^3}{R}}$$

Thus,

$$\frac{dR}{dt} = \sqrt{\frac{8\pi G \rho_0 R_0^3}{3}} \cdot \frac{1}{\sqrt{R}}$$

$$\sqrt{R}\, dR = \sqrt{\frac{8\pi G \rho_0 R_0^3}{3}}\, dt$$

$$\frac{\sqrt{R}\, dR}{\sqrt{\frac{8\pi G}{3} \rho_0 R_0^3}} = dt$$

$$\int_0^{t_0} dt = t_0 = \int_0^{R_0} \frac{1}{\sqrt{\frac{8\pi G}{3} \rho_0 R_0^3}} \cdot \sqrt{R}\, dR.$$

$$t_0 = \frac{1}{\sqrt{\frac{8\pi G}{3} \rho_0 R_0^3}} \left[\frac{R^{3/2}}{3/2}\right]_0^{R_0}$$

$$= \frac{1}{\sqrt{\frac{8\pi G}{3} \rho_0 R_0^3}} \cdot \frac{2}{3} (R_0^{3/2} - 0)$$

$$t_0 = \frac{2}{3} \cdot \frac{1}{\sqrt{\frac{8\pi G}{3} \rho_0}} \frac{1}{\cancel{R^{3/2}}} \cdot \cancel{R^{3/2}} \quad \text{—— (5)}$$

Recall that Friedmann Eqn for this case is

$$\left(\frac{\dot{R}}{R_0}\right)^2 = H^2 = \frac{8\pi G \rho}{3}$$

Thus, at the present time $t = t_0$, this implies

$$H_0^2 = \frac{8\pi G}{3}\rho_0 \quad \text{—— (6)}$$

from (6) $\longrightarrow$ $H = \sqrt{\frac{8\pi G}{3}\rho_0}$

Using this in eqn (5), we get

$$\boxed{t_0 = \frac{2}{3}H^{-1}} \quad \text{—— (7)}$$

for the matter dominated universe.

For the flat ($k=0$) radiation dominated universe from eq^n (4) setting $k=0$, we get:

$$\left(\frac{\dot{R}}{R_0}\right)^2 = \frac{8\pi G}{3}\rho_0\left(\frac{R_0}{R}\right)^2$$

$$\frac{\dot{R}}{R_0} = \sqrt{\frac{8\pi G}{3}\rho_0}\left(\frac{R_0}{R}\right)$$

$$\frac{dR}{dt} = \sqrt{\frac{8\pi G}{3}\rho_0}\ \frac{R_0^2}{R}$$

$$\frac{R\,dR}{\sqrt{\frac{8\pi G}{3}\rho_0}\ R_0^2} = dt$$

$$\int_0^{t_0} dt = t_0 = \int_0^{R_0}\frac{R\,dR}{\sqrt{\frac{8\pi G}{3}\rho_0}\ R_0^2}$$

$$t_0 = \frac{1}{\sqrt{\frac{8\pi G\rho_0}{3}}}\cdot\frac{1}{R_0^2}\cdot\frac{1}{2}(R_0^2 - 0)$$

Thus,

$$t_0 = \frac{1}{2}\frac{1}{\sqrt{\frac{8\pi G\rho_0}{3}}} \quad \text{—— (8)}$$

Again using Friedmann Eq^n

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2 = \frac{8\pi G}{3}\rho_0$$

$$\Rightarrow H_0 = \sqrt{\frac{8\pi G \rho_0}{3}}$$

Thus,

$$\boxed{t_0 = \frac{1}{2} H^{-1}} \quad \text{———} \quad (9)$$

for RD Universe.

~~for~~ Further

$$\cancel{\frac{\rho_0}{3H_0^2/8\pi G} = \Omega_0}$$

$$\cancel{\Rightarrow \frac{8\pi G}{3}\rho_0 = \Omega_0 H_0^2}$$

For the non flat ($k \neq 0$) ~~MD~~ Universe, we can start from Friedmann Eq^n

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3}\rho_0 \frac{R_0}{R}$$

and use the fact that $\frac{k}{R_0^2} = H_0^2(\Omega_0 - 1)$

and $\frac{R_0}{R} = (1+z)$, to ge

$$\left(\frac{\dot{R}}{R_0}\right)^2 + H_0^2(\Omega_0 - 1) = \frac{8\pi G}{3}\rho_0 (1+z)$$

Further, $\frac{\rho_0}{3H_0^2/8\pi G} = \Omega_0$

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$$\Rightarrow \frac{8\pi G}{3}\rho_0 = \Omega_0 H_0^2$$

Thus.

$$\left(\frac{\dot{R}}{R_0}\right)^2 + H_0^2(\Omega_0 - 1) = \Omega_0 H^2(1+z)$$

$$\left(\frac{\dot{R}}{R_0}\right)^2 = H_0^2\left(\Omega_0(1+z) + 1 - \Omega_0\right)$$

$$\frac{\dot{R}}{R_0} = H_0\left(\Omega_0(1+z) + 1 - \Omega_0\right)^{1/2}$$

Thus,

$$\frac{dR}{dt} = H_0 R_0\left(1 - \Omega_0 + \Omega_0(1+z)\right)^{1/2}$$

$$\frac{dR}{H_0 R_0\left(1 - \Omega_0 + \Omega_0(1+z)\right)^{1/2}} = dt$$

$$t = \int_0^t dt = H_0^{-1}\int_0^{R(t)} \frac{dR}{R_0\left[1 - \Omega_0 + \Omega_0(1+z)\right]^{1/2}}$$

The integral can be evaluated using the change of variables $x^{-1} = (1+z)$ to get the result.

Thus,

$$t = H_0^{-1}\frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}}\left[\cos^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0 z + \Omega_0}\right) - \frac{2(\Omega_0 - 1)^{1/2}(\Omega_0 z + 1)^{1/2}}{\Omega_0(1+z)}\right]$$

for $\Omega_0 > 1$

$$t = H_0^{-1}\frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\left[\cos^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0 z + \Omega_0}\right) - \frac{2(1 - \Omega_0)^{1/2}(\Omega_0 z + 1)^{1/2}}{\Omega(1+z)}\right]$$

for $\Omega_0 < 1$

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The present age of the Universe is given by the above expressions by taking $z=0$.

Observational Numbers

For the flat matter Dominated Universe, the Age of the Universe is related to the Hubble Constant H_0 by:

$$t_0 = \frac{2}{3} H_0^{-1}$$

Recall that $H_0 = 100\, h\ km\ s^{-1}\ mpc^{-1}$ with $h \approx 0.7$

Thus,

$$H_0^{-1} = h^{-1} (100\ km\ s^{-1}\ mpc)^{-1}$$

$$= h^{-1} \times 9.78 \times 10^9 \text{ years}$$

or

$$t_0 = \frac{2}{3} \times 9.78 \times 10^9 \times h^{-1} \text{ yrs}$$

For $h = 0.75 = \frac{3}{4}$, $h^{-1} = 1.33$

$$\Rightarrow t_0 = \frac{2}{3} \times 1.33 \times 9.78 \times 10^9 \text{ yrs}$$

$$\Rightarrow t_0 = 8.67 \times 10^9 \text{ years.}$$

Problem : Observed age is estimated to be 14×10^9 yrs

With Vacuum Energy (Dark Energy)

Let us consider the case of the flat $(k=0, \Omega=1)$ Universe but now allow both matter and vacuum energy.

In this case, $\Omega_{total}=1$ and Ω_m and Ω_v are significant

$$\Omega_m + \Omega_v = 1 \text{ for the case of interest}$$

In general, Friedmann Eqn is

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{k}{R^2} = \frac{8\pi G}{3}\rho$$

For $k=0 \Rightarrow \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho$

where $\rho = \rho_m + \rho_v = \rho_{v0} + \frac{\rho_{m0}}{R^3}$.

$$\dot{R}^2 = \frac{8\pi G R^2}{3}\left[\rho_{v0} + \frac{\rho_{m0}}{R^3}\right]$$

$$\Rightarrow \frac{dR}{dt} = \sqrt{\frac{8\pi G}{3}\rho_{v0}}\; R\left[1 + \frac{\rho_{m0}}{\rho_{v0}R^3}\right]^{1/2}$$

$$\cancel{= \sqrt{\frac{8\pi G \rho_{v0}}{3}}\; R\left[1 + \frac{\Omega_m}{\Omega_v}\cdot\frac{1}{R^3}\right]^{1/2}}$$

$$= H_0\,\Omega_v^{1/2}\, R\left[1 + \frac{\Omega_m}{\Omega_v}\cdot\frac{1}{R^3}\right]^{1/2}$$

$$\because \frac{\rho_v}{3H^2/8\pi G} = \Omega_v \Rightarrow \frac{8\pi G}{3H^2}\rho_v = \Omega_v \Rightarrow \sqrt{\frac{8\pi G}{3}\rho_v} = \Omega^{1/2} H$$

$$t_0 = \int dt = H^{-1}\Omega_v^{-1/2}\int \frac{1}{R}\frac{dR}{\left[1 + \frac{1}{R^3}\frac{\Omega_m}{\Omega_v}\right]^{1/2}}$$

Recall $\Omega_m + \Omega_v = 1$ and so we can eliminate Ω_m in favour of Ω_v.

Age of the Universe with Vacuum Energy

$$t_0 = \int dt = H_0^{-1} \cdot \Omega_v^{-1/2} \int \frac{1}{R} \frac{dR}{\left[1 + \frac{1}{R^3}\frac{\Omega_m}{\Omega_v}\right]^{1/2}}$$

Recall $\Omega_m + \Omega_v = 1$ & $\Omega_m = 1 - \Omega_v$

$$\text{Integral} = \frac{1}{3} \ln\left(\frac{\frac{1}{\Omega_v^{1/2}} + 1}{\frac{1}{\Omega_v^{1/2}} - 1}\right)$$

$$\text{where Integral} = \int_0^1 \frac{1}{R} \frac{dR}{\left[1 + \frac{1}{R^3}\frac{\Omega_m}{\Omega_v}\right]^{1/2}}$$

Thus

$$t_0 = H_0^{-1} \Omega_v^{-1/2} \frac{1}{3} \ln\left[\frac{\frac{1}{\Omega_v^{1/2}} + 1}{\frac{1}{\Omega_v^{1/2}} - 1}\right]$$

$$= H^{-1} \Omega_v^{-1/2} \frac{1}{3} \ln\left(\frac{1 + \Omega_v^{1/2}}{1 - \Omega_v^{1/2}}\right)$$

$$= H^{-1} \Omega_v^{1/2} \frac{1}{3} \ln\left(\frac{(1 + \Omega_v^{1/2})^2}{1 - \Omega_v}\right)$$

$$= H^{-1} \Omega_v^{1/2} \frac{1}{3} \ln\left[\left(\frac{(1 + \Omega_v^{1/2})}{(1 - \Omega_v)^{1/2}}\right)^2\right]$$

$$t_0 = H^{-1} \Omega_v^{1/2} \frac{2}{3} \ln\left(\frac{1 + \Omega^{1/2}}{(1 - \Omega_v)^{1/2}}\right)$$

Note: as $\Omega_v \to 1$, $t_0 \to \infty$

Thus accomodating a large age is not a problem when vacuum energy is present.

Recall that

$$H_0 = 100\, h \text{ km s}^{-1}\text{Mpc}^{-1}$$

$$H_0^{-1} = h^{-1} \times 9.78 \times 10^9 \text{ years}$$

$$H_0^{-1} = 13.6 \times 10^9 \text{ years}$$

$$t_0 = \frac{2}{3} \times 13.6 \times 10^9 \text{ years} \times \Omega_v^{-1/2} \ln\left[\frac{1+\Omega_v^{1/2}}{(1-\Omega_v)^{1/2}}\right]$$

From observations we know that

$$t_0 = 13.6 \times 10^9 \text{ years}$$

Thus, we get

$$13.6 \times 10^9 \text{ years} = 9 \times 10^9 \text{ years} \times \Omega_v^{-1/2} \ln\left[\frac{1+\Omega_v^{1/2}}{(1-\Omega_v)^{1/2}}\right] \quad \text{—(ii)}$$

Can solve eq^n (ii) numerically to get

$$\boxed{\Omega_v \approx 0.75}$$

Thus, about 3/4 of the energy density of the universe is in the form of vacuum energy or dark energy.

Recall that the Dark Energy has the equation of state which is of the form $P = w\rho$ with $w < 0$

Other Implications of Friedmann Eqn

· Scale factor as a ~~function~~ function of Time

Consider the observationally and theoretically supported case of flat ($k=0, \Omega=1$) universe. In this case Friedmann Eqn is

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_0 \cancel{\left(\frac{R_0}{R}\right)} \quad \text{——— (12)}$$

Recall also that we had the first Law of Thermodynamics:

$$d(\rho R^3) = -P\,d(R^3)$$

together with the eqn of state:

$$P = w\rho \quad \text{——— (13)}$$

which implies.

$$\rho \propto R^{-3(1+w)} \quad \text{——— (14)}$$

$$\frac{\rho}{\rho_0} = \left(\frac{R}{R_0}\right)^{-3(1+w)} \quad \text{——— (15)}$$

Putting (15) in (12) $\Rightarrow$.

$$\left(\frac{\dot{R}}{R}\right) = \frac{8\pi G\rho_0}{3}\left(\frac{R}{R_0}\right)^{-3(1+w)}$$

$$\dot{R}^2 = \frac{8\pi G\rho_0}{3}\,\frac{R^{-3-3w+2}}{R_0^{-3(1+w)}}$$

Choosing $R_0 = 1 \Rightarrow$

$$\dot{R}^2 = \frac{8\pi G\rho_0}{3}R^{-1-3w}.$$

$$\dot{R} = \sqrt{\frac{8\pi G\rho_0}{3}}\,R^{-\frac{1}{2}(1+3w)}$$

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$$\frac{dR}{dt} = \sqrt{\frac{8\pi G}{3}\rho_0}\; R^{-\frac{1}{2}(1+3w)}$$

$$\int \frac{dR}{R^{-1/2(1+3w)}} = \int\sqrt{\frac{8\pi G}{3}\rho_0}\,dt$$

$$\int R^{1/2(1+3w)}\,dR = \sqrt{\frac{8\pi G\rho_0}{3}}\int dt$$

$$R^{\frac{3}{2}(1+w)} = \frac{3}{2}(1+w)\sqrt{\frac{8\pi G\rho_0}{3}}\;t$$

$$\left[R^{\frac{3}{2}(1+w)}\right]^{\frac{2}{3(1+w)}} = \left(4(1+w)\pi G\rho_0\right)^{\frac{2}{3(1+w)}} t^{\frac{2}{3(1+w)}}$$

$$\boxed{R = k\,t^{\frac{2}{3(1+w)}}} \quad \text{——— (16)}$$

For Radiation Dominated Universe $w = \frac{1}{3}$

$$\Rightarrow \frac{2}{3(1+w)} = \frac{2}{3(1+\frac{1}{3})} = \frac{2}{3+1} = \frac{1}{2}$$

$$\boxed{R = k\,t^{1/2}} \quad \text{for RD ——— (17)}$$

~~For~~

For MD universe $w = 0$

$$\frac{2}{3(1+w)} = \frac{2}{3}$$

Thus,

$$\boxed{R = k\,t^{2/3}} \quad \text{for MD ——— (18)}$$

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For $\omega = -1$ (Vacuum Dominated)

$$\frac{2}{3(1-1)} \longrightarrow \infty$$

Expansion is actually faster that any polynomial in t (faster than any t^n with fixed n).

Consider Friedmann Eqn for the Vacuum Energy case more carefully.

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho \quad \text{—— (19)}$$

For vacuum energy $(\omega = -1)$, we know that

~~at~~ $\rho = \rho_0 = \text{constant}$ —— (20).

Putting (20) in (19)

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_0 \quad \text{—— (21).}$$

Thus, we get

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_0 = \text{Const} = H_0^2$$

$$\Rightarrow \left(\frac{\dot{R}}{R}\right) = H_0$$

$$\frac{1}{R}\frac{dR}{dt} = H_0$$

$$\int \frac{1}{R}\, dR = H_0 \int dt$$

$$\ln R = H_0 t$$

$$\boxed{R(t) = \exp(H_0 t)} \quad \text{—— (23).}$$

Summary

For RD Universe, $\omega = \frac{1}{3}$

$$R = K_1 t^{1/2}$$

For MD Universe, $\omega = 0$

$$R = K_2 t^{2/3}$$

For Dark Energy Universe, $\omega = -1$

$$R(t) = \exp(H_0 t)$$

Thermodynamics

Quantities of physical interest that thermodynamics in nature are :

$n \leftrightarrow$ Number density of particles.

$\rho \rightarrow$ Energy density of particles.

$P \rightarrow$ Pressure due to a gas of particles.

These can be given in terms of the phase space distribution (sometimes called the occupancy) function $f(\vec{p})$, where $\vec{p}$ is momentum.

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p})\, d^3p \quad \text{—— (1)}$$

$$\rho = \frac{g}{(2\pi)^3} \int E(\vec{p})\, f(\vec{p})\, d^3p \quad \text{—— (2)}$$

and $$P = \frac{g}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E} f(\vec{p})\, d^3p \quad \text{—— (3)}$$

where g = no. of internal degrees of freedom (e.g. $g=2$ for a spin $\frac{1}{2}$ particle such ~~the~~ as an e^{-})

and $E^2 = |\vec{p}|^2 + m^2$ —— (4)

i.e. E is the energy.

f → Defined and fixed in thermal equilibrium

Depends on Spin of particles

$S = 1/2, 3/2, 5/2$ ← Half-Integer

↓

Anti-Symmetric under exchange of particles

Wave function $\psi \rightarrow -\psi$

Fermions obey Fermi-dirac Statistics

Integer → $S = 0, 1, 2.$

↓

Symmetric under exchange of particles.

$\psi \rightarrow \psi$

Bosons obey Bose-Einstein statistics

Particle with integer spin ($s=0, 1, 2$) are Bosons and their phase space distribution is given by

Bose - Einstein Distribution:-

$$f(\vec{p}) = \frac{1}{\exp[(E-\mu)/T]-1} \quad \text{—— (5)}$$

Particles with half integer spin ($S = 1/2, 3/2$) are Fermions and their phase space distribution is

$$f(\vec{p}) = \frac{1}{\exp[(E-\mu)/T]+1} \quad \text{—— (6)}$$

μ is chemical potential and describes chemical ~~po~~ equilibrium.

μ: For example for the species "i" interacting with species "j, k and l" by reaction:

$$i + j \rightleftharpoons k + l$$

the chemical potentials obey the relationship:

$$\mu_i + \mu_j = \mu_k + \mu_l$$

Thus, μ captures the information about chemical equilibrium.

Frequently, we combine the description of Fermi-Dirac species and Bose Einstein species and write expressions as:

$$f(\bar{p}) = \frac{1}{\exp[(E-\mu)/T] \pm 1} \qquad (7)$$

+ : Fermi Dirac species

− : Bose Einstein species

Taking the distribution funⁿ f given by (7) and inserting into general ~~equati~~ expressions for n, ρ and P given by (1), (2) and (3) respectively, we get:

$$n = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp((E-\mu)/T) \pm 1} E\,dE$$

$$\rho = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp[(E-\mu)/T] \pm 1} E^2\,dE$$

and

$$P = \frac{g}{6\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{3/2}}{\exp[(E-\mu)/T] \pm 1} dE$$

where we have changed the variable of integration from p to E and used the relationship (4)

i.e. $E^2 = |\vec{p}|^2 + m^2$

Consider n: start with

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p})\, d^3p$$

$d^3p = dp_x\, dp_y\, dp_z$ (in cartesian coordinates)

$= p^2 dp\, d(\sin\theta)\, d\phi$ (in spherical coordinates)

$\int d^3p = \int r^2 dr \int d(\sin\theta) \int d\phi$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\Downarrow \quad\quad \Downarrow$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad 2 \quad\quad 2\pi$

$= 4\pi \int r^2 dr$

$$= \frac{g}{8\pi^3} \int f(\vec{p})\, 4\pi p^2\, dp$$

$$= \frac{g \cdot 4\pi}{8\pi^3} \int f(\vec{p})\, p^2\, dp$$

$$= \frac{g}{2\pi^2} \int f(\vec{p})\, p^2\, dp$$

$E^2 = |\vec{p}|^2 + m^2$, $\quad p = 0 \quad E = m$

$2E\, dE = 2p\, dp \Rightarrow p^2 dp = pE\, dE = (E^2 - m^2)^{1/2} E\, dE$

$$n = \frac{g}{2\pi^2} \int_m^\infty f(\vec{p})\, (E^2 - m^2)^{1/2}\, E\, dE$$

$$\boxed{n = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp((E-\mu)/T) \pm 1}\, E\, dE}$$

ρ: start with

$$\rho = \frac{g}{(2\pi)^3} \int E(\vec{p})\, f(\vec{p})\, d^3p$$

$$= \frac{g}{2\pi^2} \int_m^\infty p\, f(\vec{p})\, E^2\, dE$$

$$\boxed{\rho = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2}E^2\,dE}{\exp[(E-\mu)/T]\pm 1}}$$

P: start with

$$P = \frac{g}{(2\pi)^3}\int \frac{|\vec{p}|^2}{3E} f(\vec{p})\, d^3p$$

$$= \frac{g}{2\pi^2}\cdot\frac{1}{3}\int_m^\infty \frac{|\vec{p}|^2 f(\vec{p})\, pE\,dE}{E}$$

$$= \frac{g}{6\pi^2}\int_m^\infty |\vec{p}|^3 f(\vec{p})\, dE$$

$$\boxed{P = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2-m^2)^{3/2}}{\exp[(E-\mu)/T]\pm 1}\,dE}$$

Thus, to summarize the expressions for n, ρ and P are:

$$n = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2}}{\exp[(E-\mu)/T]\pm 1}E\,dE \qquad (8)$$

$$\rho = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2}}{\exp((E-\mu)/T]\pm 1}E^2\,dE \qquad (9)$$

$$P = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2-m^2)^{3/2}}{\exp[(E-\mu)/T]\pm 1}\,dE \qquad (10)$$

$n(T,\mu,m)$: Number density

$\rho(T,\mu,m)$: Energy density

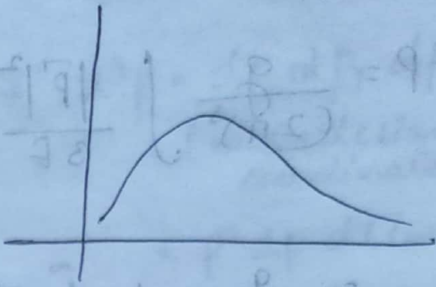
$P(T,\mu,m)$: Pressure

Limits : → High Temperature Limit
→ Low Temperature Limit

$\langle E_e \rangle \sim K_B T$

$\langle p \rangle \approx K_B T$

$E^2 = p^2 + m^2$



$(K_B)\, T >> m\,(c^2)$

High Temp → Relativistic Limit

$T >> \mu$

Heaviest particle in Standard model and Particle physics.

$m_T \sim 100$ GeV

$T >> 100$ GeV - high

$T \sim 1$ eV - low

For the High Temperature Limit (Applicable to the Early Universe)

$T >> m$ and $T >> \mu$

$$\rho = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp[(E-\mu)/T] \pm 1} E^2 dE$$

First, let us use $\mu << T \Rightarrow e^{-\mu/T} = 1$.

$$\Rightarrow \rho = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2-m^2)^{1/2} E^2 dE}{\exp(E/T) \pm 1}$$

Now use $k^2 = E^2 - m^2 \Rightarrow 2k\,dk = 2E\,dE$

$$\rho = \frac{g}{2\pi^2} \int_m^\infty \frac{k\,(k^2+m^2)^{1/2} k\,dk}{\exp\left((k^2+m^2)/T\right) \pm 1}$$

Now use $m << T$ and set $k' = k/T \Rightarrow dk = T\,dk'$

$$\rho = \frac{g}{2\pi^2} \int_m^\infty \frac{(k'T)^3\, T\,dk'}{\exp(k') \pm 1}$$

$$\boxed{\rho = \frac{g\,T^4}{2\pi^2} \int_0^\infty \frac{k'^3\,dk'}{\exp(k') \pm 1}} \quad \text{———} \quad (11)$$

For Bosons $-$ sign $\Rightarrow \boxed{\rho = \frac{\pi^2}{30} g T^4}$ ——— (12)

Fermion $+$ sign $\Rightarrow \boxed{\rho = \left(\frac{7}{8}\right)\left(\frac{\pi^2}{30}\right) g T^4}$ ——— (13)

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Consider n in high T limit

$$n = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2} E\,dE}{\exp[(E-\mu)/T]\pm 1}$$

$T \gg \mu$

$$n = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2} E\,dE}{\exp(E/T)\pm 1}$$

$E^2 = k^2 + m^2$, ~~and~~ $2E\,dE = 2k\,dk$ and $m \ll T$

$$n = \frac{g}{2\pi^2}\int_0^\infty \frac{k^2\,dk}{\exp(E/T)\pm 1}$$

let $k' = k/T \Rightarrow dk = T\,dk'$

$$n = \frac{g}{2\pi^2}\int_0^\infty T^3 \frac{k'^2\,dk'}{\exp(k')\pm 1}$$

$$n = \frac{gT^3}{2\pi^2}\underbrace{\int_0^\infty \frac{k'^2\,dk'}{\exp(k')\pm 1}}_{\text{pure number}}$$

Thus for Bosons

$$n = gT^3\left(\frac{\zeta(3)}{\pi^2}\right) \qquad (14)$$

where $\zeta(3) = 1.202\ldots$

For ~~ferf~~ Fermions (+ve sign in denominator)

$$n = gT^3\left(\frac{3}{4}\right)\left[\frac{\zeta(3)}{\pi^2}\right] \qquad (15)$$

For the pressure P

$$P = \frac{g}{6\pi^2}\int_{m}^{\infty} \frac{(E^2-m^2)^{1/2}\, dE}{\exp((E-\mu)/T) \pm 1}$$

again using $\mu << T$ & $k^2+m^2 = E^2$ & $k\,dk = E\,dE$, we get

$$P = \frac{g}{6\pi^2}\int_0^{\infty} \frac{k^3 dk}{\exp[k/T] \pm 1}$$

Let $k' = \frac{k}{T}$, $dk = T dk'$

$$P = \frac{g}{6\pi^2}\int_0^{\infty} \frac{T^4 k'^3 dk'}{\exp(k') \pm 1}$$

Thus, we get

$$P = \frac{gT^4}{6\pi^2}\int_0^{\infty} \frac{k'^3 dk'}{\exp(k') \pm 1} \quad \text{——— (16)}$$

Comparing (16) with the expression (11) for energy density ρ, we note

$$P = \frac{1}{3}\rho \quad \text{——— (17)}$$

Note that (17) is the eqn of state for Radiation or Relativistic Matter with we have now derived from fundamental thermodynamics.

Now consider the non-relativistic case $m >> T$

In this case, the number density is given by:

$$n = \frac{g}{(2\pi)^3}\int f(\vec{p})\, 4\pi p^2 dp$$

$$= \frac{g}{2\pi^2} \int f(\vec{p})\, p^2\, dp$$

with $f(\vec{p}) = \dfrac{1}{\exp\left[\frac{E-\mu}{T}\right] \pm 1}$

$$E = (p^2+m^2)^{1/2} = m\left(1+\frac{p^2}{m^2}\right)^{1/2} = m\left(1+\frac{p^2}{2m^2}\right)$$

$$f(p) = \frac{1}{\exp\left[\frac{m}{T}\left(1+\frac{p^2}{2m^2}\right) - \frac{\mu}{T}\right] \pm 1}$$

$$\frac{m}{T} >> 1 \Rightarrow \exp\left[\frac{m}{T}\left(1+\frac{p^2}{2m^2}\right) - \frac{\mu}{T}\right] >> 1$$

$$f = \frac{1}{\exp\left[\frac{m}{T}\left(1+\frac{p^2}{2m^2}\right) - \frac{\mu}{T}\right]} = \exp\left(-\frac{(m-\mu)}{T}\right)\exp\left(\frac{-p^2}{2mT}\right)$$

$$\therefore\ n = \frac{g}{2\pi^2} \exp\left(-\frac{(m-\mu)}{T}\right) \int \exp\left(\frac{-p^2}{2mT}\right) p^2\, dp$$

Let $p'^2 = \dfrac{p^2}{2mT} \Rightarrow p = p'(2mT)^{1/2}$

or $dp = (2mT)^{1/2}\, dp'$

$$n = \frac{g}{2\pi^2} \exp\left[\frac{-(m-\mu)}{T}\right] (2mT)^{3/2} \underbrace{\int_0^\infty \exp(-p'^2)\, p'^2\, dp'}_{\text{pure number}\ \left(=\frac{\sqrt{\pi}}{4}\right)}$$

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left[\frac{-(m-\mu)}{T}\right] \quad \text{—— (24)}$$

Similarly for the pressure P in $m >> T$ limit

$$P = \frac{g}{(2\pi^3)} \int \frac{|\vec{p}|^2}{3E} f(\vec{p}) d\vec{p}$$

$$P = g \exp\left[\frac{-(m-\mu)}{T}\right] \left(\frac{mT}{2\pi}\right)^{3/2} T \quad \text{——— (25)}$$

Comparing (25) and (24), we get

$$\boxed{P = nT} \quad \text{——— (26)}$$

where P is pressure, n is number density and T is temperature.

Finally, consider ρ:

$$\rho = \frac{g}{(2\pi)^3} \int E(p) f(p) d^3p.$$

Making same approximations and similar steps we get:

$$\rho = g\left(\frac{mT}{2\pi}\right)^{1/2} \exp\left[-\frac{(m-\mu)}{T}\right] . m \quad \text{——— (27).}$$

From (27) and (24), we get

$$\boxed{\rho = mn} \quad \text{——— (28).}$$

Further we note from (26) that $P = nT$ and (28) tells us $\rho = mn$

Dividing (26) by (28 $\Rightarrow$

$$\frac{P}{\rho} = \frac{nT}{mn} = \frac{T}{m}$$

Thus,

$$P = \left(\frac{T}{m}\right) \rho \quad \text{——— (29)}$$

In non-relativistic limit $\frac{T}{m} << 1$

$$P << \rho \quad \text{——— (30)}$$

or $P = \omega \rho$ with $\omega \to 0$ or $\omega = 0$ eqn of state for non relativistic matter.

Things to note and remember :

In the relativistic limit

$T >> m$ and

$\rho \sim T^4$

$n \sim T^{3}$

$P \sim T^4$

In non relativistic limit

$m >> T$ ang.

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left[-\frac{(m-\mu)}{T}\right]$$

$\rho = mn$

$P = nT << \rho$

In non relativistic limit all quantities are exponentially suppressed by $\exp\left[-\frac{(m-\mu)}{T}\right]$ factor and small compared to the similar quantities in relativistic limit.

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In general, the total energy density and Pressure for all species in equilibrium can be expressed in terms of the photon temperature T are:

$$\rho = T^4 \sum_{i=\text{all species}} \left(\frac{T_i}{T}\right)^4 \frac{g_i}{2\pi^2} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2} u^2 du}{\exp(u - y_i) \pm 1} \quad \text{—— (31)}$$

$$P = T^4 \sum_{i=\text{all species}} \left(\frac{T_i}{T}\right)^4 \frac{g_i}{6\pi^2} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{3/2} du}{\exp(u - y_i) \pm 1} \quad \text{—— (32)}$$

where $x_i = \frac{m_i}{T}$ & $y_i = \frac{\mu_i}{T}$

By introducing the quantity T_i we allow that species i may have a thermal distribution but with a temperature different than that of the photons.

One thing to note is that ρ and P of non-relativistic species (with $m >> T$) are exponentially smaller that that of relativistic species (with $m << T$).

Thus, it is a very good approximation to keep only relativistic species in the sums for ρ and P given in (31) and (32). Thus:

$$\rho = \frac{\pi^2}{30} g_* T^4 \quad \text{—— (33)}$$

and $$P = \frac{\rho}{3} = \frac{\pi^2}{90} g_* T^4 \quad \text{—— (34)}$$

where g_* counts the total number of effective massless ($m << T$) degrees of freedom.

Further g_* can be expressed as:

$$g_* = \sum_{i = bosons} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i = fermions} g_i \left(\frac{T_i}{T}\right)^4$$

The factor $\frac{7}{8}$ is due to the different distribution functions (Fermi Dirac vs Bose Einstien) for fermions and Bosons.

g_* of course depends on T.

Computed n, ρ, P from first principles

Now apply it in the context of the Universe as a whole

Examine connections between H, ρ, $R(t)$, $H(t)$.

During the early Radiation Dominated era

$$\rho \simeq \rho_R = \frac{\pi^2}{30} g_* T^4 \quad \text{———} \quad (36)$$

and $P_R = \frac{\rho_R}{3}$ (that is $w = \frac{1}{3}$) ——— (37)

$$\Rightarrow R(t) \propto t^{1/2}$$

$$\rho \propto R^{-3(1+w)} = R^{-4}$$

can now get t in terms of ρ

In RD era, we had derived the age is given by

$$t = \frac{1}{2} H^{-1} \quad \text{———} \quad (38)$$

The Friedmann Eqn for flat ($k=0$) Universe.

$$H^2 = \frac{8\pi G}{3} \rho \quad \text{———} \quad (39).$$

and $\rho = \rho_R = \frac{\pi^2}{30} g_* T^4$

Thus using eqn (36) in (39) $\Rightarrow$

$$H^2 = \frac{8\pi G}{3} \frac{\pi^2}{30} g_* T^4 \quad \text{———} \quad (40)$$

Taking the square root both sides

$$\boxed{H = K g_*^{1/2} T^2} \quad \text{———} \quad (41)$$

where g_* counts the total number of effectively massless degrees of freedom.

T is the temperature and k is the constant which can be determined from fundamental constants of nature by using eqns (5.40) and (5.41) above i.e.

$$k = \sqrt{\frac{8\pi^3 G}{90}} \quad \text{———} \quad (41b)$$

The age of the Universe can be determined by using eqns (38) and (41)

Thus,

$$t = \frac{1}{2} H^{-1} = \frac{1}{2} k^{-1} g_*^{-1/2} T^{-2}$$

or

the Age. $\longrightarrow$ $$\boxed{t = \alpha g_*^{-1/2} T^{-2}} \quad \text{———} \quad (42)$$

where g_* the total number of relativistic degrees of freedom, T is the temperature and α is a constant which can bee written down in terms of fundamental contents of nature, i.e.

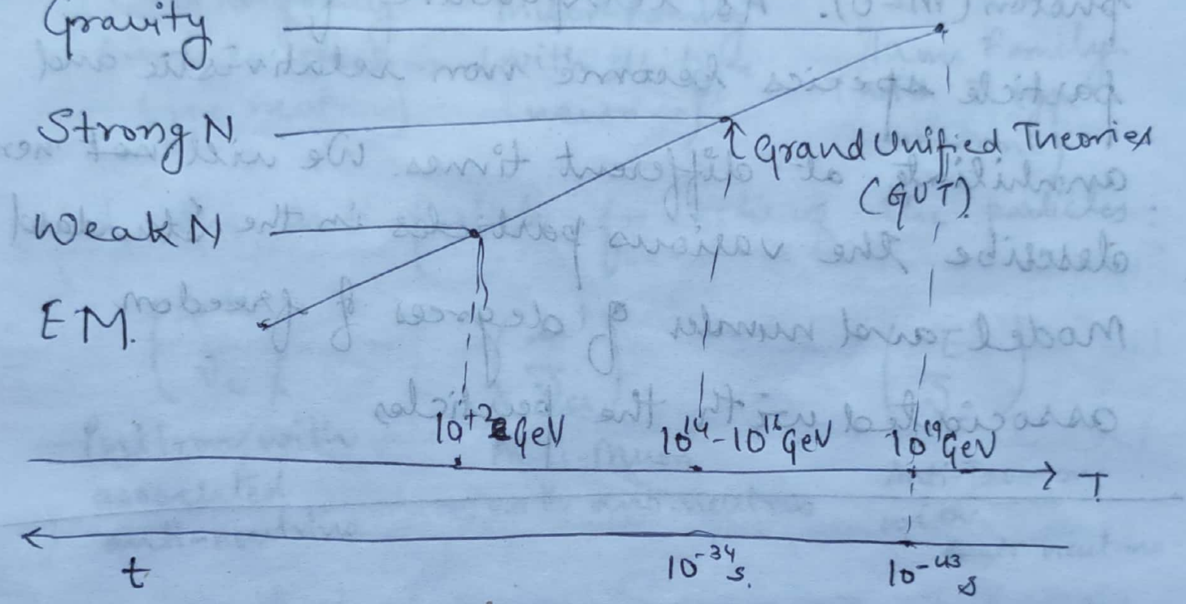
$$\alpha = \frac{1}{2} k^{-1} = \frac{1}{2}\sqrt{\frac{90}{8\pi^3 G}} \quad \text{———} \quad (42b)$$

History of the Universe

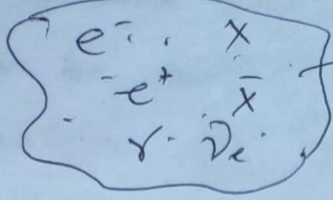
Interactions:

- Gravity.
- Electromagnetism
- Strong Nuclear Interactions
- Weak Nuclear Interactions.

Unification



The parameter that controls the onset of the Quantum Gravity involves the fundamental constants c, $\hbar$ and G in the combination with the dimensions of time $t \sim \left(\frac{G\hbar}{c^5}\right)^{1/2} \sim 10^{-43}$ sec [This is called the Planck Time]. The next event after the end of the Quantum Gravity era is the breakdown of Grand Unification. This happens when $k_B T \sim M_x, M_y$ where M_x, M_y are the masses of the Bosons mediating GUT interactions. This corresponds to time of 10^{-34} sec.



After GUT era, we can use Standard model
→ All particles → form a primordial soup

This primordial soup consists of different species of elementary particles. The masses range from the heaviest known elementary particle, the top quark ($m \sim 175$ GeV) down to the lighest particles, the e^- ($m = 511$ KeV), the neutrinos, and the photon ($m=0$). As temperature falls the various particle species become non relativistic and annhiliate at different times. We will ~~not~~ next describe the various particles in the Standard Model and number of degrees of freedom associated with the particles

Standard Model

Elementary particles come in several categories. First, there are the most familier particles such as the electrons, mesons and the neutrinos. Called the leptons. There are 3 families of leptons:

$$\begin{pmatrix} e^- \\ \nu_e \end{pmatrix} \qquad \begin{pmatrix} \mu^- \\ \nu_\mu \end{pmatrix} \qquad \begin{pmatrix} \tau^- \\ \nu_\tau \end{pmatrix}$$

Electron family (includes electron type neutrino)

muon family (with μ type neutrino)

Taon family (with τ type neutrino)

and the anti particles for each of the particles:

$$\begin{pmatrix} e^+ \\ \bar{\nu}_e \end{pmatrix}. \qquad \begin{pmatrix} \mu^+ \\ \bar{\nu}_\mu \end{pmatrix} \qquad \begin{pmatrix} \tau^+ \\ \bar{\nu}_\tau \end{pmatrix}$$

Positron with associated anti-neutrino

Anti-Muon with anti-neutrino

Anti-Taon with anti-neutrino

The leptons are spin 1/2 particles and are Fermions. The leptons in addition to electromagnetic forces (associated with electric charges) also experience Weak Nuclear Forces. The mediator of EM forces is the photon γ. The mediator of Weak Nuclear Forces are the Gauge Bosons called W^+, W^- and Z^0. They are spin 1 particles. γ is massless and W^+, W^- and Z^0 have masses in 80-91 GeV range.

The strongly interacting spin 1/2 particles are called quarks. The quarks (like the leptons) come in 3 families:

$$\begin{pmatrix} u \\ d \end{pmatrix} \qquad \begin{pmatrix} c \\ s \end{pmatrix} \qquad \begin{pmatrix} t \\ b \end{pmatrix}$$

Called $\begin{pmatrix} \text{up} \\ \text{down} \end{pmatrix} \qquad \begin{pmatrix} \text{charm} \\ \text{strange} \end{pmatrix} \qquad \begin{pmatrix} \text{top} \\ \text{bottom} \end{pmatrix}$.

Theira electric charges are

$$\begin{pmatrix} +2/3 \\ -1/3 \end{pmatrix} \qquad \begin{pmatrix} +2/3 \\ -1/3 \end{pmatrix} \qquad \begin{pmatrix} +2/3 \\ -1/3 \end{pmatrix}$$

The similarity of properties such as electric charges led to the nomenclature of 3 'generations' or 3 'families' because it looks like there are 3 copies of the same structure with most properties the same (except for mass which increases to the right as you climb the family or generation ladder.

In addition to the quarks carrying electric charge, the also carry another internal degree of freedom called 'colour'. This 'colour' is an internal degree of freedom that ~~thas~~ has to do with the strong interactions. There are 3 such 'colours' so there are 3 degrees of freedom according to colour for each of the quarks

The carriers of strong interactions are massless gauge bosons called gluons and there are 8 of these massless gauge bosons ~~&~~ called Gluons.

Finally, there is the Higgs Boson which is responsible for giving masses to all of the particles. The Higgs Boson is a spin 0 particle.

This completes the list of elementary particles. Other particles are made up of these elementary particles

Thus, the proton is made up of the uud quarks. The neutron is made up of the udd quarks. The $\pi^{\pm}$ mesons are made up of $u\bar{d}$ and ~~&~~ $d\bar{u}$ quarks and π^0 is $\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$ in terms of quarks. n, p are spin 1/2 and π meson are spin 0 particles.

At high energy densities and high temperature, the protons, neutrons and π mesons etc are broken up into the constituent quarks and the primordial soup consists of the elementary particles such as quarks, leptons, gluons, electro weak bosons and the Higgs boson.

We ~~ca~~ now summarize the properties of these particles and count the associated degrees of freedom. The degrees of freedom enter in ρ and P and determine the evolution and History of the Universe.

Particles in Standard Model

Quarks (Fermions)

Particles	anti-particles	Spin	Colour	g	Total
t	$\bar{t}$	1/2	G	$g = 2 \times 2 \times 3$ (spin, particle anti particle, colour)	= 12
b	$\bar{b}$				
c	$\bar{c}$				
s	$\bar{s}$				
u	$\bar{u}$				
d	$\bar{d}$				

$\times 6$

$72f$

Gluons (Bosons)

8 massless bosons. 1 massless $g = 2 \times 8 = 16b$

Leptons (Fermions)

τ^-	τ^+	1/2	$g = 2 \times 2 = 4$ (spin, particle anti particle)
μ^-	μ^+		
e^-	e^+		$\times 3$ = $12f$
ν_τ	$\bar{\nu_\tau}$		
ν_μ	$\bar{\nu_\mu}$		massless $g = 2 \times 1$, $\times 3$ = $6f$
ν_e	$\bar{\nu_e}$		

Particle antiparticle spin
ElectroWeak Gauge Bosons (Bosons) g Total

W^+		1		
W^-			$g=3$	$\times 3$
Z^0				
γ		massless	$g=2$	11b

Higgs Boson

H^0	0	$g=1$	1b

Total fermionic degrees of freedom $= g_f = 72+12+6 = 90$

Total bosonic degrees of freedom $= g_b = 16+11+1 = 28$

Total effective degrees of freedom $= g_* = g_b + \frac{7}{8} g_f$

$$= 28 + \frac{7}{8} \times 90 = 106.75$$

$\therefore g_* = 106.75$ for all particles in the standard model

or $g_*(T >> 100\,GeV) = 106.75$

1) Degrees of freedom associated with spin

~~g(s)~~ For massive particles
$g(s) = 2s+1$

For massless particles
$g(s) = 2s$

History of the Universe

Quark Hadron Phase Transition

After the Quark Hadron Phase Transition the only particle species left in the large numbers are pions, mesons, electrons, neutrinos and the photons

The number of degrees of freedom corresponding to these particles are:

	boson/fermion		# of species		spin		
Photon, γ	1	×	1	×	2 (massless spin 1)	=	2
electron e^-, e^+	$\frac{7}{8}$	×	2	×	2	=	$4 \times \frac{7}{8} = 3.5$
mesons μ^-, μ^+	$\frac{7}{8}$	×	2	×	2	=	3.5
Neutrinos — $\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$	$\frac{7}{8}$	×	6	×	1 (massless spin 1/2) (standard model)	=	5.25
Pions $\pi^{\pm}, \pi^0$	1	×	3	×	1 (spin 0)	=	3

Total g_* after Quark Hadron Phase Transition:

$$g_* = 2 + 3.5 + 3.5 + 5.25 + 3$$
$$= 17.25$$

Recall, before the Quark Hadron Phase Transition we had calculated:

g_* (before transition) = 61.75

$\downarrow$ $\longrightarrow$ largest drop in g_* in any event in the history of universe.

17.25

Just after the Quark Hadron Phase transition, the pions $(\pi^{\pm}, \pi^{0})$ and mesons $(\mu^{\pm})$ annihiliate. The corresponding degrees of freedom are.

$\pi^{\pm}, \pi^{0}$ $1 \times 3 \times 1 = 3$

$\mu^{\pm}$ $\frac{7}{8} \times 2 \times 2 = 3.5$

After this g_* remaining is

$$g_* = 17.25 - 6.5 = 10.75$$

The formation of light nuclei

Typical nuclear binding energies are a few MeV. Thus, when $T <$ MeV light nuclei start forming can determine the age of the Universe when this happens using the Friedmann-eq^n

$$H^2 = \frac{8\pi G}{3} \frac{\pi^2}{30} g_*(T)(T)^4 \quad \text{——} \quad (6.27)$$

together with the relationship between age of Universe and the Hubble parameter for a radiation dominated Universe:

$$t = \frac{1}{2} H^{-1}$$

Thus, a $T \sim 0.1$ MeV corresponds to age, $t \sim 1$s. Thus, light nuclei start forming when the universe is about 1 s old.

Next event (after formation of light nuclei) is matter-radiation equality followed by matter domination. The early universe is Radiation Dominated. However, since the energy density in radiation, $\rho_R \propto R^{-4}$ as a function of scale factor and $\rho_m \propto R^{-3}$, the ratio shifts in favour of matter.

Doing the calculation, it can be shown that matter domination begins at:

$$t \sim 10^3 (\Omega_{m,0} h^2)^{-2} \text{ years}$$

$$(1+z) \sim 2.3 \times 10^4 (\Omega_{m,0} h^2)$$

where z is the redshift, t is the age of the Universe, $\Omega_{m,0}$ is the radio of Matter density to critical density and h is defined by:

$$H_0 = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}$$

where H_0 is the observed value of the Hubble Constant.

After Matter Domination, the next 2 important events are Recombination and Photon Decoupling

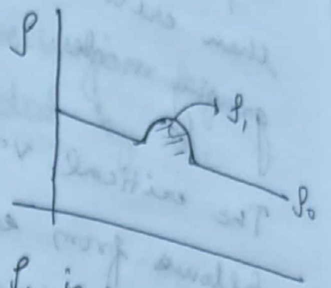
Recombination is formation of neutral atoms. When the $T_{rec} \sim 0.3$ eV then e^-s are no longer knocked away from nuclei and one gets bound neutral atoms. This happens at redshift $Z_{rec} \sim 1300$. Soon after this the the photon mean free path becomes larger than the size of the Universe and so Universe becomes transparent. This is called decoupling. It happens when temperature of the Universe $T_{dec} \sim 3000K$ and redshift of the Universe $Z_{dec} \sim 1100$. The age of the Universe at this time is $t \sim 10^5$ years.

The next event after this is Structure formation.

Growth of density fluctuation (perturbation).

$$\rho = \rho_0 + \rho_1$$

Total density — Static, homogenous part of density — Small perturbations (fluctuation)



The eqn for the density perturbation ρ_1 is:

$$\boxed{\frac{\partial^2 \rho_1}{\partial t^2} - v_s^2 \nabla \rho_1 - 4\pi G \rho_0 \rho_1 = 0} \qquad \text{——— (16)}$$

This can be solved by a soln of the form

$$\rho_1(\vec{r},t) = \delta(\vec{r},t)\rho_0 = A \exp[-i\vec{k}.\vec{r} + i\omega t]\rho_0 \qquad \text{——— (17)}$$

Substituting (17) in (16), we get

$$[A\rho_0 (i\omega)^2 - v_s^2 (-i\vec{k})(-i\vec{k})A\rho_0 - 4\pi G \rho_0 A\rho_0] e^{-i\vec{k}\vec{r}+i\omega t} = 0$$

$$\Rightarrow [-\omega^2 + v_s^2 k^2 - 4\pi G \rho_0]\rho_1(\vec{r},t) = 0 \qquad \text{——— (18)}$$

where $k = |\vec{k}|$.

Thus for $\rho_1(\vec{r},t) \neq 0$, we must have

$$\boxed{\omega^2 = v_s^2 k^2 - 4\pi G \rho_0} \qquad \text{——— (19)}$$

Recall that the soln $\rho_1(\vec{r},t)$ is of the form

$$\rho_1(\vec{r},t) = \{A\rho_0 e^{+i\vec{k}.\vec{r}}\} e^{i\omega t} \qquad \text{——— (20)}$$

If ω is real then perturbation ρ_1 will simply oscillate as sound (compressional) waves.

However, if ω is imaginary, then there will be exponentially growing modes. Thus for k less than critical value, there will be exponentially growing modes.

The critical value of k is called k_J and its value follows from eqn (19) when $\omega^2 = 0$ with the exponentially growing solutions corresponding to $\omega^2 < 0$. The critical value of k_J is

$$\boxed{k_J = \left(\frac{4\pi G}{v_s^2}\right)^{1/2}} \quad \text{---} \ (21).$$

For $k^2 << k_J^2$, ρ_1 grows exponentially on the dynamical time scale:

$$\tau_{dyn} = (\mathrm{Im}\,\omega)^{-1} \simeq (4\pi G \rho_0)^{-1/2} \quad \text{---} \ (22).$$

The length scale corresponding to the wave number k_J is

$$\boxed{\lambda_J = \frac{2\pi}{k_J}} \quad \text{---} \ (23)$$

There are 2 forces at play here :

1) Gravity which causes density perturbations to collapse ~~to~~ under attractive influence of gravity
2) Pressure which acts against the attractive gravitational forces and tries to prevent gravitational collapse. ~~Sometimes gravit~~

Sometimes gravity wins, sometimes pressure wins.

The timescale for Gravitation collapse is :

$$\tau_{grav} \approx (G\rho_0)^{-1/2} \quad \text{——— (24)}$$

The timescale of the pressure response is given by the size of the perturbation (λ) divided by the sound speed (v_s), that is:

$$\tau_{pressure} \sim \frac{\lambda}{v_s} \quad \text{——— (25)}$$

If $\tau_{pressure}$ exceeds τ_{grav}, the gravitional collapse occurs before pressure forces can respond to restore equilibrium.

Thus, for gravitational collapse; we must have that

$$\tau_{pressure} > \tau_{grav} \quad \text{—— (26)}$$

$$\lambda > \frac{v_s}{(G\rho_0)^{1/2}} \quad \text{——— (27)}$$

$$\lambda \geq \lambda_J \quad \text{——— (28)}$$