Post Midsem.

Age of the Universe

$$\dot{R} = \frac{dR}{dt}$$

$$\frac{dR}{dt} = f(R)$$

$$\int \frac{dR}{f(R)} = \int dt$$

Age can be obtained by integrating Fridmann's eq^n

Energy density scale as:

$$\frac{\rho}{\rho_0} = \left(\frac{R}{R_0}\right)^{-3} \quad \text{for Matter dominated universe} \quad (3\text{-}1)$$

$$\frac{\rho}{\rho_0} = \left(\frac{R}{R_0}\right)^{-4} \quad \text{" Radiation " universe} \quad (3\text{-}2)$$

The friedmann eq^n becomes:

$$\left(\frac{\dot{R}}{R_0}\right)^2 + \frac{k}{R_0^2} = \frac{8\pi G\rho_0}{3}\frac{R_0}{R} \quad \text{for MD} \quad (3.3)$$

&

$$\left(\frac{\dot{R}}{R_0}\right)^2 + \frac{k}{R_0^2} = \frac{8\pi G\rho_0}{3}\left(\frac{R_0}{R}\right)^2 \quad \text{for RD} \quad (3.4)$$

Note that eq^n (3.3) & (3.4) are obtained by taking the Fridman eq^n $\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G\rho}{3}$

using (3.1), (3.2) & by multiplying $\left(\frac{R}{R_0}\right)^2$ on both sides

Camlin Page
Date / /

Eqⁿ (3.3) & (3.4) can be integrated in general with some multiplication but consider first the simplest case $k=0$ this corresponds to the observationaly & theoritical favoured case of a flat ~~universe~~. universe

For the flat ($k=0$) MD universe

$$\left(\frac{\dot{R}}{R_0}\right)^2 = \frac{8\pi G}{3} \rho_0 \frac{R_0}{R}$$

$$\dot{R} = \sqrt{\frac{8\pi G}{3} \frac{\rho_0 R_0^3}{R}}$$

Thus $$\frac{dR}{dt} = \sqrt{\frac{8\pi G}{3}\rho_0 R_0^3} \cdot \frac{1}{\sqrt{R}}$$

$$\frac{\sqrt{R}\, dR}{\sqrt{\frac{8\pi G \rho_0 R_0^3}{3}}} = dt$$

$$\int_0^{t_0} dt = t_0 = \int_0^{R_0} \frac{1}{\sqrt{\frac{8\pi G\rho_0 R_0^3}{3}}} R^{1/2}\, dR$$

$$t_0 = \frac{1}{\sqrt{\frac{8\pi G\rho_0 R_0^3}{3}}} \left[\frac{R^{3/2}}{3/2}\right]_0^{R_0}$$

$$t_0 = \frac{1}{\sqrt{\frac{8\pi G \rho_0 R_0^3}{3}}} \frac{2}{3} \left(R_0^{3/2} - 0\right)$$

$$\boxed{t_0 = \frac{2}{3} \frac{1}{\sqrt{\frac{8\pi G \rho_0}{3}}} \frac{1}{\cancel{R_0^{3/2}}} \cancel{R_0^{3/2}}} \quad 3.5$$

Scanned by CamScanner

$t_0 = \frac{2}{3} H_0^{-1}$ for MD Univ.

Camlin Page
Date / /

Recall that the friedmann eqn for this case is :

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2 = \frac{8\pi G \rho}{3}$$

Thus at the present t/o $t = t_0$ this implies

Hubble constant $\leftarrow$ $$\boxed{H_0^2 = \frac{8\pi G \rho_0}{3}} \quad (3.6)$$

$$(3.6) \Rightarrow H_0 = \sqrt{\frac{8\pi G \rho_0}{3}}$$

using this in eqn (3.5) we get :

Present t/o $\leftarrow$ $$\boxed{t_0 = \frac{2}{3} H_0^{-1}}$$

for matter dominated universe.

For the flat ($k=0$) Radiation dominated universe from eqn (3.4) setting $k=0$ we get

$$\left(\frac{\dot{R}}{R_0}\right)^2 = \frac{8\pi G \rho_0}{3}\left(\frac{R_0}{R}\right)^2$$

$$\frac{\dot{R}}{R_0} = \sqrt{\frac{8\pi G \rho_0}{3}}\left(\frac{R_0}{R}\right)$$

Thus $$\frac{dR}{dt} = \sqrt{\frac{8\pi G \rho_0}{3}}\;\frac{R_0^2}{R}$$

$$\frac{R\,dR}{R_0^2}\,\frac{1}{\sqrt{\frac{8\pi G \rho_0}{3}}} = dt$$

4.3.Y.6

$t_0 = \frac{1}{2} H_0^{-1}$ for RD univ.

$$t_0 = \int_0^{t_0} dt = \frac{1}{\sqrt{\frac{8\pi G \rho_0}{3}}} \frac{1}{R_0^2} \int_0^{R_0} R\,dR$$

$$t_0 = \frac{1}{\sqrt{\frac{8\pi G \rho_0}{3}}} \frac{1}{R_0^2} \frac{1}{2} \left(R_0^2 - 0\right)$$

$$\boxed{t_0 = \frac{1}{2} \frac{1}{\sqrt{\frac{8\pi G \rho_0}{3}}}} \quad (3.8)$$

Again using the friedmann eqn

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2 = \frac{8\pi G \rho}{3}$$

$$\Rightarrow H_0 = \sqrt{\frac{8\pi G \rho_0}{3}}$$

$$\boxed{t_0 = \frac{1}{2} H_0^{-1}} \quad (3.9)$$

for RD universe

For Non-flat universe $(k \neq 0)$ MD universe we can start from the friedmann eqn:

$$\left(\frac{\dot{R}}{R_0}\right)^2 + \frac{k}{R_0^2} = \frac{8\pi G \rho_0}{3} \frac{R_0}{R}$$

we use the fact that $\frac{k}{R_0^2} = H_0^2 (\Omega_0 - 1)$

& $\frac{R_0}{R} = (1+z)$ to get:

$$\left(\frac{\dot{R}}{R_0}\right)^2 + H_0^2(\Omega - 1) = \frac{8\pi G}{3}\rho_0(1+z)$$

Further $\frac{\rho_0}{3H_0^2/8\pi G} = \Omega_0$

$$\frac{8\pi G}{3}(\rho_0) = \Omega_0 H_0^2$$

Thus $$\left(\frac{\dot{R}}{R_0}\right)^2 + H_0^2(\Omega_0 - 1) = \Omega_0 H_0^2 (1+z)$$

$$\left(\frac{\dot{R}}{R_0}\right)^2 = H_0^2\left[\Omega(1+z) + 1 - \Omega\right]$$

$$\frac{\dot{R}}{R_0} = H_0\left[\Omega_0(1+z) + 1 - \Omega_0\right]^{1/2}$$

Thus
$$\frac{dR}{dt} = H_0 R_0\left[1 - \Omega_0 + \Omega_0(1+z)\right]^{1/2}$$

$$\Rightarrow \frac{dR}{H_0 R_0\left[1-\Omega_0+\Omega_0(1+z)\right]^{1/2}} = dt$$

$$t = \int_0^t dt = H_0^{-1}\int_0^{R(t)} \frac{dR}{R_0\left[1-\Omega_0+\Omega_0(1+z)\right]^{1/2}}$$

The integral can be evaluated using the change of variables $x^{-1} = (1+z)$ to get the result.

Thus

$$t = \frac{H_0^{-1}\,\Omega_0}{2(\Omega_0 - 1)^{3/2}} \times \left[\cos^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0 z + \Omega_0} \right) - \frac{2(\Omega_0 - 1)^{1/2}(\Omega_0 z + 1)^{1/2}}{\Omega_0 (1+z)} \right]$$

for $\Omega_0 > 1$

~~for $\Omega_0 > 1$~~

and

$$t = \frac{H_0^{-1}\,\Omega}{2(1 - \Omega_0)^{3/2}} \times \left[\cos^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0 z + \Omega_0} \right) + \frac{2(1 - \Omega_0)^{1/2}(\Omega_0 z + 1)^{1/2}}{\Omega_0 (1+z)} \right]$$

for $\Omega_0 < 1$

The present age of the universe is given by the above expression by setting $z = 0$ for t_0

Observational numbers

for the flat ~~non~~ matter dominated universe, the age of the universe is related the hubble constant H_0 by:

$$t_0 = \frac{2}{3} H_0^{-1}$$

Recall that $H_0 = 100\,h$ km^{-1} Mpc^{-1}

with $h \sim 0.75$

Thus $H_0^{-1} = h^{-1} (100 \text{ km s}^{-1} \text{ Mpc}^{-1})^{-1}$

$= h^{-1} (9.78) \times 10^{9}$) years

or $t_0 = \frac{2}{3} \times 9.778 \times 10^{9} \times h^{-1}$ years.

$t_0 = 8.67 \times 10^9$ yrs.

Camlin Page
Date / /

For $h_0 = 0.75 \times \frac{3}{4}$ $h_0^{-1} = 1.33$

$t_0 = \frac{2}{3} \times 1.33 \times 9.78 \times 10^9$ years

$\Rightarrow t_0 = 8.67 \times 10^9$ years

Problem : $\underbrace{\text{observed age}}_{\text{by looking at different stars \& other objects of universe.}}$ is estimated to be 14×10^9 years

Now it turns out that there must be something else other than radiation & matter for $\boxed{\text{correct}}$ age of universe.

⊙ With Vacuum Energy (Dark energy)
Let us consider the case of the flat universe ($k = 0, \Omega = 1$) universe but now allow both matter & vacuum energy. In this case $\Omega_{total} = 1$ both $\underset{\text{matter}}{\Omega_m}$ & $\underset{\text{vacuum}}{\Omega_v}$ are significant

Thus $\Omega_m + \Omega_v = 1$ for the case of interest. In general, Friedman eq^n is :

$$\left(\frac{R'}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3} \rho$$

For $k = 0 \Rightarrow \left(\frac{R'}{R}\right)^2 = \frac{8\pi G}{3} \rho$

where $\rho = \rho_v + \rho_m = \rho_{v0} + \frac{\rho_{m0}}{R^3}$

$$\dot{R}^2 = R^2\left(\frac{8\pi G}{3}\right)\left[\rho_{v_0} + \frac{\rho_{m_0}}{R^3}\right]$$

$$\dot{R} = \left[\frac{8\pi G}{3} R \left[\rho_{v_0} + \frac{\rho_{m_0}}{R^3}\right]\right]^{1/2}$$

$$\frac{dR}{dt} = \left[\frac{8\pi G \rho_{v_0}}{3} R \left[1 + \frac{\rho_{m_0}}{\rho_{v_0}} \frac{1}{R^3}\right]\right]^{1/2}$$

$\because$ ~~$\frac{\rho_v}{3H^2/8\pi G} = \Omega_v$~~

$$\frac{dR}{dt} = H_0 \Omega_v^{1/2} R \left[1 + \frac{\Omega_m}{\Omega_v} \frac{1}{R^3}\right]^{1/2}$$

$$\because \frac{\rho_v}{3H^2/8\pi G} = \Omega_v \Rightarrow \frac{8\pi G}{3H^2}\rho_v = \Omega_v \Rightarrow \sqrt{\frac{8\pi G \rho_v}{3}} = \Omega_v^{1/2}\sqrt{H^2}$$

$$t_0 = \int dt = H_0^{-1}\Omega_v^{-1/2} \int \frac{1}{R} \frac{dR}{\left[1 + \frac{1}{R^3}\frac{\Omega_m}{\Omega_v}\right]^{1/2}}$$

Recall that $\Omega_m + \Omega_v = 1$ & so can eliminate Ω_m in favor of Ω_v.

$$\Omega_m = 1 - \Omega_v$$

$$\Rightarrow \text{Integral} = \frac{1}{3} \ln\left(\frac{1/\Omega_v^{1/2} + 1}{1/\Omega_v^{1/2} - 1}\right)$$

where $\text{Integral} = \int_0^1 \frac{1}{R} \frac{dR}{\left[1 + \frac{1}{R^3}\frac{\Omega_m}{\Omega_v}\right]^{1/2}}$

Thus, $t_0 = H_0^{-1} \Omega_v^{-1/2} \frac{1}{3} \ln \left(\frac{1/\Omega_v^{1/2} + 1}{1/\Omega_v^{1/2} - 1} \right)$

$$= H_0^{-1} \Omega_v^{-1/2} \frac{1}{3} \ln \left(\frac{1 + \Omega_v^{1/2}}{1 - \Omega^{1/2}} \right)$$

$$= H_0^{-1} \Omega_v^{-1/2} \frac{1}{3} \ln \left(\frac{(1 + \Omega_v^{1/2})^2}{(1 - \Omega_v)} \right)$$

$$= H_0^{-1} \Omega_v^{-1/2} \frac{1}{3} \ln \left(\left[\frac{(1 + \Omega_v^{1/2})}{(1 - \Omega_v)^{1/2}} \right]^2 \right)$$

$$t_0 = H_0^{-1} \Omega_v^{-1/2} \left(\frac{2}{3} \right) \ln \left[\frac{1 + \Omega_v^{1/2}}{(1 - \Omega_v)^{1/2}} \right]$$

Note as $\Omega_v \to 1$ $t_0 \to \infty$

Thus ~~according~~ accomodating a large age is not a problem when vacuum energy is present.

Recall that:

$$H_0 = 100\, h \text{ Km s}^{-1} \text{ Mpc}^{-1}$$

$$H_0^{-1} = h^{-1} \times 978 \times 10^9 \text{ years.}$$

$$H_0^{-1} = 13.6 \times 10^{-9} \text{ years.}$$

$$t_0 = \frac{2}{3} \times 13.6 \times 10^9 \text{ years} \times \Omega_v^{-1/2} \ln \left[\frac{1 + \Omega_v^{1/2}}{(1 - \Omega_v)^{1/2}} \right]$$

From observation W.K.T.

$$t_0 = 13.6 \times 10^9 \text{ years.}$$

$\Omega_V \approx 0.75$

Camlin Page
Date / /

Thus we get

$$\boxed{13.6 \times 10^9 \text{ years} = 9 \times 10^9 \text{ years} \times \Omega^{-1/2} \ln \left[\frac{[1+\Omega_V^{1/2}]}{[1-\Omega_V]^{1/2}} \right]} \quad (3.11)$$

Can solve eq^n (3.11) numerically to get

$$\boxed{\Omega_V \approx 0.75}$$

Thus about 3/4 of the energy density of the universe is in the form of vacuum energy or dark energy. recall that the dark energy has the eq^n of state which is of the form $P = \omega\rho$ with $\omega < 0$

Other implications of friedmann eq^n soln of scale factor as function of time.

Consider the observationally & theoritical supported case of the flat $K=0$ $\Omega=1$ ~~...~~ universe. In this case the friedmann eq^{n}s is

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho \quad (3.12)$$

Recall also that we had the first law of thermo

$$d(\rho R^3) = -P\,d(R^3)$$

together with the eq^n of the state $P = \omega\rho$ (3.13) which implies

$$\rho \propto R^{-3(1+w)} \qquad (3.14)$$

$$\frac{\rho}{\rho_0} = \left(\frac{R}{\rho_0}\right)^{-3(1+w)} \qquad (3.15)$$

$\cancel{\frac{\dot{R}}{R}}$ further (3.15) in (3.12) $\Rightarrow$

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho_0}{3}\left(\frac{R}{R_0}\right)^{-3(1+w)}$$

$$\dot{R}^2 = \frac{8\pi G\rho_0}{3}\,\frac{R^{-3-3w+2}}{R_0^{-3(1+w)}}$$

Choosing $R_0 = 1 \Rightarrow$

$$\dot{R}^2 = \frac{8\pi G}{3}\rho_0 R^{-1-3w}$$

$$\dot{R} = \sqrt{\frac{8\pi G\rho_0}{3}}\, R^{-\frac{1}{2}(1+3w)}$$

$$\frac{dR}{dt} = \sqrt{\frac{8\pi G\rho_0}{3}}\, R^{-1/2(1+3w)}$$

$$\Rightarrow \int \frac{dR}{R^{-1/2(1+3w)}} = \int \frac{8\pi G}{3}\rho_0\, dt$$

$$\Rightarrow \int R^{1/2(1+3w)}\, dR = \sqrt{\frac{8\pi G\rho_0}{3}} \int dt$$

$$\Rightarrow R^{3/2(1+w)} = \frac{3}{2}(1+w)\sqrt{\frac{8\pi G\rho_0}{3}}\; t$$

$$\left[R^{3/2(1+w)}\right]^{\frac{2}{3(1+w)}} = \left(4(1+w)\pi G\rho_0\right)^{\frac{2}{3(1+w)}} \cdot t^{\frac{2}{3(1+w)}}$$

$R = K t^{\frac{2}{3(1+w)}}$ $\quad R_{RD} = K_1 t^{1/2}$ $\quad R_{MD} = K_2 t^{2/3}$

1

Camlin Page
Date / /

$$\boxed{R = K t^{\frac{2}{3(1+w)}}} \quad (3.16)$$

For radiation dominated universe $w = 1/3$

$$\Rightarrow \frac{2}{3(1+w)} = \frac{2}{3\left(1+\frac{1}{3}\right)} = \frac{2}{3+1} = \frac{1}{2}$$

$$\boxed{R = K_1 t^{1/2}} \quad \text{for R.D.} \ (3.17)$$

for $w = 0$ (MD universe)

$$\frac{2}{3(1+w)} = \frac{2}{3}$$

Then $\boxed{R = K_2 t^{2/3}}$ for M.D (3.18)

For $w = -1$ vacuum dominated

$$\boxed{\frac{2}{3(1-1)} \rightarrow \infty}$$

Expansion is actualy faster than any polynomial ~~coeff~~ in t (faster than any t^n with fixed n)

Consider the fridman eq^n for the vacuum energy case more carefully.

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho \quad (3.19)$$

For vacuum ergy $(w = -1)$ we know that $\rightarrow$

$R(t) = \exp(H_0 t)$

$\rho = \rho_0 = \text{constant}$ (3.20)

Putting (3.20) in (3.19)

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_0 \quad (3.21)$$

Thus we get

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_0 = \text{Const} = H_0^2$$

$$\Rightarrow \frac{\dot{R}}{R} = H_0$$

$$\frac{1}{R}\cdot\frac{dR}{dt} = H_0 \Rightarrow \int \frac{1}{R}\,dt = H_0 \int dt$$

$$\Rightarrow \ln R = H_0 t \Rightarrow \boxed{R(t) = \exp(H_0 t)} \quad (3.23)$$

Summary

For R D universe $w = 1/3$ $R = k_1 t^{1/2}$

For MD ,, $w = 0$ $R = k_2 t^{2/3}$

For Dark energy ,, $w = -1$ $R(t) = \exp(H_0 t)$

Thermodynamics

Quantities of physical interest that are thermodynamic in nature are.

$n \rightarrow$ number density of particle , $\vec{p}$ = momentum
$\rho \rightarrow$ energy ——— " ———
$P \rightarrow$ pressure due to a gas of particles

These can be given in terms of the phase space distribution (sometimes called occupancy) function $f(\vec{p})$

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p}) \, d^3p \quad (5.1)$$

$$\rho = \frac{g}{(2\pi)^3} \int E(\vec{p}) f(\vec{p}) \, d^3p \quad (5.2)$$

$$P = \frac{g}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E} f(\vec{p}) \, d^3p \quad (5.3)$$

where g = # of internal degree of freedome

(e.g. $g=2$ for spin $1/2$ particle such as $e^{\ominus}$)

& $E^2 = |\vec{p}|^2 + m^2 \quad (5.4)$

i.e. E is the energy

n, ρ, P depends on

$\boxed{f} \rightarrow$ distribution f^n

$f \rightarrow$ defined and fixed in thermal equilibrium

~~depends on spin of particle~~

f for thermal equilibrium.

Camlin Page
Date / /

$f \rightarrow$ depends ~~on~~ and fixed in thermal equilibrium depends on spin of particle $\rightarrow S = 0, 1/2, 1, 3/2, 2, 5/2$

$S = 1/2, 3/2, 5/2 \leftarrow$ Half integer

Integer $\rightarrow S = 0, 1, 2.$

Half integer $\downarrow$ Anti-symmetric under exchange of particles $\rightarrow$ Fermions obey Fermi-Dirac statistics

Integer $\downarrow$ Symmetric under exchange of particles $\Psi \rightarrow \Psi$ $\rightarrow$ Bosons obey Bose-einstein statistics

Particle with Integer spin ($S = 0, 1, 2$) are Bosons and their ~~of~~ phase space distribution is given by the Bose-einstein distribution:

ex photon $\leftarrow$ $$f(\vec{p}) = \frac{1}{\exp[(E-\mu)/T]-1} \quad (5.5)$$

Particles with half Integer spin ($S = 1/2, 3/2, \ldots$) are fermions and their phase ~~sp~~ space distribution is

ex $e^{\ominus}$ $\leftarrow$ $$f(\vec{p}) = \frac{1}{\exp[E-\mu/T]+1} \quad (5.6)$$

μ is chemical potential & describes chemical equilibrium. T is temp.

here units are such that in $e^{-\Delta E/K_B T}$ K_B which is blotzman constant equal 1.

$$f(\vec{p}) = \frac{1}{\exp[(E-\mu)/T] \pm 1}$$

Camlin Page
Date / /

μ : For example for the species "i" interacting with species "j, k & l" by the reaction:

$$i + j \rightleftharpoons k + l \qquad \text{eg } \overset{\text{Positron}}{e^{\oplus}} + e^{\ominus} \rightleftharpoons \overset{\text{photon}}{\gamma} + \gamma$$

the chemical potential obey the relationship

$$\mu_i + \mu_j = \mu_k + \mu_l$$

Thus, μ ~~captio~~ catures the info about chemical equilibrium.

Frequently, we combine the description of Fermi Dirac species & Bose einstein species & write expression as:

$$f(\vec{p}) = \frac{1}{\exp[(E-\mu)/T] \pm 1} \qquad (5.7)$$

+ : Fermi Dirac species.

− : Bose einstein " .

Taking the distribution f^n. f given by (5.7) & inserting into general expression for n, ρ & P given by (5.1), (5.2) & (5.3) respectively, we get

$$n = \frac{g}{2\pi^2} \int_m^{\infty} \frac{(E^2 - m^2)^{1/2}}{\exp[(E-\mu)/T] \pm 1} \, E \, dE$$

$$\rho = \frac{g}{2\pi^2} \int_m^{\infty} \frac{(E^2 - m^2)^{1/2}}{\exp[(E-\mu)/T] \pm 1} \, E^2 \, dE$$

& $$P = \frac{g}{6\pi^2} \int_m^{\infty} \frac{(E^2 - m^2)^{3/2}}{\exp[(E-\mu)/T] \pm 1} \, dE$$

Scanned by CamScanner

$r \quad r d \sin\theta$

where we have changed the variable of integration from p to E & used the relationship (5.4) i.e.

$$E^2 = |\vec{p}|^2 + m^2$$

Consider n: start with

$$n = \frac{g}{(2\pi)^3}\int f(\vec{p})\, d^3p$$

$d^3p = dp_x \cdot dp_y \cdot dp_z$ in cartesian coordinate

$$= \frac{g}{8\pi^3}\int f(\vec{p})\, 4\pi p^2 dp$$

$d^3p = p^2 dp\, d\sin\theta\, d\phi$ in spherical coordinates

$$= \frac{g\, 4\pi}{8\pi^3}\int f(\vec{p})\, p^2 dp$$

$$= \frac{g}{2\pi^2}\int_0^\infty f(\vec{p})\, p^2 dp$$

x, y, z

$r^2 dr \int_0^\pi d\sin\theta \int_0^\pi d\phi$

$2 \quad 2\pi$

$$E^2 = |\vec{p}|^2 + m^2, \quad p = 0 \Rightarrow E = m$$

$$2E\,dE = 2p\,dp \Rightarrow p^2 dp = pE\,dE = (E^2 - m^2)^{1/2} E\,dE$$

$$n = \frac{g}{2\pi^2}\int_m^\infty f(\vec{p})\,(E^2 - m^2)^{1/2} E\,dE$$

$$\Rightarrow \boxed{n = \frac{g}{2\pi^2}\int_m^\infty \frac{1}{\exp[(E-\mu)/T] \pm 1}(E^2 - m^2)^{1/2} E\,dE} \quad (5.8)$$

ρ: start with

$$\rho = \frac{g}{(2\pi)^3}\int E(\vec{p})\, f(\vec{p})\, d^3p$$

High & low temp limits & significance.

$$= \frac{g}{2\pi^2}\int_m^\infty p\, f(\vec{p})\, E^2 dE$$

$$\Rightarrow \boxed{\rho = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2} E^2 dE}{\exp[(E-\mu)/T]\pm 1}} \quad (5.9)$$

P : start with

$$P = \frac{g}{(2\pi)^3}\int \frac{|\vec{p}|^2}{3E} f(\vec{p})\, d^3p$$

$$= \frac{g}{2\pi^2}\cdot\frac{1}{3}\int_m^\infty \frac{|\vec{p}|^2 f(\vec{p})\, pE\, dE}{E}$$

$$= \frac{g}{6\pi^2}\int_m^\infty |\vec{p}|^3 f(\vec{p})\, dE$$

$$\boxed{P = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2-m^2)^{3/2}}{\exp[(E-\mu)/T]\pm 1}\, dE} \quad (5.10)$$

$n(T,\mu,m)$: number density

$\rho(T,\mu,m)$: energy ,,

$P(T,\mu,m)$: Pressure ,,

→ High temp limit

→ Low temp limit

$\rho = \sum_i \rho_i$ → sum over all the particles of standard model.

ρ_{electron}

$\langle E_e \rangle \sim k_B T$

$\langle p \rangle \approx k_B T \qquad E^2 = p^2 + m^2$

$(k_B)\, T >> m(c^2)$

$k_B = 1 \quad c = 1$

~~KETc~~ → High temp → relativistic limit $T >> M$

Now in the early universe everything can be considered massless as heaviest particle was $m(c)^2 \sim 100\,\text{GeV}$ $c = 1$

& and $T >> 100\,\text{GeV}$.

For Low temp units $T \sim 1\,\text{eV}$.

For the high temperature limit

(Applicable to the early universe)

$T >> m$ & $T >> \mu$

$$\rho = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2} E^2 dE}{\exp\left[\frac{(E-\mu)}{T}\right] \pm 1}$$

First let us use $\mu << T \Rightarrow e^{-\mu/T} = 1$

$$\Rightarrow \rho = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2} E^2 dE}{\exp[E/T] \pm 1}$$

Now use ~~$p^2 = E^2 - m^2$~~

$k^2 = E^2 - m^2 \Rightarrow 2k\,dk = 2E\,dE$

n & P for high temp limit

Camlin	Page
Date / /	

Thus, $$\rho = \frac{g}{2\pi^2}\int_0^\infty \frac{k(k^2+m^2)^{1/2}\, k\,dk}{\exp\left[\frac{(k^2+m^2)^{1/2}}{T}\right] \pm 1}$$

Now use $m << T$ & set $k' = k/T \Rightarrow dk = T dk'$

$$\Rightarrow \rho = \frac{g}{2\pi^2}\int_0^\infty \frac{(k'T)^3\, T\, dk'}{\exp[k'] \pm 1}$$

$$\Rightarrow \rho = \frac{gT^4}{2\pi^2}\underbrace{\int_0^\infty \frac{k'^3\, dk'}{\exp[k'] \pm 1}}_{\text{pure number}} \quad (5.11)$$

For Boson - sign $\Rightarrow$ $\boxed{\rho = \frac{\pi^2 g T^4}{30}}$ (5.12)

For Fermi + sign $\Rightarrow$ $\boxed{\rho = \left(\frac{7}{8}\right)\left(\frac{\pi^2}{30}\right) g T^4}$ (5.13)
dirac.

→ Now taking out n & P for high temp limit

$$n = \frac{g}{2\pi^2}\int_m^E \frac{(E^2-m^2)^{1/2}\, E\,dE}{\exp[(E-\mu)/T] \pm 1}$$

$T >> \mu$

$$n = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2}\, E\,dE}{\exp[E/T] \pm 1}$$

$E^2 = k^2 + m^2$ & $2E\,dE = 2k\,dk$ & $m << T \Rightarrow$

$$n = \frac{g}{2\pi^2}\int_0^\infty \frac{k^2\, dk}{\exp[E/T] \pm 1}$$

Scanned by CamScanner

Camlin Page
Date / /

Let $k' = k/T \Rightarrow dk = T\,dk'$

$$\Rightarrow n = \frac{g}{2\pi^2}\int_0^\infty \frac{T^3 k'^2\,dk'}{\exp[k'] \pm 1}$$

$$n = gT^3 \frac{1}{2\pi^2}\int_0^\infty \frac{k'^2\,dk'}{\exp[k'] \pm 1}$$

Thus for Bosons

$$\boxed{n = gT^3\left(\frac{h(3)}{\pi^2}\right)} \quad (5.14)$$

where $h(3) = 1.202$

For Fermions (+ve sign in deno.)

$$\boxed{n = gT^3\left(\frac{3}{4}\right)\left[\frac{h(3)}{\pi^2}\right]} \quad (5.15)$$

For pressure P:

$$P = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2-m^2)^{3/2}\,dE}{\exp[(E-\mu)/T] \pm 1}$$

Again using $\mu \ll T$ & $k^2 + m^2 = E^2$
& $k\,dk = E\,dE$, we get

$$P = \frac{g}{6\pi^2}\int_0^\infty \frac{k^3\,dk}{\exp[k/T] \pm 1}$$

Let $k' = \frac{k}{T}$ & $dk = T\,dk'$

$P = \frac{1}{3}\rho$ state eq^n for radiation

Camlin Page
Date / /

$$P = \frac{g}{6\pi^2} \int_0^\infty \frac{T^4 k'^3}{\exp[k'] \pm 1} dk'$$

Thus, $P = \frac{g T^4}{6\pi^2} \int_0^\infty \frac{k'^3 dk'}{\exp[k'] \pm 1}$ (5.16)

comparing (5.16) with the expression (5.11) for the energy density ρ we note

$$\boxed{P = \frac{1}{3}\rho} \quad (5.17)$$

Note that (5.17) is the eq^n of state for radiation or relativistic matter which we have now derived from fundamental thermodynamics.

$m << T$ $E^2 = p^2 + m^2$ $E >> m \rightarrow$ radiation

⊙ Now consider the non-relativistic case the number density is given by

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p}) \, 4\pi p^2 \, dp$$

$$= \frac{g}{2\pi^2} \int f(\vec{p}) \, p^2 dp$$

$$\cancel{f(p)} \; f(p) = \frac{1}{\exp\left[\frac{E-\mu}{T}\right] \pm 1}$$

$$E = (p^2 + m^2)^{1/2} = m\left(1 + \frac{p^2}{m^2}\right)^{1/2} = m\left(1 + \frac{p^2}{2m^2}\right)$$

binomial

For non-relativistic case .

$$f = \frac{1}{\exp\left[\frac{E-\mu}{T}\right] \pm 1} = \frac{1}{\exp\left[\frac{m}{T}\left(1+\frac{p^2}{2m^2}\right) - \frac{\mu}{T}\right] \pm 1}$$

$$\frac{\mu}{T} >> 1 \Rightarrow \exp\left[\frac{m}{T}\left(1+\frac{p^2}{2m^2}\right) - \frac{\mu}{T}\right] >> 1$$

$$f = \frac{1}{\exp\left[\frac{m}{T}\left(1+\frac{p^2}{2m^2} - \frac{\mu}{T}\right)\right]} = \exp\left[-\frac{(m-\mu)}{T}\right]\exp\left[\frac{-p^2}{2mT}\right]$$

putting this in n.

$$n = \frac{g}{2\pi^2} \exp\left[\frac{-(m-\mu)}{T}\right] \cdot \int \exp\left[\frac{-p^2}{2mT}\right] p^2 dp$$

$$\text{let } p'^2 = \frac{p^2}{2mT} \Rightarrow p = p'(2mT)^{1/2}$$

$$\text{or } dp = 2mT^{1/2}\, dp'$$

$$\Rightarrow n = \frac{g}{2\pi^2} \exp\left[\frac{-(m-\mu)}{T}\right](2mT)^{3/2} \underbrace{\int_0^\infty \exp(-p'^2)\, p'^2 dp'}_{\text{pure number} = \frac{\sqrt{\pi}}{4}}$$

$$\boxed{n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left[\frac{-(m-\mu)}{T}\right]} \quad 5.24$$

similarly for the Pressure P is $m >> T$ limit

$$P = \frac{g}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E} f(\vec{p})\, d^3p$$

$P = nT \quad , \quad \rho = mn$

$$\boxed{P = g \exp\left[\frac{-(m-\mu)}{T}\right]\left(\frac{mT}{2\pi}\right)^{3/2} T} \quad (5.25)$$

comparing (5.25) & (5.24)

[expression for n] we get:

$$\boxed{P = nT} \quad (5.26)$$

where P is pressure, n is number density & T is temprature

Finally, consider ρ:

$$\rho = \frac{g}{(2\pi)^3}\int E(p)\, f(p)\, d^3p$$

Again using (f) in ρ

$$\boxed{\rho = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left[\frac{-(m-\mu)}{T}\right] m} \quad (5.27)$$

from 5.27 & 5.24 we get

$$\boxed{\rho = mn} \quad (5.28)$$

Further we note from 5.26 that $P = nT$

& (5.28) tells us $\rho = mn$

dividing (5.26) by (5.28) $\Rightarrow$

$p << \rho$ in non relativistic

$$\frac{p}{\rho} = \frac{nT}{mn} = \frac{T}{m}$$

Thus, $p = \left(\frac{T}{m}\right)\rho$ (5.29)

In the non-relativistic limit $\frac{T}{m}$ (<<)

$$\boxed{p << \rho} \quad (5.30)$$

or $p = \omega \rho$ with $\omega \to 0$ or th $\omega = 0$ eq^n of state for non-relativistic matter.

Things to note & remember

In the relativistic limit

$T >> m$ and

$\rho \sim T^4$
$n \sim T^3$
$p \sim T^4$

In the non-relativistic limit

$m >> T$ and

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left[\frac{-(m-\mu)}{T}\right]$$

$\rho = mn$
$p = nT << \rho$

In non-relativistic limit all quantities are exponentially suppressed by the $\exp\left[-\frac{(m-\mu)}{T}\right]$ factor & small compared to the similar quantities in a relativistic case.

In general the total energy density and pressure for all species in equilibrium can be expressed in terms of the photon temperature T as:

$$\boxed{\rho = T^4 \sum_{i=\text{all species}} \left(\frac{T_i}{T}\right)^4 \frac{g_i}{2\pi^2} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2} u^2 \, du}{\exp(u - y_i) \pm 1}}$$

$$\boxed{P = T^4 \sum_{i=\text{all species}} \left(\frac{T_i}{T}\right)^4 \frac{g_i}{6\pi^2} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{3/2} \, du}{\exp(u - y_i) \pm 1}}$$

(5.31)

where $x_i = \frac{m_i}{T}$ & $y_i = \frac{\mu_i}{T}$ (5.32)

By introducing the quantity T_i we allow that species may have a thermal distribution but with a temperature different than that of photon.

One thing to note is that ρ & P of non-relativistic species (with $m \gg T$) are exponentially smaller than that of relativistic species (with $m \ll T$).

Thus it is a very good approximation to keep only the relativistic species in the sum for P & ρ given in (5.31) & (5.32). Thus,

Scanned by CamScanner

$$\boxed{\rho = \frac{\pi^2}{30} g_* T^4} \quad (5.33)$$

$$\boxed{P = \frac{\rho}{3} = \frac{\pi^2}{90} g_* T^4} \quad (5.34)$$

where g_* counts the total number of effectively massless ($m << T$) degree of freedom.

Further g_* can be expressed as

$$g_* = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=\text{fermi}} g_i \left(\frac{T_i}{T}\right)^4$$

The factor $\frac{7}{8}$ is due to the different distribution functions (Fermi dirac vs Bose einstein) for Fermions & Bosons

g_* of course depends on T.

~~In RD era we derived the age is given by~~

~~$t = \frac{1}{2} H^{-1}$ (5.38)~~

Computed n, ρ, P from first principles now apply it in the context of the universe as a where

Examine connection b/w $H, \rho, R(t), H(t)$.

$H = Kg^{1/2}{}_* T^2$

During the early radiation dominated era

$$\rho \simeq \rho_R = \frac{\pi^2}{30} g_* T^4 \quad -(5.36)$$

~~Thus using eq^n (38) in (39) =>~~

and $P_R = \frac{\rho_R}{3}$ that is $w = \frac{1}{3}$ $-(5.37)$

$R(t) \propto T^{1/2}$

$$\rho \propto R^{-3(1+w)} = R^{-4}$$

~~we can now get t in terms of ρ.~~

In RD era, we ~~fixed~~ had derived the age is given by

$$t = \frac{1}{2} H^{-1} \quad (5.38)$$

The friedman eq^n for flat $(K=0)$ universe is.

$$H^2 = \frac{8\pi G\rho}{3} \quad 5.39$$

$$\& \rho = \rho_R = \frac{\pi^2}{30} g_* T^4$$

Thus using eq^n (5.36) in (5.39 =>

$$H^2 = \frac{8\pi G}{3} \frac{\pi^2}{30} g_* T^4 \quad (5.40)$$

Taking the square root of both sides =>

$$\boxed{H = Kg_*^{1/2} T^2} \quad (5.41)$$

where g_* counts the total number of effectively massless degrees of freedom.

T is ~~the~~ temperature & K is a constant which can be determined from fundamental constants of nature by using eqns (5.40) & (5.41) above i.e. $K = \sqrt{\frac{8\pi^3 G}{90}}$

5.41 b

$t = \alpha g_*^{-1/2} T^{-2}$

Camlin Page

Date / /

The age of the universe can be determined by using eqn (5.38) & (5.41).

Thus $t = \frac{1}{2} H^{-1} = \frac{1}{2} k^{-1} g_*^{-1/2} T^{-2}$

or

the age → $\boxed{t = \alpha g_*^{-1/2} T^{-2}}$ (5.41)

where g_* counts the total number of relativistic degrees of freedom, T is the temperature & α is a const which can be written down in terms of fundamental constants of nature, i.e. $\alpha = \frac{1}{2} k^{-1} = \frac{1}{2}\sqrt{\frac{90}{8\pi^3 G}}$ (5.42)

as $t\uparrow$ $T\downarrow$ so early universe was hot

g^* itself depends on t & T.

History Of the Universe

↓

"Einstein eqⁿ"

↓

Geometry ⟵⟶ Contents of the universe.

scale $R(t)$ $\underbrace{\rho, P}$

$\boxed{g_*}$

$e^{\ominus}$ $e^{\oplus}$
γ, μ^-
ν_e

⟵ standard model of particle physics determines g_*

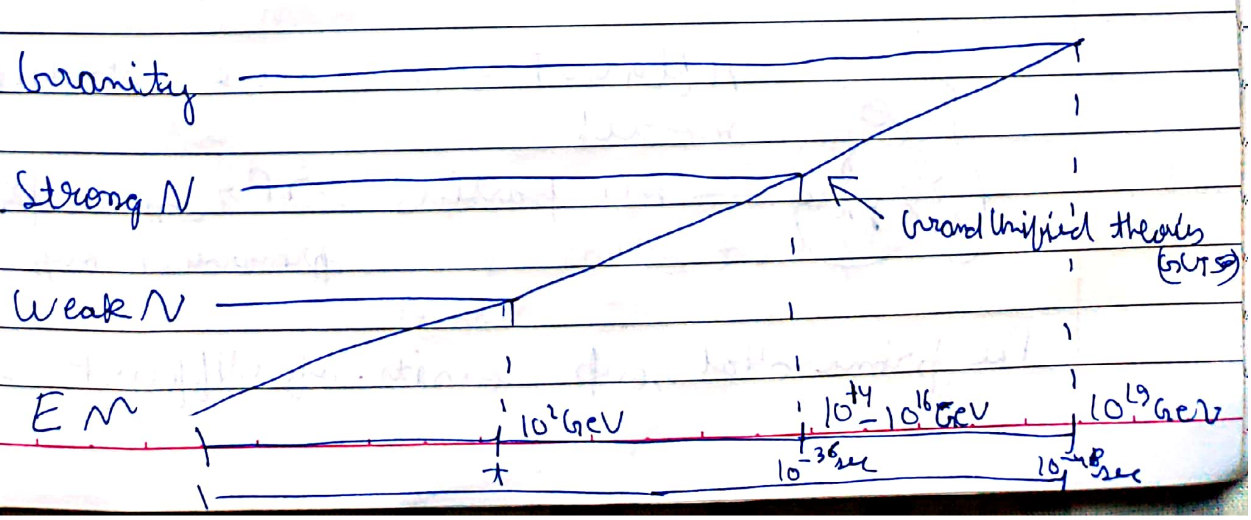
↓

⊙ Particles → Large No.

$\boxed{\odot \text{ Interactions}}$ → 4

Interactions
- Gravity
- Electromagnetism
- strong nuclear forces.
- Weak — " — .

Unification.

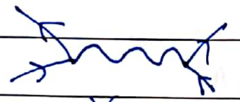


The parameters that control the onset of the quantum gravity involves the fundamental constants c, $\hbar$ & G in the combination with the dimensions of time ~~as~~

$t \sim (G\hbar/c^5)^{1/2} \sim 10^{-43}$ sec

[This is called the plank Time].

The next event after ~~the~~ the end of the quantum gravity era is the breakdown of Grand unification. This happens when $k_BT \sim M_X, M_Y$ where M_X, M_Y are the mass of bosons ~~mediating the~~ radiating GUT interactions. This corresponds to time of $t \sim 10^{-34}$ sec.



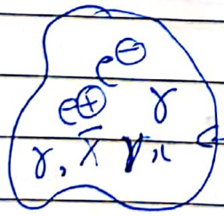
Photon is
Bosons ← carrier of EM interaction.

M_X & M_Y annihilate with time as temp drops

$$X + \bar{X} \rightleftharpoons \gamma + \gamma$$

at low temp reverse is not possible.

and the contribution of boson is trivial after GUT



After GUT era can use standard model

All particles → From a ~~prim~~ primordial soup.

The primordial soup consists of different species of elementary particles the masses range from the heaviest known ~~d~~ elementary particle the top quark ($m \sim 175\,GeV$) down to the lightest particle, the e^{-} ($m = 511\,keV$), the ~~Gev~~ neutrinos & the photon ($m = 0$). As the temperature falls ~~ted~~ the various particle becomes non-relativistic & annihilate at different times. We will ~~sept~~ next describe the various particles in the standard model & number of degrees of freedom ~~a~~ ~~assial~~ associated with this particles.

History of the universe

↓

detected by $p, \bar{p} \to \gamma$

→ contents of the universe

↓

Introduction
Particles

→ Standard Model of particle physics

⊙ Standard Model

Elementary particles comes in several categories. First, there are the most familiar particles such as the electrons, muons & the neutrinos called the Leptons. There are 3 families of leptons.

standard model
Family in standard model leptons

$\begin{pmatrix} e^- \\ \nu_e \end{pmatrix}$ $\begin{pmatrix} \mu^- \\ \nu_\mu \end{pmatrix}$ $\begin{pmatrix} \tau^- \\ \nu_\tau \end{pmatrix}$

electron family include electron type nutrinos | Muon family (with μ^--type nutrinos) | Taon family (with τ type nuetrinos)

and the anti-particle for each of the particles

$\begin{pmatrix} e^{+} \\ \bar{\nu}_e \end{pmatrix}$ $\begin{pmatrix} \mu^{+} \\ \bar{\nu}_\mu \end{pmatrix}$ $\begin{pmatrix} \tau^+ \\ \bar{\nu}_\tau \end{pmatrix}$

nutrino associated with e^{+}

Partins with associated antimatter | Antimuon with Antinutrino | Antitaon with anti nutrinos.

The leptons are spin 1/2 particles & ~~are~~ & are fermions.

The leptons in addition to electromagnatic forces (associated with electric charges) also ~~experiences~~ experince weak nuclear forces. The mediator of EM forces is the photon γ. The mediator of the weak nuclear forces are the ~~Fermi~~ Gauge Bosons called W^+, W^- & Z^0. They are spin 1 particles γ is massless & W^+, W^- & Z^0 ~~boson~~ have masses in 80-90 GeV range.

⊙ The strongly interacting spin 1/2 particles are called quarks. The quarks (~~un~~like the leptons) come in 3 families

$\begin{pmatrix} u \\ d \end{pmatrix}$ $\begin{pmatrix} c \\ s \end{pmatrix}$ $\begin{pmatrix} t \\ b \end{pmatrix}$

called $\begin{pmatrix} \text{up} \\ \text{down} \end{pmatrix}$ $\begin{pmatrix} \text{charm} \\ \text{strange} \end{pmatrix}$ $\begin{pmatrix} \text{top} \\ \text{bottom} \end{pmatrix}$

quarks   degree of freedom colour

gluons   Higgs boson

Their electric charges are $\begin{pmatrix} +2/3 \\ -1/3 \end{pmatrix}$ $\begin{pmatrix} +2/3 \\ -1/3 \end{pmatrix}$ $\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$

The similarity of particles properties such as electric charge, led to the nomenclature of 3 'generations' or 3 'families' because it looks like there are 3 copies of the same structure with most properties the same (except for mass which increases to the right as you ~~decide the~~ climb the family or generation ladder.

~~Standard mo~~

In addition to the quarks carrying electric charge they also carry another internal degree of freedom called 'colour'. This colour is an internal degree of freedom that has to do with the strong interactions. There are 3 such 'colours' so there are 3 degree of freedom corresponding to colour for each of the quarks. The carriers of strong interaction are massless gauge called gluons there are 8 of these massless gauge boson called gluons. Finally there is the Higgs boson which is responsible for giving masses to all the particles - The Higgs boson is a spin 0 particle. This complete the list of elementary particles. Other particles are made up of these elementary particles.

Thus the proton is made up of the uud quarks. The neutrons " " " " udd. The $\pi^{\pm}$ mesons are made up of $u\bar{d}$ & $d\bar{u}$ quarks & the π^0 is $\left(\frac{u\bar{u} - d\bar{d}}{\sqrt{2}} \right)$ in terms of quarks. n/p (neutron & proton) are spin 1/2 & The muons are spin 0 particles.

At high energy densities & high temperature, the proton, neutron & the ~~mesons~~ messons etc. are broken up into the ~~constant quarks~~ constituent quarks & the primordial soup consists of the elementry particles such as Quarks, leptons, gluons, electroweak gauge Bosons & the Higgs boson.

We now summarize the properties of these particles & count the associated degree of freedom. The degree of freedom enter into PDP & determine the evolution of history of the universe.

Particles in standard model

	Particles	Antiparticles	spin	colour	g	Total
Quarks (Fermions)	t	$\bar{t}$	$1/2$	3	$g = 2 \times 2 \times 3$ (spin, Particles Anti particle, Colour)	$= 12$
	b	$\bar{b}$				
	c	$\bar{c}$				
	s	$\bar{s}$				
	d	$\bar{d}$				
	u	$\bar{u}$				$\times 6$
						72+
Gluons (Bosons)	8 massless	Bosons	1		massless $g = 2 \times 8$	16
Leptons (Fermions)	τ^-	τ^+	$1/2$		$g = 2 \times 2 = 4$ (spin, Particle's Antiparticle)	
	μ^-	μ^+				
	e^-	e^+				$\times 3$
	ν_τ	$\bar{\nu}_\tau$				12+
	ν_μ	$\bar{\nu}_\mu$			massless → $g = 2$	
	ν_e	$\bar{\nu}_e$			$\times 3$	6+

	Particles	Anti-Particles	spin	colour	g	Total
electroweak	W^+		1		$g=3$	
gauge boson	W^-		1			
(Bosons)	Z^0		1			
photon	γ		1 massless		$g=2$	11 b
Higgs boson	H^0		0		$g=1$	1 b

Total fermionic degrees of freedom $= g_f = 72 + 12 + 6 = 90$

" Bosonic " ——— " $= g_b = 16 + 11 + 1 = 28$

Total effective degree of freedom $= g_* = g_b + \frac{7}{8} g_f$

$= 28 + \frac{7}{8} \times 90 = 106.25$

$\Rightarrow g_* = 106.25$

for all particles in the standard model

$$\boxed{g_*(T \gg 100\ GeV) = 106.25}$$

Degree of freedom associated with spin.

(a) for ~~ba~~ massive particles

$g(s) = 2s+1$ for ~~electron~~ $e^{\ominus}$ $s = \frac{1}{2}$ $g(s) = 2$

(b) for massless particles

$g(s) = 2s$ for γ (photon) $s = 1$ $g(s) = 2$

g^* evolution with the ... [illegible]

History of the universe → determined the contents of the universe

$$g^*(T) \longleftarrow \text{standard Model}$$

Connect to discussion on cosmology earlier.
Recall Friedmann eqn relates expansion of universe to ρ & p

The total energy density ρ is just the sum of the energy density over all the particle species $\rho = \sum_i \rho_i(T)$

where i runs over all particle species. Also note that for non-relativistic species suppressed by exponential Boltzmann factor.

Thus sum over all species to a very good approximation includes only relativistic species thus ρ is

$$\rho(T) = \frac{\pi^2}{30} g_*(T) T^4 \quad (6.1)$$

Where g_* counts the total number of effective degree of freedom $g_*(T) = g_b(T) + \frac{7}{8} g_f(T)$

& $g_b = \sum_i g_i$ over relativistic bosons

$g_f = \sum_i g_i$ " " fermions

& pressure P is $P = \frac{1}{3}\rho(T)$

Reasons for g_* to change as T changes

1) Species changes from relativistic species to non relativistic species. This is determined by the mass of the species as compared to

the temperature T. As universe evolves T drops, more & more species become non-relativistic.

2). Particle - Antiparticle annihilations also decreases the effective number of degree of freedom g_*. Thus, for example, e^{-} & e^{+} annihilate with each other through the reaction

$$e^+ + e^- \rightarrow \gamma + \gamma$$

& this leads to a decrease in g_*.

→ Let's pick up history of universe where we have left it - after the end of quantum gravity era & the end of grand unification which happens at $T \sim 10^{17}-10^{16}$ GeV. There is no dramatic change b/w 10^{14} GeV & 10^{3} GeV. [meaning g_* = constant during this period & $\rho \propto T^4$ & $\rho \propto R^{-4}$ & $R \propto T^{-1}$ as discussed earlier]

For $T > m_t$ = mass of the top quark = 175 GeV (heaviest particle in standard model).

all known particles are relativistic.
We added up their degrees of freedom to obtain:

$$g_b = 28 \quad g_f = 90$$

$$\Rightarrow g_* = g_b + \frac{7}{8} g_f = 106.75$$

By $T \sim 100$ GeV, top quark annihilation happens. This means degree of freedom corresponding to t & $\bar{t}$ disappear from the sum for g_*. The number of degree of freedom that disappeared at $T \sim 100$ GeV

Annihilation of top quarks.
Annihilation of higgs boson & gauge boson.

Camlin Page
Date / /

$$g_* = \underset{\text{Fermions}}{\frac{7}{8}} \times \underset{2\text{ spins}}{2} \times \underset{\text{spin } \frac{1}{2}}{2} \times \underset{3\text{ colours}}{3} = 10.5$$

Thus number of degrees of freedom decreases by 10.5 after top quarks and antitops (t & $\bar{t}$) annihilate since we started out with 106.75 degree of freedom. After $t-\bar{t}$ annihilation ($T \sim 100$ GeV), the effective number of degree of freedom is:

$$g_* = 106.75 - 10.5 = 96.25$$

The next set of particles to annihilate are the higgs boson & the electroweak gauge boson $W^\pm$, Z^0

The corresponding degrees of freedom which disappear after annihilation are

$$W^\pm, Z^0 : \underset{\text{No. of particles}}{3} \times \underset{\text{spin}=1}{3} = 9$$

$$H^0 : \underset{\pm 1 \text{ of particles}}{1} \times \underset{\text{spin}=0}{1} = 1$$

All bosons: Total of 10 degree of freedom disappear after Higgs Boson & electroweak gauge bosons annihilate.

$W^\pm, Z^0$ annihilate at $T \sim 80$ GeV. after $T < 80$ GeV we have $g_* = 96.25 - 10 = 86.25$

Annihilation of bottom quarks
" " charm " & leptons.

Camlin Page
Date / /

next is the annihilation of bottom & anti-bottom (b & $\bar{b}$) quarks annihilations. This happens at $T \sim 4$ GeV number of d.o.f g_* that disappear:

$$g_* = \frac{7}{8} \times 2 \times 2 \times 3 = 10.5$$

Fermions, 2 spins, 2 spin, colour.

Thus after b-$\bar{b}$ annihilate for $T < 4$ GeV we have
$g_* = 86.25 - 10.5 = 75.75$

At $T \sim 1$ GeV both the charm - anti charm (c & $\bar{c}$) quark annihilate as well as the $\tau^- - \tau^+$ leptons annihilation occurs. The number of degree of freedom that disappear are

$c - \bar{c}$ $\quad g_* = \frac{7}{8} \times 2 \times 2 \times 3 = 10.5$

$\tau^- \ \tau^+$ $\quad g_* = \frac{7}{8} \times 2 \times 2 = 3.5$

$\underline{14.0}$

After $c\bar{c}$ & $\tau^-\tau^+$ annihilation for $T < 1$ GeV, we have $g_* = 75.75 - 14.0 = 61.75$

- Next event Quark-Hadron Transition ~~Transition~~

This is a major transition
Quark hadron phase transition.

Quarks & Gluons → Baryons : n, p
$(q, \bar{q}, \ldots)$ &
Mesons : π^+, π^-, π^0

In the quark hadron phase transition the quarks get bound together to form the hadrons. Thus, for example the ~~(quarks)~~ (uud) quarks get bound together to from the proton (p), (udd) " " " neutron (n). Similarly u & d quarks get bound together to form the $\pi^{\pm}$, π^{0} mesons.

Before phase transition you have the quark gluon plasma. After phase transition you have hadrons in the form of a hadron gas. The lightest hadrons are the proton(p) & neutron(n), and the pions ($\pi^{\pm}$, π^{0}) of all the hadrons the only ones that are relativistic after the quark hadrons phase transition are the pions ($\pi^{\pm}$, π^{0}). After the Quark hadron Phase transition, the only particle species left in the large numbers are the pions, muons, electrons, neutrinos & the photon. We will thus count the number of degree of freedom corresponding to these particles to determine g_* ~~effect~~ after the quark gluon phase transition.