OPTIMIZATION TECHNIQUES for DECISION MAKING

UNIT-1

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Need of Optimization Techniques

- The ever-increasing demand on engineers to lower production costs to withstand global competition has prompted engineers to look for rigorous methods of decision making, such as optimization methods, to design and produce products and systems both economically and efficiently.
- Optimization techniques, having reached a degree of maturity in recent years, are being used in a wide spectrum of industries, including aerospace, automotive, chemical, electrical, construction, and manufacturing industries.
- With rapidly advancing computer technology, computers are becoming more powerful, and correspondingly, the size and the complexity of the problems that can be solved using optimization techniques are also increasing.

INTRODUCTION

- **Optimization is the act of obtaining the best result under given circumstances**. In design, construction, and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages.
- **The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit**. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, ***optimization*** can be defined as the process of finding the conditions that give the maximum or minimum value of a function.
- There is **no single method available for solving all optimization problems** efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems. The optimum seeking methods are also known as ***mathematical programming techniques*** and are generally studied as a part of **Operations Research**.

SCOPE OF OT / OR

O.R. has a wide scope in everyday life as it provides better solutions to various decision-making problems with great speed and competence. It finds applications in a wide range of areas including defence operations, planning, agriculture, industry (finance, marketing, personal management, production management), research and development. We now describe the applications briefly.

Areas where Optimization is applied are:

1. **Science**
2. **Engineering**
3. **Management**
4. **Finance**
5. **Business**

In Planning for Economic Development

Careful planning is necessary for economic development of any country. Operations Research is used to frame future economic and social policies.

In Agriculture

Agricultural output needs to be increased due to increasing needs for adequate quantity and quality of food for our increasing population. But there are a number of restrictions under which agricultural production is studied. Problems of agricultural production under various restrictions such as optimum allocation of land to various crops in accordance with the climatic conditions, optimum distribution of water from various resources for irrigation purposes can easily be solved by application of Operations Research techniques.

In Industry

Now-a-days, due to complexities of operations and huge sizes of industries, important decisions regarding various sections of the organisation, e.g., planning, procurement, marketing, finance, etc. have to be taken division wise. For example, the production department needs to minimise the cost of production, but maximise output; the finance department needs to optimise capital investment; the personnel department needs to appoint competent work force at minimum cost. Each department has to plan its own objectives which may be in conflict with the objectives of other departments and may not conform to the overall objectives of the organisation. For example, the sales department of an organisation may want to keep sufficient stocks in the inventory, whereas the finance department may want to have minimum investment. In that case, both departments would be in conflict with each other. The applications of O.R. techniques to such situations help in overcoming this difficulty by evolving an optimal strategy and serving efficiently the interest of the organisation as a whole.

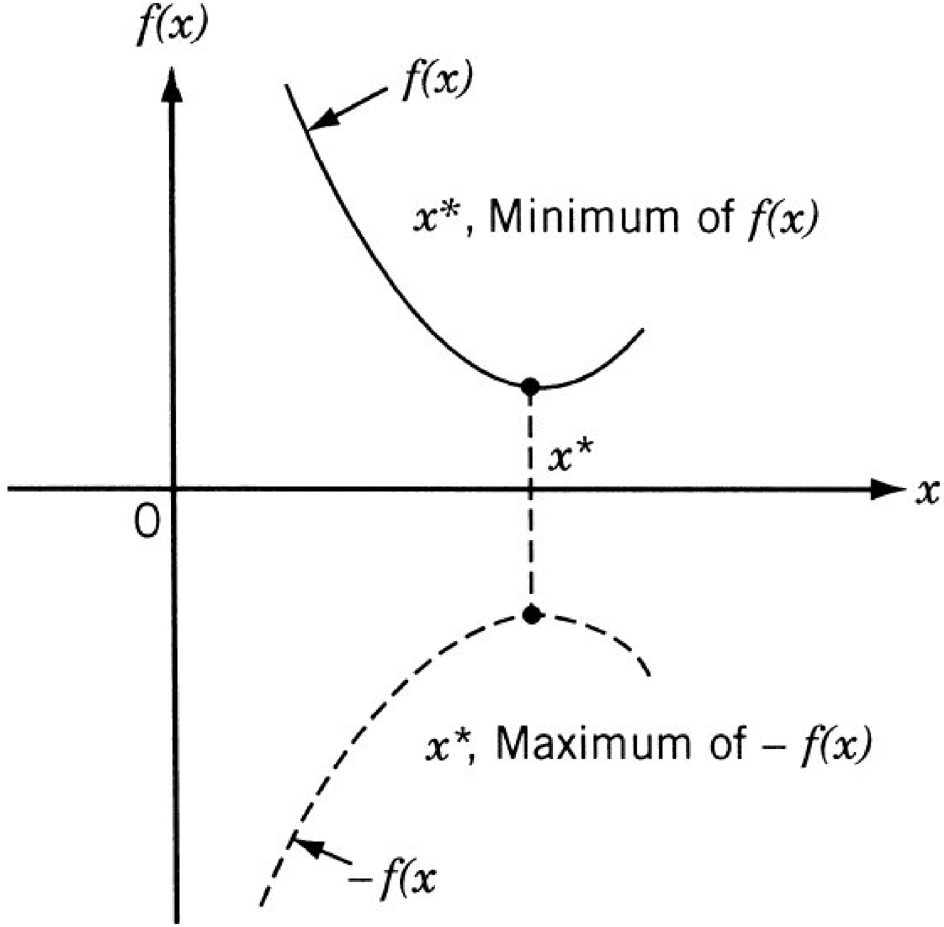

Figure 1.1 Minimum of $f(x)$ is same as maximum of $-f(x)$.

ENGINEERING APPLICATIONS OF OPTIMIZATION

Some typical applications from different engineering disciplines include :

1. Design of aircraft and aerospace structures for minimum weight

2. Finding the optimal trajectories of space vehicles

3. Minimum-weight design of structures for earthquake, wind, other types of random loading

4. Selection of machining conditions in metal-cutting for minimum production cost

5. Shortest route taken by a salesperson visiting various cities during one tour

6. Optimal production planning, controlling, and scheduling

7. Design of optimum pipeline networks for process industries

8. Selection of a site for an industry

9. Planning of maintenance and replacement of equipment to reduce operating costs

10. Inventory control management

11. Allocation of resources among several activities to maximize the profit.

12. Controlling the waiting and idle times in production lines to reduce the costs

13. Optimum design of control systems in design of electronic appliances.

GENERAL OPTIMIZATION PROBLEM

Minimize (Maximize) $f(X)$ objective function

where $f: R^n \rightarrow R$, $X = (x_1, x_2, x_3 \ldots x_n)$ decision values parameters

s.t. $X \in S \subseteq R^n$ where S is defined by

$g_k(X) \geq 0$, $k=1,2, \ldots m$ → inequality constraints

$h_j(X) = 0$, $j=1,2, \ldots l$ → equality constraints

$a_i \leq x_i \leq b_i$ → lower & upper bounds

COMPONENTS OF AN OPTIMIZATION MODEL

Decision variables

Objective function

Constraints

CLASSIFICATION

Linear Programming Problems (LPP)

Nonlinear Programming Problems (NLPP)

Unconstrained Optimization Problems

Constrained Optimization Problems

LINEAR PROGRAMMING PROBLEM

A **Linear Programming Problem** is an optimization problem where the **objective function** and all the **constraints** are linear in nature.

Objective function → A linear function of decision variables to be **maximized or minimized** (e.g., profit, cost, revenue).

Constraints → Linear equations or inequalities representing resource limitations.

Decision variables → Unknowns to be determined (e.g., number of products, allocation of resources).

The General form of LPP is as follows:

Maximize or Minimize:

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

...

$$x_i \geq 0 \quad (i = 1, 2, ..., n)$$

Where:

- Z = Objective function
- $x_1, x_2, ..., x_n$ = Decision variables
- c_i = Coefficients of objective function
- a_{ij} = Coefficients of constraints
- b_j = Resource availability

Problem:

A company produces two products A and B.
Profit from A = ₹3 per unit, profit from B = ₹2 per unit.

Constraints:

- Each unit of A requires 1 hour on machine 1 and 2 hours on machine 2.
- Each unit of B requires 1 hour on machine 1 and 1 hour on machine 2.
- Machine 1 is available for 8 hours, Machine 2 is available for 10 hours.

Formulation:

$$\text{Maximize } Z = 3x_1 + 2x_2$$

Subject to:

$$x_1 + x_2 \leq 8 \quad (Machine1)$$

$$2x_1 + x_2 \leq 10 \quad (Machine2)$$

$$x_1, x_2 \geq 0$$

Solution (Graphical):

- Plot the inequalities, find feasible region.
- Corner points = (0,0), (5,0), (2,6), (0,8).
- Evaluate Z:
    - (0,0) → 0
    - (5,0) → 15
    - (2,6) → 18
    - (0,8) → 16

Optimal Solution: Produce 2 units of A and 6 units of B → **Maximum Profit = ₹18**

NON-LINEAR PROGRAMMING

Non-Linear Programming involves optimization of a **non-linear objective function** and/or **non-linear constraints**.

General Formulation

$$\text{Maximize/Minimize } Z = f(x_1, x_2, \ldots, x_n)$$

Subject to:

$$g_j(x_1, x_2, \ldots, x_n) \leq b_j \quad \text{for } j = 1, 2, \ldots, m$$

$$x_i \geq 0$$

Where:

- $f(x)$ = non-linear objective function
- $g_j(x)$ = non-linear constraint functions

Methods of Solving NLPP

1. **Lagrange Multiplier Method** (for equality constraints)
2. **Kuhn-Tucker (KKT) Conditions** (for inequality constraints)
3. **Gradient Descent / Newton's Method** (iterative numerical methods)

CLASSIFICATION BASED ON TYPE OF DECISION VARIABLES

Dynamic Programming

Geometric Programming

Integer Programming

Quadratic Programming

Separable Programming

STATEMENT OF AN OPTIMIZATION PROBLEM

An optimization or a mathematical programming problem can be stated as follows.

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

subject to the constraints

$$\begin{aligned} g_j(\mathbf{X}) &\le 0, \qquad j = 1, 2, \ldots, m \\ l_j(\mathbf{X}) &= 0, \qquad j = 1, 2, \ldots, p \end{aligned} \tag{1.1}$$

where $\mathbf{X}$ is an n-dimensional vector called the *design vector*, $f(\mathbf{X})$ is termed the *objective function*, and $g_j(\mathbf{X})$ and $l_j(\mathbf{X})$ are known as *inequality* and *equality* constraints, respectively. The number of variables n and the number of constraints m and/or p need not be related in any way. The problem stated in Eq. (1.1) is called a *constrained optimization problem*.† Some optimization problems do not involve any constraints and can be stated as

OPTIMIZATION PROBLEM(2)

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

Such problems are called *unconstrained optimization problems*.

CONSTRAINTS IN OPTIMIZATION

- In many practical problems, the design variables cannot be chosen arbitrarily; rather, they have to satisfy certain specified functional and other requirements.
- The restrictions that must be satisfied to produce an acceptable design are collectively called *design constraints*.
- Constraints that represent limitations on the behavior or performance of the system are termed *behavior* or *functional constraints*.
- Constraints that represent physical limitations on design variables, such as availability, fabricability, and transportability, are known as *geometric* or *side constraints*.

CONSTRAINT SURFACE

For illustration, consider an optimization problem with only inequality constraints $g_j(\mathbf{X}) \leq 0$. The set of values of $\mathbf{X}$ that satisfy the equation $g_j(\mathbf{X}) = 0$ forms a hypersurface in the design space and is called a *constraint surface*. Note that this is an $(n - 1)$-dimensional subspace, where n is the number of design variables. The constraint surface divides the design space into two regions: one in which $g_j(\mathbf{X}) < 0$ and the other in which $g_j(\mathbf{X}) > 0$. Thus the points lying on the hypersurface will satisfy the constraint $g_j(\mathbf{X})$ critically, whereas the points lying in the region where $g_j(\mathbf{X}) > 0$ are infeasible or unacceptable, and the points lying in the region where $g_j(\mathbf{X}) < 0$ are feasible or acceptable. The collection of all the constraint surfaces $g_j(\mathbf{X}) = 0$, $j = 1, 2, \ldots, m$, which separates the acceptable region is called the *composite constraint surface*.

CONSTRAINT SURFACE(2)

- Figure 1.4 shows a hypothetical two-dimensional design space where the infeasible region is indicated by hatched lines.
- A design point that lies on one or more than one constraint surface is called a *bound point* , and the associated constraint is called an *active constraint* .
- Design points that do not lie on any constraint surface are known as *free points*.
- Depending on whether a particular design point belongs to the acceptable or unacceptable region, it can be identified as one of the following four types:

1. Free and acceptable point

2. Free and unacceptable point

3. Bound and acceptable point

4. Bound and unacceptable point

All four types of points are shown in Fig. 1.4.

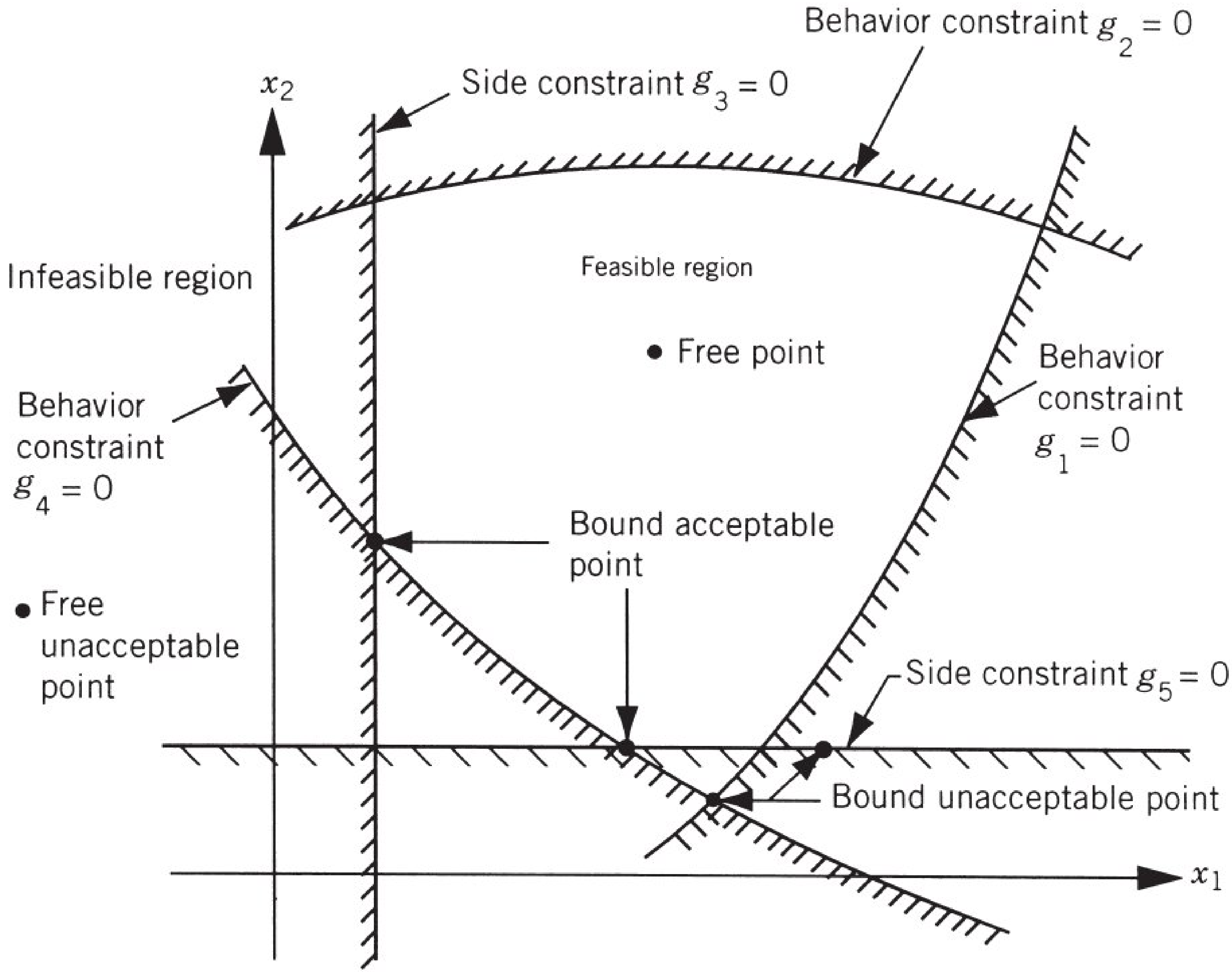

Figure 1.4 Constraint surfaces in a hypothetical two-dimensional design space.

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OBJECTIVE FUNCTION

- In general, there will be more than one acceptable design, and the purpose of optimization is to choose the best one of the many acceptable designs available.
- Thus a criterion has to be chosen for comparing the different alternative acceptable designs and for selecting the best one. The criterion with respect to which the design is optimized, when expressed as a function of the design variables, is known as the *criterion* or *merit* or *objective function*.
- The choice of objective function is governed by the nature of problem.
- The objective function for minimization is generally taken as weight in aircraft and aerospace structural design problems. In civil engineering structural designs, the objective is usually taken as the minimization of cost.
- In some situations, there may be more than one criterion to be satisfied simultaneously.
- For example, a gear pair may have to be designed for minimum weight and maximum efficiency while transmitting a specified horsepower.
- An optimization problem involving multiple objective functions is known as a *multiobjective programming problem*.

PROBLEM 1

1. Reddy Mikks produces both interior and exterior paints from two raw materials M1 and M2. Following table provides the basic data of problem:
2. A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton.
3. Also, the maximum daily demand for interior paint is 2 tons.
4. Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit?
5. Find the optimal solution to this problem

Tons of raw material per ton			
	Exterior paint	Interior paint	Max daily availability
Raw material, M1	6	4	24
Raw material, M2	1	2	6
profit per ton ($1000)	5	4	

SOLUTION

- The LP model, has three basic components:

1. Decision variables that we seek to determine.

2. Objective that we need to optimize (maximize or minimize).

3. Constraints that solution must satisfy.

The **decision variables** of model are defined as

$x1$= tons produced daily of exterior paint

$x2$= tons produced daily of interior paint

Objective function : The company wants to maximize (i.e., increase as much as possible) the total daily profit of both paints.

Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

Total profit from exterior paint = $5x_1$

Total profit from interior paint = $4x_2$

Let Z denote the total daily profit, then objective of the company is

Maximize $\mathbf{Z = 5x_1 + 4x_2}$

CONSTRAINTS : To construct the constraints that restrict raw material usage and product demand.

The raw material restrictions are expressed verbally as

(usage of a raw material by both paints) ≤ (max. raw material availability)

Daily usage of raw material M1 is 6 tons of exterior paint and 4 tons of interior paint.

So, Usage of raw material M1 by exterior paint = $6x_1$ tons/day ,

Usage of raw material M1 by interior paint = $4x_2$ tons/day

Therefore, Usage of M1 by both paints = $6x_1 + 4x_2$ tons/day

By similar logic,

Usage of M2 by both paints = $1x_1 + 2x_2$ tons/day

Because the daily availabilities of raw materials M1 and M2 are limited to 24 and 6 tons respectively, the associated restrictions are given as

$$6x_1 + 4x_2 \leq 24 \quad \textbf{(Raw material M1)}$$

$$x_1 + 2x_2 \leq 6 \quad \textbf{(Raw material M2)}$$

Demand restriction stipulates that the excess of daily production of interior over exterior paint, x_2-x_1, should not exceed 1 ton, which translates to $x_2 - x_1 \leq 1$ **(market limit)**

Second demand restriction- the max. daily demand of interior paint is limited to 2 tons, which translates to $x_2 \leq 2$ **(demand limit)**.

An implicit (or "understood- to –be") restriction is that variables x_1 and x_2 cannot assume negative values. So $x_1 \geq 0$, $x_2 \geq 0$ **(nonnegativity restrictions)**

The complete Reddy Mikks Model is

Maximize $\mathbf{Z = 5x_1 + 4x_2}$

Subject to:

$$6x_1 + 4x_2 \leq 24 \qquad (1)$$

$$x_1 + 2x_2 \leq 6 \qquad (2)$$

$$x_2 - x_1 \leq 1 \qquad (3)$$

$$x_2 \leq 2 \qquad (4)$$

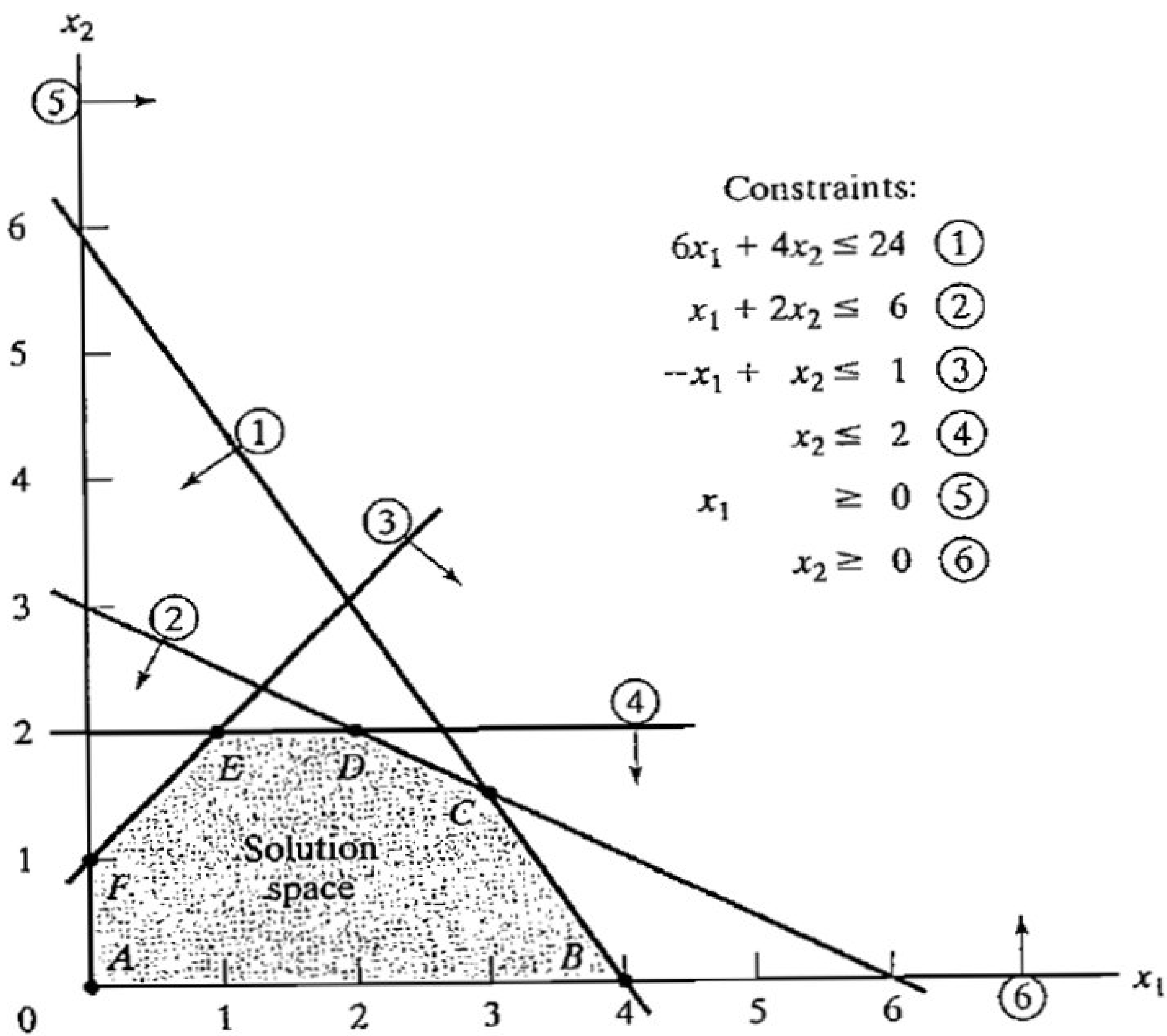
$$x_1,\ x_2 \geq 0 \qquad (5)$$

Any values of x_1 and x_2 that satisfy all five constraints constitute a feasible solution. Otherwise, the solution is infeasible.

For example, the solution, $x_1 = 3$ tons per day and $x_2 = 1$ ton per day, is feasible because it does not violate any of the constraints.

The goal of the problem is to find the best feasible solution, or the optimum, that maximizes the total profit. Before we can do that, we need to know how many feasible solutions the Reddy Mikks problem has. The answer, as we will see from the the graphical solution is an infinite number, which makes it impossible the solve the problem by enumeration.

GRAPHICAL SOLUTION OF LPP



The graphical procedure includes 2 steps:

1. Determination of the feasible solution space.

2. Determination of the optimum solution from among all the feasible points in the solution space

Determination of Optimum Solution

- The **feasible space** in figure is delineated by the line segments joining the points A, B, C, D, E, and F. **Any point within or on the boundary of the space ABCDEF is feasible.** Because the feasible space ABCDEF consists of an infinite number of points, we need a systematic procedure to identify the optimum solution.
- **An important characteristic of the optimum LP solution is that it is always associated with a corner point of the solution space** (where two lines intersect). This is true even if the objective function happens to be parallel to a constraint. For example, if the objective function is $z = 6x_1 + 4x_2$, which is parallel to constraint 1, we can always say that the optimum occurs at either corner point B or comer point C.
- The observation that the LP optimum is always associated with a corner point means that **the optimum solution can be found simply by enumerating all the corner points** as the following table shows:

Determination of Optimum Solution(2)

- As the number of constraints and variables increases, the number of corner points also increases.
- Nevertheless, the idea shows that, from the standpoint of determining the LP optimum, the solution space ABCDEF with its infinite number of solutions can, in fact, be replaced with a finite number of promising solution points-namely, the corner points, A, B, C, D, E, and F.
- The optimum solution is $x_1 = 3$ and $x_2 = 1.5$ with $Z = (5 * 3) + (4 * 1.5) = 21$. **This calls for a daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint. The associated daily profit is $21,000.**

Corner point	(x_1, x_2)	z
A	(0, 0)	0
B	(4, 0)	20
C	**(3, 1.5)**	**21 (OPTIMUM)**
D	(2, 2)	18
E	(1, 2)	13
F	(0, 1)	4

PRACTICE PROBLEM 1

A company produces two products, A and B. The sales volume for A is at least 80% of the total sales of both A and B.

However, the company cannot sell more than 100 units of A per day. Both products use one raw material, of which the maximum daily availability is 240 lb. The usage rates of the raw material are 2 lb per unit of A, and 4 lb per unit of B.

The profit units for A and B are $20 and $50, respectively.

Formulate the LPP for the same.

SOLUTION TO PRACTICE PROBLEM

If we let A = units of product A and B =units of product B, then we'll

$$\text{maximize } z = 20A + 50B$$

subject to

$$\begin{aligned} 2A + 4B &\leq 240 && \text{(raw material availability)} \\ A &\leq 100 && \text{(sales limit of A)} \\ -0.2A + 0.8B &\leq 0 && \text{(sales of A at least 80\%)} \\ A, B &\geq 0 && \text{(sign restrictions)} \end{aligned}$$

The sales volume for A is at least 80% of the total sales of both A and B. So, $A \geq 0.8\,(A + B)$ which gives us $0 \geq -0.2\,A + 0.8\,B$

PRACTICE PROBLEM 2

A company produces two types of items P and Q that require gold and silver.

Each unit of type P requires 4g silver and 1g gold while that of type Q requires 1g silver and 3g gold. The company can produce 8g silver and 9g gold.

If each unit of type P brings a profit of `44 and that of type Q `55, determine the number of units of each type that the company should produce to maximise the profit. What is the maximum profit?

Degeneracy in Linear Programming

- Degeneracy in LPP occurs when a basic feasible solution has one or more basic variables equal to zero, which can lead to the Simplex method cycling or failing to reach an optimal solution in a finite number of steps.
- It manifests when there is a tie in the minimum ratio test while selecting an outgoing variable in the Simplex table (**Ties in Replacement Ratios)**. Also degeneracy can arise if a**t least one of the constraints has a zero value on the right-hand side (Zero-Valued Basic Variables).** Degeneracy can also arise from **redundant constraints** that overly restrict the solution space.
- In the graphical method for LPPs, degeneracy occurs when **a basic feasible solution has at least one basic variable equal to zero,** , or when **multiple constraints intersect at the same corner point**, resulting in a degenerate corner point. Graphically, it means that more than one constraint line passes through a single corner of the feasible region.
- **Degeneracy can lead to cycling or difficulty in identifying unique optimal solutions.**

Consequences of Degeneracy

1. The primary consequences of degeneracy in LPP are the potential for the simplex algorithm to experience **cycling** (repeatedly visiting the same set of basic feasible solutions without improving the objective function) or **stalling** (failing to make progress toward the optimal solution).
2. **Increased iterations**: Even if it doesn't lead to cycling, degeneracy can significantly increase the number of iterations required for the simplex algorithm to converge to the optimal solution.
3. Degeneracy typically occurs when a pivot operation results in **no improvement to the objective function value**, often due to a tie in determining the outgoing variable in a simplex tableau, making it difficult (potentially requiring more iterations) to reach the optimal solution.

- Degeneracy does not affect the **existence of an optimal solution**.
- **Feasibility:** Degeneracy does not make a basic feasible solution infeasible.

Duality in LPP

- In LPPs, **duality is the concept that every LPP, called the Primal, has an associated LPP called the Dual**, derived from the same data and sharing the same solution. **Dual LP problem** provides useful economic information about worth of resources to be used.

Relationship between Primal and Dual :

- **Variables and Constraints:** The variables in the primal problem become the constraints in the dual problem, and vice-versa.
- **Objective Functions:** The objective function coefficients of primal become the RHS constants of the dual's constraints, and the RHS constants of the primal constraints become the objective function coefficients of the dual.
- **Optimization Direction:** If the primal is a maximization problem, its dual will be a minimization problem, and vice versa.

Duality Theorem: This theorem states that if the primal problem has an optimal solution, the dual problem also has an optimal solution, and the optimal values of their objective functions are equal.

Observations in Duality

- The number of constraints in the primal problem is equal to the number of variables in the dual problem.
- Similarly, the number of variables in the primal problem corresponds to the number of constraints in the dual problem.
- When primal is in maximization form, the dual is in minimization form.
- The coefficients in the objective function of the primal problem become the right-hand side(RHS) of the constraints in the dual problem.
- The right-hand side of the primal problem becomes the coefficients in the objective function of the dual problem.
- The coefficients of the variables in the constraints of the primal

Primal-Dual Relationship

	If Primal		*Then Dual*
(i)	Objective is to maximize	(i)	Objective is to minimize
(ii)	jth primal variable, x_j	(ii)	jth dual constraint
(iii)	ith primal constraint	(iii)	ith dual variable, y_i
(iv)	Primal variable x_j unrestricted in sign	(iv)	Dual constraint j is = type
(v)	Primal constraint i is = type	(v)	Dual variable y_i is unrestricted in sign
(vi)	Primal constraints $\leq$ type	(vi)	Dual constraints $\geq$ type

Primal Problem (LPP)

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 4x_2 \\ \text{subject to} \quad & \frac{1}{2}x_1 + 2x_2 \leq 30 \\ & 3x_1 + x_2 \leq 25 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Dual LPP

$$\begin{aligned} \text{Minimize} \quad & Z = 30y_1 + 25y_2 \\ \text{subject to} \quad & \frac{1}{2}y_1 + 3y_2 \geq 3 \\ & 2y_1 + y_2 \geq 4 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Primal-Dual Relationship

Normal Primal Problem	Normal Dual Problem
$\begin{aligned} \text{Maximize} \quad & Z = \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$	$\begin{aligned} \text{Minimize} \quad & W = \mathbf{b}^\top \mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^\top \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned}$

$$
\begin{aligned}
\text{Minimize} \quad & Z = 2x_1 + 3x_2 + 4x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + x_3 \geq 5 \\
& 3x_1 + x_2 + 2x_3 \geq 8 \\
& -3x_1 - x_2 - 2x_3 \geq -8 \\
& -x_1 - 4x_2 - 3x_3 \geq -10 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Minimize} \quad & Z = 2x_1 + 3x_2 + 4x_3 \\
\text{subject to} \quad & x_1y_1 + 2x_2y_1 + x_3y_1 \geq 5y_1 \\
& 3x_1y_2 + x_2y_2 + 2x_3y_2 \geq 8y_2 \\
& -3x_1y_3 - x_2y_3 - 2x_3y_3 \geq -8y_3 \\
& -x_1y_4 - 4x_2y_4 - 3x_3y_4 \geq -10y_4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Maximize} \quad & \boxed{Z = 5y_1 + 8y_2 - 8y_3 - 10y_4} \\
\text{subject to} \quad & \boxed{y_1 + 3y_2 - 3y_3 - y_4} \leq 2 \\
& \boxed{2y_1 + y_2 - y_3 - 4y_4} \leq 3 \\
& \boxed{y_1 + 2y_2 - 2y_3 - 3y_4} \leq 4 \\
& y_1, y_2, y_3, y_4 \geq 0
\end{aligned}
$$

Primal Problem (or Dual Problem)		Dual Problem (or Primal Problem)
Maximize Z (or W)		Minimize W (or Z)
Constraint i:		Variable y_i (or x_i):
$\leq$ form	$\longleftrightarrow$	$y_i \geq 0$
$=$ form	$\longleftrightarrow$	Unconstrained
$\geq$ form	$\longleftrightarrow$	$y_i' \leq 0$
Variable x_j (or y_j):		Constraint j:
$x_j \geq 0$	$\longleftrightarrow$	$\geq$ form
Unconstrained	$\longleftrightarrow$	$=$ form
$x_j' \leq 0$	$\longleftrightarrow$	$\leq$ form

Obtain the dual of the following LPP:

Maximize $Z_x = x_1 - 2x_2 + 3x_3$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 = 2$, (ii) $2x_1 + 3x_2 + 4x_3 = 1$

and $x_1, x_2, x_3 \geq 0$

Solution Since both the primal constraints are of the equality type, the corresponding dual variables y_1 and y_2, will be unrestricted in sign. Following the rules of duality formulation, the dual of the given primal LP problem is

Minimize $Z_y = 2y_1 + y_2$

subject to the constraints

(i) $-2y_1 + 2y_2 \geq 1$, (ii) $y_1 + 3y_2 \geq -2$, (iii) $3y_1 + 4y_2 \geq 3$

and y_1, y_2 unrestricted in sign.

Benefits of Duality in LPP

- Study of duality helps to identify only an increase (or decrease) in the value of objective function due to per unit variation in the amount of resources available.

1. **Alternative Formulations:** Provides another way to view and solve the same problem.
2. **Solution Bounds:** Helps in establishing upper or lower bounds for the optimal solution of the primal problem.
3. **Sensitivity Analysis:** Facilitates the calculation of shadow prices, which indicate the value of additional units of a resource.
4. **Feasibility and Optimality:** Helps in evaluating whether a solution is feasible or optimal.

Fundamental Theorem of Linear Programming

If a linear programming problem (LPP) has an optimal solution, then at least one optimal solution occurs at a **corner point (vertex)** of the feasible region.

•Implications:

- Search for optimal solutions can be restricted to corner points of the feasible region.
- There may be:
  - **Unique solution** at one vertex.
  - **Multiple optimal solutions** if the objective function is parallel to a constraint.
  - **Unbounded solution** if feasible region is open in the direction of optimization.
  - **Infeasible problem** if feasible region is empty.

Degenerate Solutions in LPP

A solution is **degenerate** if one or more basic variables take the value zero at a basic feasible solution (BFS).

- **Causes:**
  - Redundant constraints.
  - Intersection of more than ‘m’ constraints at a BFS (where *m = number of constraints*).
- **Implications:**
  - May lead to **stalling** in the simplex method.
  - Could cause **cycling** (repetition of same BFS).

Simplex-Based Methods

- **Purpose:** Solve LPPs by moving from one BFS to another, improving the objective function until optimality.
- **Key Components:**
  - **Initial Basic Feasible Solution (IBFS):** Obtained using slack/surplus/artificial variables.
  - **Pivot Operations:** Exchange of basic and non-basic variables.
  - **Optimality Test:** When all reduced costs are ≥ 0 (for maximization).
  - **Unboundedness Check:** If entering variable has no positive ratio for leaving variable test.

Cycling in Simplex

- **Problem:** Simplex method may revisit the same set of BFS repeatedly due to degeneracy.
- **Result:** Infinite loop, no progress toward optimality.
- **Prevention Techniques:**
  - **Bland’s Rule:** Always choose entering and leaving variables with smallest index.
  - **Perturbation Technique:** Slightly adjust constraints to remove degeneracy.

SENSITIVITY ANALYSIS in LPP

- The process of modifying an OR model to observe the effect upon its outputs is called **Sensitivity Analysis**. Purpose is to evaluate the effect on the optimal solution of an LP problem due to variations in the input coefficients (also called parameters), one at a time.
- In an LP model, the coefficients (also known as parameters) such as: (i) profit (cost) contribution (c_j) per unit of a decision variable, x_j (ii) availability of a resources (b_i), and (iii) consumption of resource per unit of decision variables (a_{ij}), are assumed to be constant and known with certainty.
- *Sensitivity analysis determines the sensitivity range (both lower and upper limit) within which the LP model parameters can vary (one at a time) without affecting the optimality of the current optional solution.*
- This analysis reveals the magnitude of change in the optimal solution of an LP model due to discrete variations (changes) in its parameters. The possible change in the parameter values, can range from zero to a substantial change.
- Thus, aim of sensitivity analysis is to determine the range within which the LP model parameters can change without affecting the current optimal solution.

SENSITIVITY ANALYSIS (2)

- The sensitivity analysis is also referred to as *post-optimality analysis* because it does not begin until the optimal solution to the given LP model has been obtained.
- Different parametric changes in an LPP are:

1. Profit (or cost) per unit (c_j) associated with both basic and non-basic decision variables (i.e., coefficients in the objective function).
2. Availability of resources (i.e., right-hand side constants, b_i in constraints).
3. Consumption of resources per unit of decision variables x_j (i.e., coefficients of decision variables in the constraints, a_{ij}).
4. Addition of a new variable to the existing list of variables in LP problem.
5. Addition of a new constraint to the existing list of constraints in the LP problem.

- *So, what happens to the optimal solution value when we have a* **change in the Objective Function Coefficient** ($\boldsymbol{c_j}$)? Analysing this is termed as **SENSITIVITY ANALYSIS**.

Line Search Methods in Optimization

Definition: Techniques to determine optimal step size α_k along a search direction.

- **Basic Idea:**
  - Start at x_k.
  - Choose search direction d_k.
  - Update:

$$x_{k+1} = x_k + \alpha_k d_k$$

where α_k is step size.

Stationarity of Limit Points in Steepest Descent

- If:
  1. The objective function $f(x)$ is continuously differentiable.
  2. Step sizes α_k are chosen by exact or inexact line search (satisfying descent conditions).

Then:

- Any **accumulation point** of the sequence $\{x_k\}$ generated by steepest descent is a **stationary point** (i.e., satisfies $\nabla f(x^*) = 0$).
- In practice: steepest descent may take many iterations to reach acceptable accuracy.

Backtracking is simple and widely used.
Exact line search (finding optimal α analytically) is rare in practice because it may require solving another optimization problem.

Exact Line Search Method

- Find the best α that minimizes along the direction d_k.
- Usually involves solving a 1-variable optimization problem

Line Search Method Concept

- Need to find good step size α.
- Steepest descent converges to stationary points.
- Backtracking reduces α until sufficient decrease.

Problem 1: Exact Line Search

Minimize $f(x) = (x-2)^2$ using steepest descent, start $x_0 = 0$.

Solution:

- Gradient: $2(x-2)$. Direction: $d = -\nabla f(x_0) = +4$.
- Line search: $f(0 + \alpha \cdot 4) = (4\alpha - 2)^2$.
- Derivative wrt α: $2(4\alpha - 2)(4) = 0 \Rightarrow \alpha = 0.5$.
- Update: $x_1 = 0 + 0.5 \cdot 4 = 2$.
- Reached optimum in 1 step.

$$x_{k+1} = x_k + \alpha_k d_k$$

Successive Step-Size Reduction Algorithms

- Instead of fixing step size, start large and **reduce until progress is adequate.**
- Common rules:

(a) Backtracking Line Search

1. Choose initial $\alpha = 1$.
2. While:

$$f(x_k + \alpha d_k) > f(x_k) + c\alpha \nabla f(x_k)^T d_k$$

(Armijo condition not satisfied), reduce α (e.g., $\alpha \leftarrow \beta\alpha$, with $\beta \in (0, 1)$).

3. Accept the reduced α.

- Guarantees sufficient decrease in each step.

❑ **Backtracking is simple and widely used method in Line Search Optimization problems.**

❑ **Exact line search (finding optimal α analytically) is rare in practice because it may require solving another optimization problem.**

❑ **Another method is Wolfe condition**

(a) Backtracking Line Search

1. Choose initial $\alpha = 1$.
2. While:

$$f(x_k + \alpha d_k) > f(x_k) + c\alpha \nabla f(x_k)^T d_k$$

(Armijo condition not satisfied), reduce α (e.g., $\alpha \leftarrow \beta\alpha$, with $\beta \in (0, 1)$).

Problem 2: Backtracking Line Search

Minimize

$$f(x) = x^2$$

starting from $x_0 = 1$, direction $d = -\nabla f(x_0) = -2$, initial $\alpha = 1$, $\beta = 0.5$.

- Gradient at $x_0 = 1$: 2. Direction = -2.
- Trial step: $x = 1 - 2(1) = -1$. $f(-1) = 1$.
- Armijo: $f(-1) \leq f(1) + c\alpha\nabla f(1)d$. (With $c = 0.5$).
- RHS = $1 + 0.5(1)(2)(-2) = -1$. LHS = 1 > -1 → not satisfied.
- Reduce step: $\alpha = 0.5$.
- New point: $x = 0$. $f(0) = 0$. Condition satisfied.
- Accept $\alpha = 0.5$.

$$x_{k+1} = x_k + \alpha_k d_k$$

Problem: Minimize $f(x) = x^2$ starting from $x_0 = 1$, direction $d = -\nabla f(x_0) = -2$. Initial step $\alpha_0 = 1$, shrink factor $\beta = 0.5$. Use the Armijo (sufficient decrease) condition in the backtracking loop.

1. compute gradient and direction

$$f(x) = x^2, \qquad \nabla f(x) = 2x.$$

At $x_0 = 1$:

$$\nabla f(1) = 2, \qquad d = -\nabla f(1) = -2.$$

The trial point for step α is

$$x(\alpha) = x_0 + \alpha d = 1 + \alpha(-2) = 1 - 2\alpha.$$

The value along the line is

$$\phi(\alpha) \equiv f(x(\alpha)) = (1 - 2\alpha)^2.$$

2. Armijo (sufficient decrease) condition

Armijo condition requires

$$\phi(\alpha) \le \phi(0) + c\,\alpha\,\nabla f(x_0)^\top d,$$

where $\phi(0) = f(x_0) = 1$ and $\nabla f(x_0)^\top d = (2) \cdot (-2) = -4$. So the right-hand side is

$$\phi(0) + c\alpha\nabla f(x_0)^\top d = 1 + c\alpha(-4) = 1 - 4c\alpha.$$

Thus Armijo inequality becomes

$$(1 - 2\alpha)^2 \le 1 - 4c\alpha.$$

3. Test $\alpha = \alpha_0 = 1$

Left-hand side:

$$\phi(1) = (1 - 2 \cdot 1)^2 = (-1)^2 = 1.$$

Right-hand side:

$$1 - 4c \cdot 1 = 1 - 4c.$$

Inequality is $1 \leq 1 - 4c$, i.e. $0 \leq -4c$. That is **false** for any $c > 0$.

So $\alpha = 1$ **fails** Armijo and we reduce α.

4. Reduce α: $\alpha \leftarrow \beta\alpha = 0.5$

Now $\alpha = 0.5$.

Left-hand side:

$$\phi(0.5) = (1 - 2 \cdot 0.5)^2 = (1 - 1)^2 = 0.$$

Right-hand side:

$$1 - 4c \cdot 0.5 = 1 - 2c.$$

Inequality is $0 \leq 1 - 2c$, i.e. $c \leq 0.5$. For any typical Armijo constant c (e.g. $c = 10^{-4}$ up to $c = 0.1$), this holds. So $\alpha = 0.5$ **satisfies** Armijo and is accepted.

5. Numerical check with $c = 0.1$ (concrete)

- For $\alpha = 1$: LHS $= 1$, RHS $= 1 - 4(0.1) = 0.6$. $1 \leq 0.6$ → fails.
- For $\alpha = 0.5$: LHS $= 0$, RHS $= 1 - 2(0.1) = 0.8$. $0 \leq 0.8$ → holds.

So backtracking accepts $\alpha = 0.5$.

---

6. Update iterate

$$x_1 = x_0 + \alpha d = 1 + 0.5 \cdot (-2) = 0.$$

Evaluate objective: $f(x_1) = 0$, which is the global minimum for $f(x) = x^2$. (In fact, an exact line search on this quadratic also gives $\alpha = 0.5$.)

final answer

- First trial $\alpha = 1$ fails Armijo.
- After one shrink ($\beta = 0.5$) we get $\alpha = 0.5$ which satisfies Armijo (for any common choice of $c \leq 0.5$, in particular $c = 0.1$).
- Accepted step $\alpha = 0.5$ → new iterate $x_1 = 0$.
- The method reached the exact minimizer in one accepted backtracking step.

Problem 3: Stationarity of Limit Points

Show that steepest descent for

$$f(x) = (x-1)^2$$

leads to a stationary point.

Solution:

- Gradient: $2(x-1)$.
- Update: $x_{k+1} = x_k - \alpha(2(x_k - 1))$.
- As $k \to \infty$, $x_k \to 1$.
- At $x = 1$, gradient = 0 → stationary point.

Problem 3: Backtracking Line Search

$f(x) = x^2$, start at $x_0 = 1$, direction $d = -\nabla f(1) = -2$.

Parameters: $c = 0.1, \beta = 0.5$.

We try $\alpha = 1$:

Check Armijo: $f(1-2) \leq f(1) + c\alpha\nabla f(1)d$.

LHS $= f(-1) = 1$. RHS $= 1 + 0.1(1)(-2)(-2) = 1 + 0.4 = 1.4$.

Condition holds → accept $\alpha = 1$.

Update: $x_1 = -1$.

$$x_{k+1} = x_k + \alpha_k d_k$$

Unconstrained Optimization

- Unconstrained optimization is a mathematical process that finds the minimum or maximum value of an objective function without any restrictions on the decision variables
- Unconstrained optimization plays a crucial role in the training of neural networks.
- Unlike constrained optimization, where the solution must satisfy certain constraints, unconstrained optimization seeks to minimize (or maximize) an objective function **without any restrictions on the variable values**.
- **Common Unconstrained Optimization Techniques**
  - **Gradient descent method (Steepest Descent Method) :** move in the direction of the negative gradient.
  - **Newton's method:** use both gradient and Hessian for faster convergence

1. What is Unconstrained Optimization?

An **unconstrained optimization problem** is one where we want to **minimize (or maximize) an objective function** without any restrictions (constraints) on the variables.

Formally:

$$\min_{x \in \mathbb{R}^n} f(x)$$

where:

- $f(x)$ is the objective function,
- $x = (x_1, x_2, \ldots, x_n)$ is the decision vector,
- No equality or inequality constraints are imposed.

2. First-Order Condition (Stationary Point)

A necessary condition for x^* to be an **optimal point** (minimum or maximum) is:

$$\nabla f(x^*) = 0$$

This means the **gradient** of $f(x)$ must vanish at x^*. Such a point is called a **stationary point**.

3. Second-Order Condition

To classify a stationary point x^*, we examine the **Hessian matrix** $H(x) = \nabla^2 f(x)$:

- If $H(x^*)$ is **positive definite** → x^* is a **local minimum**.
- If $H(x^*)$ is **negative definite** → x^* is a **local maximum**.
- If $H(x^*)$ is **indefinite** (both positive and negative eigenvalues) → x^* is a **saddle point** (neither min nor max).

4. Methods for Unconstrained Optimization

Since there are no constraints, algorithms focus directly on improving the objective function:

(a) Analytical methods

- Solve $\nabla f(x) = 0$ directly if possible.

(b) Iterative numerical methods

- **Steepest Descent Method**: move in the direction of the negative gradient.
- **Newton's Method**: use both gradient and Hessian for faster convergence.

Example 1 (1D Function)

$$f(x) = x^2 - 4x + 5$$

Step 1: First derivative (gradient in 1D).

$$f'(x) = 2x - 4$$

Set equal to zero:

$$2x - 4 = 0 \;\Rightarrow\; x^* = 2$$

Step 2: Second derivative.

$$f''(x) = 2 > 0$$

So $x^* = 2$is a **local (and global) minimum**.

Function value:

$$f(2) = 2^2 - 4(2) + 5 = 1$$

Minimum value of $f(x)$ is 1 at x = 2

6. Example 2 (2D Function)

$$f(x,y) = x^2 + y^2 - 2x - 4y + 5$$

Step 1: Gradient.

$$\nabla f(x,y) = \begin{bmatrix} 2x-2 \\ 2y-4 \end{bmatrix}$$

Set $\nabla f = 0$:

$$2x-2 = 0 \;\Rightarrow\; x^* = 1$$

$$2y-4 = 0 \;\Rightarrow\; y^* = 2$$

Step 2: Hessian.

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

This is positive definite (all eigenvalues = 2 > 0).

So $(1,2)$ is a **local minimum**.

Function value:

$$f(1,2) = 1^2 + 2^2 - 2(1) - 4(2) + 5 = 2$$

Important Points :

- Unconstrained optimization = **no restrictions** on variables.
- Stationary points are found by setting $\nabla f\,(x)\,(\text{derivative})\,=0.$
- Second-order test (Hessian) classifies stationary points.
- Numerical iterative methods are used when analytical solutions are difficult.

Problem 2 (1D Cubic)

$$f(x) = x^3 - 3x^2 + 2$$

Solution:

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Stationary points: $x = 0, 2$.

Second derivative:

$$f''(x) = 6x - 6$$

- At $x = 0$: $f''(0) = -6 < 0 \Rightarrow$ local **maximum**.
- At $x = 2$: $f''(2) = 6 > 0 \Rightarrow$ local **minimum**.

Function values:

$$f(0) = 2 \quad (\text{local max}), \quad f(2) = 8 - 12 + 2 = -2 \quad (\text{local min})$$

Problem 3: 2D Quadratic with Cross Term

$$f(x, y) = (x-1)^2 + 2(y+2)^2 + xy$$

(a) Find the stationary points by solving $\nabla f = 0$.

(b) Compute the Hessian and classify the stationary point(s).

(c) State whether the stationary point is global and why.

Step 1: Expand the function (for clarity)

$$f(x, y) = (x^2 - 2x + 1) + 2(y^2 + 4y + 4) + xy$$

Simplify:

$$f(x, y) = x^2 + 2y^2 + xy - 2x + 8y + 9$$

Step 2: Compute the first derivatives

$$\frac{\partial f}{\partial x} = 2x + y - 2$$

$$\frac{\partial f}{\partial y} = 4y + x + 8$$

Step 3: Stationary point condition ($\nabla f = 0$)

Set both partial derivatives to zero:

$$\begin{cases} 2x + y - 2 = 0 \\ x + 4y + 8 = 0 \end{cases}$$

Step 4: Solve the linear system

From the first equation:

$$y = 2 - 2x$$

Substitute into the second:

$$x + 4(2 - 2x) + 8 = 0$$

Simplify:

$$x + 8 - 8x + 8 = 0$$

$$-7x + 16 = 0$$

$$x = \frac{16}{7}$$

Now substitute back into $y = 2 - 2x$:

$$y = 2 - 2\left(\frac{16}{7}\right) = 2 - \frac{32}{7} = \frac{14 - 32}{7} = -\frac{18}{7}$$

✅ Stationary point:

$$(x^*, y^*) = \left(\frac{16}{7}, -\frac{18}{7}\right)$$

Step 5: Compute the Hessian matrix

Second derivatives:

$$f_{xx} = 2, \quad f_{yy} = 4, \quad f_{xy} = f_{yx} = 1$$

So,

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

Step 6: Test positive definiteness of Hessian

For a 2×2 symmetric matrix

$$H = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

it is **positive definite** if:

1. $a > 0$
2. Determinant $ac - b^2 > 0$

Compute:

$$a = 2 > 0$$

$$\det(H) = (2)(4) - (1)^2 = 8 - 1 = 7 > 0$$

Hence, $H$is **positive definite**.

Step 7: Classify the stationary point

Since H is positive definite,

$$\Rightarrow (x^*, y^*) = \left(\frac{16}{7}, -\frac{18}{7}\right)$$

is a **local minimum.**

Step 8: Compute the minimum value

Substitute $x = \frac{16}{7}, y = -\frac{18}{7}$ into $f(x, y) = x^2 + 2y^2 + xy - 2x + 8y + 9$:

$$f = \left(\frac{16}{7}\right)^2 + 2\left(\frac{-18}{7}\right)^2 + \left(\frac{16}{7}\right)\left(\frac{-18}{7}\right) - 2\left(\frac{16}{7}\right) + 8\left(\frac{-18}{7}\right) + 9$$

Simplify step-by-step:

$$f = \frac{256}{49} + 2\left(\frac{324}{49}\right) - \frac{288}{49} - \frac{32}{7} - \frac{144}{7} + 9$$

Convert everything to denominator 49:

$$f = \frac{256 + 648 - 288}{49} - \frac{176 \times 7}{49} + \frac{9 \times 49}{49}$$

$$f = \frac{616}{49} - \frac{1232}{49} + \frac{441}{49} = \frac{-175}{49} = -\frac{25}{7}$$

Minimum value of f is : - 25/7 = - 3.57

Step 9: Check if global

Since $f(x,y)$ is a **quadratic function** with a **positive definite Hessian**, it is **strictly convex**.

→ The stationary point is the **unique global minimum**.

Quantity	Result
Function	$f(x,y) = (x-1)^2 + 2(y+2)^2 + xy$
Stationary point	$(x^*, y^*) = \left(\frac{16}{7}, -\frac{18}{7}\right)$
Hessian	$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$, Positive definite
Classification	Local **and** global minimum
Minimum value	$f_{min} = -\frac{25}{7} \approx -3.571$

Problem 4 — 2D Non-Convex Polynomial

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

Find all stationary points, use the Hessian to classify them, and identify which are minima, maxima, or saddle points.

Step 1: Compute the gradient (first-order partial derivatives)

$$\frac{\partial f}{\partial x} = 4x^3 - 4y$$

$$\frac{\partial f}{\partial y} = 4y^3 - 4x$$

Stationary points satisfy $\nabla f = 0$:

$$4x^3 - 4y = 0, \qquad 4y^3 - 4x = 0$$

or equivalently,

$$x^3 = y, \qquad y^3 = x$$

Step 2: Solve for stationary points

Substitute $y = x^3$ into $y^3 = x$:

$$(x^3)^3 = x \quad \Rightarrow \quad x^9 = x$$

$$x(x^8 - 1) = 0$$

Hence

$$x = 0 \quad \text{or} \quad x^8 = 1$$

For real x:

$$x = 0, \quad x = \pm 1$$

Now obtain corresponding y values from $y = x^3$:

x	$y = x^3$
0	0
1	1
−1	−1

Stationary points: (0,0), (1,1), (−1,−1)

Step 3: Compute the Hessian matrix

$$f_{xx} = 12x^2, \quad f_{yy} = 12y^2, \quad f_{xy} = f_{yx} = -4$$

Thus

$$H(x,y) = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

Step 4: Evaluate Hessian at each stationary point

(a) At (0,0):

$$H = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

Determinant $D = (0)(0) - (-4)^2 = -16 < 0.$

→ **Indefinite Hessian ⇒ Saddle point.**

(b) At (1,1):

$$H = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

Compute principal minors:

$$a_{11} = 12 > 0, \quad D = (12)(12) - (-4)^2 = 144 - 16 = 128 > 0$$

→ **Positive definite ⇒ Local minimum.**

Function value:

$$f(1,1) = 1 + 1 - 4(1)(1) + 1 = -1$$

Local minimum at (1,1) with $f = -1$.

(c) At (−1,−1):

$$H = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

(same structure as above)

→ **Positive definite ⇒ Local minimum.**

Function value:

$$f(-1,-1) = (-1)^4 + (-1)^4 - 4(-1)(-1) + 1 = 1 + 1 - 4 + 1 = -1$$

Local minimum at (-1, -1) with $f = -1$.

Step 5: Summary of classification

Point	Hessian	Determinant	Type	$f(x,y)$
(0, 0)	$\begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$	−16	Saddle point	1
(1, 1)	$\begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$	128	Local min	−1
(−1, −1)	same as (1,1)	128	Local min	−1

Step 6: Global behavior

For large $|x|, |y|$, the quartic terms $x^4 + y^4$ dominate $-4xy$.
Since $x^4 + y^4 \geq 0$ and grows to infinity, $f(x,y) \to \infty$ as $|(x,y)| \to \infty$.

Hence both local minima at $(\pm 1, \pm 1)$ are also **global minima**.

Step 7: Visual intuition

- The function has **two symmetric global minima** at (1, 1) and (−1, −1).
- The origin (0, 0) is a **saddle point** (a “mountain pass” between two valleys).
- The surface is non-convex overall because the Hessian is not positive definite everywhere.

Final Results

Quantity	Value / Interpretation
Stationary points	(0, 0), (1, 1), (−1, −1)
Classification	(0, 0): Saddle; (±1, ±1): Local & Global Minima
Minimum function value	$f_{min} = -1$
Non-convex?	Yes — Hessian not positive definite for all (x,y)

Unbounded Solution in a Linear Programming Problem

In Linear Programming, we aim to:

$$\text{Maximize (or Minimize)} \quad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$Ax \leq b, \quad x \geq 0$$

The **feasible region** is the set of all points x satisfying the constraints.

An **LPP has an unbounded solution** if the objective function can increase (for maximization) or decrease (for minimization) **indefinitely** without violating any constraint, i.e.

$$Z \to +\infty \text{ (for maximization)} \quad \text{or} \quad Z \to -\infty \text{ (for minimization)}$$

while still remaining inside the feasible region.

Geometric Explanation

In two dimensions:

•The feasible region (intersection of constraints) may **open infinitely** in one or more directions.

•If the **objective function line** (isoprofit or isocost line) can move indefinitely in that open direction without leaving the feasible region, the problem is **unbounded**.

Example sketch idea:

•If you maximize $Z = 3x + 2y$

•and feasible region extends infinitely in direction of increasing x, y, then Zhas no finite maximum — it is **unbounded**.

Algebraic (Simplex Method) Condition for Unboundedness

During the **Simplex algorithm**, we work with the **tableau** and repeatedly select:

1. **Entering variable:**

   The non-basic variable with the **most negative coefficient** (for maximization) in the Z-row.

2. **Leaving variable:**

   Determined using the **minimum ratio test:**

$$\text{Ratio} = \frac{\text{RHS}}{\text{positive entry in pivot column}}$$

- Only **positive** pivot column entries** are considered.
- The **smallest positive ratio** decides which variable leaves the basis.

Condition for Unbounded Solution (in Simplex) : If in the selected **pivot (entering) column**, **all entries are ≤ 0**, then the LPP is **unbounded**.

Reason:

- The entering variable can **increase indefinitely**
- Since all entries in that column are ≤ 0, increasing that variable will **not violate any constraint** (the RHS values won't become negative).
- Therefore, Zcan be increased (in maximization) or decreased (in minimization) infinitely — i.e., no finite optimum exists.