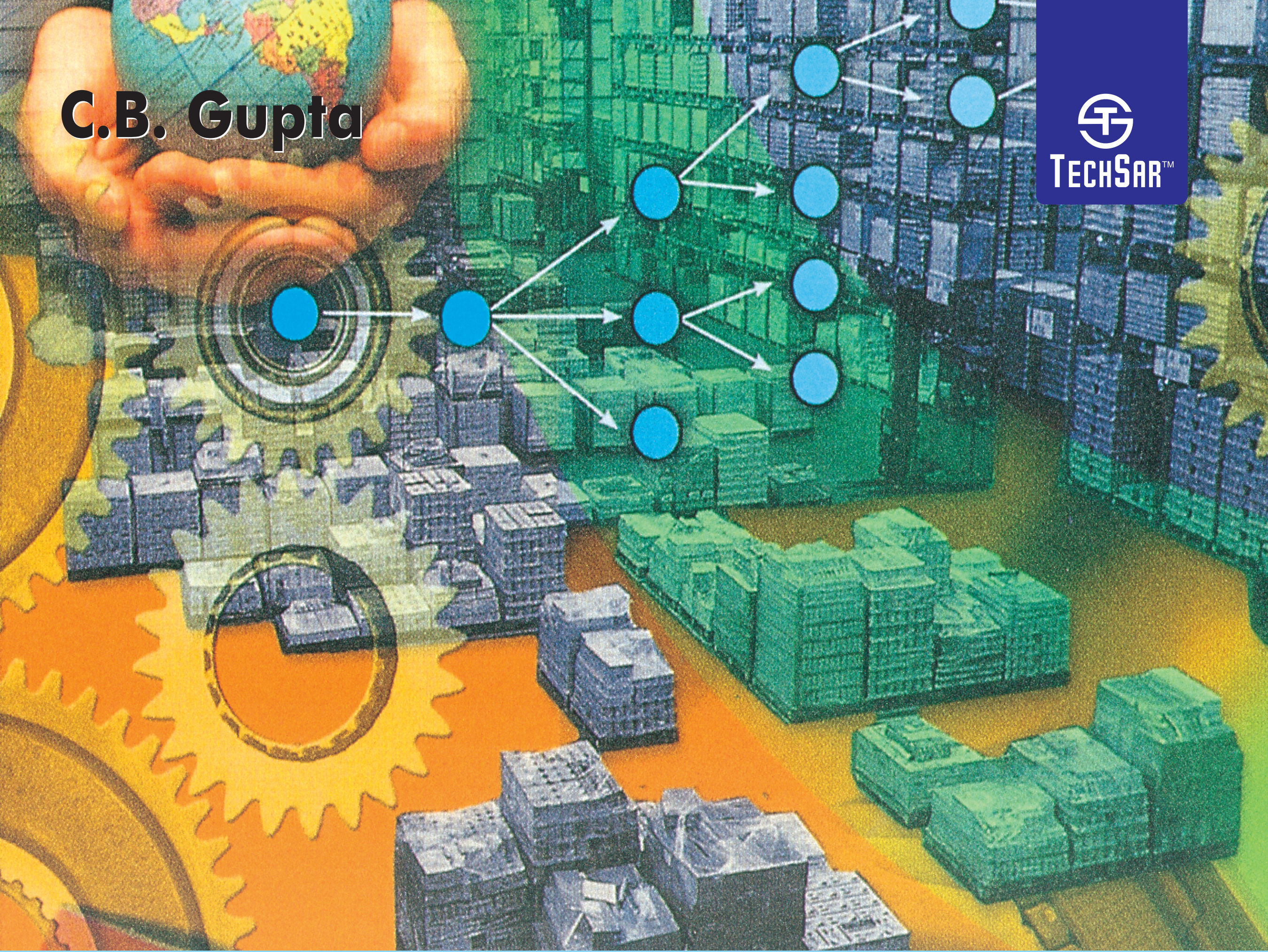

C.B. Gupta

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Optimization Techniques in Operations Research

Second Edition

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Optimization Techniques in Operations Research

Second Edition

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Preface to the Second Edition

Two chapters, “Introduction to Optimization” and “Classical Optimization Techniques” have been added to the present edition. Some more solved, unsolved examples and a new article on processing 2-jobs through *k*-machines are also added in Chapter 13. I really appreciate those readers who have given their valuable suggestions and comments for the addition of these chapters. I am highly thankful to Dr. A.K. Malik of BK BIET, Pilani for his valuable suggestions.

C.B. Gupta

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Preface to the First Edition

The main objective of writing this book is to provide a textbook to the graduate and postgraduate students of Engineering/Science and Management who deal with the different optimization techniques to optimize various problems. In this book mainly optimization techniques related to deterministic mathematical models are discussed.

The book is divided in eleven chapters. Chapter 1 deals with linear programming problem. Formulation and solution of a linear programming problem by graphical method, simplex method and artificial techniques (Big M and Two-Phase methods) is discussed.

In chapter 2 duality of a linear programming problem is discussed. The relation between the solution of a primal and dual is also dealt in this chapter.

Chapter 3 deals with sensitivity analysis in which it is discussed that what will be effect on optimal solution of the given linear programming problem if various changes are made in the given problem after optimal solution of the problem.

In chapter 4 transportation with transhipment and assignment problem are dealt. In this chapter various methods to find starting basic feasible solution for transportation problem and Hungarian algorithm to solve an assignment problem is discussed.

Integer linear programming and travelling salesman problems are discussed in chapter 5. Various techniques to solve mixed ILPP, pure ILPP and branch and bound technique to solve salesman problem is discussed here.

Chapter 6 deals with dynamic programming in which various examples including Capital budgeting, Cargo loading, Reliability, etc. are solved with the help of forward and backward relations.

In chapter 7 game theory is discussed. Various game problems are solved by different methods including graphical and simplex method.

Chapter 8 deals with non-linear programming problem including quadratic programming. Kuhn-Tucker conditions are used to find solutions of various non-linear problems.

In chapter 9 network analysis including Critical Path Method (CPM) and Programming Evaluation and Review Technique (PERT) is discussed.

Chapters 10 and 11 respectively deal with goal programming and sequencing.

Suggestions and comments from all quarters will be appreciated and duly acknowledged.

C.B. GUPTA

Acknowledgements

I take this opportunity to express my sincere gratitude to Prof. L.K. Maheshwari, Vice-Chancellor, Birla Institute of Technology and Science (BITS), Pilani who always been a source of inspiration and encouragement. I am also thankful to Prof. K.E. Raman, Deputy Director (Administration) and Prof. G. Raghurama, Deputy Director (Academic), BITS, Pilani for providing me all necessary facilities and time-to-time help and encouragement. Prof. B.R. Natarajan, Dean distance Learning Programmes Division and Prof. R.K. Mittal, Dean Academic Registration and Counselling, BITS, Pilani deserve for special thanks for their moral support and inspiration. I am also sincerely thankful to Prof. R.N. Saha, Dean FD III and Education Development Division, BITS, Pilani for his all round support and facilities provided during completion of this project.

At last but not the least I am highly thankful to all my family members specially my mother Mrs. Bharpai Devi, elder brother Shsi Raj Kumar Gupta, wife Mrs. Anita Gupta, daughter Miss Chhavi Gupta and son Master Ankur Gupta for their unflagging support, inspiration and encouragement and without these it could not had been possible for me to complete this project. It is to them I dedicate this book.

C.B. GUPTA

Contents

Chapter 1

Introduction to Optimization

In this chapter, we shall discuss the introduction to optimization, and operations research, optimization techniques, various applications of optimization techniques, optimization problems and their classification.

1.1 INTRODUCTION

Optimization techniques have a great importance in education, industries, five-star hotels, hospitals, banking, computer science, electronics, mechanical, electrical, automotive, civil, medical science, biotechnology, aerospace, management and information technology, etc. Optimization means obtaining the best output which may be maximum or minimum value of the criterion. Optimization techniques were used in solving various mathematical, biological, physical and chemical problems, etc. from the time of Newton, Cauchy and Lagrange. Newton and Leibnitz developed some important optimization methods in differential calculus. Lagrange developed a method of optimization for constrained problems, which involves the addition of unknown multiples known by the name of Lagrange's method of undetermined multipliers.

Cauchy was the first mathematician who gave the first application of unconstrained minimization problems solved by steepest desent method. Optimization techniques are generally studied as a part of operations research. Operations research is a branch of mathematics concerned with the application of scientific methods and techniques to decision-making problems and with the best or optimal solution.

1.2 INTRODUCTION TO OPERATIONS RESEARCH

Operations Research has become the most important instrument in the organization and management in various institutions. In recent years service organizations such as airlines, railways, banks, libraries and hospitals have started recognizing the usefulness of operations research for improving efficiency. The application of mathematical models for solving such business problems first attracted the attention of a Russian mathematician, L.V. Kantorovich, in 1939, who published a monograph "Mathematical methods in the organization and planning of production". The term of operations research was first coined by McCloskey and Trefthen in 1940. They used the term "Operations Research" as a result of military operations during World War II.

In 1914-1915 Thomas Edison made an effect to use a tactical game board to find a way to minimize shipping losses from enemy submarines instead of risking ships in actual war conditions. He used a particular model and techniques of operations research for this purpose. An Operations Research Club was established in England in 1948, later known as the Operational Research Society of U.K. Operations Research in India came into existence in 1949 with the development of an operations research unit at regional research laboratory, Hyderabad for the purpose of planning and organizing research of equal importance was the setting up of an operations research team at Defence Science Laboratory by Prof. R.S. Verma for the specific purpose of solving the problems of store, purchase and planning. Simplex algorithm developed by G.B. Dantzig is a practical tool for decision-making.

A big step in the field of operational research was the establishment of Operations Research Society of America (ORSA) in 1952 and publications of its first journal *Operations Research* in 1953. Operations Research received a further boost with the setting up of operations research team in the Indian Statistical Institute, Kolkata by Prof. P.C. Mahalanobis in 1953 for the purpose of solving the problems related to national planning and survey. Later the Operation Research Society of India (ORSI) was established in 1957. This society started publishing its journal *Opsearch* from 1964. Operations Research techniques provide effective base for management decisions. Some definitions of operations research which are commonly used are as follows:

> "OR is the application of scientific methods, techniques and tools to problems involving the operation of a system so as to provide those in control of the system with optimum solution to the problem."

(Churchman, Ackoff and Arnoff)

> "OR is a scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources."

(H.A. Taha)

> "OR is applied decision theory. It uses many scientific mathematical or logical means to attempt to cope with the problems that confront the executive when he tries to achieve through-going rationality in dealing with his decision problems."

(D.W. Miller and M.K. Starr)

> "OR may be described as a scientific approach to decision-making that involves the operations of organizational system."

(F.S. Hiller and G.T. Lieberman)

> "OR is a scientific method of providing executive departments with a quantitative basis for decisions under their control."

(P.M. Morse and G.E. Kimball)

"OR is the systematic application of quantitative methods, techniques and tools to the analysis of problem involving the operation of systems."

(Dallenbach and George)

"OR is the art of giving bad answers to problems which otherwise have worse answers."

(T.L. Saaty)

"OR is the art of winning wars without actually fighting them."

(Arthur Clarke)

"OR is a scientific approach to problems solving for executive management."

(H.M. Wagner)

1.3 INTRODUCTION TO OPTIMIZATION TECHNIQUES

Optimization techniques are also known as mathematical programming techniques which are generally studied as a part of operations research. Mathematical programming techniques are very useful in finding the maximum or minimum of a function of several variables under the given set of constraints. The various mathematical programming techniques along with other well-defined area of operations research are given below.

(a) **Mathematical Programming Techniques**

- (i) Linear programming
- (ii) Integer programming
- (iii) Multiobjective programming
- (iv) Geometric programming
- (v) Dynamics programming
- (vi) Non-linear programming
- (vii) Quadratic programming
- (viii) Goel programming
- (ix) Stochastic programming
- (x) Separable programming
- (xi) Sequencing theory
- (xii) Game theory
- (xiii) Information theory
- (xiv) Transportation methods
- (xv) Simulated annealing
- (xvi) Neural networks
- (xvii) Assignment methods
- (xviii) Inventory control methods
- (xix) Calculus of variations

(xx) Network scheduling methods (CPM and PERT)
(xxi) Differential calculus methods

(b) **Stochastic Process Techniques**
(i) Queueing theory
(ii) Statistical decision theory
(iii) Markov processes
(iv) Simulation methods
(v) Reliability theory.

(c) **Statistical Methods**
(i) Design of experiments
(ii) Cluster analysis
(iii) Correlation analysis
(iv) Regression analysis
(v) Factor analysis

1.4 APPLICATIONS OF OPTIMIZATION IN ENGINEERING

The applications of optimization techniques have a great importance in solving various engineering problems, some are given below:

(i) Optimal designing of control system.
(ii) Optimal inventory cost.
(iii) Optimal planning, scheduling and controlling.
(iv) Optimal designing of aircraft and aerospace structures for minimum weight.
(v) Optimal planning to get maximum profit in the presence of one or more competitor.
(vi) Optimal designing of electrical networks.
(vii) Optimal designing of water resources system for maximum benefit.
(viii) Optimal designing of plastic structures.
(ix) Optimal selection of a new site for an industry.
(x) Optimal designing of pipeline networks for process industries
(xi) Optimal designing of chemical processing equipment and plants.
(xii) Optimal controlling the waiting and idle times and queueing in production lines to reduce the costs.
(xiii) Obtaining the optimal trajectories of space vehicles.
(xiv) Optimal planning of maintenance and replacement of equipment to reduce operating costs.
(xv) Obtaining the optimal shortest route taken by a salesmen visiting various cities during one tour.
(xvi) Optimal designing of electrical machinery such as generators, transformers and motors.
(xvii) Optimal designing of civil engineering structures such as bridge, tower, frames, dam, etc. for minimum cost.
(xviii) Optimal designing of material handling equipment such as conveyors, trucks and cranes for minimum cost.

(xix) Optimal designing of computer structurers for minimum cost.
(xx) Optimal designing of pumps, heat transfer and turbines equipment for maximum efficiency.
(xxi) Optimal designing of earthquake structures for minimum weight.
(xxii) Optimal designing of mechanical components such as gears, cams and machine tools, etc.

1.5 OPTIMIZATION PROBLEM

An optimization or mathematical programming problem can be written as

Optimize (Max. or Min.) $\quad Z = f(X)$

subject to the constraints

$$g_j(X) \geq = \leq b; \quad j = 1, 2, 3, \ldots, m$$

and $\quad X \geq 0$

where $\quad X = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix}$ is an n-dimensional

vector called the design vector, $f(x)$ is termed objective function and $g_j(x)$ are known as constraints. This type of problem is called a constrained optimization problem, but in some optimization problems constraints are not involved. Such type of problems are called unconstrained optimization problems.

Design Vector

Every engineering system is defined by a set of quantities, in which some are fixed and others are not. The quantities that are not fixed, i.e., treated as variables in the design process is called design or decision variables

$$x_i, \; i = 1, 2, 3, \ldots, n.$$

Objective Function

Every optimization problem has an aim to be satisfied—the functional and other requirement of the problem. For example, maximization of mechanical efficiency, minimization of cost, aerospace structure design and to produce the acceptable design, etc. The criterion with respect to which the design is optimized, when expressed as a function of the design variables is called objective function. But in some cases, there may be more than one criterion to be satisfied. Such type of problem which has multiple objective function is called multiobjective programming problem. For example, a pair of gears may be designed to be minimum in weight and maximum in efficiency while transmitting a specified *horsepower*. If $f_1(X)$ and $f_2(X)$ are two objective functions then a new objective function for optimization problem is

$$f(X) = a_1 f_1(X) + a_2 f_2(X)$$

where a_1 and a_2 are constants.
The decision variables are collectively represented as a design vector

$$x = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix}$$

Design Constraints

The design variables cannot be chosen arbitrarily in various practical problems. They not only satisfy certain specified functions but also some other requirements. The restrictions that must be satisfied by design (decision) vectors are called the design constraints.

Constraint Surface

Let us consider an optimization problem with only inequality constraints $g_j(X) \le 0$ or ≥ 0. The set all points (values of X) satisfying the equation $g_j(X) = 0$ forms a hypersurface in the design space which is called the constraint surface.

Objective Function Surface

Suppose $f(x)$ is a objective function, where X is a design vector. The locus of all points which satisfies

$$f(X) = C \text{ (constant)}$$

from a hypersurface in the design space and each value of C gives the different member of the family of surfaces, which is called objective function surfaces.

Non-negative Restrictions

If any optimization problem for each variable $x_i \ge 0$, $i = 1, 2, 3, \ldots, n$ satisfies this condition, i.e., non-negative then the condition $x_i \ge 0$ is called non-negative restrictions.

1.6 CLASSIFICATION OF OPTIMIZATION PROBLEMS

Optimization or mathematical programming problems can be classified in different ways as discussed below.

1.6.1 Classification Based on the Number of Objective Functions

(i) ***Single objective programming problem:*** An optimization problem in which only single objective function is involved is called a single objective programming problem.

(ii) ***Multiobjective programming problem***: An optimization problem in which two or more objective functions are involved is called a multiobjective programming problem.

1.6.2. Classification Based on the Existence of Constraints

(i) ***Constrained optimization problem:*** If the optimization problem has constraints then it is called constrained optimization problem. Formulation is

Optimize (Max. or Min.) $Z = f(X)$

subject to the constraints

$$g_j(X) \geq \; = \; \leq b; \quad j = 1, 2, 3, \ldots, m$$

and $X \geq 0$

(ii) ***Unconstrained optimization problem:*** If the optimization problem has no constraints, then it is called unconstrained optimization problem. Formulation is

Optimize (Max. or Min.) $Z = f(X)$

and $X \geq 0$

1.6.3 Classification Based on the Physical Structure of the Problem

(i) ***Optimal control problem:*** An optimization problem involving a number of stages in which each stage evolves from the preceding stage in a specified manner is called optimal control problem.

(ii) ***Non-optimal control problem:*** An optimization problem which is not optimal control problem (classified based on the physical structure) is called non-optimal problem.

1.6.4. Classification Based on the Nature of the Design Variables

(i) ***Static (parameter) optimization problem:*** If in the optimization problem, we find the value of a set of design parameters, in which some prescribed function of these parameters made that optimizes subject to certain constraint is called static optimization problem. For example, minimization of weight of a rectangular beam subject to a limitation on the maximum deflection.

(ii) ***Dynamic (trajectory) optimization problem:*** If in the optimization problem, the objective is to determine a set of design parameters, in which all are continuous functions of some other parameters that optimize an objective function subject to the prescribed constraints is called dynamic optimization problem. For example, as in above case of rectangular beam, the cross-sectional dimensions are allowed to vary along its length. Hence, such type of problem in which each design variable is a function of one or more parameters is known as dynamic optimization problem.

1.6.5 Classification Based on the Nature of the Equations Involved

The classification of optimization problems based on the nature of the expressions for the objective function and the constraints is very important. The optimization problem can be classified into the following:

(i) ***Linear programming problem:*** If the objective function $f(X)$ and all the constraints $g_j(X)$ are linear in an optimization problem or mathematical programming problem it is called a linear programming problem.

(ii) ***Non-linear programming problem:*** If the objective function $f(X)$ or at least one of the constraints $g_j(X)$ are not linear in an optimization problem or mathematical programming problem it is called a non-linear programming problem.

(iii) ***Geometric programming problem:*** If the objective function $f(X)$ and constraint $g_j(X)$ are expressed as polynomial in an optimization problem it is called a geometric programming problem.

(iv) ***Quadratic programming problem:*** If the objective function $f(X)$ is quadratic and constraints $g_j(X)$ are linear in an optimization problem it is called a quadratic programming problem.

1.6.6 Classification Based on the Deterministic Nature of the Variables

(i) ***Deterministic programming problem:*** If all the design variables $x_1, x_2, x_3, \ldots, x_n$ in an optimization problem are deterministic then the optimization problem is called a deterministic programming problem.

(ii) ***Stochastic programming problem:*** If some or all the design variables $x_1, x_2, x_3, \ldots, x_n$ are probabilistic in an optimization problem then the optimization problem is called a stochastic programming problem.

1.6.7 Classification Based on the Permissible Values of the Design

(i) ***Integer programming problem:*** In a linear programming problem, if some or all the design variables $x_1, x_2, x_3, \ldots x_n$ in a optimization problem are restricted to take only integer values then the optimization problem is called an integer programming problem.

(ii) ***Real valued programming problem:*** If all the design variables $x_1, x_2, x_3, \ldots, x_n$ in an optimization problem are permitted to take any real value, then the optimization problem is called a real valued programming problem.

1.6.8 Classification Based on the Separability of the Functions

(i) ***Separable programming problem:*** If the objective function $f(X)$ and the constraints $g_j(X)$ are separable in a optimization problem, then it is called a separable programming problem.

(ii) ***Non-separable programming problem:*** If the objective function $f(X)$ and the constraints $gi(X)$ are non-separable in an optimization problem (classified based on the separability of function), then it is called a non-separable programming problem.

SOLVED EXAMPLES

Example 1: A paper mill produces two grades of paper, namely *A* and *B*. It cannot produce more than 400 tons of grade *A* and 300 tons of *B* in a week. There are 160 production hours in a week. It takes 0.2 and 0.4 hours to produce a ton of products *A* and *B* respectively with corresponding profit of Rs. 200 and Rs. 500 per ton. Formulate the above as a linear programming problem.

Solution: Let x_1 and x_2 be the tons of paper of grade *A* and grade *B* produced every week respectively. The problem can be formulated as

Maximize $$Z = 200\,x_1 + 500\,x_2$$

subject to the constraints

$$\begin{aligned} x_1 &\leq 400 \\ x_2 &\leq 300 \\ 0.2\,x_1 + 0.4\,x_2 &\leq 160 \end{aligned}$$

and non-negative restrictions

$$x_1,\ x_2,\ \geq 0.$$

Example 2: A medical practitioner recommends the constituents of a balanced diet for a patient which satisfies the daily minimum requirements of proteins *P* units, Fats *F* units, and carbohydrates *C* units at a minimum cost. Choice from five different types of foods can be made. The yield per unit of these foods is given by:

Food Type	1	2	3	4	5
Protein	p_1	p_2	p_3	p_4	p_5
Fats	f_1	f_2	f_3	f_4	f_5
Carbohydrates	c_1	c_2	c_3	c_4	c_5
Cost per unit	d_1	d_2	d_3	d_4	d_5

How the patient should select the items so that he has to pay maximum?

Solution: Let x_1, x_2, x_3, x_4 and x_5 be the number of units of food which the patient selects. The problem can be formulated as:

Minimize $Z = d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 + d_5x_5$

subject to the constraints

$$\begin{aligned} p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 + p_5x_5 &\geq P \\ f_1x_1 + f_2x_2 + f_3x_3 + f_4x_4 + f_5x_5 &\geq F \\ c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 &\geq C \end{aligned}$$

and non-negative restrictions

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0.$$

Example 3: A company produces two types of leather cricket balls *C* and *D*. The respective profits are Rs. 10 and Rs. 5 per ball. The supply of raw material is sufficient for making 850 cricket balls per day. For C, a special type of buckle is required and 500 are available per day. There are 700 buckles available for ball D per day. Ball C requires twice as much time as that required for ball D. The company can produce 500 balls if all of them were of type C. Formulate a model for the above problem.

Solution: Let x_1 and x_2 be the number of balls C and D respectively, produced by the company. The linear programming problem model can be formulated as

Maximize $Z = 10x_1 + 5x_2$

subject to the constraints

$$x_1 + x_2 \le 850$$
$$x_1 \le 500$$
$$x_2 \le 700$$
$$2x_1 + x_2 \le 1000$$

and non-negative restrictions

$$x_1, x_2 \ge 0.$$

Example 4: A firm produces 3 products A, B and C with profit per unit Rs. 3, Rs. 2 and Rs. 4 respectively. The firm has two machines M_1 and M_2. The required processing time in minutes for each machine on each product is given below:

Product / Machine	A	B	C
M_1	3	2	4
M_2	2	1	3

Machine M_1 and M_2 have 1800 and 2400 machines minutes respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but not more than 150 A's. Formulate a linear programming problem to maximize the profit.

Solution: Let x_1, x_2 and x_3 be the number of products A, B and C respectively. The problem can be formulated as

Maximize $Z = 3x_1 + 2x_2 + 4x_3$

subject to the constraints

$$3x_1 + 2x_2 + 4x_3 \le 1800$$
$$2x_1 + x_2 + 3x_3 \le 2400$$
$$100\ x_1, \le 150$$
$$x_2 \ge 200$$
$$x_3 \ge 5$$

and non-negative restrictions

$$x_1, x_2, x_3 \ \ge 0$$

Example 5: A city hospital has the following minimal daily requirement for nurses.

Period	Time-Intervals	Minimum number of nurses required
1	6 a.m. – 10 a.m.	2
2	10 a.m. – 2 p.m.	7
3	2 p.m. – 6 p.m.	15
4	6 p.m. – 10 p.m.	8
5	10 p.m. – 2 a.m.	20
6	2 a.m. – 6 a.m.	6

Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimum number of nurses to be employed so that there will be sufficient number of nurses available for each period. Formulate the problem.

Solution: Suppose x_1, x_2, x_3, x_4, x_5 and x_6 is the number of nurses since each nurse has to work for 8 consecutive hours, the x_i nurses who where employed during the first period shall still be on duty when second period starts. Thus, during the second period there will be $x_1 + x_2$ nurses. But the minimum number of nurses required during the second period is specified to be 7. We have $x_1 + x_2 \geq 7$ and similarly others. The problem can be formulated as

Minimize $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

subject to the constraints

$$\begin{aligned} x_1 + x_2 &\geq 7 \\ x_2 + x_3 &\geq 15 \\ x_3 + x_4 &\geq 8 \\ x_4 + x_5 &\geq 20 \\ x_5 + x_6 &\geq 6 \\ x_6 + x_1 &\geq 2 \end{aligned}$$

and non-negative restrictions

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

EXERCISE 1.1

1. Write a short note an optimization techniques.
2. Give ten engineering applications of optimization techniques.
3. What is optimization? Explain applications of optimization in engineering.
4. Write a short note on the historical development of optimization.
5. Write a short note on classification of optimization problems.
6. A manufacturer produces two types of machines M_1 and M_2. Each machine of type M_1, requires 4 hours of grinding and 2 hours of polishing, whereas each machine of type M_2

requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works 60 hours a week. Profit on machine M_1 is Rs. 30 and on M_2 is Rs. 40. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of machines, so that he may make the maximum profit in a week?

(Ans: Maximize $Z = 30x_1 + 40x_2$
subject to constraints $x_1 + 2x_2 \leq 80$
$2x_1 + 5x_2 \leq 180$
and non-negative restrictions
$x_1, x_2 \geq 0$)

7. A company manufactures two products U and V. These products are processed in the same machine. It takes 10 minutes to process one unit of product U and 2 minutes for each unit of product V and the machine operates for a maximum of 35 hours a week. Product U requires 1 kg and V 0.5 kg of raw material per unit. The supply of raw material is 600 kg per week. Market requires at least 800 units of product V every week. Product U costs Rs. 5 per unit and sold at Rs. 10, whereas V costs Rs. 6 per unit and sold at Rs. 8. Formulate a model for the above problem.

(Ans: Maximize $Z = 5x_1 + 2x_2$
subject to constraints
$10x_1 + 2x_2 \leq 2100$
$x_1 + 0.5x_2 \leq 600$
$x_2 \leq 800$
and non-negative restrictions
$x_1, x_2 \geq 0$)

8. A cargo load is to be made of three types of articles A, B, C with respective weights (in tons) volumes (in metric tons) and costs (in thousands Rs. per ton with respect to) is as given below.

Articles	Weight	Volumes	Cost
A	5	4	8
B	8	5	9
C	9	4	12

The total weight and volume of the cargo is not to exceed 2000 tons and 2500 metric tons. Formulate the problem for maximum cost.

(Ans: Maximize $Z = f(x) = 8x_1 + 9x_2 + 12x_3$
subject to the constraints
$5x_1 + 8x_2 + 9x_3 \leq 2000$
$4x_1 + 5x_2 + 4x_3 \leq 2500$
and non-nagative restrictions
$x_1 + x_2 + x_3 \geq 0.$

Linear Programming Problem

2.1 INTRODUCTION

All of us always want to do a particular job in the best possible ways. What are the best or optimal ways to do a job?

For example:

(a) *Assignment:* Let there be m persons and n jobs are to be assigned to these persons. How should these jobs be assigned, so that efficiency of work is maximum?

(b) *Transportation:* This problem arises when the material is to be transported from places of manufacturing to places of requirements. Let there be m places of manufacturing and n places of requirements. How the material should be transported from m places of manufacturing to n places of requirements, so that the total cost of transportation is minimum?

(c) *Inventory:* The problem arises when it is necessary to stock different commodities to meet the demand of customers over a specific period of time. Here one has to decide how much quantity of the commodity and at what time it should be ordered.

These are not only the problems where we do optimisation. In addition to the above, there are many other problems where we optimise time, money, etc. in our day-to-day life under certain restrictions. All of these are formulated mathematically and we call it mathematical programming. Now in the following section, we shall be explaining, what is mathematical programming.

2.2 MATHEMATICAL PROGRAMMING

When we give a precise mathematical language to our thoughts, subject to some conditions and then optimise it, then we call it mathematical programming. Mathematical programming has applications almost everywhere i.e., sciences, engineering, medicines, social sciences, and management, etc. Therefore, mathematical programming or a mathematical model of an optimisation problem consists of an objective function (which is to be optimised) and constraints.

A mathematical programming in general form can be written as follows:

Maximize or Minimize $Z = f(X)$, $X = (x_1, x_2, \dots x_n)$

Subject to the constraints

$$gi(X) \lesseqgtr b,\ i = 1, 2, \dots, m$$

$$X \geq 0$$

The conditions $gi(x) \lesseqgtr b$ are normally known as constraints, $X \geq 0$ are called non-negativity conditions or non-negativity restrictions (NNR).

Mathematical programming can be classified into the following three categories:

(i) *Linear Programming Problem (LPP):* If the objective function $f(X)$ and constraints, all are linear functions, then mathematical programming is called Linear Programming Problem.

For example,

Maximize $\quad Z = 2x_1 + 3x_2 + x_3$

Subject to

$$x_1 + 4x_2 + 5x_3 \leq 40$$
$$2x_1 + 3x_2 + x_3 \leq 20$$
$$x_1, x_2, x_3 \geq 0$$

represents an LPP.

(ii) *Quadratic Programming Problem (QPP):* If the objective function $f(X)$ is a quadratic function and all constraints $gi(X) \lesseqgtr b$ are linear functions, then mathematical programming is known as Quadratic Programming Problem.

For example,

Minimize $\quad Z = x_1^2 + x_2^2$

Subject to

$$2x_1 + 3x_2 \geq 3$$
$$3x_1 + 6x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

represents a QPP.

(iii) *Non-Linear Programming Problem (NLPP):* If some or all of $f(X)$ and $gi(X)$ of a mathematical programming are non-linear functions, then it is called Non-linear Programming Problem.

For example,

Maximize $\quad Z = x_1^2 + 2x_2 + 3x_3^3$

Subject to

$$2x_1 + 2x_2^2 + x_3 \leq 10$$
$$x_1^2 + 2x_2 + 3x_3 \geq 15$$
$$x_1, x_2, x_3 \geq 0$$

represents an NLPP.

A quadratic programming problem is also a NLPP but converse is not true. We will now discuss all the above three one by one but first of all we shall discuss LPP.

2.3 LINEAR PROGRAMMING PROBLEM (LPP)

The general form of a LPP is

$$\text{opt. } Z = f(X) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

Subject to

$$g_1(X) = \alpha_{11} x_1 + \alpha_{12} x_2 + \ldots + \alpha_{1n} x_n \gtreqless b_1$$

$$g_2(X) = \alpha_{21} x_1 + \alpha_{22} x_2 + \ldots + \alpha_{2n} x_n \gtreqless b_2$$

$$\vdots$$

$$g_m(X) = \alpha_{m1} x_1 + \alpha_{m2} x_2 + \ldots + \alpha_{mn} x_n \gtreqless b_m$$

$$x_1, x_2, \ldots, x_n \geq 0$$

Where, $c_1, c_2, \ldots, c_n$ are costs of $x_1, x_2, \ldots, x_n$. If C, X and b stand for the column vectors

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}; \; X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}; \; b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

and,

A stands for $m \times n$ matrix

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \end{bmatrix} = (\alpha_{ij})_{m \times n}$$

then the LPP, in matrix notation, can be expressed as

$$\text{opt. } Z = f(X) = C^T X$$

Subject to

$$AX \gtreqless b$$

$$X \geq 0.$$

C is known as cost vector, entries of A are known as technological coefficients, b is known as availability vector and X is known as variable vector.

Sometimes, we want to change a maximisation problem into a minimisation problem or < (>) constraint into a > (<) constraint and vice versa. We can change it in the following way:

(a) *Change of maximisation problem into a minimisation problem:* Since Max. $f(X) = C^T X$ is same as Min. $h(X) = -f(X) = -C^T X$, a maximisation problem can be changed into a minimisation problem by multiplying the objective function by –1.

Thus,

$$\text{Max. } f(X) = C^T X;\; AX \gtreqless b;\; X \geq 0, \text{ and}$$

$$\text{Min } -f(X) = h(x) = -C^T X;\; AX \gtreqless b;\; X \geq 0$$

have the same solutions except that, if the minimum value comes out be $+\alpha$ then the maximum value would be $-\alpha$.

Similarly, a minimisation problem can be converted into a maximisation problem.

(b) *Change of < (>) constraint into a > (<) constraint:* A < (>) constraint can be converted into a > (<) constraint by multiplying both sides by (–1). Thus,

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \le b_1$$
$$\equiv -\alpha_1 x_1 - \alpha_2 x_2 - \ldots - \alpha_n x_n \ge -b_1$$

Similarly,

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \ge b_1$$
$$\equiv -\alpha_1 x_1 - \alpha_2 x_2 - \ldots - \alpha_n x_n \le -b_1$$

Slack-surplus Variables

An inequation

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \le b_1 \qquad \ldots(1)$$

can be converted into an equation by adding a variable 'S'. So,

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n + S = b_1.$$

The left hand side of eqn. (1) is smaller than or equal to the right hand side, so we have added a variable 'S'. 'S' is called a *Slack Variable.*

Similarly, an inequation

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \ge b_1$$

can be converted into an equation by subtracting a variable 'S' as left hand side is greater than or equal to right hand side.

So $$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n - S = b_1.$$

In this case 'S' is called a *Surpus Variable.*

Standard form of a LPP: A LPP is said to be in standard form, if it is of the following form.

$$\text{opt. } f(X) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

Subject to
$$\begin{aligned}
&\alpha_{11} x_1 + \alpha_{12} x_2 + \ldots + \alpha_{1n} x_n = b_1\\
&\alpha_{21} x_1 + \alpha_{22} x_2 + \ldots + \alpha_{2n} x_n = b_2\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
&\alpha_{m1} x_1 + \alpha_{m2} x_2 + \ldots + \alpha_{mn} x_n = b_m\\
&x_1, x_2, \ldots, x_n \ge 0;\ b_1, b_2, \ldots, b_m \ge 0
\end{aligned}$$

i.e,

$$\text{opt. } f(X) = C^T X$$

Subject to
$$AX = b$$
$$X \ge 0,\ b \ge 0.$$

Where,
$$\begin{aligned}
C &= (c_1, c_2, \ldots, c_n)^T\\
X &= (x_1, x_2, \ldots, x_n)^T\\
b &= (b_1, b_2, \ldots, b_m)^T\\
A &= (\alpha_{ij})_{m \times n} \text{ matrix.}
\end{aligned}$$

Thus, if any inequation has b_i as a negative number, it can be changed to positive by multiplying by (–1) and then by adding slack or surplus variable (as the case may be) it may be changed into an equation. Thus, every LPP can be brought into the standard form if all variables follow the non-negativity restrictions *i.e.*, $X \ge 0$.

Change of condition $x_i \geq \alpha$ into the condition $x'_i \geq 0$

In case, a variable x_i satisfies the condition $x_i \geq \alpha$, Thus α may be negative or positive, can be converted into '≥ 0' by substituting

$$y_i = x_i - \alpha, \text{ i.e., } x_i = y_i + \alpha$$

i.e., in the whole problem x_i is replaced by the new variable y_i. In this case $y_i \geq 0$.

The above will bring the condition as desired. It is applicable irrespective of whether α is negative or positive.

There is another method to handle this case, but it is least preferred because of certain complexity and due to the fact that it makes the problem lengthy. We discuss it in two different conditions.

Case I When $\alpha > 0$

If $\alpha > 0$ and $x_i \geq \alpha$ then $x_i > 0$ is already implied. Therefore, we take $x_i \geq \alpha$ as one extra constraint and in non-negativity conditions, we take $x_i \geq 0$. Because if $x_i \geq \alpha$ as one of the constraint, our solution will definitely have $x_i \geq \alpha$.

In this case, an extra constraint is added which will definitely increase the calculations and so time.

Case II When $\alpha < 0$

In this case, the above method does not work, because if we add an extra constraint $x_i \geq \alpha$ and also take $x_i \geq 0$, then $x_i \geq \alpha$ becomes redundant as $x_i \geq 0$ takes over. In this case, if we proceed to find a solution then either the answer would be 'no solution' or if the solution is obtained it may not be optimal as the optimal solution may be for $\alpha < x_i < 0$. So, this method cannot be applied if $\alpha < 0$.

Change of the condition of the type $x_i \leq \alpha$ into the condition of the type $x_i \geq 0$

In case x_i satisfies $x_i \leq \alpha$, α may be negative or positive, we substitute

$$\alpha - x_i = y_i \quad \text{or} \quad x_i = \alpha - y_i$$

i.e., in the whole problem x_i is replaced by $\alpha - y_i$. In this case $y_i \geq 0$.

Change of the condition 'x_i unrestricted in sign' by non-negativity conditions

Sometimes a variable x_i may not satisfy any of the conditions of the following types:

$$x_i \geq 0;\ x_i \leq 0;\ x_i \geq \alpha;\ x_i \leq \alpha$$

but may have the flexibility in sign *i.e.*, it may be unrestricted in sign.

In this case, we introduce two variables, normally, taken as x_i^+ and x_i^- and replace x_i by $x_i^+ - x_i^-$; $x_i^+, x_i^- \geq 0$.

If the solution obtained has $0 < x_i^+ < x_i^-$, then x_i is negative. If $0 < x_i^- < x_i^+$, then x_i is positive. Thus, in this case number of variables increases by those number of variables which are unrestricted in sign.

If k variables are unrestricted in sign, then number of variables increases by k. There is another way to handle such problems. By this method the, number of variables increases by only one irrespective of that k_1 or k_2 number of variables are unrestricted in sign.

Let $x_1, x_2, \ldots, x_k$ variables be unrestricted in sign. Then we introduce the variables $y, y_1, y_2, \ldots, y_k$ such that $y, y_i \geq 0$, $i = 1, 2, \ldots, k$ and

$$x_i = y_i - y.$$

Example 1: Convert the following problem into a maximisation problem.

Minimize $$Z = f(X) = 2x_1 - x_2 + \frac{1}{2}x_3$$

Subject to
$$\begin{aligned} x_1 + x_2 - x_3 &\leq 5 \\ 2x_1 + 3x_3 &\geq 6 \\ x_1 + 3x_2 &\leq -7 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution: It is a minimisation problem.

The maximisation problem, we will get by multiplying the objective function by -1 and constraints will remain unchanged. Therefore, the maximisation problem is

Maximize $$Z = -f(X) = h(X) = -2x_1 + x_2 - \frac{1}{2}x_3$$

Subject to
$$\begin{aligned} x_1 + x_2 - x_3 &\leq 5 \\ 2x_1 + 3x_3 &\geq 6 \\ x_1 + 3x_2 &\leq -7 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Example 2: Write the above minimisation problem described in example 1 in standard form.

Solution: Standard form is

Minimize $Z = f(X) = 2x_1 - x_2 + \dfrac{1}{2}x_3$

Subject to
$$\begin{aligned} x_1 + x_2 - x_3 + s_1 &= 5 \\ 2x_1 + 3x_3 - s_2 &= 6 \\ -x_1 - 3x_2 - s_3 &= 7 \\ x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0 \end{aligned}$$

(Here s_1 is a slack variable and s_2, s_3 are surplus variables).

Example 3: Write the following problem in standard form.

Max. $Z = 3x_1 - x_2 + 7x_3$

Subject to
$$\begin{aligned} 2x_1 - x_2 - x_3 &\leq 7 \\ x_1 - 2x_2 + x_3 &\geq -3 \\ x_1 \geq -3,\ x_2 \geq 6,\ x_3 &\geq 0 \end{aligned}$$

Solution: Putting $x_1 = y_1 - 3$, $x_2 = y_2 + 6$ in the given problem we obtain

Max. $Z = 3(y_1 - 3) - (y_2 + 6) + 7x_3$

$= 3y_1 - y_2 + 7x_3 - 15$

Subject to $2(y_1 - 3) - (y_2 + 6) - x_3 \le 7 \quad \Rightarrow \quad 2y_1 - y_2 - x_3 \le 19$

$(y_1 - 3) - 2(y_2 + 6) + x_3 \ge -3 \quad \Rightarrow \quad y_1 - 2y_2 + x_3 \ge 12$

$y_1 \ge 0,\ y_2 \ge 0,\ x_3 \ge 0.$

The problem in standard form is

Max. $Z = 3y_1 - y_2 + 7x_3 - 15$

Subject to
$$2y_1 - y_2 - x_3 + s_1 = 19$$
$$y_1 - 2y_2 + x_3 - s_2 = 12$$
$$y_1, y_2, x_3, s_1, s_2 \ge 0$$

(Here s_1 is a slack variable and s_2 is a surplus variable).

Example 4: Write the following problem in standard form.

Min. $Z = -x_1 + 3x_2 + 4x_3$

Subject to
$$x_1 - 7x_2 + 3x_3 \le 24$$
$$-x_1 + 4x_2 - 5x_3 \ge -12$$
$x_1 \ge 2$, x_2 unrestricted in sign, $x_3 \ge 0$

Solution: Putting $x_1 = y_1 + 2$, $x_2 = x_2^+ - x_1^-$ in the given problem, we get

Min. $Z = -y_1 + 3x_2^+ - 3x_1^- + 4x_3 - 2$

Subject to
$$y_1 - 7x_2^+ + 7x_2^- + 3x_3 \le 22$$
$$-y_1 + 4x_2^+ - 4x_2^- - 5x_3 \ge -10$$
$$y_1, x_2^+, x_2^-, x_3 \ge 0$$

Now introducing slack variable s_1 and surplus variable s_2, the standard form of the above problem is

Min. $Z = -y_1 + 3x_2^+ - 3x_2^- + 4x_3 - 2$

Subject to
$$y_1 - 7x_2^+ + 7x_2^- + 3x_3 + s_1 = 22$$
$$y_1 - 4x_2^+ + 4x_2^- + 5x_3 + s_2 = 10$$
$$y_1, x_2^+, x_2^-, x_3, s_1, s_2 \ge 0$$

Example 5: Show how the following objective function can be presented in equation form.

Min. $Z = \text{Max.}\ \{|x_1 - 2x_2 + 2x_3|, |-2x_1 + 3x_2 - 2x_3|\}$

$x_1, x_2, x_3 \ge 0$

Solution: Let Max. $\{|x_1 - 2x_2 + 2x_3|, |-2x_1 + 3x_2 - 2x_3|\} = y$

Then Min. $Z = y$

Then, either $|x_1 - 2x_2 + 2x_3| \le y$ …(1)

or, $|-2x_1 + 3x_2 - 2x_3| \le y$ …(2)

From (1)
$$x_1 - 2x_2 + 2x_3 \le y$$
$$-x_1 + 2x_2 - 2x_3 \le y$$

and from (2)
$$-2x_1 + 3x_2 - 2x_3 \le y$$
$$2x_1 - 3x_2 + 2x_3 \le y$$

Then the problem is
Min. $Z = y$
Subject to

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &\le y \\ -x_1 + 2x_2 - 2x_3 &\le y \\ -2x_1 + 3x_2 - 2x_3 &\le y \\ 2x_1 - 3x_2 + 2x_3 &\le y \\ x_1, x_2, x_3, y &\ge 0 \end{aligned}$$

EXERCISE 2.1

1. Convert the following LPP into standard form.
   Minimize $Z = x_1 - 2x_2 + x_3$
   Subject to
   $$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\ge -4 \\ 3x_1 + 5x_2 + 2x_3 &\ge 7 \end{aligned}$$
   $x_1, x_2 \ge 0$, x_3 is unrestricted in sign.
2. Convert the following LPP into standard form.
   (a) Maximize $Z = 3x_1 - 2x_2 + 4x_3$
   Subject to
   $$\begin{aligned} 2x_1 + x_2 + 2x_3 &\le 12 \\ x_1 - 2x_2 - x_3 &\ge -6 \\ 3x_2 - 2x_3 &\le 1 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$
   (b) Part (a) with the requirements $x_2 \ge 0$ changed to the requirements $x_2 \le 0$.
   (Hint: Let $y_2 = -x_2$)
3. In part (a) of exercise 2, the requirements $x_1, x_2, x_3 \ge 0$ are changed to the requirements $x_1 \ge 4$, $x_2 \ge 2$, $x_3 \ge 6$.
   (a) Convert the above LPP into the standard form having six constraints, and
   (b) Convert the above LPP into the standard form having three constraints.
   (Hint: $y_1 = x_1 - 4$, $y_2 = x_2 - 2$, $y_3 = x_3 - 6$).
4. Linearize the following objective function.
   Max. $Z = \text{Min. } \{|3x_1 - 4x_2|, |2x_1 - 6x_2|\}$.

2.4 INTEGER LINEAR PROGRAMMING PROBLEM (ILPP)

An LPP, in addition to non-negativity conditions may also have conditions, that one or more than one, or all decision variables are integers. If at least one variable is integer but not all, we call an LPP as mixed ILPP. If all variables are integers, we call it ILPP.

For example,

Min. $Z = 2x_1 + 3x_2$

Subject to

$$x_1 + x_2 \le 6$$
$$x_1 - 2x_2 \le 5$$
$$3x_1 - x_2 \ge 2$$
$$x_1, x_2 \ge 0 \text{ and integers is an ILPP}$$

while

Min. $Z = 2x_1 + 3x_2$

Subject to

$$x_1 + x_2 \le 6$$
$$x_1 - 2x_2 \le 5$$
$$3x_1 - x_2 \ge 2$$
$$x_1, x_2 \ge 0 \text{ and } x_1 \text{ is an integer is a mixed ILPP.}$$

2.5 FORMULATION OF LP PROBLEM

In general, in optimisation theory, after identification of the problem, collection of relevant data, the given problem should be translated into appropriate mathematical model. This process of translation is called formulation. The mathematical model of LPP includes the following three basic elements.

1. Decision variables that we seek to determine.
2. Objective function that we aim to optimize.
3. Constraints (restrictions) that we need to satisfy.

The first step to develop a model is to identify the decision variables, once decision variables are defined, then, constructing the objective function and constraints are not difficult.

Thus, the mathematical formulation of an LPP consists of the following five major steps.

1. Identifying the objective underlying the problem.
2. Identifying the decision variables.
3. Construction of the objective function.
4. Construction of the constraints using $\lesseqgtr$ sign.
5. Non-negativity restrictions.

Example 1: A manufacturer wishes to determine the number of tables and chairs to be made by him in order to optimise the use of his available resources. These products utilize two different types of timber and he has on hand 2000 board feet of the first type and 1500 board feet of the second type. He has 1000 manhours available for the total job. Each table and chair requires 4 and 2 board feet respectively of the first type of timber, and 3 and 5 board feet of the second type. 5 manhours are required to make a table and 3 manhours are needed to make a chair. The manufacturer makes a profit of Rs. 50 on a table and Rs. 30 on a chair. Formulate the above as an LPP to maximize the profit.

Solution: Let x_1, x_2 be the number of tables and chairs respectively manufactured.

Then the LPP of above problem is

Max. $Z = 50x_1 + 30x_2$

Subject to
$$4x_1 + 3x_2 \leq 2000$$
$$2x_1 + 5x_2 \leq 1500$$
$$x_1, x_2 \geq 0 \text{ and integers.}$$

Example 2: Ram wants to decide the constituents of a diet which will fulfil his daily requirement of fats, proteins and carbohydrates at the minimum cost. The choice is to be made from three different types of food. The yield per unit of these foods is given in the following table.

	Yield/Unit			
Food Type	*Fats*	*Proteins*	*Carbohydrates*	*Cost/Unit (Rs.)*
1	3	4	8	60
2	2	3	6	50
3	5	6	4	80
Minimum Requirement	180	850	750	

Formulate the above as an LPP.

Solution: Let x_1, x_2, x_3 respectively represent the three food types. Then the LPP of the above problem is

Min. $Z = 60x_1 + 50x_2 + 80x_3$

Subject to
$$3x_1 + 2x_2 + 5x_3 \geq 180$$
$$4x_1 + 3x_2 + 6x_3 \geq 850$$
$$8x_1 + 6x_2 + 4x_3 \geq 750$$
$$x_1, x_2, x_3 \geq 0$$

Example 3: A firm manufactures two types of products P_1 and P_2 and sells them on a profit of Rs. 3 on type P_1 and Rs. 4 on type P_2. Each product is processed on two machines A and B. Type P_1 requires 2 minutes of processing time on A and one minute on B; type P_2 requires 3 minutes on A and 2 minutes on B. The machine A is available for not more than 7 hours 30 minutes while machine B is available for 12 hours during any working day. Formulate the problem as an LPP.

Solution: Let x_1 = number of products of type P_1
and x_2 = number of products of type P_2

then the LPP of the above problem is

Max. $Z = 3x_1 + 4x_2$

Subject to
$$2x_1 + 3x_2 \leq 450$$
$$3x_1 + 2x_2 \leq 720$$
$$x_1, x_2 \geq 0$$

Example 4: The purchasing section of a company has purchased sufficient amount of curtain cloth to meet the requirements of the company. The curtain cloth is in pieces, each of length 15 feet. The curtain requirements is as follows:

Curtain of length (in feet)	*Number required*
6	2000
7	1500
8	3250

The problem is how to cut the pieces to meet the above requirements, so that the trim loss is minimised. The width of required curtains is same as that of available pieces. A piece of 15 feet curtain cloth can be cut in p_1, p_2, p_3, and p_4 patterns according to the following table.

Curtains of length (feet)	*Number of curtains of different sizes in pattern.*			
	p_1	p_2	p_3	p_4
6	2	0	1	0
7	0	2	0	1
8	0	0	1	1
Trim loss (in feet)	3	1	1	0

(This problem is known as Trim loss problem).

Solution: Let x_1, x_2, x_3 and x_4 be the number of pieces cut according to patterns p_1, p_2, p_3 and p_4, respectively.

The constraints of the problem are.

$$2x_1 + x_3 - s_1 \geq 2000$$
$$x_2 + x_4 - s_2 \geq 1500$$
$$x_3 + x_4 - s_3 \geq 3250$$
$$x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0 \text{ and integers.}$$

The objective function is

Min. $Z = 3x_1 + x_2 + x_3 + 6s_1 + 7s_2 + 8s_3$

Example 5: Peter wants to keep some hens with him and has Rs. 2000. The old hens can be bought for Rs. 50 but young ones cost Rs. 100. The old hens lay 20 eggs per week and the young ones 60 eggs per week, each worth being Re. 1.00. The feed for young and old hens costs Rs. 10 and Rs 6 per hen per week. He can keep 30 hens with him. How many of each kind of hens Peter should buy to get a profit of more than Rs. 500. Formulate as an LPP.

Solution: Let x_1 = number of old hens
and x_2 = number of young hens

Then the LPP is

Max. $Z = (20 \times 1 - 6)\, x_1 + (60 \times 1 - 10)\, x_2$

Max. $Z = 14x_1 + 50x_2$

Subject to
$$50x_1 + 100x_2 \leq 2000$$
$$x_1 + x_2 \leq 30$$
$$14x_1 + 50x_2 \geq 500$$
$$x_1, x_2 \geq 0 \text{ and integers.}$$

EXERCISE 2.2

1. A company has three operational departments (weaving, processing and packing) with capacity to produce three different types of clothes namely suitings, shirtings and woollens yielding a profit of Rs. 2, Rs. 4 and Rs. 3 per metre, respectively. One metre suiting requires 3 minutes in weaving 2 minutes in processing and 1 minute in packing. One metre of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing while one metre of woollen requires 3 minutes in each department. In a week, total run time of each department is 60, 40 and 80 hours for weaving, processing and packing departments, respectively.

   Formulate as LPP to maximize the profit.

   Ans: (Max. $Z = 2x_1 + 4x_2 + 3x_3$ Subject to $3x_1 + 4x_2 + 3x_3 \le 3600$, $2x_1 + x_2 + 3x_3 \le 2400$, $x_1 + 3x_2 + 3x_3 \le 4800$, $x_1, x_2, x_3 \ge 0$).

2. The owner of Cosmo Sports wishes to determine how many advertisements to place in the selected quarterly magazines *A*, *B* and *C*. His objective is to advertise in such a way that the total exposure to principal buyers of expensive sports good is maximised. Percentage of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements placed multiplied by the number of principal buyers. The following data may be used.

	Magazine		
	A	*B*	*C*
Readers	1 Lakh	0.6 Lakh	0.4 Lakh
Principal buyers	20%	15%	8%
Cost per advt. (Rs.)	8000	6000	5000

   The budgeted amount is at most Rs. 1 lakh for the advertisements. The owner has already decided that magazine *A* should have no more than 15 advertisements and that *B* and *C* each have at least 8 advertisements.

   Formulate the problem as an LPP.

   Ans: (Max $Z = 20000\ x_1 + 9000\ x_2 + 3200\ x_3$, Subject to $8000\ x_1 + 6000\ x_2 + 5000,\ x_3 \le 100000$, $x_1 \le 15$, $x_2 \ge 8$, $x_3 \ge 8$ $x_1, x_2, x_3 \ge 0$, where x_1, x_2 and x_3 are number of advertisements in magazines *A*, *B* and *C*, respectively).

3. A farmer has 100 acre farm. He can sell all tomatoes, cabbage or radish he can raise. The price he can obtain Rs. 1.00 per kg for tomatoes, Rs 0.75 per cabbage and Rs. 2.00 per kg for radishes. The average yield per acre is 2000 kg of tomatoes, 3000 heads of cabbage and 1000 kg of radishes. Fertilizer is available at Rs. 0.50 per kg and the amount required per acre is 100 kg each for tomato and cabbage and 50 kg for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes and 6 man-days for cabbage. A total of 400 man-days of labour are available at Rs. 20 per man-day. Formulate the problem as an LPP to maximize the farmer's profit.

Ans: (Max. $Z = 1850\ x_1 + 2080\ x_2 + 1875\ x_3$, Subject to $x_1 + x_2 + x_3 \le 100$, $5x_1 + 6x_2 + 5x_3 \le 400$, $x_1, x_2, x_3 \ge 0$ where, x_1, x_2, x_3 is the number of acres to grow tomatoes, cabbages and radishes, respectively).

4. A company has to manufacture the circular tops of cans. Two sizes one of diameter 10 cm and the other of diameter 20 cm are required. They are to be cut from metal sheets of dimensions 20 cm by 50 cm. The requirement of smaller size is 15000 and of larger size is 10000. How to cut the tops from metal sheets so that the number of sheets used is minimised. Formulate as an LPP.

   (Hint: The plates can be cut in three patterns. The first pattern has 10 tops of smaller size, the second pattern has 2 tops of smaller size and 2 tops of larger size, the third pattern has 6 tops of smaller size and 1 top of large size.)

   Ans: (Min $Z = x_1 + x_2 + x_3$, subject to $10x_1 + 2x_2 + 6x_3 \ge 15000$, $2x_2 + x_3 \ge 10000$, $x_1, x_2, x_3 \ge 0$ and integers where x_1, x_2, x_3 be the number of sheets cut according to first, second and third pattern, respectively.)

5. A firm manufactures 3 products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4, respectively. The firm has 2 machines and below is the required processing time in minutes for each machine on each product.

Product

Machine		A	B	C
	M_1	4	3	5
	M_2	2	2	4

   Machines M_1 and M_2 have 2000 and 2500 machine minutes, respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but no more than 150 A's. Formulate the above as an LPP to maximize the profit.

   Ans: (Max. $Z = 3x_1 + 2x_2 + 4x_3$, subject to $4x_1 + 3x_2 + 5x_3 \le 2000$, $2x_1 + 2x_2 + 4x_3 \le 2500$, $100 \le x_1 \le 150$, $200 \le x_2 \ge 0$, $50 \le x_3 \ge 0$, where x_1, x_2 and x_3 be the number of products of A, B and C, respectively).

6. Three grades of coal A, B and C contain ash and phosphorous as impurities. In a particular industrial process a fuel obtained by blending the above grades containing not more than 25% ash and 0.03% phosphorous is required. The maximum demand of the fuel is 100 tonnes. Percentage impurities and costs of the various grades of coal are shown below. Assuming that there is an unlimited supply of each grade of coal and there is no loss in blending. Formulate this as an LPP to minimize the cost.

Coal Grade	*% ash*	*% phosphorous*	*Cost per tonne in Rs.*
A	30	0.02	240
B	20	0.04	300
C	25	0.03	280

   Ans: (Minimize $Z = 240x_1 + 300x_2 + 280x_3$, subject to $x_1 - x_2 + 2x_3 \le 0$, $-x_1 + x_2 \le 0$, $x_1 + x_2 + x_3 \le 100$, $x_1, x_2, x_3 \ge 0$ where x_1, x_2, x_3 are tonnes of grade A, B and C coal, respectively).

7. A ship has three cargo holds: forward, aft and centre; the capacity limits are:

Forward	2000 tonnes	100,000 m^3
Centre	3000 tonnes	135,000 m^3
Aft	1500 tonnes	30,000 m^3

The following cargoes are offered; the ship owners may accept all or any part of each commodity.

Commodity	*Amount (tonnes)*	*Volume per tonne* (m^3)	*Profit per tonne (Rs.)*
A	6,000	60	60
B	4,000	50	80
C	2,000	25	50

In order to preserve the trim of the ship, the weight in each hold must be proportional to the capacity in tonnes. The objective is to maximize the profit. Formulate as an LPP.

Ans: (Max $Z = 60(x_{1A} + x_{2A} + x_{3A}) + 50(x_{2A} + x_{2B} + x_{2C}) + 25(x_{3A} + x_{3B} + x_{3C})$, subject to $x_{1A} + x_{2A} + x_{3A} \leq 6000$, $x_{2A} + x_{2B} + x_{2C} \leq 4000$, $x_{3A} + x_{3B} + x_{3C} \leq 2000$, $x_{1A} + x_{1B} + x_{1C} \leq 2000$, $x_{2A} + x_{2B} + x_{2C} \leq 3000$, $x_{3A} + x_{3B} + x_{3C} \leq 1500$, $60\, x_{1A} + 50\, x_{1B} + 25\, x_{1C} \leq 100{,}000$, $60x_{2A} + 50x_{2B} + 25x_{2C} \leq 135{,}000$, $60x_{3A} + 50x_{3B} + 25x_{3C} \leq 30{,}000$, all *variables* ≥ 0, where x_{iA}, x_{iB}, x_{iC} $(i = 1, 2, 3)$ be the weights (in kg) of commodities *A*, *B* and *C*, respectively).

2.6 SOLUTION OF A LINEAR PROGRAMMING PROBLEM

A solution of an LPP is the set of values of the variables $x_1, x_2, \ldots, x_n$; i.e., the vector $(x_1, x_2, \ldots, x_4)$ that satisfy the conditions and gives the optimal value of the objective function.

There are many vectors X which would satisfy the conditions $AX \gtreqless b$; $X \geq 0$. But only few would give the optimal value of the objective function $f(X)$.

Therefore, in order to find the solution of the LPP we would first find out the set of all solutions of conditions $AX \gtreqless b$; $X \geq 0$. The required optimal solution would be one from it. No point outside this set can be a solution of the problem.

The solution set of the conditions (inequations) $AX \lesseqgtr b$; $X \geq 0$ is called set of feasible solutions and is denoted by S_F. Thus, first we should find S_F and then pick that point of S_F which gives the optimal value of $f(X)$.

The S_F may be empty, that would mean solution does not exist. If S_F is not empty, then S_F may be bounded or unbounded. S_F is bounded means there exists vectors A and B such that $A \leq X \leq B$ $\forall$ $X \varepsilon S_F$. In this case solution of the LPP would exist. In case either A does not exist or B does not exist or none of the two exist, we say S_F is bounded above but not below, or bounded below but not above, or unbounded from both sides. In this case solution of the LPP may or may not exist.

Geometry of S_F, Graphical Solution

Each constraint, non-negative condition, is an equation. This when converted in equation form represents a hyperplane in E_n, a plane in E_3 and a line in E_2.

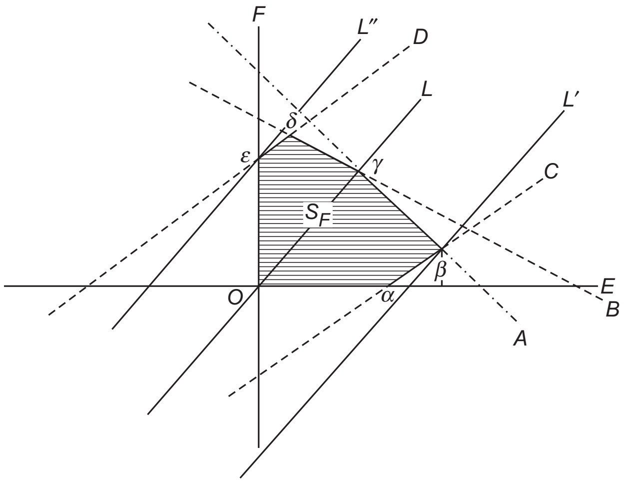
The inequation represents one of two half (hyper) spaces, which is the solution set of the inequations.

The solution set of all the *m* constraints and non-negativity conditions is the intersection of the half spaces with edges as lines, planes, hyperplanes as the case may be.

In order to understand the inequations in E_2, i.e., in two variables, let the inequations be

$$
\begin{aligned}
x_1 + x_2 &\le 5 \quad A\\
4x_1 + 7x_2 &\le 28 \quad B\\
2x_3 - 3x_2 &\le 6 \quad C\\
-3x_1 + 4x_2 &\le 12, \quad D\\
x_1 \ge 0,\ x_2 &\ge 0\ E,\ F
\end{aligned}
$$

Then S_F is shaded portion as shown below, where



lines are represented by the so called converted inequations.

Thus, in E_2 it is a region. In general, it is called a polytope. This polytope S_F may be bounded (as in the above case) or unbounded as the case when constraints *A* and *B* are not there.

Definition: The set of all convex linear combination (C.L.C.) of finite number of points is called a *polyhedron*.

By definition of C.L.C., a polyhedron is always a convex set.

The points 0, α, β, γ, δ, ε are vertices of S_F. Actually vertices are solutions of two equations.

Now we illustrate a method for solving an optimisation problem. It is known as **Graphical-method**, which can be easily applied in E_2 and to some extent in E_3. Beyond E_3 it is not possible.

Illustration: Find the solution of the following LPP

Max. $Z = 8x_1 - 7x_2$

Subject to $\qquad x_1 + x_2 \le 5$

$$4x_1 + 7x_2 \leq 28$$
$$2x_1 - 3x_2 \leq 6$$
$$-3x_1 + 4x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

***Solution*:** We shall first sketch S_F. It is same as above. To find optimum solution, i.e., the maximum value of $f(X) = 8x_1 - 7x_2$, literally and algebraically, we take each and every point of S_F and substitute in $8x_1 - 7x_2$ and pick the maximum value.

It is impossible. Therefore, we first draw the line $f(X) = 8x_1 - 7x_2 = 0$. In figure it is L. Any line parallel to L has the equation of the form

$$8x_1 - 7x_2 = c, \ c \neq 0$$

If $c > 0$, it is towards α, β and if $c < 0$, it is towards δ, ε.

So we move the line L keeping it parallel towards α, β to get maximum value and towards δ, ε to get minimum value, so that at least one point of S_F remains on the line.

In order to get maximum of $8x_1 - 7x_2$, L' is the final position of L and L'' is the final position in order to get minimum value. Thus, the maximum value of the objective function occurs at the vertex β and the minimum value at the vertex ε.

Thus, maximum occurs at $\left(\frac{21}{5}, \frac{4}{5}\right)$ and Max $f(X) = 28$

If it is a minimisation problem, it occurs at (0, 3) and the Min $f(X) = -21$

Looking at the above illustration, we notice that Maximum (Minimum) occurs at a vertex of S_F and S_F is a convex set.

Every half plane represented by an inequation is a convex set. S_F is the intersection of all these convex sets and hence a convex set.

It is not peculiar. But it is always true. We shall prove these results.

Because of non-negativity conditions, S_F is at least bounded below, hence by a theorem it has at least one vertex.

2.7 SOME EXCEPTIONAL CASES IN GRAPHICAL METHOD

In the preceding example we have seen that solution of an LPP occurs at a vertex of S_F. But it is not always true, there may be an LPP for which no solution exists or for which the only solution obtained is an unbounded one or an LPP may have more than one solution. In this section we shall discuss the following three special cases that arise in the application of graphical method.

(a) Alternative optimal solution.
(b) Infeasible (or non-existing) solution.
(c) Unbounded solution.

2.7.1 Alternative Optimal Solution

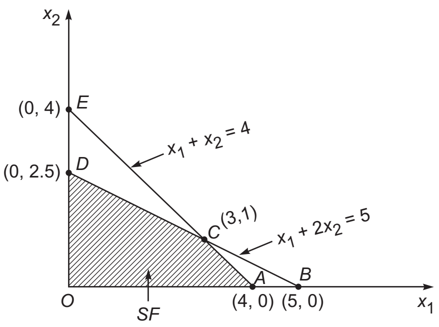
When the objective function assumes the same optimum value at more than one vertex of S_F, then we say that the LPP has an alternative optimal solution.

For example:

Maximize $Z = 2x_1 + 4x_2$

Subject to

$$x_1 + 2x_2 \leq 5$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$



Solution: Maximum occurs at C (3, 1) and D (0, 2.5) and the value of $Z = 10$

2.7.2 Infeasible Solution

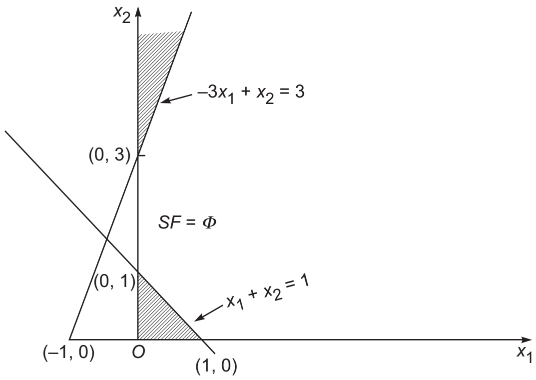
When the constraints are not satisfied simultaneously, the LPP has no feasible solution. This implies if $S_F = \Phi$. This situation can never occur if all the constraints are of the $\leq$ type.

For example:

Maximize $Z = x_1 + x_2$

Subject to

$$x_1 + x_2 \leq 1$$
$$-3x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$



As $S_F = \Phi$, the problem has no feasible solution.

2.7.3 Unbounded Solution

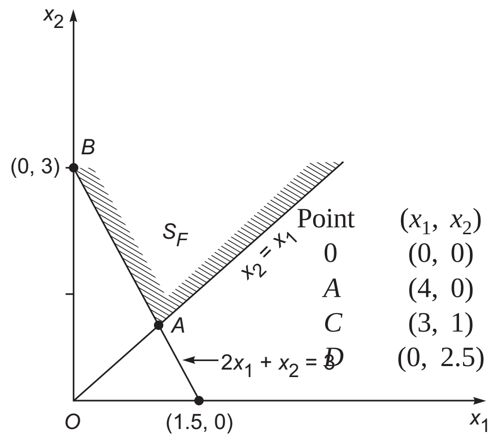
When the value of the decision variables may be increased indefinitely without violating any of the constraints the solution space S_F is unbounded. The value of objective function, in such cases, may increase (for maximization) or decrease (for minimization) indefinitely. Thus, both the solution space and the objective function value are unbounded.

For example:

Maximize $\quad Z = 6x_1 + x_2$

Subject to

$$2x_1 + x_2 \geq 3$$
$$x_2 - x_1 \geq 0$$
$$x_1, x_2 \geq 0$$



Point	(x_1, x_2)	$Z = 2x_1 + 4x_2$
0	(0, 0)	0
A	(4, 0)	8
C	(3, 1)	10 ← Max.
D	(0, 2.5)	10 ← Max.

The graphical solution of the given LPP is depicted in the above figure. The two vertices of the feasible region are A and B. We observe, that the feasible region S_F is unbounded. The value of the objective function at the vertex A (1, 1) and B (0, 3) are 7 and 3, respectively.

But there exist number of points in feasible region for which the value of the objective function is more than 7. For example, the point (3, 6) lies in the feasible region and the objective function value at this point is 24 which is more than 7. Thus, both the variables x_1 and x_2 can be made arbitrarily large and the value of Z also increases. Hence, the problem has an unbounded solution.

Remark: An unbounded solution means that there exist an infinite number of solutions to the problem.

EXERCISE 2.3

1. Use graphical method to solve

   Maximize $\quad Z = 4x_1 + 3x_2$

   Subject to

   $$2x_1 + x_2 \leq 1000$$
   $$x_1 + x_2 \leq 800$$
   $$x_1 \leq 400$$
   $$x_2 \leq 700$$
   $$x_1, x_2 \geq 0$$

   (**Ans:** $x_1 = 200$, $x_2 = 600$ Max. $Z = 2600$)

2. Solve the following LPP graphically.

Minimize $Z = 5x_1 + 3x_2$

Subject to
$$x_1 + x_2 \le 6$$
$$2x_1 + 3x_2 \ge 3$$
$$0 \le x_1 \le 3,\ 0 \le x_2 \le 3$$

(**Ans:** $x_1 = 3 = x_2$, Min $Z = 24$)

3. Use graphical method to solve

   Maximize $Z = x_1 + 2x_2$

   Subject to
   $$x_1 - x_2 \le 1$$
   $$x_1 + x_2 \ge 3$$
   $$x_1, x_2 \ge 0$$

   (**Ans:** Infeasible sol.)

4. Solve the following LPP graphically

   Maximize $Z = 6x_1 - 3x_2$

   Subject to
   $$x_1 + x_2 \le 1$$
   $$2x_1 - x_2 \le 1$$
   $$-x_1 + 2x_2 \le 1$$
   $$x_1, x_2 \ge 0$$

   $\left(\textbf{Ans: } x_1 = \frac{2}{3}, x_2 = \frac{1}{3}, Z = 3 \text{ or } x_1 = \frac{1}{2}, x_2 = 0, Z = 3, \text{Alternative sol.}\right)$

5. Use graphical method to solve

   Minimize Z = $-4x_1 + x_2$

   Subject to
   $$x_1 - 2x_2 \le 2$$
   $$-2x_1 + x_2 \le 2$$
   $$x_1, x_2 \ge 0$$

   (**Ans:** Unbounded solution space and unbounded solution)

6. Solve the problem 5 by changing objective function to maximization.

   (**Ans:** Unbounded solution space but bounded solution)

7. Solve the following graphically

   Minimize $Z = x_1 - 2x_2$

   Subject to
   $$-x_1 + x_2 \le 1$$
   $$2x_1 + x_2 \le 2$$
   $$x_1, x_2 \ge 0$$

   $\left(\textbf{Ans: } x_1 = \frac{1}{3}, x_2 = \frac{4}{3}, Z = -\frac{7}{3}\right)$

2.8 CONVEX SETS AND LINEAR PROGRAMMING PROBLEM

2.8.1 Introduction

It is assumed here that we are familiar with Real Vector Spaces, Inner Product Space, Euclidean Space, linear dependence, linear independence, linear combination, subspaces, etc.

E_n, the set of all n-tuples of real numbers, is an Euclidean space. We shall confine ourselves to E_n only.

A vector $X \varepsilon E_n$ is called a linear combination of vectors $X_1, X_2, \ldots, X_k$, if X can be expressed as

$$X = \alpha_1 X_1 + \ldots + \alpha_k X_k$$

2.8.2 Convex Linear Combination

In order to make a linear combination (l.c.), the choice of scalars too large. A special type of l.c. is called a convex linear combination. To be precise, we define

Definition: Let $X_1, X_2, \ldots, X_k$ be vectors of E_n. A vector $X \varepsilon E_n$ is called a convex linear combination (C.L.C.) of $X_1, X_2, \ldots, X_k$, if

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_k X_k,$$
$$\alpha_i \geq 0, \alpha_1 + \alpha_2 + \ldots \alpha_k = 1$$

The expression $\alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_k X_k$, $\alpha_i \geq 0$, $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 1$ itself is called C.L.C. of $X_1, X_2, \ldots, X_k$.

A C.L.C. of two vectors X_1, X_2 can also be written as $(1 - \lambda)X_1 + \lambda X_2$, $(\lambda \geq 0, \lambda \leq 1)$ or $0 \leq \lambda \leq 1$

In E_2, $X = (1 - \lambda)X_1 + \lambda X_2$, $0 \leq \lambda \leq 1$ means

$$(x, y) = (1 - \lambda)(x_1, y_1) + \lambda (x_2, y_2)$$

i.e., $$x = \frac{(1-\lambda)x_1 + \lambda x_2}{(1-\lambda)+\lambda},\ y = \frac{(1-\lambda)y_1 + \lambda y_2}{(1-\lambda)+\lambda},\ 0 \leq \lambda \leq 1$$

i.e., points on the line segment joining the points (x_1, y_1) and (x_2, y_2). Thus, set of all C.L.C. of X_1, X_2 in E_2 is a line segment joining X_1, X_2.

In general, in E_n, set of all C.L.C. of X_1, X_2 is the 'line segment' joining the points X_1, X_2. To be precise, in E_n,

$$\{X \varepsilon E_n \mid X = (1 - \lambda)X_1 + \lambda X_2, 0 \leq \lambda \leq 1 \mid\}$$
= 'line segment' joining X_1, X_2

$$\{X \varepsilon E_n \mid X = (1 - \lambda)X_1 + \lambda X_2 \mid \lambda > 0\}$$
= 'half line' from X_1 towards X_2; X_1 excluded

$$\{X \varepsilon E_n \mid X = (1 - \lambda)X_1 + \lambda X_2, \lambda \geq 0\}$$
= 'ray' from X_1 in the direction of X_2, and so on.

Example 1: Is $\left(1, -\frac{1}{4}\right)$ a C.L.C. of (1, 0), (1, 1) and (1, –2)?

If yes, express it

Solution: $$\left(1, -\frac{1}{4}\right) = \alpha (1, 0) + \beta (1, 1) + \gamma (1, -2)$$

$$\alpha + \beta + \gamma = 1$$

$$\beta - 2\gamma = -\frac{1}{4}$$

Let $$\alpha = \frac{1}{3},\ \beta + \gamma = \frac{2}{3},\ \beta - 2\gamma = -\frac{1}{4}$$

or, $$\beta = \frac{13}{36},\ \gamma = \frac{11}{36}$$

Thus, $$\left(1, -\frac{1}{4}\right) = \frac{1}{3}(1,\ 0) + \frac{13}{36}(1,\ 1) + \frac{11}{36}(1,\ -2)$$

It is a required C.L.C. and the above is the required expression.

Example 2: Is (1.5, .6) a C.L.C. of (0, 0), (2, 0), (1, 1)? If yes, express it.

Solution: Let
$$(1.5,\ .6) = \alpha(0,\ 0) + \beta(2,\ 0) + \gamma(1,\ 1)$$
$$2\beta + \gamma = 1.5$$
$$\gamma = .6$$
$$\alpha + \beta + \gamma = 1$$

On solving we get $\gamma = .6$, $\beta = .45$, $\alpha = -.05$.

Thus, it is not a C.L.C. of the required points.

2.8.3 Convex Set

Definition: A non-empty subset $S \subset E_n$ is said to be convex if and only if, the set of all C.L.C. of any two given points of S, is a subset of S, i.e.,

$$\{X = (1 - \lambda)\ X_1 + \lambda X_2 |\ X_1, X_2\ \varepsilon\ S,\ 0 \le \lambda \le 1\} \subset S$$

or, iff $X = (1 - \lambda)\ X_2 + \lambda X_2,\ 0 \le \lambda \le 1,\ \varepsilon\ S\ \ \forall\ X_1, X_2\ \varepsilon\ S.$

Example 3: A line in E_2, E_3 or in E_n is a convex set.

Example 4: A plane $\alpha_1\ x_1 + \alpha_2\ x_2 + \alpha_3\ x_3 = \beta$ is a convex set in E_3.

Example 5: A hyperplane $\alpha_1\ x_1 + \alpha_2\ x_2 + \ldots + \alpha_n\ x_n = \beta$ is a convex set in E_n.

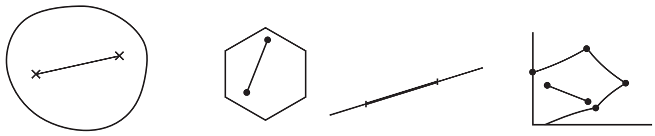
In matrix notations, $\alpha_1\ x_1 + \ldots + \alpha_n\ x_n = \beta$, can be expressed as

$$(\alpha_1\ \alpha_2\ \ldots\ \alpha_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \beta$$

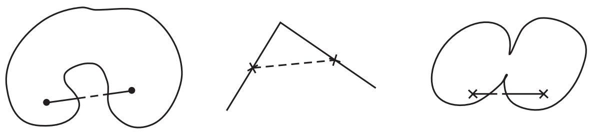
or, $$C^T X = \beta, \text{ where } C = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Thus, $C^T X = \beta$ is a hyperplane in E_n.

Example 6: In E_2, E_3, a set S, is a convex set, if any two points of S can be joined by a 'line segment' contained in S. Thus,



are convex sets, while



are not convex sets.

Definition: The sets

$$\{X \;\varepsilon\; E_n \mid C^T X < \alpha\} \qquad (1)$$
$$\{X \;\varepsilon\; E_n \mid C^T X \leq \alpha\} \qquad (2)$$
$$\{X \;\varepsilon\; E_n \mid C^T X > \alpha\} \qquad (3)$$
$$\{X \;\varepsilon\; E_n \mid C^T X \geq \alpha\} \qquad (4)$$

are called hyperspaces in E_n.

In order to understand it, let us come down to E_3. In E_3

$$C^T X = \alpha_1\, x_1 + \alpha_2\, x_2 + \alpha_3\, x_3 = \beta$$

is a plane P which divides the whole space in two parts, both called half spaces or simply space. The points in these spaces satisfy the inequalities.

$$\alpha_1\, x_1 + \alpha_2\, x_2 + \alpha_3\, x_3 < \beta \text{ or } \alpha_1\, x_1 + \alpha_2\, x_2 + \alpha_3\, x_3 > \beta$$

If the plane P is included in these spaces, then inequalities are

$$\alpha_1\, x_1 + \alpha_2\, x_2 + \alpha_3\, x_3 < \beta \text{ or } \alpha_1\, x_1 + \alpha_2\, x_2 + \alpha_3\, x_3 \geq \beta.$$

We shall not be going in the depth of topological concepts, but we shall confine ourselves with the understanding of 'boundary' which also has its literal meaning. 'Boundary point' would mean a point on the boundary.

Taking 'boundary' as an undefined term, we define the following.

Definition: A set S, is said to be *open*, if it contains *no* boundary point.

In other words, if no point of boundary of S is in S, then S is open.

Definition: A set S is said to be *closed* if it contains *all* its boundary points, i.e., whole boundary.

Thus, half spaces $C^T X < \alpha$, $C^T X > \alpha$ are open sets while $C^T X \leq \alpha$ and $C^T X \geq \alpha$ are closed sets.

In example 3, we have shown that hyperplane is a convex set and also we prove something more.

Theorem 1: *A hyperplane* $S : C^T X = \alpha$ *is a closed convex set.*

Proof: No point in $C^T X < \alpha$ and $C^T X > \alpha$ is a boundary point of $C^T X = \alpha$. Thus, all boundary points of the hyperplane are in it. Hence, it is closed.

Now, let $X_1, X_2 \,\varepsilon\, S$

Therefore, $C^T X_1 = \alpha$ and $C^T X_2 = \alpha$

Let $X = (1 - \lambda)\, X_1 + \lambda X_2,\ 0 \le \lambda \le 1$ be any C.L.C. of X_1, X_2

Then
$$
\begin{aligned}
C^T X &= C^T[(1 - \lambda)\, X_1 + \lambda X_2] \\
&= (1 - \lambda)\, C^T X_1 + \lambda\, C^T X_2, \text{ since matrix multiplication is distributive over '+'} \\
&= (1 - \lambda)\, \alpha + \lambda\alpha = \alpha
\end{aligned}
$$

Hence $X \,\varepsilon\, S$. So S is a convex set.

Hence, the result.

Theorem 2: *A closed half space in E_n is a closed convex set.*

Proof: Let $S = \{X \,\varepsilon\, E_n \mid C^T X \le \alpha\}$ be a closed half space.
Let $X_1, X_2 \,\varepsilon\, S$. Then $C^T X_1 \le \alpha$, $C^T X_2 \le \alpha$
Let $X = (1 - \lambda)\, X_1 + \lambda X_2$ be a C.L.C. of X_1, X_2.
Then
$$
\begin{aligned}
C^T X &= (1 - \lambda)\, C^T X_1 + \lambda\, C^T X_2 \\
&\le (1 - \lambda)\, \alpha + \lambda\alpha = \alpha
\end{aligned}
$$

$\therefore$ $X \,\varepsilon\, S$. So S is convex

Hence, the result.

Lines in E_2 or E_3 are convex sets. By theorem 1, XY-plane and YZ-plane are convex sets but their union is not a convex set as (1, 0, 0) is in XY-plane, (0, 0, 1) is in YZ-plane but their C.L.C., $\frac{1}{2}(1, 0, 0) + \frac{1}{2}(0, 0, 1) = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ is in neither. Thus, union of two convex sets is not a convex set. But intersection of two convex sets is a convex set as is proved below.

Theorem 3: *Intersection of two convex sets is a convex set.*

Proof: Let S_1, S_2 be convex sets and $S = S_1 \cap S_2$ be their intersection.

Let $X_1, X_2 \,\varepsilon\, S$.

So $X_1, X_2 \,\varepsilon\, S_1$ and $X_1, X_2 \,\varepsilon\, S_2$

Let $X = (1 - \lambda)\, X_1 + \lambda\, X_2,\ 0 \le \lambda \le 1$, be a C.L.C. of X_1, X_2 Since S_1, S_2 are convex sets, by definition, $X \,\varepsilon\, S_1$ and $X \,\varepsilon\, S_2$. Hence, $X \,\varepsilon\, S$. Thus, S is convex.

We have defined convex sets in terms of C.L.C. of two points. We have also defined C.L.C. of more than 2 points. Thus, proved

Theorem 4: *A set S is convex iff every C.L.C. of points in S is in S.*

Proof: Let every C.L.C. of points in S be in S.

Therefore, every C.L.C. of two points in S is in S.

Hence, S is convex.

Now, let S be convex and $X_1, X_2, X_3, \ldots, X_m \,\varepsilon\, S$

Let $X = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_m X_m,\ \alpha_i \ge 0,$

and $\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$

be a C.L.C. of $X_1, X_2, \ldots, X_m$. We shall now show that $X \,\varepsilon\, S$.

We shall prove it by induction.

Obviously, it holds for $k = 1$, as $X = X_1 \,\varepsilon\, S$

Also, it holds for $k = 2$, as $X = \alpha'_1 X_1 + \alpha'_2 X_2$, $\alpha'_1 + \alpha'_2 = 1$, $\alpha'_i \geq 0$ belong to S by definition of convex set.

Let, now, it hold for $k = k$, *i.e.*,

$$\alpha'_1 X_1 + \alpha'_2 X_2 + \ldots + \alpha'_k X_k \,\varepsilon\, S$$

$$\alpha'_i \geq 0,\ \alpha'_1 + \alpha'_2 + \ldots + \alpha'_k = 1.$$

Now consider a C.L.C. of $k + 1$ points,

$$\beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k + \beta_{k+1} X_{k+1},\ \beta_1 + \beta_2 + \ldots + \beta_k + \beta_{k+1} = 1$$

$$= \frac{\beta_1 + \beta_2 + \ldots + \beta_k}{\beta_1 + \beta_2 + \ldots + \beta_k} (\beta_1 X_1 + \ldots + \beta_k X_k) + \beta_{k+1} X_{k+1}$$

$$= (\beta_1 + \beta_2 + \ldots + \beta_k)(\alpha_1 X_1 + \ldots + \alpha_k X_k) + \beta_{k+1} X_{k+1}$$

Where, $\alpha_i = \dfrac{\beta_i}{\beta_1 + \ldots + \beta_k}$

Since $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 1$, $\alpha_1 X_1 + \ldots + \alpha_k X_k = Y$, is a C.L.C. of $X_1, X_2, \ldots, X_k$ which is in S by assumption.

Thus, $\beta_1 X_1 + \ldots + \beta_k X_k + \beta_{k+1} X_{k+1} = (\beta_1 + \ldots + \beta_k) Y + \beta_{k+1} X_{k+1}$ is a C.L.C. of $Y, X_{k+1} \,\varepsilon\, S$

Hence, it belongs to S. Hence, the result.

Example 7: The set $S = \{X \,\varepsilon\, E_n \mid \|X - X_0\| \leq \alpha\}$ is a convex set, $\|\cdot\|$ is the norm (usual), the 'distance' of X from X_0.

Proof: Let $X_1, X_2 \,\varepsilon\, S$. Therefore, $\|X_1 - X_0\| \leq \alpha$ and $\|X_2 - X_0\| \leq \alpha$.

Let $X = (1 - \lambda) X_1 + \lambda X_2$, $0 \leq \lambda \leq 1$, be a C.L.C. of X_1, X_2.

Then

$$\begin{aligned}
\|X - X_0\| &= \|(1 - \lambda) X_1 + \lambda X_2 - X_0\| \\
&= \|(1 - \lambda) X_1 - (1 - \lambda) X_0 + \lambda X_2 - \lambda X_0\| \\
&\leq \|(1 - \lambda)(X_1 - X_0)\| + \|\lambda (X_2 - X_0)\| && \text{(norm property)} \\
&= (1 - \lambda)\|X_1 - X_0\| + \lambda \|X_2 - X_0\| && \text{(norm property)} \\
&\leq (1 - \lambda)\alpha + \lambda\alpha && \text{(given)} \\
&\leq \alpha
\end{aligned}$$

So $X \,\varepsilon\, S$. Hence, proved.

In E_2, S is the circle, with interior, centred at X_0 and of radius α, while in E_3, S is the sphere with interior, of radius α and centred at X_0.

Example 8: The solution set of the m inequations in n-variables is a convex set.

Let the inequations be

$$\alpha_{11} x_1 + \alpha_{12} x_2 + \ldots + \alpha_{1n} x_n \le \beta_1$$
$$\alpha_{21} x_1 + \alpha_{22} x_2 + \ldots + \alpha_{2n} x_n \le \beta_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\alpha_{i1} x_1 + \alpha_{i2} x_2 + \ldots + \alpha_{in} x_n \le \beta_i$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\alpha_{m1} x_1 + \alpha_{m2} x_2 + \ldots + \alpha_{mn} x_n \le \beta_m$$

The solution set S_i, of each inequation is a half space $C_i^T X \le \beta_i$ is a convex set.

The solution set of these m inequations is the intersection of $S_1, S_2, \ldots, S_m$, which by theorem is a convex set.

Notation:

$$X = (x_1, x_2, \ldots, x_n) \ge (\alpha_1, \alpha_2, \ldots, \alpha_n) = A$$

means $\quad x_i \ge \alpha_i,\ i = 1, 2, \ldots, n$

Definition:

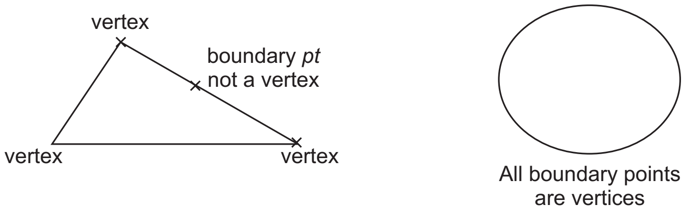
(a) A subset $S \subset E_n$ is said to be bounded below if $\exists$ a point $Y \,\varepsilon\, E_n \ni X \ge Y \ \forall\ X \,\varepsilon\, S$.

(b) A subset $S \subset E_n$ is said to be bounded above if $\exists$ a point $Y \,\varepsilon\, E_n \ni X \le Y \ \forall\ X \,\varepsilon\, S$.

(c) A subset $S \subset E_n$ is said to be bounded if it is bounded below as well as bounded above *i.e.*, $\exists\ Y_1, Y_2 \,\varepsilon\, E_n \ni Y_1 \le X \le Y_2 \ \forall\ X \,\varepsilon\, S$.

Definition: Let S be a convex subset of E_n. A point $X \,\varepsilon\, S$ is called a vertex of S if it cannot be expressed as a C.L.C. of two other (other than X) points of S, *i.e.*, if it is not possible to find X_1, $X_2 \,\varepsilon\, S \ni X = (1 - \lambda) X_1 + \lambda X_2,\ 0 < \lambda < 1$.

Vertex is a boundary point but a point on boundary need not be a vertex. It is also called an extreme point.



Definition: Let S be a closed subset of E_n. A plane P: $C^T X = \alpha$ is called a separating hyperplane if S is contained in one of the half spaces determined by P, *i.e.*, $C^T X \le \alpha \ \forall\ X \,\varepsilon\, S$ or $C^T X \ge \alpha$ $\forall\ S$.

Definition: Let S be a closed convex subset of E_n. A plane P: $C^T X = \alpha$ is called a supporting hyperplane at $X_0 \,\varepsilon\, S$ if $C^T X_0 = \alpha$ and $C^T X \le \alpha \ \forall\ X \,\varepsilon\, S$ or $C^T X \ge \alpha \ \forall\ X \,\varepsilon\, S$, *i.e.*, $X_0 \,\varepsilon\, S$ lies on the plane P and S is contained in one of the two half spaces determined by P.

Remark: It is obvious that X_0 is a boundary point of S. It cannot be an interior point.

It is obvious that given a closed convex set S, we can find a separating hyperplane passing through a given exterior point, *i.e.*, given point outside S. We shall now give a formal proof.

Theorem 5: Let S be a non-empty closed convex subset of E_n and $X_0 \notin S$. Then there exists a separating hyperplane through X_0, *i.e.*, passing through X_0.

Proof:

$$X_0 \, \varepsilon \, S.$$

Consider $T = \{||Y - X_0|| \mid Y \varepsilon S\}$. Since $||\cdot|| \geq 0$, the set T is bounded below and it is a set of real non-negative numbers. Hence, minimum (greatest lower bound) of T exists. Let min T occurs for the point $Z \varepsilon S$. Thus,

$$||Z - X_0|| = \min_{Y \in S} ||Y - X_0||$$

or, $$||Z - X_0|| \leq ||Y - X_0|| \;\; \forall \; Y \varepsilon S \text{ and a given } Z \varepsilon S.$$

Let X be any point in S. Also $Z \varepsilon S$ and since S is convex,

$$(1 - \lambda) Z + \lambda X \, \varepsilon \, S, \; 0 \leq \lambda \leq 1$$

Therefore,

$$||(1 - \lambda) Z + \lambda X - X_0|| \geq ||Z - X_0||$$

or, $$||(1 - \lambda) Z + \lambda X - X_0||^2 \geq ||Z - X_0||^2$$

Using the definition of norm, we have

$$((1 - \lambda) Z + \lambda X - X_0)\cdot(1 - \lambda) Z + \lambda X - X_0) \geq (Z - X_0)\cdot(Z - X_0)$$

'$\cdot$' is the inner product in E_n

or, $$(Z - X_0) + \lambda (X - Z))\cdot((Z - X_0) + \lambda(X - Z)) \geq (Z - X_0)\cdot(Z - X_0)$$

Using the properties of Inner Product in E_n, we get

$$(Z - X_0)\cdot(Z - X_0) + \lambda^2(X - Z)\cdot(X - Z) + 2\lambda(Z - X_0)\cdot(X - Z)$$
$$\geq (Z - X_0)\cdot(Z - X_0)$$

or, $$\lambda^2(X - Z)\cdot(X - Z) + 2\lambda(Z - X_0)\cdot(X - Z) \geq 0$$

Since it is true even for all $\lambda > 0$, we take $\lambda \neq 0$, and

$$\lambda(X - Z)\cdot(X - Z) + 2(Z - X_0)\cdot(X - Z) \geq 0$$

λ can be taken arbitrary small, so it holds even when
$\lambda \to 0$, which in turn gives

$$(Z - X_0)\cdot(X - Z) \geq 0$$

Let $$C = Z - X \text{ and } ||C|| \geq 0 \text{ as } Z \neq X$$

Thus, $$C\cdot(X - Z) \geq 0$$

Using Matrix notations, we obtain

$$C^T(X - Z) \geq 0$$

or, $$C^TX - C^TZ \geq 0$$

or, $$C^TX \geq C^TZ$$

Let $C^TZ = \alpha$. Then, $C^TX \geq \alpha \;\; \forall \; X \in S$.

Hence, $C^TX = \alpha$ is the required separating hyperplane. Hence, the result.

It is evident that every supporting plane is a separating plane. In other words supporting plane is a limiting case of a separating plane. We shall now prove the existence of a supporting plane through a boundary point of the closed convex set.

Theorem 6: Let S be a non-empty closed convex subset of E_n and X_0 a boundary point of S. Then there exists a supporting hyperplane at X_0.

Proof: As mentioned earlier, it is a limiting case and can be easily proved as a limiting case of the above theorem.

Let X_0 be a boundary point of S. Thus each open ball centred at X_0 has a point of S and also a point exterior (outside) to S, *i.e.*, for each $\in > 0$, $||X - X_0|| < \in$ has a point not in S.

Let $Y \in$ open ball $||X - X_0|| < \in$ and $Y \notin S$.

By theorem 5, $\exists$ a separating plane $C^T X = \alpha$ passing through Y, *i.e.*, $C^T Y = \alpha$.

As $\in$ goes on reducing, *i.e.*, $\in \to 0$, the above result holds, *i.e.*, separating plane exists. As $\in \to 0$, points in open ball come nearer to X_0. Therefore, separating plane comes closer to S and in limit it passes through X_0 and hence becomes a supporting plane through X_0. Hence, the theorem.

The above theorem assures that there exists supporting plane passing through a boundary point and hence through a vertex. What about the converse? Whether every supporting plane has a boundary point? The answer is in affirmative by definition. But whether every supporting plane of S has a vertex of X, does not follow from definition. We shall prove this result below under a condition.

Theorem 7: Let S be a non-empty, closed, convex subset of E_n, which is bounded below. Then every supporting plane of S has a vertex (an extreme point) of S.

Proof: Supporting hyperplane is defined with respect to a boundary point. Let X_0 be a boundary point of S. Then by theorem 5, there exists a supporting hyperplane

$$H = \{X \in E_n \mid C^T X = \alpha\}$$

through X_0, *i.e.*, $C^T X_0 = \alpha$.

Let $T = S \cap H$.

Since S and H are closed, convex sets, so is T. Since S is bounded below, T is also bounded below. Thus, there exists a $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in E_n \ni$

$$(x_1, x_2, \ldots, x_n) \geq (\alpha_1, \alpha_2, \ldots, \alpha_n) \;\forall\; (x_1, x_2, \ldots, x_n) \in T.$$

Choose a point X^* in T which has smallest first coordinate, smallest 2nd coordinate, …, smallest n^{th} coordinate. Let it be

$$X^* = (x^*_1, x^*_2, \ldots, x^*_{j-1}, x^*_j, x^*_{j+1}, \ldots, x^*_n)$$

This would be an unique point in T because only one point can have all smallest components, *i.e.*, there is no tie in one of the components, say x^*_j.

$$x_i \geq x^*_i,\; i = 1, 2, \ldots, n \;\forall\; X = (x_1, x_2, \ldots, x_n) \in T \qquad (*)$$

We now show that this $X^* \in T$ is an extreme point (vertex).

Let it not be a vertex. Then $\exists$ $Y, Z \in T$

Such that

$$X^* = (1 - \lambda)\, Y + \lambda Z,\; 0 < \lambda < 1,\; Y \neq Z$$

i.e., $$x^*_i = (1 - \lambda)\, y_i + \lambda Z_i,\; i = 1, 2, \ldots, n.,\; 0 < \lambda < 1.$$

Let $y_i > Z_i$. Then,

$$x_i^* = Z_i + (1 - \lambda)\ (y_i - Z_i) > Z_i$$

which contradicts, (*), $x_i^* \leq Z_i$,

Let $y_i < Z_i$. Then

$$x_i^* = y_i + \lambda(Z_i - y_i) > y_i$$

which also contradicts, *, $x_i^* \leq y_i$.

Thus, $\quad Y_i = Z_i = x_i^*,\ i = 1, 2, \ldots, n$

i.e., $\quad X^* = Y = Z,$

a contradiction. Hence, X^* is a vertex of T.

Now we shall prove that X^* is a vertex of S too. In order to prove this, we shall show that a point of T which is not a vertex of S is not a vertex of T.

Let $X \in T$ and not a vertex of S, *i.e.*, $\exists\ Y\ \&\ Z \in S, \ni$

$$X = (1 - \lambda)\ Y + \lambda Z,\ 0 < \lambda < 1,\ Y \neq Z$$

or, $\quad C^T X = (1 - \lambda)\ C^T Y + \lambda\ C^T Z$

Where C^T corresponds to H.

Since $Y, Z \in S$, $C^T Y$, $C^T Z$ are both $\geq \alpha$ or both $\leq \alpha$. Let $C^T Y \geq \alpha$, $C^T Z \geq \alpha$. Then,

$$C^T X \geq \alpha$$

But $\quad X \in H,\ C^T X = \alpha$, therefore $C^T Y = C^T Z = \alpha$

Thus, $\quad Y, Z \in H;\ Y,\ Z \in T$

Therefore, X is not a vertex of T, which proves the result.

Now as a corollary to the above theorem, we prove the following.

Corollary

Let S be a non-empty, closed, convex subset of E_n. If S is bounded below (above), then S has at least one vertex.

Proof: S is a closed set. So it has a boundary point $X_0 \in S$.

By Theorem 5 there exists a supporting plane through X_0.

By theorem 7 this supporting plane has a vertex as it is bounded below too. Hence, the result.

The theorem 7 can also be proved on similar lines in case S is bounded above and consequently the above corollary will follow with ‘below’ replaced by ‘above’.

In case the set S is bounded then the above result follows but something more also follows. It is proved in the following result.

Theorem 8: Let S be a non-empty, closed, convex subset of E_n. If S is bounded, then it has at least one vertex and every point of S is a C.L.C. of its vertices.

The proof is left to the reader.

Theorem 9: The optimum of $f(X)$, the objective function, of LPP occurs at a vertex of S_F, provided S_F is bounded.

Proof: Let the LPP be a maximization problem. S_F is bounded polytope, so it has finite number of vertices. Let $X_1, X_2, \ldots, X_m$ be m vertices.

Let $\quad f(X)$ assume maximum at $X_0 \in S_F$.

i.e., $\quad f(X_0)$ is maximum, *i.e.*,

$$f(X_0) \geq f(X) \quad \forall \; X \in S_F$$

Also $\quad f(X_0) \geq f(X_i) \quad \forall \; i = 1, 2, \ldots, m$ (vertices)

We shall now prove that $f(X_0) = f(X_k)$ for some $k = 1, 2, \ldots, m$

Since S_F is non-empty, closed, convex, by a theorem, every point of S_F is a C.L.C. of $X_1, X_2, \ldots, X_m$. So

$$X_0 = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_m X_m, \; 0 \leq \alpha_i \leq 1,$$

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

or,

$$\begin{aligned} f(X_0) &= f(\alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_m X_m) \\ &= \alpha_1 f(X_1) + \alpha_2 f(X_2) + \ldots + \alpha_m f(X_m) \end{aligned}$$

(since f is linear)

Since $\{f(X_1), \ldots, f(X_m)\}$ is a finite set, it has a maximum, let $f(X_k)$ be maximum.

Then $\quad f(X_i) \leq f(X_k) \quad \forall \; i = 1, 2, \ldots, k_m$. So

$$\begin{aligned} f(X_0) &\leq \alpha_1 f(X_k) + \alpha_2 f(X_k) + \ldots + \alpha_m f(X_k) \\ &= (\alpha_1 + \alpha_2 + \ldots + \alpha_k) f(X_k) \\ &\leq f(X_k) \end{aligned}$$

But $\quad f(X_0) \geq f(X_k)$

Therefore, $\quad f(X_0) = f(X_k)$.

Hence, the maximum occurs at X_k, a vertex of S_F.

Hence, the theorem.

From the above theorem optimum occurs at a vertex. We shall now prove that if f is not a constant function, $f(X)$ will not attain maximum at an interior point.

Theorem 10: In an LPP, if the objective function $f(X)$ is non-constant, then f does not attain optimum at an interior point.

Proof: Let f attain maximum at an interior point $X_0 \in S_F$

Then

$$f(X_0) \geq f(X) \quad \forall \; X \in S_F$$

Let $X_1 \in S_F$. Since X_0 is an interior point of S_F, $\exists \; X_2 \in S_F \ni X_0 = \lambda X_1 + (1 - \lambda)X_2$, $0 < \lambda < 1$, X_1, X_2 interior points of S_F.

$$\begin{aligned} f(X_0) &= f(\lambda X_1 + (1 - \lambda)(X_2)) \\ &= \lambda f(X_1) + (1 - \lambda) f(X_2) \end{aligned}$$

But $\quad f(X_0) \geq f(X_1)$ and $f(X_0) \geq f(X_2)$

$\therefore \quad \lambda f(X_1) + (1 - \lambda) f(X_2) \geq f(X_1)$

or, $\quad f(X_2) \geq f(X_1)$

Similarly, $\quad \lambda f(X_1) + (1 - \lambda) f(X_2) \geq f(X_2)$

$$f(X_1) \geq f(X_2)$$

Which implies $\quad f(X_1) = f(X_2)$

$$\begin{aligned} f(X_0) &= \lambda f(X_1) + (1 - \lambda) f(X_2) \\ &= \lambda f(X_1) + (1 - \lambda) f(X_2) \\ &= f(X_1) \end{aligned}$$

Since X_1 is arbitrary, f is constant function which is a contradiction. Hence, the theorem.

As mentioned earlier S_F is bounded below, if it is not bounded above then either $\exists$ a vertex of S_F where optimum occurs or solution becomes unbounded.

EXERCISE 2.4

1. (a) Express $\left(\frac{1}{2}, \frac{1}{8}\right)$ as a C.L.C. of the points (0, 0), (2, 1) and (4, 0)

$$\left(\textbf{Ans: } \frac{13}{16}(0,0) + \frac{1}{8}(2,1) + \frac{1}{16}(4,0)\right)$$

   (b) Express $\left(\frac{1}{6}, \frac{1}{12}\right)$ as a C.L.C. of the points (0, 0), (3, 0) and (1, 1).

$$\left(\textbf{Ans: } \frac{8}{9}(0,0) + \frac{1}{36}(3,0) + \frac{1}{12}(1,1)\right)$$

   (c) Express $\left(\frac{1}{3}, \frac{1}{9}\right)$ as a C.L.C. of the points (2, 0), (0, 0) and (2, 2)

$$\left(\textbf{Ans: } \frac{1}{9}(2,0) + \frac{5}{6}(0,0) + \frac{1}{18}(2,2)\right)$$

2. (a) Is $X = \left(-\frac{1}{2}, \frac{1}{2}, \frac{7}{4}\right)$ a C.L.C. of X_1, X_2, X_3?

   (b) Is $X = (1, 1, 1)$ a C.L.C. of X_1, X_2, X_3?

   (**Ans:** (a) Yes (b) No)
3. Prove that closed half space $C^T X \geq \alpha$ is a convex set.
4. Prove that open half spaces $C^T X < \alpha$ and $C^T X > \alpha$ are convex set.
5. If S_1 and S_2 are convex sets in E_n. Then show that $S = S_1 + S_2$ is also a convex set.
6. By giving a counter example, show that region between two concentric circles in E_2 is not a convex set.
7. Prove that the minimum of the objective function $f(X)$ of an LPP occurs at a vertex of S_F, provided S_F is bounded.
8. Prove that if optimum occurs at more than one vertex of S_F, then it also occurs at a C.L.C. of these vertices.
9. In an LPP, find the minimum of the function $f(X) = 6x_1 - 4x_2$ in the convex region whose vertices are (0, 1), (1, 2), (3, 2), (4, 1), (2, 0), (1, 0)

   (**Ans:** Minimum $f(X) = -4$ at (0, 1))
10. Let X_0 be any optimal solution of the LPP

    Maximize $\quad f(X) = C^T X$

    Subject to $\quad AX = b,\ X \geq 0$

in standard form and X^* is any optimal solution when C is replaced by C^*, then prove that

$$(C^* - C)^T (X^* - X_0) \geq 0.$$

2.9 ALGEBRAIC FORMULATION OF THE LPP THEORY

From the above theorems 9 and 10 of section 1.8, it is clear that optimum occurs at a vertex. Thus, theoretically speaking, a bounded S_F (polytope) has finite number of vertices, so we can find the values of objective function and then find the optimum value. This would give us optimum value of the function as well as the vertex (coordinates of which are the values of variables) at which optimum occurs. It looks simple in saying and certainly not difficult in E_2, but in higher dimension, it is not simple as there is no geometry.

In this case we proceed algebraically. First we need to equip with certain definitions and concepts.

Normally constraints are inequations. We introduce slack, surplus variables to convert them into equations which increases the number of variables and represent hyperplanes. Vertices are points of intersection of these hyperplanes.

Let constraint equations (after introducing Slack/Surplus variables) be

$$\begin{aligned}
\alpha_{11} x_1 + \alpha_{12} x_2 + \ldots + \alpha_{1n} x_n &= b_1 \\
\alpha_{21} x_1 + \alpha_{22} x_2 + \ldots + \alpha_{2n} x_n &= b_2 \\
\ldots \quad \ldots \quad \ldots \quad & \ldots \\
\alpha_{m1} x_1 + \alpha_{m2} x_2 + \ldots + \alpha_{mn} x_n &= b_m
\end{aligned}$$

Which can be written in matrix form as

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & & \alpha_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or

$$AX = b.$$

Let $A_1, A_2, \ldots, A_j, \ldots A_n$ denote the columns of A, *i.e.*,

$$A_j = \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{bmatrix}$$

Then the matrix A can be expressed as

$$A = [A_1 \; A_2 \; \ldots \; A_n]$$

and the system of equations as

$$[A_1 \; A_2 \; \ldots \; A_n] \; X = b$$

In almost all problems, after introducing slack surplus variables, number of unknowns are more than number of equations, so for our discussion, we can safely take $m < n$.

Case I We have learnt earlier that, if $r(A) \neq r(A, b)$ then the system is inconsistent and there would not be any solution. Thus, S_F would be empty and the LPP will not have any feasible solution.

Case II If $r(A) = r(A, b)$, the system will have a solution and consequently the LPP will have a solution.

If $r(A) = r(A, b) = r < m$, then it means that the system has $m–r$ equations dependent on r equations. These $m–r$ equations are redundant and can be removed without affecting the solution and then $r(A) = r(A, b) = r$ would be equal to the number of equations. Therefore, without loss of generality, we can, for our discussion purposes, assume that $r(A) = r(A, b) = m$, *i.e.*, all the m equations are independent.

Since $r(A) = r(A, b) = m$, A will have a submatrix B of size mxm which would be non-singular and hence invertible.

Also, since $r(A) = r(A, b) = m$, m unknowns can be evaluated in terms of remaining $n–m$ unknowns, values of which can be arbitrarily chosen and can be chosen as zero.

We also know that only those variables can be obtained in terms of others, whose columns make the non-singular submatrix B.

Thus, in order to obtain a solution of the system, we assume $n–m$ variables as zero such that the columns of remaining m variables form a non-singular submatrix B, and solve the remaining m-equations in m-unknowns, *i.e.*, the system $BX_B = b$, which has a unique solution because B is non-singular.

A solution, in which n-m unknowns have been taken to be zero is called a BASIC solution. The variables which have been assumed to be zero are called NON-BASIC variables. The variables which have been obtained by solving $BX_B = b$ are called BASIC variables.

Now, we know that the solution obtained above *i.e.*, basic solution satisfies the constraints. Now considering the non-negativity conditions, we discard those basic solutions in which basic variables have negative values (non-basic variables are already zero) and retain those in which basic variables are ≥ 0. Thus, a basic solution in which all basic variables are ≥ 0 (non-basic variables are already zero) satisfy $AX = b$ and $X \geq 0$ and therefore are called BASIC FEASIBLE SOLUTION, in short BFS.

Theorem 1: A *BFS* of an LPP is a vertex of S_F, the convex set of feasible solution.

Proof: Let X_V be a *BFS*. It has m-basic variables. Let us assume, without loss of generality, first m-variables are basic variables. Let

$$X_V = (v_1, v_2, \ldots, v_m, 0, 0, \ldots, 0)^T,$$

where, $v_1, v_2, \ldots, v_m \geq 0$ and

$$AX_V = b$$

or, $$v_1 A_1 + v_2 A_2, + \ldots + v_m A_m = b,$$

where, $A_1, A_2, \ldots, A_m$ are columns of A corresponding to basic variables, hence linear independent.

Since, X_V is a *BFS*, $X_V \in S_F$.

We shall now prove that X_V is a vertex. Let us assume the contrary.

Let, if possible, X_V be not a vertex. Then, $\exists\ X_1, X_2 \in S_F \ni X_V \neq X_1 \neq X_2$ and

$$X_V = \lambda X_1 + (1 - \lambda) X_2, 0 < \lambda < 1.$$

Let $X_1 = (y_1, y_2, \ldots, y_n)^T$ and $X_2 = (z_1, z_2, \ldots, z_n)^T$.

and, $X_1, X_2 \geq 0$

Then,

$$(v_1, v_2, \ldots, v_m, 0, 0, \ldots 0)^T = \lambda (y_1, y_2, \ldots, y_m, y_{m+1}, \ldots, y_n)^T + (1 - \lambda) (z_1, z_2, \ldots, z_m, z_{m+1}, \ldots, z_n)^T$$

or, $v_i = \lambda y_i + (1 - \lambda) z_i; i = 1, 2, \ldots, m$

and, $0 = \lambda y_i + (1 - \lambda) z_i; i = m + 1, m + 2, \ldots, n.$

Since, $y_i, z_i \geq 0 \ \forall \ i$ and also $\lambda, 1 - \lambda > 0$,

$$\lambda y_i + (1 - \lambda) z_i = 0, i = m + 1, \ldots, n$$

gives that $y_i, z_i = 0 \ \forall \ i = m + 1, m + 2, \ldots, n.$

Therefore,

$$X_1 = (y_1, y_2, \ldots, y_m, 0, 0, \ldots, 0)^T$$

and, $X_2 = (z_1, z_2, \ldots, z_m, 0, 0, \ldots, 0)^T$.

Since, $X_1, X_2 \in S_F$, we get

$$AX_1 = b \text{ and } AX_2 = b,$$

i.e.,

$$y_1 A_1 + y_2 A_2 + \ldots + y_m A_m = b$$

and, $Z_1 A_1 + Z_2 A_2 + \ldots + Z_m A_m = b$

or, $(y_1 - z_1) A_1 + (y_2 - z_2) A_2 + \ldots + (y_m - z_m) A_m = 0$

Since $A_1, A_2, \ldots, A_m$ are linearly independent, we obtain

$$y_1 - z_1 = 0, y_2 - z_2 = 0, \ldots, y_m - z_m = 0$$

or, $y_1 = z_1, y_2 = z_2, \ldots, y_m = z_m$

i.e., $X_1 = X_2$, a contradiction.

Hence, X_V is a vertex, *i.e.*, a *BFS* is a vertex.

Hence, the theorem.

The converse of the above result also holds.

Theorem 2: A vertex of S_F, the convex set of feasible solution of the LPP is a BFS of LPP.

Proof:

Let $X_V = (v_1, v_2, \ldots, v_m)^T$ be a vertex of S_F.

Therefore, $X_V \in S_F$, Thus

$$AX_V = b$$

or, $v_1 A_1 + v_2 A_2 + \ldots + v_n A_n = b$

and, $v_1, v_2, \ldots, v_n \geq 0.$

We wish to prove that it is a BFS, *i.e.*, at least *n*-*m* variables are zero.

Let r ($\leq n$) v_i*'s* be non-zero. We assume without loss of generality, the first $v_1, v_2, \ldots, v_r \neq 0$. *i.e.*, $v_1, v_2, \ldots, v_r > 0$. Therefore, we get

$$v_1 A_1 + v_2 A_2 + \ldots + v_r A_r = b$$

The columns $A_1, A_2, \ldots, A_r$ are vectors of E_m. E_m can have utmost m linearly independent vectors. So, if we can prove that $A_1, A_2, \ldots, A_r$ are linearly independent then it would mean $r \leq m$, *i.e.*, at least n-m variables are zero. Hence, X_V is a BFS.

Thus, it remains to prove that $A_1, A_2, \ldots, A_r$ are linearly independent

Let us assume the contrary. Let $A_1, A_2, \ldots, A_r$ be linearly dependent. So there exists $\alpha_1, \alpha_2, \ldots, \alpha_r$ scalars not all zero such that

$$\alpha_1 A_1 + \alpha_2 A_2 + \ldots + \alpha_r A_r = 0 \qquad (1)$$

We also have

$$v_1 A_1 + v_2 A_2 + \ldots + v_r A_r = b. \qquad (2)$$

From the above two equations, we obtain

$$(v_1 + c\alpha_1) A_1 + (v_2 + c\alpha_2) A_2 + \ldots + (v_1 + c\alpha_r) A_r = b \qquad (3)$$

and, $$(v_1 - c\alpha_1) A_1 + (v_2 - c\alpha_2) A_2 + \ldots + (v_r - c\alpha_r) A_r = b \qquad (4)$$

for any c. If we assume $c \leq \min\limits_i \left\{ \dfrac{v_i}{|\alpha_i|} \right\}$, then

$$v_i \pm c\ \alpha_i \geq 0 \ \ \forall \ \ i = 1, 2, \ldots, r.$$

Thus, we take $c \leq \min\limits_i \left\{ \dfrac{v_i}{|\alpha_i|} \right\}$.

Take the vectors

$$X_1 = (v_1 + c\alpha_1, v_2 + c\alpha_2, \ldots, v_r + c\alpha_r, 0, 0, \ldots, 0)^T$$

and, $$X_2 = (v_1 - c\alpha_1, v_2 - c\alpha_2, \ldots, v_r - c\alpha_r, 0, 0, \ldots, 0)^T.$$

In view of (3) and (4) we have,

$$AX_1 = b \text{ and } AX_2 = b$$

and also $X_1, X_2 \geq 0$. Thus, $X_1, X_2 \in S_F$, and $X_1 \neq X_2$.

Also

$$X_V = \frac{1}{2} X_1 + \frac{1}{2} X_2$$

i.e., X_V is a C.L.C. of X_1, X_2 where $X_1, X_2 \in S_F$ and $X_1 \neq X_2$. Therefore, X_V is not a vertex, which is a contradiction. Hence, $A_1, A_2, \ldots, A_r$ are linearly independent So X_V is a BFS. Hence, the theorem.

On summarizing the above two theorems, we find that BFS $\Leftrightarrow$ vertex.

Thus, in order to obtain all vertices, we obtain all its BFS which are ${}^nc_m$ in number. Thus, total number of vertices would also be ${}^nc_m$.

We have noticed that values of non-basic variables in a BFS are zero and values of basic variables a BFS are non-negative which means basic variables can be zero or positive. So we define.

Definition:

(a) A BFS in which all basic variables are > 0 is called a NON-DEGENERATE BFS.

(b) A BFS in which at least one basic variable is zero, is called a DEGENERATE BFS.

In a degenerate case a basic variable which is zero can be treated as non-basic and a non-basic can take its place. These two, though same solution, would be regarded as two different BFS. This implies that a vertex can correspond to more than one BFS.

EXERCISE 2.5

1. Without sketching, find the vertices of the set of feasible solutions S_F for the following LPP

$$\begin{aligned} -x_1 + x_2 &\leq 2 \\ 2x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$\left(\textbf{Ans: } (0, 0), (0, 1), (1, 0), \left(\frac{1}{3}, \frac{4}{3}\right)\right)$$

2. Consider the following system

$$\left.\begin{aligned} 3x_1 + x_2 - x_3 + 2x_4 - x_5 + s_1 &= 2 \\ x_1 - x_2 + 2x_3 - 2x_4 - 3x_5 + s_2 &= -2 \end{aligned}\right\} \quad (1)$$

(a) Can x_2 and x_4 be basic variables. Given reason

[Sol. Equating all variables equal to zero except x_2 & x_4 equation (1) becomes.

$$\left.\begin{aligned} x_2 + 2x_4 &= 2 \\ -x_2 - 2x_4 &= 2 \end{aligned}\right\} \quad (2)$$

The coefficient matrix A of eq. (1) is

$$A = \begin{bmatrix} 3 & 1 & -1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 2 & -2 & -3 & 0 & 1 \end{bmatrix}$$

Submatrix of A

$$B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

The columns of B are linearly dependent or det (B) = 0 or B^{-1} does not exist or B is a singular matrix. Thus, the variables x_2 and x_4 are not basic variables.

Also observe both eqns. in (2) are identical. Hence, there exist infinite number of solutions of system (2) given by $x_2 = 2 - 2x_4$ for each real x_4. But these are not basic solutions of (1).

(b) Answer (a) if the right hand side of 2nd equation in (1) is 2.

(**Ans:** No)

(c) Can x_3 and S_2 be basic variables?

[**Ans:** Yes and the basic solution of (1) is $(x_1, x_2, x_3, x_4, x_5, s_1, s_2)$ $= (0, 0, -2, 0, 0, 0, 2)$]

(d) Can x_2 and s_1 be basic variables?

[**Ans:** Yes and the basic solution of (1) is $(x_1, x_2, x_3, x_4, x_5, s_1, s_2)$ $= (0, 2, 0, 0, 0, 0, 0)$]

2.10 SOLUTION OF A LINEAR PROGRAMMING PROBLEM BY SIMPLEX METHOD

2.10.1 Introduction

We have seen that optimum of an LPP occurs at a vertex. Since vertex is BFS and BFS is a vertex, we shall not differentiate the two. So we can say that optimum of an LPP occurs at a BFS.

Therefore, we can find all its BFS, as they are finite in number as long as S_F is bounded and then can find values of objective function at all BFS and pick the optimum.

But, finite in number does not mean small in number, There are actually ${}^n c_m$ BFS. We shall now develop algorithm (method) by which we would avoid this treachery of finding all BFS.

In this method, known as ***Simplex method***, we shall find a BFS, which we will call ***starting BFS***, and then move to another BFS in such a way that value of objective function improves. We shall continue the steps till optimum is reached.

This method is applicable to an LPP only. If a problem is an LPP, then only simplex method can be applied. Some of the NLPP can be approximated by LPP and hence can be solved by simplex method but it would give only approximate solution.

Some of the problems, apparently are not LPP but can be converted into LPP and hence can be solved by simplex method. One such problem conversion is given in the next section.

NLPP approximated by LPP will be discussed later.

2.10.2 Conversion of Typical Problems into LPP

(i) Look at the problem

Max Z = Min $\{-8x_1 + 8x_2 - 2x_3, -10x_1 - 3x_2 + 10x_3, 10x_1 - 5x_2 - 10x_3\}$

Subject to $x_1 + x_2 + x_3 = 3420$; $x_1, x_2, x_3 \geq 0$ & integers.

It can be converted into LPP as follows:

Let y = Min $\{-8x_1 + 8x_2 - 2x_3, -10x_1 - 3x_2 + 10x_3, 10x_1 - 5x_1 - 10x_3\}$

Then, we have

Max $Z = y$

Subject to

$$\begin{aligned} y &\leq -8x_1 + 8x_2 - 2x_3 \quad &\text{or} \quad 8x_1 - 8x_2 + 2x_3 + y \leq 0 \\ y &\leq -10x_1 - 3x_2 + 10x_3 \quad &\text{or} \quad 10x_1 + 3x_2 - 10x_3 + y \leq 0 \\ y &\leq 10x_1 - 5x_2 - 10x_3 \quad &\text{or} \quad 10x_1 - 5x_2 - 10x_3 - y \geq 0 \\ & & x_1 + x_2 + x_3 = 3420 \\ & & x_1, x_2, x_3 \geq 0 \text{ \& integers.} \end{aligned}$$

(ii) In some of the problems constraints could be either or type.

Let the problem be

Max $Z = 2x_1 + x_2$

$$\text{Subject to} \qquad \left.\begin{aligned} x_1 + x_2 \leq 1 \\ |x_1 - 2x_2| \geq \frac{1}{2} \\ x_1, x_2 \geq 0 \end{aligned}\right\} (*)$$

Here (*) implies either $x_1 - 2x_2 \geq \frac{1}{2}$ or $x_1 - 2x_2 \leq -\frac{1}{2}$

We handle this problem in the following manner.

Let M be a very big positive number. By 'big', we mean a finite number but big enough in comparison to any number involved in the problem.

Then

$$\text{either} \qquad x_1 - 2x_2 \geq \frac{1}{2}$$

$$\text{or,} \qquad -x_1 + 2x_2 \geq \frac{1}{2}\text{"}$$

can be replaced by

$$\text{"}M\,y + x_1 - 2x_2 \geq \frac{1}{2}$$

$$\text{and} \quad M\,(1 - y) - x_1 + 2x_2 \geq \frac{1}{2};\ y = 0 \text{ or } 1\text{"}$$

If y assumes the value '0', then first inequation becomes active and second inequation

$$M + (-x_1 + 2x_2) \geq \frac{1}{2}$$

becomes redundant as it would hold irrespective of the values of x_1, x_2. If $y = 1$, the second inequation becomes active and first becomes redundant. Hence, the problem becomes

Max $Z = 2x_1 + x_2$

Subject to $\quad x_1 + x_2 \leq 1$

$$M\,y + x_1 - 2x_2 \geq \frac{1}{2}$$

$$M\,(1 - y) - x_1 + 2x_2 \geq \frac{1}{2}$$

$$x_1, x_2 \geq 0, \ \boxed{y = 0 \text{ or } 1} \ \text{ or } \ \boxed{y \geq 0, y \leq 1 \text{ and } y \text{ an integer}}$$

where, M is a big number.

2.10.3 Simplex Method

As mentioned in the Introduction, in simplex method, we obtain a starting BFS and move to another BFS, so that value of objective function improves; continue the steps till we get the solution.

But the questions before us are: how to get starting BFS? How to move to another BFS so that value of $f(X)$ improves? When to stop, *i.e.*, how to know, optimum has occurred?

In short, we say, in simplex method keeping feasibility we move towards optimality.

We answer the above questions below and develop the simplex algorithm.

Let the problem be:

Max $\qquad Z = f(X) = C^T X$

subject to $\qquad AX = b;\ X \geq 0,$

Here we have written that the LPP in standard form and have assumed that it is a maximization problem and also that it is neither ILPP nor mixed ILPP.

Here,

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{j} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2j} & \dots & \alpha_{2n} \\ - & - & - & - & - & - \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mj} & \dots & \alpha_{mn} \end{bmatrix} = [A_1\ A_2\ \dots\ A_j\ \dots\ A_n],$$

Where, $A_j = \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{bmatrix}$, the j^{th} column of A, and

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

First of all, we shall answer the second question. Knowing a BFS, how to move to another BFS, so that value improves.

We assume all the m rows are linearly independent. So $r(A) = m$. Thus, we put n-m variables equal to zero and solve the equations for the remaining m equations for m variables (basic variables).

For our discussion, without loss of generality, we assume first m variable $x_1, x_2, \dots, x_m$ be basic variables. Then, the corresponding BFS is

$$X^* = (x_1,\ x_2,\ \dots,\ x_m,\ 0,\ 0,\ \dots,\ 0)^T$$

X^* satisfies $AX = b$, so

$$AX^* = A_1\, x_1 + A_2\, x_2 + \dots + A_m\, x_m = b$$

Where, $A_1, A_2, \dots A_m$ are the columns of A corresponding to the variables $x_1, x_2, \dots, x_m$ the basic variables.

These basic variables are those variables for which m-equations with $x_{m+1} = x_{m+2} = \dots = x_n = 0$ have a unique solution. Therefore, the square matrix

$$B = [A_1\ A_2\ \dots\ A_m]$$

is non-singular, *i.e.*, $A_1, A_2, \dots, A_m$ are linearly independent. B is formed by writing columns of A corresponding to the basic variables in that order. Now

$$x_1\, A_1 + x_2\, A_2 + \dots + x_m\, A_m = b$$

can be written as

$$[A_1\ A_2\ \ldots\ A_m]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = b$$

or, $$B\ X_B = b,\ \text{or}\ X_B = B^{-1}\ b.$$

where, $X_B = (x_1, x_2, \ldots, x_m)^T$ which is practically same as X^* for solution purposes.

Corresponding to this BFS, the value of objective function is

$$\begin{aligned} f(X_B) &= c_1\ x_1 + c_2\ x_2 + \ldots + c_m\ x_m \\ &= C_B^T\ X_B, \end{aligned}$$

Where, $C_B^T = (c_1, c_2, \ldots, c_m)$, the vectors formed by the costs of basic variables.

Since $A_1, A_2, \ldots, A_m$ are linearly independent in E_m, each column A_j of A which is a vector of E_m, can be expressed uniquely as linear combination of $A_1, A_2, \ldots, A_m$. Thus,

$$A_j = \alpha_1^j\ A_1 + \alpha_2^j\ A_2 + \ldots + \alpha_m^j\ A_m,\ j = 1, 2, \ldots, n,$$

where, $\alpha^j = (\alpha_1^j, \alpha_2^j, \ldots, \alpha_m^j)$ is the coordinate vector of A_j relative to the basis $\{A_1, A_2, \ldots, A_m\}$.

Obviously $\alpha^1 = e_1$, $\alpha^2 = e_2$, …, $\alpha^m = e_m$, m-dimensional vectors. But α^{m+1}, α^{m+2}, …, α^n are not e_is. It is clear that, once we know B, *i.e.*, B^{-1}, we can find α^j as

$$A_j = B\ \alpha^j$$

or, $$\alpha^j = B^{-1}\ A_j.$$

So far we have assumed that a BFS is known and equipped ourselves with notations and relevant things to take up the question. How to move to another BFS, so that value improves?

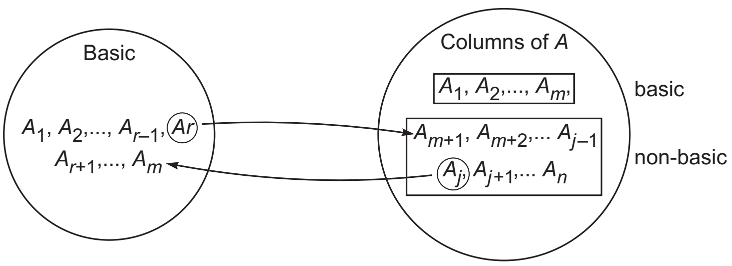
What do we mean by moving to another BFS? It means finding another BFS. We shall proceed step by step. We shall make one non-basic variable as basic variable and consequently one basic as non-basic.

In other words we shall enter one non-basic, in the set of basic variable, so that the solution remains feasible and allow a basic to become non-basic so that solution (value) improves.

Variables can be identified by the columns of A. Therefore, we can also talk in terms of A_j's as follows.

We shall enter one A_j (corresponding to a non-basic variable), $j = m + 1, m + 2, \ldots, n$ in the basis $\{A_1, A_2, \ldots, A_m\}$ so that the solution remains feasible and then take one column out from $A_1, A_2, \ldots, A_m$ so that solution improves.

Let A_j, j is one of the $m + 1, m + 2, \ldots, n$, enters the basis, and the column A_r, r is one of the $1, 2, \ldots, m$ leaves the basis.



A_j, $j = m + 1, m + 2, \ldots, n$ can be expressed as linear combination of $A_1, \ldots, A_m$. Now all columns of non-basic variables would be expressed as a linear combination of new basis, *i.e.*, $A_1, \ldots, A_{r-1}, A_j, A_{r+1}, A_m$. In other words A_r should be expressed in terms of the above new basis. We know that

$$A_j = \alpha_1^j A_1 + \alpha_2^j A_2 + \ldots + \alpha_{r-1}^j A_{r-1} + \alpha_r^j A_r + \alpha_{r+1}^j A_{r+1} + \ldots + \alpha_m^j A_m.$$

A_r can be expressed in terms of new basis $A_1, \ldots, A_{r-1}, A_j, A_{r+1}, \ldots, A_m$ only when $\alpha_r^j \neq 0$. In other words, after entering A_j, only those columns A_r can leave for which r-th coordinate $\alpha_r^j \neq 0$.

Earlier BFS gave

$$x_1 A_1 + x_2 A_2 + \ldots + x_{r-1} A_{r-1} + x_r A_r + x_{r+1} A_{r+1} + \ldots + x_m A_m = b$$

Replacing A_r by (assuming $\alpha_r^j \neq 0$)

$$A_r = -\frac{\alpha_1^j}{\alpha_r^j} A_1 - \frac{\alpha_2^j}{\alpha_r^j} A_2 - \ldots - \frac{\alpha_{r-1}^j}{\alpha_r^j} A_{r-1}$$
$$+ \frac{1}{\alpha_r^j} A_j - \frac{\alpha_{r+1}^j}{\alpha_r^j} A_{r+1} \ldots \frac{\alpha_m^j}{\alpha_r^j} A_m$$

We get

$$\left(x_1 - \frac{\alpha_1^j x_r}{\alpha_r^j}\right) A_1 + \left(x_2 - \frac{\alpha_2^j}{\alpha_r^j} x_r\right) A_2 + \ldots + \left(x_{r-1} - \frac{\alpha_{r-1}^j}{\alpha_r^j} x_r\right) A_{r-1}$$
$$+ \frac{x_r}{\alpha_r^j} A_j + \ldots + \left(x_{r+1} - \frac{\alpha_{r+1}^j x_r}{\alpha_r^j}\right) A_{r+1} + \ldots + \left(x_m - \frac{\alpha_m^j x_r}{\alpha_r^j}\right) A_m = b$$

This gives that

$$\hat{x}_B = \left(\left(x_1 - \frac{\alpha_1^j}{\alpha_r^j} x_r\right), \left(x_2 - \frac{\alpha_2^j x_r}{\alpha_r^j}\right), \ldots, \left(x_{r-1} - \frac{\alpha_{r-1}^j x_r}{\alpha_r^j}\right),\right.$$
$$\left.\frac{x_r}{\alpha_r^j}, \left(x_{r+1} - \frac{\alpha_{r+1}^j x_r}{\alpha_r^j}\right), \ldots \left(x_m - \frac{\alpha_m^j}{\alpha_r^j} x_r\right)\right)$$

is a basic solution. The leaving variable x_r should be so chosen that $\hat{x}_B$ remains a Basic Feasible solution, *i.e.*, $\hat{x}_B \geq 0$ or

$$x_i - \frac{\alpha_i^j}{\alpha_r^j} x_r \geq 0; \; i = 1, 2, \ldots, r-1, r+1, \ldots, m \text{ and } \frac{x_r}{\alpha_r^j} \geq 0.$$

This gives that $\alpha_r^j > 0$ since $x_r \geq 0$. Also, since $x_i' s \geq 0$, if any $\alpha_i^j = 0$, it is okay. So if $\alpha_i^j \neq 0$, we must have

$$\alpha_i^j \left(\frac{x_i}{\alpha_i^j} - \frac{x_r}{\alpha_r^j}\right) \geq 0, \; i = 1, 2, \ldots, r-1, r+1, \ldots, m$$

It is clear, if $\alpha_i^j < 0$, the inequality is satisfied. Only problem are those cases for which $\alpha_i^j >$ 0. In this case, we need

$$\frac{x_i}{\alpha_i^j} - \frac{x_r}{\alpha_r^j} \geq 0 \text{ for all those } i \text{ for which } \alpha_i^j > 0.$$

This is possible, if we take

$$\frac{x_r}{\alpha_r^j} = \operatorname*{Min}_i \left(\frac{x_i}{\alpha_i^j}, \alpha_i^j > 0 \right) = \theta_j \geq 0$$

i.e., we select that variable to leave for which $\frac{x_i}{\alpha_i^j}$, $\alpha_i^j > 0$, $i = 1, 2, \ldots, m$, is minimum. It would guarantee that the new solution would be a BFS.

This tells us that leaving variable affects feasibility. It should be so chosen that the solution remains feasible.

For this, as developed above, after deciding about the entering variable, we look at its coordinate vector. We calculate $\frac{x_i}{\alpha_i^j}$ only for those for which $\alpha_i^j > 0$ (x_i is the value of that basic variable). We pick the one for which $\frac{x_i}{\alpha_i^j}$ is minimum. This decision about leaving variable guarantees feasibility.

Now selection of entering variable. We should enter that variable (*i.e.*, move to that BFS) which improves the value. New BFS is

$$\hat{X}_B = ((x_1 - \alpha_1^j\ \theta_j), (x_2 - \alpha_2^j\ \theta_j), \ldots, (x_{r-1} - \alpha_{r-1}^j\ \theta_j),$$
$$\theta_j, (x_{r+1} - \alpha_{r+1}^j\ \theta_j), \ldots, (x_m - \alpha_m^j\ \theta_j))$$

Earlier the value of the objective function was

$$Z = f(X_B) = C_B^T\ X_B$$
$$= c_1\ x_1 + c_2\ x_2 + \ldots + c_{r-1}\ x_{r-1} + c_r\ x_r + c_{r+1}\ x_{r+1} + \ldots + c_m\ x_m$$

Now the value of the objective function is

$$\hat{Z} = f(\hat{X}_B) =$$
$$c_1\ (x_1 - \alpha_1^j\ \theta_j) + c_2\ (x_2 - \alpha_2^j\ \theta_j) + \ldots + c_{r-1}\ (x_{r-1} - \alpha_{r-1}^j\ \theta_j) + \ldots + c_j\ \theta_j + c_{r+1}\ (x_{r+1} - \alpha_{r+1}^j\ \theta_j) + \ldots + c_m\ (x_m - \alpha_m^j\ \theta_j)$$

because now the variables are $x_1, x_2, \ldots, x_{r-1}, x_j, x_{r+1}, \ldots, x_m$

Also,

$$\hat{Z} - Z = -\alpha_1^j\ c_1\ \theta_j - \alpha_2^j\ c_2\ \theta_j \ldots \alpha_{r-1}^j\ c_{r-1}\ \theta_j + c_j\ \theta_j - c_r\ x_r - \alpha_{r+1}^j\ c_{r+1}\ \theta_j \ldots c_m\ \alpha_m^j\ \theta_j$$
$$= -\theta_j \left(c_1\,\alpha_1^j + c_2\,\alpha_2^j + \ldots + c_{r-1}\,\alpha_{r-1}^j - c_j + c_r\,\frac{x_r}{\theta_j} + c_{r+1}\,\alpha_{r+1}^j \ldots + c_m\,\alpha_m^j \right)$$
$$= -\theta_j\ (Z_j - c_j),$$

Where, $Z_j = c_1\ \alpha_1^j + c_2\ \alpha_2^j + \ldots + c_{r-1}\ \alpha_{r-1}^j + c_r\ \alpha_r^j + c_{r+1}\ \alpha_{r+1}^j + \ldots + c_m\ \alpha_m^j$
$= C_B^T\ \alpha^j$

But $\theta_j \geq 0$

Therefore, since it is a maximization problem, the value improves, *i.e.*, $\hat{Z} \geq Z$ if

$$Z_j - c_j \leq 0.$$

It means that value of objective function would have maximum improvement if the variable x_j (column A_j) to enter is so chosen that

$$\theta_j\ (Z_j - c_j) \text{ is most negative,}$$

and in case of minimisation problem, *i.e.*, $\hat{Z} < Z$

$$\theta_j\ (Z_j - c_j) \text{ is most positive.}$$

Thus, in order to find the non-basic variable x_j to enter the basis (to be precise the column A_j to enter the basis) we should find $\theta_j\ (Z_j - c_j)$ for each variable (column) and choose the one that has most negative value for maximisation problem and has most positive value for minimisation problem.

Since $\theta_j \geq 0$, if we consider the sign of $Z_j - c_j$ only. But whenever there are more than one variable having the same value (most negative/most positive) of $Z_j - c_j$ then we should find θ_j also for these variables and check the value of $\theta_j\ (Z_j - c_j)$. Such situations are normally referred to as "tie between entering variables".

In case of tie amongst entering variable, we consider $\theta_j\ (Z_j - c_j)$. Even if we take the attitude of finding $Z_j - c_j$ only and choosing arbitrarily then also it is okay. At the most it would increase the number of iterations by one which would be easier to handle in place of finding $\theta_j\ (Z_j - c_j)$.

Further if there is a tie even in $\theta_j\ (Z_j - c_j)$, we choose arbitrarily.

Summarising the above, we find that, in order to move from one vertex to another BFS, we first select the non-basic variable (column) to enter the basis by seeing $Z_j - c_j$ and pick the one for which $Z_j - c_j$ is most negative or most positive as the case may be. After this we select the basic variable (column) to leave the basis by picking the one for which $\dfrac{x_i}{\alpha_i^j}$, $\alpha_i > 0$ is minimum. In case of tie, we select arbitrarily and then calculate new values of α^j, $Z_j - c_j$, $x_i's$, and $f(X)$. Now we go on repeating the steps till optimum occurs.

In order to do this, we perform the steps in tabular form. We first write the starting table, as follows.

We first list all the variables in a horizontal row, keeping in mind that all basic variables appear towards the end. In the second row we write the value of $Z_j - c_j$ for that variable. After this, from 3rd row to $(m + 3)$th row we write α^j as column below each non-basic variable and below m-basic variable, we write B^{-1}, the inverse of the submatrix formed by the columns of A corresponding to basic variable. To this table we add two more columns, one in the beginning listing the basic variables in the same order as in the first horizontal row and the other column at the end giving the values of basic variable, *i.e.*, $B^{-1}b$ as column and the value of $f(x)$ in the second row of this column.

The table would look like as given below:

Basic variables	x_{m+1}	x_{m+2} … x_n		x_1 x_2 … … x_m	Sol.
	$Z_{m+1}-c_{m+1}$	$Z_{m+2}-c_{m+2}$ …	Z_n-c_n	Z_1-c_1 Z_2-c_2 … … Z_m-c_m	$f(X)$
x_1 x_2 ⋮ x_m	$\begin{bmatrix} \alpha^{m+1} \end{bmatrix}$	$\begin{bmatrix} \alpha^{m+2} \end{bmatrix}$	$\begin{bmatrix} \alpha^{n} \end{bmatrix}$	$\begin{bmatrix} B^{-1} \end{bmatrix}$	$\begin{bmatrix} B^{-1} b \\ = X_B \end{bmatrix}$

We notice that writing first table would itself be a terse problem. First identify basic variables, then B, then calculate B^{-1}, α^j $(= B^{-1} A_j)$, $B^{-1} b$, $f(X)$ $(= C_B^T B^{-1} b)$ and also $Z_j - c_j$ for each variables.

The problem of writing starting table would be quite simple, if we choose basic variables in a proper fashion, *i.e.*, the one whose columns in the matrix A form the identity matrix, *i.e.*,

$$B = I$$

$\therefore$
$$B^{-1} = I$$

and,
$$\alpha^j = B^{-1} A_j = Z A_j = A_j$$
$$X_B = B^{-1}b = Ib = b$$
$$f(X_B) = C_B^T b$$

and,
$$Z_j - c_j = C_B^T \alpha^j - c_j \text{ for all variables.}$$

for *basic variable*
$$\alpha^j = e_j.$$

$\therefore$
$$Z_j - c_j = C_B^T e_j - c_j = c_j - c_j = 0;\ j = 1, 2, \ldots, m$$

for *non-basic variables*
$$Z_j - c_j = C_B^T \alpha^j - c_j$$

In order to simplify it further, we eliminate basic variables from $f(X)$ with the help of constraints, *i.e.*, we make $f(X)$ free of basic variables, so that cost of the basic variables in $f(X)$ are zero, *i.e.*, $C_B^T = \bar{0}$. In this situation

$$Z_j - c_j = -c_j.$$

It will not effect the values of $z_j - c_j$ for basic variables, as $C_B^T = 0$, c_j is also 0, so $Z_j - c_j =$ 0.

Also
$$f(X_B) = C_B^T X_B = 0$$

In this case table would be as follows

Basic variables	x_{m+1}	x_{m+2} … x_n		x_1 x_2 … x_m	Solution
Z	$-c_{m+1}$	$-c_{m+2}$ …	$-c_n$	0 0 0	0
x_1 x_2 ⋮ x_m	$\begin{bmatrix} A_{m+1} \end{bmatrix}$	$\begin{bmatrix} A_{m+2} \end{bmatrix}$ …	$\begin{bmatrix} A_n \end{bmatrix}$	$\begin{bmatrix} e_1 & e_2 & & e_m \end{bmatrix}$	$\begin{bmatrix} b \end{bmatrix}$

Since, $f(X)$ is taken free of basic variable, we write second row $f(X)$ taking their coefficients and changing their sign and 0 in the column of solution.

From 3rd row to $(m + 3)$th row we write matrix A as it is except that we rearrange the columns if need be so that below basic variable we have B^{-1} *i.e.*, $B = I$ which we have already assumed and b as it is in the last column.

We illustrate the above table writing below:

Example 1: Let the LPP be

$$\text{Min} \quad Z = -6x_1 + 2x_2 - 4x_3$$

$$\text{Subject to} \quad \begin{aligned} 2x_1 - 3x_2 + x_3 &\le 14 \\ -4x_1 + 5x_2 - 9x_3 &\le 43 \\ 2x_1 + 2x_2 - 4x_3 &\le 39 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Solution: This LPP in standard from is

$$\text{Min} \quad Z = -6x_1 + 2x_2 - 4x_3$$

$$\text{Subject to} \quad \begin{aligned} 2x_1 - 3x_2 + x_3 + s_1 &= 14 \\ -4x_1 + 5x_2 - 9x_3 + s_2 &= 43 \\ 2x_1 + 2x_2 - 4x_3 + s_3 &= 39 \\ x_1, x_2, x_3, s_1, s_2, s_3 &\ge 0. \end{aligned}$$

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
Z	6	–2	4	0	0	0	0
s_1	$\boxed{2}$	–3	1	1	0	0	14
s_2	–4	5	–9	0	1	0	43
s_3	2	2	–4	0	0	1	39

Here all are slack variables. If each equation has slack variable, its corresponding columns will give identity matrix. So we take s_1, s_2, s_3 at the end. Objective function is normally free of slack/ surplus variables. So to write 2nd row, we transfer everything on right to left and write it will change the signs of costs. It will give $Z_j - c_j$ below each variable. We write α^j (= A_j) below each variable that amounts same matrix A and below solution we write b.

Thus, we get the starting table. It gives starting BFS as $(x_1, x_2, x_3, s_1, s_2, s_3)$ as (0, 0, 0, 14, 43, 39) as x_1, x_2, x_3 are non-basic variables and s_1, s_2, s_3 are basic variables.

Now to move to other BFS for a better solution, we proceed as follows:

We first decide about entering variable.

Since it is a minimisation problem we pick the variable which has most positive $Z_i - c_i$. It is 'x_1' $Z_1 - c_1$ is 6. So 'x_1' enters and now we decide leaving variable.

$$\operatorname*{Min}_i \left(\frac{x_i}{\alpha_i^j}, \alpha_i^j > 0 \right) = \text{Min} \left(\frac{14}{2}, \frac{39}{2} \right) = 7$$

which is for 's_1'. Thus, 's_1' leaves.

The entry in 'x_1' column and s_1 row is called pivotal entry. It is enclosed in a square in the table.

Now we have to change the matrix B^{-1}, X_B, α^j, $f(x)$ $z_j - c_j$ in the table. But to do it.

Here the basic variables are now x_1, s_2, s_3. So from the problem, B is

$$B = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

In order to find B^{-1}, we proceed as follows

$$\left[\begin{array}{rrr|rrr} 2 & 0 & 0 & 1 & 0 & 0 \\ -4 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

and apply row operation to obtain identity matrix on the left. We obtain

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Actually we have converted the column $(2, -4, 2)^T$ into $(1, 0, 0)^T$ by row operations which converted I into B^{-1}.

For new α^j, $(= B^{-1} A_j)$ we have to perform the same row operations on old α^j. For new X_B $(= B^{-1}b)$, we have to perform the same row operations on b, *i.e.*, old X_B. Also for new $Z_j - c_j$, we have to free $f(X)$ from x_1 *i.e.*, to bring $Z_1 - c_1 = 0$. We shall perform the row operations in such a way that entry below x_1 becomes zero, *i.e.*, column of x_1 now becomes (0 : 1, 0, 0). This row operation will bring the new value of $f(X) = C_B^T X_B$ in the column of solution and row of $Z_i - c_i$.

Performing the row operations as mentioned above. We obtain

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	Solution
Z	0	7	1	–3	0	0	–42
x_1	1	–3/2	1/2	1/2	0	0	7
s_2	0	–1	–7	2	1	0	71
s_3	0	5	–5	–1	0	1	25

Thus, new BFS is (7, 0, 0, 0, 71, 39) and the value is –42. It is a better solution as the earlier value is ‘0’. It is important to note that:

$$B^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

appears in the columns of s_1, s_2, s_3 (initial basic variables) though it is the inverse of the matrix of columns of x_1, s_2, s_3. It will hold every time, *i.e.*, new B^{-1} will be in the columns of initial basic variables even if in the new BFS no initial basic variable may appear as a basic variable.

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -1 \\ 5 \end{bmatrix}; \begin{bmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -9 \\ -4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -7 \\ -5 \end{bmatrix}$$

and,
$$\begin{bmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 2 \\ -1 \end{bmatrix}$$

are the new columns (which are α^j) for non-basic variables.

Also,
$$\begin{bmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 14 \\ 43 \\ 39 \end{bmatrix} = \begin{bmatrix} 7 \\ 71 \\ 25 \end{bmatrix}$$

is the new X_B.

Also, new $\quad C_B^T = (-6, 0, 0)$

So,
$$f(X) = C_B^T X_B = (-6, 0, 0)\begin{pmatrix} 7 \\ 71 \\ 25 \end{pmatrix} = -42$$

Also, for non-basic variables,

$$\begin{aligned} Z_2 - c_2 &= C_B^T \alpha^2 - c_2 \\ &= [-6, 0, 0]\begin{bmatrix} -3/2 \\ -2 \\ 5 \end{bmatrix} - 2 \\ &= 9 - 2 = 7 \\ Z_3 - c_3 &= [-6, 0, 0]\begin{bmatrix} 1/2 \\ -7 \\ -5 \end{bmatrix} + 4 \\ &= -3 + 4 = 1 \\ Z_4 - c_4 &= [-6, 0, 0]\begin{bmatrix} 1/2 \\ 2 \\ -1 \end{bmatrix} - 0 = -3 \end{aligned}$$

For basic variables, $Z_i - c_i$ would be zero and columns would be $e_i's$.

Now, what? Question is, whether the problem is over? If yes, how to know it? If not, how to proceed?

We answer the problem and leave the proof, till this illustrative example is complete. Also, we shall prove that row-operations would change all the columns, X_B, solution and $Z_i - c_i$ to the desired one.

The solution obtained above is certainly a better solution, but not an optimum solution.

In case of a minimisation (maximisation) problem the values of $z_i - c_i$ for all values should be ≤ 0 (≥ 0).

In the above example; the optimum has not yet reached, because $z_i - c_i$ are not all ≤ 0. So we repeat the steps, till optimum is attained *i.e.*, all $z_i - c_i$ are ≤ 0.

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	Solution
Z	0	7	1	–3	0	0	–42
x_1	1	$-3/2\downarrow$	1/2	1/2	0	0	7
s_2	0	–1	–7	2	1	0	71
$\leftarrow s_3$	0	$\boxed{5}$	–5	–1	0	1	25
Z	0	0	$\frac{-16}{5}$	–8/5	0	–7/5	–77
x_1	1	0	–1	4/5	0	3/10	29/2
s_2	0	0	–8	9/5	1	1/5	76
x_2	0	1	–1	–1/5	0	1/5	5

In the last table, we find that all $Z_i - c_i$ are ≤ 0 and since it is a minimisation problem, the optimal has reached and this table, we shall refer to as optimal table. The optimal solution is

$$x_1 = 29/2,\ x_2 = 05,\ x_3 = 0,\ s_1 = 0,\ s_2 = 76,\ s_3 = 0$$

or, $\left(\frac{29}{2}, 5, 0, 0, 76, 0\right)$ and the optimal value is –77.

Now we summarise this method in steps form. This method is known as *simplex method.*

Step 1: Write the LPP in standard form.

Step 2: Check whether the matrix A has the identity matrix as submatrix

If yes, we can start simplex iteration.

If not, we have to search for another method.

In case identity matrix exist,

Rearrange columns so identity matrix appears towards the end and write basic variables in order.

Step 3: Free the objective function from basic variables.

Step 4: Write the starting table. It gives the starting BFS.

Step 5: Decide the entering variable

For maximisation problem – most negative $Z_i - c_i$.

For minimisation problem – most positive $Z_i - c_i$

Step 6: Decide the leaving variable

It is the one for which

$$\frac{x_i}{\alpha_i^j}, \ \alpha_i^j > 0$$

is minimum, and mark the pivot element

Step 7: Apply iteration, *i.e.*, row operation so as to make the column of entering variable as

$$(0: 0, 0, .0, \underset{\substack{\uparrow \\ \text{Pivot element}}}{1}, 0, \ldots 0)^T$$

Pivot row: New pivot row = current pivot row ÷ pivot element

All other rows including Z row = current row – (Its pivot column coefficient × new pivot row).

Step 8: Continue: Repeat the steps, till the optimal table is reached, *i.e.*,
For maximisation problem – All $Z_i - c_i$ are ≥ 0
For minimisation problem – All $Z_i - c_i$ are ≤ 0

Remark: In case of tie in the entering variable find θ_i $(Z_i - c_i)$ and decide. Instead of finding θ_i, we reduce the labour choose arbitrarily. Utmost it increases the number of steps of operation.

Remark: In case of tie in leaving variable, choose arbitrarily. It may be noted here that in this case new BFS would degenerate solution.

Now we shall prove certain results, which we have assumed while illustrating the method. The first one is about the optimal table, *i.e.*, all $Z_i - c_i \geq 0$ (≤ 0) gives the optimal *i.e.*, maximal (minimal) solution.

Theorem 1: Let the LPP be

Max $\qquad f(X) = C^T X$

Subject to $\qquad AX = b;\ X \geq 0.$

If for a BFS, X_B^*, $Z_i^* - C_i \geq 0 \ \forall \ i = 1, 2, \ldots, n$ then X_B^* is the optimal solution

Proof:

Let
$$X_B^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$$
$$B^* = (A_1^*, A_2^*, \ldots, A_m^*)$$
$$C_B^* = (C_1^*, C_2^*, \ldots, C_m^*)^T$$

Then,
$$f(X_B^*) = C_B^{*\mathrm{T}} X_B^*,$$

where, x_i^* is some x_j, A_i^* is the column of x_i^* in A, and c_i^* is the cost of x_i^*.

Let Y be any other feasible solution, *i.e.*,

$$Y = (y_1, y_2, \ldots, y_n)^T$$

Since columns of B^*, *i.e.*, $A_1^*, A_2^*, \ldots, A_m^*$ is a basis, we have

$$A_j = \alpha_1^j A_1^* + \alpha_2^j A_2^* + \ldots + \alpha_m^j A_m^*, \ j = 1, 2, \ldots, n$$

Since Y is a feasible solution, we have

$$AY = y_1 A_1 + y_2 A_2 + \ldots + y_n A_n = b$$

Also
$$y_i \geq 0, \ i = 1, 2, \ldots, n.$$

Substituting the values of $A_1, A_2, \ldots, A_n$, we get

$$(\alpha_1^1 y_1 + \alpha_1^2 y_2 + \ldots + \alpha_1^n y_n) A_1^* + (\alpha_2^1 y_1 + \alpha_2^2 y_2 + \ldots + \alpha_2^n y_n) A_2^* + \ldots$$
$$+ (\alpha_m^1 y_1 + \alpha_m^2 y_2 + \ldots + \alpha_m^n y_n) A_m^* = b$$

Also, since X_B^* is a BFS, we have

$x_1^* A_1^* + x_2^* A_2^* + \ldots + x_m^* A_m^* = b$

But, since $\{A_1^*, A_2^*, \ldots, A_m^*\}$ is a basis, the expression for b would be unique. So,

$$x_j^* = \sum_{i=1}^{n} \alpha_j^i y_n.$$

Now, $$f(Y) = \sum_{k=1}^{n} C_k Y_k$$

But, $$Z_i^* - C_i \geq 0 \ \forall \ i, \text{ i.e., } C_i \leq Z_i^* \ \forall \ i, \text{ we get}$$

$$f(Y) \leq \sum_{k=1}^{n} Z_k^* Y_k$$
$$= \sum_{k=1}^{n} (C_1^* \alpha_1^k + C_2^* \alpha_2^k + \ldots + C_m^* \alpha_m^k) y_k$$
$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{m} C_j^* \alpha_j^k \right) y_k$$
$$= \sum_{j=1}^{m} C_j^* \left(\sum_{k=1}^{n} \alpha_j^k y_k \right)$$
$$= \sum_{j=1}^{n} C_j^* x_j^* = f(X_B^*)$$

i.e., $$f(Y) \leq f(X_B^*)$$

Since Y is an arbitrary feasible solution, we have that X_B^* is the maximal *i.e.*, optimal solution. Hence, the theorem.

Similarly, we can prove the result when an LPP is a minimisation problem.

Now we shall prove that desired changes occur everywhere by row operation.

Theorem 2: Let the coordinate vector of column of entering variable x_k, *i.e.*, of A_k be

$$(\alpha_1^k, \alpha_2^k, \ldots, \alpha_m^k)^T = \alpha^k$$

and the matrix $B = [\beta_1, \beta_2, \ldots, \beta_{r-1}, \beta_r, \beta_{r+1}, \ldots, \beta_m]^T$

Also, let β_r leaves the basis, then the new basis matrix $\hat{B}$ is given by

$$\hat{B} = [\beta_1, \beta_2, \ldots, \beta_{r-1}, A_k, \beta_{r+1}, \ldots, \beta_m]^T$$

or $$\hat{B} E = B,$$

Where, $$E = [e_1, e_2, \ldots, e_{r-1}, \zeta, e_{r+1}, \ldots, e_m]^T,$$

$$\zeta = \left(-\frac{\alpha_1^k}{\alpha_r^k}, \frac{-\alpha_2^k}{\alpha_r^k}, \ldots, -\frac{\alpha_{r-1}^k}{\alpha_r^k}, \frac{1}{\alpha_r^k}, \frac{-\alpha_{r+1}^k}{\alpha_r^k}, \ldots, -\frac{\alpha_m^k}{\alpha_r^k}\right)^T$$

and, $\hat{B}^{-1} = EB^{-1}$

$$\hat{X}_B = EX_B$$

$$\hat{Z}_j - C_j = (Z_j - C_j) - \left(\frac{\alpha_r^j}{\alpha_r^k}\right) (Z_k - C_k).$$

Proof:

$$B = (\beta_1, \ldots, \beta_{r-1}, \beta_r, \beta_{r+1}, \ldots, \beta_m)$$

$$\hat{B} = (\beta_1, \ldots, \beta_{r-1}, A_k, \beta_{r+1}, \ldots, \beta_m)$$

The coordinate vector of the column of x_k be

$$(\alpha_1^k, \alpha_2^k, \ldots \alpha_m^k)^T$$

Then, $A_k = B\alpha^k$

$$= \alpha_1^k \beta_1 + \alpha_2^k \beta_2 + \ldots + \alpha_{r-1}^k \beta_{r-1} + \alpha_r^k \beta_r + \alpha_{r+1}^k \beta_{r+1} + \ldots + \alpha_m^k \beta_m$$

Since β_r leaves, $\alpha_r^k \neq 0$, we have

$$\beta_r = \left(-\frac{\alpha_1^k}{\alpha_r^k}\right) \beta_1 + \left(-\frac{\alpha_2^k}{\alpha_r^k}\right) \beta_2 + \ldots + \left(-\frac{\alpha_{r-1}^k}{\alpha_r^k}\right) \beta_{r-1} + \frac{1}{\alpha_r^k} A_k + \left(\frac{-\alpha_{r+1}^k}{\alpha_r^k}\right) \beta_{r+1} + \ldots + \left(-\frac{\alpha_m^k}{\alpha_r^k}\right) \beta_m.$$

Let $$\zeta = \left(-\frac{\alpha_1^k}{\alpha_r^k}, -\frac{\alpha_2^k}{\alpha_r^k}, \ldots, -\frac{\alpha_{r-1}^k}{\alpha_r^k}, \frac{1}{\alpha_r^k}, -\frac{\alpha_{r+1}^k}{\alpha_r^k}, \ldots, -\frac{\alpha_m^k}{\alpha_r^k}\right)^T$$

Then, $\beta_r = [\beta_1, \beta_2 \ldots \beta_{r-1} A_k \beta_{r+1} \ldots \beta_m] \zeta$

$= \hat{B} \zeta$

Let $E = [e_1\ e_2 \ldots e_{r-1}\ \zeta\ e_{r+1} \ldots e_m]$

Then, $\hat{B}E = [\hat{B}e_1, \hat{B}e_2 \ldots \hat{B}e_{r-1}\ \hat{B}\zeta\ \hat{B}e_{r+1} \ldots \hat{B}e_m]$

$= [\beta_1\ \beta_2 \ldots \beta_{r-1}\ \beta_r\ \beta_{r+1} \ldots \beta_m]$

$= B$

Hence, $\hat{B}^{-1} = EB^{-1}$.

Now, let $\hat{B} = [\hat{\beta}_1 \quad \hat{\beta}_2 \quad \ldots \quad \hat{\beta}_m]$

Where, $\hat{\beta}_i = \beta_i,\ i \neq r$

$$\hat{\beta}_r = A_k$$

$$A_j = \hat{\alpha}_1^j \hat{\beta}_1 + \hat{\alpha}_2^j \hat{\beta}_2 + \ldots + \hat{\alpha}_m^j \hat{\beta}_m$$

Also, $$A_j = \alpha_1^j \beta_1 + \alpha_2^j \beta_2 + \ldots + \alpha_m^j \beta_m$$

Substituting the value of β_r, we obtain

$$A_j = \sum_{i=1}^{r-1} \left(\alpha_i^j - \frac{\alpha_i^k \alpha_r^j}{\alpha_r^k} \right) \beta_i + \frac{\alpha_r^j}{\alpha_r^k} A_k + \sum_{i=r+1}^{m} \left(\alpha_i^j - \frac{\alpha_i^k \alpha_r^j}{\alpha_r^k} \right) \beta_i.$$

Since, $\beta_i = \hat{\beta}_i$, $i \neq r$, $A_k = \hat{\beta}_r$ and $\hat{\beta}_1, \ldots, \hat{\beta}_m$ form a basis, we obtain

$$\hat{\alpha}_i^j = \alpha_i^j - \frac{\alpha_i^k \alpha_r^j}{\alpha_r^k}, \; i \neq r$$

$$\hat{\alpha}_r^j = \alpha_r^j / \alpha_r^k.$$

This is what we get after row operation.

Further

$$X_B = (x_1, x_2, \ldots, x_m)^T$$

and, $$\hat{X}_{\hat{B}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m)^T$$

Then, $$\hat{X}_{\hat{B}} = \hat{B}^{-1} b = (E\, B^{-1})\, b = E\, (B^{-1} b) = EX_B.$$

It gives

$$\hat{x}_i = \left(0, 0, \ldots 1, 0, \ldots, \frac{-\alpha_i^k}{\alpha_r^k}, 0, \ldots, 0 \right) \begin{pmatrix} x_1 \\ \\ x_m \end{pmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ i\text{th place} & r\text{th place} \end{array}$$

$$= x_i - \frac{\alpha_i^k}{\alpha_r^k} x_r, \; i \neq r$$

and, $$\hat{x}_r = \frac{1}{\alpha_r^k} x_r$$

This is what we get after row-operation.

Now, let

$$\hat{C}_{\hat{B}} = (\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_m)^T$$

and, $$C_B = (C_1, C_2, \ldots, C_m)^T$$

Where, $$\hat{C}_i = C_i, \; i \neq r, \; \hat{C}_r = C_K.$$

Then, $$\hat{Z}_j - C_j = \hat{C}_{\hat{B}}^T \hat{\alpha}^j - C_j$$

$$= \hat{C}_1 \hat{\alpha}_1^j + \hat{C}_2 \hat{\alpha}_2^j + \ldots + \hat{C}_{r-1} \hat{\alpha}_{r-1}^j + \hat{C}_r \hat{\alpha}_r^j + \ldots + \hat{C}_m \hat{\alpha}_m^j - C_j$$

$$= C_1 \left(\alpha_1^j - \frac{\alpha_1^k \alpha_r^j}{\alpha_r^k}\right) + \ldots + (C_{r-1}) \left(\alpha_{r-1}^j - \frac{\alpha_{r-1}^k \alpha_r^j}{\alpha_r^k}\right)$$

$$+ C_k \frac{\alpha_r^j}{\alpha_r^k} + C_{r+1} \left(\alpha_{r+1}^j - \alpha_{r+1}^k \frac{\alpha_r^j}{\alpha_r^k}\right) + \ldots + C_m \left(\alpha_m^j - \frac{\alpha_m^k \alpha_r^j}{\alpha_r^k}\right) - C_j$$

Since $C_r \left(\alpha_r^j - \dfrac{\alpha_r^k \alpha_r^j}{\alpha_r^k}\right) = 0$, we can write it as

$$\hat{Z}_j - C_j = \sum_{i=1}^{m} C_i \left(\alpha_i^j - \frac{\alpha_i^k \alpha_r^j}{\alpha_r^k}\right) + C_k \frac{\alpha_r^j}{\alpha_r^j} - C_j$$

$$= \sum_{i=1}^{m} C_i \alpha_i^j - C_j - \left[\frac{\alpha_r^j}{\alpha_r^k}\left(\sum_{i=1}^{m} C_i \alpha_i^k - C_k\right)\right]$$

$$= (Z_j - C_j) - \frac{\alpha_r^j}{\alpha_r^k} (Z_k - C_k)$$

This is what we would get after row-operations.

2.11 EXCEPTIONAL CASES

There are some exceptional cases which arise in simplex method of solving LPP. We cite four of them here.

(i) ***Non-existing Feasible solution:*** This means S_F is empty. In the case discussed above, the starting BFS is given by slack variables, *i.e.*, constraints are all '≤' type. In this case S_F can never be empty at $x_1 = x_2 = \ldots = x_n = 0$ shall always satisfy the constraints. So in this case it will never occur.

It may occur when there is surplus variable, *i.e.*, some constraint is '≥' type, which we shall discuss later.

(ii) ***Degeneracy:*** A BFS is called degenerate if any of the basic variables is zero otherwise it is called non-degenerate solution.

If at any step of simplex iteration, there is a tie between leaving variables, and we select one to leave arbitrarily, the other variables having tie become zero after iteration and that BFS becomes degenerate.

In this situation, three cases arise,

(a) ***A degenerate optimal solution may exist.*** This means that optimal solution may itself turn out to be a degenerate solution.

(b) ***Degeneracy may disappear at some step and optimal solution may be a non-degenerate one.*** This means that after a few step, the BFS may again become non-degenerate and finally the optimal solution may be a non-degenerate solution. In this case it is called TEMPORARY DEGENERACY.

(c) ***Cycling may occur.*** This means that after a particular step, leaving and entering variables may go on changing *i.e.*, same BFS may go on appearing again and again and optimality is not reached. In such situation we say cycling has appeared.

In case of cycling, the method discussed above never leads to the optimal solution. There are methods to break the cycling, which we shall discuss later.

(iii) ***Unbounded Solutions:*** Unbounded solution may occur when S_F is unbounded. It does not mean that when S_F is unbounded, solution will be unbounded.

This situation is identified in simplex table by the following method. The optimal solution will be unbounded if below a non-basic variable all entries are ≤ 0, and $Z_i - C_i$ is

(a) negative in a maximisation problem, and

(b) positive in a minimisation problem. (proof is omitted)

(iv) ***Alternative Optimal Solution*:** This means that optimal solution may occur at more than one vertex, *i.e.*, more than one BFS may be the optimal solution.

This situation arises when the value of $Z_i - C_i$ of a non-basic variable in the optimal table becomes zero.

In this situation, this non-basic variable, if forced to enter the basis then the solution, *i.e.*, BFS changes but the optimal value remains unchanged.

Now we take one example to illustrate the simplex iteration, once again.

Example 1:

Min $Z = 2x_1 + 3x_2$

Subject to
$$-x_1 + 2x_2 \leq 2$$
$$2x_1 - x_2 \leq 2$$
$$-x_1 - x_2 \leq 2$$

x_1, x_2 unrestricted in sign but bounded.

Solution:

Put
$$x_1 = x_1^+ - x_1^-$$
$$x_2 = x_2^+ - x_2^-$$

and write the LPP in standard form

Min
$$Z = 2x_1^+ + 3x_2^+ - 2x_1^- - 3x_2^-$$

Subject to
$$-x_1^+ + 2x_2^+ + x_1^- - 2x_2^- + s_1 = 2$$
$$2x_1^+ - x_2^+ - 2x_1^- + x_2^- + s_2 = 2$$
$$-x_1^+ - x_2^+ + x_1^- + x_2^- + s_3 = 2$$
$$x_1^+, x_2^+, x_1^-, x_2^- \geq 0.$$

Starting table

Basic variable	x_1^+	x_2^+	x_1^-	x_2^-	s_1	s_2	s_3	Solution
z	–2	–3	2	3↓	0	0	0	0
s_1	–1	2	1	–2	1	0	0	2
$\leftarrow s_2$	2	–1	–2	$\boxed{1}$	0	1	0	2
s_3	–1	–1	1	1	0	0	1	2

Basic	x_1^+	x_2^+	x_1^-↓	x_2^-	s_1	s_2	s_3	Solution
z	–8	0	8	0	0	–3	0	–6
s_1	3	0	–3	0	1	2	0	6
x_2^-	2	–1	–2	1	0	1	0	2
s_3	–3	0	$\boxed{3}$	0	0	–1	1	0
						(a degenerate solution)		
z	0	0	0	0	0	–1/3	–8/3	–6
s_1	0	0	0	0	1	1	1	6
x_2^-	0	–1	0	1	0	1/3	2/3	2
x_1^-	–1	0	1	0	0	–1/3	1/3	0
						(It is an optimal Table)		

Solution: $s_1 = 6, x_2^- = 2, x_1^- = 0, x_1^+ = x_2^+ = s_2 = s_3 = 0$

or, $x_1 = 0, x_2 = -2$

Optimal value = –6

Non-basic variables x_1^+, x_2^+ have $z_i - c_i = 0$. It amounts that it has alternate solution if these are forced to enter the basis, then there is nothing to leave, and the solution will become non-feasible if x_2^+ enters but if we force x_1^+ to enter, we get the following table:

z	0	0	0	0	0	1/3	–8/3	–6
s_1	0	0	0	0	1	1	1	6
x_2^-	0	–1	0	1	0	1/3	2/3	2
x_1^+	1	0	–1	0	0	+1/3	–1/3	0

Here it is again an optimal table.

Solutions are $x_1^+ = 0, x_2^- = 2, s_1 = 6, x_1^- = x_2^+ = s_2 = s_3 = 0$

or $x_1 = 0, x_2 = -2$ and optimal value is –6. Final BFS in terms of x_1, x_2 is same but in terms of $x_1^+, \ldots$ etc., we have

(0, 0, 0, 2, 6, 0, 0) and (0, 0, 0, 2, 6, 0, 0)

Though they are same but in one case x_1^+ is zero as basic variable and $x_1^- = 0$ as non-basic variable while in other case $x_1^+ = 0$ as non-basic and $x_1^- = 0$ as basic.

EXERCISE 2.6

1. Consider the following LPP.

   Maximize $Z = x_1 - x_2 + 2x_3$

   Subject to
$$x_1 + x_2 + 3x_3 \le 15$$
$$2x_1 - x_2 + x_3 \le 2$$
$$-x_1 + x_2 + x_3 \le 4$$
$$x_1, x_2, x_3 \ge 0$$

   One of the simplex iteration table is

Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	Soln.
z	1				0	3/2	$\frac{1}{2}$	
s_1	0			0			-2	
x_3	0			1			$\frac{1}{2}$	
x_2	0			0			$\frac{1}{2}$	

   (Where s_1, s_2, s_3 are slack variables)

   Without performing simplex iterations, find the following missing entries in the above table.

   (a) s_1 – column

   (b) x_1 – column

   (c) Entry below x_2 in z-row.

   (d) The BFS corresponding to above table

   [**Ans:** (a); $(1, 0, 0)^T$, (b); $\left(1, \frac{1}{2}, -3/2\right)^T$ (c); 0 (d); $(s_1, x_3, x_2)^T$ $= (21, 3, 1)^T$]

2. Use simplex method to solve

   Maximize $Z = 8x_1 + 9x_2$

   Subject to
$$2x_1 + 3x_2 \le 50$$
$$2x_1 + 6x_2 \le 80$$
$$3x_1 + 3x_2 \le 70$$
$$x_1, x_2 \ge 0$$

   $\left(\textbf{Ans:}\ x_1 = 20, x_2 = \frac{10}{3}, Z = 190\right)$

3. Use simplex method to solve the following LPP

   Maximize $Z = 3x_1 + 5x_2$

   Subject to
$$x_1 + 2x_2 \leq 2000$$
$$x_1 + x_2 \leq 1500$$
$$x_2 \leq 600$$
$$x_1, x_2 \geq 0.$$

(**Ans:** $x_1 = 1000$, $x_2 = 500$, $Z = 5500$)

4. Consider the following LPP

   Minimize $Z = x_1 - 3x_2 + 2x_3$

   Subject to
$$3x_1 - x_2 + 2x_3 \leq 7$$
$$-2x_1 + 4x_2 \leq 12$$
$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

   one of the simplex iteration table is

Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	1	0	0		$\frac{-1}{5}$	$\frac{-4}{5}$		
x_1	0					$\frac{1}{10}$		
x_2	0					$\frac{3}{10}$		
s_3	0					$-\frac{1}{2}$		

   (Where s_1, s_2, s_3 are slack variables). Without performing simplex iterations, find the following missing entries in the above table.

   (a) x_2 – column
   (b) s_3 – column
   (c) Entry below x_3 in z-row
   (d) The BFS corresponding to above table
   (e) The value of objective function corresponding to above BFS obtained in (d).

$$\left[\textbf{Ans:}\ \text{(a);} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{(b);} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{(c); 0, (d);} \begin{pmatrix} x_1 \\ x_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 11 \end{pmatrix} Z = -11 \right]$$

5. Use simplex method to
   (a) Solve the following LPP.

   Maximize $Z = 10x_1 + 15x_2 + 20x_3$

   Subject to $x_1 + 2x_2 + 3x_3 \le 12$

   $x_1 + 3x_2 + 2x_3 \le 10$

   $x_1, x_2, x_3 \ge 0$

   (b) Does above problem has a degenerate soln.

   (**Ans:** (a) $(x_1, x_2, x_3)^T = (6, 0, 2)$, $Z = 100$) (b) Yes.)

6. Solve the following LPP using simplex method:

   Minimize $Z = -4x_1 + x_2$

   Subject to $x_1 - 2x_2 \le 2$

   $-2x_1 + x_2 \le 2$

   $x_1, x_2 \ge 0$

   (**Ans:** unbounded solution)

7. Use simplex method to solve the following LPP

   Maximize $Z = 6x_1 - 3x_2$

   Subject to $x_1 + x_2 \le 1$

   $2x_1 - x_2 \le 1$

   $-x_1 + 2x_2 \le 1$

   $x_1, x_2 \ge 0$

   Does the above problem has any alternate solution. If yes, find all the possible solutions of the problem.

   (**Ans:** $x_1 = 1/2$, $x_2 = 0$, $z = 3$; $x_1 = 2/3$, $x_2 = 1/3$, $z = 3$

   $X = (x_1, x_2, s_1, s_2, s_3)^T = \lambda\alpha_1 + (1 - \lambda)\,\alpha_2$, $0 \le \lambda \le 1$)

   $$\left(\text{Where } \alpha_1 = \left(\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{3}{2}\right) \text{ \& } x_2 = \left(\frac{2}{3}, \frac{1}{3}, 0, 0, 1\right)^T\right)$$

8. Show that for a basic variable $z_j - c_j = 0$
9. Use simple method to solve the following LPP.

   Maximize $Z = 5x_1 + 4x_2$

   Subject to $x_1 \le 7$

   $x_1 - x_2 \le 8$

   $x_1, x_2 \ge 0$

   (**Ans:** Unbounded soln.)

10. Solve the following LPP by simplex method.

    Maximize $Z = 3x_1 + 9x_2$

    Subject to $x_1 + 4x_2 \le 8$

$$x_1 + 2x_2 \le 4$$
$$x_1, x_2 \ge 0$$

What type of solutions we get?

(Degenerate soln. $x_1 = 0$, $x_2 = 2$, $Z = 18$)

11. Maximize $Z = 2x_1 + 4x_2$

Subject to $x_1 + 2x_2 \le 5$

$$x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$

Does this problem has alternate solution, if yes find all the alternate solutions?

(Yes, (a) $x_1 = 0$, $x_2 = 5/2$, $Z = 10$

(b) $x_1 = 3$, $x_2 = 1$, $Z = 10$

$\alpha_1^* = \lambda\ (0) + (1 - \lambda)\ 3 = 3 - 3\lambda$

$x_2^* = \lambda\ (3) + (1 - \lambda)\ 1 = 2\ \lambda + 1,\ 0 \le \lambda \le 1$)

2.12 ARTIFICIAL VARIABLE METHOD

2.12.1 Introduction

In simplex method, we have seen that the presence of an identity matrix is required. So far we have considered only those cases in which identity matrix is present. But it is not always possible. What is to be done in such cases?

One method is to find any non-singular matrix B and proceed as described in the earlier section. But it would be quite cumbersome. So we adopt another method, known as Artificial Variable Method.

2.12.2 BIG M-Method

As we know, normally, the identity matrix is present because of slack variables. If, instead of slack variables, surplus variables are present in some constraints, then identity matrix will not be present.

Therefore, we insert artificial variable R_i with plus sign in each of the constraint having surplus variable. 'i' is the number of constraint in which it was added.

Also, in case of maximisation problem, we add ($-MR_i$) in the objective function corresponding to each artificial variable introduced in the constraints. M is a big positive number. Bigness is defined as biggest so that all numbers involved in steps are smaller than M. And in case of minimisation problem we add ($+MR_i$).

This is done in order to achieve the following:

It is quite evident that since it is an artificial variable, its value must come out to be zero. As, if it is not so, then it means that we are changing the constraints, *i.e.*, the problem. In maximisation problem, since MR_i present in the objective function, with M a big positive number, it would be possible only if $R_i = 0$ otherwise it cannot be maximised as M is arbitrary. Thus, M is to be taken arbitrarily big. For example, let the LPP be

$$\text{Max. } Z = f(X) = x_1 + 2x_2 + 3x_3$$

$$\text{Subject to} \quad \begin{aligned} x_1 - x_2 + 5x_3 &\le 4 \\ -2x_1 - 3x_2 - x_3 &\le -5 \\ x_1 + x_2 + x_3 &\ge -3 \\ 5x_1 + 7x_2 + 11x_3 &\ge 26 \\ x_1, x_2, x_3 &\ge 0. \end{aligned}$$

Writing it in standard form, we get

$$\text{Max } Z = f(X) = x_1 + 2x_2 + 3x_3$$

$$\text{Subject to} \quad \begin{aligned} x_1 - x_2 + 5x_3 + s_1 &= 4 \\ 2x_1 + 3x_2 + x_3 - s_2 &= 5 \\ -x_1 - x_2 - x_3 + s_3 &= 3 \\ 5x_1 + 7x_2 + 11x_3 - s_4 &= 26 \\ x_1, x_2, x_3, x_4, s_1, s_2, s_3, s_4 &\ge 0 \end{aligned}$$

Since, 2nd and 4th constraints do not permit to get identity matrix, we add artificial variables R_2 and R_4 in 2nd and 4th equation, respectively and $-MR_2 - MR_4$ in the objective function. So now, we get

$$\text{Max } Z = f(X) = x_1 + 2x_2 + 3x_3 - MR_2 - MR_4$$

$$\text{Subject to} \quad \begin{aligned} x_1 - x_2 + 5x_3 + s_1 &= 4 \\ 2x_1 + 3x_2 + x_3 - s_2 + R_2 &= 5 \\ -x_1 - x_2 - x_3 + s_3 &= 3 \\ 5x_1 + 7x_2 + 11x_3 - s_4 + R_4 &= 26 \\ x_1, x_2, x_3, s_1, s_2, s_3, s_4, R_2, R_4 &\ge 0 \end{aligned}$$

If it is minimisation problem, the objective function would be

$$\text{Min. } Z = f(X) = x_1 + 2x_3 + 3x_3 + MR_2 + MR_4.$$

Now we apply simplex method and get the solution. Since R'_is are artificial variables, its values should come out to be zero. If optimal solution is obtained with at least one $R_i \neq 0$, it means that it is not a solution of the problem, rather we say that the LPP has no solution.

Example 1:

Min $\quad Z = f(X) = 2x_1 + x_2$

Subject to $\quad 3x_1 + x_2 \ge 9$

$\quad x_1 + x_2 \le 1$

Solution: $\quad x_1, x_2 \ge 0$

This problem in standard form with artificial variable, is

Min $\qquad Z = f(X) = 2x_1 + x_2 + MR_1$

$$3x_1 + x_2 - s_1 + R_1 = 9$$
$$x_1 + x_2 + s_2 = 1$$
$$x_1, x_2, R_1, s_1, s_2 \geq 0$$

In order to apply simplex, we have to first free the objective function from basic variables. In this case initial basic variables are R_1 and s_2. So we have free objective function from R_1. This can be done by replacing R_1 by $9 - 3x_1 - x_2 + s_1$ from 1st constraint.

This can also be achieved by the following process. Write the simplex table as mentioned earlier. Then obviously below basic variable, some of $Z_i - C_i \neq 0$. Make them zero by row operations. That means make $Z_i - C_i$ below each basic variable as zero by row operations. Then only simplex method can start. So first we get starting table and after the above operations, we get initial table, to which simplex method can be applied.

B.V.	x_1	x_2	s_1	R_1	s_2	Solution	
Z	-2	-1	0	$-M$	0	0	
R_1	3	1	-1	1	0	9	Starting
s_2	1	1	0	0	1	1	Table
Z	$-2 + 3M$	$-1 + M$	$-M$	0	0	$9M$	
R_1	3	1	-1	1	0	9	Initial
s_2	$\boxed{1}$	1	0	0	1	1	Table
Z	0	$1 - 2M$	$-M$	0	$2 - 3M$		$2 + 6M$
R_1	0	-2	-1	1	-3	6	
x_1	1	1	0	0	1	1	

It is the optimal table but $R_1 \neq 0$, which clearly means that this LPP has no solution. Let us now take another example.

Example 2:

Max $Z = -x_1 + 3x_2$

Subject to
$$x_1 + 2x_2 \geq 2$$
$$3x_1 + x_2 \leq 3$$
$$x_1 \leq 4;\ x_1, x_2, x_3 \geq 0$$

Solution:

Max $Z = -x_1 + 3x_2 - MR_1$

$$x_1 + 2x_2 - s_1 + R_1 = 2$$
$$3x_1 + x_2 + s_2 = 3$$
$$x_1 + s_3 = 4$$
$$x_1, x_2, x_3, s_1, s_2, s_3, R_1 \geq 0$$

Basic	x_1	x_2	s_1	R_1	s_2	s_3	Solution
Z	1	–3	0	M	0	0	0
R_1	1	2	–1	1	0	0	2
s_2	3	1	0	0	1	0	3
s_3	1	0	0	0	0	1	4
z	$1 - M$	$-3 - 2M$	M	0	0	0	$-2M$
R_1	1	$\boxed{2}$	–1	1	0	0	2
s_2	3	1	0	0	1	0	3
s_3	1	0	0	0	0	1	4
z	5/2	0	–3/2	3/2 + M	0	0	3
x_2	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	1
s_2	$\frac{5}{2}$	0	$\boxed{\frac{1}{2}}$	$-\frac{1}{2}$	1	0	2
s_3	1	0	0	0	0	1	4
z	10	0	0	M	3	0	9
x_2	3	1	0	0	1	0	3
s_1	5	0	1	–1	2	0	4
s_3	1	0	0	0	0	1	4

$\therefore$ Solution is $x_1 = 0$, $x_2 = 3$, $f(x) = 9$

There is another method of solving such problems involving artificial variables. In this method we solve the problem in two steps. Therefore, this method is known as two-phase methods.

2.12.3 Two-Phase Method

In this method, we introduce artificial variables as above. But the objective function is taken different in phase I. It is done to avoid 'M' which is big enough. Since bigness of M cannot be ascertained, the big M-method is not workable on computers.

In phase I, we solve the following LPP:

$$\text{Min } R = \sum R_i$$

Subject to constraints and non-negativity conditions.

Since $R_i \geq 0$, it is a must that each R_i comes out to be zero as a solution of the above Phase I problem. If it is not so, then it amounts that we have to go out of S_F. Hence, the problem would have no solution. The construction of phase I is to guarantee the existence of a BFS (vertex) of S_F.

After getting the solution from Phase I, we superimpose it on the given objective function of the LPP and apply simplex iteration to get the optimal solution.

Example 3:

Min $Z = f(X) = 2x_1 + x_2$

Subject to $3x_1 + x_2 \geq 9$

$x_1 + x_2 \leq 1$

$x_1, x_2 \geq 0$

Solution:

Phase-I

Min $R = R_1$

$$3x_1 + x_2 - s_1 + R_1 = 9$$
$$x_1 + x_2 + s_2 = 1$$
$$x_1, x_2, s_1, s_2, R_1 \geq 0.$$

Basic	x_1	x_2	s_1	R_1	s_2	Solution
R	0	0	0	–1	0	0
R_1	3	1	–1	1	0	9
s_2	1	1	0	0	1	1
R	3	1	–1	0	0	9
R_1	3	1	–1	1	0	9
s_2	$\boxed{1}$	1	0	0	1	1
R	0	–02	–01	0	–03	6
R_1	0	–2	–1	1	–3	6
x_1	1	1	0	0	1	1

It is optimal table, but $R_1 = 6 \neq 0$.
Hence, LPP has no solution. Actually there is no feasible solution.

Example 4:

Max $Z = -x_1 + 3x_2$

Subject to $x_1 + 2x_2 \geq 2$

$3x_1 + x_2 \leq 3$

$x_1 \leq 4;\ x_1, x_2, x_3 \geq 0$

Phase I

Min $Z = R_1$

Subject to $x_1 + 2x_2 - s_1 + R_1 = 2$

$3x_1 + x_2 + s_2 = 3$

$x_1 + s_3 = 4$

$x_1, x_2, x_3, s_1, s_2, s_3, R_1 \geq 0$

Basic	x_1	x_2	s_1	R_1	s_2	s_3	Solution
R	0	0	0	-1	0	0	0
R_1	1	2	-1	1	0	0	2
s_2	3	1	0	0	1	0	3
s_3	1	0	0	0	0	1	4
R	1	2	-1	0	0	0	2
R_1	1	$\boxed{2}$	-1	1	0	0	2
s_2	3	1	0	0	1	0	3
s_3	1	0	0	0	0	1	4
R	0	0	0	-1	0	0	0
x_2	1/2	1	$-1/2$	1/2	0	0	1
s_2	5/2	0	1/2	$-1/2$	1	0	2
s_3	1	0	0	0	0	1	4

It is optimal table of Phase I which has all $R_i = 0$. Thus, Phase I has a solution in S_F.

We now move to Phase II by taking BFS of Phase I with the given objective function. Thus, starting table is by removing column of R_i's.

Phase II

Basic	x_1	x_2	s_1	s_2	s_3	Solution
Z	1	-3	0	0	0	0
x_2	1/2	1	$-1/2$	0	0	1
s_2	5/2	0	1/2	1	0	2
s_3	1	0	0	0	1	4
Z	5/2	0	$-3/2$	0	0	3
x_2	1/2	1	$-1/2$	0	0	1
s_2	5/2	0	$\boxed{1/2}$	1	0	2
s_3	1	0	0	0	1	4
Z	10	0	0	3	0	9
x_2	3	1	0	1	0	3
s_1	5	0	1	2	0	4
s_3	1	0	0	0	1	4

It is optimal table of Phase II. Thus, we have optimal solution of the LPP.

$$x_1 = 0,\ x_2 = 3 \text{ and Max } Z = f(X) = 9.$$

Note: There are some important points to note.

1. Artificial variable is never considered for reentry into the basic. The reasons are obvious.
2. If all R_i's are not zero then problem has no solution (no feasible solution).

3. If original problem has a solution, big M-method or Phase I method would yield a solution with all R_i's equal to zero.
4. In Phase I, it is always a minimisation problem.
5. The redundancy in the system is possible only if artificial variables are present, as we are assuming that A has the identity matrix as submatrix. For, suppose at any stage R_K appears as ith basic variable then, in the column of R_K, except the ith entry, other entries are zero. If $R_K = 0$, and $\alpha_i^k = 0$ for all non-basic variables, except for $\alpha_i^i = 1$. Hence, all columns are expressible as a linear combination of the remaining basic variables. Thus, the rank of $A = m - 1$ which shows that one row is redundant.

We have already discussed the exceptional cases. The situation of non-existence of feasible solution occurs in case of artificial variable techniques. Here, whenever all R_i's are not zero, it amounts that the problem has no feasible solution.

EXERCISE 2.7

1. Use Big-M method to solve the following LPP.

   Minimize $Z = 3x_1 + 2x_2$

   Subject to
   $$\begin{aligned} x_1 + x_2 &\geq 2 \\ x_1 + 3x_2 &\leq 3 \\ x_1 - x_2 &= 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

   (**Ans:** $x_1 = 3/2$, $x_2 = 1/2$, $Z = 11/2$)
2. Solve exercise 1 by two-phase method.
3. Solve the following LPP by two-phase method

   Maximize $Z = 3x_1 - x_2$

   Subject to
   $$\begin{aligned} 2x_1 + x_2 &\geq 2 \\ x_1 + 3x_2 &\leq 3 \\ x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

   (**Ans:** $x_1 = 3$, $x_2 = 0$, $Z = 9$)
4. Use Big-M method to solve the following LPP

   Maximize $Z = 6x_2$

   Subject to
   $$\begin{aligned} x_1 - x_2 &\leq 0 \\ 2x_1 + 3x_2 &\leq -6 \end{aligned}$$

   x_1, x_2 are unrestricted in sign

   $$\left(\textbf{Ans: } x_1 = x_2 = -\frac{6}{5}, Z = -\frac{36}{5}\right)$$
5. Solve exercise 4 by two-phase method.
6. Solve the following problem using Big-M and two-phase methods.

   Maximize $Z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \le 2$

$$3x_1 + 4x_2 \ge 12$$
$$x_1, x_2 \ge 0$$

(**Ans:** No feasible solution)

7. Show that the following LPP has unbounded solution. (Use Big-M method)

   Maximize $Z = 107x_1 + x_2 + 2x_3$

   Subject to $14x_1 + x_2 - 6x_3 + 3x_4 = 8$

$$16x_1 + \frac{x_2}{2} - 6x_3 \le 6$$
$$3x_1 - x_2 - x_3 \le 0$$
$$x_1, x_2, x_3, x_4 \ge 0$$

8. Using Big-M method to solve the following LPP, show that the problem has alternate solutions.

   Maximize $Z = x_1 + x_2 + x_4$

   Subject to $x_1 + x_2 + x_3 + x_4 = 4$

$$x_1 + 2x_2 + x_3 + x_5 = 4$$
$$x_1 + 2x_2 + x_3 = 4$$
$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

9. Consider the following LPP

   Maximize $Z = 6x_1 + 4x_2$

   Subject to $2x_1 + 3x_2 \le 30$

$$3x_1 + 2x_2 \le 24$$
$$x_1 + x_2 \ge 3$$
$$x_1, x_2 \ge 0$$

   Solve the above problem by Big-M method and show that this has infinite solutions.

10. Show that the following LPP has no feasible solution

    Minimize $Z = x_1 + x_2$

    Subject to $2x_1 + x_2 \ge 2$

$$-x_1 - x_2 \ge 1$$
$$x_1, x_2 \ge 0$$

Duality in Linear Programming

3.1 INTRODUCTION

It can be seen as the number of constraints increases, the calculations, number of BFS and the number of steps increases. In such cases, we use the concept of duality. The concept of duality which would be explained in the articles below not only helps in above cases alone but have many more other advantages.

3.2 CANONICAL FORM OF AN LPP

An LPP is said to be in canonical form, if

(a) it is a minimisation problem.
(b) all constraints are '≥' type (no restriction on b_i's).
(c) all decision variables are non-negative.

Thus, the LPP

Min $\quad Z = f(X) = C^T X$

Subject to $\quad AX \geq b,\ X \geq 0$

is in canonical form.

We know that,

(a) Max $Z = f(X) = C^T X$ can be converted into
Min $Z = f_1(X) = -C^T X$

(b) $\alpha_{i1}\, x_1 + \alpha_{i2}\, x_2 + \ldots + \alpha_{in}\, x_n \leq b_i$ can be converted into
$-\alpha_{i1}\, x_1 - \alpha_{i2}\, x_2 \ldots - \alpha_{in}\, x_n \geq -b_i$,
and

(c) $\alpha_{i1}\, x_1 + \alpha_{i2}\, x_2 + \ldots + \alpha_{in}\, x_n = b_i$
can be replaced by two constraints,
$\alpha_{i1}\, x_1 + \alpha_{i2}\, x_2 + \ldots + \alpha_{in}\, x_n \geq b_i$, and
$\alpha_{i1}\, x_1 + \alpha_{i2}\, x_2 + \ldots + \alpha_{in}\, x_n \leq b_i$, *i.e.*,
$-\alpha_{i1}\, x_1 - \alpha_{i2}\, x_2 \ldots + \alpha_{in}\, x_n \geq -b_i$.

This suggests that every LPP can be transformed into another LPP in canonical form.

Example 1: Transform the following LPP in canonical form.

Max $Z = 3x_1 - 2x_2 + 4x_3$

Subject to
$$x_1 + x_2 + x_3 \le 230$$
$$2x_1 + 3x_2 - 4x_3 \ge 170$$
$$x_1 - x_2 - x_3 = 50$$
$$x_1, x_2, x_3 \ge 0.$$

Solution: Its canonical form is

Min $Z = -3x_1 + 2x_2 - 4x_3$

Subject to
$$-x_1 - x_2 - x_3 \ge -230$$
$$2x_1 + 3x_2 - 4x_3 \ge 170$$
$$x_1 - x_2 - x_3 \ge 50$$
$$-x_1 + x_2 + x_3 \ge -50$$
$$x_1, x_2, x_3 \ge 0$$

3.3 DUAL OF AN LPP

As seen above, every LPP can be transformed into canonical form. We now define dual of an LPP.

Definition: Let an LPP be in canonical form

$$\left.\begin{array}{ll} \text{Min} & Z = f(X) = C^T X \\ \text{Subject to} & AX \ge b,\ X \ge 0 \end{array}\right] *$$

The dual of (*) is the LPP

$$\left.\begin{array}{ll} \text{Max} & W = \phi(Y) = b^T Y \\ \text{Subject to} & A^T Y \le C, Y \ge 0 \end{array}\right] **$$
$$Y = (y_1, y_2, \ldots, y_m)^T$$

(**) is called dual of (*) and (*) is called primal.

We notice that dual is a maximisation problem obtained from a minimisation problem in canonical form. Objective function is obtained by multiplying b^T with $Y = (y_1, y_2, \ldots, y_m)^T$ (m is the number of constraints) and constraints are $A^T y \le C$. Thus if the primal has n variables and m constraints, its dual will have m variables and n constraints.

Example 2: Formulate the dual of the LPP

Max $Z = 5x_1 + 6x_2$

Subject to
$$x_1 + 9x_2 \ge 60$$
$$2x_1 + 3x_2 \le 45$$
$$x_1, x_2 \ge 0.$$

Solution: Its canonical form is

Min $\quad Z = -5x_1 - 6x_2$

Subject to
$$x_1 + 9x_2 \ge 60 \quad y_1$$
$$-2x_1 - 3x_2 \ge -45 \ y_2$$
$$x_1, x_2 \ge 0$$

Its dual is

Max $W = 60y_1 - 45y_2$

Subject to
$$y_1 - 2y_2 \le -5$$
$$9y_1 - 3y_2 \le -6$$
$$y_1, y_2 \ge 0$$

We now prove an interesting result, which says dual of a dual is primal.

Theorem 1: The dual of a dual is primal.

Proof: Let the primal (in canonical form) be

Min $$Z = f(X) = C^T X \qquad (*)$$
$$AX \ge b,\ X \ge 0;\ \text{no restriction on } b.$$

Dual is

Max $$W = \phi(Y) = b^T Y$$
$$A^T Y \le C,\ Y \ge 0 \qquad (**)$$

To write dual of dual, we first write canonical form of (**)

Min $$-\phi(Y) = -b^T Y$$
$$-A^T Y \ge -C,\ Y \ge 0 \qquad (***)$$

Dual of dual *i.e.*, dual of (***) is

Max $$V = \psi(Z) = (-C)^T Z \qquad (****)$$
$$(-A^T)^T Z \le -b^T,\ Z \ge 0$$

i.e., $$\text{Min } -\psi(Z) = C^T Z$$
$$AZ \ge b,\ Z \ge 0.$$

Which is same as primal (*) except for the variable notation Z. Hence, the result.

Thus, we find that

If primal is (*) its dual is (**), and if primal is taken to be (**), its dual is (*). In other words, if primal is in the form

(a) Minimization problem, constraints '≥' type or (b) Maximization problem, constraints '≤' type then dual of (a) type primal is (b) type and dual of (b) type primal is (a) type.

This suggests that dual can be directly written without converting it into canonical form.

Example 3: Write the dual of

Max $Z = 5x_1 + 6x_2$

Subject to
$$x_1 + 9x_2 \ge 60$$
$$2x_1 + 3x_2 \le 45,\ x_1, x_2 \ge 0$$

or,
$$-x_1 - 9x_2 \le -60 \quad y_1$$
$$2x_1 + 3x_2 \le 45 \quad y_2$$
$$x_1, x_2 \ge 0$$

Solution: Dual is Min $W = -60y_1 + 45y_2$
$$-y_1 + 2y_2 \ge 5$$
$$-9y_1 + 3y_2 \ge 6$$
$$y_1, y_2 \ge 0$$

Let us now take an example, in which there is a constraint with '=' sign.

Example 4: Write the dual of

Min $Z = 2x_1 + 3x_2 + 4x_3$

Subject to

$$
\begin{aligned}
2x_1 + 2x_2 + 3x_3 &\le 4 \\
3x_1 + 4x_2 + 5x_3 &\ge 5 \\
x_1 + x_2 + x_3 &= 7 \\
x_i &\ge 0 \quad i = 1, 2, 3
\end{aligned}
$$

Solution: Its canonical form is

Min $Z = 2x_1 + 3x_2 + 4x_3$

$$
\begin{aligned}
-2x_1 - 2x_2 - 3x_3 &\ge -4 \quad y_1 \\
3x_1 + 4x_2 + 5x_3 &\ge 5 \quad y_2 \\
x_1 + x_2 + x_3 &\ge 7 \quad y_3 \\
-x_1 - x_2 - x_3 &\ge -7 \quad y_4 \\
x_i &\ge 0
\end{aligned}
$$

So the dual is

Max $W = -4y_1 + 5y_2 + 7y_3 - 7y_4$

$$
\begin{aligned}
-2y_1 + 3y_2 + y_3 - y_4 &\le 2 \\
-2y_1 + 4y_2 + y_3 - y_4 &\le 3 \\
-3y_1 + 5y_2 + y_3 - y_4 &\le 4 \\
y_i &\ge 0
\end{aligned}
$$

In this dual problem, we find that $y_3 - y_4$ appears as one entity, so if we replace $y_3 - y_4$ by y_5, we obtain

Max $W = -4y_1 + 5y_2 + 7y_5$

$$
\begin{aligned}
-2y_1 + 3y_2 + y_5 &\le 2 \\
-2y_1 + 4y_2 + y_5 &\le 3 \\
-3y_1 + 5y_2 + y_5 &\le 4 \\
y_1 \ge 0,\ y_2 &\ge 0
\end{aligned}
$$

but y_5 is unrestricted as $y_3 - y_4 = y_5$ may have any sign though $y_3, y_4 \ge 0$. Thus, we find that dual can be easily written without changing '=' into '≤' and '≥' signs. Only thing we have to bear in mind that the dual variables associated with '=' constraints have to be kept unrestricted in sign.

Now we shall move towards establishing relations between the solutions of primal and dual. For our discussion, the primal is taken to be in canonical form. Let

Primal:

$$
P: \left[\begin{array}{l}
\text{Min } Z = f(X) = C^T X \\
\quad AX \ge b,\ X \ge 0, \text{ where} \\
\qquad X = (x_1, x_2, \ldots, x_n)^T; b = (b_1, b_2, \ldots, b_m)^T \\
\qquad C = (c_1, c_2, \ldots, c_n)^T; A = (\alpha_{ij})_{m \times n}
\end{array}\right.
$$

Dual:

$$D: \left[\begin{array}{l} \text{Max } W = \phi(Y) = b^T Y \\ \quad A^T Y \le C, Y \ge 0, \text{ where} \\ \qquad Y = (y_1, y_2, \dots, y_m)^T \end{array}\right.$$

There is no need of emphasising, as it is clear S_F in both the cases are different. Our first result is

Theorem 2: If X is a feasible solution of P, the primal and Y a feasible solution of D, its dual, then

$$f(X) \ge \phi(Y), \text{ i.e., } C^T X \ge b^T Y.$$

Proof: Since, X and Y are feasible solutions of P and D, respectively, we have

$$A^T Y \le C$$

and, $$AX \ge b$$

since, $Y \ge 0$, multiplying $AX \ge b$ by Y^T, we get

$$Y^T(AX) \ge Y^T b.$$

or, $$Y^T(AX) \ge b^T Y$$

as $Y^T b = b^T Y$. Matrix multiplication is associative. So,

$$(Y^T A)X \ge b^T Y$$

On taking transpose of $A^T Y \le C$, we get

$$Y^T A \le C^T$$

Therefore,

$$b^T Y \le (Y^T A)X \le C^T X$$

i.e., $$\phi(Y) \le f(X).$$

Hence, the result.

The above result can be interpreted as: at any feasible solutions, values of objective function of dual is less than or equal to the value of the primal.

The next result says that if the values of the objective functions are same, then these solutions are optimal solutions. To be precise:

Theorem 3: If X^* is a feasible solution of the primal P and Y^* a feasible solution of the dual D such that $f(X^*) = \phi(Y^*)$, then X^* and Y^* are optimal solutions of P and D, respectively.

Proof: Since, X^* Y^* are feasible solutions of P and D, we get

$$AX^* \ge b, \; A^T Y^* \le C$$

Also, we have $C^T X^* = b^T Y^*$.

Let X be any feasible solution of P.

Since, Y^* is a feasible solution of D, we have by an earlier theorem

$$b^T Y^* \le C^T X$$

or, $$C^T X \ge b^T Y^* = C^T X^*$$

i.e., $$C^T X \ge C^T X^* \;\forall\; X \in S_F$$

Since, P is a minimisation problem, X^* is an optimal solution. Similarly, let Y be any feasible solution of D. Since, X^* is a feasible solution of P, we have, by
an earlier result,

$$b^T Y \leq C^T X^* = b^T Y^* \ \forall \ Y \in S_F$$

Since, D is a maximisation problem, Y^* is an optimal solution. Hence, the theorem.

Thus we find that values of objective functions of P and D are same at some feasible solutions then they are respective optimal solutions and the value, the optimal value. But this result does not speak about the existence of the optimal solution. So, we have:

Theorem 4: If primal P has an optimal solution, then its dual D also has an optimal solution.

Proof: We shall prove this result by actually constructing the solution of dual on the assumption that primal has solution.

Let primal P be
Min $Z = f(X) = C^T X$

$$AX \geq b, \ X \geq 0$$

'no' restriction on b.

We break the proof in two parts. One $b \geq 0$, *i.e.*, all $b_i \geq 0$ and the other all $b_i \ngeq 0$.

Case-I $b \geq 0$ *i.e.*, all $b_i \geq 0$, $i = 1, 2, \ldots, m$.

Writing the LPP in standard form, we get, since $b_i \geq 0$, $i = 1, 2, \ldots, m$.

Min $$Z = f(X) = C^T X$$

and,

$$\alpha_{11} x_1 + \alpha_{12} x_2 + \ldots + \alpha_{1n} x_n - s_1 = b_1$$
$$\alpha_{21} x_1 + \alpha_{22} x_2 + \ldots + \alpha_{2n} x_n - s_2 = b_2$$
- -
$$\alpha_{m1} x_1 + \alpha_{m2} x_2 + \ldots + \alpha_{mn} x_n - s_m = b_m$$

Let X_B be the optimal solution of P. It implies that artificial variables are zero. So,

$$Z_i - C_i \leq 0 \ \forall i = 1, 2, \ldots, n, n + 1, n + 2, \ldots, n + m$$

as there are $n + m$ variables.

For columns of A, $A_1, A_2, \ldots, A_n$

$$Z_i - C_i = C_B^T \alpha^i - C_i \leq 0, \ i = 1, 2, \ldots, n.$$

or, $$C_B^T \alpha^i \leq C_i$$

or, $$C_B^T B^{-1} A_i \leq C_i \ \forall \ i = 1, 2, \ldots, n$$

or, $$C_B^T B^{-1} A \leq C^T$$

$C_B^T B^{-1}$ is a matrix of the size $1 \times m$. Let us denote it $Y^T = (y_1, y_2, \ldots, y_m)^T$. So

$$Y^T A \leq C^T$$

or, $$A^T Y \leq C$$

This shows Y defined as $(C_B^T B^{-1})^T$ is a solution of dual constraints. Let us now show that $Y \geq 0$ *i.e.*, it is a feasible solution of dual D.

Since, X_B is an optimal solution of P, for the surplus variables too,

$$Z_i - C_i \leq 0, \ i = 1, 2, \ldots, m$$

or, $$C_B^T\,\alpha^i \le C_i,\ i = 1, 2,\ldots, m$$
or, $$-C_B^T\,B^{-1}e_i \le C_i,\ i = 1, 2,\ldots, m$$
as column of $s_i = (0, 0, \ldots\, 0, \underset{\substack{\uparrow \\ i\text{th entry}}}{-1}, 0, \ldots, 0) = -e_i$. Also

cost of S_i is zero. So

$$-C_B^T\,B^{-1}\,e_i \le 0,\ i = 1, 2,\ldots, m$$
or, $$C_B^T\,B^{-1}\,e_i \ge 0,\ i = 1, 2,\ldots, m$$
or, $$Y^T\,e_i \ge 0,\ i = 1, 2,\ldots, m$$
or, $$y_i \ge 0,\ i = 1, 2,\ldots, m$$
or, $$Y \ge 0.$$

Thus, it is a feasible solution. Thus, we have obtained a feasible solution of D. Now remains to prove that it is an optimal solution.

We have

$$\boxed{\phi(Y) = b^TY = Y^Tb = C_B^T\,B^{-1}b = C_B^T\,X_B = f\,(X_B)}$$

Thus, we have feasible solution of primal, X_B and of dual Y such that $\phi(y) = f(X_B)$. Hence, by the earlier theorem

$$\boxed{Y^T = C_B^T\,B^{-1}}$$

is the optimal solution of D, which proves the result.

Case-II At least one b_i, say, $b_k < 0$.

In such cases, we first multiply kth constraint by (-1) and obtain

$$\alpha_{11}\,x_1 + \alpha_{12}\,x_2 + \ldots + \alpha_{1n}\,x_n - s_1 = b_1$$
$$-\alpha_{k1}\,x_1 - \alpha_{k2}\,x_2\ \ldots\ \alpha_{kn}\,x_n + s_k = -b_k,\ -b_k > 0$$
$$\alpha_{m1}\,x_1 + \alpha_{m2}\,x_2 + \ldots + \alpha_{mn}\,x_n - s_m = b_m.$$
$$x_i,\ s_j \ge 0$$

In order to solve this, we have to add artificial variable in each constraint except k^{th}.

Proceeding as in case 1, we find that in the matrix B, the kth row would be multiplied by (–1) and consequently kth column of B^{-1} would (-1) times the initial B^{-1}. Thus, in

$$Y^T = C^TB\ B^{-1},$$

y_k would be negative. This can also be seen that from $Z_i - C_i$ below s_k.

$$Z_k - C_k \le 0$$
$$Z_k \le 0$$
$$C^TB\ B^{-1}\ e_k \le 0$$
$$Y^Te_k \le 0$$
$$y_k \le 0.$$

Thus, in this case, too, we find that optimal solution of dual exists and that is $C_B^T\,B^{-1} = Y^T$, with the sign of y_k changed. Hence, the result.

Remark: It is to mentioned here without proof that, if

Primal		**Dual**
	then	
has bounded optimal solution		has bounded optimal solution
has unbounded solution		has no feasible solution
has no feasible solution		has no feasible solution
and has no feasible solution		has unbounded solution

Now we prove an interesting theorem, known as complementary slackness theorem which brings closer the solutions of primal and dual.

Theorem 5: (complementary slackness theorem)

(a) If, in a primal, any slack/surplus variable, say s_k, appears as the basic variable in optimal solution, then corresponding dual variable y_k is zero in optimal dual solution.

(b) If, in a primal, any decision variable, say x_k, appears as the basic variable in optimal solution, then corresponding kth constraint in dual holds as equality, *i.e.*, k-th dual slack/surplus variable is zero.

Proof: Let the primal P be

Min $$f(X) = C^T X$$

$$AX \geq b,\ X \geq 0$$

(a) Since s_k appears as basic variable, the value of $Z_k - C_k$ in optimal table is zero, *i.e.*,

$$Z_k - C_k = 0$$

C_k is already zero for s_k, so

$$Z_k = 0$$

or, $$C_B^T B^{-1} \alpha^k = 0$$

or, $$Y^T e_k = 0 \text{ or } y_k = 0$$

(b) Since x_k appears in the basis, we have below x_k,

$$Z_k - C_k = 0$$

or, $$C^T B\ B^{-1} A_K = C_k$$

or, $$Y^T A_K = C_k$$

or, $$(y_1\ y_2\ \ldots\ y_m)\begin{bmatrix} \alpha_{1k} \\ \alpha_{2k} \\ \vdots \\ \alpha_{mk} \end{bmatrix} = C_k$$

or, $$\alpha_{1k}\, y_1 + \alpha_{2k}\, y_2 + \ldots + \alpha_{mk}\, y_k = C_k.$$

The k-th dual constraint is

$$Y^T A_K \leq C_k$$

i.e., $\alpha_{1k}\, y_1 + \alpha_{2k}\, y_2 + \ldots + \alpha_{mk}\, y_m \leq C_k$.

Hence, the result.

Example 5: Let the LPP be

$$\left.\begin{array}{ll} \text{Max } Z = 5x_1 + 6x_2 & \\ \text{Subject to} & x_1 + 9x_2 \geq 60 \\ & 2x_1 + 3x_2 \leq 45 \\ & x_1, x_2 \geq 0 \end{array}\right] P$$

its dual is

$$\left.\begin{array}{ll} \text{Min } W = -60y_1 + 45y_2 & \\ \text{Subject to} & -y_1 + 2y_2 \geq 5 \\ & -9y_1 + 3y_2 \geq 6 \\ & y_1, y_2 \geq 0 \end{array}\right] D.$$

We now solve both the problems by Big-M method. In standard form, we have

P: Max $Z = 5x_1 + 6x_2 - MR_1$	D: Min $W = -60y_1 + 45y_2 + MR_1 + MR_2$
$x_1 + 9x_2 - s_1 + R_1 = 60$	$-y_1 + 2y_2 - s_1 + R_1 = 5$
$2x_1 + 3x_2 + s_2 = 45$	$-9y_1 + 3y_2 - s_2 + R_2 = 6$
$x_1, x_2, s_1, s_2, R_1 \geq 0$	$y_1, y_2, s_1, s_2, R_1, R_2 \geq 0$

Primal

BV	x_1	x_2	s_1	R_1	s_2	Solution	
Z	-5	-6	0	$0M$	0	0	
R_1	1	9	-1	1	0	60	
s_2	2	3	0	0	1	45	
Z	$-5-M$	$-6-9M$	M	0	0	$-60M$	
R_1	1	$\boxed{9}$	-1	1	0	60	SPT
s_2	2	3	0	0	1	45	(Starting Primal Table)
Z	$-13/3$	0	$-2/3$	$2/3 + M$	0	40	
x_2	1/9	1	$-1/9$	1/9	0	20/3	PT1
s_2	$\boxed{5/3}$	0	1/3	$-1/3$	1	25	(Primal Table 1)
Z	0	0	1/5	$-\frac{1}{5} + M$	13/45	105	
x_2	0	1	$-2/15$	2/15	$-1/15$	5	OPT
x_1	1	0	1/5	$-1/5$	3/5	15	(Optimal Primal Table)

Solution: is $x_1 = 15$, $x_2 = 5$, $s_1 = s_2 = 0$, $R_1 = 0$

Opt. Value 105

Dual

BV	y_1	y_2	s_1	s_2	R_1	R_2	Soln	
W	60	-45	0	0	$-M$	$-M$	0	
R_1	-1	2	-1	0	1	0	5	
R_2	-9	3	0	-1	0	1	6	
W	$60-10M$	$-45+5M$	$-M$	$-M$	0	0	$11M$	
R_1	-1	2	-1	0	1	0	5	SDT
R_2	-9	$\boxed{3}$	0	-1	0	1	6	(Starting Primal Table)
W	$-75+5M$	0	$-M$	$-15+\frac{2}{3}M$	0	$15-\frac{5}{3}M$	$90+M$	
R_1	$\boxed{5}$	0	-1	2/3	1	$-2/3$	1	DT1
y_2	-3	1	0	$-1/3$	0	1/3	2	(Dual Table 1)
W	0	0	-15	-5	$15-M$	$5-M$	105	
y_1	1	0	$-1/5$	2/15	1/5	$-2/15$	1/5	ODT
y_2	0	1	$-3/5$	1/15	3/5	$-1/15$	13/5	(Optimal Dual Table)

Solution: is $y_1 = 1/5$, $y_2 = 13/5$, $s_1 = s_2 = 0$, $R_1 = R_2 = 0$

Opt. value 105

From the above solutions of P and D, we note the following:

1. In any simplex table, the matrix under starting basis is B^{-1} at that step.

$$\text{In PT1} \begin{bmatrix} \frac{1}{9} & 0 \\ -1/3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 3 & 1 \end{bmatrix}^{-1}$$

$$\text{In OPT} \begin{bmatrix} \frac{2}{15} & -\frac{1}{15} \\ -1/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 3 & 2 \end{bmatrix}^{-1}$$

$$\text{In DT1} \begin{bmatrix} 1 & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^{-1}$$

$$\text{In ODT} \begin{bmatrix} \frac{1}{5} & -\frac{2}{15} \\ 3/5 & -1/15 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -9 & 3 \end{bmatrix}^{-1}$$

Before discussing further, we define, ***simplex multiplier*** as

$$C_B^T\, B^{-1} = \Pi = (\Pi_1, \Pi_2, \ldots, \Pi_m)^T$$

at any step. It is clear that if an LPP has a solution (bounded), then the simplex multiplier of optimal table is the required solution of the dual.

2. In PT1, $\Pi = (6,\ 0)\begin{bmatrix} \frac{1}{9} & 0 \\ -1/3 & 1 \end{bmatrix} = \left(\frac{2}{3}, 0\right)$

   In OPT, $\Pi = (6,\ 5)\begin{bmatrix} 2/15 & -1/15 \\ -1/5 & 3/5 \end{bmatrix} = \left(-\frac{1}{5}, \frac{13}{5}\right)$

   In DT1, $\Pi = (M,\ 45)\begin{bmatrix} 1 & -2/3 \\ 0 & 1/3 \end{bmatrix} = \left(M, 15 - \frac{2}{3}M\right)$

   In ODT, $\Pi = (-60,\ 45)\begin{bmatrix} \frac{1}{5} & -\frac{2}{15} \\ \frac{3}{5} & -\frac{1}{15} \end{bmatrix} = (15,\ 5)$

Thus, Π of OPT is an optimal solution of dual with sign of y_1 changed because b_1 in P is negative. Similarly, Π of ODT is an optimal solution of primal. Of course optimal value of P and D are same as 105.

3. The simplex multiplier is used in many more ways.

   Entries in '$Z - C$' row:

$$Z_i - C_i = C_B^T\, B^{-1}\, A_i - C_i = \Pi\, A_i - C_i$$

In PT1: below x_1: $\left(\frac{2}{3}, 0\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} - (+5) = \frac{2}{3} - 5 = -13/3$

below x_2: $\left(\frac{2}{3}, 0\right)\begin{bmatrix} 9 \\ 3 \end{bmatrix} - 6 = 0$

below s_1: $\left(\frac{2}{3}, 0\right)\begin{bmatrix} -1 \\ 0 \end{bmatrix} - 0 = -\frac{2}{3}$

below s_2: $\left(\frac{2}{3}, 0\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = 0$

below R_1: $\left(\frac{2}{3}, 0\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - (-M) = \frac{2}{3} + M$

In OPT: below x_1: $\left(-\frac{1}{5}, \frac{13}{5}\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 5 = 0$

$$\text{below } x_2: \left(-\frac{1}{5}, \frac{13}{5}\right)\begin{bmatrix} 9 \\ 3 \end{bmatrix} - 6 = 0$$

$$\text{below } s_1: \left(-\frac{1}{5}, \frac{13}{5}\right)\begin{bmatrix} -1 \\ 0 \end{bmatrix} - 0 = \frac{1}{5}$$

$$\text{below } s_2: \left(-\frac{1}{5}, \frac{13}{5}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = \frac{13}{5}$$

$$\text{below } R_1: \left(-\frac{1}{5}, \frac{13}{5}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + M = -\frac{1}{5} + M$$

Same thing can be easily seen on dual tables.

4. As stated earlier, Π of optimal table is an optimal solution of the dual. But Π at any step of simplex iteration need not be a feasible solution of dual.

$$\text{In PT1, } \Pi = \left(\frac{2}{3}, 0\right)$$

1st constraint: $-y_1 + 2y_2 \geq 5$. So $-\dfrac{2}{3} \geq 5$

2nd constraint: $-9y_1 + 3y_2 \geq 6$. $-6 \geq 6$

Thus, it does not even satisfy the dual constraints.

5. Solution at any step is $C_B^T B^{-1}b = \Pi\ b$.

$$\text{In PT1: } \left(\frac{2}{3}, 0\right)\begin{bmatrix} 60 \\ 45 \end{bmatrix} = 40$$

$$\text{In OPT: } \left(-\frac{1}{5}, \frac{13}{5}\right)\begin{bmatrix} 60 \\ 45 \end{bmatrix} = 105$$

$$\text{In DT1: } \left(M, 15 - \frac{2}{3}M\right)\begin{bmatrix} 5 \\ 6 \end{bmatrix} = 90 + M$$

$$\text{In ODT: } (15, 5)\begin{bmatrix} 5 \\ 6 \end{bmatrix} = 105$$

Example 6: Find, if possible, an LPP which is self Dual.

Solution: Let the LPP be

$$\left.\begin{array}{ll} \text{Min} & Z = f(X) = C^T X \\ & AX \geq b,\ X \geq 0 \end{array}\right] P$$

$$\text{Its dual is Max } \left.\begin{array}{l} W = \phi(Y) = b^T Y \\ A^T Y \leq C, Y \geq 0 \end{array}\right] D$$

D will be P, if

D: Min $-W = -\phi(Y) = (-b^T)Y$

$$(-A^T)Y \geq -C, \; Y \geq 0$$

$$C^T = -b^T \text{ or } C = -b$$

and, $A = -A^T$

i.e., A is a square, skew symmetric matrix and $C = -b$.

Example 7: With the help of the following example, show using definition of dual that if in the primal problem a variable is unrestricted in sign then the corresponding dual constraint is an equality.

Min $Z = 2x_1 - 3x_2$

Subject to
$$x_1 + x_2 \geq 3$$
$$-x_1 + 2x_2 \leq 2 \quad \Rightarrow \quad x_1 - 2x_2 \geq -2$$
$$2x_1 + x_2 \geq -1$$

$x_1 \geq 0$, x_2 unrestricted in sign.

Solution: dual of the above problem is

Max $W = 3y_1 - 2y_2 - y_3$

Subject to
$$y_1 + y_2 + 2y_3 \leq 2$$
$$y_1 - 2y_2 + y_3 \leq -3 \quad \Rightarrow \quad -y_1 + 2y_2 - y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$

Example 8: Find the solution of the following primal problem by solving its dual problem.

Max $x_1 - 4x_2$

Subject to
$$x_1 - x_2 \leq 2$$
$$x_1 - 2x_2 \leq -3$$
$$x_1, x_2 \geq 0$$

Solution:

Min $2y_1 - 3y_2$

Subject to
$$y_1 + y_2 \geq 1 \quad \Rightarrow \quad y_1 + y_2 - s_1 + R_1 = 1$$
$$-y_1 - 2y_2 \leq -4 \quad \Rightarrow \quad y_1 + 2y_2 + s_2 = 4$$
$$y_1 \, y_2 \geq 0$$

Solution of dual: Two-Phase Method

Phase I.

Min $R = R_1$

Subject to
$$y_1 + y_2 - s_1 + R_1 = 1$$
$$y_1 + 2y_2 + s_2 = 4$$

all variable ≥ 0

BV	R	y_1	y_2	s_1	R_1	s_2	Solution
R	1	$\cancel{0}$ (1)	$\cancel{0}$ (1)$\downarrow$	$\cancel{0}$ (–1)	$-\cancel{1}$(0)	0	$\cancel{0}$ 1
$\leftarrow R_1$	0	1	$\boxed{1}$	–1	1	0	1
s_2	0	1	2	0	0	1	4
r_0	1	0	0	0	–1	0	0
y_2	0	1	1	–1	1	0	1
s_2	0	–1	0	2	–2	1	2

Alternate Phase I

R	1	0	0	0	–1	0	0
y_1	0	1	1	–1	1	0	1
s_2	0	0	1	1	–1	1	3

Phase II.

BV	W	y_1	y_2	s_1	R_1	s_2	Solution	
W	1	$-\cancel{2}$(– 5)	$\cancel{3}$(0)	$\cancel{0}$ (3)$\downarrow$	–4	0	–3	
y_2	0	1	1	–1	1	0	1	
$\leftarrow s_2$	0	–1	0	$\boxed{2}$	–2	1	2	
y_0	1	–7/2	0	0	0	–3/2	–6	
y_2	0	1/2	1	0	0	1/2	2	$y_1 = 0$
s_1	0	–1/2	0	1/0	–1	1/2	1	$y_2 = 2$

Alternate Phase II

BV	W	y_1	y_2	s_1	R_1	s_2	Solution
W	1	–20(0)	$\cancel{3}$(5)$\downarrow$	$\cancel{0}$ (–)2		0	$\cancel{0}$ (2)
$\leftarrow y_1$	0	1	1	–1		0	1
s_2	0	0	1	1		1	3
W	1	–5	0	3		0	–3
y_2	0	1	1	–1$\downarrow$		0	1
$\leftarrow s_2$	0	–1	0	$\boxed{2}$		1	2
W	1	–7/2	0	0		–3/2	–6
y_2	0	y_2	1	0		1/2	2
s_1	0	–1/2	0	1		1/2	1

$$(x_1, -x_2)^T = C_B^T B^{-1} = (-3,\ 0) \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & 1/2 \end{pmatrix} = \left(0, -\frac{3}{2}\right)$$

$$x_1 = 0,\ x_2 = 3/2 \text{ Max } Z = -6$$

EXERCISE 3.1

1. Write the dual of the following problem and verify the theorem that dual of dual is primal.

   Maximize $Z = 6x_1 + 5x_2$

   Subject to
   $$6x_1 + 4x_2 \geq 35$$
   $$x_1 + 2x_2 \leq 15$$
   $$x_1, x_2 \geq 0$$

2. Write the dual of Exercise 1, if x_1 and x_2 are unrestricted in sign.
3. Write the dual of the following LPP without converting equality constraint into inequality constraints.

   Minimize $Z = 5x_1 + 2x_2 + 3x_3$

   Subject to
   $$2x_1 + 5x_2 + 2x_3 = 40$$
   $$x_1 - 4x_2 - 5x_3 \leq 30$$
   $$x_1, x_2, x_3 \geq 0$$

4. Consider the following LPP

   Minimize $Z = 3x_1 - 2x_2$

   Subject to
   $$x_1 + x_2 \geq 4$$
   $$-2x_1 + x_2 \leq 1$$
   $$x_1 + 2x_2 \geq -1$$
   $$x_1 \text{ unrestricted in sign, } x_2 \geq 0$$

   Show by using definition of dual that if in the primal problem a variable is unrestricted in sign then the corresponding dual constraint is an equality.

5. Consider the following LPP

   Minimize $Z = 2x_1 - 3x_2$

   Subject to
   $$\left.\begin{array}{r} x_1 + x_2 \geq 1 \\ -x_1 - 2x_2 \geq -4 \\ x_1, x_2, \geq 0 \end{array}\right\} \qquad (1)$$

   (a) Write the dual of (1).

   (b) Solve (1) by two-phase method.

   (c) From the optimal table of (1), find the optimal solution of the dual obtained in (a).

6. Under what conditions on A, C and b the following LPP

   Minimize $Z = C^T X$

   Subject to
   $$AX \geq b$$
   $$X \geq 0$$

is self dual.

(**Ans:** A is a square matrix & $C = -b$)

7. Consider the following LPP

   Maximize $Z = x_1 - 3x_2 - 7x_3 + 5x_4$

   Subject to
   $$5x_1 - x_2 + x_3 - 4x_4 \le 10$$
   $$-5x_1 + x_2 + 4x_4 \le 5$$
   $$-3x_1 - x_3 + x_4 \le 5$$
   $$x_1, x_2, x_3, x_4 \ge 0$$

   (a) Write the dual of the above problem.

   (b) Solve the given primal by simplex method.

   (c) From optimal table of (b) find the optimal solution of (a).

   (**Ans:** Unbounded solution as primal has no solution)

8. Consider the following LPP

   Maximize $Z = 9x_1 + 6x_2$

   Subject to
   $$-2x_1 - x_2 \le -3$$
   $$x_1 + x_2 \le 1$$
   $$x_1, x_2 \ge 0$$

   (a) Write the dual of the above problem.

   (b) Solve the above problem by two-phase method.

   (c) From optimal table of (b), find the optimal solution of (a).

   (**Ans:** As the primal has no feasible solution, then dual either has no feasible solution or unbounded solution).

9. Consider the following LPP

   Maximize $Z = -2x_1 - 2x_2$

   Subject to
   $$x_1 - 2x_2 \ge 4$$
   $$2x_1 - x_2 \le 1$$
   $$x_1, x_2 \ge 0$$

   (a) Write the dual of the above problem.

   (b) Solve (a).

   (c) From the optimal table of (b), find the optimal solution of (a).

   (**Ans:** (b): unbounded solution (c): no feasible solution)

3.4 DUAL SIMPLEX METHOD

3.4.1 Introduction

We know how the simplex method works. We start with a BFS and go on moving to other BFS so as to optimise the solution. In other words, keeping the feasibility, we move towards optimality.

The duality concept helps in deriving a method which is just similar to simplex method, which gives that if optimality has reached but feasibility lost then keeping optimality row to move towards feasibility. This method is known as **Dual Simplex method.**

3.4.2 Dual Simplex Method

We shall forgo the theory of development of this method but shall give the method or procedure to apply this method.

This method is applicable only if the simplex table is optimal, *i.e.*, on the application of simplex iteration, optimality has reached but the feasibility has lost, *i.e.*, in the optimal table, at least one of the basic variables is negative in value.

In simplex method, we first decide about the entering variable and then about the leaving variable, but in dual simplex method, the first decision is taken about the leaving variable and then it is decided about the entering variable. Step-wise-step procedure is as follows:

(i) ***Leaving variable:*** The variable x_r whose value amongst all negative x_i is minimum, *i.e.*, the variable which is most negative in value leaves, *i.e.*, the variable x_r such that

$$x_r = \operatorname*{Min}_i \{x_i \mid x_i < 0\}$$

leaves the basis.

It ensures the movement towards feasibility.

(ii) ***Entering variable:*** The variable x_k for which $\left|\dfrac{z_i - c_i}{\alpha_r^i}\right|$, $\alpha_r^i < 0$ is minimum, *i.e.*, for which

$$\left|\frac{z_k - c_k}{\alpha_r^k}\right| = \min\left\{\left|\frac{z_k - c_k}{\alpha_r^k}\right|, \alpha_r^i < 0\right\}$$

It ensures optimality.

(iii) Now perform the operations as in simplex method.

(iv) Continue till feasibility is reached.

Example 1:

Max $Z = -4x_1 - 6x_2 - 18x_3$

Subject to
$$x_1 + 3x_3 \geq 3$$
$$x_2 + 2x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

Solution: If we look at this problem, we find that we will have to add artificial variable for solving it. So we solve it by big M-method.

Max $$Z = -4x_1 - 6x_2 - 18x_3 - MR_1 - MR_2$$
$$x_1 + 3x_3 - s_1 + R_1 = 3$$
$$x_2 + 2x_3 - s_2 + R_2 = 5$$
i.e., $$x_1, x_2, x_3, s_1, s_2, R_1, R_2 \geq 0$$

Max $Z = -4x_1 - 6x_2 - 18x_3$

Subject to
$$x_1 + 3x_3 \geq 3 \quad \Rightarrow \quad -x_1 - 3x_3 + s_1 = -3$$
$$x_2 + 2x_3 \geq 5 \quad \Rightarrow \quad -x_2 - 2x_3 + s_2 = -5$$
$$x_1, x_2, x_3 \geq 0 \text{ all var} \geq 0$$

BV	x_1	x_2	x_3	s_1	s_2	R_1	R_2	Solution
Z	4	6	18	0	0	M	M	0
R_1	1	0	3	–1	0	1	0	3
R_2	0	1	2	0	–1	0	1	5
Z	$4 - M$	$6 - M$	$18 - 5M$	M	M	0	0	$-8M$
R_1	1	0	3	–1	0	1	0	3
R_2	0	1	2	0	–1	0	1	5
Z	$-2+\frac{2}{3}M$	$6 - M$	0	$+6 - \frac{2}{3}M$	M	$\frac{5}{3}M - 6$	0	$-3M - 18$
x_3	1/3	0	1	–1/3	0	1/3	0	1
R_2	–2/3	01	0	2/3	–1	–2/3	1	3
Z	2	0	0	2	6	$-2 + M$	0	–36
x_3	1/3	0	1	–1/3	0	1/3	0	1
x_2	–2/3	1	0	2/3	–1	–2/3	1	3

Thus, the solution is $x_1 = 0$, $x_2 = 3$, $x_3 = 1$; Opt. value is –36. While doing this problem, we noticed that identity matrix is present in the columns of x_1 and x_2, so there is no need of artificial variable. So this problem can also be done as follows:

Max $Z = -4x_1 - 6x_2 - 18x_3$

$$3x_3 - s_1 + x_1 = 3$$

$$2x_3 - s_2 + x_2 = 5$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0.$$

BV	x_3	s_1	s_2	x_1	x_2	Solution
Z	18	0	0	4	6	0
x_1	3	–1	0	1	0	3
x_2	2	0	–1	0	1	5
Z	–6	4	6	0	0	–42
x_1	3	–1	0	1	0	3
x_2	2	0	–1	0	1	5
BV	x_3	s_1	s_2	x_1	x_2	Solution
	0	2	6	2	0	–36
x_3	1	–1/3	0	1/3	0	1
x_2	0	2/3	–1	–2/3	1	3

Thus, it is the optimal table, hence the solution is $x_1 = 0$, $x_2 = 3$, $x_3 = 1$; optimal value is –36

If we look at the problem deeply, we find that it would have had slack in place of surplus variable then since objective function is free from slack/surplus variable, the starting table would be optimal table. Also while converting surplus into slack variables, the solution would have become non-feasible. But it is no problem as we would have used dual simplex method. So we do as follows:

Max $Z = -4x_1 - 6x_2 - 18x_3$

$$x_1 + 3x_3 \geq 3$$
$$x_2 + 2x_3 \geq 5,\ x_1, x_2, x_3 \geq 0$$

or Max $Z = -4x_1 - 6x_2 - 18x_3$

$$-x_1 - 3x_3 + s_1 = -3$$
$$-x_2 - 2x_3 + s_2 = -5;\ x_1, x_2, x_3, s_1, s_2 \geq 0$$

BV	x_1	x_2	x_3	s_1	s_2	Solution
Z	4	6	18	0	0	0
s_1	–1	0	–3	1	0	–3
s_2	0	$\boxed{-1}$	–2	0	1	–5

Since the table is o ptimal, we apply the dual simplex method.

Z	4	0	6	0	6	–30
s_1	–1	0	$\boxed{-3}$	1	0	–3
x_2	0	1	2	0	–1	5
Z	2	0	0	2	6	–36
x_3	1/3	0	1	–1/3	0	1
x_2	–2/3	1	0	2/3	–1	3

Solution is $x_1 = 0$, $x_2 = 3$, $x_3 = 1$; optimal value is –36.
Thus,

(a) If the table is optimal *i.e.*, the solution is optimal and solution is not feasible, we apply *dual simplex method.*

(b) If the table is not optimal but solution is feasible, we apply *simplex method.*

(c) If the table is optimal and also solution is feasible, we have nothing to do, it is an optimal feasible solution.

(d) If neither the table is optimal nor the solution is feasible, then right now we have no method and we cannot proceed.

EXERCISE 3.2

1. Use dual simplex method to solve the following LPP.
   Minimize $Z = 4x_1 + 9x_2$
   Subject to
   $$x_1 + x_2 \geq 6$$
   $$2x_1 + 3x_2 \geq 18$$
   $$x_1, x_2 \geq 0$$
2. Solve the following problem by dual simplex method.

Minimize $Z = 2x_1 + 2x_3 + 2x_5 + 4s_1 + 6s_2 + 8s_3$

Subject to $-3x_1 - 2x_2 - x_3 + s_1 = -1500$

$-x_2 - x_4 - 2x_5 + s_2 = -1000$

$-x_3 - x_4 + s_3 = -3000$

all variables ≥ 0

3. Use dual simplex method to solve the following LPP.

Maximize $Z = -15x_1 - 5x_2 - 6x_3$

Subject to $5x_1 - 2x_2 + 3x_3 \geq 6$

$10x_1 + 3x_2 + 3x_3 \geq 5$

$x_1 + x_2 + x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$ ($x_1 = 0 = x_2$, $x_3 = 2$, $Z = -12$)

Chapter 4

Sensitivity Analysis

4.1 INTRODUCTION

In case in an LPP m and or n are (is) too large, then we can easily visualise that application of simplex method and getting an optimal solution will be a herculean task. There, if after getting the optimal table (*i.e.*, optimal solution), one comes to know that he has made slip(s) in making the starting table or writing the initial table in terms of writing wrong $b_i's$, $c_i's$, $\alpha_{ij}'s$, etc., then what should be done? Also, if after getting the optimal solution, one comes to know that due to reasons, the problem constraints, variables, requirements, etc. have changed, then what should be done?

A crude answer to this problem would be, okay start afresh, which we know is going to be another herculean task. So, in order to save time and not to allow our earlier efforts to go entirely waste, we have method to start from the optimal table and incorporate the required change. We give below methods of handling these cases.

These changes may be of the following types:

(i) Change in b_i's (availabilities), right hand side of constraints.

(ii) Change in c_i's (costs), coefficient(s) of decision variables in objective function.

(iii) Change in α_{ij}'s (technological coefficients), coefficient(s) of decision variables in constraints.

(iv) Addition of a new variable.

(v) Addition of a new constraint.

(vi) Deletion of a variable.

(vii) Deletion of a constraint.

4.2 HOW TO FIND SOLUTION OF AN LPP AFTER CHANGES

4.2.1 Change in Availabilities

Change in availabilities, means change in b, *i.e.*, b_i's. It is obvious from the simplex iteration that this change effects only X_B, the solution and the optimal value, $C_B^T X_B$. If, on taking new b, we would have performed same operation then we would have got new value and some other basic solution.

B.V.	Variables	Sol.
Z	$Z_j - C_j = C_B^T B^{-1} A_j - C_j$	$C_B^T X_B$
Basic Var.	$\alpha^j = B^{-1} A_j$	$X_B = B^{-1}b$

Thus, since B^{-1} would not change, we take B^{-1} and C_B^T from the optimal table and obtain new solution and value by the following formulae.

New $X_B' = B^{-1} b'$, where b' is new b

and, $f(X_B') = C_B^T X_B'$.

Nothing can be said about X_B', except that it would be a basic solution but it may be feasible or it may not be feasible.

If solution becomes infeasible, then we apply dual simplex method to move towards feasibility and obtain a BFS, since the table is optimal.

Thus, this change may effect only feasibility.

Example 1: Let the LPP be

Max $Z = -x_1 + 3x_2$

$$x_1 + 2x_2 \geq 2$$
$$3x_1 + x_2 \leq 3$$
$$x_1 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

Solution: We solve this problem by Big-M method.

The optimal table is

BV	x_1	x_2	s_1	R_1	s_2	s_3	Solution	New
Z	10	0	0	M	3	0	9	15
x_2	3	1	0	0	1	0	3	5
s_1	5	0	1	–1	2	0	4	6
s_3	1	0	0	0	0	1	4	6

If $b = (2, 3, 4)^T$ is changed to $(4, 5, 6)^T$, then

$$X_B' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 6 \end{bmatrix}, \text{ and}$$

$$f(X_B') = (3, 0, 0)\,(5, 6, 6)^T = 15$$

Thus, we find that new solution remains optimal and feasible. New solution is

$$x_1 = 0, x_2 = 5; \text{ Optimal Value} = 15$$

Let us take another example.

Example 2: Let the LPP be

Min $Z = 2x_1 + x_2$

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3;\; x_1, x_2 \geq 0$$

Solution: We solve it by Big-M method. Let the optimal table be

BV	x_1	x_2	s_2	R_1	R_2	s_3	Solution	New
Z	0	0	–1/5	$\frac{2}{5} - M$	$\frac{1}{5} - M$	0	$\frac{12}{5}$	3
x_1	1	0	1/5	3/5	–1/5	0	3/5	2
x_2	0	1	–3/5	–4/5	3/5	0	6/5	–1
s_3	0	0	1	1	–1	1	0	3

If we change $b = (3, 6, 3)^T$ to $(5, 5, 3)^T$, then

$$X'_B = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ +1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

and, $$f(X'_B) = (2, 1, 0) \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 3$$

In this case new solution is not feasible but the table is optimal, so we apply dual simplex method and obtain the following table.

	x_1	x_2	s_2	R_1	R_2	s_3	Solution
	0	–1/3	0	$2/5 - M$	$1/5 - M$	0	10/3
x_1	1	1/3	0	1/3	0	0	5/3
s_2	0	–5/3	1	$\frac{4}{3}$	–1	0	5/3
s_3	0	5/3	0	–1/3	0	1	4/3

Now it is optimal table and solution is feasible so we obtain

$$x_1 = \frac{5}{3},\; x_2 = 0,\; \text{Opt. Val.} = \frac{10}{3}.$$

4.2.2 Change in Costs

As clear from the method of simplex iteration, change in costs affects $z_i - c_i$, *i.e.*, the optimality and also $f(X) = C_X^T$, the optimal value.

Thus, if cost (coefficient) of a decision variable has changed, we assume that $C = (c_1, c_2, \ldots, c_n)$ has changed and calculate the entries in $z_i - c_i$ row by the formulae:

$$z_i' - c_i' = C_B'^T B^{-1} A_i - C_i$$
$$= \Pi' A_i - C_i$$

and,

$$f(X_B) = C_B'^T X_B = C_B'^T B^{-1} b$$
$$= \Pi' b,$$

Where, $C_B'^T$ is the new cost vector, c_i' is the new cost of x_i and π' is new simplex multiplier.

By this the initial optimal table may become non-optimal. If optimality has not been lost, we feel happy and the new table will give optimal solution and optimal value. If optimality has been lost, we apply simplex method and move towards optimality. To illustrate this, we take the earlier example.

Example 3: Let the LPP be

Max $Z = -x_1 + 3x_2$

$$x_1 + 2x_2 \geq 2$$
$$3x_1 + x_2 \leq 3$$
$$x_1 \leq 4; \; x_i \geq 0$$

Solution: We introduce artificial variable, apply Big-M method and get the optimal table

BV	x_1	x_2	s_1	R_1	s_2	s_3	Solution
Z	10	0	0	M	3	0	9
x_2	3	1	0	0	1	0	3
s_1	5	0	1	–1	2	0	4
s_3	1	0	0	0	0	1	4

Let the cost of x_1, x_2 have changed from $(-1, 3)^T$ to $(2, 5)^T$, then

BV	x_1	x_2	s_1	R_1	s_2	s_3	Solution
Z	13	0	0	M	5	0	15
x_2	3	1	0	0	1	0	3
s_1	5	0	1	–1	2	0	4
s_3	1	0	0	0	0	1	4

$$\Pi' = (5, 0, 0)\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (0, 5, 0)$$

and $z'_i - c'_i$, $f(X_B)$ are given in table. Thus, the table remains optimal. Solution remains same but optimal value changes to 15. On the other hand, if C^T is changed from $(-1, 3)^T$ to $(4, 1)^T$, then

$$\Pi' = (1,\ 0,\ 0)\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (0,\ 1,\ 0)$$

and the new table becomes

BV	x_1	x_2	s_1	R_1	s_2	s_3	Solution
Z	–1	0	0	M	1	0	3
x_2	3	1	0	0	1	0	3
s_1	5	0	1	–1	2	0	4
s_3	1	0	0	0	0	1	4

It is not optimal. We apply simplex method x_1 enters and s_1 leaves. We obtain

BV	x_1	x_2	s_1	R_1	s_2	s_3	Solution
Z	0	0	1/5	M – 1/5	7/5	0	19/5
x_2	0	1	–3/5	3/5	–1/5	0	3/5
x_1	1	0	1/5	–1/5	2/5	0	4/5
s_3	0	0	–1/5	1/5	–2/5	1	16/5

It is optimal table. New solution is

$$x_1 = \frac{4}{5},\ x_2 = \frac{3}{5};\ \text{Optimal value is } \frac{19}{5}.$$

4.2.3 Change in Technological Coefficients

Change in technological coefficients, *i.e.*, α_{ij}, the coefficients of decision variables can be regarded as the change of column (s) A_i of matrix A.

If more than one column has changed then we would take them step-by-step. In other words, at one time only one column would be changed.

Change in A_i's amounts the changes in α^i and $z_i - c_i$, thus it effects optimality.

It may be noted this change could be in columns of a basic variable or of a non-basic variable. Thus, we shall take them as two cases. Change of A_i's of a non-basic variable, which is easy to handle by the method described below. Change of A_i's of a basic variable cannot be handled by the same method. The reason is obvious as it may change $z_i - c_i$ which is zero.

The new $\alpha'^{\,i} = B^{-1} A_i'$;

$$z_i' - c_i = C_B^T B^{-1} A_i' - c_i$$

Case I Change of A_i of a non-basic variable:

Let the LPP be as given below

Example 4:

Max $Z = -x_1 + 3x_2$

$$x_1 + 2x_2 \geq 2$$
$$3x_1 + x_2 \leq 3$$
$$x_1 \leq 4,\ x_i's \geq 0$$

Solution:

OPTIMAL TABLE IS:

BV	x_1	x_2	s_1	R_1	s_2	s_3	Solution
Z	10	0	0	M	3	0	9
x_2	3	1	0	0	1	0	3
s_1	5	0	1	–1	2	0	4
s_3	1	0	0	0	0	1	4

Let the column of x_1, the non-basic variable be changed from $(1, 3, 1)^T$ to $(3, 5, 1)^T$. New table becomes:

$$\alpha'_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}$$

$$z'_1 - c_1 = (3, 0, 0)\begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix} = 15 - (-1) = 16$$

BV	x_1	x_2	s_1	R_1	s_2	s_3	Solution
Z	16	0	0	M	3	0	9
x_2	5	1	0	0	1	0	3
s_1	7	0	1	–1	2	0	4
s_3	1	0	0	0	0	1	4

It is an optimal table, *i.e.*, optimality has not been lost. Solution is same $x_1 = 0$, $x_2 = 3$; optimal value 9.

If the column $(1, 3, 1)^T$ is changed to $(1, -1, 1)^T$, then the table becomes

BV	x_1	x_2	s_1	R_1	s_2	s_3	Solution
Z	–2	0	0	M	3	0	9
x_2	–1	1	0	0	1	0	3
s_1	–3	0	1	–1	2	0	4
s_3	1	0	0	0	0	1	4

It is not optimal. We apply simplex method.

Z	0	0	0	M	3	2	17
x_2	0	1	0	0	1	1	7
s_1	0	0	1	–1	2	3	16
x_1	1	0	0	0	0	1	4

It is optimal table. Solution is

$x_1 = 4$, $x_2 = 7$; Optimal value is 17.

The second case would need other parts of sensitivity analysis, so we defer it till then.

4.2.4 Addition of a Variable

A variable which is added in an LPP will appear in objective function as well as constraints. Thus, we introduce that variable in the optimal table and calculate $z_i - c_i$ for it and also α^i for it, by

$$z_i^{(n)} - c_i^{(n)} = C_B^T B^{-1} A_i^{(n)} - C_i^{(n)}$$

and,
$$\alpha^{(n)i} = B^{-1} A_i^{(n)}$$

Where, $A_i^{(n)}$ is the column of new variable and $C_i^{(n)}$ is the cost of new variable; $\alpha^{(n)i}$ is the column of new variable in the optimal table.

If the addition of new variable in optimal table leaves the table optimal it is done and the solution is as it was and the new variable would be a non-basic variable. But if it is not optimal, then we apply simplex method and move towards optimality. We illustrate it by examples below.

Example 5: Let the LPP be

Max $\quad Z = -x_1 + 3x_2$

$$x_1 + 2x_2 \geq 2$$
$$3x_1 + x_2 \leq 3$$
$$x_1 \leq 4;\ x_1, x_2 \geq 0$$

Solution: Its optimal table is:

BV	x_1	x_2	s_1	R_1	s_2	s_3	Solution
Z	10	0	0	M	3	0	9
x_2	3	1	0	0	1	0	3
s_1	5	0	1	–1	2	0	4
s_3	1	0	0	0	0	1	4

Let the variable x_3 be added with cost ‘2’ and the column $(3, 1, -1)^T$. The table becomes

BV	x_1	x_2	x_3	s_1	R_1	s_2	s_3	Solution
Z	10	0	–5	0	M	3	0	9
x_2	3	1	–1	0	0	1	0	3
s_1	5	0	–5	1	–1	2	0	4
s_3	1	0	1	0	0	0	1	4

$$\alpha^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix}$$

and, $$z_3 - c_3 = (3, 0, 0) \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - 2$$

$$= (3, 0, 0)\ (-1, -5, 1)^T - 2$$
$$= -3 - 2 = 5$$

In this case the optimality is lost. We apply simplex iteration and obtain.

BV	x_1	x_2	x_3	s_1	R_1	s_2	s_3	Solution
Z	15	0	0	0	M	3	5	29
x_2	4	1	0	0	0	1	1	7
s_1	10	0	0	1	–1	2	5	24
x_3	1	0	1	0	0	0	1	4

It is optimal table: Solution is $x_1 = 0$, $x_2 = 7$, $x_3 = 4$; optimal value is 29. But it we add variable x_3 with cost '–4' and the same column then $z_i - c_i$ would have come as 1. Thus, the table would have remained optimal *i.e.*, new variable x_3 would have become non-basic variable. Solution in that case would have become

$$x_1 = 0,\ x_2 = 3,\ x_3 = 0;$$

Optimal value = 9.

4.2.5 Deletion of a Variable

It is clear from the optimal table as also the LPP, that if the variable to be deleted has value 0, either because it is non-basic or otherwise then its deletion will have no effect on the solution.

Therefore, if the deleted variable is non-basic then we can remove it just as it is by removing its existence from the table. But if it is a basic variable, we first make it non-basic by forcing it to leave the basis. This can be achieved by multiplying its row by (–1) thus making it a non-feasible solution and by applying dual simplex method after removing the column of that variable.

We illustrate it by an example.

Example 6: Let the LPP be

Max $$Z = 3x_1 - 2x_2 + 4x_3$$

$$x_1 + 2x_2 + x_3 \le 430$$
$$3x_1 + 2x_3 \le 460$$
$$x_1 + 4x_2 \le 420;\ x_i \ge 0$$

Solution: Its optimal table is

BV	x_1	x_2	x_3	s_1	s_2	s_3	Solution
Z	3	2	0	0	2	0	920
s_1	–1/2	2	0	1	–1/2	0	200
x_3	3/2	0	1	0	1/2	0	230
s_3	0	4	0	0	0	1	420

Suppose x_2 is removed. It is a non-basic variable. So the problem and optimal table becomes

$$\text{Max} \quad Z = 3x_1 + 4x_3$$
$$x_1 + x_3 \le 430$$
$$3x_1 + 2x_3 \le 460$$
$$x_1 \le 420;\ x_1,\ x_3 \ge 0$$

and optimal table becomes

BV	x_1	x_3	s_1	s_2	s_3	Solution
Z	3	0	0	2	0	920
s_1	–1/2	0	1	–1/2	0	200
x_3	3/2	1	0	1/2	0	230
s_3	0	0	0	0	1	420

Solution: $x_1 = 0$, $x_3 = 230$, Opt. value = 920

But if x_3 is deleted then we first make solution infeasible as it is a basic variable and force it to leave the basis and then apply dual simplex method after cancelling x_3

BV	x_1	x_2	x_3	s_1	s_2	s_3	Solution
Z	3	2	0	0	2	0	920
s_1	–1/2	2	0	1	–1/2	0	200
x_3	–3/2	0	–1	0	–1/2	0	–230
s_3	0	4	0	0	0	1	420
z	0	2		0	1	0	460
s_1	0	2		1	–1/3	0	830/3
x_1	1	0		0	1/3	0	460/3
s_3	0	4		0	0	1	420

So the solution is $x_1 = \dfrac{460}{3}$, $x_2 = 0$; Optimal value = 460

4.2.6 Addition of a Constraint

As we know that addition of a constraint affects the S_F. It slashes (shrinks) S_F. In this process if the BFS giving the optimal solution gets deleted, then BFS becomes infeasible and if BFS giving optimal solution does not get deleted, same solution satisfies the newly added constraint and thus same solution and value remains optimal.

Whenever we wish to add a constraint, we check whether the optimal solution satisfies it. If so we write the same solution.

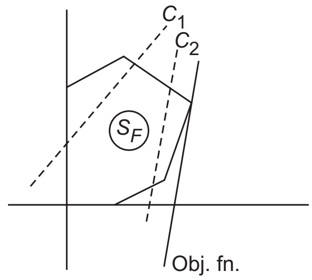
If not then we proceed as follows:

Let the constraint added be

$$\alpha_{m+1,1}\, x_1 + \alpha_{m+1,2}\, x_2 + \ldots + \alpha_{m+1,n}\, x_n + s_{m+1} = b_{m+1}$$

Then we add a row and a new variable s_{m+1} as

$$s_{m+1}\ \left|\, \alpha_{m+1,1}\ \ \alpha_{m+1,2}\ \ldots\ \alpha_{m+1,n} \right|\, b_{m+1}$$



Initially, the table will not be a simplex table. We make it a simplex table and apply dual simplex method because feasibility would be lost.

Example 7: Let the LPP be

Max $\qquad Z = 3x_1 - 2x_2 + 4x_3$

$$x_1 + 2x_2 + x_3 \le 430$$
$$3x_1 + 2x_3 \le 460$$
$$x_1 + 4x_2 \le 420;\ x_i \ge 0$$

Its optimal table is

BV	x_1	x_2	x_3	s_1	s_2	s_3	Solution
Z	3	2	0	0	2	0	920
s_1	$-\frac{1}{2}$	2	0	1	–1/2	0	200
x_3	3/2	0	1	0	1/2	0	230
s_3	1	4	0	0	0	1	420

Solution: $x_1 = 0,\ x_2 = 0,\ x_3 = 230$; Opt. Val. = 920

Let the constraint to be added be

$$3x_1 + 2x_3 \le 500$$

Since, the optimal solution satisfies this constraint, the solution given by the optimal table remains optimal feasible solution.

On the other hand, if we add the constraint

$$3x_1 + 2x_3 \le 400$$

then the optimal solution of the LPP does not satisfy it. So we add the constraint

$$3x_1 + 2x_3 + s_4 = 400$$

With s_4 as basic variable and also an extra variable and make the table as follows.

BV	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Solution
Z	3	2	0	0	2	0	0	920
s_1	–1/2	2	0	1	–1/2	0	0	200
x_3	3/2	0	1	0	1/2	0	0	230
s_3	1	4	0	0	0	1	0	420
s_4	3	0	2	0	0	0	1	400

But it is not starting table. The x_3 column should have been $(0; 0, 1, 0, 0)^T$. So we make it a starting simplex table.

BV	x_2	x_1	x_3	s_1	s_2	s_3	s_4	Solution
Z	3	2	0	0	2	0	0	920
s_1	–1/2	2	0	1	–1/2	0	0	200
x_3	3/2	0	1	0	1/2	0	0	230
s_3	1	4	0	0	0	1	0	420
s_4	0	0	0	0	–1	0	1	–60

Here the solution is optimal but not feasible so we apply dual simplex method. s_4 leaves and s_2 enters.

BV	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Solution
Z	3	2	0	0	0	0	2	800
s_1	–1/2	2	0	1	0	0	–1/2	230
x_3	3/2	0	1	0	0	0	1/2	200
s_3	1	4	0	0	0	1	0	420
s_2	0	0	0	0	1	0	–1	60

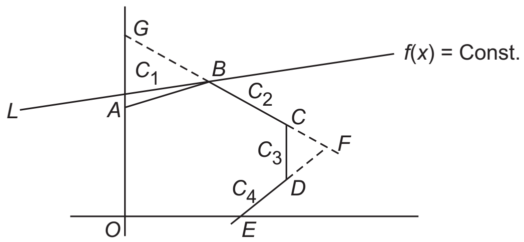
It gives the optimal feasible solution. So the solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 200$; Optimal value = 800

4.2.7 Deletion of a Constraint

The deletion of a constraint, as is clear, enlarges S_F. Each BFS lies on one or more than one boundary surface, *i.e.*, hyperplane, *i.e*, constraint with equality.

If BFS does not lie on the hyperplane represented by the constraint to be deleted, then it is okay. The optimal solution, given by the optimal table remains. On the other hand, if it is not so, *i.e.*, hyperplane representing the constraint to be deleted does contain the BFS where optimality has reached then it affects the solution, not the feasibility but the optimality.

Let us understand it in the following manner.



Let OABCDE be the S_F and L the position of the objective function giving the optimal value. If the constraint C_3 is removed, the S_F becomes OABFE and position of L remains unchanged, *i.e.*, optimal solution remains unchanged. On the other hand, if c_1 is deleted, the S_F becomes OGCDE and the vertex B is out and we have to move towards optimality.

In technical terms, we call that

(i) C_1 (and also C_2) is a binding on optimal solution, and (ii) C_3 (and also C_4) is not a binding on optimal solution.

Thus, if a constraint that is not a binding on the optimal solution is deleted, solution remains unchanged but it changes if a constraint that is a binding on the optimal solution is deleted. Now the question is how to recognise whether a given constraint is a binding or not a binding on the optimal solution, from the optimal table?

If the BFS giving optimal solution lies on a constraint (hyperplane), *i.e.*, satisfies the constraint with equality, then it is a binding otherwise not. In other words, in the optimal table, if a slack/surplus variable is zero then it means that corresponding constraint is satisfied as equality. Thus, all those constraints whose, slack/surplus variable is zero in optimal table are binding on the optimal solution and those whose slack/surplus variable are not zero but > 0 are not binding on the optimal solution.

Thus, a constraint whose slack/surplus variable is not zero but > 0 can be deleted without affecting the optimal solution, and the solution remains unchanged. If we have to delete a constraint which is binding on the optimal solution, *i.e.*, of which slack/surplus variable is zero, we proceed as follows:

The aim of this method is to make that constraint defunct, *i.e.*, if the constraint is

$$\alpha_{k1}\, x_1 + \alpha_{k2}\, x_2 + \ldots + \alpha_{kn}\, x_n \le b_k$$

then we allow that left hand side may be less than or equal b_k or it can be greater than equal to b_k, *i.e.*, in

$$\alpha_{k1}\, x_1 + \alpha_{k2}\, x_2 + \ldots + \alpha_{kn}\, x_n + s_k = b_k$$

We permit s_k to have any value either positive or negative. By while getting the optimal table we have already taken $s_k \ge 0$, therefore we add '$-s$', a new variable s to this constraint only, so that combination '$s_k - s$' will take care of + or –. Recall that s_k is already '0', so s has to enter the basis after introducing s.

Thus,

(a) if the constraint to be deleted is

$$\alpha_{k1}\, x_1 + \alpha_{k2}\, x_2 + \ldots + \alpha_{kn}\, x_n + s_k = b_k,\ (s_k = 0)$$

We add $-s$, ($s \ge 0$) to this constraint only, *i.e.*, with cost '0' and column (0, 0, …0, –1, 0, … 0)^T

↑

k.th

(b) if the constraint to be deleted is

$$\alpha_{k1}\, x_1 + \alpha_{k2}\, x_2 + \ldots + \alpha_{kn}\, x_n - s_k = b_k,\ (s_k = 0)$$

We add $+s$, ($s \ge 0$) to this constraint only, *i.e.*, with cost '0' and column (0, 0, …, 0, 1, 0, …, 0)^T = e_k.

(c) if the constraint to be deleted is

$$\alpha_{k1}\, x_1 + \alpha_{k2}\, x_2 + \ldots + \alpha_{kn}\, x_n + R = b_k$$

We add two variables $-s$, s', $(s, s' \geq 0)$ to this constraint with costs '0' and columns $-e_k$ and e_k, respectively.

and proceed as in the case of addition of variables. This will ensure that this constraint is defunct which amounts to the deletion of this constraint.

We shall now take one example

Example 8: Let the LPP be

Max $Z = -x_1 + 3x_2$

$$x_1 + 2x_2 \geq 2$$

$$3x_1 + x_2 \leq 3$$

$x_1 \leq 4$; $x_i \geq 0$, and the optimal table is

BV	x_1	x_2	s_1	R_1	s_2	s_3	Solution
Z	10	0	0	M	3	0	9
x_2	3	1	0	0	1	0	3
s_1	5	0	1	–1	2	0	4
s_3	1	0	0	0	0	1	4

Here s_1, s_3 are not binding on optimal solution, so deletion of first and 3rd constraint will not change the optimal solution already obtained. But, since $s_2 = 0$, second constraint is a binding on optimal solution. So let us see what happens if it is deleted.

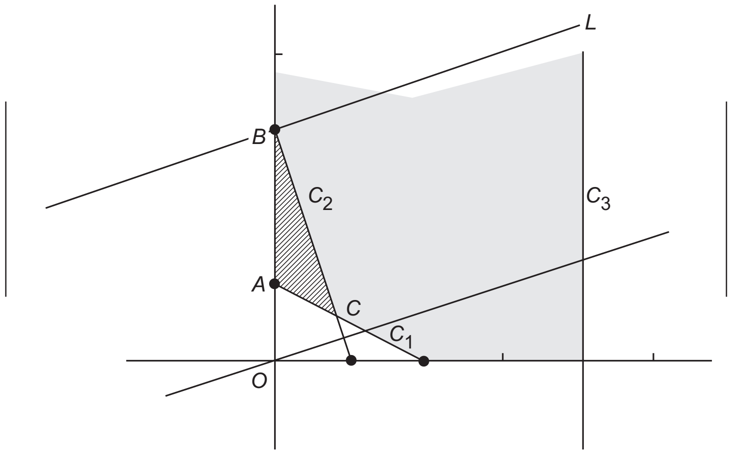
So we add $-s$ to second constraint, *i.e.*, a variable s, $(s \geq 0)$ with zero cost and the column $(0, -1, 0)^T$. We obtain

BV	x_1	x_2	s_1	s	R_1	s_2	s_3	Solution
Z	10	0	0	–3	M	3	0	9
x_2	3	1	0	–1	0	1	0	3
s_1	5	0	1	–2	–1	2	0	4
s_3	1	0	0	0	0	0	1	4

We now apply simplex iteration, and obtain.

We cannot proceed further as the solution would become unbounded as s enters, but there is nothing to leave.

We look it as follows:



C_1, C_2, C_3 give S_F *i.e.*, the region ABC. Actually C_3 does not contribute anything to S_F. L is the optimal position of the objective function. If C_2 region becomes the shaded portion which is not bounded, L has to be moved further beyond limit which means unbounded solution.

We now take another example

Example 9:

Max $Z = 3x_1 - 2x_2 + 4x_3$

Subject to $\quad x_1 + 2x_2 + x_3 \leq 430$

$$3x_1 + 2x_3 \leq 460$$

$$x_1 + 4x_2 \leq 420;\ x_i \geq 0$$

Optimal table is

BV	x_1	x_2	x_3	s_1	s_2	s_3	Solution
Z	3	2	0	0	2	0	920
s_1	$-\frac{1}{2}$	2	0	1	$-\frac{1}{2}$	0	200
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
s_3	1	4	0	0	0	1	420

In this case second constraint is binding, so suppose we delete it.

We add a variable $-s$ to this second constraint with cost '0' and column $(0, -1, 0)^T$.

So the new table is as given below. It is not optimal s enters and s_1 leaves.

BV	x_1	x_2	x_3	s	s_1	s_2	s_3	Solution
Z	3	2	0	–2	0	2	0	920
s_1	$-\frac{1}{2}$	2	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	200
x_3	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	230
s_3	1	4	0	0	0	0	1	420
Z	1	10	0	0	4	0	0	1720
s	–1	4	0	1	2	–1	0	400
x_3	1	2	1	0	1	0	0	430
s_3	1	4	0	0	0	0	1	420

It is optimal. Thus, without second constraint the LPP has the solution $x_1 = x_2 = 0$, $x_3 = 430$; Optimal value 1720.

Remark: The replacement of a constraint or variable is done is two stages, one by adding and the second by deleting.

4.2.8 Change in Technological Coefficient

This we have already taken for the case of non-basic variable. Now we take up the case of basic variable.

Let the variable x_i be a basic variable. Its column A_i has changed to A_i' and cost x_i be c_i.

To do this, we add a new variable x_i' with cost c_i and column A_i' and then delete the variable x_i by

(i) treating x_i as artificial variable and solving by Phase I, and then Phase II.

(ii) by replacing c_i in x_i column by big-M and treating x_i as artificial variable and solving by big-M method.

(iii) by deleting variable x_i by the method discussed above.

Example 10: Let the LPP be

Min $Z = 2x_1 + x_2$

Subject to
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3;\ x_i \geq 0$$

Its Optimal Table is

BV	x_1	x_2	s_2	R_1	R_2	s_3	Solution
Z	0	0	–1/5	$2/5 - M$	$1/5 - M$	0	12/5
x_1	1	0	1/5	3/5	–1/5	0	3/5
x_2	0	1	–3/5	–4/5	3/5	0	6/5
s_3	0	0	1	1	–1	1	0

Let x_1, which is a basic variable, column is changed to $(6, 0, -7)^T$

So we proceed as follows: (1st Tech.)

BV	x_1'	x_1	x_2	s_2	R_1	R_2	s_3	Solution
Z	0	–1	0	0	0	0	0	0
x_1	18/5	1	0	1/5	3/5	–1/5	0	3/5
x_2	–24/5	0	1	–3/5	–4/5	3/5	0	6/5
s_3	–1	0	0	1	1	–1	1	0
Z	18/5	0	0	1/5	3/5	–1/5	0	3/5
$\leftarrow x_1$	$\boxed{18/5}$ $\downarrow$	1	0	1/5	3/5	–1/5	0	3/5
x_2	–24/5	0	1	–3/5	–4/5	3/5	0	6/5
s_3	–1	0	0	1	1	–1	1	0
Z	0	–1	0	0	0	0	0	0
x_1'	1	5/18	0	1/18	1/6	–1/18	0	1/6
x_2	0	4/3	1	–1/3	0	1/3	0	2
s_3	0	5/18	0	19/18	7/6	–19/18	1	3/5
Z	0		0	$-\frac{1}{12}$			0	7/3
x_1'	1		0	1/8			0	1/6
x_2	0		1	–1/3			0	2
s_3	0		0	19/18			1	3/5

It is optimal solution

$$x_1' = x_1 = 1/6,\ x_2 = 2 \text{ Opt. Val} = 7/3$$

Similarly, we can use the other two techniques.

Sometimes original problem is not known, only optimal table is known and changes are known in terms of increment then also some cases can be handled.

Case 1 b is changed to $b + b^*$, and costs are known.

In this case now

$$X_B' = B^{-1}(b + b^*) = B^{-1}b + B^{-1}b^* = X_B + B^{-1}b^*$$

and, $$f(X_B') = C_B^T X_B' = C_B^T X_B + C_B^T B^{-1}b^* = f(X_B) + C_B^T B^{-1}b^*$$

Then apply dual simplex method, if needed.

Case 2 If A_i is changed to $A_i + A_i^*$, costs are known

In this case now

$$\alpha^{i\prime} = B^{-1} (A_i + A_i^*)$$
$$= \alpha^i + B^{-1} A_i^*$$

and, $$z_i - c_i = C_B^T B^{-1}(A_i + A_i^*) - C_i$$
$$= (z_i - c_i) + C_B^T B^{-1} A_i^*$$

Then apply simplex method, if needed,

Case 3 c_i is changed to $c_i + \alpha = c_i^*$

In this case now $z_i - c_i$ is

$$z_i - c_i = C_B^T B^{-1} A_i - (C_i + \alpha) = (z_i - c_i) - \alpha$$

if x_i is non-basic.

If x_i is a basic variable, then

$$C_B^{T\prime} = C_B^T + \alpha e_k$$

and,
$$z_i' - c_i^* = C_B^T B^{-1} A_i + \alpha e_k B^{-1} A_i - C_i - \alpha$$
$$= (z_i - c_i) + \alpha (e_k B^{-1} A_i - 1)$$
$$= (z_i - c_i) + \alpha (\beta - 1),$$

Where, β is the coordinate in x_i column and x_i row.

Example 11: The optimal table of an LPP is

BV	Z	x_1	x_2	x_3	s_1	s_2	Solution
Z	1	7/2	0	0	1/2	7/2	43/2
x_2	0	3/2	1	0	1/2	1/2	13/2
x_3	0	5/2	0	1	1/2	3/2	23/2

Using sensitivity analysis technique, answer the following:

(a) Add the constraint $-2x_2 + x_3 \geq 15$ to the LPP whose optimal table is given above. Find optimal solution of the new LPP.

(b) In the LPP whose optimal table is above, it is desired that $s_1 + s_2 \geq 2$.

(c) Delete the first constraint from the LPP where optimal table is above.

Solution: If we add the constraint $-2x_2 + x_3 \geq 15$ the given optimal solution does not satisfy the constraint. So to add it $+2x_2 - x_3 + s_3 \leq -15$

We get

BV	Z	x_1	x_2	x_3	s_1	s_2	s_3	Solution	
Z	1	7/2	0	0	1/2	7/2	0	43/2	
x_2	0	3/2	1	0	1/2	1/2	0	13/2	
x_3	0	5/2	0	1	1/2	3/2	0	23/2	Initial
s_3 0 0	2	–1	0	0	1	–15	table		
Z	1	7/2	0	0	1/2	7/2	0	43/2	
x_2	0	3/2	1	0	1/2	1/2	0	13/2	Starting
x_3	0	5/2	0	1	1/2	3/2	0	23/2	table
$\leftarrow s_3$	0	–1/2	0	0	$\boxed{-1/2}$	1/2	1	–33/2	
Z	1	3	0	0	0	4	1	5	
$\leftarrow x_2$	0	1	1	0	0	1	1	–10	
x_3	0	2	0	1	0	2	1	–5	
s_1	0	1	0	0	1	–1	–2	33	

x_2 to leave but nothing to enter so no feasible solution

(b) Add $-s_1 - s_2 + s_3 \le -2$

BV	Z	x_1	x_2	x_3	s_1	s_2	s_3	Solution
Z	1	7/2	0	0	1/2↓	7/2	0	43/2
x_2	0	3/2	1	0	1/2	1/2	0	13/2
x_3	0	5/2	0	1	1/2	3/2	0	23/2
← s_3	0	0	0	0	$\boxed{-1}$	–1	1	–2
Z	1	7/2	0	0	0	3	1/2	41/2
x_2	0	3/2	1	0	0	0	1/2	11/2
x_3	0	5/2	0	1	0	1	1/2	21/2
s_1	0	0	0	0	1	1	–1	2

Sol. $x_1 = 0$, $x_2 = 11/2$, $x_3 = 21/2$ Max $Z = 41/2$

(c) If we delete the first constraint, it is binding on the optimal solution add a new variable $-s \ge 0$. Add a new column s to optimal table with cost 0 and column $(0, -1, 0)^T$

$$\alpha^s = B^{-1} A^s = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}$$

$$2_s - C_s = C_B^T B^{-1} A^s = C_B^T \alpha^s = -(2_{s_1} - C_{s_1}) = -1/2$$

BV	Z	x_1	x_2	x_3	s	s_1	s_2	Solution
Z	1	7/2	0	0	–1/2	1/2	7/2	43/2
x_2	0	3/2	1	0	–1/2↓	1/2	1/2	13/2
x_3	0	5/2	0	1	–1/2	1/2	3/2	23/2

s to leave nothing to enter.
⇒ Unbounded solution.

EXERCISE 4.1

1. Consider the following LPP

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

Subject to
$$x_1 + 2x_2 + x_3 \le 430$$
$$3x_1 + 2x_3 \le 460$$
$$x_1 + 4x_2 \le 420$$
$$x_1, x_2, x_3 \ge 0$$

The optimal table of the above problem is

Basic	Z	x_1	x_2	x_3	s_1	s_2	s_3	Solution
Z	1	4	0	0	1	2	0	1350
x_2	0	–1/4	1	0	1/2	–1/4	0	100
x_3	0	3/2	0	1	0	1/2	0	230
s_3	0	2	0	0	–2	1	1	20

(a) If b_1 is changed to $430 + D_1$, find the range of D_1 for which solution remains feasible.

(b) If C_1 is changed to $3 + d_1$, find the range of d_1 for which solution remains feasible. (**Ans:** (a) $-200 \le D_1 \le 10$, (b) $d_1 \le 4$)

2. Consider the following LPP

Minimize $\quad Z = 2x_1 + x_2$

Subject to
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \ge 6$$
$$x_1 + 2x_2 \le 3$$
$$x_1, x_2 \ge 0$$

The optimal table of the above problem by solving Big-M method is

Basic	Z	x_1	x_2	s_2	R_1	R_2	s_3	Solution
Z	1	0	0	–1/5	$2/5 - M$	$\frac{1}{5} - M$	0	12/5
x_1	0	1	0	1/5	3/5	–1/5	0	3/5
x_2	0	0	1	–3/5	–4/5	3/5	0	6/5
s_3	0	0	0	1	1	–1	1	0

Using sensitivity analysis, answer the following:

(a) If $(b_1, b_2, b_3)^T$ is changed from $(3, 6, 3)^T$ to $(5, 5, 3)^T$ then what is the optimal solution of the changed problem?

(**Ans:** $x_1 = 5/3$, $x_2 = 0$, Min. $Z = 10/3$)

(b) If C_1 is changed from 2 to 4, find the optimal solution of the changed problem. (**Ans:** $x_1 = 3/5$, $x_2 = 6/5$, $x_3 = 0$, Min $z = 18/5$)

3. Let the optimal table for a maximization problem (with all constraints $\le$ type) be

Basic	Z	x_1	x_2	x_3	s_1	s_2	Solution
Z	1	0	0	3/5	29/5	2/5	141/5
x_2	0	0	1	–1/5	2/5	–1/5	8/5
x_1	0	1	0	7/5	1/5	2/5	9/5

Let a new variable $x_4 \ge 0$ be added to the problem with a cost 30 assigned to it in the objective function and with the column $(5, 7)^T$. State whether

(a) Optimality Changes
(b) Feasibility Changes
(c) or both (a) and (b)
Give reasons to your answers.

4. Consider the following LPP

Maximize $Z = 15x_1 + 45x_2$

Subject to
$$x_1 + 16x_2 \le 240$$
$$5x_1 + 2x_2 \le 162$$
$$x_2 \le 50$$
$$x_1, x_2 \ge 0$$

The optimal table of the above LPP is

Basic	Z	x_1	x_2	s_1	s_2	s_3	Solution
Z	1	0	0	5/2	5/2	0	1005
x_2	0	0	1	5/78	1/78	0	173/15
x_1	0	1	0	–1/39	8/39	0	352/13
s_3	0	0	0	–5/78	1/78	1	477/13

If Maximize $Z = C_1x_1 + C_2x_2$ and C_2 is fixed to 45, determine how much changes can be made in C_1 so the above solution remains unchanged.

$\left(\textbf{Ans: } 0 \le \Delta C_1 \le \dfrac{195}{2}\right)$ Where, ΔC_1 is change in C_1.

5. Consider the following LPP

Maximize $Z = 3x_1 + x_2$

Subject to
$$3x_1 + 2x_2 \le 18$$
$$x_1 \le 4$$
$$x_2 \le 6$$
$$x_1, x_2 \ge 0$$

(a) Solve the above problem.
(b) If the objective function of the above problem is changed to Maximize $Z = 3x_1 + 5x_2$, discuss the effect on the optimality.

6. Consider the following LPP

Maximize $Z = 10x_1 + 3x_2 + 6x_3 + 5x_4$

Subject to
$$x_1 + 2x_2 + x_4 \le 6$$
$$3x_1 + 2x_3 \le 5$$
$$x_2 + 4x_3 + 5x_4 \le 3$$
$$x_1, x_2, x_3, x_4 \ge 0$$

(a) Solve the above problem.
(b) How much changes should be made in C_1 and b_1 so that the solution remains optimal?

7. The optimal table of an LPP is given below:

Basic	Z	x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	Sol.
Z	1	0	0	0	2	0	2	$\frac{1}{10}$	2	17
x_1	0	1	0	0	–1	0	$\frac{1}{2}$	$\frac{1}{5}$	–1	3
x_2	0	0	1	0	2	1	–1	–1	$\frac{1}{2}$	1
x_3	0	0	0	1	–1	–1	5	3/10	2	7

If a constraint $2x_1 + 3x_2 - x_3 + 2x_4 + 4x_5 \le 5$ is added to the problem, what would be the change in the optimal solution? (Give reason for your answer)

8. Let the LPP be

Minimize $\quad Z = x_1 - 2x_3 + x_3$

Subject to
$$x_1 + 2x_2 - 2x_3 \le 4$$
$$x_1 - x_3 \le 3$$
$$2x_1 - x_2 + 2x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0$$

(a) Solve the above problem by simplex method.

(b) Using sensitivity analysis and optimal table of (a) find the optimal solution of problem (i) if the third constraint is deleted. (ii) if C_3 (the cost coefficient of x_3) in objective function is changed from 1 to –1.

9. Consider the following LPP

Maximize $\quad Z = 8x_1 + 6x_2$

Subject to
$$6x_1 + 8x_2 \le 48$$
$$6x_1 + 5x_2 \le 54$$
$$x_1, x_2 \ge 0.$$

(a) By using sensitivity analysis and the optimal table of the above problem, find the optimal solution of the problem, if a new variable x_3 is added to with cost 7 as column $(3, 5)^T$ to the problem.

(b) Find the solution of the changed problem, if a constraint is added to $4x_1 + 2x_2 + 8x_3 \le 50$ is added to (b).

10. Let the LPP be

Minimize $\quad Z = -5x_1 - 6x_2$

Subject to
$$3x_1 + 4x_2 \le 18$$
$$2x_1 + x_2 \le 7$$
$$x_1, x_2 \ge 0$$

(a) Solve the above problem.

(b) Using sensitivity analysis and optimal table of (a) find the solution of the changed problem.

(i) If b_2 is changed from 7 to 17 and a new constraint $2x_1 + 2x_2 \leq 9$ is added.

(ii) If the cost C_1 of x_1 in objective function is changed from –5 to 5 and add a new variable x_3 with cost 3 and column $(1, 1)^T$.

(iii) If b_2 is changed to 17 and C_1 to 5.

(**Ans:** (a) $x_1 = 2$, $x_2 = 3$, Min $Z = -28$
(b) (i) $x_1 = 0$, $x_2 = 9/2$, Min $Z = -27$
(ii) $x_1 = 0$, $x_2 = 9/2$, $x_3 = 0$, Min $Z = -27$
(iii) $x_1 = 0$, $x_2 = 9/2$, Min $Z = -27$)

Chapter 5

Transportation and Assignment Problems

5.1 TRANSPORTATION PROBLEM

5.1.1 Introduction

This problem first arose in optimising the transportation cost of shipment of goods from various sources to different destinations. These problems when formulated or say when mathematical models of these problems were formulated, then it was noticed that these are LPPs of a special type. Later, any LPP when had that character was also termed transportation problem. We discuss these problems below.

5.1.2 Transportation Problem

Let there be m sources, namely, $S_1, S_2, \ldots, S_m$ and n destinations $D_1, D_2, \ldots, D_n$. The sources could be industries, warehouses, cargo offices, distributors, etc. and destinations could be shops, godowns, warehouses, etc. The goods are to be transported from S_i to D_j ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$).

We are given with transportation costs c_{ij} from source S_i to destination D_j. Also given the quantities of goods available, say a_i, at source S_i to be transported and quantities of goods required, say b_j, at destination D_j. a_i's are known as availabilities and b_j's are known as requirements.

The above data can be written in tabular form as:

	D_1	D_2	D_3	.	.	.	.	.	D_n	Availability
S_1	c_{11} x_{11}	c_{12} x_{12}	c_{13} x_{13}						c_{1n} x_{1n}	a_1
S_2	c_{21} x_{21}	c_{22} x_{22}	c_{23} x_{23}						c_{2n} x_{2n}	a_2
.										a
.										
S_m	c_{m1} x_{m1}	c_{m2} x_{m2}	c_{m3} x_{m3}						c_{mn} x_{mn}	a_n
Requirement	b_1	b_2	b_3	.	.	.	.	.	b_n	

At the right hand upper corner of each square is the transportation cost.

Let x_{ij} be the quantity of goods transported from S_i to D_j. Then the LP problem is

$$\text{Min} \quad Z = c_{11}\, x_{11} + c_{12}\, x_{12} + \ldots + c_{1n}\, x_{1n} + c_{21}\, x_{22} + c_{22}\, x_{22} + \ldots + c_{2n}\, x_{2n} + \ldots + c_{m1}\, x_{m1} + c_{m2}\, x_{m2} + \ldots + c_{mn}\, x_{mn}$$

$$\text{or,} \quad \text{Min} \quad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

where c_{ij} is the cost of transportation of one unit from source S_i to destination D_j

$$\text{Subject to} \quad \left.\begin{aligned} &\sum_{j=1}^{n} x_{ij} = \underset{a_i > 0}{a_i}, i = 1, 2, \ldots, m \\ &\sum_{i=1}^{m} x_{ij} = \underset{b_j > 0}{b_j}, j = 1, 2, \ldots, n, \end{aligned}\right] \qquad (*)$$

where a_i is the quantity available at source S_i and b_j is the quantity required at destination D_j.

$x_{ij} \geq 0 \ \ \forall \ i, j$ and integer.

This is an LPP. Any LPP having this characteristic is known as transportation problem.

It has mn variables and $m + n$ constraints.

Balanced Transportation Problem

If in a transportation problem total availability is equal to total requirement, then it is called Balanced Transportation problem. In this case

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

Also, the above relation gives

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = A$$

Which allows to eliminate one of the constraints in terms of others in the constraints (*). Thus, in case of balanced transportation problem having m sources and n variables there are mn variables and '$m + n - 1$' linearly independent constraints.

Unbalanced Transportation Problem

If the total availability, $\sum_{i=1}^{m} a_i$, and the total requirement, $\sum_{j=1}^{n} b_j$, are unequal then it is called unbalanced transportation problem.

In this case

either, $$\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j \text{ or } \sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j.$$

An unbalanced transportation problem can be converted into a balanced transportation problem. This can be achieved as follows:

Case 1 $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$

In this, availability is less than the requirement, which we normally term 'short supply'. In such case, we create a new source, normally called fictitious source or dummy source S_F with the availability $\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$.

Transportation from S_F to D_j would practically mean that supply to D_j would be in short. We would take the cost of transportation, C_{Fj}, $j = 1, 2, \ldots, n$, equal to 0 in order to minimise the transportation cost.

Case 2 $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$

In this case, availability is more than the requirement, which we normally term 'short demand'. In such case, we create a fictitious destination D_F with the requirement equal to $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$.

Transportation from S_i to D_F would practically mean supply from S_i of this much goods won't take place.

Transportation cost S_i to D_F, *i.e.*, C_{iF} would be taken as zero in order to minimise the cost of transportation.

Remark 1: If the problem requires no transportation from S_i to D_j, then we replace cost c_{ij} by big-M.

Remark 2: If the problem requires transportation to D_j only from S_i, or, say from S_i to D_j only, we replace c_{ij} by zero for optimal solution but for optimal cost we add $c_{ij}\, x_{ij}$.

Remark 3: In case of short supply, *i.e.*, $\Sigma a_i < \Sigma b_j$. We introduce S_F, with cost zero to each D_j. There may arise two situations.

(i) Destination(s) may impose penalties for short supply on each item supplied short. If by destination D_j, penalty is P_j, then C_{Fj} is to be taken as P_j.

(ii) Demand at D_j must be fulfilled. This case can be treated as S_F does not supply anything to D_j, *i.e.*, C_{Fj} is to taken as big-M.

Remark 4: In case of short demand, *i.e.*, $\Sigma a_i > \Sigma b_j$, we introduce D_F with cost zero from each S_i. There may arise two situations.

(i) Source(s) would be left with some quantities of goods which were sent to D_F, *i.e.*, which were not transported. If sources are going to incur some expenses on their storage then it has to be included. If source S_i incurs an expense of q_i on each item then C_{iF} is to be taken as q_i.

(ii) If the problem requires that source S_i must transport all its goods, then we assume that S_i does not supply any thing to D_F, *i.e.*, C_{iF} to be taken as big-M.

Thus, each transportation problem, after incorporating the above remarks, can be converted into a balanced transportation problem.

We shall now discuss method for solving a balanced transportation problem. We can now apply simplex method to get the optimal solution. The method which is going to discuss is simplex method but presentation or table is given in different form.

5.1.3 Solution of a Balanced Transportation Problem

Insert S_F, D_F if needed, replace costs by 0 or M in the light of the above remarks. After doing the steps, we have the balanced transportation problem. Let the sources be $S_1, S_2, \ldots, S_m$ and destinations be $D_1, D_2, \ldots, D_n$. Write them in the tabular form as shown earlier or as given below.

	D_1	D_2	.	.	.	D_n	Availability
S_1	c_{11}	c_{12}				c_{1n}	a_1
S_2	c_{21}	c_{22}				c_{2n}	a_2
.							.
.							.
S_m	c_{m1}	c_{m2}				c_{mn}	a_m
Requirement	b_1	b_2	.	.	.	b_n	

It has mn variables and $m + n$ constraints, but since it is balanced, it has $m + n - 1$ linearly independent constraints.

It will have a solution consisting of basic and non-basic variables. Each BFS would have at the most $m + n - 1$ non-zero variables. If BFS has $< m + n - 1$ non-zero variables that BFS would be a non-degenerate solution. Now we apply simplex method, but the table would be as given above.

We have to first find a starting BFS, then we would move towards optimality. In the treatment given below same steps are followed.

Getting a Starting BFS

There are three commonly applicable methods to get a starting BFS. These are:

(i) North-West Corner Method (NWCM)
(ii) Least Cost Entry Method (LCM)
(iii) Vogel's Approximation Method (VAM)

(i) *North-West Corner Method (NWCM):* We shall select variables which would be our basic variables and assign them values. In this NWCM we shall start with the North-West corner variable, *i.e.*, x_{11} then proceed to x_{12}, x_{13}, …, x_{1n}, x_{21}, x_{22}, …, x_{2n}, …, x_{m1}, x_{m2}, …, x_{mn}.

Assigning a value to x_{ij} would mean to determine the quantity of goods to be transported from S_i to D_j which is obviously the minimum of availability at S_i and the requirement at D_j.

After assigning x_{ij}, say the value A, we would decrease the value of the requirement at D_j by A and the availability at S_i by A because of the natural reason and move to next variable.

When all availabilities and thus all requirements reduce to zero, *i.e.*, get filled, we stop and the boxes inside which some entry is made become basic variable, and which are left blank are non-basic variable.

Thus, we arrive at a BFS, which is starting BFS and the value of objective function is obtained by adding the products of basic variable with its cost.

We illustrate this concept (method) by an example.

Example 1: Find a BFS for the following transportation problem by NWCM.

	D_1	D_2	D_3	D_4	
S_1	1	5	6	4	90
S_2	3	2	3	3	50
S_3	2	6	1	5	60
S_4	6	4	2	2	60
	70	60	60	70	

Solution:

	D_1	D_2	D_3	D_4	
S_1	70 [1]	20 [5]	[6]	[4]	~~90~~ ~~20~~
S_2	[3]	40 [2]	10 [3]	[3]	~~50~~ ~~10~~
S_3	[2]	[6]	50 [1]	10 [5]	~~50~~ 10
S_4	[6]	[4]	[2]	60 [2]	~~60~~
	~~70~~	~~60~~ ~~40~~	~~60~~ ~~50~~	~~70~~ ~~60~~	

Thus, we have the starting BFS as

$$x_{11} = 70,\ x_{12} = 20,\ x_{22} = 40,\ x_{23} = 10,\ x_{33} = 50$$
$$x_{34} = 10,\ x_{44} = 60;\ \text{rest zero.}$$
$$\text{Value} = 70 + 100 + 80 + 30 + 50 + 50 + 120 = 500$$

In Physical sense, it gives

$S_1 \to D_1$	70	70
$S_1 \to D_2$	20	100
$S_2 \to D_2$	40	80
$S_2 \to D_3$	10	30
$S_3 \to D_3$	50	50
$S_3 \to D_4$	10	50
$S_4 \to D_4$	60	120
		500 = Total cost

(ii) ***Least Cost Entry Method (LCM):*** This method also gives the starting BFS. It works in the same manner as NWCM except that we do not start form North-West Corner, *i.e.*, x_{11} and proceed to $x_{12}, x_{13}, \ldots, x_{1n}, x_{21}, x_{22}, \ldots, x_{2n}, \ldots, x_{m1}, x_{m2}, \ldots, x_{mn}$ but at each step, we pick the box with the least cost.

In case of a tie, we pick the one where maximum number can be assigned. In case of further tie, we break it arbitrarily.

Example 2: Find BFS of the following transportation problem by LCM.

	D_1	D_2	D_3	
S_1	1	5	6	90
S_2	3	2	3	10
S_3	2	6	0	20
S_4	5	4	2	40
	60	50	50	

Solution:

	D_1	D_2	D_3	
S_1	60 $\boxed{1}$	30 $\boxed{5}$	$\boxed{6}$	~~90~~ ~~30~~
S_2	$\boxed{3}$	10 $\boxed{2}$	$\boxed{3}$	~~10~~
S_3	$\boxed{2}$	$\boxed{6}$	20 $\boxed{0}$	~~20~~
S_4	$\boxed{5}$	10 $\boxed{4}$	30 $\boxed{2}$	~~40~~ ~~10~~
	~~60~~	~~50~~ ~~40~~ ~~30~~	~~50~~ ~~30~~	

Thus, the solution is

$S_1 \to D_1$	: $x_{11} = 60$	;	60
$S_1 \to D_2$	: $x_{12} = 30$	;	150
$S_2 \to D_2$	: $x_{22} = 10$	;	20

$S_3 \to D_3 \quad : \quad x_{33} = 20 \quad ; \quad 0$
$S_4 \to D_2 \quad : \quad x_{42} = 10 \quad ; \quad 40$
$S_4 \to D_3 \quad : \quad x_{43} = 30 \quad ; \quad 60$

Rest variables are non-basic, hence their values are zero.

Transportation cost: 330

Let us do the same problem by NWCM.

	D_1	D_2	D_3	
S_1	60 $\boxed{1}$	30 $\boxed{5}$	$\boxed{6}$	~~90~~ ~~30~~
S_2	$\boxed{2}$	10 $\boxed{2}$	$\boxed{3}$	~~10~~
S_3	$\boxed{2}$	10 $\boxed{6}$	10 $\boxed{0}$	~~20~~ ~~10~~
S_4	$\boxed{5}$	$\boxed{4}$	40 $\boxed{2}$	~~40~~
	~~60~~	~~50~~ ~~20~~ ~~10~~	~~50~~ ~~40~~	

Here the solution is

$S_1 \to D_1 \quad : \quad x_{11} = 60 \quad ; \quad 60$
$S_1 \to D_2 \quad : \quad x_{12} = 30 \quad ; \quad 150$
$S_2 \to D_2 \quad : \quad x_{22} = 10 \quad ; \quad 20$
$S_3 \to D_2 \quad : \quad x_{32} = 10 \quad ; \quad 60$
$S_3 \to D_3 \quad : \quad x_{33} = 10 \quad ; \quad 00$
$S_4 \to D_3 \quad : \quad x_{43} = 40 \quad ; \quad 80$

Rest variables are non-basic, hence zero.

Transportation Cost = 370

Thus, it emerges that LCM gives a better solution then NWCM.

(iii) ***Vogel's Approximation Method (VAM):*** VAM works on the principle of taking into account the penalty (extra cost) one has to pay, if least cost is not chosen. We first define the penalty.

Penalty of a row (column) is the difference between the lowest cost and the next higher cost in that row (column). If the least cost appears at two places in a row (column), then to calculate penalty of that row (column), we shall take only the next higher cost, *i.e.*, penalty would not be zero, unless all costs in that row (column) is same.

We shall follow the following steps in VAM:

(i) Calculate the penalty of each row (column).
(ii) Choose a row (column) having largest penalty and allot the maximum in the least cost box.
(iii) Reduce the availability and requirement.
(iv) Calculate penalty again after suppressing row (column) satisfied.
(v) Continue, till all rows (columns) are satisfied.

Remark: If the largest penalty at any step is not unique then select the one having the least cost box, otherwise break the tie arbitrarily.

Remark: If the least cost in a row (column) is not unique choose the box where maximum allocation can be made, otherwise break the tie arbitrarily.

Example 3: Find a BFS by VAM for the following transportation problem.

	D_1	D_2	D_3	
S_1	1	5	6	90
S_2	3	2	3	10
S_3	2	6	0	20
S_4	5	4	2	40
	60	50	50	

Solution: We first make the table as made earlier and make space for writing penalties at each step.

	D_1	D_2	D_3	Availability	← Penalty →				
S_1	60 $\boxed{1}$	30 $\boxed{5}$	$\boxed{6}$	~~90~~ ~~30~~	4	1	1	5	
S_2	$\boxed{3}$	10 $\boxed{2}$	$\boxed{3}$	~~10~~	1	1	1	2	
S_3	$\boxed{2}$	$\boxed{6}$	20 $\boxed{0}$	~~20~~	2	6	–	–	
S_4	$\boxed{5}$	10 $\boxed{4}$	30 $\boxed{2}$	~~40~~ ~~10~~	2	2	2	4	
Requirement	~~60~~	~~50~~ ~~20~~ ~~10~~	~~50~~ ~~30~~						
Penalty	1	2	2						
	–	2	2						
	–	2	1						
	–	2	–						

Hence, the solution is

$S_1 \to D_1$	:	$x_{11} = 60$	;	60
$S_1 \to D_2$	:	$x_{12} = 30$	;	150
$S_2 \to D_2$	:	$x_{22} = 10$	;	20
$S_3 \to D_3$	:	$x_{33} = 20$	;	00
$S_4 \to D_2$	:	$x_{42} = 10$	;	40
$S_4 \to D_3$	:	$x_{43} = 30$	;	60

Rest variables, being non-basic are zero.

Transportation cost = 330

Remark 1: The same problem has been done by all the three methods and we find that VAM gave the same solution as given by LCM and a better solution then that obtained by NWCM, but actually, it was a mere coincidence. Actually VAM gives better solution than by both methods LCM and NWCM. So generally we apply VAM to obtain the starting BFS unless otherwise the method is specified.

Remark 2: Whenever shipment of all goods from source S_i is over, we say *i*-th row is satisfied. It also means *i*th constraint of availability is satisfied. Similarly, if shipment to destination D_j is over, we say *j*-th column is satisfied. It also means *j*-th constraint of requirement constraint is satisfied.

Remark 3: Whenever a row and a column are satisfied simultaneously then it would mean two constraints are satisfied simultaneously and it would amount to reduction of a basic variable. Therefore, in this case we assume that only one either a row or column is satisfied and the other row (column) still has to despatch (receive) '0' items. This ultimately would mean that in one box we would enter '0' or say one basic variable would be zero or say that BFS would be degenerate.

Now we take examples where the problems are unbalanced problems, and would illustrate how to obtain the starting BFS.

Example 4: Let there be 3 destinations and 3 sources, costs of transportation, availabilities, requirements are given by the following table. Find a starting BFS.

	D_1	D_2	D_3	
S_1	1	5	6	90
S_2	3	2	3	10
S_3	2	6	1	20
	60	50	50	

Solution: We create a fictitious source S_F with availability 40 and cost of transportation '0' to each destination and solve it by VAM.

	D_1	D_2	D_3				
S_1	60 [1]	30 [5]	[6]	~~90~~ ~~30~~	4	1	1
S_2	[3]	10 [2]	[3]	~~10~~	1	1	1
S_3	[2]	[6]	20 [1]	~~20~~	1	5	–
S_4	[0]	10 [0]	30 [0]	~~40~~ ~~10~~	0	0	0
	~~60~~	~~50~~	~~50~~ ~~30~~				
	1	2	1				
	–	2	1				
	–	2	3				
	–	2	–				

Solution:

$S_1 \to D_1$	:	$x_{11} = 60$	60
$S_1 \to D_2$	:	$x_{12} = 30$	150
$S_2 \to D_2$	:	$x_{22} = 10$	20
$S_3 \to D_3$	:	$x_{32} = 20$	20
$S_F \to D_2$	:	$x_{F2} = 10$	00
$S_F \to D_3$	:	$x_{F3} = 30$	00

Thus, short supply to D_2 by 10 and to D_3 by 30. Cost = 250.

Let us add another condition. For short supply there is a penalty by D_j, say 2, 4, 6 by D_1, D_2, D_3, respectively. Thus, starting BFS.

	D_1	D_2	D_3						
S_1	60 $\boxed{1}$	10 $\boxed{5}$	20 $\boxed{6}$	~~90~~ 30	4✓	1	1	1	
S_2	$\boxed{3}$	$\boxed{2}$	10 $\boxed{3}$	~~10~~	1	1	1	–	–
S_3	$\boxed{2}$	$\boxed{6}$	20 $\boxed{1}$	~~20~~	1	5✓	–	–	–
S_F	$\boxed{2}$	40 $\boxed{4}$	$\boxed{6}$	~~40~~	2	2	2	2✓	–
	~~60~~	~~50~~ ~~10~~	~~50~~ ~~30~~ ~~20~~						
	1	2	2						
	–	2	2						
	–	2	3✓						
	–	2	0						

Solution:

$S_1 \to D_1$	:	$x_{11} = 60$	60
$S_1 \to D_2$	:	$x_{12} = 10$	50
$S_1 \to D_3$	:	$x_{13} = 20$	120
$S_2 \to D_3$	:	$x_{23} = 10$	30
$S_3 \to D_3$	:	$x_{33} = 20$	20
$S_F \to D_2$	:	$x_{F2} = 40$	160

Short supply to D_2 by 40

Cost: 440

Example 5: Find a BFS for the following transportation problem storage costs at each source of each unshipped item is also given.

	D_1	D_2	D_3	*Availability*	*Storage cost*
S_1	1	2	1	20	3
S_2	0	4	5	40	4
S_3	2	3	2	30	5
Requirement	30	20	20		

Solution: We make transportation model and get a BFS by VAM.

	D_1	D_2	D_3	D_F							
S_1	$\boxed{1}$	$\boxed{2}$	20 $\boxed{1}$	0 $\boxed{3}$	~~20~~ 0	1	✓ 1	1	3		
S_2	30 $\boxed{0}$	$\boxed{4}$	$\boxed{5}$	10 $\boxed{4}$	~~40~~ 0	✓ 4	1	0	4		
S_3	$\boxed{2}$	20 $\boxed{3}$	$\boxed{2}$	10 $\boxed{5}$	~~30~~ 10	1	1	✓ 2	5		
	~~30~~	~~20~~	~~20~~	~~20~~							
	1	1	1	1							
	–	1	1	1							
	–	1	–	1							
	–	–	–	1							

$S_1 \to D_3$: $x_{13} = 20$ 20
$S_1 \to D_F$: $x_{14} = 00$ 00
$S_2 \to D_1$: $x_{21} = 30$ 00
$S_2 \to D_F$: $x_{2F} = 10$ 40
$S_3 \to D_2$: $x_{32} = 20$ 60
$S_3 \to D_F$: $x_{3F} = 10$ 50

It is a degenerate solution. S_2 & S_3 would be left with goods to store. Cost = 170

Now we take another step. So far we have given method for getting a BFS. Now we give methods to move towards optimality.

This method is based upon the concept of duality, *i.e.*, transforming the problem into its dual and then solving it.

As we know that transportation problem is an LPP.

Min $Z = c_{11}\, x_{11} + c_{12}\, x_{12} + \dots + c_{1n}\, x_{1n} + \dots + c_{m1}\, x_{m1} + c_{m2}\, x_{m2} + \dots + c_{mn}\, x_{mn}$

$x_{11} + x_{12} + \dots + x_{1n} = a_1 \dots u_1$
$x_{21} + x_{22} + \dots + x_{2n} = a_2 \dots u_2$

$x_{m1} + x_{m2} + \dots + x_{mn} = a_m \dots u_m$
$x_{11} + x_{21} + \dots + x_{m1} = b_1 \dots v_1$
$x_{12} + x_{22} + \dots + x_{m2} = b_2 \dots v_2$

$x_{1n} + x_{2n} + \dots + x_{mn} = b_n, \dots v_n;\ x_{ij} \geq 0$

We now find its dual and associate dual variable $u_1, u_2, \dots, u_n$ to its row constraints, *i.e.*, rows and $v_1, v_2, \dots, v_n$ to its column constraints, *i.e.*, columns, then its dual would be

Max $W = a_1\, u_1 + a_2\, u_2 + \dots + a_m\, u_m + b_1\, v_1 + b_2\, v_2 + \dots + b_n\, u_n$

$u_1 + v_1 = c_{11} \dots x_{11}$
$u_1 + v_2 = c_{12} \dots x_{12}$

$u_i + v_j = c_{ij} \dots x_{ij}$

$u_m + v_n = c_{mn}$, u_i, v_j unrestricted in sign.

Then solve its dual by assigning the values of dual variable satisfying the dual constraints.

If x_{ij} is a basic variable, then $u_i + v_j - c_{ij} = 0$. If x_{ij} is a non-basic variable $x_{ij} \geq 0$ or $x_{ij} \leq 0$. Thus, we calculate $u_i + v_j - c_{ij}$ for each (non-basic variable) box where allocation was not made, and write it in the south-west corner. Since it is a minimisation problem, each of these entries should be ≤ 0. Thus, if each of the south-west corner entry is ≤ 0, it is optimal table and the solution can be read.

If any of these entry is ≥ 0, then it is not an optimal table, we need iteration to move towards optimality.

In order to do this we need entering variable, leaving variable and iteration.

We choose the non-basic variable x_{ij} for which $u_i + v_j - c_{ij}$ is most positive. In other words, we pick the box where the south-west corner entry is most positive and allocate maximum possible goods. This gives entering variable.

In order to allocate maximum possible goods in this box at least one of the box is going to be empty. One of the box, or say, variable attached to it is taken to be the leaving variable.

In order to do this, after deciding entering variable, we find the basic variables (filled boxes) which are going to be affected by it. This is done by constructing a loop by the following method.

Start from the box where entry is to be made move horizontally or vertically and return to the starting point with the following restrictions:

(i) Loop would have straight lines as edges with corners.

(ii) Corners should be only in the box of a basic variable *i.e.*, in a box where entry has already been made.

(iii) The direction of the loop is immaterial.

(iv) The loop could be self intersecting.

Now assume allocating θ in the starting box, then move in one direction to corner, this θ amount should be reduced in this box, continuing in the same direction, go to next corner θ amount is to be increased at this corner and continue till we reach to the starting point.

Now we pick the boxes where θ amount has been reached. Find θ, which is minimum of the entries in these boxes, so that entry in all the corner boxes is ≥ 0.

This gives the maximum allocation in this starting box.

The box which becomes empty becomes leaving variable box.

Caution: If more than one box becomes empty, then we make empty only one box and leave '0' in other boxes.

Now allocate the amount so obtained in the starting box and reduce/increase at other corner boxes. This completes one iteration.

Again assign values to dual variables and calculate afresh the values of $u_i + v_j - c_{ij}$ and write in south-west corner of each box and continue.

Remark: It may be noted that $z_{ij} - c_{ij}$ of each variable x_{ij} is given by

$$C_B^T B^{-1} A_j - C_{ij}$$
$$= Y^T A_j - C_{ij}$$

$$\begin{array}{c} \\ = (u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n)\,(0, 0, \ldots, \end{array}\!\!\begin{array}{c} i\text{th} \\ \downarrow \\ 1, \end{array}\!\! \begin{array}{c} \\ \\ 0, \ldots, 0, 0, 0, \ldots, \end{array}\!\!\begin{array}{c} i\text{th} \\ \downarrow \\ 1, \end{array}\!\!\begin{array}{c} \\ \\ \ldots\ 0)^T - c_{ij} \end{array}$$

$$= u_i + v_j - c_{ij}$$

i.e., why we calculate this and write in south-west corner of each box.

Now we illustrate this concept by taking the example already taken for which starting BFS has already been found by three methods:

(i) North-West Corner Method (NWCM)
(ii) Least Cost Entry Method (LCM)
(iii) Vogel's Approximation Method (VAM)

Example 6: Let us take the example whose starting BFS has been obtained by NWCM.

	$v_1 = -4$ D_1	$v_2 = 0$ D_2	$v_3 = -6$ D_3
S_1 $u_1 = 5$	$\boxed{1}$ 60 $\boxed{0}$	$\boxed{5}$ 30 $\boxed{0}$	$\boxed{6}$ $\boxed{-7}$
S_2 $u_2 = 2$	$\boxed{3}$ $\boxed{-5}$	$\boxed{2}$ 10 $\boxed{0}$	$\boxed{3}$ $\boxed{-7}$
S_3 $u_3 = 6$	$\boxed{2}$ $\boxed{0}$	$\boxed{6}$ 10 $\boxed{0}$	$\boxed{0}$ 10 $\boxed{0}$
S_4 $u_4 = 8$	$\boxed{5}$ $\boxed{-1}$	$\boxed{4}$ $\boxed{4}$	$\boxed{2}$ 40 $\boxed{0}$

Total cost = 370

We start assigning value of dual variable of that row (column) in which maximum number of boxes have been allocated. In this case it is v_2. In case of a tie, choose any. We put $v_2 = 0$. Our aim is to assign other u, $v's$ such that for the boxes where allocation has been made, $u + v - c = 0$ *i.e.*, $u + v = 0$. So, we get

$$u_1 = 5,\ u_2 = 2,\ u_3 = 6;$$

Now with the help of these $u's$, we get

$$v_1 = -4,\ v_3 = -6$$

Now with the help of these v's, we get

$$u_4 = 8$$

Thus, we get all u's, v's. Now we calculate $u_i + v_j - c_{ij}$ for all boxes. Obviously for basic variable boxes it would be zero.

Remark: If a non-basic variable box has '0' in south-west corner it means for this transportation, there is an alternate solution.

It is not an optimal table. x_{42} enters. Now we form a loop.

$$\begin{array}{ccc} 10 - \theta & \text{- - - - - -} & 10 + \theta \\ \theta & \text{- - - - - -} & 40 - \theta \end{array}$$

We enter θ in x_{42} and reduce θ in x_{32}, increase θ in x_{33}, reduce θ in x_{43}.

Max θ = Min (10, 40) = 10

So we allocate θ in x_{42} and reduce θ/ increase θ in x_{32}, x_{33} and x_{43} and get new BFS.

	$v_1 = -4$ D_1	$v_2 = 0$ D_2	$v_3 = -2$ D_3
S_1 $u_1 = 5$	$\boxed{1}$ 60 $\boxed{0}$	$\boxed{5}$ 30 $\boxed{0}$	$\boxed{6}$ $\boxed{-3}$
S_2 $u_2 = 2$	$\boxed{3}$ $\boxed{-5}$	$\boxed{2}$ 10 $\boxed{0}$	$\boxed{3}$ $\boxed{-3}$
S_3 $u_3 = 2$	$\boxed{2}$ $\boxed{-4}$	$\boxed{6}$ $\boxed{-4}$	$\boxed{0}$ 20 $\boxed{0}$
S_4 $u_4 = 4$	$\boxed{5}$ $\boxed{-5}$	$\boxed{4}$ 10 $\boxed{0}$	$\boxed{2}$ 30 $\boxed{0}$

This is the solution obtained by VAM. Thus, we find VAM (LCM) gives better solution then by NWCM.

Again proceeding, we find that it is an optimal solution. Thus, VAM sometime may even give an optimal solution.

Take another example.

Example 7: Find optimal solution of the following transportation problem.

	D_1	D_2	D_3	D_4	
S_1	9	16	15	9	15
S_2	2	1	3	5	25
S_3	6	4	7	3	20
	10	15	25	10	

Solution: We find first BFS by VAM.

	$v_1 = -1$ D_1	$v_2 = 0$ D_2	$v_3 = 2$ D_3	$v_4 = -1$ D_4						
S_1 $u_1 = 10$	$\boxed{9}$ 10 $\boxed{0}$	$\boxed{16}$ $\boxed{-6}$	$\boxed{15}$ $\boxed{-3}$	$\boxed{9}$ 5 $\boxed{0}$	~~15~~ ~~5~~	✓ 6	✓ 6	–	–	–
S_2 $u_2 = 1$	$\boxed{2}$ $\boxed{-2}$	$\boxed{1}$ 0 $\boxed{0}$	$\boxed{3}$ 25 $\boxed{0}$	$\boxed{5}$ $\boxed{-5}$	~~25~~ ~~0~~	1	2	2	✓ 4	–
S_3 $u_3 = 4$	$\boxed{6}$ $\boxed{-3}$	$\boxed{4}$ 15 $\boxed{0}$	$\boxed{7}$ $\boxed{-1}$	$\boxed{3}$ 5 $\boxed{0}$	~~20~~	1	1	1	1	1
	~~10~~	~~15~~ ~~15~~	~~25~~	~~10~~ ~~5~~						
	4	3	4	2						
	–	3	4	2						
	–	3	4 ✓	2						
	–	3	–	2						
	–	4	–	3						

T.C. = 90 + 45 + 0
+ 75 + 60 + 15
= 285

It is a BFS. We now see. Whether it is optimal. If not, we move towards optimality.

It comes out to be an optimal solution. No iteration is needed. But if it is done by LCM, then, we obtain:

	$v_1 = 2$ D_1	$v_2 = 1$ D_2	$v_3 = 3$ D_3	$v_4 = -1$ D_4	
S_1 $u_1 = 12$	$\boxed{9}$ $\boxed{5}$	$\boxed{16}$ $\boxed{-3}$	$\boxed{15}$ 15 $\boxed{0}$	$\boxed{9}$ $\boxed{2}$	15
S_2 $u_2 = 0$	$\boxed{2}$ 10 $\boxed{0}$	$\boxed{1}$ 15 $\boxed{0}$	$\boxed{3}$ $\boxed{0}$	$\boxed{5}$ $\boxed{-6}$	~~25~~ ~~10~~
S_3 $u_3 = 3$	$\boxed{6}$ 0 $\boxed{0}$	$\boxed{4}$ $\boxed{1}$	$\boxed{7}$ 10 $\boxed{0}$	$\boxed{3}$ 10 $\boxed{0}$	~~20~~ ~~10~~ ~~10~~
	~~0~~ ~~10~~	~~15~~	~~25~~ 15	~~10~~	

T.C. = 360

It is not optimal, we construct a loop x_{11} enters. Loop is

θ	$15 - \theta$
$0 - \theta$	$10 + \theta$

Thus, we find that new BFS, which does not affect the optimal value as it is shifting of '0' only. So we get

	$v_1 = 9$ D_1	$v_2 = 8$ D_2	$v_3 = 15$ D_3	$v_4 = 11$ D_4
S_1 $u_1 = 0$	$\boxed{9}$ 0 $\boxed{0}$	$\boxed{16}$ $\boxed{-8}$	$\boxed{15}$ 15 $\boxed{0}$	$\boxed{9}$ $\boxed{2}$
S_2 $u_2 = -7$	$\boxed{2}$ 10 $\boxed{0}$	$\boxed{1}$ 15 $\boxed{0}$	$\boxed{3}$ $\boxed{5}$	$\boxed{5}$ $\boxed{-1}$
S_3 $u_3 = -8$	$\boxed{6}$ 0 $\boxed{-5}$	$\boxed{4}$ $\boxed{-4}$	$\boxed{7}$ 10 $\boxed{0}$	$\boxed{3}$ 10 $\boxed{0}$

Now x_{23} enters. Loop is

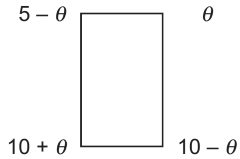
$0 + \theta$	$15 - \theta$
$10 - \theta$	θ

$\theta = 10$. So we get a new BFS

	$v_1 = 9$ D_1	$v_2 = 13$ D_2	$v_3 = 15$ D_3	$v_4 = 11$ D_4
S_1 $u_1 = 0$	$\boxed{9}$ 10 $\boxed{0}$	$\boxed{16}$ $\boxed{-3}$	$\boxed{15}$ 5 $\boxed{0}$	$\boxed{9}$ $\boxed{2}$
S_2 $u_2 = -12$	$\boxed{2}$ $\boxed{-5}$	$\boxed{1}$ 15 $\boxed{0}$	$\boxed{3}$ 10 $\boxed{0}$	$\boxed{5}$ $\boxed{-6}$
S_3 $u_3 = -8$	$\boxed{6}$ $\boxed{-5}$	$\boxed{4}$ $\boxed{1}$	$\boxed{7}$ 10 $\boxed{0}$	$\boxed{3}$ 10 $\boxed{0}$

T.C. = 310

Now x_{14} enters. Loop is



θ = 5. So we get a new BFS

	$v_1 = 9$ D_1	$v_2 = 11$ D_2	$v_3 = 13$ D_3	$v_4 = 9$ D_4
S_1 $u_1 = 0$	$\boxed{9}$ 10 $\boxed{0}$	$\boxed{16}$ $\boxed{-5}$	$\boxed{15}$ $\boxed{-2}$	$\boxed{9}$ 5 $\boxed{0}$
S_2 $u_2 = -10$	$\boxed{2}$ $\boxed{-3}$	$\boxed{1}$ 15 $\boxed{0}$	$\boxed{3}$ 10 $\boxed{0}$	$\boxed{5}$ $\boxed{-6}$
S_3 $u_3 = -6$	$\boxed{6}$ $\boxed{-3}$	$\boxed{4}$ $\boxed{1}$	$\boxed{7}$ 15 $\boxed{0}$	$\boxed{3}$ 5 $\boxed{0}$

It is not optimal. x_{32} enters. Loop is

θ = 15. New BFS is

	$v_1 = 0$ D_1	$v_2 = 1$ D_2	$v_3 = 3$ D_3	$v_4 = 0$ D_4
S_1 $u_1 = 9$	$\boxed{9}$ 10 $\boxed{0}$	$\boxed{16}$ $\boxed{-6}$	$\boxed{15}$ $\boxed{-3}$	$\boxed{9}$ 5 $\boxed{0}$
S_2 $u_2 = 0$	$\boxed{2}$ $\boxed{-2}$	$\boxed{1}$ 0 $\boxed{0}$	$\boxed{3}$ 25 $\boxed{0}$	$\boxed{5}$ $\boxed{-5}$
S_3 $u_3 = 3$	$\boxed{6}$ $\boxed{-3}$	$\boxed{4}$ 15 $\boxed{0}$	$\boxed{7}$ $\boxed{-1}$	$\boxed{3}$ 5 $\boxed{0}$

T.C. = 285

It is an optimal Table. Hence, solution is

$S_1 \to D_1$: $x_{11} = 10$; 90
$S_1 \to D_4$: $x_{14} = 5$; 45
$S_2 \to D_2$: $x_{22} = 0$; 00
$S_2 \to D_3$: $x_{23} = 25$; 75
$S_3 \to D_2$: $x_{32} = 15$; 60
$S_3 \to D_4$; $x_{34} = 5$; 15

T.C. = 285.

Example 8: Solve the following transportation problem. Demand at D_1 must be met from S_4 only

	D_1	D_2	D_3	
S_1	5	1	0	20
S_2	3	2	4	10
S_3	7	5	2	15
S_4	9	6	0	15
	5	10	15	

Solution: It is an unbalanced problem. We create D_F with cost 0 and replace costs to D_1 from S_1, S_2, S_3 by M. We purposely solve by LCM for the starting BFS.

	$v_1 = 9$ D_1	$v_2 = 6$ D_2	$v_3 = 5$ D_3	$v_4 = 0$ D_F	
S_1, $u_1 = -5$	$\boxed{M}$ $\boxed{4-M}$	$\boxed{1}$ 5 $\boxed{0}$	$\boxed{0}$ 15 $\boxed{0}$	$\boxed{0}$ $\boxed{-5}$	~~20~~ ~~5~~
S_2, $u_2 = 0$	$\boxed{M}$ $\boxed{9-M}$	$\boxed{2}$ $\boxed{4}$	$\boxed{4}$ $\boxed{1}$	$\boxed{0}$ 10 $\boxed{0}$	~~10~~
S_3, $u_3 = 0$	$\boxed{M}$ $\boxed{9-M}$	$\boxed{5}$ $\boxed{1}$	$\boxed{2}$ $\boxed{3}$	$\boxed{0}$ 15 $\boxed{0}$	~~15~~
S_4, $u_4 = 0$	$\boxed{9}$ 5 $\boxed{0}$	$\boxed{6}$ 5 $\boxed{0}$	$\boxed{0}$ $\boxed{5}$	$\boxed{0}$ 5 $\boxed{0}$	~~15~~ ~~10~~ ~~5~~
	~~5~~	~~10~~ ~~5~~	~~15~~	~~30~~ ~~15~~ ~~5~~ T.C. = 80	

It is not optimal. x_{43} enters. Loop is:

$$\begin{array}{cc} 5+\theta & 15-\theta \\ 5-\theta & \theta \end{array}$$

$\theta = 5$. So new BFS is as given by the following table:

	$v_1 = 9$ D_1	$v_2 = 6$ D_2	$v_3 = 5$ D_3	$v_4 = 0$ D_F	
S_1, $u_1 = -5$	$\boxed{M}$ $\boxed{4-M}$	$\boxed{1}$ 5 $\boxed{0}$	$\boxed{0}$ 15 $\boxed{0}$	$\boxed{0}$ $\boxed{-5}$	~~20~~ ~~5~~
S_2, $u_2 = 0$	$\boxed{M}$ $\boxed{9-M}$	$\boxed{2}$ $\boxed{4}$	$\boxed{4}$ $\boxed{1}$	$\boxed{0}$ 10 $\boxed{0}$	~~10~~
S_3, $u_3 = 0$	$\boxed{M}$ $\boxed{9-M}$	$\boxed{5}$ $\boxed{1}$	$\boxed{2}$ $\boxed{3}$	$\boxed{0}$ 15 $\boxed{0}$	~~15~~
S_4, $u_4 = 0$	$\boxed{9}$ 5 $\boxed{0}$	$\boxed{6}$ 5 $\boxed{0}$	$\boxed{0}$ $\boxed{5}$	$\boxed{0}$ 5 $\boxed{0}$	~~15~~ ~~10~~ ~~5~~
	~~5~~	~~10~~ ~~5~~	~~15~~	~~30~~ ~~15~~ ~~5~~ T.C. = 80	

It is an optimal solution. Hence, the solution is

$S_1 \to D_2$	:	$x_{12} = 10$	:	10
$S_1 \to D_3$	:	$x_{13} = 10$	:	00
$S_2 \to D_F$	:	$x_{2F} = 10$	:	00
$S_3 \to D_F$	:	$x_{3F} = 15$	:	00
$S_4 \to D_1$	:	$x_{41} = 5$	:	45
$S_4 \to D_3$	:	$x_{43} = 5$	:	00
$S_4 \to D_F$	:	$x_{4F} = 5$	:	00
			Total Cost :	55

From this we find that only S_1 and S_4 would supply to meet the demand, S_2 and S_3 would not supply any item.

5.1.4 Degeneracy in Transportation Problems

The variable attached to box(es) where allocation is made is a basic variable. If the basic variable at any step is zero, solution is said to be a degenerate solution. Degeneracy occurs in two cases.

(i) If in NWCM, LCM or VAM, a row and a column are satisfied simultaneously, we get one variable short in basis and there a variable with value '0' is to be taken as basic variable. For this purpose, in this case we assume that only one, either a row or a column is satisfied and at the other there are '0' items to be despatched or received.

(ii) If in the iteration by forming a loop, two or more boxes become empty at the same time, it also yields one variable short in basis, thus a degenerate solution. In this case, we assume that only one box has become empty and in the other boxes, there are still '0' allocation and proceed as usual.

In case of a degenerate solution, while performing the iteration, which has '0' in one of the boxes and is not an optimal solution, a situation may arise that one may have the value of $\theta = 0$, which should be incorporated. This is the case where allocating '0' has shifted from a basic variable to a non-basic variable (empty box) without affecting the transportation cost at this step though the solution, thus an alternate solution.

5.1.5 Alternate Solution

In a transportation problem, situations of alternate solution arise in the following cases.

(i) As mentioned above, while at any step of iteration an allocation $\theta = 0$ is shifted from one box to another.

(ii) In an optimal table, an empty box has '0' in south-west corner, *i.e.*, the value of $z_{ij} - c_{ij}$ for a non-basic variable, we can force this variable to enter.

5.1.6 Special Cases

(i) In a balanced transportation problem, if each row has same cost c_i, then the optimal transportation cost would be $\Sigma c_i\, a_i$, where a_i's are availabilities of sources s_i.

(ii) In a balanced transportation problem, if each column has same cost c_i, then the optimal transportation cost would be $\Sigma c_i\, b_i$, where b_i's are requirements of destinations D_i.

5.2 TRANSPORTATION WITH TRANSHIPMENT

So far we have considered transportation direct from source to destination. If transportation is permitted via another source, destination or a combination, then the problem is called transportation with transhipment. It would involve three more types of transportation, namely, transportation:

(i) from source to source,
(ii) from destination to source, and
(iii) from destination to destination.

In such cases, for all practical purposes each source is a source as well as destination. Similarly, each destination is a source as well as destination. Therefore, we consider them as sources as well as destinations. If there are m-sources and n-destinations, we consider there are $m + n$ sources and $m + n$ destinations.

With this in mind we convert this problem into a balanced ordinary transportation problem and solve it as usual. This conversion is done in the following manner. We skip theory and its development.

Remark 1: It is but natural to expect that transportation cost from S_i to S_j; $i, j = 1, 2, \ldots, m$ and from D_i to D_j, $i, j = 1, 2, \ldots, n$ are known.

Remark 2: It is expected that transportation cost from A to B and from B to A would be same. But, in special circumstances, they may be different. This would be taken into account.

In order to convert the problem in ordinary transportation problem, we assume first that it is a problem without transhipment, make it balanced, consider all special requirements.

Make a table of $m + n$ by $m + n$ boxes and write $S_1, S_2, \ldots, S_m, D_1, D_2, \ldots, D_n$ in order horizontally and vertically, and enter the transportation costs.

We know that it is a balanced transportation problem, so $\Sigma a_i = \Sigma b_j = B$, say. Availabilities and requirements are taken as follows:

Availability:

At each S_i it is $a_i + B$
At each D_j it is B

Requirement:

At each S_i it is B
At each D_j it is $b_j + B$.

This converts a transportation problem with transhipment into an ordinary transportation problem.

Remark 3: As mentioned earlier, first we have to make it a balanced problem. The following points are to be kept in mind.

(i) Transportation cost from S_i or D_j to S_F is to kept infinite, *i.e.*, big M.
(ii) Transportation cost from D_F to S_i or D_j is also to be kept infinite, *i.e.*, big M.

This is done in the light that S_F (D_F) cannot act as destination (source).

(iii) If problem has S_F, then (S_F, S_F) box would have cost '0'. Other costs in S_F row are zero and other costs in S_F column are big M.

Now we illustrate it by an example.

Example 1: Solve the following transportation problem with transhipment.

	D_1	D_2	
S_1	2	4	100
S_2	4	6	250
	200	200	

	S_1	S_2
S_1	0	1
S_2	1	0

	D_1	D_2
D_1	0	2
D_2	2	0

Solution: It is not a balanced problem. No special requirement. The converted problem is as follows. We apply VAM.

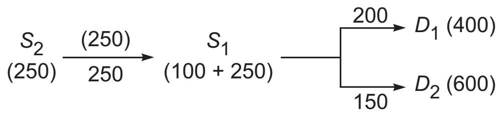
	$u_1 = -4$ S_1	$u_2 = -5$ S_2	$u_3 = 0$ S_F	$u_4 = -2$ D_1	$u_5 = 0$ D_2								
S_1, $u_1 = 4$	$\boxed{0}$ 150 $\boxed{0}$	$\boxed{1}$ $\boxed{-2}$	$\boxed{M}$ $\boxed{4-M}$	$\boxed{2}$ 200 $\boxed{0}$	$\boxed{2}$ 150 $\boxed{0}$	~~150~~ ~~500~~ ~~300~~	1	1	1	1	1	1	1
S_2, $u_2 = 5$	$\boxed{1}$ 250 $\boxed{0}$	$\boxed{0}$ 400 $\boxed{0}$	$\boxed{M}$ $\boxed{5-M}$	$\boxed{4}$ $\boxed{-1}$	$\boxed{6}$ $\boxed{-1}$	650	1	1	1	1	1	1	1
S_F, $u_3 = 0$	$\boxed{0}$ $\boxed{-4}$	$\boxed{0}$ $\boxed{-5}$	$\boxed{0}$ 400 $\boxed{0}$	$\boxed{0}$ $\boxed{-2}$	$\boxed{0}$ 50 $\boxed{0}$	~~50~~ ~~450~~	0	0	0	0	–	–	–
D_1, $u_4 = 2$	$\boxed{2}$ $\boxed{-4}$	$\boxed{4}$ $\boxed{-7}$	$\boxed{M}$ $\boxed{2-M}$	$\boxed{0}$ 400 $\boxed{0}$	$\boxed{2}$ $\boxed{0}$	~~400~~	2	2✓	–	–	–	–	–
D_2, $u_5 = 0$	$\boxed{4}$ $\boxed{-8}$	$\boxed{6}$ $\boxed{-11}$	$\boxed{M}$ $\boxed{-M}$	$\boxed{2}$ $\boxed{-4}$	$\boxed{0}$ 400 $\boxed{0}$	~~400~~	2	2	2	–	–	–	–
	250 ~~400~~	400	~~400~~	~~600~~ ~~200~~	~~600~~ ~~200~~ ~~150~~								
	1	1	M✓	2	2								
	1	1	–	2	2								
	1	1	–	2	4✓								
	1	1	–	2	4✓								
	1	1	–	2✓	2								
	1	1	–	–	2✓								
	1✓	1	–	–	–								

We now associate dual variables and calculate south-west corner entries (*i.e.*, $z_{ij} - c_{ij}$)

We find that it is an optimal table. It also shows that it has an alternate solution. Solution is

1. $S_1 \rightarrow S_1$: $x_{11} = 150$: Cost = 000
2. $S_1 \rightarrow D_1$: $x_{14} = 200$: Cost = 400
3. $S_1 \rightarrow D_2$: $x_{15} = 150$: Cost = 600
4. $S_2 \rightarrow S_1$: $x_{21} = 250$: Cost = 250
5. $S_2 \rightarrow S_2$: $x_{22} = 400$: Cost = 000
6. $S_F \rightarrow S_F$: $x_{FF} = 400$: Cost = 000
7. $S_F \rightarrow D_2$: $x_{F5} = 50$: Cost = 000
8. $D_1 \rightarrow D_1$: $x_{44} = 400$: Cost = 000
9. $D_2 \rightarrow D_2$: $x_{55} = 400$: Cost = 000

1, 5, 6, 8, 9, yield nothing. They transport to themselves. So can be neglected. Combining 2, 3, 4, we get the following picture:



(7) tells that D_2 would get short supply by 50.
Total optimal transportation cost is 1250.

EXERCISE 5.1

1. Obtain the initial basic feasible solution of the following problem by

21	16	25	13	11
17	18	14	23	13
32	27	18	41	19
6	10	12	15	

(a) North West Corner Method.
(b) Least Cost Entry Method.
(c) Vogal's Approximation Method.

(**Ans:** (a) $x_{11} = 6$, $x_{12} = 5$, $x_{22} = 5$, $x_{23} = 8$, $x_{33} = 4$, $x_{44} = 15$, TP Cost = 1095
(b) $x_{14} = 11$, $x_{21} = 1$, $x_{23} = 12$, $x_{31} = 5$, $x_{32} = 10$, $x_{34} = 4$, TP Cost = 922
(c) $x_{11} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$, $x_{33} = 12$, TP Cost = 796)

2. Consider the following transportation problem

4	2	3	2	6	8
5	4	5	2	1	12
6	5	4	7	3	14
4	4	6	8	8	

Find the initial basic feasible solution of the above problem by
(a) North West Corner method.
(b) Least Cost Entry Method.
(c) Vogal's Approximation Method.

(**Ans:** (a) $x_{11} = 4$, $x_{12} = 4$, $x_{23} = 6$, $x_{24} = 6$, $x_{34} = 2$, $x_{35} = 8$, $x_{3F} = 6$, TP Cost = 104
(b) $x_{14} = 2$, $x_{1F} = 6$, $x_{24} = 4$, $x_{25} = 8$, $x_{31} = 4$, $x_{32} = 4$, $x_{33} = 6$, $x_{34} = 2$, TP Cost = 102

(c) $x_{12} = 4$, $x_{14} = 4$, $x_{24} = 4$, $x_{25} = 8$,
$x_{31} = 4$, $x_{33} = 6$, $x_{3F} = 4$, TP Cost = 80)

3. A company has three factories I, II, III and four warehouses A, B, C & D. The transportation cost (in Rs.) per unit from each factory to each warehouse, the availability of goods at each factory and requirements of each warehouse are given below:

Warehouse — Availability

Factory	A	B	C	D	Availability
I	42	48	38	37	160
II	40	49	52	51	150
III	39	38	40	43	190
Requirements	80	90	110	160	

Determine the optimum schedule to minimize the transportation cost (use VAM to find initial BFS) and answer the following:

(i) Does the problem has degenerate solution?

(ii) Does an alternate solution exists for the problem, if yes find all the solutions.

(**Ans:** $x_{14} = 160$, $x_{21} = 80$, $x_{22} = 10$, $x_{32} = 80$, $x_{33} = 110$, TP Cost = 17050)

4. Using VAM to find initial BFS of the following transportation problem, show that the initial BFS itself is optimal solution.

6	8	4	14
4	0	8	12
1	2	6	5
6	10	15	

(**Ans:** $x_{13} = 14$, $x_{21} = 1$, $x_{22} = 10$, $x_{23} = 1$, $x_{31} = 5$, TP Cost = 73)

5. A person has three factories I, II, III which supply goods to three godowns E, F and G. Daily factory capacities are 10, 80 & 15 units, respectively while the daily requirements of godowns are 75, 25 and 50 units, respectively. Unit shipping cost (in Rs.) are given below:

	Godown		
Factory	E	F	G
I	5	1	7
II	6	4	6
III	3	2	5

The penalty cost for not satisfying demand at the godowns E, F and G are Rs. 5, Rs. 3 and Rs. 2 per unit respectively. Determine the optimum shipping schedule to minimize the cost.

(**Ans:** $x_{12} = 10$, $x_{21} = 60$, $x_{22} = 15$, $x_{23} = 5$, $x_{31} = 15$
and $x_{F3} = 40$, Min Cost = Rs 505)

6. Solve the following transportation problem. The demand at destination 2 must be shipped from source 2 only. The entries are transportation cost per unit.

5	1	0	20
3	2	4	10
6	5	0	15
5	10	15	

(**Ans:** $x_{11} = 5$, $x_{1F} = 15$, $x_{22} = 10$, $x_{33} = 15$, TP Cost = 35)

Problem has alternate solution also.

7. Consider the following transportation problem

	D_1	D_2	D_3	
S_1	2	6	7	90
S_2	4	3	4	10
S_3	3	7	2	20
	60	50	50	

Let the penalty costs per unit of unsatisfied demand be 7, 5, 3 for destinations D_1, D_2 and D_3, respectively. Find the optimal solution (Use VAM to find initial BFS)

(**Ans:** $x_{11} = 60$, $x_{12} = 30$, $x_{22} = 10$, $x_{33} = 20$, $x_{F2} = 10$, $x_{F3} = 30$
(D_2 & D_3 will be 10 & 30 units short of demands). TP Cost = 510)

8. In exercise 8, let there be no penalties but the demand at D_3 must be satisfied exactly. Find the optimal solution.

(**Ans:** $x_{11} = 60$, $x_{12} = 0$, $x_{13} = 30$, $x_{22} = 10$,
$x_{33} = 20$, $x_{F2} = 40$, TP Cost = 400)

9. There are four villages V_1, V_2, V_3 and V_4 which are affected by floods. Foodgrain is to be dropped in these villages by three aircraft A_1, A_2 and A_3. The following matrix is given.

	V_1	V_2	V_3	V_4	a_i
A_1	9	7	5	2	60
A_2	7	9	4	2	40
A_3	1	4	8	9	50
bj	30	50	60	40	

In the above matrix, a_i denote the total number of trips that aircraft A_i can make in one day. b_j denote the number of trips required to village V_j in one day, the cost c_{ij} denote the amount of foodgrains that aircraft A_i can carry to village V_j in one trip. Find the number of trips that aircraft A_i should make to village V_j so that the total quantity of food dropped in a day is maximized. (Use VAM to find initial BFS).

Hint: As it is maximization problem, first convert all c_{ij} to $-c_{ij}$ to make it minimization problem and then solve in usual manner.

(**Ans:** $x_{11} = 30$, $x_{12} = 10$, $x_{13} = 20$, $x_{22} = 40$, $x_{33} = 10$, $x_{34} = 40$
V_3 falls short of 30 trips. Maximize value = 1240)

10. Consider the following problem

	D_1	D_2	
S_1	3	5	100
S_2	5	7	250
	200	200	

	S_1	S_2
S_1	1	2
S_2	2	1

	D_1	D_2
D_1	1	3
D_2	3	1

Solve the above transportation problem with transhipment.

(**Ans:** (a) 200 units from S_1 to D_1, 150 units from S_1 to D_2, 250 units from S_2 to S_1: Cost = 1400. Alternate solution exists.)

5.3 ASSIGNMENT PROBLEM

5.3.1 Introduction

The assignment problem is nothing but is a special case of transportation problem in which the objective is to assign a number of sources to the equal number of destinations at a minimum cost.

Assignment problem is a completely degenerate form of transportation problem. The units available at each source is 1 and also the units required at each destination is equal to 1. It means that in each row and each column there will be exactly one cell not like transportation problem ($n + n - 1 = 2n - 1$) cell.

5.3.2 Mathematical Formulation of the Assignment

Problem: Let there be n workers and n jobs and the problem is to assign n jobs to n workers in such a way that the total cost is minimum.

The cost matrix (c_{ij}) is given below:

		Jobs				Availability
		J_1	J_2	…	J_n	
Workers	W_1	c_{11}	c_{12}	…	c_{1n}	1
	W_2	c_{21}	c_{22}	…	c_{2n}	1
	⋮	⋮				⋮
	W_n	c_{n1}	c_{n2}	…	c_{nn}	1
Requirement		1	1	…	1	

Let x_{ij} is the assignment of i^{th} worker to j^{th} job such that

$$x_{ij} = 1, \text{ if worker } i \text{ is assigned to job } j$$
$$= 0, \text{ otherwise.}$$

Then the mathematical formulation of the assignment problem is

$$\text{Minimize} \quad Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, x_{ij} = c_{11}\, x_{11} + c_{12}\, x_{12} + \dots + c_{nn}\, x_{nn}$$

Subject to $\sum_{i=1}^{n} x_{ij} = 1 \;\forall\; j = 1, 2, \ldots, n$

$\sum_{j=1}^{n} x_{ij} = 1 \;\forall\; i = 1, 2, \ldots, n$

$x_{ij} = 0$ or $1, \;\forall\; i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$

Example 1: Consider the following assignment problem

	D_1	D_2	D_3
S_1	2	3	4
S_2	5	6	2
S_3	3	2	5

Write the above problem as transportation problem.

Solution: Let x_{ij} = Assignment of S_i ($i = 1, 2, 3$) to D_j ($j = 1, 2, 3$) such that

$x_{ij} = 1$, if S_i is assigned to D_j
$= 0$, otherwise

Then the transportation problem is

Minimize $Z = 2x_{11} + 3x_{12} + 4x_{13} + 5x_{21} + 6x_{22} + 2x_{23} + 3x_{31} + 2x_{32} + 5x_{33}$

Subject to
$x_{11} + x_{12} + x_{13} = 1$
$x_{21} + x_{22} + x_{23} = 1$
$x_{31} + x_{32} + x_{33} = 1$
$x_{11} + x_{21} + x_{31} = 1$
$x_{12} + x_{22} + x_{32} = 1$
$x_{13} + x_{23} + x_{33} = 1$
$x_{ij} = 0$ or 1 for $i = 1, 2, 3$ and $j = 1, 2, 3$

Theorem: In an assignment problem if we add or subtract a number which is constant to every element of a row (or column) of the cost matrix $\{c_{ij}\}$, then an assignment which minimizes the total cost on one matrix also minimizes the total cost on the other matrix. Mathematically we can say that if $x_{ij} = x_{ij}^*$ is an optimal solution of the problem

Minimize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$,

Subject to $\sum_{j=1}^{n} x_{ij} = 1 = \sum_{i=1}^{n} x_{ij}$,

$x_{ij} = 0$ or 1 then x_{ij}^* is also an optimal solution of Minimize $Z^* = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^*\, x_{ij}$, where $c_{ij}^* = c_{ij} - a_i - b_j$ for all $i = 1, 2, \ldots, n$ and a_i, b_j are some real numbers.

Proof:

$$Z^* = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^{*}\, x_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} (c_{ij} - a_i - b_j)\, x_{ij} \;(\because\; c_{ij}^{*} = c_{ij} - a_i - b_j \text{ given})$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, x_{ij} - \sum_{i=1}^{n} a_i \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} b_j \sum_{i=1}^{n} x_{ij}$$

$$Z^* = Z - \sum_{i=1}^{n} a_i - \sum_{j=1}^{n} b_j \;(\because\; \sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1)$$

This shows that the minimization of the new objective function Z^* gives the same solution as the minimization of the original function Z. (Because $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{n} b_j$ are independent of x_{ij}).

This theorem is known as Reduction Theorem.

5.3.3 Hungarian Algorithm to Solve an Assignment Problem

An efficient method to solve an assignment problem was developed by an Hungarian mathematician D. Konig, which is given as below:

Step 1: From the cost matrix, check whether number of sources is equal to the number of destinations or not, if yes then go to the next step, otherwise add a dummy source or dummy destination with cost zero to make the number of sources equal to the number of destinations.

Step 2: After making number of sources equal to the number of destinations, identify the smallest element in each row and then subtract the same from each element of the row. This will give at least one zero in each row.

Step 3: In the reduced matrix obtained in step 2, identify the smallest element in each column and subtract the same from each element of each column. This assumes at least one zero in each column.

Step 4: By step 2 and step 3 we get a reduced matrix in which at least one zero is in each row and each column. Now we can go for optimal assignment as follows:

(i) Examine each row and see which row has a single zero. Enrectangle this zero (☐) and cross off all other zeros in its column. Continue until all the rows have been taken care of.

(ii) Similarly, repeat the procedure 4(i) for each column of the reduced matrix.

(iii) If a row or/and a column has two or more than two zeros and one cannot be chosen by inspection then assign arbitrary any one of these rows and cross off all other zeros of that row/column.

(iv) Repeat (i) to (iii) above until the chain of assigning (☐) or cross (X) ends.

Step 5: If the number of assignments (☐) is equal to n (the order of cost matrix), then we got the optimum solution. If the number of assignments is less than n, then go to the next step 6.

Step 6: Draw the minimum number of horizontal and/or vertical lines to cover all the zeros of the reduced matrix. This can be done by using following procedure.

(i) Mark (✓) rows that do not have any assigned zero.
(ii) Mark (✓) columns that have assigned zeros in the marked rows.
(iii) Mark (✓) rows that have assigned zeros in the marked columns.
(iv) Repeat (ii) and (iii) above until the chain of marking is completed.
(v) Draw lines through all the unmarked rows and marked columns.

Step 7: Find the new revised cost matrix as follows:

(i) Find the smallest element of the reduced matrix not covered by any of the lines.
(ii) Subtract this element from all the uncovered elements and add the same to all the elements lying at the intersection of any two lines.

Step 8: Go to step 5 and repeat the procedure until we get optimal solution.

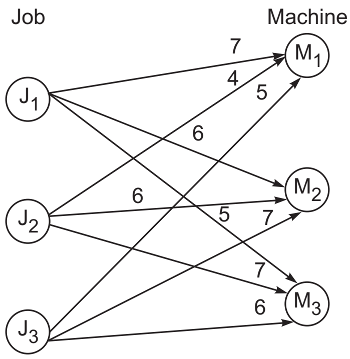
Example 2: An engineer wants to assign 3 jobs J_1, J_2, J and J_3 to 3 machines M_1, M_2 and M_3 in such a way that each job is assigned to some machines and no machine works on more than one job. The cost of assigning job i to machine j is given below:

		M_1	M_2	M_3
Jobs	J_1	7	6	5
	J_2	4	6	7
	J_3	5	7	6

(a) Draw the associated network.
(b) Formulate the network LPP.

Solution:

(a) Network



(b) LPP

Minimize $Z = 7x_{11} + 6x_{12} + 5x_{13} + 4x_{21} + 6x_{22} + 7x_{23} + 5x_{31} + 7x_{32} + 6x_{33}$

Subject to
$$x_{11} + x_{12} + x_{13} = 1$$
$$x_{21} + x_{22} + x_{23} = 1$$
$$x_{31} + x_{32} + x_{23} = 1$$
$$x_{11} + x_{21} + x_{31} = 1$$
$$x_{12} + x_{22} + x_{32} = 1$$
$$x_{13} + x_{23} + x_{33} = 1$$
$$x_{ij} = 0 \text{ or } 1,\ i = 1, 2, 3,\ j = 1, 2, 3$$

Example 3: A Dean of an Engineering College wants four tasks to be performed by four heads of the department. The heads differ in efficiency, and the tasks differ in their intrinsic difficulty. His estimate of the time each head would take to perform each task is given below:

	Heads			
Tasks	*A*	*B*	*C*	*D*
I	20	28	19	13
II	15	30	16	28
III	40	21	20	17
IV	21	28	26	12

How should the tasks be allocated one to a head, so as to minimise the total man-hour?

Solution:

Step 1: Here number of tasks = number of heads = 4

Step 2: Subtracting the smallest element of each row from every element of the corresponding row, we get the following reduced matrix.

7	15	6	0
0	15	1	13
23	4	3	0
9	16	14	0

Step 3: Subtract the smallest element of each column of the reduced matrix from every element of the corresponding column, we get the reduced matrix as under.

7	11	5	0
0	11	0	13
23	0	2	0
9	12	13	0

Step 4: Starting with row 1, enrectangle (☐) *i.e* make assignment a single zero, if any and cross (X) all other zeros in the column as marked. Then we get

$$\begin{array}{|cccc|}\hline 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & \boxed{\cancel{0}} & 13 \\ 23 & \boxed{0} & 2 & \boxed{\cancel{0}} \\ 9 & 12 & 13 & \boxed{\cancel{0}} \\ \hline\end{array}$$

In the above matrix, we arbitrarily enrectangled a zero in column 1, because row 2 had two zeros. Now we can note that column 3 & row 4 have no assignment. So we move to the next step.

Step 5:

(i) Since row 4 has no assignment, we mark it (v).

(ii) There is a zero in the column 4 of the marked row. So, we mark column 4 (✓).

(iii) There is an assignment in the row 1 of ticked column. So we mark row 1 (✓).

(iv) Draw straight lines through all unmarked rows and marked columns. Thus, we get.

$$\begin{array}{|cccc|c}\cline{1-4} 7 & 11 & 5 & \boxed{0} & \checkmark \\ \boxed{0} & 11 & \cancel{0} & 13 & \\ 23 & \boxed{0} & 2 & \cancel{0} & \\ 9 & 12 & 13 & \cancel{0} & \checkmark \\ \cline{1-4}\end{array}$$

Step 6: In step 6, we observe that the minimum number of lines, drawn is 3 which is less than the order of cost matrix 4. Therefore, this is not the optimum assignment.

To increase the minimum number of lines, we generate new zeros in the modified matrix.

The smallest element not covered by the lines is 5. Subtracting 5 from all the uncovered elements and adding 5 to all the elements 13, 0 (lying at the intersection of the lines), we obtain the new reduced cost matrix as follows:

$$\begin{array}{|cccc|}\hline 2 & 6 & 0 & 0 \\ 0 & 11 & 0 & 18 \\ 23 & 0 & 2 & 5 \\ 4 & 7 & 8 & 0 \\ \hline\end{array}$$

Step 7: Repeating step 4 on the reduced matrix, we get

$$\begin{array}{|cccc|}\hline 2 & 6 & \boxed{0} & \cancel{0} \\ \boxed{0} & 11 & \cancel{0} & 18 \\ 23 & \boxed{0} & 2 & 5 \\ 4 & 7 & 8 & \boxed{0} \\ \hline\end{array}$$

Now the above matrix shows that each row and each column has one and only one assignment. We get the optimal solution. The optimum assignment is

$$\text{I} \to \text{C, II} \to \text{A, III} \to \text{B, IV} \to \text{D}$$

The minimum total cost = 19 + 15 + 21 + 12 = 67 man-hours.

Example 4: A company wishes to assign 4 jobs to 3 machines. The estimates of the times (in minutes) each machine would take to complete a job is given below. How should the jobs should be allocated to the machines, so that the total cost is minimum?

	Machines		
Jobs	M_1	M_2	M_3
I	8	25	14
II	12	26	5
III	34	19	14
IV	17	29	19

Solution: Since, the given problem is unbalanced, we add a dummy machine M_4 with all the entries zero and use Hungarian algorithm to find optimal solution.

Now reduce the balanced cost matrix and make assignment in rows and columns having single zeros. We have,

$$\begin{array}{c|cccc|}
 & M_1 & M_2 & M_3 & M_4 \\
\hline
\text{I} & \boxed{0} & 6 & 9 & \cancel{0} \\
\text{II} & 4 & 7 & \boxed{0} & \cancel{0} \\
\text{III} & 26 & \boxed{0} & 9 & \cancel{0} \\
\text{IV} & 9 & 10 & 14 & \boxed{0} \\
\hline
\end{array}$$

The optimum assignment is

I $\to M_1$, II $\to M_3$, III $\to M_2$ and job IV should be assigned to dummy machine M_4 *i.e.*, job IV remains incompleted. The minimum time is 8 + 5 + 19 = 32 minutes.

5.3.4 Special Cases in Assignment Problems

(a) ***Maximization Case:*** In some cases the objective function is maximization in nature instead of minimization. For example, the elements of the matrix are revenues, profits, etc. instead of costs, in such cases, we would like to maximize total revenue or profit not to minimize. To solve such problem, we can still use Hungarian method by changing the objective function to minimization from maximization *i.e.*, multiply each element of the given matrix by –1 and then solve it in usual manner by the Hungarian algorithm but finding the value of the objective function we take positive values of c_{ij}.

(b) ***Prohibited Assignment:*** Sometimes due to some reason we can put a condition that a particular person (machine) cannot be assigned a particular activity (job). In such cases, the cost of performing that particular activity by a particular person is considered to be very large (M, a positive number) as large as to prohibit the entry of this pair.

Example 5: The following is the cost matrix of assigning 4 persons to 4 jobs. Find the optimal assignment if person 1 cannot be assigned to job A.

	Jobs			
Persons	*A*	*B*	*C*	*D*
1	–	6	3	1
2	5	8	6	7
3	6	9	5	4
4	4	7	7	3

What is the minimum total cost?

Solution: First put $c_{11} = M$, then
reduce the cost matrix by subtracting the smallest element of each row (column) from the corresponding row (column). In the reduced matrix make assignment in rows and columns that have single zero.

Draw the minimum number of lines to cover all the zeros of the reduced matrix. See Table 1

$$\begin{array}{|cccc|}\hline M & 2 & 1 & \boxed{0} \\ \xcancel{0} & \boxed{0} & \xcancel{0} & 2 \\ 2 & 2 & \boxed{0} & \xcancel{0} \\ 1 & 2 & 3 & \xcancel{0} \\ \hline\end{array}$$

Table 1

Modify the reduced cost matrix by subtracting 1 from all the elements not covered by the lines and adding 1 at the intersecting two lines. We get Table 2.

$$\begin{array}{|cccc|}\hline M & 1 & \xcancel{0} & \boxed{0} \\ \boxed{0} & \xcancel{0} & \xcancel{0} & 3 \\ 2 & 2 & \boxed{0} & 1 \\ 0 & \boxed{0} & 2 & \xcancel{0} \\ \hline\end{array}$$

The optimum solution is.

Person 1 $\rightarrow$ Job *D*
Person 2 $\rightarrow$ Job *A*
Person 3 $\rightarrow$ Job *C*
Person 4 $\rightarrow$ Job *B*

or,

Person 1 $\rightarrow$ Job *D*
Person 2 $\rightarrow$ Job *B*

Person 3 $\rightarrow$ Job C
Person 4 $\rightarrow$ Job A, Minimum cost = 18

EXERCISE 5.2

1. Solve the following assignment problem.

	A	B	C	D
I	2	5	7	4
II	10	8	11	10
III	5	6	12	8
IV	9	8	9	6

(**Ans:** I $\rightarrow$ A, II $\rightarrow$ C, III $\rightarrow$ B, IV $\rightarrow$ D: Cost = 25)

2. Use Hungarian method to solve the following assignment problem.

	1	2	3	4
A	9	24	14	19
B	14	29	4	14
C	34	19	11	23
D	16	24	23	19

(**Ans:** A $\rightarrow$ 1, B $\rightarrow$ 3, C $\rightarrow$ 2, D $\rightarrow$ 4, Cost = 51)

3. A software company that has three projects with the departments of health, education and housing of U.P. Govt. Based on the background and experience of the project leaders, they differ in terms of their performance of various projects. The performance score matrix is given below:

Project leaders	*Projects*		
	Health	*Education*	*Housing*
A	22	28	44
B	26	34	52
C	34	36	46

Find the optimal assignment that maximizes the total performance score.
Hint: Multiply all the entries of the matrix by –1 and then solve in usual manner. The optimal solution is

(**Ans:** A $\rightarrow$ Education, B $\rightarrow$ Housing, C $\rightarrow$ Health:
Max. Performance = 114)

4. Find the optimal assignment schedule for the following problem.

Salesmen	*Markets* M_1	M_2	M_3	M_4
I	81	71	76	73
II	76	76	81	86
III	79	79	83	79

What is the total maximum sale?

(**Ans:** I $\rightarrow$ M_2, II $\rightarrow$ M_1, III $\rightarrow$ M_4, Maximum sales = 226).

Chapter 6

Integer Linear Programming and Travelling Salesman Problem

6.1 INTEGER LINEAR PROGRAMMING PROBLEM

6.1.1 Introduction

If an LPP has extra condition that decision variables are integers, then that problem is called Integer Linear Programming Problem (ILPP). In this, there are two cases.

(i) When all decision variables are required to be integers. In this situation, we call it all integer ILPP.

(ii) When not all decision variables are required to be integers. In this case we call it Mixed ILPP.

The simplex method discussed earlier may give the required integer solution but, in general, the solution obtained by simplex method may not be the required solution. We are now going to discuss method to solve all integer ILPP and also Mixed ILPP. First. we discuss all integer case.

6.1.2 All Integer ILPP

Let there be an all integer ILPP,

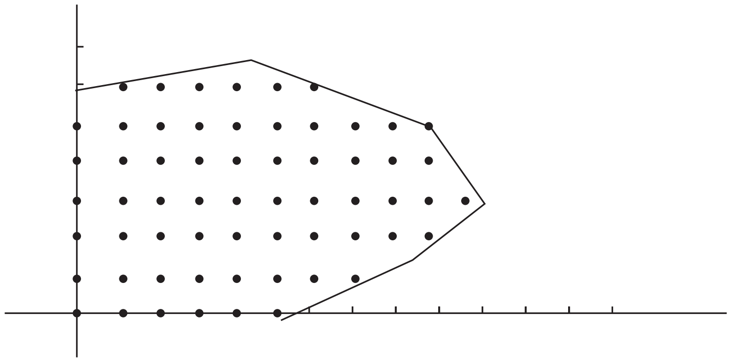
Opt $$Z = f(X) = C^T X$$

Subject to $$AX \gtreqless b, \; X \geq 0 \text{ and } x_i's \text{ are integer.}$$

We can find S_F' by excluding the condition x_i's are integer and then we can pick only those points X from S_F' which has all integer coefficients. Then S_F for all integers ILPP would be

$S_F = \{X \in S_F' | \text{all coordinates of } X \text{ are integers}\}$

This S_F would be obviously finite if S_F' is a bounded set. Geometrically, we look in E_2.



We can now substitute these points of S_F in the objective function and find the one which gives the optimal solution.

It seems to be simple in saying, but in practice, it will not be easy even to find all possible integer solutions in S_F', *i.e.*, S_F.

So we have to look for another method. Let us assume that simplex method has been applied and optimal solution has been obtained. If it has given us all integer solution then it is okay, we have the required solution. If x_i is not an integer, then can we say its approximation, *i.e.*, $[x_i]$ or $[x_i] + 1$ will work as optimal solution?

To answer this, we look at the following example:

Min $Z = -12x_1 - 15x_2$

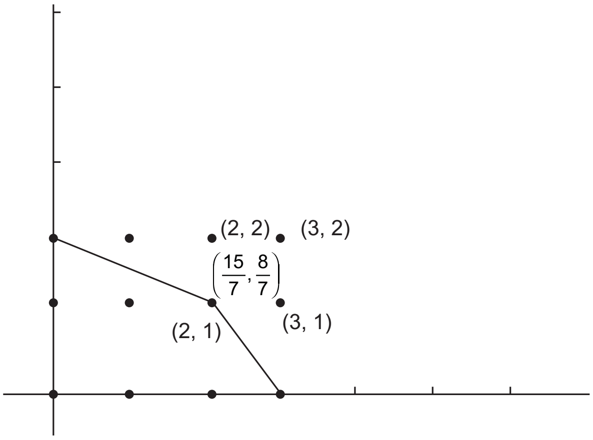
Subject to $\quad 4x_1 + 3x_2 \leq 12$

$\quad 2x_1 + 5x_2 \leq 10;\ x_1, x_2 \geq 0$ & integers.

If we solve it by simplex method by ignoring x_1, x_2 integers, we obtain the optimal table

B.V.	x_1	x_2	S_1	S_2	Solution
Z	0	0	$-\frac{15}{7}$	$-\frac{12}{7}$	$-\frac{300}{7}$
x_1	1	0	$\frac{5}{14}$	$-\frac{3}{14}$	$\frac{15}{7}$
x_2	0	1	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{8}{7}$

Then we find that $x_1 = \dfrac{15}{7}$ and $x_2 = \dfrac{8}{7}$ are solutions which are not integer. Approximating them by integers and taking all possible combinations, we get (2, 1), (2, 2), (3, 2), (3, 1). Is any one of them, the required solution? Let us see geometrically and answer it.



We find that out of these 4 possible approximate integer solution, three even do not belong to S_F. Only one (2, 1) belong to S_F.

There are eight integer solutions in S_F, the value of objective function at them are

(0, 2)	:	–30
(0, 1)	:	–15
(1, 1)	:	–27
(2, 1)	:	–39
(0, 0)	:	00
(1, 0)	:	–12
(2, 0)	:	–24
(3, 0)	:	–36

Thus, (2, 1) is the optimal solution with value –39. This, by chance, also happens to be one of the approximated integer solutions.

Thus, approximation may not be a member of S_F and if it is, it may not be the optimal solution. Therefore, we want to get the solution from S_F'. For this, we would go on slashing S_F' so that no point of S_F is removed but the optimal solution obtained in S_F' is removed. This is what is done in the method discussed below and is known as GOMORY's CUTTING PLANE METHOD.

6.1.3 Gomory's Cutting Plane Method

The related LPP of ILPP is solved by simplex method. If the solution reached is the required all integer solution, it is okay, otherwise we apply a cut, *i.e.*, to add a constraint so that the

(i) optimal solution reached is removed, *i.e.*, becomes infeasible, and

(ii) no integer feasible solution is removed, *i.e.*, no integer feasible solution becomes infeasible.

This is arrived by Gomory by the following method.

Let the optimal table of the related LPP be

B.V.	x_1	x_2	$\dots$	x_i	$\dots$	x_n	S_1	S_2	$\dots$	S_m	Solution
x_1											
$\vdots$											
x_i	0			1							α
$\vdots$											
s											

If a decision variable is non-basic, its value is zero so there is no problem. If a decision variable is basic then we look at its value in the table. If it is an integer then there is no problem. If it is not, we have to find a cut (constraint).

Let the decision variable x_i be a basic variable whose value is non-integer α. x_i-row in optimal table will have '1' below x_i; zero below each other basic variables; and below each non-basic variables it will have $\alpha_1, \alpha_2, \dots, \alpha_0$ which may have any value. The corresponding equation would be

$$x_i + \alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_P y_P = \alpha$$

where, y_i's are non-basic variables (new names, right now). Let us write each α as sum of its integral part and fraction. Let

$$[\alpha_i] = \beta_i \ , \ \beta_i \geq 0$$

$$[\alpha] = \beta \ , \ \beta \geq 0$$

then,
$$\alpha_i = \beta_i + f_i$$

$$\alpha = \beta + f,$$

where, β_i, β are integral parts and f_i, f fractional parts of α_i, α, respectively. Obviously

$$0 \leq f_i < 1$$

$$0 < f < 1$$

because α is assumed to be a fraction.

The equation now becomes

$$x_i + \sum_{i=1}^{p} (\beta_i + f_i) y_i = \beta + f$$

or,
$$x_i = \left(\beta - \sum_{i=1}^{p} \beta_i \, y_i \right) + \left(f - \sum_{i=1}^{p} f_i \, y_i \right)$$

We want x_i to be an integer. For this purpose solution must change, some non-basic become basic, *i.e.*, some y_i become greater than '0', and integers.

In such case first bracket, namely,

$$\beta - \sum_{i=1}^{p} \beta_i \, y_i \qquad (*)$$

is an integer. Hence, second bracket, namely,

$$f - \sum_{i=1}^{p} f_i \, y_i \qquad (**)$$

must become an integer. Since, $y_i \geq 0$ and $0 \leq f_i < 1$, the term

$$\sum_{i=1}^{p} f_i \, y_i \geq 0$$

Since, $0 < f < 1$, $f - \sum_{i=1}^{p} f_i \, y_i \geq 0$ cannot make it an integer. But

$$f - \sum_{i=1}^{p} f_i \, y_i \leq 0$$

will make it integer, though negative. Thus, we add the constraint

$$f \leq \sum_{i=1}^{p} f_i \, y_i$$

to the LPP. It is a cut. It makes the optimal solution already obtained infeasible as all y_i are non-basic, hence zero. So $f \leq 0$, a contradiction. Also, since fractional parts are used, no integer solution becomes infeasible. Thus, the cut is generated as follows:

$$\sum_{i=1}^{p} (\alpha_i - [\alpha_i]) \, y_i \geq \alpha - [\alpha]$$

i.e.,

$$(1 - [1]) \, x_i + \Sigma(0 - [0]) \, z_i + \Sigma(\alpha_i - [\alpha_i]) \, y_i \geq \alpha - [\alpha]$$

Thus, if x_i is not an integer and required to be an integer, we take the x_i row, replace each coefficient a by $a - [a]$ and right hand side by $x_i - [x_i]$ and '=' sign by '≥'.

Remark 1: In case more than one decision variables have non-integral values, we select the one which has largest integral part.

Remark 2: In case of a tie, we break it arbitrarily or select the one which has largest strength, where the strength of the cut is determined as follows:

If $a_1 \, x_1 + a_2 \, x_2 + \ldots + a_n \, x_n \geq b > 0$

is the cut, then its strength is defined by

$$\frac{b}{a_1 + a_2 + \ldots + a_n}$$

A cut having largest strength reduces S_F maximum.

Remark 3: The condition whether slack/surplus variables are to be integer or not can be easily determined by inspecting the constraints, e.g., in the constraint

$$2x_1 + 3x_2 - 5x_3 + S_1 = 7$$

S_1 has to be an integer, because x_i's are integer, while in the constraint

$$\frac{1}{2}x_1 - 2x_2 + \frac{3}{2}x_3 + S_1 = \frac{5}{7}$$

S_1 is not integer. Also in the constraint

$$\frac{1}{2}x_1 + \frac{1}{4}x_2 + x_3 + S_1 = 3$$

S_1 may or may not be an integer.

If all s_i's are also required to be integers, it is all integer ILPP, otherwise mixed ILPP.

6.1.4 Mixed Integer Linear Programming Problems

For mixed ILPP, we proceed with the same idea except that the cut is determined by the following method:

Let x_i be the decision variable which is required to be an integer. It is a basic variable, then as earlier x_i-row gives

$$x_i + \sum_{k=1}^{p} \alpha_k y_k = \alpha,$$

where, y_k are non-basic variable. By rearranging the terms of summation sign, first positive, then negative we get

$$x_i + \Sigma \alpha_k^+ y_k + \Sigma \alpha_k^- y_k = \alpha$$

where, α_k^+ and α_k^- are the positive or negative coefficients. Let $\alpha = \beta + f$, where $\beta = [\alpha]$. Then the cut is given by

$$\Sigma \alpha_k^+ y_k + \frac{f}{f-1} \Sigma \alpha_k^- y_k \geq f$$

For example, if the row is

$$x_1 + 5x_3 + \frac{7}{10}S_1 - \frac{8}{7}S_2 - \frac{9}{2}S_3 = \frac{15}{7}$$

then the cut is

$$5x_3 + \frac{7}{10}S_1 + \frac{1/7}{1/7-1}\left(-\frac{8}{7}S_2 - \frac{9}{2}S_3\right) \geq \frac{1}{7}$$

or, $$5x_3 + \frac{7}{10}S_1 - \frac{1}{6}\left(-\frac{8}{7}S_2 - \frac{9}{2}S_3\right) \geq \frac{1}{7}$$

or, $$5x_3 + \frac{7}{10}S_1 + \frac{4}{21}S_2 + \frac{3}{4}S_3 \geq \frac{1}{7}.$$

Now we illustrate this concept by examples

Example 1:

Max $Z = x_1 - x_2$

$$x_1 + 2x_2 \le 4$$
$$6x_1 + 2x_2 \le 9$$
$$x_1, x_2 \ge 0 \text{ and integers}$$

Solution: Max $Z = x_1 - x_2$

$$x_1 + 2x_2 + S_1 = 4$$
$$6x_1 + 2x_2 + S_2 = 9$$
$$x_1, x_2 \ge 0, \text{ Also } S_1, S_2 \ge 0$$

All are integers.

Solving the related LPP, we obtain the following optimal table

BV	x_1	x_2	S_1	S_2	Solution
Z	0	4/3	0	1/6	3/2
S_1	0	5/3	1	–1/6	5/2
x_1	1	1/3	0	1/6	3/2

All variables have to be integers. Both have same fractional part

S_1-cut is $\frac{2}{3}x_2 + \left(\frac{4}{6} - (-1)\right)S_2 \ge \frac{1}{2}$

or, $$\frac{2}{3}x_2 + \frac{5}{6}S_2 \ge \frac{1}{2}$$

x_1-cut is $\frac{1}{3}x_2 + \frac{1}{6}S_2 \ge \frac{1}{2}$

Strength: S_1-cut $\frac{1/2}{2/3+5/6} = 1/3$

Strength: x_1-cut $\frac{1/2}{1/3+1/6} = 1$

So we select x_1-cut to make x_1 integer.

Actually, when there is a tie between x & S, we should take x, as if x is integer would force S to be integer but the converse need not hold. So we add the constraints

$$2x_2 + S_2 \ge 3$$

or, $$2x_2 + S_2 = 9 - 6x_1 \ge 3$$

or, $$x_1 \le 1$$

or, $$x_1 + S_3 = 1.$$

BV	x_1	x_2	S_1	S_2	S_3	Solution
Z	0	4/3	0	1/6	0	3/2
S_1	0	5/3	1	–1/6	0	5/2
x_1	1	1/3	0	1/6	0	3/2
S_3	1	0	0	0	1	1

BV	x_1	x_2	S_1	S_2	S_3	Solution	
Z	0	4/3	0	1/6	0	3/2	
S_1	0	5/3	1	–1/6	0	5/2	
x_1	1	1/3	0	1/6	0	3/2	Apply
S_3	0	–1/3	0	$\boxed{-1/6}$	1	–1/2	Dual simplex method
Z	0	1	0	0	1	1	
S_1	0	2	1	0	–1	3	
x_1	1	0	0	0	1	1	
S_2	0	2	0	1	–6	3	

It is optimal table and all variables are ≥ 0 and integers

Thus, solution is

$$x_1 = 1,\ x_2 = 0,\ S_1 = 3,\ S_2 = 0,\ S_3 = 3$$

and optimal value is 1.

Example 2:

Max $Z = .25x_1 + x_2$

$$.50x_1 + x_2 \le 1.75$$
$$x_1 + 0.30x_2 \le 1.50$$
$$x_1, x_2 \ge 0 \text{ and integers}$$

Solution: Max $Z = .25x_1 + x_2$

$$.5x_1 + x_2 + S_1 = 1.75$$
$$x_1 + 3x_2 + S_2 = 1.50$$
$$x_1, x_2, S_1, S_2 \ge 0,\ x_1, x_2 \text{ integers.}$$

Obviously S_1, S_2 need not be integers.

We solve it as mixed ILPP. Optimal table of related LPP is

BV	x_1	x_2	S_1	S_2	Solution
Z	.25	0	1	0	1.75
x_2	.5	1	01	0	1.75
S_2	.85	0	–.3	1	.975

x_2 is not an integer. Cut is

$$.5x_1 + S_1 \ge .75$$

or, $$1.75 - x_2 \ge .75$$

or, $$x_2 \le 1 \text{ or } x_2 + S_3 = 1$$

So, we get

B.V.	x_1	x_2	S_1	S_2	S_3	Solution	
Z	.25	0	1	0	0	1.75	
x_2	.5	1	1	0	0	1.75	
S_2	.85	0	–.3	1	0	.975	
S_3	0	1	0	0	1	1	
	.25	0	1	0	0	1.75	
x_2	.5	1	1	0	0	1.75	Apply
S_2	.85	0	–.3	1	0	.975	DSM (Dual simplex
← S_3	$\boxed{-.5}$	0	–1	0	1	–.75	method)
Z	0	0	.5	0	.5	1.375	
x_2	0	1	0	0	1	1.00	
← S_2	0	0	$\boxed{-2}$	1	1.70	–.30	Apply DSM
x_1	1	0	2	0	–2	1.50	
Z	0	0	0	.25	.925	1.30	
x_2	0	1	0	0	1	1.00	
S_1	0	0	1	–1/2	–.85	.15	
x_1	1	0	0	1	–.3	1.20	

x_1 is not integer Apply Cut $S_2 + \dfrac{.20}{.20-1}(-.3)\, S_3 \geq .20$

$$\text{or } -S_2 - \frac{3}{40} S_3 + S_4 = -.20$$

So we get

BV	x_1	x_2	S_1	S_2	S_3	S_4	Solution	
Z	0	0	0	.25	.925	0	1.30	
x_2	0	1	0	0	1	0	1	
S_1	0	0	1	–1/2	–.85	0	.15	
x_1	1	0	0	1	–.3	0	1.2	
← S_4	0	0	0	$\boxed{-1}$	–.075	1	–.2	Apply DSM
Z	0	0	0	0	.90625	.25	1.25	
x_2	0	1	0	0	1	0	1	
S_1	0	0	1	0	–.70	–1/2	.05	
x_1	1	0	0	0	–.375	1	1	
S_2	0	0	0	1	.075	–1	.2	

Thus we have the optimal solution

$$x_1 = 1,\ x_2 = 1$$

$$\text{Opt. Val.} = 1.25.$$

6.1.5 Branch and Bound Technique

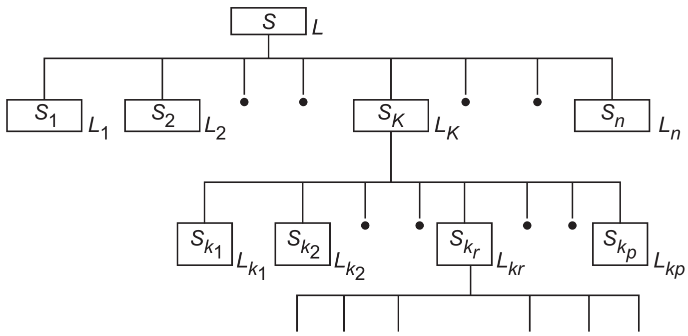
The Branch and Bound Technique method is applicable to those optimisation problems that have finite number of feasible solutions. The number of feasible solutions go on increasing exponentially as the number of decision variables go on increasing.

In this method, we go on filtering the set of feasible solutions S_F by some rule and examine only a subset of S_F and the process continues till we get required solution.

This method is very good for solving (i) All integer ILPP, (ii) Mixed ILPP, (iii) Travelling Salesman Problem, (iv) Cargo Loading Problem, etc. This method proceeds as follows:

Let the problem be Minimisation (Maximisation) problem, and S be the set of all feasible solutions. We divide S in subsets $S_1, S_2, \ldots, S_n \ni \bigcup_{i=1}^{n} S_i = S$ by some rule and attach a bound called lower bound (LB) in case of a minimisation problem and upper bound (UB) to each subset in case of maximisation problem and these are denoted by $L_1, L_2, \ldots, L_n$ or $U_1, U_2, \ldots, U_n$ as the case may be, *i.e.*, $f(X) \geq L_i \;\; \forall \;\; X \in S_i \; (f(X) \leq U_i \;\; \forall \;\; X \in S_i)$.

We now compare each L_i (U_i) and pick the minimum (maximum). Let L_k (U_k) be the minimum (maximum). We now further subdivide subset S_k as above and assign LB (UB) to each S_{ki}, *i.e.*, subsets are $S_{k1}, S_{k2}, \ldots, S_{kP}$ and LB (UB) are $L_{k1}, L_{k2}, \ldots, L_{kP}$ ($U_{k1}, U_{k2}, \ldots, U_{kP}$). Now we compare all LB (UB), $L_1, L_2, \ldots, L_{k-1}, L_{k1}, L_{k2}, \ldots, L_{kP}, L_{k+1}, \ldots, L_n$ ($U_1, U_2, \ldots, U_{k-1}, U_{k1}, U_{k2}, \ldots, U_{kP}, U_{k+1}, \ldots, U_n$) and pick the set with minimum (maximum) LB (UB) and continues till the required solution is reached. We depict this in the following figure.



In the process some subsets at each stage are discarded as they will not contain the required solution. At each stage a filtration of this type is done and the process continues till required optimal solution is reached.

6.1.6 Branch and Bound Algorithm for ILPP

In Gomory's cutting plane method solution arrives after many iterations and sometime they may become indefinitely large and therefore this method has become obsolete nowadays and only Branch and Bound Technique is used.

In case of ILPP, this method proceeds as follows:

We solve the related LPP by simplex method and write the solution inside the boxes (nodes) and assign the value of objective function as its bound. Now divide S_F of related LPP by the following rule:

If $x_k = x_k^*$ is required to be an integer and is not an integer, we divide S_F in three parts, one with condition $x_k \le [x_k^*]$, second with condition $[x_k^*] < x_k < [x_k^*] + 1$ and the third with the condition $x_k \ge [x_k^*] + 1$. Obviously, required solution will not be in second subset and so filter it out and do not consider it. We introduce an extra constraint $x_k \le [x_k^*]$ in related LPP which becomes modified LPP for this subset and find its solution and write it inside the node and value as a bound. Also we solve the related LPP with extra constraint $x_k \ge [x_k^*] + 1$ which becomes the modified LPP for this second subset (third in actual as the second is discarded) and write the solution inside the node and value as a bound. We now compare bounds (if solution is not reached) of all terminal nodes and proceed as above.

If a node with LB (UB) = α is further partitioned and we get a node with LB (UB) = β, $\beta > \alpha$ ($\beta < \alpha$), then, it is not further partitioned because we won't improve the value. Such nodes are called FATHOMED NODE.

Example 3:

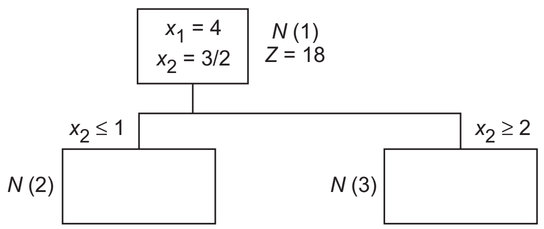
$$\left.\begin{aligned} &\text{Max } Z = 3x_1 + 4x_2 \\ &\qquad 7x_1 + 16x_2 \le 52 \quad \text{or} \quad 7x_1 + 16x_2 + S_1 = 52 \\ &\qquad 3x_1 - 2x_2 \le 9 \quad \text{or} \quad 3x_1 - 2x_2 + S_2 = 9 \\ &\qquad x_1, x_2, S_1, S_2 \ge 0 \text{ and integers} \end{aligned}\right] \qquad (*)$$

Solution: We solve the related LPP by simplex method and get the optimal table (T1).

T1

BV	x_1	x_2	S_1	S_2	Solution
Z	0	0	$\frac{9}{31}$	$\frac{10}{31}$	18
x_2	0	1	$\frac{3}{62}$	$-\frac{7}{62}$	$\frac{3}{2}$
x_1	1	0	$\frac{1}{31}$	$\frac{8}{31}$	4

x_2 is not an integer. We divide one with $x_2 \le 1$ and the other $x_2 \ge 2$



We insert constraint $x_2 \le 1$ in T1 for solution for $N(1)$ and insert $x_2 \ge 2$ in T1 for solution for N(2).

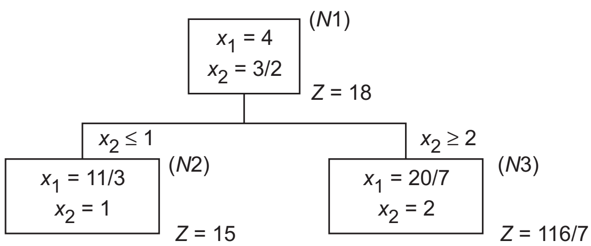
T2 (N2)

BV	x_1	x_2	S_1	S_2	S_3	Solution
Z	0	0	0	1	6	15
x_2	0	1	0	0	1	1
x_1	1	0	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{11}{3}$
S_1	0	0	1	$-\frac{7}{3}$	$-\frac{62}{3}$	$\frac{31}{3}$

T3 (N3)

BV	x_1	x_2	S_1	S_2	S_3	Solution
Z	0	0	3/7	0	$\frac{20}{7}$	116/7
x_2	0	1	0	0	–1	2
x_1	1	0	$\frac{1}{7}$	0	$\frac{16}{7}$	$\frac{20}{7}$
S_2	0	0	$-\frac{3}{7}$	1	$\frac{62}{7}$	$\frac{31}{7}$

Thus, we get



We bifurcate from (N2) with $x_1 \leq 3$ and $x_1 \geq 4$ as x_1 is not an integer. We get the following Tables T4 (N4) and T5 (N5)

T4 (N4)

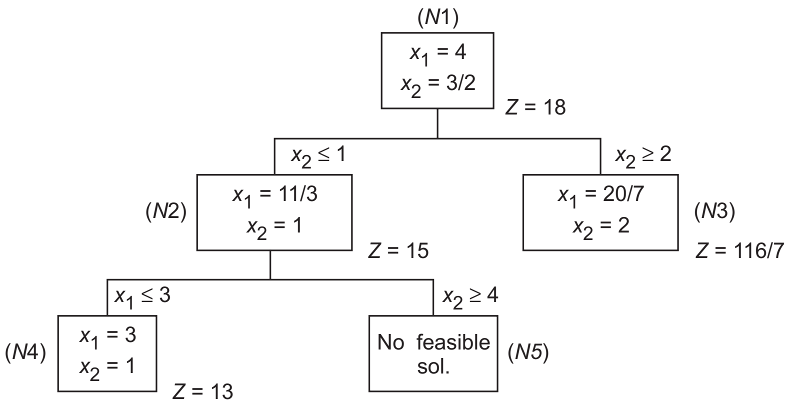
BV	x_1	x_2	S_1	S_2	S_3	S_4	Solution
Z	0	0	0	0	4	3	13
x_2	0	1	0	0	1	0	1
x_1	1	0	0	0	0	1	3
S_1	0	0	1	0	16	–7	15
S_2	0	0	0	1	2	–3	2

T5 (N5)

BV	x_1	x_2	S_1	S_2	S_3	S_4	Solution
Z	0	0	0	1	6	0	15
x_2	0	1	0	0	1	0	1
x_1	1	0	0	1/3	2/3	0	11/3
S_1	0	0	1	−7/3	−62/3	0	31/3
S_4	0	0	0	1/3	2/3	1	−1/3

No feasible solution

Thus, we get



As we go on partitioning, we go on getting reduced value of objective function. Though (N4) gives integer solution, is it the required solution? It may happen that partitioning of (N3) may give a better solution as it is a maximisation problem. Actually we should have first partitioned (N3). Since, it is a maximisation problem, we should partition from the node having largest bound. It would be partitioned from $x_1 \le 2$ and $x_1 \ge 3$, we get tables T6, T7 for Nodes (N6) & (N7), respectively.

T6 (N6)

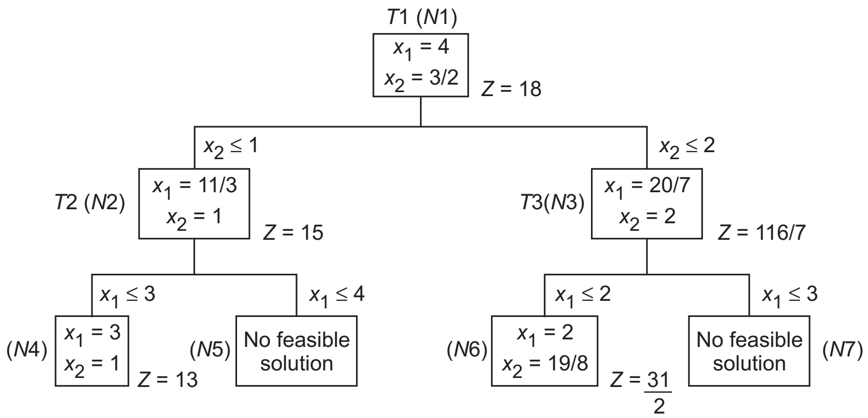
BV	x_1	x_2	S_1	S_2	S_3	S_4	Solution
Z	0	0	1/4	0	0	5/4	$\frac{31}{2}$
x_2	0	1	1/16	0	0	$-\frac{7}{16}$	19/8
x_1	1	0	0	0	0	1	2
S_2	0	0	1/8	1	0	$-\frac{31}{8}$	31/4
S_3	0	0	1/16	0	1	−7/16	3/8

T7 (N7)

BV	x_1	x_2	S_1	S_2	S_3	S_4	Solution
Z	0	0	3/7	0	20/7	0	116/7
x_2	0	1	0	0	–1	0	2
x_1	1	0	1/7	0	16/7	0	20/7
S_2	0	0	–3/7	1	–62/7	0	31/7
S_4	0	0	1/7	0	16/7	1	–1/7

No feasible solution

Thus, we have



Now we have to partition for (N6) for $x_2 \leq 2$ and $x_2 \geq 3$. The corresponding tables are

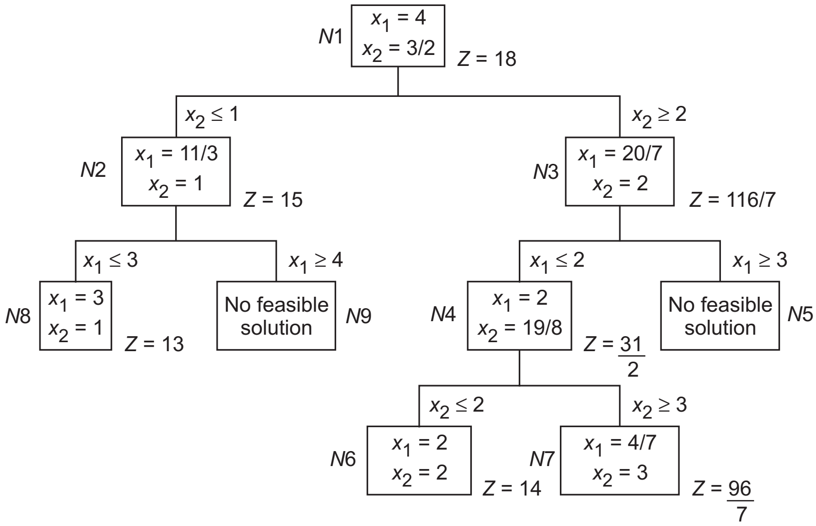
T8 (N8)

BV	x_1	x_2	S_1	S_2	S_3	S_4	S_5	Solution
Z	0	0	0	0	0	3	4	14
x_2	0	1	0	0	0	0	1	2
x_1	1	0	0	0	0	1	0	2
S_2	0	0	0	1	0	–3	2	7
S_3	0	0	0	0	1	0	1	0
S_1	0	0	1	0	0	7	–16	6

T9 (N9)

BV	x_1	x_2	S_1	S_2	S_3	S_4	S_5	Solution
Z	0	0	$\frac{3}{7}$	0	0	0	$\frac{20}{7}$	96/7
x_2	0	1	0	0	0	0	–1	3
x_1	1	0	1/7	0	0	0	16/7	4/7
S_2	0	0	–3/7	1	0	0	–62/7	93/7
S_3	0	0	0	0	1	0	–1	1
S_4	0	0	–1/7	0	0	1	–16/7	10/7

Thus we get



It is a maximisation problem. Therefore, (N7), (N8) are fathomed nodes in comparison to (N6) which gives integer solution. Hence, the solution is

$$x_1 = 2$$
$$x_2 = 2$$

and optimal value = 14.

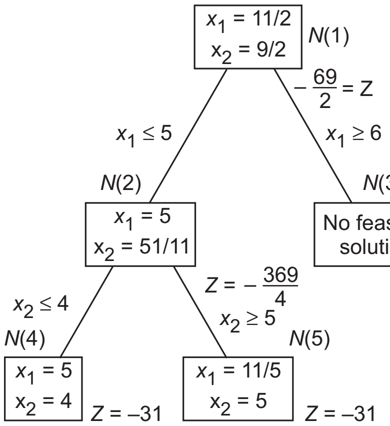
Example 4:

Min. $Z = -3x_1 - 4x_2$

Subject to
$$3x_1 - 1x_2 \le 12$$
$$3x_1 + 11x_2 \le 66$$
$$x_1, x_2 \ge 0 \text{ and integers}$$

Solution of related LPP is

BV	Z	x_1	x_2	S_1	S_2	Solution
Z	1	3	4	0	0	0
S_1	0	3	– 1	1	0	12
S_2	0	3	$\boxed{11}$	0	1	66
Z	1	21\11	0	0	–4\11	– 24
S_1	0	$\boxed{36\backslash 11}$	0	1	1\11	18
x_2	0	3\11	1	0	$\frac{1}{11}$	6
Z	1	0	0	–7\12	–19\132	–69/2
x_1	0	1	0	$\frac{11}{36}$	$\frac{1}{36}$	11\2
x_2	0	0	1	–1/12	1/12	9/2



N(5) is a fathomed nodes as $\alpha = -31$ and $\beta = -31$.

Opt Sol is $x_1 = 5$

$x_2 = 4$

Min $Z = -31$

EXERCISE 6.1

1. Use Branch and Bound technique to solve

   Minimize $Z = -3x_1 - 4x_2$

   Subject to $3x_1 - x_2 \le 12$

   $3x_1 + 11x_2 \le 66$

   $x_1, x_2 \ge 0$ and integers

   (**Ans:** $x_1 = 5$, $x_2 = 4$, Minimize $Z = -31$)

2. Solve the following by branch and bound method.

   Maximize $Z = 7x_1 + 9x_2$

   Subject to $-x_1 + 3x_2 \le 6$

   $7x_1 + x_2 \le 35$

   $x_2 \le 7$

   $x_1, x_2 \ge 0$ and integers

   (**Ans:** $x_1 = 4$, $x_2 = 3$ Maximize $Z = 55$)

3. Use branch and bound method to solve

   Maximize $Z = 3x_1 + 4x_2$

   Subject to $7x_1 + 16x_2 \le 52$

   $3x_1 - 2x_2 \le 9$

   $x_1, x_2 \ge 0$ and integers

   (**Ans:** $x_1 = 2$, $x_2 = 2$, Maximize $Z = 14$)

4. Consider the following problem

   Minimize $Z = -5x_1 - 7x_2$

Subject to $-x_1 + 3x_2 \leq 5$

$5x_1 + x_2 \leq 15$

$x_1, x_2 \geq 0$ and x_1 is an integer

$\left(\textbf{Ans: } x_1 = 2, x_2 = 7/3, \text{ Minimize } Z = -\dfrac{79}{3}\right)$

5. Solve exercise 4 by adding one more condition that x_2 is also an integer.

(**Ans:** $x_1 = 3$, $x_2 = 0$, Minimize $Z = -15$)

6. Solve the following by Gomory's cutting plane method.

Minimize $Z = -5x_1 - 7x_2$

Subject to $-x_1 + 3x_2 \leq 5$

$5x_1 + x_2 \leq 15$

$x_1, x_2 \geq 0$ and integers

(**Ans:** $x_1 = 2$, $x_2 = 2$, Minimize $Z = -24$)

6.2 TRAVELLING SALESMAN PROBLEM

6.2.1 Introduction

Many manufacturers have salesmen who visit various cities in order to procure orders. Industries have to plan their tour programmes. This planning is done keeping in mind their objective and also the cost. Obviously, they see that the cost of tour is minimum.

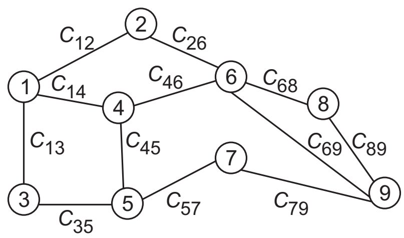
For this purpose, the cost of the travel from each city to each other city must be known. Normally, the cost of travel from city 'i' to city 'j' and from city 'j' to city 'i' is same. But it is not universally true. We may have the situation where they may be different. In the first case it is known as symmetric case and in the other it is known as non-symmetric case. We shall present the method and also the problem in the article below.

6.2.2 Travelling Salesman Problem

The problem can be represented diagrammatically, normally known as network diagram.

In this diagram, we denote the city names/numbers in circles and join each city with other cities by lines. These lines are drawn only between those cities which can be directly accessed from each other. If city 'j' cannot be accessed from city 'i' or vice-versa, then these two cities are not joined together. We write the cost of the travel from city 'i' to city 'j' on the line segment joining the two cities.

A sample network looks like the one given in the figure below.

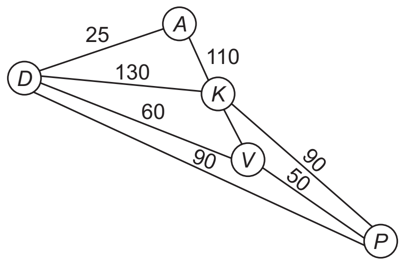


The network is self explanatory. For example, from city 6 one can travel to either city 2, 4, 8 or 9 but not to 1, 3, 5 & 7 and c_{69} is the cost of travel from city 6 to city 9.

Example 1: A salesman located at Delhi has to travel to market his products. He has to visit his distributors at Agra, Kolkata, Varanasi, Patna. Delhi is connected to Agra, Varanasi, Patna and Kolkata directly. Agra is connected to Delhi, Kolkata directly. Varanasi is connected to Delhi, Patna, Kolkata directly. Patna is connected directly to Delhi, Varanasi and Kolkata. Lastly Kolkata is connected to Delhi, Agra, Varanasi and Patna directly.

The cost of travel Delhi to Agra, Delhi to Varanasi, Delhi to Patna and Delhi to Kolkata are respectively Rs. 25, Rs. 60, Rs. 90 and Rs. 130. The cost of travel from Agra to Kolkata is 110. The travel cost from Varanasi to Patna and Varanasi to Kolkata are respectively Rs. 50/- and Rs. 90/-. Assume the case to be symmetric, make a network diagram.

Solution: D, A, K, V and P represent respectively Delhi, Agra, Kolkata, Varanasi and Patna.



This problem can also be represented in the matrix (tabular) form. If there are n cities, we make a table with boxes in n-rows and n-columns. Mark city names (numbers) over each box at the top and also vertically on the right hand side outside the table with same order. Enter the travel cost c_{ij} from city 'i' to city 'j' in the box at the intersection of i-th row and j-th column as shown below.

	1	2	3	.	.	.	n
1	c_{11}	c_{12}	c_{13}	.	.	.	c_{1n}
2	c_{21}	c_{22}	c_{23}	.	.	.	c_{2n}
3	c_{31}	c_{32}	c_{33}	.	.	.	c_{3n}
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
n	c_{n1}	c_{n2}	c_{n3}	.	.	.	c_{nn}

It is also called travel cost matrix. It is always a square matrix. If the problem is symmetric (non-symmetric), the matrix is symmetric (non-symmetric). If journey from city 'i' to city 'j' is not feasible, then c_{ij} is taken to be ∞. Obviously $c_{ii} = \infty$.

Whenever a travel salesman problem is to be solved, we first make this matrix. It is obvious that the cost would be minimised if a city is reached only once except the starting city. A complete tour programme would consist of sets of direct journey from one city to another. A journey from city '*i*' to city '*j*' is denoted by the ordered pair (i, j). This total tour programme T is

$$T = \{(i_1, i_2), (t_2, i_3), (i_3, i_4), \ldots, (i_{n-1}, i_n), (i_n, i_1)\}$$

The following points are to be noted:

(i) If the journey has been taken from city '*i*' to city '*j*', *i.e.*, (i, j) is included in T, then from city '*i*' we cannot go anywhere and also city '*j*' cannot be approached from any other city, *i.e.*, (i, k), $k \neq j$ and (k, j), $k \neq i$ are not included in T.

In order to formulate the problem, we assume that x_{ijk} denotes whether travel from city '*i*' to 'city '*j*' as the *k*-th city visited in the city visited has been taken or not. Thus,

$$x_{ijk} = 1 \text{ if answer is yes}$$
$$= 0 \text{ if answer is no'.}$$

Thus the objective function is

Min $$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} C_{ij}\, x_{ijk} = \sum_{i,j,k=1}^{n} C_{ij}\, x_{ijk}$$

Constraints are

$$x_{ij1} + \sum_{i=1}^{n} \sum_{k=2}^{n} x_{ijk} = 1, \; j = 2, 3, \ldots, n$$

(*j*th city is visited once)

$$\sum_{i=2}^{n} x_{i1n} = 1$$

(Last city is city 1)

$$\sum_{j=2}^{n} x_{ij1} = 1$$

(First city is city 1)

$$x_{i1n} + \sum_{j=2}^{n} \sum_{k=1}^{n-1} x_{ijk} = 1$$

(Each city is visited once)

$$\sum_{i=1}^{n} \sum_{j=2}^{n} x_{ijk} = 1$$

(Any city except 1st can be *k*-th city visited)

Finally, if the *k*-th city visited is city '*j*' then $(k + 1)$th city visited must be approached from city '*j*'. For this, we have the constraints

$$\sum_{i=1}^{n} x_{ijk} = \sum_{P=1}^{n} x_{jP(k+1)}, \; k = 2, 3, \ldots, n-1; \; j = 2, 3, \ldots, n$$

$$x_{ij1} = \sum_{P=1}^{n} x_{jP2} \text{ , } j = 2, 3, \ldots, n$$

$$x_{j1n} = \sum_{i=1}^{n} x_{ij(n-1)} \text{ , } j = 2, 3, \ldots, n.$$

Also we have $x_{ijk} \geq 0$ and it is either '0' or '1'.

It is also called zero-one problem. In this non-negative condition has no relevance.

6.2.3 Methodology

In order to solve a travelling salesman problem whether symmetric on non-symmetric, we first make the travel cost matrix. It is the first step.

Entries in *i*-th row are the cost of travel from city '*i*' to each city. If each entry in this row is reduced by a constant, *C*, then travel from city '*i*' to each other city has become cheaper by amount *C*. This operation does not affect the tour plan but would affect the cost of tour. Similarly, *j*-th column gives the travel cost from each city to city '*j*'. Reduction of each entry by *C* of this column makes the travel to city '*j*' from each city cheaper by amount *C*. This would also not affect the tour plan but it would certainly affect the tour cost.

Thus, we subtract from each row 'R_i' smallest positive number C_i, so that there is a zero in each row. This new matrix would give a tour plan which would also be the solution of the initial problem except for the travel cost which is reduced by $\sum_{i=1}^{n} C_i$.

Similarly, we subtract from each column S_i, smallest positive number d_i so that there is a zero in each column. It will also not affect the tour plan accept for the tour cost which is further reduced by $\sum_{i=1}^{n} d_i$.

As a second step, we bring at least one zero in each row and each column, by subtracting C_i and d_i. This new matrix is called **reduced matrix.** So, we find reduced matrix and write the amount $\sum_{i=1}^{n} C_i + \sum_{i=1}^{n} d_i = A$ on the right hand side outside the matrix. The solution obtained from this reduced matrix would be the required solution except the tour cost in which we have to add *A* to get the travel cost of the problem.

Now we proceed to the next step. Now we would be using the branch and bound technique. Let *S* be the set of all direct journey's from one city to another, *i.e.*, the set of all ordered pairs (i, j) ; $i, j = 1, 2, \ldots, n$. This set *S* is to be partitioned.

Choose an ordered pair which is to be included in *T*. Let the ordered pair be (i, j). So *S* is partitioned by the following method. Let

$$S_1 = \{(r, s) | r \neq i\}$$
$$S_2 = \{(i, 1), (i, 2), \ldots, (i, j), (i, j + 1), \ldots, (i, n)\}$$

Then portions of S are

$$S' = \{(i, j), (r, s) | r \neq i\}$$
$$S'' = \{(r, s), (i, t) | r \neq i, t \neq j\}$$

Obviously $S' \cap S'' \neq \phi$. S' includes the pair (i, j), that is journey from 'i' to 'j'. While S'' does not include (i, j). We shall denote S' by (i, j) and S'' by $\overline{(i, j)}$ meaning that (i, j) is included or not included and write the 'reduced amount' as its bound which would serve as its lower bound and continue. How to get the **'reduced amount'** at each step would be explained below.

But a major question remains, how to choose the travel pair (i, j) for our tour plan T? This is done in the following manner.

The matrix is the redued matrix. So there is a zero in each row and in each column. We should select the one which has the minimum cost, *i.e.*, '0'. Since, there are many zeros, which one we should take? For this we find the penalty which is known ***Least Cost of Exclusion (LCE)*** of each '0' entry by which we mean if this is not selected what is the 'least cost' we have to bear extra for our tour plan. Then we include that pair which has the largest LCE.

This LCE is calculated as follows. Let the pair (i, j) have zero entry. If we do not include it then it means that we are not travelling from city 'i' to city 'j', *i.e.*, from city 'i' we have to go somewhere else, so what is least cost for it, *i.e.*, minimum amount in this row other than zero in (i, j) box. Let it be c. Similarly, since we are not going to j from i, we have to travel to j from some other city, so the least cost for it, *i.e.*, minimum amount in j-th column other than zero in (i, j) box. Let it be d. So L.C.E is $c + d$ *i.e.*, sum of the minimum costs in i-th row and j-th column other than (i, j)-th box entry.

Remark: In case of a tie, we break it arbitrarily.

Now we would explain how to get the LB, *i.e.*, reduced amount at each partitioned node. For this we first find the matrix of each node. To find this we proceed as follows. Take the reduced matrix with its $LB = A$.

Let the node be of (i, j) type, *i.e.*, without bar. It means (i, j) is included. We have travelled from city 'i' to city 'j'. Thus, (i) We cannot go from city 'i' to any other city so remove i-th row, (ii) We cannot reach city 'j' from other cities, so remove j^{th} column, and (iii) We cannot go from city 'j' to city 'i', so make (j, i) infeasible *i.e.*, replace the entry in (j, i) box by ∞. Now reduce this new matrix. Let the amount be A_1. This amount is the known cost of inclusion of the candidate pair (i, j). Then the LB of this node (partition) will be $A + A_1$.

Now take the bar type node, *i.e.*, $\overline{(i, j)}$. It means (i, j) is not included, *i.e.*, (i, j) is infeasible. So replace this entry by ∞. Now reduce this matrix. Obviously, this amount would be the LCE of the pair (i, j). Thus, LB of $\overline{(i, j)}$ would be A, + LCE of this pair (i, j).

We now illustratate this method by an example.

Example 2: Let the matrix of travelling salesman be given in Table S'. Find the optimal tour and its cost.

Table S'

	1	2	3	4	5
1	∞	5	8	4	5
2	3	∞	5	2	3
3	9	8	∞	9	7
4	1	1	5	∞	5
5	3	3	4	6	∞

Solution:

Table S

	1	2	3	4	5
1	∞	1	3	0_1	1
2	1	∞	2	0_1	1
3	2	1	∞	2	0_2
4	0_0	0_0	3	∞	4
5	0_0	0_0	0_2	3	∞

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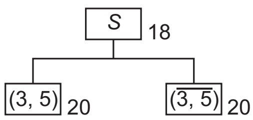
Let the reduced matrix with its lower bound be given in Table '*S*'. Also LCE of each candidate pair be given. (3, 5) and (5, 3) have largest LCE. We break arbitrarily and include (3, 5). Thus, we have



Where Table of (3, 5), and $(\overline{3,5})$ are given below.

Table (3, 5)

	1	2	3	4	5
1	∞	1	3	0	
2	1	∞	2	0	
3					
4	0	0	3	∞	
5	0	0	∞	3	

18

Reduced (R) Table (3, 5)

	1	2	3	4	5
1	∞	1	1	0_1	
2	1	∞	0_1	0_0	
3					
4	0_0	0_0	3	∞	
5	0_0	0_0	∞	3	

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Table $(\overline{3,5})$

Reduced (R) Table $(\overline{3,5})$

	1	2	3	4	5
1	∞	1	3	0	1
2	1	∞	2	0	1
3	2	1	∞	2	∞
4	0	0	3	∞	4
5	0	0	0	3	∞

18

	1	2	3	4	5
1	∞	1	3	0_0	0_0
2	1	∞	2	0_0	0_0
3	1	0_1	∞	1	∞
4	0_0	0_0	3	∞	3
5	0_0	0_0	0_2	3	∞

20

As it is clear that at each step tour cost would be the *LB* + something more, So *LB* at each step would be increasing. So we have to partition from the minimum *LB* in order to keep the tour cost minimum. At this step we partition from the (3, 5) node as both have same *LB*. In case of a tie for selecting node for branching off, we break it arbitrarily. Branching off from a bar type node should be taken as the last choice because of the following reasons.

(i) It is slightly difficult from a bar-type node.

(ii) It would take more iterations as it is not a part of tour programme.

So we branch off from node (3, 5). From *RN* (3, 5) we find that there are two candidate pairs (1, 4) and (2, 3), we select (2, 3). New nodes are now (2, 3) and $(\overline{2, 3})$. The tables (2, 3), and $(\overline{2, 3})$ and also their reduced form are given below.

Table (2, 3)

	1	2	3	4	5
1	∞	1		0	
2					
3					
4	0	0		∞	
5	0	∞		3	

20

R Table (2, 3)

	1	2	3	4	5
1	∞	1		0_4	
2					
3					
4	0_0	0_0		∞	
5	0_0	∞		3	

20

Table $(\overline{2, 3})$

	1	2	3	4	5
1	∞	1	1	0	
2	1	∞	∞	0	
3					
4	0	0	3	∞	
5	0	0	∞	3	

20

R Table $(\overline{2, 3})$

1	2	3	4	5
∞	1	0_2	0_0	
1	∞	∞	0_1	
0_0	0_0	2	∞	
0_0	0_0	∞	3	

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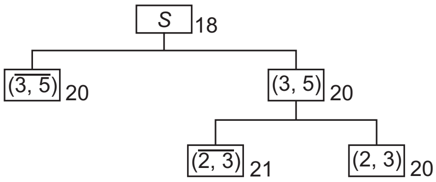
Now the branching figure is as given below. Notice that in Table (2, 3), we have made (5, 2) entry ∞ because (3, 5), (2, 3) and (5, 2) would result in a subtour.



There are 3 terminal nodes. There is a tie between (2, 3) and $(\overline{3,5})$ nodes. For obvious reasons, we branch off from (2, 3) node and select (1, 4) as it has largest LCE. So tables of new nodes (1, 4), $(\overline{1,4})$ are given below.

Table (1, 4)

	1	2	3	4	5
1					
2					
3					
4	∞	0_∞			
5	0_∞	∞			

20

R Table (1, 4)

	1	2	3	4	5
1					
2					
3					
4	∞	0_∞			
5	0_∞	∞			

20

Table $(\overline{1,4})$

	1	2	3	4	5
1	∞	1		∞	
2					
3					
4	0	0		∞	
5	0	∞		3	

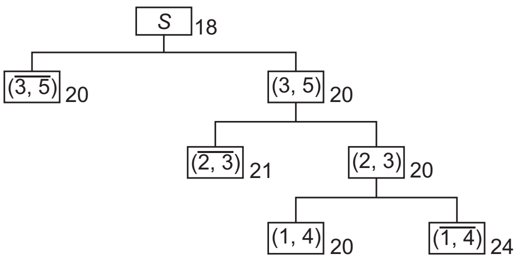
20

R Table $(\overline{1,4})$

	1	2	3	4	5
1	∞	0_∞		∞	
2					
3					
4	0_0	0_0		∞	
5	0_0	∞		0_∞	

24

and the branching figure is



From four terminal nodes, we branch off from node (1, 4). There are two candid pairs, we select (4, 2). Tables of two new nodes (4, 2) $(\overline{4,2})$ and branching figure is

Table (4, 2)

	1	2	3	4	5
1					
2					
3					
4					
5	0				

20

R Table (4, 2)

	1	2	3	4	5
1					
2					
3					
4					
5	0				

20

Table $(\overline{4,2})$

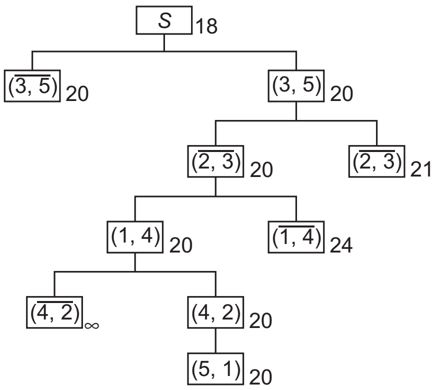
	1	2	3	4	5
1					
2					
3					
4	∞	∞			
5	0	∞			

20

R Table $(\overline{4,2})$

	1	2	3	4	5
1					
2					
3					
4	0	0			
5	0	∞			

∞



Thus, the problem is over. The solution is the tour *T* consisting of the journeys (3, 5), (2, 3), (1, 4), (4, 2) and (5, 1) with the tour cost 20. This tour is also written as, starting from city '1',

(1, 4), (4, 2), (2, 3), (3, 5), (5, 1)

or also (1, 4, 2, 3, 5, 1) : Tour cost = 20.

The nodes $(\overline{2,3})$, $(\overline{1,4})$, $(\overline{4,2})$ are fathomed nodes. We have not gone from $(\overline{3,5})$. We could have gone from here also. We would have reached to a better solution or not is a question. Since, the optimal tour cost is 20 and *LB* of $(\overline{3,5})$, we would not have reached to a better solution. At the most, we could have got an alternate solution.

In this problem, we find that, had we started from $(\overline{3,5})$, our tour would come as

$$(\overline{3,5}),\ (5,\ 3),\ (3,\ 2),\ (4,\ 1),\ (2,\ 4),\ (1,\ 5)$$

or, on excluding $(\overline{3,5})$,

$$(1,\ 5),\ (5,\ 3),\ (3,\ 2),\ (2,\ 4),\ (4,\ 1)$$

or (1, 5, 3, 2, 4, 1); Tour cost = 20

i.e., the same tour except that direction has changed.

EXERCISE 6.2

1. Solve the following travelling salesman problem

	1	2	3	4
1	∞	4	6	6
2	3	∞	12	3
3	7	11	∞	6
4	2	5	7	∞

(**Ans:** Tour ($1 \to 3 \to 4 \to 2 \to 1$) with cost 20)

2. Let the cost matrix of a travelling salesman be given in table S'. Find the optimal tour.

	1	2	3	4
1	∞	7	2	5
2	7	∞	6	1
3	2	6	∞	9
4	5	1	9	∞

Table S'

(**Ans:** Tour ($1 \to 3 \to 2 \to 4 \to 1$) with cost = 14)

3. Solve the following travelling salesman problem by branch and bound algorithm.

	1	2	3	4
1	∞	3	5	5
2	2	∞	11	2
3	6	10	∞	5
4	1	4	6	∞

(**Ans:** Tour $1 \to 3 \to 4 \to 2 \to 1$ with cost 16)

Chapter 7

Dynamic Programming

7.1 INTRODUCTION

Dynamic programming is a technique for solving various types of problems. Even an LPP can be subjected to dynamic programming techniques.

In this, a problem is broken up in a finite number of states. At each state, ignoring how one has reached to that state, the decision for optimal solution is taken.

This technique can be used to solve a variety of problems. To name a few, finding the shortest path from one station to another, cargo loading problem, reliability problem, capital budgeting problem, inventory problem, etc.

7.2 DYNAMIC PROGRAMMING

Dynamic programming is based on **"Bellman's Principle of Optimality"**, which is stated as follows:

An optimal policy has the property that, whatever be the initial state and initial decision, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

Thus, according to the Bellman's principle of optimality, a problem is split in stages and optimal solution is obtained by taking decisions sequentially. Look at the following two figures.

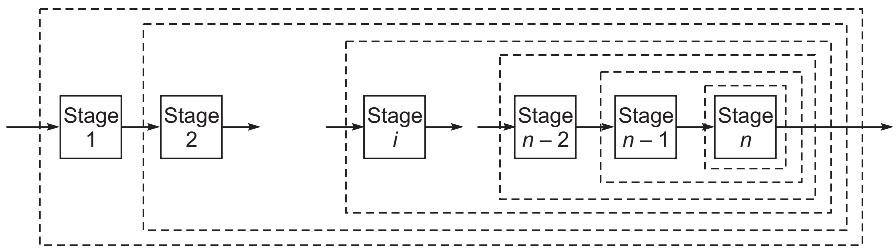


Figure 1

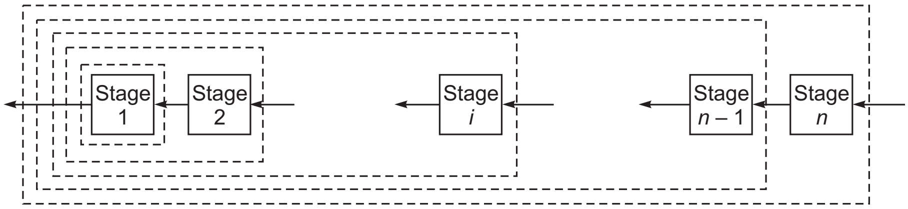


Figure 2

Now take any of the two figures, say figure 1. Here, we assume that we are at stage n. This we call as state. At stage $n - 1$ we would decide how to come to stage n, the final stage. We would forget how did we arrive at $(n - 1)$th stage but would take the best decision to arrive at stage n. Similarly at stage $n - 2$, ignoring how actually we arrived here, we would take best decision to arrive at stage $n - 1$, and so on.

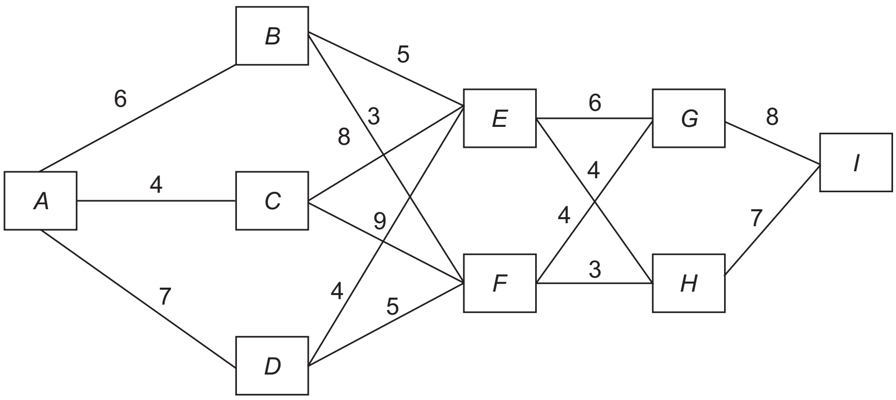
At each stage, we would be having different states. Thus, we associate a variable, called state variable whose probable values are different. For deciding the value of state variable at a stage, we would be using some decision variable.

Dynamic programming is not a general alogorithm in the sense of simplex method of an LPP but it is a kind of approach to solve certain linear and non-linear programming problems. Stages and states are not so obvious.

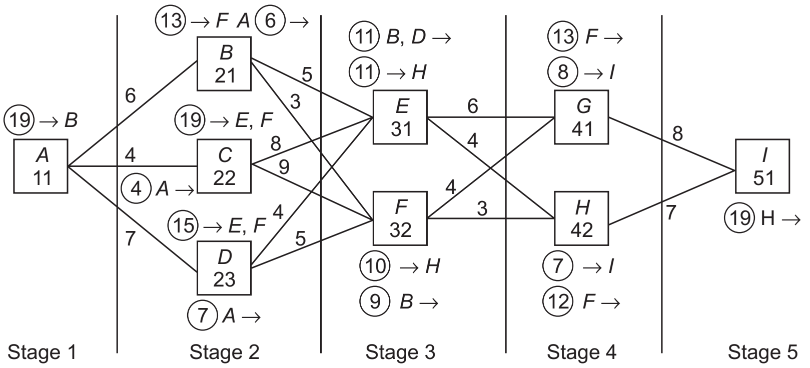
In figure 1 we have the backward approach. So it is called backward dynamic programming. Figure 2, we have the forward approach, so it is called Forward dynamic programming.

We now illustrate the concept. We shall first take shortest distance problem.

Example 1: Let the cities A to I be connected by road as per the following network. Numbers on the line represent the distance in km. Find the shortest route from A to I.



Solution:



Number inside the block which is a two-digit number represent stage (1st digit) and state (2nd digit). Let X_1, X_2, X_3, X_4, X_5 be state variable at stage 5, 4, 3, 2 and 1, respectively. These state variables can take the following values.

$$x_5 = A$$
$$x_4 x_2 = B \text{ or } C \text{ or } D$$
$$x_3 = E \text{ or } F$$
$$x_2 x_4 = G \text{ or } H$$
$$x_1 x_5 = I$$

At each stage, to decide upon the value of state variable. We would be using value of the state variable reached at the last stage and also some decision variables at this stage. Let y_1, y_2, y_3, y_4 and y_5 be the decision variable at stage 5, 4, 3, 2 and 1, respectively. Thus, possible values of these variables are

$$y_1 = 8,\ 7$$
$$y_2 = 6,\ 4,\ 4,\ 3$$
$$y_3 = 5,\ 3,\ 8,\ 9,\ 4,\ 5$$
$$y_4 = 6,\ 4,\ 7$$
$$y_5 = -$$

Let us use backward dynamic programming.

It is a minimisation problem. Stage 5 needs no decision. Let us be at stage 4. Whether $x_2 = G$ or H. If G we would travel 8 to reach I and if H we would travel 7 to reach I. We write these numbers over G, H in a circle and put '$\rightarrow I$' to mean 'to reach I'.

At stage 3, if E, would travel 14 ($G \rightarrow I$) or 11 ($H \rightarrow I$) to reach I through G and H, respectively. So best decision is 11 through H. So we write 11 $\rightarrow H$ over E. Similarly over F we write 10 $\rightarrow H$.

At stage 2, over B we write 13 $\rightarrow F$ which is minimum distance to I through F. Similarly over C we write 19 $\rightarrow E$, F. Here either we go through E or F we move the same distance. At D we write 15 $\rightarrow E$, F.

At stage 1, we write 19 $\rightarrow B$. Thus, we have the answer.

The shortest distance is 19 and route is $A \to B \to F \to H \to I$.

If instead of using backward programming, we use forward programming and start from stage 2 as stage 1 is the initial stage, we write

Over B, 6 A $\to$ Which means distance 6 coming from A, as it is the shortest path

Over C, 4 A $\to$ Which means distance 4 coming from A, as it is the shortest path

Over D, 7 A $\to$ Which means distance 7 coming from A, as it is the shortest path

Over E, 11 $B, D \to$ Which means distance 11 coming from either B or D as it is the shortest path.

Over F, 9 B $\to$ Which means coming from B with distance 9, as it is the shortest path.

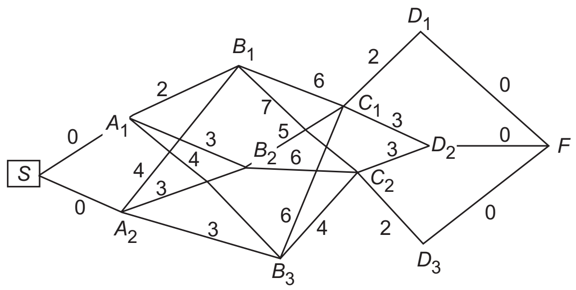
Over G, 13 F $\to$ The shortest path upto G is from F, distance 13.

Over H, 12 F $\to$ Shortest path upto H is from F, distance 12

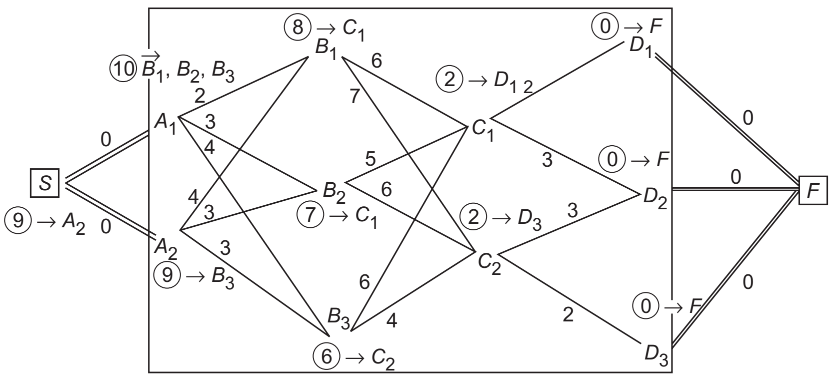
Over I, 19 H $\to$ Shortest path upto I is from H, distance 19.

Thus, shortest distance is 19 and path is $I \leftarrow H \leftarrow F \leftarrow B \leftarrow A$. We have the same answer.

Example 2: Find the shortest distance from stage A to stage D.



Solution:



Here we create S and F, starting and final stages with distances zero. Using backward dynamic programming, we get the shortest path

$$\boxed{S \to}\ A_2 \to B_3 \to C_2 \to D_3\ \boxed{\to F}$$

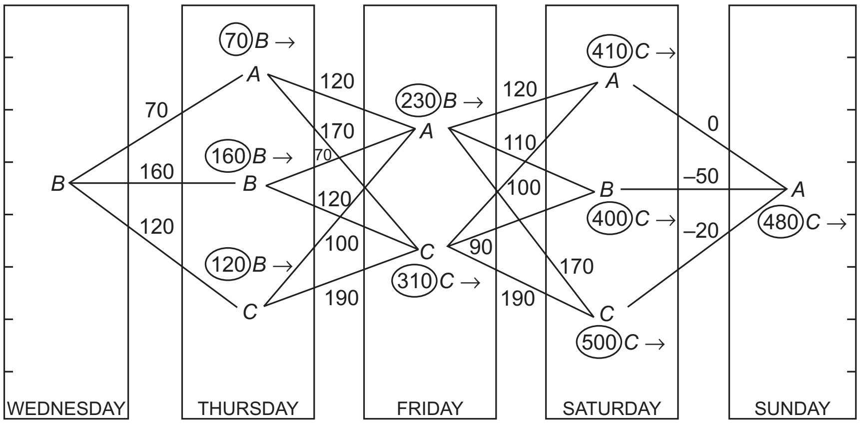
or simply $A_2 \xrightarrow[3]{} B_3 \xrightarrow[4]{} C_2 \xrightarrow[2]{} D_3$ with distance 9

Example 3: A scientist lives in town *B* and has to be in town *A* next Sunday. On each of the days Thursday, Friday and Saturday he can give one talk in any of the towns *A*, *B* or *C* except that he cannot give a talk in *B* on Friday. He can give more than one talk in a town, but on different days. The fee for talks is Rs. 120 in *A*, Rs. 160 in *B* and Rs. 190 in *C*, plus expenses for an overnight stay in a town where he gave his talk.

Where should he spend the last 3 days and nights of the week so as to maximise his income from talks. Travel costs are to be borne by him and are given below.

From \ To	*A*	*B*	*C*
A	–	50	20
B	50	–	70
C	20	70	–

Solution: The problem can be done by dynamic programming.



Let the stages be categorised as Wednesday, Thursday, etc. In each stage, the states are cities *A*, *B*, etc. Since, on Friday he cannot be in city *B*, so it is missing in the stage Friday. State variable can assume value either *A* or *B* or *C* on Thursday and so on. The notion of 'distance' from, say, *B* to *A* is taken as fee of delivering lecture at *A* minus the cost of travel from *B* to *A*. So, the problem is done as in the last example except that here we have to maximise the earning. $B \to C \to C \to C \to A$ Max. income is Rs. 480.

7.3 RECURSIVE RELATIONS

If a problem can be broken up in stages and using optimality principle, stage-wise decisions can be taken, then that problem can be solved by dynamic programming.

A variety of problem can be solved by dynamic programming as we shall see through various illustrative examples. These illustrative examples and its solution should be seen in theme and a parallel should be drawn for other similar problems which may not be same in name. But we shall see a general mathematical model of a problem.

Let the objective function be expressible as

$$F(Y) = \sum_{i=1}^{n} f_i(y_i) \text{ or } \prod_{i=1}^{n} f_i(y_i)$$

i.e., either as a sum or product of n functions of single variable y_i. These functions may be linear or non-linear. The values of decision variables depend upon the states which are normally derived from constraints.

Let a constraint be also expressible as

$$\sum_{i=1}^{m} g_i(y_i) \gtreqless b \text{ or } \prod_{i=1}^{m} g_i(y_i) \gtreqless b.$$

Now we define state variable as follows:

Let $f_i(y_i)$ – return function due to decision y_i.

$x_i = T_i\ (x_i + 1, y_i + 1)$, $i = 1$ to $n - 1$ called state transformation functions

(i) in case of 'Σ' type constraint

$$x_n = \sum_{i=1}^{n} g_i(y_i) \gtreqless b$$

$$x_{n-1} = \sum_{i=1}^{n-1} g_i(y_i) = x_n - g_n(y_n) = T_{n-1}(x_n, y_n)$$

$$x_{n-2} = \sum_{i=1}^{n-2} g_i(y_i) = x_{n-1} - g_{n-1}(y_{n-1}) = T_{n-2}(x_{n-1}, y_{n-1})$$

$$x_2 = \sum_{i=1}^{2} g_i(y_i) = x_3 - g_3(y_3) = T_2(x_3, y_3)$$

$$x_1 = g_1(y_i) = x_2 - g_2(y_2) = T_1(x_2, y_2)$$

(ii) in case of 'Π' type constraint

$$x_n = \prod_{i=1}^{n} g_i(y_i) \gtreqless b$$

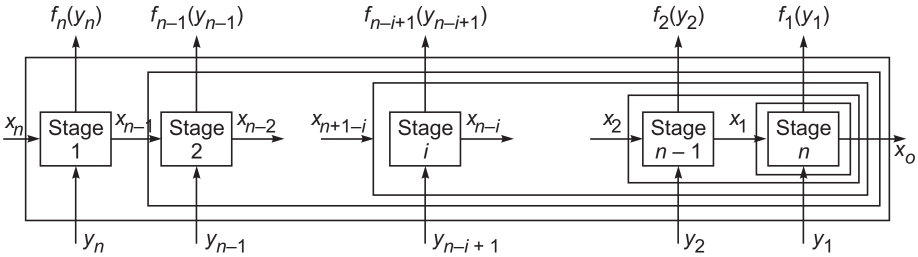
$$x_{n-1} = \prod_{i=1}^{n-1} g_i(y_i) = x_n/g_n(y_n) = T_{n-1}(x_n, y_n)$$

$$x_{n-2} = \prod_{i=1}^{n-2} g_i(y_i) = x_{n-1}/g_{n-1}(y_{n-1}) = T_{n-2}(x_{n-1}, y_{n-1})$$

$$x_2 = g_1(y_1)\ g_2(y_2) = x_3/g_3(y_3) = T_2(x_3, y_3)$$

$$x_1 = g_1(y_1) = x_2/g_2(y_2) = T_1(x_2, y_2)$$

This would break the problem in stages. It would be done by dynamic programming.



At stage 1

From the domain of decision variable y_1, we shall obtain the value of

$$\text{Opt } \{f_1(y_1)\}.$$

This value of y_1 will yield the value of the state variable x_1

$$x_1 = g_1(y_1) = T_1(x_2, y_2) = x_2 - g_2(y_2)$$

and the value of

$$F_1(x_1) = \underset{y_1}{\text{Opt}} \; \{f_1(y_1)\}$$

Now at stage 2, we shall use the value of x_1, thus, obtained and $F_1(x_1)$ and also y_2, we shall obtain

$$F_2(x_2) = \underset{y_2}{\text{Opt}} \; \{f_2(y_2) + F_1(x_1)\}$$

The value of y_2 for which $F_2(x_2)$ is obtained, would yield the value of

$$x_2 = g_1(y_1) + g_2(y_2) = T_2(x_3, y_3) = x_3 - g_3(y_3)$$

and continue. At the last stage we would get

$$F_n(x_n) = \underset{y_n}{\text{Opt}} \; \{f_n(y_n) + F_{n-1}(x_{n-1})\} = \text{Opt} \sum_{i=1}^{n} f_i(y_i)$$

and, also $x_n = \sum_{i=1}^{n} g_i(y_i) \gtreqless b.$

In case x_n is specified, we use backward dynamic programming and whenever x_0 is specified, we use forward dynamic programming.

We now illustrate the dynamic programming by examples. We shall first take the case of continuous type decision variables.

Example 1:

Max $\quad F(y) = (4y_1 - y_1^2) + (4y_2 - 2y_2^2) + (4y_3 - 3y_3^2)$

Subject to $\quad y_1 + y_2 + y_3 = 11, \; y_i \geq 0$

Solution:

$$x_3 = y_1 + y_2 + y_3 = 11$$
$$x_2 = y_1 + y_2 = x_3 - y_3$$
$$x_1 = y_1 = x_2 - y_2$$
$$F_3(x_3) = \underset{y_3}{\text{Max}} \; \{(4y_3 - 3y_3^2) + F_2(x_2)\}$$
$$F_2(x_2) = \underset{y_2}{\text{Max}} \; \{(4y_2 - 2y_2^2) + F_1(x_1)\}$$

$$F_1(x_1) = \underset{y_1}{\text{Max}} \ \{(4y_1 - y_1^2)\}, \ y_1 = x_2 - y_2$$

So $$F_2(x_2) = \underset{y_2}{\text{Max}} \ \{(4y_2 - 2y_2^2) + 4(x_2 - y_2) - (x_2 - y_2)^2\}$$

Its maximum can be obtained by calculus, since it is a continuous case

$$\frac{d}{dy_2}((4y_2 - 2y_2^2) + 4(x_2 - y_2) - (x_2 - y_2)^2) = 0$$

or, $$\cancel{4} - 4y_2 - \cancel{4} + 2(x_2 - y_2) = 0, \ \frac{d^2}{dy_2^2} = -6, \text{ so max.}$$

or, $$2x_2 = 6y_2 \text{ or } y_2 = \frac{x_2}{3}$$

$\therefore$ $$F_2(x_2) = \left\{\left(\frac{4x_2}{3} - \frac{2x_2^2}{9}\right) + 4\left(x_2 - \frac{x_2}{3}\right) - \left(x_2 - \frac{x_2}{3}\right)^2\right\}$$

$$= \left\{\frac{4x_2}{3} + \frac{8x_2}{3} - \frac{2x_2^2}{9} - \frac{4x_2^2}{9}\right\} = \left\{4x_2 - \frac{2x_2^2}{3}\right\}$$

Now, $$F_3(x_3) = \underset{y_3}{\text{Max}} \ \left\{4y_3 - 3y_3^2 + 4x_2 - \frac{2x_2^2}{3}\right\}, \ x_2 = x_3 - y_3$$

$$= \text{Max} \left\{4y_3 - 3y_3^2 + 4x_3 - 4y_3 - \frac{2}{3}(x_3 - y_3)^2\right\}^{**}$$

or, $$\frac{d}{dy_3}\left[-3y_3^2 + 4x_3 - \frac{2}{3}(x_3 - y_3)^2\right] = 0$$

or, $$-6y_3 + \frac{4}{3}(x_3 - y_3) = 0; \ \frac{d^2}{dy_3^2} = -6 - \frac{4}{3}, \text{ so max}$$

or, $$-18y_3 + 4x_3 - 4y_3 = 0$$

or, $$y_3 = \frac{2x_3}{11}$$

$$F_3(x_3) = -3x\frac{4x_3^2}{121} + 4x_3 - \frac{2}{3}\left(x_3 - \frac{2x_3}{11}\right)^2$$

$$= 4x_3 - \frac{12x_3^2}{121} - \frac{162x_3^2}{363} = 4x_3 - \frac{6x_3^2}{11}$$

But, $x_3 = 11$

So, $y_3 = 2, \ x_2 = 9, \ y_2 = 3, \ x_1 = 6, \ y_1 = 6$

and solution is $y_1 = 6; \ y_2 = 3, \ y_3 = 2$

and, Max $$F(y) = 4 \times 11 - \frac{6}{11} \times 11 \times 11 = 44 - 66 = -22.$$

Example 2: (Cargo Loading Problem)

Let us take cargo loading problem, and solve it by dynamic programming.

Item No.	*Weight (kg)*	*Value/Unit Weight*
1	46	1.5
2	44	1.25
3	42	1.00
4	40	0.75

Weight constraint 110 kg.

Solution: Let y_1, y_2, y_3, y_4 stand for items no. 1, 2, 3 and 4. If item 'i' is to be included, value of $y_i = 1$. If not included $y_i = 0$.

So, the problem is

Max $F(Y) = 69y_1 + 55y_2 + 42y_3 + 30y_4$

Subject to $46y_1 + 44y_2 + 42y_3 + 40y_4 \le 110$, $y_i = 0$ or 1.

Here the domain of each variable y_i is {0, 1}

$$\left.\begin{aligned} x_4 &= 46y_1 + 44y_2 + 42y_3 + 40y_4 \le 110 \\ x_3 &= 46y_1 + 44y_2 + 42y_3 = x_4 - 40y_4 \\ x_2 &= 46y_1 + 44y_2 = x_3 - 42y_3 \\ x_1 &= 46y_1 = x_2 - 44y_2 \end{aligned}\right] \quad *$$

and,

$$F_4(x_4) = \underset{y_4}{\text{Max}} \ \{30y_4 + F_3(x_3)\}$$

$$F_3(x_3) = \underset{y_3}{\text{Max}} \ \{42y_3 + F_2(x_2)\}$$

$$F_2(x_2) = \underset{y_2}{\text{Max}} \ \{55y_2 + F_1(x_1)\}$$

$$F_1(x_1) = \underset{y_1}{\text{Max}} \ \{69y_1\}$$

We have,

$$\left.\begin{aligned} y_1 &= 0, 1 \\ y_2 &= 0, 1 \\ y_3 &= 0, 1 \\ y_4 &= 0, 1 \end{aligned}\right] \quad **$$

Possible values of x_1, x_2, x_3, x_4 obtained from (*) and (**) are as follows:

y_1	y_2	y_3	y_4	x_4	x_3	x_2	x_1
1	1	1	1	–	–	–	–
1	1	1	0	–	–	–	–
1	1	0	1	–	–	–	–
1	1	0	0	90	90	90	–
1	0	1	1	–	–	–	–
1	0	1	0	88	88	–	–
1	0	0	1	76	–	–	–
1	0	0	0	46	46	46	46
0	1	1	1	–	–	–	–
0	1	1	0	86	86	–	–
0	1	0	1	74	–	–	–
0	1	0	0	44	44	44	–
0	0	1	1	72	–	–	–
0	0	1	0	42	42	–	–
0	0	0	1	30	–	–	–
0	0	0	0	0	0	0	0

Thus, the domains of x_1, x_2, x_3, x_4 are {0, 46}, {0, 44, 46, 90}, {0, 42, 44, 46, 86, 88, 90} and {0, 30, 42, 44, 46, 72, 74, 76, 86, 88, 90} respective. Now we make table for defining state transformation function.

The state transformation functions

$$x_3 = x_4 - 30y_4 = T_3(x_4, y_4)$$

$$x_2 = x_3 - 42y_3 = T_2(x_3, y_3)$$

and, $$x_1 = 46y_1 = x_2 - 44y_2 = T_1(x_2, y_2)$$

are given in Tables 1, 2, 3, respectively.

$x_4 \diagup y_4$	0	1
90	90	–
88	88	–
86	86	–
76	–	46
74	–	44
72	–	42
46	46	–
44	44	–
42	42	–
30	–	0
0	0	–

$x_3 = x_4 - 30y_4$
$= T_3(x_4, x_4)$

$x_3 \diagup y_3$	0	1
90	90	–
88	–	46
86	–	44
46	46	–
44	44	–
42	–	0
0	0	–

$x_2 = x_3 - 42y_3$
$= T_2(x_3, y_3)$

Table 2

$x_2 \diagup y_2$	0	1
90	–	46
46	46	–
44	–	0
0	0	–

$x_1 = x_2 - 44y_2$
$= T_1(x_2, y_2)$

Table 3

Table 1

These tables give the values of x_i for different values of x_{i+1} and y_{i+1}. If the value is out of domain, it has been shown by '–'. Now we divide the problem into subproblems.

In each subproblem we would take best decision at every state.

Solution for subproblem 1

	y_1	0	1
	x_1	0	46
$f_1(y_1) = 69y_1 =$	$F_1(x_1)$	0	69*

i.e., if $y_1 = 0 = x_1$ then $F_1(x_1) = 0$

and $y_1 = 1$, i.e., $x_1 = 46$, $F_1(x_1) = 69$.

Solution for subproblem 2

	$f_2(y_2) = 55y_2$		$F_1(x_1)$		$55y_2 + F_1(x_1) = f_2(y_2) + f_1(x_1)$		$F_2(x_2) = \underset{y_2}{\text{Max}}\,[f_2(y_2) + F_1(x_1)]$
x_2/y_2	0	1	0	1	0	1	
90	–	55	–	69*	–	124*	124*
46	0	–	69	–	69	–	69
44	–	55	–	0	–	55	55
0	0	–	0	–	0	–	0
(1)	(2)		(3)		(4)		(5)

Column (1) gives values of x_2. Column (2) gives the value of $55y_2$. Since, it is independent of x_2, we get its value as 0 or 55 at $y_2 = 0$ or 1. Wherever, we have value of x_1 in domain, given by Table 3, we write 0 below $y_2 = 0$ and 55 below $y_2 = 1$.

Now coming to column (3), where we write the value of $F_1(x_1)$ obtained in earlier table. Table 3 gives x_1 at different location. We write the value of $F_1(x_1)$ for the value of x_1 at that location. For example, when $x_2 = 46$, $y_2 = 0$ we have from table 3, $x_1 = 46$. For $x_1 = 46$, $F_1(x_1) = 69$. So we enter 69 in $x_2 = 46$ row and $y_2 = 0$ in column (3).

In column (4) we calculate $55y_2 + F_1(x_1)$ by entry wise adding the (2) and (3) columns. In (5) column, we take maximum of the entries in column (4) in each row. This is because it is a maximisation problem and it gives maximum at each state, i.e., the best decision at each state.

Similarly, we give solutions of subproblems 3 and 4.

Solution of Subproblem 3

x_3 / y_3	$f_3(y) = 42y_3$ 0	1	$F_2(x_2)$ 0	1	$f_3(y_3) + F_2(x_2) = 42y_3 + F_2(x_2)$ 0	1	$F_3(x_3) = \text{Max}\ [f_3(y_3) + F_2(x_2)]$
90	0	–	124*	–	124*	–	124*
88	–	42	–	69	–	111	111
86	–	42	–	55	–	97	97
46	0	–	69	–	69	–	69
44	0	–	55	–	55	–	55
42	–	42	–	0	–	42	42
0	0	–	0	–	0	–	0

Similarly, we give solution of subproblem 4.

Solution of Subproblem 4

x_4 / y_4	$f_4(y_4)$ $30y_4$ 0	1	$F_3(x_3)$ 0	1	$f_4(y_4) + F_3(x_3)$ $30y_4 + F_3(x_3)$ 0	1	$F_4(x_4) = \text{Max}\ \{f_4(y_4) + f_3(y_3)\}$
90	0	–	124*	–	124*	–	124*
88	0	–	111	–	111	–	111
86	0	–	97	–	97	–	97
76	–	30	–	69	–	99	99
74	–	30	–	55	–	85	85
72	–	30	–	42	–	72	72
46	0	–	69	–	69	–	69
44	0	–	55	–	55	–	55
42	0	–	42	–	42	–	42
30	–	30	–	0	–	30	30
0	0	–	0	–	0	–	0

Now we take the best decisions. Amongst the values at each state, since it is a maximisation problem, the best value is 124. We put a * over 124 in $F_4(x_4)$ column.

Tracing back, it has come from 124 in column (4) and as a sum of 124 and 0 in columns (3) and (2). We put (*) over 124 in column (4) and over 124 in column (3). Column (3) gives the value of $F_3(x_3)$. So we go to solution of subproblem 3, and put * over 124 in $F_3(x_3)$ column and continue till we reach to the solution of subproblem 1.

Thus, drawing the solution from *'s, we obtain

From subproblem 1, $y_1 = 1$ $x_1 = 46$
From subproblem 2, $y_2 = 1$ $x_2 = 90$
From subproblem 3, $y_3 = 0$ $x_3 = 90$
From subproblem 4, $y_4 = 0$ $x_4 = 90$

and Maximum value is 124.

Thus, on interpreting the solution, we get that load items 1 and 2. Total weight (x_4) would be 90 and cost of ship 124.

Example 3:

Max $F(Y) = y_1^3 + y_2^3 + y_3^3$
Subject to $y_1 y_2 y_3 \le 5$, $y_i > 0$ and integers.

Solution: Solving as above, we define state variables (transformation functions)

$$x_3 = y_1 y_2 y_3 \le 5$$
$$x_2 = y_1 y_2 = x_3/y_3 = T_2(x_3, y_3)$$
$$x_1 = y_1 = x_2/y_2 = T_1(x_2, y_2)$$

Domains: $y_1 = 1, 2, 3, 4, 5$ $x_1 = 1, 2, 3, 4, 5$
$y_2 = 1, 2, 3, 4, 5$ $x_2 = 1, 2, 3, 4, 5$
$y_3 = 1, 2, 3, 4, 5$ $x_3 = 1, 2, 3, 4, 5$

Subproblems: $F_1(x_1) = \underset{y_1}{\text{Max}}\ y_1^2$

$$F_2(x_2) = \underset{y_2}{\text{Max}}\ [y_2^2 + F_1(x_1)]$$
$$F_3(x_3) = \underset{y_3}{\text{Max}}\ [y_3^2 + F_2(x_2)]$$

State transformation functions are defined by the tables.

$x_3 \diagup y_3$	1	2	3	4	5
1	1	–	–	–	–
2	2	1	–	–	–
3	3	–	1	–	–
4	4	2	–	1	–
5	5	–	–	–	1

$x_2 = x_3/y_3 = T_2(x_3, y_3)$

Table 1

$x_2 \diagup y_2$	1	2	3	4	5
1	1	–	–	–	–
2	2	1	–	–	–
3	3	–	1	–	–
4	4	2	–	1	–
5	5	–	–	–	1

$x_1 = x_2/y_2 = T_1(x_2, y_2)$

Table 2

Recursive calculations are given by the following tables.

y_1	1	2	3	4	5
x_1	1	2	3	4	5
$F_1(x_1) = y_1^3$	1*,**′	8	27	64	125**

Table 3

x_2/y_2	$f_2(y_2) = y_2^3$					$F_1(x_1)$					$y_2^3 + F_1(x_1)$					$F_2(x_2)$
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	
1	1	–	–	–	–	1*	–	–	–	–	2*	–	–	–	–	2*
2	1	8	–	–	–	8	1	–	–	–	9	9	–	–	–	9
3	1	–	27	–	–	27	–	1	–	–	28	–	28	–	–	28
4	1	8	–	64	–	64	8	–	1	–	65	16	–	65	–	65
5	1	–	–	–	125	125**	–	–	–	1**′	126**	–	–	–	126**′	126**

Table 4

Also

x_3/y_3	y_3^3					$F_2(x_2)$					$y_3^3 + F_2(x_2)$					$F_3(x_3)$
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	
1	1	–	–	–	–	2	–	–	–	–	3	–	–	–	–	3
2	1	8	–	–	–	9	2	–	–	–	10	10	–	–	–	10
3	1	–	27	–	–	28	–	2	–	–	29	–	29	–	–	29
4	1	8	–	64	–	65	9	–	2	–	66	17	–	66	–	66
5	1	–	–	–	125	126**	–	–	–	2*	127**	–	–	–	127*	127*

It has alternate solutions.

First Solution

$y_1 = 1 \qquad x_1 = 1$
$y_2 = 1 \qquad x_2 = 1$
$y_3 = 5 \qquad x_3 = 5$

Second Solution

$y_1 = 5 \qquad x_1 = 5$
$y_2 = 1 \qquad x_2 = 5$
$y_3 = 1 \qquad x_3 = 5$

Third Solution

$y_1 = 1 \qquad x_1 = 1$
$y_2 = 5 \qquad x_2 = 5$
$y_3 = 1 \qquad x_3 = 5$

and Maximum Value is 127.

Example 4:

Max $\quad f_1(y_1)\ f_2(y_2)\ f_3(y_3)\ f_4(y_4)$

Subject to $\quad y_1y_2y_3y_4 = 12,\ y_1,\ y_2,\ y_3,\ y_4 > 0$ and integers

Where the functions are defined in the following tabular form.

y_1	$f_1(y_1)$
1	2
3	6
5	7

y_2	$f_2(y_2)$
2	3
4	6
6	8

y_3	$f_3(y_3)$
1	3
2	4
4	8

y_4	$f_4(y_4)$
2	4
3	5
5	9

Solution:

$$x_4 = y_1 y_2 y_3 y_4 = 12$$
$$x_3 = y_1 y_2 y_3 = x_4/y_4$$
$$x_2 = y_1 y_2 = x_3/y_3$$
$$x_1 = y_1 = x_2/y_2$$

Domains $y_1 = 1, 3, 5$ $\quad x_1 = 1, 3, 5$
$y_2 = 2, 4, 6$ $\quad x_2 = 2, 4, 6, 10, 12$
$y_3 = 1, 2, 4$ $\quad x_3 = 2, 4, 6, 8, 10, 12$
$y_4 = 2, 3, 5$ $\quad x_4 = 4, 6, 8, 10, 12$

Subproblems: $F_1(x_1) = \text{Max } [f_1(y_1)]$

$$F_2(x_2) = \underset{y_2}{\text{Max}} \; [f_2(y_2)\; F_1(x_1)]$$

$$F_3(x_3) = \underset{y_3}{\text{Max}} \; [f_3(y_3)\; F_2(x_2)]$$

$$F_4(x_4) = \underset{y_1}{\text{Max}} \; [f_4(y_4)\; F_3(x_3)]$$

State transformation functions are given by the following tables.

$x_4 \diagup y_4$	2	3	5
4	2	–	–
6	–	2	–
8	4	–	–
10	–	–	2
12	6	4	–

Table 1

$x_3 \diagup y_3$	1	2	4
2	2	–	–
4	4	2	–
6	6	–	–
8	–	4	2
10	10	–	–
12	12	6	–

Table 2

$x_2 \diagup y_2$	2	4	6
2	1	–	–
4	–	1	–
6	3	–	1
10	5	–	–
12	–	3	–

Table 3

Recursive calculations are given by the following tables.

	y_1	1	3	5
	x_1	1	3	5
$F_1(x_1) =$	$f_1(y_1)$	2	6*	7

Table 4

and,

x_2/y_2	$f_2(y_2)$ 2	4	6	$F_1(x_1)$ 2	4	6	$f_2(y_2)F_1(x_1)$ 2	4	6	$F_2(x_2)$
2	3	–	–	2	–	–	6	–	–	6
4	–	6	–	–	2	–	–	12	–	12
6	3	–	8	6*	–	2	18*	–	16	18*
10	3	–	–	7	–	–	21	–	–	21
12	–	6	–	–	6	–	–	36	–	36

Table 5

and,

x_3/y_3	$f_3(y_3)$ 1	2	4	$F_2(x_2)$ 1	2	4	$f_3(y_3)F_2(x_2)$ 1	2	4	$F_3(x_3)$
2	3	–	–	6	–	–	18	–	–	18
4	3	4	–	12	6	–	36	24	–	36
6	3	–	–	18*	–	–	54*	–	–	54*
8	–	4	8	–	12	6	–	48	48	48
10	3	–	–	21	–	–	63	–	–	63
12	3	4	–	36	18	–	108	72	–	108

Table 6

and,

x_4/y_4	$f_4(y_4)$ 2	3	5	$F_3(x_3)$ 2	3	5	$f_4(y_4)F_3(x_3)$ 2	3	5	$F_4(x_4)$
4	4	–	–	18	–	–	72	–	–	70
6	–	5	–	–	18	–	–	90	–	90
8	4	–	–	36	–	–	144	–	–	144
10	–	–	9	–	–	18	–	–	162	162
12	4	5	–	54*	36	–	216*	180	–	216*

Table 7

Tracing back the solution and putting stars, we get the following solution:

$$y_1 = 3, y_2 = 2, y_3 = 1, y_4 = 2$$
$$x_1 = 3, x_2 = 6, x_3 = 6, x_4 = 12$$

Maximum Value = 216

Example 5:

Minimize $\quad y = f_1(y_1)\ f_2(y_2)\ f_3(y_3)$

Subject to $\quad y_1 + y_2 + y_3 = 10,\ y_1,\ y_2,\ y_3, > 0$ and integers

Where the functions are defined in the following tabular form.

y_1	$f_1(y_1)$
1	2
2	4

y_2	$f_2(y_2)$
2	3
3	5

y_3	$f_3(y_3)$
3	5
5	6

Solution: State Variables

$$x_3 = y_1 + y_2 + y_3 = 10$$
$$x_2 = y_1 + y_2 = x_3 - y_3,\ x_1 = y_1 = x_2 - y_2$$

Domain

$y_1 = 1, 2$	$x_1 = 1, 2$
$y_2 = 2, 3$	$x_2 = 3, 4, 5$
$y_3 = 3, 5$	$x_3 = 6, 7, 8, 9, 10$

Subproblems.

$$F_3(x_3) = \underset{y_3}{\text{Min}}\ [f_3(y_3) \cdot F_2(x_2)]$$
$$F_2(x_2) = \underset{y_2}{\text{Min}}\ [f_2(y_2) \cdot F_1(x_1)]$$
$$F_1(x_1) = \underset{y_1}{\text{Min}}\ [f_1(y_1)]$$

State transformations are given by

$x_3 \diagup y_3$	3	5
6	3	–
7	4	–
8	5	3
9	–	4
10	–	5

$x_2 = y_3 - x_3$

$x_2 \diagup y_2$	2	3
3	1	–
4	2	1
5	–	2

$x_1 = x_2 - y_2$

Recursive relations

I[st] Subproblem

y_1	1	2
x_1	1	2
$F_1(x_1) = f_1(y_1)$	2*	4

2nd Subproblem

	$f_2(y_2)$		$F_1(x_1)$		$f_2(y_2) \cdot F_1(x_1)$		Min $F_2(x_2)$
$x_2 \diagup y_2$	2	3	2	3	2	3	
3	3	–	2*	–	6*	–	6*
4	3	5	4	2	12	10	10
5	–	5	–	4	–	20	20

3rd Subproblem

	$f_3(y_3)$		$F_2(x_2)$		$f_3(y_3) \cdot F_2(x_2)$		$F_3(x_3)$
$x_3 \diagup y_3$	3	5	3	5	3	5	$= \text{Min}\ f_3(y_3) \cdot F_2(x_2)$
6	5	–	6*	–	30*	–	30*
7	5	–	10	–	50	–	50
8	5	6	20	6	100	36	36
9	–	6	–	10	–	60	60
10	–	6	–	20	–	120	120

$$y_3 = 3 \qquad x_3 = 6$$
$$y_2 = 2 \qquad x_2 = 3$$
$$y_1 = 1 \qquad x_1 = 1$$
$$\text{Min } y = 30$$

Example 6: (Capital Budgeting)

A businessman has four houses in a hill station. He intends to spend Rs. 8 lakhs in converting these houses into hotels. *i*th house requires Rs. y_i to convert into *C*, *B* or *A* type hotel and the corresponding returns are given in the following tables. Maximise the total return. (All data in lakhs of ruppes)

	y_1	Return $f_1(y_1)$	y_2	Return $f_2(y_2)$	y_3	Return $f_3(y_3)$	y_4	Return $f_4(y_4)$
No change	0	0	0	0	0	0	0	0
C type	2	3	1	2	1	2	3	6
B type	3	5	4	6	2	3	4	7
A type	5	7	–	–	4	6	5	8
	House No. 1		House No. 2		House No. 3		House No. 4	

Problem is:

Max $$F(Y) = f_1(y_1) + f_2(y_2) + f_3(y_3) + f_4(y_4)$$

Subject to $y_1 + y_2 + y_3 + y_4 \le 8$, $y_1, y_2, y_3, \ge 0$ and integers

Solution:
$$x_4 = y_1 + y_2 + y_3 + y_4 \le 8$$
$$x_3 = y_1 + y_2 + y_3 = x_4 - y_4$$
$$x_2 = y_1 + y_2 = x_3 - y_3$$
$$x_1 = y_1 = x_2 - y_2$$

Subproblems
$$F_1(x_1) = \underset{y_1}{\text{Max}} \; [f_1(y_1)]$$
$$F_2(x_2) = \underset{y_2}{\text{Max}} \; [f_2(y_2) + F_1(x_1)]$$
$$F_3(x_3) = \underset{y_3}{\text{Max}} \; [f_3(y_3) + F_2(x_2)]$$
$$F_4(x_4) = \underset{y_4}{\text{Max}} \; [f_4(y_4) + F_3(x_3)]$$

Domains

$$y_1 = 0, 2, 3, 5 \qquad x_1 = 0, 2, 3, 5$$
$$y_2 = 0, 1, 4 \qquad x_2 = 0, 1, 2, 3, 4, 5, 6, 7$$
$$y_3 = 0, 1, 2, 4 \qquad x_3 = 0, 1, 2, 3, 4, 5, 6, 7, 8$$
$$y_4 = 0, 3, 4, 5 \qquad x_4 = 0, 1, 2, 3, 4, 5, 6, 7, 8$$

State transformation functions are given by the following tables.

$x_3 = x_4 - y_4$

x_4 \ y_4	0	3	4	5
0	0	–	–	–
1	1	–	–	–
2	2	–	–	–
3	3	0	–	–
4	4	1	0	–
5	5	2	1	0
6	6	3	2	1
7	7	4	3	2
8	8	5	4	3

Table 1

$x_2 = x_3 - y_3$

x_3 \ y_3	0	1	2	4
0	0	–	–	–
1	1	0	–	–
2	2	1	0	–
3	3	2	1	–
4	4	3	2	0
5	5	4	3	1
6	6	5	4	2
7	7	6	5	3
8	–	7	6	4

Table 2

$x_1 = x_2 - y_2$

x_2 \ y_2	0	1	4
0	0	–	–
1	–	0	–
2	2	–	–
3	3	2	–
4	–	3	0
5	5	–	–
6	–	5	2
7	–	–	3

Table 3

Recursive tables are

Table 4

y_1	0	2	3	5
x_1	0	2	3	5
$F_1(x_1) = f_1(y_1)$	0	3	5*	7

Table 5

	$f_2(y_2)$			$F_1(x_1)$			$F_1(x_1) + f_2(y_2)$			
x_2 \ y_2	0	1	4	0	1	4	0	1	4	$F_2(x_2)$
0	0	–	–	0	–	–	0	–	–	0
1	–	2	–	–	0	–	–	2	–	2
2	0	–	–	3	–	–	3	–	–	3
3	0	2	–	5	3	–	5	5	–	5
4	–	2	6	–	5*	0	–	7*	6	7*
5	0	–	–	7	–	–	7	–	–	7
6	–	2	6	–	7	3	–	9	9	9
7	–	–	6	–	–	5	–	–	11	11

Table 6

x_3 \ y_3	$f_3(y_3)$ 0	1	2	4	$F_2(x_2)$ 0	1	2	4	$f_3(y_3)+F_2(x_2)$ 0	1	2	4	$F_3(x_3)$
0	0	–	–	–	0	–	–	–	0	–	–	–	0
1	0	2	–	–	2	0	–	–	2	2	–	–	2
2	0	2	3	–	3	2	0	–	3	4	3	–	4
3	0	2	3	–	5	3	2	–	5	5	5	–	5
4	0	2	3	6	7	5	3	0	7	7	6	6	7
5	0	2	3	6	7	7*	5	2	7	9*	8	8	9*
6	0	2	3	6	9	7	7	3	9	9	10	9	10
7	0	2	3	6	11	9	7	5	11	11	10	11	11
8	–	2	3	6	–	11	9	7	–	13	12	13	13

Table 7

x_4 \ y_4	$f_4(y_4)$ 0	3	4	5	$F_3(x_3)$ 0	3	4	5	$f_4(y_4)+F_3(x_3)$ 0	3	4	5	$F_4(x_4)$
0	0	–	–	–	0	–	–	–	0	–	–	–	0
1	0	–	–	–	2	–	–	–	2	–	–	–	2
2	0	–	–	–	4	–	–	–	4	–	–	–	4
3	0	6	–	–	5	0	–	–	5	6	–	–	6
4	0	6	7	–	7	2	0	–	7	8	7	–	8
5	0	6	7	8	9	4	2	0	9	10	9	8	10
6	0	6	7	8	10	5	4	2	10	11	11	10	11
7	0	6	7	8	11	7	5	4	11	13	12	12	13
8	0	6	7	8	13	9*	7	5	13	15*	14	13	15*

Tracing back the solution, we get

$$y_1 = 3,\ y_2 = 1,\ y_3 = 1,\ y_4 = 3$$

i.e., Convert House 1 in *B* type, House 2 in *C* type, House 3 in *C* type and House 4 in *C* type.

Total return would be 15 lakhs and total expenditure would be 8 lakhs.

Example 7: (Reliability problem)

An electric equipment has four components 1, 2, 3 and 4 which are connected in series. To improve the reliability of the equipment, each component is supplied with parallel units, so that in case of a failure of a component, other parallel unit takes over and the equipment remains working. Following data give the cost of parallel units of each component and reliability at each component. The total money available is 15 lakhs.

Find the number of parallel units of each component, so that reliability is maximum.

No. of Parallel Units	Component 1		Component 2		Component 3		Component 4	
	Cost y_1	Reli. $f_1(y_1)$	Cost y_2	Reli. $f_2(y_2)$	Cost y_3	Reli. $f_3(y_3)$	Cost y_4	Reli. $f_4(y_4)$
1	2	.6	1	.4	1	.3	2	.4
2	4	.7	3	.5	2	.4	3	.5
3	6	.8	5	.7	3	.5	4	.6
4	8	.9	7	.8	4	.6	6	.8

Solution: [Use the fact that reliability of equipment is the product of reliability of each component]

Let y_i be the amount spent on ***i***-th component and $f_i(y_i)$ is the reliability. So

Max $$F(Y) = f_1(y_1)\ f_2(y_2)\ f_3(y_3)\ f_4(y_4)$$

Subject to $y_1 + y_2 + y_3 + y_4 \le 15.$

State variables (transformations) are

$$x_4 = y_1 + y_2 + y_3 + y_4 \le 15$$

$$x_3 = y_1 + y_2 + y_3 = x_4 - y_4$$

$$x_2 = y_1 + y_2 = x_3 - y_3$$

$$x_1 = y_1 = x_2 - y_2$$

Subproblems are: $F_1(x_1) = \underset{y_1}{\text{Max}}\ [f_1(y_1)]$

$$F_2(x_2) = \underset{y_2}{\text{Max}}\ [f_2(y_2)\ F_1(x_1)]$$

$$F_3(x_3) = \underset{y_3}{\text{Max}}\ [f_3(y_3)\ F_2(x_2)]$$

$$F_4(x_4) = \underset{y_4}{\text{Max}}\ [f_4(y_4)\ F_3(x_3)]$$

Domains:

$y_1 = 2, 4, 6, 8$ $\quad x_1 = 2, 4, 6, 8$

$y_2 = 1, 3, 5, 7$ $\quad x_2 = 3, 5, 7, 9, 11, 13, 15$

$y_3 = 1, 2, 3, 4$ $\quad x_3 = 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14,15$

$y_4 = 2, 3, 4, 6$ $\quad x_4 = 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.$

State transformation functions are given by the following tables.

$x_3 = x_4 - y_4$

$x_4 \diagup y_4$	2	3	4	6
6	4	–	–	–
7	5	4	–	–
8	6	5	4	–
9	7	6	5	–
10	8	7	6	4
11	9	8	7	5
12	10	9	8	6
13	11	10	9	7
14	12	11	10	8
15	13	12	11	9

Table 1

$x_2 = x_3 - y_3$

$x_3 \diagup y_3$	1	2	3	4
4	3	–	–	–
5	–	3	–	–
6	5	–	3	–
7	–	5	–	3
8	7	–	5	–
9	–	7	–	5
10	9	–	7	–
11	–	9	–	7
12	11	–	9	–
13	–	11	–	9
14	13	–	11	–
15	–	13	–	11

Table 2

$x_1 = x_2 - y_2$

$x_2 \diagup y_2$	1	3	5	7
3	2	–	–	–
5	4	2	–	–
7	6	4	2	–
9	8	6	4	2
11	–	8	6	4
13	–	–	8	6
15	–	–	–	8

Table 3

Recursive tables are:

y_1	2	4	6	8
x_1	2	4	6	8
$F_1(x_1) = f_1(y_1)$	.6*	.7	.8	.9

Table 4

	$f_2(y_2)$				$F_1(x_1)$				$F_1(x_1) f_2(y_2)$				
$x_2 \diagup y_2$	1	3	5	7	1	3	5	7	1	3	5	7	$F_2(x_2)$
3	.4	–	–	–	.6	–	–	–	.24	–	–	–	.24
5	.4	.5	–	–	.7	.6	–	–	.28	.30	–	–	.30
7	.4	.5	.7	–	.8	.7	.6*	–	.32	.35	.42*	–	.42*
9	.4	.5	.7	.8	.9	.8	.7	.6	.36	.40	.49	.48	.49
11	–	.5	.7	.8	–	.9	.8	.7	–	.45	.56	.56	.56
13	–	–	.7	.8	–	–	.9	.8	–	–	.63	.64	.64
15	–	–	–	.8	–	–	–	.9	–	–	–	.72	.72

Table 5

	$f_3(y_3)$				$F_2(x_2)$				$F_2(x_2)f_3(y_3)$				
x_3 / y_3	1	2	3	4	1	2	3	4	1	2	3	4	$F_4(x_4)$
4	.3	–	–	–	.24	–	–	–	.072	–	–	–	.072
5	–	.4	–	–	–	.24	–	–	–	.096	–	–	0.96
6	.3	–	.5	–	.30	–	.24	–	.090	–	.120	–	.120
7	–	.4	–	.6	–	.30	–	.24	–	.120	–	.144	.144
8	.3	–	.5	–	.42	–	.30	–	.126	–	.150	–	.150
9	–	.4	–	.6	–	.42	–	.30	–	.168	–	.180	.180
10	.3	–	.5	–	.49	–	.42	–	.147	–	.210	–	.210
11	–	.4	–	.6	–	.49	–	.42*	–	.196	–	.252*	.252*
12	.3	–	.5	–	.56	–	.49	–	.168	–	.245	–	.245
13	–	.4	–	.6	–	.56	–	.49	–	.224	–	.294	.294
14	.3	–	.5	–	.64	–	.56	–	.192	–	.280	–	.280
15	–	.4	–	.6	–	.64	–	.56	–	.256	–	.336	.336

Table 6

	$f_4(y_4)$				$F_3(x_3)$				$F_3(x_3)f_4(y_4)$				
x_4 / y_4	2	3	4	6	2	3	4	6	2	3	4	6	$F_4(x_4)$
6	.4	–	–	–	.072	–	–	–	.0288	–	–	–	0.288
7	.4	.5	–	–	.096	.072	–	–	.0384	.0360	–	–	.0384
8	.4	.5	.6	–	.120	.096	.072	–	.0480	.0480	.0432	–	.0480
9	.4	.5	.6	–	.144	.120	.096	–	.0576	.0600	.0576	–	.0600
10	.4	.5	.6	.8	.150	.144	.120	.072	.0600	.0720	.0720	.0576	.0720
11	.4	.5	.6	.8	.180	.150	.144	.096	.0720	.0750	.0864	.0768	.0864
12	.4	.5	.6	.8	.210	.180	.150	.120	.0840	.0900	.0900	.0960	.0960
13	.4	.5	.6	.8	.252	.210	.180	.144	.1008	.1050	.1080	.1152	.1152
14	.4	.5	.6	.8	.245	.252	.210	.150	.0980	.1260	.1260	.1200	.1260
15	.4	.5	.6	.8	.294	.245	.252*	.180	.1176	.1225	.1512*	.1440	.1512*

Table 7

Tracking back, we get the solution

$$y_1 = 2,\ y_2 = 5,\ y_3 = 4,\ y_4 = 4$$
$$x_1 = 2,\ x_2 = 7,\ x_3 = 11,\ x_4 = 15$$

and we should provide

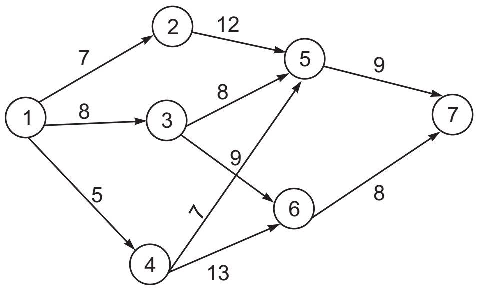
	No. of Parallel Units	*Cost*	*Reliability*
Component 1	1	2	.6
Component 2	3	5	.7
Component 3	4	4	.6
Component 4	3	4	.6

Total Expenditure 15

Reliability = .6 × .7 × .6 × .6 = .1512

EXERCISE 7.1

1. Use dynamic programming to find the shortest path from city 1 to city 7 of the following route network. (Distance between cities are given in kilometres).



(**Ans:** Path is: 1 → 4 → 5 → 7 with 21 kilometres)

2. Find three non-negative real numbers such that the sum of squares of these is minimum with the restriction that their sum is not less than 27.

(**Ans:** $y_1 = y_2 = y_3 = 9$ & Minimum $y_1^2 + y_2^2 + y_3^2 = 243$)

3. **(Cargo Loading Problem):** We have to load a vessel with three items. The maximum allowable weight is 7. The weight per unit of different items and their values are given below. It is required to find the loading which maximises the values of the vessel without exceeding the weight constraint of 7.

Item	Wt/Unit	Value/Unit
1	1	20
2	3	90
3	2	70

(**Ans:** $y_1 = 0$, $y_2 = 1$, $y_3 = 2$ Max. Value = 230)

4. **(Reliability Problem):** An electric equipment has three components 1, 2 and 3 which are connected in series. To improve the reliability of the equipment, each component is to be supplemented by one or two or three parallel units (in case the component fails then the supplementary unit works and consequently the system remains working). The following data gives the cost of parallel units and reliability at each component. The total money available for improvement of device is Rs. 6 lakh.

No. of parallel	Component 1		Component 2		Component 3	
Units	*C*	*R*	*C*	*R*	*C*	*R*
1	1	.4	2	.5	3	.7
2	3	.6	4	.6	4	.8
3	4	.8	5	.7	5	.9

Where C: Cost in thousands of rupees and R: Reliability of the equipment. Let y_i denotes the money spent on attaching supplementary units to component i. Find y_i, such that the reliability of equipment is maximised. (Ans: $y_1 = 3$, $y_2 = 1$, $y_3 = 2$, Max. reliability = 0.32)

5. Use dynamic programming to solve

Minimize $\quad y_1^2 + y_2^2 + y_3^2$

Subject to $\quad y_1 y_2 y_3 = 6$

$\quad y_i > 0$ and integers

(**Ans:** $y_1 = 1$, $y_2 = 3$, $y_3 = 2$, Min $F = 14$)

6. **(Capital Budgeting):** A manufacturing company has to improve working of three plants A_1, A_2 and A_3 with available capital of 5 units of money (in lakhs of rupees). The possible plans with costs and corresponding returns are as follows (in Lakhs of Rs.)

Plan	Plant A_1		Plant A_2		Plant A_3	
No.	C	R	C	R	C	R
1	0	0	0	0	0	0
2	1	5	2	8	1	3
3	2	6	3	9	–	–
4	–	–	4	12	–	–

Where C: Cost & R: return.

Find the optimal policy for budgeting the available capital, if y_i is the amounts allocated to ith plant.

(**Ans:** $y_1 = 1$, $y_2 = 3$, $y_3 = 1$ and Max = 17)

Chapter 8

Game Theory

8.1 INTRODUCTION

A game which is defined as set of rules for playing is played between more than one party. Each party is called a player. No player knows the move choice of others unless he himself has moved. The rules of the game describe when and who to make a move? What alternate possible move exists? When the game terminates, etc.

The game may be a chess game, games of cards, conflict situations in military activities, marketing strategies amongst rival companies, etc.

At the end of the game there are pay-offs to each player which could be in terms of stakes or only winning and losing. Winning is taken as positive pay-offs while loosing is taken as negative pay-offs.

Game theory deals with playing strategies, so that winning (losing) is maximum (minimum) to each player.

If the sum of total gain is equal to the sum of total losses, i.e., net gain is zero, the problem is called zero sum problem. As is clear, there may be more than two parties in a game.

We shall restrict ourselves with only zero-sum two persons game.

8.2 ZERO-SUM TWO PERSONS GAME

Let there be two persons A and B. We are given with or rather say, we know the set of strategies of each player A and B. Let their strategies be A_1, $A_{2,}$ ..., A_m and B_1, B_2, …, B_n, respectively.

If A chooses strategy A_i and consequently B chooses B_j, then, let a_{ij} be the pay-off from B to A or say to A from B. All these pay-offs are also known. The matrix $(a_{ij})_{m \times n}$ is known as A′s pay-off matrix. Obviously, positive a_{ij} mean pay-off to A from B and negative a_{ij} would mean pay-off to B from A.

We denote it in the following tabular form.

$$
\begin{array}{c|cccccc|}
 & B_1 & B_2 & B_3 & \ldots & \ldots & B_n \\
\hline
A_1 & a_{11} & a_{12} & a_{13} & \ldots & \ldots & a_{1n} \\
A_2 & a_{21} & a_{22} & a_{23} & \ldots & \ldots & a_{2n} \\
A_3 & a_{31} & a_{32} & a_{33} & \ldots & \ldots & a_{3n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A_m & a_{m1} & a_{m2} & a_{m3} & \ldots & \ldots & a_{mn} \\
\hline
\end{array}
$$

A's pay-off matrix.

Since, it can be represented in the above form, a zero, sum two persons game is also called rectangular game.

The objective of a game is to maximise the gain. There are two types of strategies.

(i) ***Pure Strategy:*** If chances do not determine any move of any player, then we have a deterministic situation. Strategies in this situation are called PURE STRATEGIES.

(ii) ***Mixed Strategy:*** If chances affect moves then the situation is probabilistic and in this situation, the objective is to maximise the expected. These situations are called MIXED STRATEGIES. A mixed strategy is a selection of pure strategies with fixed probabilities.

The mathematical formulation of Game theory is based upon Maximin or Minimax criterion developed by J. Von Neumann. Dantzig, G. B., was the first person to use simplex method to solve a game problem. We develop the Minimax criterion in the next section.

8.3 MINIMAX (MAXIMIN) CRITERION

Consider the pay-off matrix as given in section 2. Let A choose the strategy 'i' then depending upon the strategy B_j, that B chooses, he is sure to get

$$\underset{j=1,\ldots,n}{\text{Min}} \{a_{ij}\}$$

Thus, we write this minimum of each row outside the box. Thus, to maximise the gain A should choose the maximum of these minimum, *i.e.*, the strategy for which the entry on the right outside the box is maximum. Let this entry be denoted by $\underline{a}$ and the strategy be As. Thus,

$$\underset{i}{\text{Max}}\ \underset{j}{\text{Min}}\ \{a_{ij}\} = \underline{a}$$

On the other hand, if B chooses the strategy j, then B is sure that A will not gain more than

$$\underset{i}{\text{Max}}\ \{a_{ij}\}$$

We write this maximum of each column outside the box at the bottom of each column. Naturally B will choose the move for which these maximum is minimum, *i.e.*, the strategy for which bottom outside box entry is minimum. Let this entry be denoted by $\bar{a}$ and move of B be $B_{t.}$
Thus,

$$\underset{j}{\text{Min}}\ \underset{i}{\text{Max}}\ \{a_{ij}\} = \bar{a}$$

These define Maximin and Minimax criterion. If $\bar{a} = \underline{a}$, then the game is said to have a **saddle point**. In this case optimal solution of the game exists. Let, in this case $\bar{a} = \underline{a} = a$, or

$$a_{st} = \underset{j}{\text{Min}}\ \underset{i}{\text{Max}}\ \{a_{ij}\} = \underset{i}{\text{Max}}\ \underset{j}{\text{Min}}\ \{a_{ij}\} = \bar{a} = \underline{a} = a$$

and the optimal strategies are A_s and B_t for players A and B, respectively. These are optimal as if B choose B_t he cannot lose more than a_{st} and if A choose A_s, he gains at least a_{st}, and optimal value of game is 'a'.

We know that

$$a_{ij} \le \underset{i}{\text{Max}}\ \{a_{ij}\} \text{ for each } i \text{ and fixed } j$$

and, $$a_{ij} \ge \underset{j}{\text{Min}}\ \{a_{ij}\} \text{ for each } j \text{ and fixed } i.$$

Let $\underset{i}{\text{Max}}\ \{a_{ij}\} = a_{pj}$ and $\underset{j}{\text{Min}}\ \{a_{ij}\} = a_{iq}$.

Thus,

$$a_{pj} \ge a_{ij} \ge a_{iq}\ \ \forall\ i, j.$$

Hence,

$$\underset{j}{\text{Min}}\ \{a_{pj}\} \ge a_{ij} \ge \underset{i}{\text{Max}}\ \{a_{iq}\}\ \ \forall\ i, j$$

or, $$\underset{i}{\text{Max}}\ \underset{j}{\text{Min}}\ \{a_{ij}\} \le \underset{j}{\text{Min}}\ \underset{i}{\text{Max}}\ \{a_{ij}\}$$

or, $$\underline{a} \le \bar{a}$$

But, whenever $\underline{a} = \bar{a} = a$ then it has a saddle and optimal solution exists and optimal value is a which is pay-off to A from B. It determines the move of each player.

Example 1: Below is given the A's pay-off matrix. Find the optimal solution.

	B_1	B_2	B_3	B_4	Row Min.
A_1	0	0	−2	5	−2
A_2	1	2	3	1	$\boxed{1}$ $\underline{a}$ Maximin
A_3	−3	−4	2	0	−4
A_4	1	3	−2	4	−2
A_5	0	1	−1	2	−1
Col. Max.	$\boxed{1}$ $\bar{a}$ minimax	3	3	5	

In this case $\underline{a} = 1 = \bar{a}$. Thus, a saddle point exists and an optimal solution exists. *Solution:*

A adops strategy A_2

and, B adops strategy B_1

optimal value = pay-off to A from B is 1.

8.4 MIXED STRATEGY

We have seen in the previous section that $\underline{a} \le \bar{a}$ and also seen how to get optimal solution, in case $\underline{a} = \bar{a}$ using maximin and minimax criteria. Let us see the following example.

Example 1: Let A's pay-off matrix be

$$\begin{array}{c|c|c|c|l} & B_1 & B_2 & B_3 & \text{row min} \\ \hline A_1 & 2 & 0 & 0 & 0 \leftarrow \underline{a} \\ \hline A_2 & 0 & 0 & 4 & 0 \\ \hline A_3 & 0 & 3 & 0 & 0 \\ \hline \text{Col. max} & \underset{\bar{a}}{\overset{2}{\uparrow}} & 3 & 4 & \end{array}$$

In this case $\underline{a} \neq \bar{a}$. Hence, saddle point does not exist, so we have to use mixed strategies, rather than pure strategies.

In such cases players do not select pure strategies as his strategies would depend upon chance. In other words his strategy is determined by *a* game of chance.

Assume for A some chance device selects strategy i with probability p_i and for B another game of chance selects strategy j with probability q_i. In this way chance devices determine the strategy each player will use and neither knows what the other's strategy will be. In fact, none knows what his own strategy will be until it is determined by the chance device. We denote it in tabular form as,

$$\begin{array}{c|ccccc|} & q_1 & q_2 & q_3 & \dots & q_n \\ & B_1 & B_2 & B_3 & \dots & B_n \\ \hline p_1, A_1 & a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ p_2, A_2 & a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & & \\ p_m, A_m & a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \\ \hline \end{array}$$

Where,

p_i = probability that player A selects strategy A_i

q_j = probability that player B selects strategy B_j *i.e*

$P(A_i) = p_i,\ i = 1$ to m

$P(B_j) = q_j,\ j = 1$ to n

Obviously, $\Sigma p_i = 1$ and $\Sigma q_j = 1$

and, $p_i \geq 0$

$q_j \geq 0.$

The vectors $P = (p_1, p_2, \dots, p_m)^T$ and $Q = (q_1, q_2, \dots, q_n)^T$ define mixed strategies for A and B, respectively.

When each player decides to use mixed strategies, we can no longer be sure about the outcome of the game. It is only possible to speak of expected winning of player A. If A uses mixed strategy P and B uses Q, then

$$\begin{aligned} E(P, Q) = {} & p_1 a_{11} q_1 + p_1 a_{12} q_2 + p_1 a_{13} q_3 + \dots + p_1 a_{1n} q_n + \\ & p_2 a_{21} q_1 + p_2 a_{22} q_2 + p_2 a_{23} q_3 + \dots + p_2 a_{2n} q_n + \\ & - + \\ & p_m a_{m1} q_1 + p_m a_{m2} q_2 + p_m a_{m3} q_3 + \dots + p_m a_{mn} q_n \end{aligned}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} p_i \; a_{ij} \; q_j = P^T AQ$$

It should be noted here that uncertainty can creep into game in two different ways, namely, (*i*) nature may make some of the moves, (*ii*) players may choose their strategies by chance devices.

We have not yet said anything about the determination of P and Q. This is the central problem in solving games.

A would like to find P to maximise his winning irrespective of what B selects. For any P, A selects, he is sure that his winning would be at least

$$\min_{Q} \; E(P, Q)$$

He would then like to maximise it. Let

$$\underline{a}^* = \max_{P} \min_{Q} \; E(P, Q)$$

Similarly, B can be sure that the expected winning of A is not more than

$$\max_{P} \; E(P, Q)$$

He would then like to minimise it. Let

$$\bar{a}^* = \min_{Q} \max_{P} \; E(P, Q)$$

If there exists P^* and Q^* such that $\underline{a}^* = \bar{a}^*$, then we have a generalised saddle point and in such case A should use the mixed strategy P^* and B should use Q^* and expected winning of player A is $a^* = \underline{a}^* = \bar{a}^*$.

P^*, Q^* always exist such that

$$\underline{a}^* = \max_{P} \min_{Q} \; E(P, Q) = \min_{Q} \max_{P} \; E(P, Q) = \bar{a}^*$$

Such that $\quad P^* \geq O, \; Q^* \geq O$

and, $\quad \Sigma p_i = 1, \; \Sigma q_j = 1.$

8.5 REDUCTION OF A GAME TO LPP

If B uses pure strategy j and A uses mixed strategy $P = (p_1, p_2, \ldots, p_m)^T$, then expected pay-off to A is

$$\sum_{i=1}^{m} a_{ij} \; P_i$$

A can be sure that his minimum pay-off would be $\underset{\sim}{a}$, for his mixed strategy P, where

$$\underset{\sim}{a} = \min_{j} \left\{ \sum_{i=1}^{m} a_{ij} \; p_i \right\}$$

A would like to select his mixed strategy P such that he maximises a, say

$$\max_{P} \; \underset{\sim}{a} \qquad (1)$$

Subject to $\Sigma p_i = 1,\ P \geq 0$

Similarly, if A selects pure strategy i and B selects mixed strategy $Q = (q_1, q_2, \ldots, q_n)^T$ then expected pay-off from B to A would be

$$\sum_{j=1}^{n} a_{ij}\, q_j$$

B can be sure this maximum pay-off from him would be $\tilde{a}$, for his mixed strategy Q, where

$$\tilde{a} = \operatorname*{Max}_{i} \left\{ \sum_{j=1}^{n} a_{ij}\, q_j \right\}$$

B would select Q such that he minimises $\tilde{a}$, so,

$$\operatorname*{Min}_{Q} \tilde{a} \qquad (2)$$

Subject to $\Sigma q_j = 1,\ Q \geq 0$

Remark: A pure strategy j is the mixed strategy e_j.

Game theory and Linear programming are closely related. Every two person zero-sum problem can be converted into a LPP as is done below.

We have,

$$\underset{\sim}{a} = \text{Min} \left\{ \sum_{i=1}^{m} a_{i1}\, p_i, \sum_{i=1}^{m} a_{i2}\, p_i, \ldots, \sum_{i=1}^{m} a_{in}\, p_i \right\}$$

Then [1] becomes

$$\text{Max } Z = a$$

where, $$\underset{\sim}{a} \leq \sum_{i=1}^{m} a_{ij}\, p_i,\ j = 1, 2, \ldots, n$$

$$\sum_{i=1}^{m} p_i = 1,$$

$$p_i \geq 0,\ i = 1, 2, \ldots, m$$

If $\underset{\sim}{a} > 0$, it is okay. If $\underset{\sim}{a} \leq 0$ then we add a constant c_1 to each pay-off in the matrix, as it won't change the game. Hence, it won't change the solution except that optimal value would be changed. We would then subtract c_1 from the optimal value. It is done so that $\underset{\sim}{a}$ becomes strictly positive.

So, we add c_1 so that $\underset{\sim}{a} > 0$, if needed. Without loss of generality, we can now assume $\underset{\sim}{a} > 0$.

Now, since $\underset{\sim}{a}$ is a positive constant, we use substitution

$$x_i = \frac{p_i}{\underset{\sim}{a}}$$

and obtain,

$$\text{Max } Z = \underset{\sim}{a}$$

Subject to $\sum_{i=1}^{m} a_{ij}\, x_i \geq 1,\ j = 1, 2, \ldots, n$

$$\sum_{i=1}^{m} x_i = \frac{1}{\underset{\sim}{a}}$$

$$x_i \geq 0,\ i = 1, 2, \ldots, m.$$

We know that Max $\underset{\sim}{a}$ = Min $\left(\dfrac{1}{\underset{\sim}{a}}\right)$, therefore, we get

$$\text{Min } Z = \frac{1}{\underset{\sim}{a}} = x_1 + x_2 + \ldots + x_m,$$

Subject to $\sum_{i=1}^{m} a_{ij}\, x_i \geq 1,\ j = 1, 2, \ldots, n$ (3)

$$x_i \geq 0$$

Similarly, we obtain

$$\tilde{a} = \text{Max}\left\{\sum_{j=1}^{n} a_{1j}\, q_j, \sum_{j=1}^{n} a_{2j}\, q_j, \ldots, \sum_{j=1}^{n} a_{mj}\, q_j\right\}$$

Then (3) becomes

$$\text{Min } Z = \tilde{a}$$

Subject to $\sum_{j=1}^{n} a_{ij}\, q_j \leq \tilde{a},\ i = 1, 2, \ldots, m.$

$$\sum_{j=1}^{n} q_j = 1,$$

$$q_j \geq 0,\ j = 1, 2, \ldots, n.$$

If $\tilde{a} > 0$, it is okay, otherwise as discussed above, we add a positive constant c_2 such that $\tilde{a} > 0$. It won't affect the solution except the optimal value.

In fact, we add a constant C such that both $\underset{\sim}{a}$ and $\tilde{a}$ become positive. This addition does not affect the solution except the optimal value and subtract C from the optimal value.

So, we assume $\tilde{a} > 0$ and substitute $q_j = \tilde{a} y_j$,

We obtain

$$\text{Min } Z = \tilde{a}$$

Subject to $\sum_{j=1}^{n} a_{ij}\, y_j \leq 1,\ i = 1, 2, \ldots, m$

and, $$\sum_{j=1}^{n} y_j = \frac{1}{\tilde{a}},$$

$$y_j \geq 0,\ j = 1, 2, \ldots, n$$

Also, since Min $\tilde{a}$ = Max $1/\tilde{a}$, we obtain

$$\text{Max } W = y_1 + y_2 + \ldots\ y_n$$

Subject to $$\sum_{j=1}^{n} a_{ij}\, y_j \leq 1,\ i = 1, 2, \ldots, m \qquad (4)$$

$$y_j \geq 0.$$

Thus, [1] reduces to [3] and [2] reduces to [4]. If we look at closely, we notice that [3] and [4] are dual to each other, as we assume that Max $\underset{\sim}{a}$ = Min $\tilde{a}$ = a.

$$\text{The value of the game} = \frac{1}{W} - C = \frac{1}{Z} - C$$

$$\text{Mixed strategies for } A,\ p_i = \frac{x_i}{Z}$$

and, $$\text{Mixed strategies for } B,\ q_j = \frac{y_j}{W}$$

Since players can use pure strategies, it is feasible to obtain p_i and q_j. Thus, [3] and [4] have feasible solution. If either has unbounded solution, then the other has no feasible solution, which is a contradiction. Thus, they have bounded solution.

We can therefore solve any one of them. It will yield the same solution. We select the one that has less number of constraints.

Example 1: Let A′s pay-off matrix be

	B_1	B_2	B_3	B_4	B_5	Row's Min
A_1	2	−1	5	−2	6	$\boxed{-2}$
A_2	−2	4	−3	1	0	−3
Col. Max	2	4	5	$\boxed{1}$	6	

Solve the above as LPP or find mixed strategies for players A and B.

Solution: Convert into an LPP. We add 3 in each entry so that both $\tilde{a}$, $\underset{\sim}{a}$ are > 0

	B_1	B_2	B_3	B_4	B_5	Row Min
A_1	5	2	8	1	9	1
A_2	1	7	0	4	3	0
Col. Max	5	7	8	4	9	

LPP of A's problem

$$\text{Min } Z = x_1 + x_2$$

$$5x_1 + x_2 \geq 1$$
$$2x_1 + 7x_2 \geq 1$$
$$8x_1 \geq 1$$
$$x_1 + 4x_2 \geq 1$$
$$9x_1 + 3x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

LPP of B's problem

$$\text{Max } W = y_1 + y_2 + y_3 + y_4 + y_5$$

$$5y_1 + 2y_2 + 8y_3 + y_4 + 9y_5 \leq 1$$
$$y_1 + 7y_2 + 4y_4 + 3y_5 \leq 1$$
$$y_1, y_2, y_3, y_4, y_5, \geq 0$$

We would prefer to solve the later one

BV	y_1	y_2	y_3	y_4	y_5	s_1	s_2	*Sol.*
W	–1	–1	–1	–1	–1	0	0	0
s_1	5	2	8	1	9	1	0	1
s_2	1	7	0	$\boxed{4}$	3	0	1	1
W	–3/4	3/4	–1	0	–1/4	0	1/4	1/4
s_1	19/4	1/4	$\boxed{8}$	0	33/4	1	–1/4	3/4
y_4	1/4	7/4	0	1	3/4	0	1/4	1/4
W	–5/32	25/32	0	0	25/32	1/8	7/32	11/32
y_3	$\boxed{19/32}$	1/32	1	0	33/32	1/8	–1/32	3/32
y_4	1/4	7/4	0	1	3/4	0	1/4	1/4
W	0	15/19	5/19	0	20/19	3/19	4/19	7/19
y_1	1	1/19	32/19	0	33/19	4/19	–1/19	3/19
y_4	0	33/19	–8/19	1	6/19	–1/19	5/19	4/19

It is an optimal table. Thus, we have

$$y_1 = \frac{3}{19}, y_2 = 0, y_3 = 0, y_4 = \frac{4}{19}, y_5 = 0$$

and optimal value 7/19 which is of $\frac{1}{\tilde{a}}$.

Hence, $\quad \text{Min } \tilde{a} = \dfrac{19}{7}.$

Thus, $\quad q_1 = \tilde{a} y_1 = \dfrac{3}{7},\ q_2 = 0,\ q_3 = 0,\ q_4 = \dfrac{4}{7},\ q_5 = 0.$

and optimal value is $\dfrac{19}{7} - 3 = -\dfrac{2}{7}$

From this we can get the solution of A's LPP, i.e., primal, we have

$$(x_1, x_2)^T = C_B^T B^{-1}$$

$$= (1, 1) \begin{bmatrix} 4/89 & -1/19 \\ -1/19 & 5/19 \end{bmatrix}$$

$$= \left(\frac{3}{19}, \frac{4}{19}\right)^T$$

or, $\quad x_1 = \dfrac{3}{19},\ x_2 = \dfrac{4}{19}$

or, $\quad p_1 = \dfrac{3}{7},\ p_2 = \dfrac{4}{7}$

This gives that pay-off to A would be $-\dfrac{2}{7}$ i.e., he will lose and probabilities of strategies of A's are 3/7, 4/7 while that of B's are $\dfrac{3}{7}$, 0, 0, $\dfrac{4}{7}$, 0

Let us take another example, the one which we illustrated in the beginning of the section.

Example 2: Solve the following game problem.

	B_1	B_2	B_3
A_1	2	1	1
A_2	1	1	5
A_3	1	4	1

Solution: LPP of both A and B's problem would have same number of constraints and variable. We write B's problem which would have advantage of slack variables. We get

$$\text{Max } W = y_1 + y_2 + y_3$$

$$2y_1 + y_2 + y_3 \le 1$$

$$y_1 + y_2 + 5y_3 \le 1$$

$$y_1 + 4y_2 + y_3 \le 1,\ y_1, y_2, y_3 \ge 0.$$

Solving, we obtain

BV	y_1	y_2	y_3	s_1	s_2	s_3	*Sol.*
W	–1	–1	–1	0	0	0	0
s_1	$\boxed{2}$	1	1	1	0	0	1
s_2	1	1	5	0	1	0	1
s_3	1	4	1	0	0	1	1
W	0	–1/2	–1/2	1/2	0	0	1/2
y_1	1	1/2	1/2	1/2	0	0	1/2
s_2	0	1/2	9/2	–1/2	1	0	1/2
s_3	0	$\boxed{7/2}$	1/2	–1/2	0	1	1/2
W	0	0	$-\frac{3}{7}$	$\frac{3}{7}$	0	$\frac{1}{7}$	4/7
y_1	1	0	3/7	$\frac{4}{7}$	0	$-\frac{1}{7}$	3/7
s_2	0	0	$\boxed{\frac{31}{7}}$	$-\frac{3}{7}$	1	$-\frac{1}{7}$	3/7
y_2	0	1	1/7	–1/7	0	$\frac{2}{7}$	$\frac{1}{7}$
W	0	0	0	$\frac{12}{31}$	$\frac{3}{31}$	$\frac{4}{31}$	19/31
y_1	1	0	0	$\frac{19}{31}$	$-\frac{3}{31}$	$-\frac{4}{31}$	12/31
y_3	0	0	1	$-\frac{3}{31}$	$\frac{7}{31}$	$-\frac{1}{31}$	$\frac{3}{31}$
y_2	0	1	0	$-\frac{4}{31}$	$-\frac{1}{31}$	$\frac{9}{31}$	$\frac{4}{31}$

It is optimal table. Thus,

$$y_1 = \frac{12}{31}, y_2 = \frac{4}{31}, y_3 = \frac{3}{31}$$

and,
$$\text{Max } W = \frac{1}{\tilde{a}} = \frac{19}{31}$$

or,
$$\tilde{a} = \frac{31}{19}$$

Thus, $$q_1 = \frac{12}{19},\ q_2 = \frac{4}{19},\ q_3 = \frac{3}{19}$$

and optimal expected value is $\frac{31}{19} - 1 = \frac{12}{31}$

$$(x_1,\ x_2,\ x_3)^T = \frac{1}{31}\ (1,\ 1,\ 1)\begin{bmatrix} 19 & -3 & -4 \\ -3 & 7 & -1 \\ -4 & -1 & 9 \end{bmatrix} = \frac{1}{31}(12,\ 3,\ 4)^T$$

or, $$p_1 = \frac{12}{19},\ p_2 = \frac{3}{19},\ p_3 = \frac{4}{19}$$

Thus, expected pay-off to A would be 12/31 and the probabilities to A's and B's strategies would be as obtained.

EXERCISE 8.1

1. The following games give A's pay-off. Determine the values of a and b that will make the entry (2, 2) of each game a saddle point.

(a)

	B_1	B_2	B_3
A_1	2	b	7
A_2	a	6	11
A_3	7	3	4

(**Ans:** $a \geq 5,\ b \leq 5$)

(b)

	B_1	B_2	B_3
A_1	1	3	4
A_2	9	6	b
A_3	3	a	5

(**Ans:** $a \leq 6,\ b \geq 6$)

2. Consider the following pay-off of player A. Find the range for the value of the game.

	B_1	B_2	B_3	B_4
A_1	4	12	9	3
A_2	5	6	11	7
A_3	−2	1	13	0
A_4	10	7	1	−2

(**Ans:** Value of game lies between 5 and 7 (both inclusive))

3. Using Maximin, Minimax Criterion find the optimal strategies and value of the game for the following pay-off of player A.

$$\begin{array}{c|cccc|} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 6 & 4 & 0 & 6 \\ A_2 & 6 & 7 & 2 & 3 \\ A_3 & 5 & 3 & 1 & 4 \\ \hline \end{array}$$

(**Ans:** Value of game = 2. Optimal strategy for $A = A_2$
& optimal strategy for $B = B_3$)

4. Solve the following game by LP.

$$\begin{array}{c|ccc|} & B_1 & B_2 & B_3 \\ \hline A_1 & 3 & -1 & -3 \\ A_2 & -2 & 4 & -1 \\ A_3 & -5 & -6 & 2 \\ \hline \end{array}$$

(**Ans:** $x_1 = \dfrac{39}{100}$, $x_2 = \dfrac{8}{25}$, $x_3 = \dfrac{29}{100}$

$y_1 = \dfrac{8}{25}$, $y_2 = \dfrac{2}{25}$, $y_3 = \dfrac{3}{5}$

Value of the game $= \dfrac{91}{100}$)

5. Consider the following pay-off matrix of player A

$$\begin{array}{c|ccc|} & B_1 & B_2 & B_3 \\ \hline A_1 & 4 & 3 & 7 \\ A_2 & 2 & 4 & 1 \\ \hline \end{array}$$

(a) Write the player A's problem.
(b) Write the player B's problem.
(c) Solving the player B's problem, find the mixed strategies for player A and B.
(d) What is the value of game?

8.6 GRAPHICAL METHOD TO SOLVE A (2 × *n*) AND (*m* × 2) GAME PROBLEM

If the game problem does not have a saddle point and player A has 2 strategies and player B has n strategies, then this problem can be solved by graphical method. Similarly, if player A has m strategies and player B has 2 strategies can be solved graphically.

$$\begin{array}{cc|c|c|c|c|} & & \multicolumn{4}{c}{B} \\ & & y_1 & y_2 & \dots & y_3 \\ \cline{3-6} A & x_1 & a_{11} & a_{12} & \dots & a_{1n} \\ \cline{3-6} & x_2 = 1 - x_1 & a_{21} & a_{22} & \dots & a_{2n} \\ \cline{3-6} \end{array}$$

Consider the $(2 \times n)$ games. Since, player A has two strategies x_1 and x_2, then $x_2 = 1 - x_1$, $x_1 \geq 0$, $x_2 \geq 0$ (because $x_1 + x_2 = 1$). Thus, for each of the pure strategies available to the player B, the expected pay-off for the player B, the expected pay-off for the player A would be as follows:

B's pure strategies	A's Expected pay-off
1	$a_{11}\, x_1 + a_{21}(1 - x_1) = (a_{11} - a_{21})x_1 + a_{21}$
2	$a_{12}\, x_1 + a_{22}(1 - x_1) = (a_{12} - a_{22})x_1 + a_{22}$
$\vdots$	
n	$a_{1n}\, x_1 + a_{2n}(1 - x_1) = (a_{1n} - a_{2n})x_1 + a_{2n}$

This shows that player A's expected pay-off varies linearly with x_1.

According to minimax criterion, the player A should select the value of x_1 so as to maximise his minimum expected pay-off. This may be done by plotting the straight lines as follows:

$$
\begin{aligned}
E(v) &= (a_{11} - a_{21})\, x_1 + a_{21} \\
E(v) &= (a_{12} - a_{22})\, x_1 + a_{22} \\
&\vdots \\
E(v) &= (a_{1n} - a_{2n})\, x_1 + a_{2n}
\end{aligned}
$$

as functions of x_1. The lower boundary of these lines will give the minimum expected pay-off as function of x_1 and the highest point on this lower boundary would then give the maximin expected pay-off and the optimum value of x_1. Now determine only two strategies of player B corresponding to these two lines which pass through the maximin point. This way we can reduce a $2 \times n$ game to 2×2 games.

Similarly, we can solve a $(m \times 2)$ games except that minimax point is the lowest point on the upper boundary instead of highest point on the lower boundary.

Based on the above discussion, we can say that any $(2 \times n)$ or $(m \times 2)$ game can be reduced in a (2×2) game.

Example 1: Solve the following game graphically.

$$
\begin{array}{cc}
 & \text{Player } B \\
\text{Player } A & \begin{array}{c|c|c|c|}
 & \text{I} & \text{II} & \text{III} \\
\hline
\text{I} & 1 & 3 & 11 \\
\hline
\text{II} & 8 & 5 & 2 \\
\hline
\end{array}
\end{array}
$$

Solution: The above game does not have a saddle point. Thus, the A's expected pay-off corresponding to the player B's pure strategies are given in the following table:

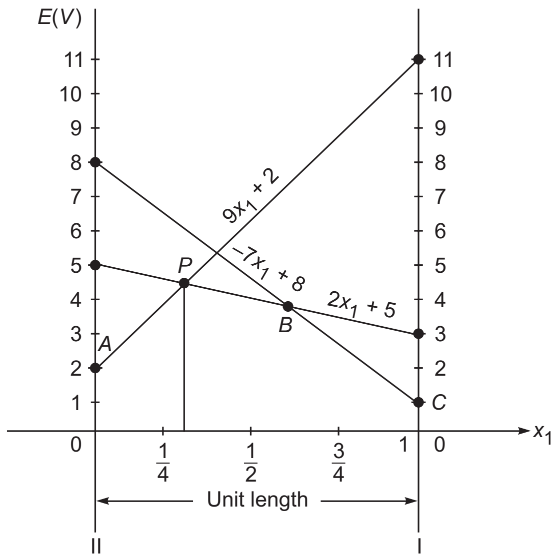
B's pure strategy	A's expected pay-off.
I	$1 \cdot x_1 + 8(1 - x_1) = -7x_1 + 8$
II	$3x_1 + 5(1 - x_1) = -2x_1 + 5$
III	$11(x_1 + 2(1 - x_1)) = 9x_1 + 2$

Three expected pay-off lines are

$$-7x_1 + 8, -2x_1 + 5 \quad \text{and} \quad 9x_1 + 2.$$

These lines can be plotted on a graph as follows: First draw two parallel lines one unit apart and mark a scale on each. These two lines will represent two strategies available to the player *A*. Then draw lines to represent each of the player *B*′s strategies.

For example, to represent the player *B*′s I strategy, join mark 1 on scale I and 8 on scale II; to represent the player *B*′s II strategy join mark 3 on I and 5 on II. Similarly, to represent the player *B*′s III strategy join mark 11 on scale I and 2 on scale II. Since, the expected pay-off is a function of x_1 alone so these three expected pay-off lines can be drawn by taking x_1 as *x*-axis *E*(*v*) as *y*-axis.



(Graphical representation for the above example (2 × 3) game)

Point *A*, *P*, *B* and *C* on the lowest boundary which is shown by a thick line in the above figure represent the lowest possible expected gain to the player *A* for any value of x_1 between 0 and 1. According to the maximin criterion, the player *A* chooses the best of these worst outcomes. The highest point *P* on the lowest boundary will give the largest expected gain *PN* to *A*. So best strategies for player *B* are those which pass through the print *P*. Thus, the game reduced to (2 × 2) as follows:

		Player *B* I	Player *B* II
Player *A*	I	3	11
	II	5	2

Now solving simultaneous equations:

$$3x_1 + 5x_2 = V$$
$$11x_1 + 2x_2 = V$$
$$x_1 + x_2 = 1,\ x_1,\ x_2 \geq 0$$
$$3y_2 + 11y_3 = V$$
$$5y_2 + 2y_3 = V$$
$$y_2 + y_3 = 1$$

we get $x_1 = 3/11,\ x_2 = \dfrac{8}{11}\ y_2,\ y_3 \geq 0$

and, $y_2 = \dfrac{9}{11}$ and $y_3 = \dfrac{2}{11}$

Value of game $= \dfrac{49}{11}$

The solution of the game is as follows:

(i) The player A chooses optimal mixed strategies $(x_1, x_2) = \left(\dfrac{3}{11}, \dfrac{8}{11}\right)$

(ii) The player B chooses optimal mixed strategies $(y_1, y_2, y_3) = \left(0, \dfrac{9}{11}, \dfrac{2}{11}\right)$

(iii) Value of the game $= \dfrac{49}{11}$.

Example 2: Solve the following by graphical method.

		Player B 1	2
Player A	1	2	7
	2	3	5
	3	11	2
		y_1	$1-y_1$

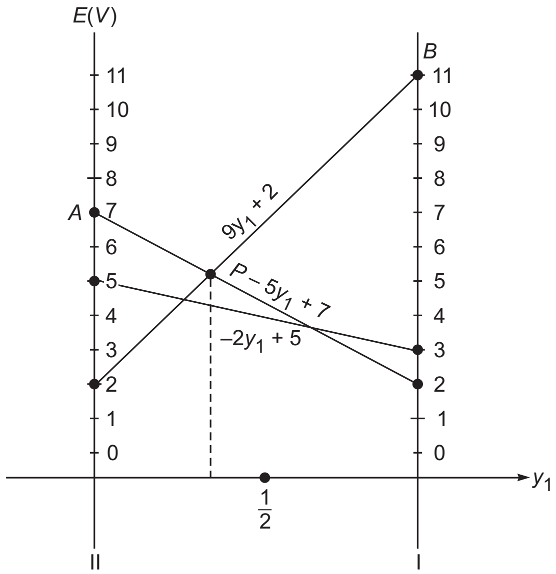
Solution: It is a (3×2) game.

Expected pay-off player B when player A chooses mixed strategies:

Player A's strategies	Player B's pay-off
1	$2y_1 + 7(1 - y_1) = -5y_1 + 7$
2	$3y_1 + 5(1 - y_1) = -2y_1 + 5$
3	$11y_1 + 2(1 - y_1) = 9y_1 + 2$

The expected pay-off lines are

$-5y_1 + 7$, $-2y_1 + 5$ and $9y_1 + 2$. Plot the graph as follows.



Optimal strategies of A are 1 and 3 so the game reduces to

		Player B	
		1	2
Player A	2	2	7
	3	11	2

The simultaneous equations are:

$$\begin{aligned} 2y_1 + 7y_2 &= v \\ 11y_1 + 2y_2 &= v \\ y_1 + y_2 &= 1 \\ y_1, y_2 &\geq 0 \end{aligned}$$

and,

$$\begin{aligned} 2x_1 + 11x_3 &= v \\ 7x_1 + 2x_3 &= v \\ x_1 + x_3 &= 1 \\ x_1, x_3 &\geq 0 \\ x_2 &= 0 \end{aligned}$$

Solving above equations, we get

$$y_1 = \frac{5}{14}, \; y_2 = \frac{9}{14}$$

$$x_1 = \frac{19}{14}, \; x_2 = 0, \; x_3 = \frac{5}{14}$$

$$\text{Value of game} = \frac{73}{14}.$$

EXERCISE 8.2

1. Use graphical method to solve the following two persons zero, sum game whose pay-off matrix is given below:

Player A

		I	II	III	IV	V
Player B	I	1	3	−1	4	2	−5
	II	−3	5	6	1	2	0

(**Ans:** Player A's mixed strategies = $\left(0, \frac{3}{5}, 0, \frac{2}{5}, 0 \text{ and } 0\right)$

Player B's mixed strategies = $\left(\frac{4}{5}, \frac{1}{5}\right)$

Value of game = $\frac{17}{5}$)

2. Solve the following (2 × 4) game by graphical method.

B

		1	2	3	4
A	1	2	2	3	−1
	2	4	3	2	0

$\left(\textbf{Ans: } x_1 = \frac{1}{2} = x_2,\ y_1 = 0 = y_2, y_3 = 7/8,\ y_4 = 1/8, \text{ Value of game } = 5/2\right)$

3. Solve the following game whose pay-off matrix is given by graphical method.

A

		I	II	III	IV
B	I	2	2	3	−2
	II	4	3	2	6

(**Ans:** The player A chooses the optimal mixed

strategy $(x_1, x_2, x_3, x_4) = \left(\frac{1}{3}, 0, \frac{2}{3}, 0\right)$

The player B chooses the optimal mixed strategies

$(y_1, y_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$

Value of game = $\frac{8}{3}$

4. Solve the following game graphically.

$$\begin{array}{cc} & B \\ & \begin{array}{ccc} \mathrm{I} & \mathrm{II} & \mathrm{III} \end{array} \\ A \begin{array}{c} \mathrm{I} \\ \mathrm{II} \end{array} & \begin{array}{|c|c|c|} \hline 4 & -1 & 0 \\ \hline -1 & 4 & 2 \\ \hline \end{array} \end{array}$$

$\left(\textbf{Ans: } x_1 = 3/7, x_2 = 4/7, y_1 = \dfrac{2}{7}, y_2 = 0, y_3 = 5/7, \text{Value of game} = 8/7\right)$

Chapter 9

Classical Optimization Techniques

9.1 INTRODUCTION

In this chapter, we deal with classical optimization techniques. In classical optimization techniques, we consider the unconstrained optimization problems with single and multivariable optimization problems, constrained multivariable optimization problem with equality constrained using the direct substitution method and Lagrange multipliers method, and unequality constrained using Kuhn-Tucker condition to find optimum solution. Convex and concave functions are also discussed. The classical optimization techniques are very important in obtaining the optimum solution of continuous and differentiable function involving problems. Such type of techniques are analytical in nature and very important for differential calculus to find maxima and minima.

9.2 UNCONSTRAINED OPTIMIZATION PROBLEMS

We shall discuss single variable and multivariable optimization (unconstrained) problems.

9.2.1 Single Variable Optimization Problems

Let us consider a continuous function $f(x)$ of single variable x_1 defined in interval $[a, b]$.

Local Maxima: A function $f(x)$ with single variable is said to have a local (relative) maxima at

$$x = x_0 \text{ if}$$

$$f(x_0) \geq f(x_0 + h)$$

for all sufficiently small positive and negative value of h.

Local Minima: A function $f(x)$ with single variable is said to have a local (relative) minima at

$$x = x_0 \text{ if}$$

$$f(x_0) \leq f(x_0 + h)$$

for all sufficiently small positive and negative value of h.

Global Maxima: A function $f(x)$ with single variable is said to have a global (absolute) maxima at

$$x = x_0 \text{ if}$$

$$f(x_0) \geq f(x)$$

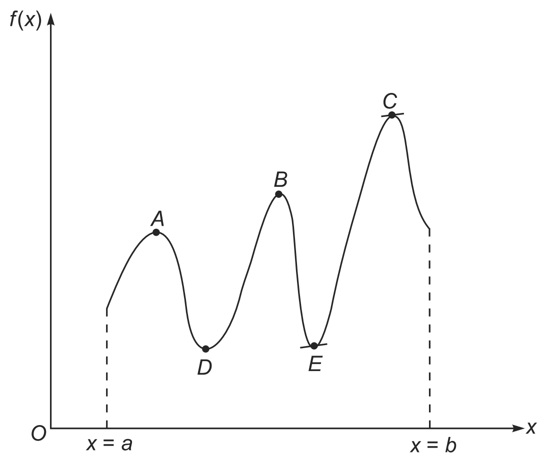
for all x defined in the interval $[a, b]$.

Global Minima: A function $f(x)$ with single variable is said to have a global (absolute) minima at

$$x = x_0 \text{ if}$$

$$f(x_0) \leq f(x)$$

for all x defined in the interval $[a, b]$.



Here A, B, C are local maxima, D and E are local minima; C is global maxima and E is global minima.

Condition for Local Maxima or Minima of single variable function.

Necessary Condition

If a function $f(x)$ is defined in the interval $[a, b]$ and has a local maxima or local minima at $x = x_0$, where $a < x_0 < b$ and if $f'(x)$ exists as a finite number at $x = x_0$ then $f'(x_0) = 0$.

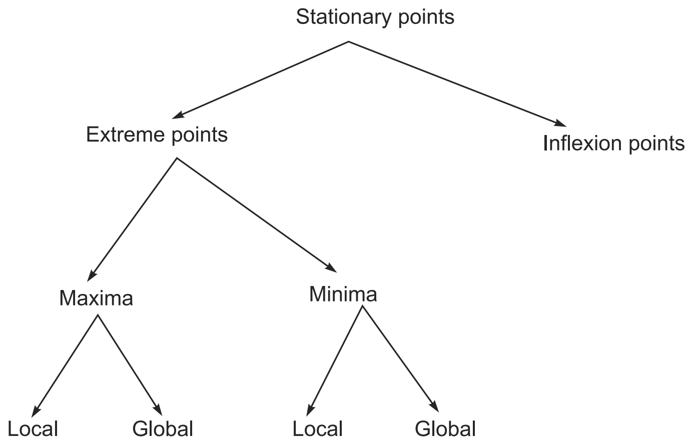
Sufficient Condition

If $f'(x_0) = f''(x_0) = f'''(x_0) = \ldots = f^{(n-1)}_{(x_0)} = 0$, $f^{n}_{(x_0)} = \neq 0$ then $f(x_0)$ has

(i) a minimum value of $f(x)$ if $f^{n}_{(x_0)} > 0$ and n is even

(ii) a maximum value of $f(x)$ if $f^{n}_{(x_0)} < 0$ and n is even

(iii) neither maxima nor minima, i.e., point of inflexion of $f^{n}_{(x_0)} \neq 0$ and n is add.



Determination of stationary (critical) points.

Working rule to finding the extreme point of functions of one variable
Consider a single variable function

$$y = f(x) \tag{1}$$

Step 1: Differentiating equation (1) with respect to x both sides, we get

$$\frac{dy}{dx} = f'(x) \tag{2}$$

Step 2: For extreme point, we have

$$\frac{dy}{dx} = 0$$

$\Rightarrow$ $$f'(x) = 0$$

we get values of x which satisfied $f'(x) = 0$

Let $$x = x_0, x_1, \ldots$$

Step 3: Differentiating equation (2) again with respect to x both sides, we get

$$\frac{d^2y}{dx^2} = f''(x)$$

Step 4: If $f''(x_0) > 0$ then $f(x)$ has minima at

$$x = x_0$$

Step 5: If $f''(x_0) < 0$ then $f(x)$ has maxima at

$$x = x_0$$

Step 6: If $f''(x_0) = 0$ then find $f'''(x)$ and we see that $f'''(x_0) \neq 0$, then there is neither maxima nor minima at $x = x_0$, i.e. $x = x_0$ is called point of inflexion.

Step 7: If $f'''(x_0) = 0$ then find $f''''(x)$ or $f^{(\text{iv})}_{(x)}$ and use the steps 4 and 5.

9.2.2 Multivariable Optimization Problems

Working rule to obtain the extreme points of functions of two varables

Consider $U = f(x_1, x_2)$ is a function of x_1 and x_2. (1)

Step 1: Differentiating partially equation (1) with respect to x_1 and x_2 both sides respectively, we get

$$\frac{\partial f}{\partial x_1} \text{ and } \frac{\partial f}{\partial x_2} \tag{2}$$

Step 2: For extreme point, we have

$$\frac{\partial f}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial f}{\partial x_2} = 0$$

Solving these equations, we get (a_1, b_1), (a_2, b_2) …

Step 3: Differentiating again partially (2) and we get

$$r = \frac{\partial^2 f}{\partial x_1^2}, \; s = \frac{\partial^2 f}{\partial x_1 \partial x_2} \quad \text{and } t = \frac{\partial^2 f}{\partial x_2^2}$$

Step 4: At (a_1, b_1) obtain $rt - s^2$

Case 1 If $rt - s^2 > 0$ and $r > 0$, then $f(x_1, x_2)$ is minimum at (a_1, b_1).

Case 2 If $rt - s^2 > 0$ and $r < 0$, then $f(x_1, x_2)$ is maximum at (a_1, b_1).

Case 3 If $rt - s^2 < 0$, then $f(x_1, x_2)$ is neither maxima nor minima at (a_1, b_1).

Case 4 If $rt - s^2 > 0$, then $f(x_1, x_2)$ has point of inflexion at (a_1, b_1).

Working rule to obtain the extreme points of functions of n variables.

Let $$U = f(x_1, x_2, x_3, \ldots; x_n)$$

be a function of $x_1, x_2, x_3, \ldots, x_n$ (1)

Necessary conditions

$$\frac{\partial f}{\partial x_1} = 0, \; \frac{\partial f}{\partial x_2} = 0, \; \frac{\partial f}{\partial x_3} = 0, \; \ldots, \; \frac{\partial f}{\partial x_n} = 0$$

Sufficient Conditions

The Hessian matrix at P for n variables will be

$$H = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{vmatrix}$$

Its leading minors are defined as

$$H_1 = \left| \dfrac{\partial^2 f}{\partial x_1^2} \right|$$

$$H_2 = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{vmatrix}$$

$$H_3 = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_3} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_3} \\ \dfrac{\partial^2 f}{\partial x_3 \partial x_1} & \dfrac{\partial^2 f}{\partial x_3 \partial x_2} & \dfrac{\partial^2 f}{\partial x_3^2} \end{vmatrix}$$

Here some cases will arise, which are as follows:

Case 1 If $H_1, H_2, H_3, \ldots$ are positive (i.e., positive definite), then $f(x_1, x_2, x_3, \ldots, x_n)$ is minimum at P.

Case 2 If $H_1, H_2, H_3 \ldots$ are alternatively negative, positive, negative (i.e., negative definite), then $f(x_1, x_2, x_3, \ldots x_n)$ is maximum at P.

Case 3 If H_1 and $H_3 \ldots$ are not of same sign and $H_2 = 0$ (i.e., semi-definite or indefinite), then $f(x_1, x_2, x_3, \ldots, x_n)$ has a saddle point at P.

SOLVED EXAMPLES

Example 1. A beam of length l is supported at one end. If w is its uniform load per unit length, the bending moment, M at a distance x from the end, is given by $M = \frac{1}{2}lx - \frac{1}{2}wx^2$.

Find the maximum bending moment and the point where it acts.

Solution: Given that the bending moment

$$M = \frac{1}{2}lx - \frac{1}{2}wx^2 \tag{1}$$

Differentiating equation (1) with respect to x both sides, we get

$$\frac{dM}{dx} = \frac{1}{2}\,1 - wx$$

For maxima, we have

$$\frac{dM}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{2}l - wx = 0$$

$$\Rightarrow \quad x = \frac{l}{2w}$$

Differentiating equation (2) again with respect to x both sides, we get

$$\frac{d^2M}{dx^2} = -w \qquad \text{(+ve)}$$

Therefore the bending moment M is maximum at $x = \frac{l}{2w}$ from the end.

$$\text{At } x = \frac{l}{2w},$$

$$M = \frac{l}{2}l\left(\frac{l}{2w}\right) - \frac{1}{2}w\left(\frac{l}{2w}\right)^2$$

$$= \frac{l^2}{8w}$$

Example 2. Find the maximum and minimum values of $y = 3x^5 - 5x^3$.

Solution: Given that

$$y = 3x^5 - 5x^3 \tag{1}$$

Differentiating equation (1) with respect to x both sides, we get

$$\frac{dy}{dx} = 15x^4 - 15x^2 \tag{2}$$

For maxima and minima, we have

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad 15x^4 - 15x^2 = 0$$

$$\Rightarrow \qquad 15x^2\,(x^2 - 1) = 0$$

$$\Rightarrow \qquad x = 0, -1, 1.$$

Differentiating equation (2) again with respect to x both sides, we get

$$\frac{d^2 y}{dx^2} = 60x^3 - 30x$$

At $x = 0$, $\dfrac{d^2 y}{dx^2} = 0$ i.e., x is a point of inflexion.

At $x = -1$, $\dfrac{d^2 y}{dx^2} = -30 > 0$ i.e., y is maximum at $x = -1$

At $x = 1$, $\dfrac{d^2 y}{dx^2} = 30 > 0$ i.e., y is minimum at $x = 1$.

Example 3. The profit earned P, by a company is function of the units produced (x) and is given by

$$P = 800x - 2x^2$$

If the company's expenditure or interest, rent and salary of the staff be Rs. 1 lac. Show that the company will always be in loss.

Solution: Given that

$$P = 800x - 2x^2 \tag{1}$$

Differentiating equation (1) with respect to x both sides, we get

$$\frac{dp}{dx} = 800 - 4x \tag{2}$$

For maxima and minima, we have

$$\frac{dp}{dx} = 0$$

$$\Rightarrow \qquad 800 - 4x = 0$$

$$\Rightarrow \qquad x = 200$$

Differentiating equation (2) again with respect to x both sides, we get

$$\frac{d^2 p}{dx^2} = -4 \text{ (–ve)}$$

Therefore, P is maximum for $x = 200$

The net profit $= P -$ expenditure

$$= 800 \times 200 - 2(200)^2 - 1,00,000$$
$$= -20,000$$

Hence, the company will always be in loss.

Example 4. In a submarine telegraph cable, the speed of signalling varies as $x^2 \log\left(\frac{1}{x}\right)$ where x is the ratio of the radius of the cube to that of the covering. Show that the greatest speed is attained when this ratio is $1 : \sqrt{e}$.

Solution: Suppose U is the speed of signally then

$$U = \lambda x^2 \log\left(\frac{1}{x}\right), \quad \lambda > 0,\ x \neq 0$$

$$\Rightarrow \qquad U = -\lambda x^2 \log x \qquad (1)$$

Differentiating equation (1) with respect to x both sides, we get

$$\frac{dU}{dx} = -2\,\lambda x \log x - \lambda x^2\,\frac{1}{x}$$

$$= -\lambda\,[2x \log x + x] \qquad (2)$$

For maxima and minima, we have

$$\frac{dU}{dx} = 0$$

$$\Rightarrow \qquad -\lambda\,[2x \log x + x] = 0$$

$$\Rightarrow \qquad \log x = -1/2$$

$$\Rightarrow \qquad x = e^{-/2} = \frac{1}{\sqrt{e}}$$

Differentiating equation (2) again with respect to x both sides, we get

$$\frac{d^2U}{dx^2} = -\lambda\left[2x\cdot\frac{1}{x} + 2\log x + 1\right]$$

$$= -\lambda\,[2 \log x + 3]$$

At $\qquad x = \dfrac{1}{\sqrt{e}}, \quad \dfrac{d^2u}{dx^2} = -2\lambda$ (–ve)

i.e., U is maximum

Hence, U is maximum for ratio $x = 1 : \sqrt{e}$.

Example 5. Prove that the minimum radius vector of the curve

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1 \text{ is of length } (a + b).$$

Solution: Given that

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1 \qquad (1)$$

Changing the variable co-ordinate, cartesian to polars by taking

$x = r\cos\theta$, $y = r\sin\theta$ in (1), we get

$$\frac{a^2}{r^2\cos^2\theta} + \frac{b^2}{r^2\sin^2\theta} = 1$$

or
$$r^2 = a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta \qquad (2)$$

Let
$$R = a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta \qquad (3)$$

Differentiating equation (3) with respect to θ both sides, we get

$$\frac{dR}{d\theta} = 2\,a^2\sec\theta. + \sec\theta\tan\theta + 2b^2\operatorname{cosec}\theta\,(-\operatorname{cosec}\theta\cot\theta)$$

or
$$\frac{dR}{d\theta} = 2\,a^2\sec^2\theta\tan\theta - 2b^2\operatorname{cosec}^2\theta\cot\theta \qquad (4)$$

For maxima and minima, we have

$$\frac{dR}{d\theta} = 0$$

$\Rightarrow$ $$2\,a^2\sec^2\theta\tan\theta - 2b^2\operatorname{cosec}^2\theta\cot\theta = 0$$

$\Rightarrow$ $$\tan^4\theta = b^2/a^2$$

$\Rightarrow$ $$\tan^2\theta = b/a$$

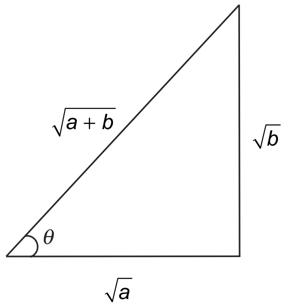
$\Rightarrow$ $$\tan\theta = \sqrt{b/a}$$

Differentiating equation (4) again with respect to θ both sides, we get

$$\frac{d^2R}{d\theta^2} = 2a^2\,[\sec^2\theta\sec^2\theta + 2\sec^2\theta + \tan\theta.\tan\theta]$$
$$-\,2b^2\,[\text{cose}^2\theta\,(-\operatorname{cosec}^2\theta) - 2\operatorname{cosec}\theta\operatorname{cosec}\theta\cot^2\theta]$$
$$= 2a^2\,[\sec^4\theta + 2\sec^2\theta\tan^2\theta] + 2b^2\,[\operatorname{cosec}^4\theta + 2\operatorname{cosec}^2\theta\cot^2\theta] > 0$$

i.e., R or r^2 is minimum at $\tan\theta = \sqrt{b/a}$

$\Rightarrow$ $$\sin\theta = \sqrt{\frac{b}{a+b}}, \quad \cos\theta = \sqrt{\frac{b}{a+b}}$$



Using equation (2), we have

$$r^2 = a^2 \left(\frac{a+b}{a}\right) + b^2 \left(\frac{a+b}{b}\right)$$

$$= a(a + b) + b\ (a + b)$$

$$= (a + b)^2$$

At $\tan \theta = \sqrt{b/a}$, the value of r is $(a + b)$.

Example 6. A given quantity of metal is to be casted into a half cylinder, i.e., with rectangular base and semicircular ends. Show that in order to have minimum surface area, the ratio of the height of the cylinder to the diameter of semicircular ends is π: $\pi + 2$.

Solution: Consider the volume of the half cylinder is

$$V = \frac{1}{2}\ \pi r^2 h \tag{1}$$

where r and h are the radius and height of the half cylinder respectively.

Surface area of rectangular base $= 2rh$

curved surface $= \pi rh$

Two semicircular ends $= \pi r^2$

The total surface area

$$S = 2rh + \pi rh + \pi r^2$$

$$= rh\ 2 + \pi + \pi r^2 \tag{2}$$

Using (1), we have

$$h = \frac{2V}{\pi r^2} \tag{3}$$

Then

$$S = r\ \frac{2V}{\pi r^2}\ (2 + \pi) + \pi r^2$$

$$= \frac{2V}{\pi r}\ (\pi + 2) + \pi r^2 \tag{4}$$

Differentiating equation (4) with respect to r both sides, we get

$$\frac{ds}{dr} = 2\pi r - \frac{2V}{\pi r^2}(\pi + 2) \tag{5}$$

For maxima and minima, we have

$$\frac{ds}{dr} = 0$$

$$\Rightarrow \quad 2\pi r - \frac{2V}{\pi r^2}(\pi + 2) = 0$$

from (1), we have

$$2\pi r - h(\pi + 2) = 0$$

$$\Rightarrow \quad \frac{h}{2r} = \frac{\pi}{\pi + 2} \tag{5}$$

using (3), we have

$$\Rightarrow \quad \frac{2V}{2r \cdot \pi r^2} = \frac{\pi}{\pi + 2}$$

$$\Rightarrow \quad \frac{V}{\pi r^3} = \frac{\pi}{\pi + 2} \tag{6}$$

Differentiating equation (5) again with respect to r both sides we get

$$\frac{d^2 s}{dr^2} = 2\pi + \frac{4V}{\pi r^3}(\pi + 2)$$

$$= 2\pi + \frac{4\pi}{\pi + 2}(\pi + 2) \text{ [using (7)]}$$

$$= 6\pi \text{ (+ve)}$$

i.e., S is minimum

Hence, surface area S is minimum at $\dfrac{h}{2r} = \dfrac{\pi}{\pi + 2}$.

Example 7. A rectangular sheet of metal has four equal square portions removed at the corners and the sides are then turned up so as to form an open rectangular box. Show that when the volume contained in the box is maximum the depth will be

$$\frac{1}{6}[(a + b) - (a^2 - a\,b + b^2)^{1/2}]$$

where a and b are sides of the original dimensions of the rectangular.

Solution: Suppose x is the length of each side of the square removed at the corners. Then the dimensions of the box will be $(a - 2x)$, $(b - 2x)$ and x. Let V be the volume of the box. Then we have

$$V = (a - 2x), (b - 2x)\, x$$

$$= 4x^3 - 2x^2 (a + b) + abx \tag{1}$$

Differentiating equation (1) with respect to x both sides, we get

$$\frac{dV}{dx} = 12x^2 - 4x (a + b) + ab \tag{2}$$

For maxima and minima, we have

$$\frac{dV}{dx} = 0$$

$$\Rightarrow \quad 12x^2 - 4x(a + b) + ab = 0$$

$$\Rightarrow \quad x = \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{24}$$

$$\Rightarrow \quad x = \frac{1}{6}\left[(a+b) \pm \sqrt{(a-b)^2 + ab}\right]$$

Differentiating equation (2) again with respect to x both sides, we get

$$\frac{d^2V}{dx^2} = 24\, x - 4\, (a + b)$$

At
$$x = \frac{1}{6}\left[(a+b) \pm \sqrt{(a+b)^2 + ab}\right]$$

if
$$x = \frac{1}{6}\left[(a+b) \pm \sqrt{(a+b)^2 + ab}\right] \quad \text{then } \frac{d^2V}{dx^2} \text{ is (–ve)}$$

i.e. V is maximum.

Example 8. The efficiency of a screw jack is given by $\eta = \tan \alpha \cot (\alpha + \phi)$, where ϕ is constant. Prove that the efficiency is maximum at

$$\alpha = \frac{\pi}{4} - \frac{\phi}{2} \text{ and } \eta_{\max} = \frac{1 - \sin\phi}{1 + \sin\phi}$$

Solution: The efficiency of a screw jack is

$$\eta = \tan \alpha \cdot \cot (\alpha + \phi) \tag{1}$$

Differentiating equation (1) with respect to α both sides, we get

$$\frac{dx}{d\alpha} = -\tan \alpha \operatorname{cosec}^2 (\alpha + \phi) + \sec^2\alpha \cdot \cot (\alpha + \phi)$$

$$= \sec^2\alpha \cot(\alpha + \phi) - \tan \alpha \operatorname{cosec}^2 (\alpha + \phi)$$

$$= \frac{1}{\cos^2\alpha \sin^2(\alpha + \phi)} [\sin(\alpha + \phi) \cos(\alpha + \phi) - \sin \alpha \cos \alpha]$$

for maxima and minima, we have

$$\frac{dx}{d\alpha} = 0$$

$$\Rightarrow \quad \frac{1}{\cos^2\alpha \sin^2(\alpha + \phi)} [\sin (\alpha + \phi) \cos (\alpha + \phi) - \sin \alpha \cos \alpha] = 0 \tag{2}$$

$\Rightarrow$ $\sin 2(\alpha + \phi) - \sin 2\alpha = 0$ $[\sin(\pi - \phi) = \sin \phi]$

$\Rightarrow$ $2(\alpha + \phi) = \pi - 2\alpha$

$\Rightarrow$ $\alpha = \dfrac{\pi}{4} - \dfrac{\phi}{2}$

Using (2), we have

$$\cos^2\alpha \sin^2(\alpha + \phi) \quad \frac{dx}{d\alpha} = \frac{1}{2}[\sin 2(\alpha + \phi) - \sin 2\alpha]$$

$$= \cos(2\alpha + \phi)\cos\phi \tag{3}$$

Differentiating equation (3) again with respect to α both sides, we get

$$\cos^2 \sin^2(\alpha + \phi)\frac{d^2x}{d\alpha^2} + \frac{d}{d\alpha}(\cos^2 \sin^2(\alpha + \phi))\frac{dx}{d\alpha} = -\sin(2\alpha + \phi)\cos\phi$$

At $\alpha = \dfrac{\pi}{4} - \dfrac{\phi}{2}, \dfrac{dx}{d\alpha} = 0$ then we get

$\dfrac{d^2x}{d\alpha^2}$ is negative, i.e. x is maximum

From (1), we have

$$\eta = \frac{\sin\alpha\cos(\alpha+\phi)}{\cos\alpha\sin(\alpha+\phi)}$$

$$= \frac{\sin(2\alpha+\phi) - \sin\phi}{\sin(2\alpha+\phi) + \sin\phi}$$

At $\alpha = \dfrac{\pi}{4} - \dfrac{\phi}{2},\ \eta = \dfrac{1-\sin\phi}{1+\sin\phi}$.

Example 9. A DC generator has an interval resistance of R ohms and has an open circuit voltage of V volts. Find the load resistance r for which the power delivered by the generator is maximum.

Solution: We know that the Ohm's law

$$V = i(R + r)$$

$\Rightarrow$ $i = \dfrac{V}{R + r}$

The power generated $P = i^2r$

$$= \frac{V^2r}{(R+r)^2} \tag{1}$$

where V, R being constant.

Differentiating equation (1) with respect to r both sides, we get

$$\frac{dp}{dr} = V^2 \left[\frac{(R+r)^2 \cdot 1 - r \cdot 2(R+r)}{(R+r)^4}\right]$$

$$= V^2 \left[\frac{(R+r) - 2r}{(R+r)^3}\right]$$

$$= V^2 \left[\frac{(R-r)}{(R+r)^3}\right] \tag{2}$$

For maxima and minima, we have

$$\frac{dp}{dr} = 0$$

$$\Rightarrow \qquad V^2 \frac{R-r}{(R+r)^3} = 0 \Rightarrow R = r$$

Differentiating equation (2) again with respect to r both sides, we get

$$\frac{d^2 p}{dr^2} = V^2 \left[\frac{(R+r)^3 \cdot (-1) - (R-r)\,3\,(R+r)^2}{(R+r)^6}\right]$$

$$= V^2 \left[\frac{-(R+r) - 3\,(R-r)}{(R+r)^4}\right] = V^2 \left[\frac{-4R + 2r}{(R+r)^4}\right]$$

At $\qquad R = r, \; \dfrac{d^2 p}{dr^2} = V^2 \left[\dfrac{-2R}{(2R)^4}\right] = -\dfrac{V^2}{8R^3}$ (–ve)

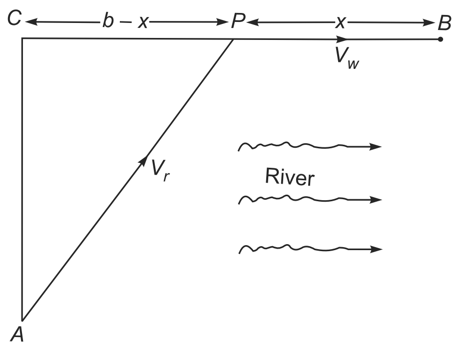
i.e., P is maximum for $\qquad r = R$

Hence, $\qquad P_{\max} = V^2 \dfrac{R}{4R^2}$

$$= \frac{V^2}{4R}.$$

Example 10. A person being in a boat ‘a’ miles from the nearest point of the beach, wishes to reach as quickly as possible a point ‘b’ miles from that point along the seashore. The ratio of his rate of walking to his rate of rowing is sec α. Prove that he should land at a distance b–a cot α from the place to be reached.

Solution: Let the person stand at A and reach Point B. Let C be the point where AC and BC meet.



Suppose $BC = b$ and let him row the distance AP in the boat and land at P where $PB = x$. If t is the time of the journey then

$$t = \frac{AP}{V_r} + \frac{BP}{V_w}, \quad \text{where } AP = \sqrt{a^2 + (b-x)^2}$$

and $$BP = x$$

Given that $$\frac{V_w}{V_r} = \sec \alpha$$

so $$t = \frac{1}{V_r}\left[\sqrt{a^2 + (b-x)^2} + \frac{x}{\sec \alpha}\right] \tag{1}$$

Differentiating equation (1) with respect to x both sides, we get

$$\frac{dt}{dx} = \frac{1}{V_r}\left[\frac{-(b-x)}{\sqrt{a^2 + (b-x)^2}} + \frac{1}{\sec \alpha}\right] \tag{2}$$

For maxima and minima, we have

$$\frac{dt}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{V_r}\left[\frac{-(b-x)}{\sqrt{a^2 + (b-x)^2}} + \frac{1}{\sec \alpha}\right] = 0$$

$$\Rightarrow \quad -(b-x) \sec \alpha + \sqrt{a^2 + (b-x)^2} = 0$$

$$\Rightarrow \quad (b-x)^2 \sec a^2 = a^2 + (b-x)^2$$

$\Rightarrow$ $$(b - x)^2 \tan^2 \alpha = a^2$$
$\Rightarrow$ $$b - x = a \cot \alpha$$
$\Rightarrow$ $$x = b - a \cot \alpha$$

Using equation (2), we have

$$\sec^2 V_r \sqrt{a^2 + (b-x)^2} \frac{dt}{dx} = -(b - x) \sec \alpha + \sqrt{a^2 + (b-x)^2} \tag{3}$$

Differentiating equation (3) again with respect to x both sides, we get

$$\sec \alpha \, V_r \sqrt{a^2 + (b-x)^2} \frac{d^2t}{dx^2} + \frac{d}{dt}\left(\sec^2 \alpha V_r \sqrt{a^2 + (b-x)^2}\right)$$

$$= \sec \alpha - \frac{b-x}{\sqrt{a^2 + (b-x)^2}}$$

At $\quad x = b - a \cot \alpha, \quad \dfrac{dt}{dx} = 0$ then

$$\frac{d^2t}{dx^2} \text{ is +ve}$$

i.e., t is minimum at $\quad x = b - a \cot \alpha.$

Example 11. Show that the minimum value of $U = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$ – is $3a^2$.

Solution: Given that $U = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$ (1)

Differentiating partially equation (1) with respect to x and y both sides respectively, we get

$$\frac{\partial U}{\partial x} = y - \frac{a^3}{x^2}$$

and
$$\frac{\partial U}{\partial y} = x - \frac{a^3}{y^2} \tag{2}$$

For maxima and minima, we have

$$\frac{\partial U}{\partial x} = 0 \Rightarrow y - \frac{a^3}{x^2} = 0$$

and
$$\frac{\partial U}{\partial y} = 0 \Rightarrow x - \frac{a^3}{y^2} = 0$$

Solving these, we get $x = a$, $y = a$.

Differentiating partially equation (2) again with respect to x and y both sides respectively, we get

$$r = \frac{\partial^2 U}{\partial x^2} = \frac{2a^3}{x^3}, \quad s = \frac{\partial^2 U}{\partial x \partial y} = 1 \text{ and } t = \frac{\partial^2 U}{xy^2} = \frac{2a^3}{y^3}$$

At (a, a), we get $r = 2$, $s = 1$, $t = 2$ then

$$rt - s^2 = 3 > 0$$

since at (a, a), $rt - s^2 > 0$ and $r > 0$

then U is minimum at (a, a)

The minimum value of $U = a \cdot a + \frac{a^3}{a} + \frac{a^3}{a}$

$= 3a^2$.

Example 12. Find the extreme points $f(x_1, x_2) = 20\,x_1 + 26\,x_2 + 4x_1x_2 - 4x_1^2, - 3x_2^2$.

Solution: Given that

$$f(x_1, x_2) = 20\,x_1 + 26\,x_2 + 4x_1x_2 - 4x_1^2, - 3x_2^2 \tag{1}$$

Differentiating partially equation (1) with respect to x_1 and x_2 both sides respectively, we get

$$\frac{\partial f}{\partial x_1} = 20 + 4x_2 - 8x_1$$

and

$$\frac{\partial f}{\partial x_2} = 26 + 4x_1 - 6x_2 \tag{2}$$

For extreme points, we have

$$\frac{\partial f}{\partial x_1} = 0 \text{ and } \frac{\partial f}{\partial x_2} = 0$$

$\Rightarrow$ $20 + 4x_2 - 8x_1 = 0$

and $26 + 4x_1 - 6x_2 = 0$

Solving these, we get $x_1 = 7$, $x_2 = 9$

Differentiating partially equation (2) again and we get

$$r = \frac{\partial^2 f}{\partial x_1^2} = -8, \; s = \frac{\partial^2 f}{\partial x_1 \partial x_2} = 4 \text{ and } t = \frac{\partial^2 f}{\partial x_2^2} = -6$$

we have $$rt - s^2 = (-8) \times (-6) - (4)^2$$
$$= 32$$

At (7, 9), $rt - s^2 > 0$ and $e < 0$

i.e., f is maximum at (7, 9).

Example 13. Find the point (x_1, x_2, x_3) at which is functions $f(x_1, x_2, x_3) = -x_1^2 - x_2^2 - x_3^2 + x_2 x_3 + x_1 + 2x_3$ has optimum values.

Solution: Given that

$$f(x_1, x_2, x_3) = -x_1^2 - x_2^2 - x_3^2 + x_2 x_3 + x_1 + 2x_3 \tag{1}$$

Differentiating partially equation (1) with respect to x_1, x_2 and x_3 both sides respectively, we get

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} &= -2x_1 + 1 \\ \frac{\partial f}{\partial x_2} &= -2x_2 + x_3 \\ \frac{\partial f}{\partial x_3} &= -2x_3 + x_2 + 2 \end{aligned}\right\} \tag{2}$$

and

For extreme points, we have

$$\frac{\partial f}{\partial x_1} = 0, \ \frac{\partial f}{\partial x_2} = 0 \text{ and } \frac{\partial f}{\partial x_3} = 0$$

$\Rightarrow$ $$-2x_1 + 1 = 0$$

$\Rightarrow$ $$-2x_2 + x_3 = 0$$

$\Rightarrow$ $$-2x_3 + x_2 + 2 = 0$$

Solving these, we get $$x_1 = \frac{1}{2}, \ x_2 = \frac{2}{3}, \ x_3 = \frac{4}{3}$$

Let the point P be $\left(\frac{1}{2}, \frac{2}{3}, \frac{4}{3}\right)$.

Differentiating partially equation (2) again and we get

$$\frac{\partial^2 f}{\partial x_1^2} = -2, \ \frac{\partial^2 f}{\partial x_2^2} = -2, \ \frac{\partial^2 f}{\partial x_3^2} = -2, \ \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0, \ \frac{\partial^2 f}{\partial x_2 \partial x_3} = 1, \ \frac{\partial^2 f}{\partial x_3 \partial x_1} = 0,$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_3} = 0, \ \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0, \ \frac{\partial^2 f}{\partial x_3 \partial x_2} = 1$$

The Hessian matrix of $f(x_1, x_2, x_3)$ is $$H = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

The leading minors of H are $$H_1 = |-2| = 2,\ H_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4,$$

$$H_3 = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} = -6$$

Here H_1, H_2, H_3 are negative definite, i.e., alternating negative, positive and negative. So f is maximum at $P\left(\frac{1}{2}, \frac{2}{3}, \frac{4}{3}\right)$.

Example 14. Find the dimension of a box of largest volume that can be inscribed in a sphere of radius 3 metres.

Solution: Let the volume of the box be

$$V = 2x \cdot 2y \cdot 2z$$
$$= 8\,xyz \qquad (1)$$

Changing the variables Cartesian to spherical polar co-ordinates as $x = r \sin\theta \cos\phi$, $y = r\sin\theta \sin\phi$, $z = r\cos\theta$ and using $r = 3$ (given)

$$V = 216 \sin^2\theta \cos\theta \cos\phi \sin\phi$$
$$= 108 \sin^2\theta \cos\theta \sin 2\phi \qquad (2)$$

Differentiating partially equation (2) with respect to θ and ϕ both sides, we get

$$\frac{\partial V}{\partial \theta} = 108 \sin 2\phi\, [\sin^2\theta\,(-\sin\theta) + 2 \sin \cos^2\theta]$$

$$\left.\begin{aligned} \Rightarrow \quad \frac{\partial V}{\partial \phi} &= 108 \sin 2\phi \sin\theta\, [2\cos^2\theta - \sin^2\theta] \\ \text{and} \quad \frac{\partial V}{\partial \phi} &= 216 \cos\theta \sin^2\theta \cos\theta \end{aligned}\right\} \qquad (3)$$

For maxima and minima, we have

$$\frac{\partial V}{\partial \theta} = 0 \text{ and } \frac{\partial V}{\partial \phi} = 0$$

$$\Rightarrow \quad 108 \sin 2\phi \sin\phi\, (2\cos^2\theta - \sin^2\theta) = 0$$

i.e., $\sin\theta = 0$ or $\tan\theta = \sqrt{2}$ or $\sin\phi = 0$

i.e., $\phi = 0,\ \theta = \tan^{-1}\sqrt{2},\ \phi = 0$

and $216\cos 2\phi \sin^2\theta\cos\theta = 0$

i.e., $\theta = 0,\ \phi = \dfrac{\pi}{4}$

Because $\theta = 0$ or $\phi = 0$ gives $V = 0$, so we take only

$\theta = \tan^{-1}\sqrt{2}$ and $\phi = \pi/4$.

Differentiating partially equation (3) again and we get

$$\frac{\partial^2 V}{\partial\theta^2} = 108\sin 2\phi\,[\sin\theta(-4\cos\theta\sin\theta - 2\sin\theta\cos\theta) + (2\cos^2\theta - \sin^2\theta)\cos\theta]$$

At $\left(\tan^{-1}\sqrt{2}, \dfrac{\pi}{4}\right)$, $\dfrac{\partial^2 V}{\partial\theta^2} = -\dfrac{432}{\sqrt{3}}$

$$\frac{\partial^2 V}{\partial\theta r\phi} = 216\cos 2\phi\,[\sin^2\theta(-\sin\theta) + \cos^2\theta\, 2\sin\theta]$$

$$= 216\cos^2\phi\sin\theta(2\cos^2\theta - \sin^2\phi)$$

At $\left(\tan^{-1}\sqrt{2}, \dfrac{\pi}{4}\right)$, $\dfrac{\partial^2 V}{\partial\theta\partial\phi} = 0$

$$\frac{\partial^2 V}{\partial\phi^2} = -432\sin 2\phi\sin^2\theta\cos\theta$$

At $\left(\tan^{-1}\sqrt{2}, \dfrac{\pi}{4}\right)$, $\dfrac{\partial^2 V}{\partial\phi^2} = -\dfrac{864}{3\sqrt{3}}$

Now at $\left(\tan^{-1}\sqrt{2}, \dfrac{\pi}{4}\right)$, we have

$$\frac{\partial^2 V}{\partial\theta^2}\cdot\frac{\partial^2 V}{\partial\phi^2} - \left(\frac{\partial^2 V}{\partial\theta\partial\phi}\right)^2 = \left(\frac{-432}{\sqrt{3}}\right)\left(\frac{-864}{3\sqrt{3}}\right) - 0 > 0$$

i.e., V is maximum

The dimensions of the maximum inscribed box are

$$2x = 2r\sin\theta\cos\phi$$

$$= 2\,(3)\,\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}$$

$$= 2\sqrt{3}$$

$$2y = 2r \sin\theta \sin\phi$$

$$= 2(3) \cdot \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}}$$

$$= 2\sqrt{3}$$

$$2z = 2r \cos\theta$$

$$= 2\ (3) \cdot \frac{1}{\sqrt{3}}$$

$$= 2\sqrt{3}$$

Example 15. Find the extremum points of the function

$$U = x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 16yz$$

Solution: Given that

$$U = x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 16yz \tag{1}$$

Differentiating partially equation (1) with respect to x, y and z both sides respectively, we get

$$\left.\begin{aligned} \frac{\partial U}{\partial x} &= 2x + 4y + 4z \\ \frac{\partial U}{\partial y} &= 8y + 4x + 16z \\ \frac{\partial U}{\partial z} &= 8z + 4x + 16y \end{aligned}\right\} \tag{2}$$

For extreme points, we have

$$\frac{\partial U}{\partial x} = 0, \ \frac{\partial U}{\partial y} = 0 \text{ and } \frac{\partial U}{\partial z} = 0$$

$$\Rightarrow \quad 2x + 4y + 4z = 0 \text{ or } 2\ (x + 2y + 2z) = 0$$

$$8y + 4x + 16z = 0 \text{ or } 4\ (x + 2y + 4z) = 0$$

$$8z + 4x + 16y = 0 \text{ or } 4(x + 4y + 2z) = 0$$

Solving these, we get $x = 0,\ y = 0,\ z = 0.$

Let the point P be (0, 0, 0)

Differentiating partially equation (2) again and we get

$$\frac{\partial^2 U}{\partial x^2} = 2, \ \frac{\partial^2 U}{\partial y^2} = 8, \ \frac{\partial^2 U}{\partial z^2} = 8, \ \frac{\partial^2 U}{\partial x \partial y} = 4, \ \frac{\partial^2 U}{\partial x \partial z} = 4,$$

$$\frac{\partial^2 U}{\partial y \partial x} = 4,\ \frac{\partial^2 U}{\partial y \partial z} = 16,\ \frac{\partial^2 U}{\partial z \partial x} = 4,\ \frac{\partial^2 U}{\partial z \partial y} = 16.$$

The Hessian matrix of $f(x_1, x_2, x_3)$ is $$H = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 8 & 16 \\ 4 & 16 & 8 \end{bmatrix}$$

The leading minors of H are $$H_1 = |2| = 2,\ H_2 = \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} = 0,$$

$$H_3 = \begin{vmatrix} 2 & 4 & 4 \\ 4 & 8 & 16 \\ 4 & 16 & 8 \end{vmatrix} < 0$$

Here H_1 and H_3 are not of same sign and $H_2 = 0$ (i.e., semi-definite). Hence, U has a saddle point at (0, 0, 0).

Example 16. Find the extreme points

$$f(x_1, x_2) = x_1^3 + x_2^3 + x_2^2 + 2x_1^2 + 4x_2^2 + 6$$

Solution: Given that

$$f(x_1, x_2) = x_1^3 + x_2^3 + 2\ x_1^2 + 4x_2^2 + 6 \tag{1}$$

Differentiating partially equation (1) with respect to x_1 and x_2 both sides respectively, we get

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} &= 3x_1^2 + 4x_1 \\ \text{and} \quad \frac{\partial f}{\partial x_2} &= 3x_2^2 + 8x_2 \end{aligned}\right\} \tag{2}$$

For extreme points, we have

$$\frac{\partial f}{\partial x_1} = 0 \text{ and } \frac{\partial f}{\partial x_2} = 0$$

$\Rightarrow$ $$3x_1^2 + 4x_1 = 0 \text{ or } x_1(3x_1 + 4) = 0 \text{ i.e., } x_1 = 0,\ -\frac{4}{3}.$$

and $$3x_2^2 + 8x_2 = 0 \text{ or } x_2\ (3x_2 + 8) = 0 \text{ i.e., } x_2 = 0,\ -\frac{8}{3}.$$

Then the stationary points are (0, 0), $\left(0, -\frac{8}{3}\right)$, $\left(-\frac{4}{3}, 0\right)$ and $\left(-\frac{4}{3}, -\frac{8}{3}\right)$.

Differentiating partially equation (2) again and we get

$$r = \frac{\partial^2 f}{\partial x_1^2} = 6x_1 + 4$$

$$s = \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

and $$t = \frac{\partial^2 f}{\partial x_2^2} = 6x_2 + 8.$$

At (0, 0), $$rt - s^2 = (6x_1 + 4)\,(6x_2 + 8) - 0$$
$$= 32 > 0$$

and $r > 0$ i.e., f is minimum at (0, 0).

At $$\left(0, -\frac{8}{3}\right),\ rt - s^2 = (6x_1 + 4)\,(6x_2 + 8) - 0$$
$$= -32 < 0$$

$\Rightarrow$ no extreme point

i.e., f has a saddle point at $\left(0, -\frac{8}{3}\right)$.

At $$\left(-\frac{4}{3}, 0\right),\ rt - s^2 = (6x_1 + 4)\,(6x_2 + 8) - 0$$

$$= -32 > 0$$

$\Rightarrow$ no extreme point

i.e. f has a saddle point at $\left(-\frac{4}{3}, 0\right)$

At $$\left(-\frac{4}{3}, -\frac{8}{3}\right),\ rt - r^2 = (6x_1 + 4)\,(6x_2 + 8) - 0$$

$$= 32 > 0$$

and $r < 0$ i.e., f is maximum at $\left(-\frac{4}{3}, -\frac{8}{3}\right)$.

Example 17. Find the extreme points of the function $f(x_1, x_2) = x_1^3 + 2x_2^3 + 3x_1^2 + 12x_2^2 + 24$ and determine their nature also.

Solution: Given that

$$f(x_1, x_2) = x_1^3 + 2x_2^3 + 3x_1^2 + 12x_2^2 + 24 \tag{1}$$

Differentiating partially equation (1) with respect to x_1 and x_2 both sides respectively, we have

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} &= 3x_1^2 + 6x_1 \\ \text{and} \quad \frac{\partial f}{\partial x_2} &= 6x_2^2 + 24x_2 \end{aligned}\right\} \tag{2}$$

For extreme points, we have

$$\frac{\partial f}{\partial x_1} = 0 \text{ and } \frac{\partial f}{\partial x_2} = 0$$

$\Rightarrow$ $\quad 3x_1^2 + 6x_1 = 0$ or $3x_1 (x_1 + 2)$ i.e., $x_1 = 0, -2.$

and $\quad 6x_2^2 + 24x_2 = 0$ or $6x_2 (x_2 + 4)$ i.e., $x_2 = 0, -4$

Hence, the stationary points are (0, 0), (0, –4), (–2, 0) and (–2, –4).

Differentiating partially equation (2) again and we get

$$r = \frac{\partial^2 f}{\partial x_1^2} = 6x_1 + 6$$

$$s = \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

and $$t = \frac{\partial^2 f}{\partial x_2^2} = 12x_2 + 24.$$

At (0, 0), $\quad rt - s^2 = (6x_1 + 6)(12x_2 + 24) - 0$

$\quad = 72 > 0$

and $r > 0$ i.e., f is minimum at (0, 0).

At (0, –4), $\quad rt - s^2 = (6x_1 + 6)(12x_2 + 24)$

$\quad = -144 < 0$

$\Rightarrow$ no extreme point

i.e., f has a saddle point at (0, –4).

At (–2, 0), $\quad rt - s^2 = (6x_1 + 6)(12x_2 + 24)$

$\quad = -144 > 0$

$\Rightarrow$ no extreme point

i.e., f has a saddle point at (–2, 0).

At (–2, –4), $\quad rt - s^2 = (6x_1 + 6)(12x_2 + 24)$

$\quad = -144 > 0$

and $r < 0$ i.e., f is maximum at (–2, –4).

Example 18. Obtain the maximum and minimum value of U where

$$U = \sin x \sin y \sin (x + y).$$

Solution: Given that

$$U = \sin x \sin y \sin (x + y) \tag{1}$$

Differentiating partially equation (1) with respect to x and y both sides respectively, we get

$$\frac{\partial U}{\partial x} = \sin y\,[\sin x \cos (x + y) + \cos x \sin (x + y)] \tag{2}$$

and $$\frac{\partial U}{\partial x} = \sin x\,[\sin y \cos (x+y) + \cos y \sin (x+y)] \quad (3)$$

For maxima and minima, we have

$$\frac{\partial U}{\partial x} = 0 \text{ and } \frac{\partial U}{\partial y} = 0$$

$\Rightarrow$ $$\sin y\,[\sin x \cos (x+y) + \cos x \sin (x+y)] = 0$$

and $$\sin x\,[\sin y \cos (x+y) + \cos y \sin (x+y)] = 0$$

Solving these equations, we get

$$\tan (x+y) = -\tan x \quad (4)$$

$$\tan (x+y) = -\tan y \quad (5)$$

Now we have $$\tan x = \tan y \Rightarrow x = y$$

and $$\tan 2x = -\tan x$$

$$= \tan (\pi - x)$$

$$2x = \pi - x$$

$$x = \frac{\pi}{3} = y$$

Also, $$\sin y = 0 \Rightarrow y = 0$$

and $$\sin x = 0 \Rightarrow x = 0$$

Thus, the stationary points are (0, 0), $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

Differentiating partially equations (2) and (3) again, we get

$$r = \frac{\partial^2 U}{\partial x^2} = 2 \sin y \cos (2x + y)$$

$$s = \frac{\partial^2 U}{\partial x \partial y} = \sin 2(x+y)$$

and $$t = \frac{\partial^2 U}{\partial y^2} = 2 \sin x \cos (2y + x)$$

At (0, 0), we get $$r = 0,\ s = 0,\ t = 0$$

$\Rightarrow$ $rt - s^2 = 0$ i.e., U has a saddle point at (0, 0).

Now at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, we get $$r = 2 \sin \frac{\pi}{3} \cos \pi = -\sqrt{3}$$

$$s = \sin\left(\frac{4\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$t = 2\sin\frac{\pi}{3}\sin\pi = -\sqrt{3}$$

$$\Rightarrow \qquad rt - s^2 = \frac{9}{4} > 0 \text{ and } r < 0$$

Hence, U is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

EXERCISE 9.1

1. The demand function of a commodity is described by the exponential function $d = 10.59\, e^{0.012}$, where q is the quantity demand. Determine the quantity for which the total revenue is maximum.

   (**Ans:** 100.)
2. Prove that the function $y = x^3$ has a point of inflexion at (0, 0).
3. Find the output x which maximizes profit P, given by the relationship

   $$P = 5000 + 1200\, x - x^2.$$

   (**Ans:** $x = 600$, $P = 3{,}65{,}000$)
4. Assume the following relationship for revenue and cos + functions. Find out at what level of output x, where x is measured in tons per week. Profit are maximum

   $$R(x) = 1000\, x - 2x^2 \text{ and } C(x) = x^3 - 59\, x^2 + 1315x + 5000.$$

   (**Ans:** 35)
5. Obtain the maximum and minimum values of the function

   $$f(x) = 12x^5 - 45x^4 - 40x^3 + 5$$

   (**Ans:** At $x = 0$, inflexion; at $x = 1$, maxima and value 12; at $x = 2$, minima and value 11)
6. Show that the right circular cylinder of given surface (including its ends) and maximum volume is such that its height is equal to twice its radius.
7. Show that the semi-vertical angle of a core of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.
8. Find maxima or minima If any of the function
   $f(x) = x^4\,(x - 1)^{-1}\,(x - 3)^{-1}$, $x \neq 1$ and $x \neq 3$.

   (**Ans:** $x = \frac{6}{5}$ give maxima)
9. Prove that $\frac{x}{1 + x\tan x}$ is a maximum when x is given by the equation $x = \cos x$.
10. The average cost per unit of manufacturing for Bharat Food Products is given by the following relationship.

$$C\ (x) = 4000 - 16x - 0.2x^2$$

(**Ans:** $x = 400$)

where x is the quantity produced. Find the value of x that will minimize average cost per unit.

11. Find the local maxima or minima of the function

$$f(x) = \frac{x^5}{5} - 7\,\frac{x^4}{4} + 17\,\frac{x^3}{3} - 17\,\frac{x^2}{2} + 6x + 10 \text{ for } xe\ R.$$

(**Ans:** At $x = 1$ inflexion, at $x = 2$, local maxima and $x = 3$, local minima)

12. Find the volume of the greatest right circular cone described by the revolution of a triangle of hypotenuse C about, one of its sides.

(**Ans:** $h = C/\sqrt{3}$, Volume $= \dfrac{2\pi C^3}{g\sqrt{3}}$)

13. Find the area of the largest rectangle, if its perimeter is to be constant.

(**Ans:** $y = x = C/4$, i.e. square)

14. The total revenue R of a firm is given by $R = 20x - 2x^2$, where x represents the quantity sold and $C = x^2 - 4x + 20$.

    Find the value of x for which the revenue will be maximum also find the profit, price and total revenue.

(**Ans:** $x = 5$, total revenue = 50, profit = 25, price = 5 per unit)

15. Assuming that the petrol burnt (per hour), in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of C kmph is $\dfrac{3C}{2}$ km per hour.

16. Obtain the maxima and minima of the function

$$U = x^3 + y^3 - 3x - 12y + 25.$$

(**Ans:** Maxima at (–1, –2) and minima at (1, 2))

17. Obtain the maxima and minima of U, where

$$U = x^2 + y^2 + z^2 + x - 2z - xy.$$

(**Ans:** Minima at $\left(-\dfrac{2}{3}, -\dfrac{1}{3}, 1\right)$)

18. Obtain the maxima and minima of the function

$$U = 2\ (x - y)^2 - x^4 - y^4.$$

(**Ans:** Maxima at $(\sqrt{2}, -\sqrt{2})$)

19. Find the maximum value of $U = \dfrac{xyz}{(a+x)(x+y)(y+z)(z+b)}$.

(**Ans:** Maximum at (ar, ar^2, ar^3), $U = \dfrac{1}{(a^{1/4} + b^{1/4})^4}$)

20. Find the maxima or minima of the function

$$f(x_1, x_2) = 8x_1x_2 + 3x^2{}_2.$$

(**Ans:** Saddle point at (0, 0))

21. Obtain the extremum points of the function

$$f(x_1, x_2, x_3) = \sin x_1 \sin x_2 \sin x_3$$

(**Ans:** Maxima at $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$)

where x_1, x_2 and x_3 are the vertex angles of a triangle.

22. Consider the function $f(x_1, x_2) = x_1 + 2x_2 + x_1x_2 - x_1{}^2 - x_2{}^2$. Obtain the maximum or minimum point if any of the function.

(**Ans:** Maxima at $\left(\dfrac{4}{3}, \dfrac{5}{3}\right)$)

23. Find the stationary point of U, where

$$U = x^2 - y^2$$

(**Ans:** Saddle point at (0, 0))

24. A rectangle box open at the top is to have a given capacity. Find the dimensions of the box requiring least meterial for its construction.

(**Ans:** Minima at $[x = y\ 2z = (2v_0)^{1/3}]$)

25. Find the maximum and minimum values of $xy\,(a - x - y)$.

(**Ans:** At $(a/3, a/3)$ minimum if a is negative and maximum if a is positive)

9.3 CONSTRAINED MULTIVARIABLE OPTIMIZATION PROBLEMS WITH EQUALITY CONSTRAINTS

The optimization problem of a continuous and differentiable function subject to equality constraints:

Optimize (max or min) $\quad Z = f(x)$

subject to constraints

$$g_j(X) = 0; \quad j = 1, 2, 3, \ldots, m$$

where
$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Here m is less than or equal to n. There are several methods for solving this type of problem. Here we discuss only two methods; Direct substitution method and Lagrange multipliers method.

Direct Substitution Method

In this method, the value of any variable from the constraint set is put into objective function. The new objective function is like that as has no constraint and we get the optimum value by unconstrained optimization method.

Lagrange Multiplier Method

Consider a general problem with n variables and m equality constraints

Optimize $\quad Z = f(X)$

subject to constraints

$$g_j(X) = 0; \quad j = 1, 2, 3, \dots m \ (m < n)$$

and $X \geq 0$, where
$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Now define a function

$$L\ (x_1, x_2, x_3, \dots, x_n; \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m) = f(x) + \sum_{j=1}^{m} \lambda_j\, g_j(X) \tag{3}$$

Here $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m$ are known as Lagrange's undetermined multipliers.
The necessary conditions for extreme of L are

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \tag{4}$$

and
$$\frac{\partial L}{\partial \lambda_j} = 0; \quad \begin{pmatrix} i = 1, 2, \dots, n \\ j = 1, 2, \dots, m \end{pmatrix} \tag{5}$$

Solving these equations (4) and (5), we get

$$X^* = \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix} \text{ and } \lambda^X = \begin{bmatrix} \lambda_1^* \\ \lambda_2^* \\ \vdots \\ \lambda_n^* \end{bmatrix}$$

The sufficient condition for an extreme point X^* is $rt - s^2 > 0$ is replaced by the condition that the eigenvalues k must be of same sign. If all the eigenvalues k are negative, then it is a maximum and

$$\begin{vmatrix} L_{11}-k & L_{12} & \dots & L_{1n} & g_{11} & g_{21} & \dots & g_{m1} \\ L_{21} & L_{22-k} & \dots & L_{2n} & g_{12} & g_{22} & \dots & g_{m2} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & & \vdots \\ L_{n1} & L_{n2} & \dots & L_{nn-k} & g_{1n} & g_{2n} & \dots & g_{mn} \\ g_{11} & g_{12} & \dots & g_{1n} & 0 & 0 & \dots & 0 \\ g_{21} & g_{22} & \dots & g_{2n} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ g_{m1} & g_{m2} & \dots & g_{mn} & 0 & 0 & \dots & 0 \end{vmatrix} = 0$$

if all the eigenvalues k are positive then it is a minima. But if some eigenvalues are zero or of different sign then that is a saddle point, in above L_{ij} and g_{ji} denoted by $\dfrac{\partial^2 L}{\partial x_i\, \partial x_j}$ and $\dfrac{\partial g_j}{\partial x_i}$ respectively.

Kuhn-Tucker conditions

Consider the optimization problem

Optimize (Max or min) $\qquad Z = f(X) \qquad (1)$

subject to constraints

$$g_j(X) \le 0; \quad j = 1, 2, 3, \dots m \qquad (2)$$

Converting the inequality constraints in equality constraints by adding slack variables y_j^2, we have

$$g_j(X) + y_j^2 = 0 \qquad (3)$$

Now define a Lagrangian function

$$L\,(x_1, x_2, x_3, \dots x_n; \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m) = f(X) + \sum_{j=1}^{m} \lambda_j\ [g_j(X) + y_j^2] \qquad (4)$$

The Kuhn-Tucker necessary conditions to be satisfied at extreme points are

$$\frac{\partial L}{\partial x_i} = 0; \quad i = 1, 2, 3, \dots, n$$

$$\Rightarrow \qquad \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} \quad (i = 1, 2, 3, ..., n) \qquad (5)$$

$$\frac{\partial L}{\partial \lambda_j} = 0$$

$\Rightarrow \qquad g_j + y_j^2 = 0; \quad j = 1, 2, \ldots, m \qquad (6)$

and $$\frac{\partial L}{\partial y_j} = 0$$

$\Rightarrow \qquad 2\,\lambda_j + y_j = 0; \quad j = 1, 2, \ldots\ m \qquad (7)$

By (6) and (7), we have

$$\lambda_j\ g_j(X) = 0$$

$\Rightarrow \qquad \lambda_j = 0$ or $g_j(X) = 0$

Case 1. If $g_j(X) = 0$ at the optimum point, then it is called active constraint and we can find optimum solution.

Case 2. If $\lambda_j = 0$ at the optimum point, then it is called inactive constraint.

Note: If the given optimization problem is of minimization or if the constraints are of the form $g_j(X) \geq 0$, then $\lambda_j < 0$ but if the problem is of maximization with constraints of the form $g_j(X) \leq 0$ then $\lambda_j \geq 0$.

Some form of maximization and minimization are as follows :

(i) Maximize $Z = f(X)$

subject to constraints

$$g_j(X) \leq 0\ ; \quad j = 1, 2, 3, \ldots, m$$

This problem will be maximum when

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} l_j \frac{\partial g_j}{\partial x_i} = 0; \quad i = 1, 2, \ldots, n$$

$$\lambda_j\ g_j\ (X) = 0; \quad j = 1, 2, \ldots, m$$

and $\lambda_j \leq 0$.

(ii) Maximize $Z = f(X)$

subject to constraints

$$g_j\ (X) \geq 0; \quad j = 1, 2, \ldots, m$$

This problem will be maximum when

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} l_j \frac{\partial g_j}{\partial x_j} = 0; \quad i = 1, 2, \ldots, n$$

$$\lambda_j\ g_j(X) = 0$$

and $\lambda_j \geq 0$

(iii) Minimize $Z = f(X)$

subject to constraints

$$g_j\ (X) \leq 0; \quad j = 1, 2, ..., m$$

This problem will be minimum when

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} l_j \frac{\partial g_j}{\partial x_i} = 0; \quad i = 1, 2, \ldots, n$$

$$\lambda_j \, g_j(X) = 0; \quad j = 1, 2, \ldots, m$$

and $\lambda_j \geq 0.$

(iv) Minimize $Z = f(X)$

subject to constraints

$$g_j(X) \geq 0; \quad j = 1, 2, \ldots, m$$

This problem will be minimum when

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0; \quad i = 1, 2, \ldots, n$$

$$l_j \, g_j(X) = 0; \quad j = 1, 2, \ldots, m$$

and $\lambda_j \leq 0.$

SOLVED EXAMPLES

Example 1. Find the optimum solution of the following constrained multivariable problem:

Minimize $Z = x_1^2 + (x_2 + 1)^2 + (x_3 - 1)^2$

subject to constraint $x_1 + 5x_2 - 3x_3 = 6.$

Solution: Given that

$$Z = x_1^2 + (x_2 + 1)^2 + (x_3 - 1)^2 \tag{1}$$

and $x_1 + 5x_2 - 3x_3 = 6$

$$\Rightarrow \quad x_3 = \frac{x_1 + 5x_2 - 6}{3} \tag{2}$$

From equations (1) and (2), we get

$$Z = x_1^2 + (x_2 + 1)^2 + \frac{1}{9}(x_1 + 5x_2 - 9)^2 \tag{3}$$

Differentiating partially equation (3) with respect to x_1 and x_2 both sides respectively, we get

$$\left.\begin{aligned} \frac{\partial Z}{\partial x_1} &= 2x_1 + \frac{2}{9}(x_1 + 5x_2 - 9) \\ \text{and} \quad \frac{\partial Z}{\partial x_2} &= 2(x_2 + 1) + \frac{10}{9}(x_1 + 5x_2 - 9) \end{aligned}\right\} \tag{4}$$

For maxima and minima, we have

$$\frac{\partial Z}{\partial x_1} = 0 \text{ and } \frac{\partial Z}{\partial x_2} = 0$$

$$\Rightarrow \quad 2x_1 + \frac{2}{9}(x_1 + 5x_2 - 9) = 0$$

and $2(x_2 + 1) + \dfrac{10}{9}(x_1 + 5x_2 - 9) = 0$

Solving these, we get $x_1 = \dfrac{2}{5}$ and $x_2 = 1$

Differentiating partially equation 4 with respect to x_1 and x_2 both sides respectively, we get

$$r = \frac{\partial^2 Z}{\partial x_1^2} \; 2 + \frac{2}{9} = \frac{20}{9}, \quad s = \frac{\partial^2 Z}{\partial x_1 \partial x_2} = \frac{10}{9}$$

and $$t = \frac{\partial^2 Z}{\partial x_2^2} = 2 + \frac{50}{9} = \frac{68}{9}.$$

At $$\left(\frac{2}{5}, 1\right),\ rt - s^2 = \left(\frac{20}{9}\right)\left(\frac{68}{9}\right) - \left(\frac{10}{9}\right)^2$$

$$= 1260 > 0$$

and $r > 0$ i.e., Z is minimum at $\left(\dfrac{2}{5}, 1\right)$

The minimum value of $Z = \dfrac{28}{5}$.

Example 2. Optimize $Z = x^2 + y^2 + z^2$
subject to constraint

$$4x + y^2 + 2z = 14.$$

Solution: Given that $$Z = x^2 + y^2 + z^2 \tag{1}$$

and $4x + y^2 + 2z = 14$

$$\Rightarrow \quad g(x, y, z) = 4x + y^2 + 2z - 14 = 0 \tag{2}$$

Construct the Lagrangian function L is

$$L\ (x, y, z; \lambda) = x^2 + y^2 + z^2 + \lambda\ (4x + y^2 + 2z - 14) \tag{3}$$

The necessary condition for extreme of L are

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2x + 4\lambda = 0 \\ \frac{\partial L}{\partial y} &= 2y + 2y\lambda = 0 \\ \frac{\partial L}{\partial z} &= 2z + 2\lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= 4x + y^2 + 2z - 14 = 0 \end{aligned}\right\} \tag{4}$$

Solving these, we get $x = -2\lambda,\ z = -\lambda$ and $\lambda = -1$

$\Rightarrow$ $x_1 = 2,\ z = 1$ and $y^2 = 14 - 8 - 2 = 4$ i.e. $y = \pm 2$

Putting $x = -2\lambda$, $z = -\lambda$, $y = 0$ in (2), we get $\lambda = -1.4$. Here (2, 2, 1, – 1), (2, –2, 1, –1) and (– 2λ, 0, – λ, λ) or (2.8, 0, 1.4, – 1.4) are extreme points.

Differentiating partially equation 4 again and we get

$$\frac{\partial^2 L}{\partial x^2} = 2; \quad \frac{\partial^2 L}{\partial x \partial y} = 0; \quad \frac{\partial^2 L}{\partial x \partial z} = 0; \quad \frac{\partial g}{\partial x} = 4;$$

$$\frac{\partial^2 L}{\partial y \partial x} = 0; \quad \frac{\partial^2 L}{\partial y^2} = 2 + 2\lambda; \quad \frac{\partial^2 L}{\partial y \partial z} = 0; \quad \frac{\partial g}{\partial xy} = 2y;$$

$$\frac{\partial^2 L}{\partial zx \partial x} = 0; \quad \frac{\partial^2 L}{\partial z \partial y} = 0; \quad \frac{\partial^2 L}{\partial z^2} = 2; \quad \frac{\partial g}{\partial z} = 2$$

The sufficient condition for extreme point is

$$H = \begin{vmatrix} \dfrac{\partial^2 L}{\partial x^2} - k & \dfrac{\partial^2 L}{\partial x \partial y} & \dfrac{\partial^2 L}{\partial x \partial z} & \dfrac{\partial g}{x} \\ \dfrac{\partial^2 L}{\partial y \partial x} & \dfrac{\partial^2 L}{\partial y^2} - k & \dfrac{\partial^2 L}{\partial y \partial z} & \dfrac{\partial g}{\partial y} \\ \dfrac{\partial^2 L}{\partial z \partial x} & \dfrac{\partial^2 L}{\partial z \partial y} & \dfrac{\partial^2 L}{\partial z^2} - k & \dfrac{\partial g}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} & 0 \end{vmatrix} = 0$$

i.e.,

$$H = \begin{vmatrix} 2-k & 0 & 0 & 4 \\ 0 & 2+2\lambda-k & 0 & 2y \\ 0 & 0 & 2-k & 2 \\ 4 & 2y & 2 & 0 \end{vmatrix} = 0$$

i.e., $4(2 - k)\,[-10 + 5k - 10\,\lambda + 2y^2 + y^2 k] = 0$

At (2, 2, 1, – 1) from (5), we have $k = 2,\ 8/9$

i.e. value of k are positive then there is minima.

at (2 –2, 1, –1) from 5, we have $k = 2\ ,\ 8/9$

Also value of k are positive then there is minima.

At (2 –8, 0, 1.4 – 1.4) from 5, we have $k = 2,\ -4/5$

i.e. values of k are positive and negative (neither maxima nor minima) i.e., saddle point.

Example 3. Find the dimensions of a box of largest volume that can be inscribed in a sphere of radius a.

Solution: Let x, y and z be the dimensions of the box with respect to origin 0 and $0X$, $0Y$, $0Z$ are reference axes. The volume of the box is

$$V = 8xyz \tag{1}$$

Given that the box is to be inscribed in a sphere of radius 'a'

i.e., $$x^2 + y^2 + z^2 = a^2 \tag{2}$$

Eliminating Z from (1) and (2), we get

$$V = 8xy(a^2 - x^2 - y^2)^{1/2} \tag{3}$$

Differentiating partially equation (1) with respect to x and y both sides respectively, we get

$$\frac{\partial V}{\partial x} = 8y\left[x \cdot \frac{1}{2}(a^2 - x^2 - y^2)^{-1/2}(-2x) + (a^2 - x^2 - y^2)^{1/2}\right]$$

$$= 8y\left[\frac{x^2}{(a^2 - x^2 - y^2)^{1/2}} + (a^2 - x^2 - y^2)^{1/2}\right]$$

or $$\left.\begin{aligned} \frac{\partial V}{\partial y} &= 8y\left[\frac{a^2 - 2x^2 - y^2}{(a^2 - x^2 - y^2)^{1/2}}\right] \\ \text{Similarly,} \quad \frac{\partial V}{\partial y} &= 8x\left[\frac{a^2 - x^2 - y^2}{(a^2 - x^2 - y^2)^{1/2}}\right] \end{aligned}\right\} \tag{4}$$

For maxima and minima, we have

$$\left.\begin{aligned} \frac{\partial V}{\partial x} &= 0 \Rightarrow 8y\left[\frac{a^2 - 2x^2 - y^2}{(a^2 - x^2 - y^2)^{1/2}}\right] \Rightarrow a^2 - 2x^2 - y^2 = 0 \\ \frac{\partial V}{\partial y} &= 0 \Rightarrow 8x\left[\frac{a^2 - x^2 - y^2}{(a^2 - x^2 - y^2)^{1/2}}\right] \Rightarrow a^2 - x^2 - 2y^2 = 0 \end{aligned}\right\} \tag{5}$$

Solving these, we get $$x = \frac{a}{\sqrt{3}} \text{ and } y = \frac{a}{\sqrt{3}}$$

Differentiating partially equation 1 again with respect to x and y both sides respectively, we get

$$r = \frac{\partial^2 V}{\partial x^2} = 8y\left[\frac{(a^2 - x^2 - y^2)^{1/2}(-4x) - (a^2 - 2x^2 - y^2)\frac{1}{2}(a^2 - x^2 - y^2)^{-1/2}(-2x)}{(a^2 - x^2 - y^2)}\right]$$

$$= 8y\left[\frac{x(a^2-2x^2-y^2)-4x(a^2-x^2-y^2)}{(a^2-x^2-y^2)^{3/2}}\right]$$

$$= \frac{-32xy}{(a^2-x^2-y^2)^{1/2}} \qquad \{\text{using equation 5}\}$$

$$t = \frac{-32xy}{(a^2-x^2-y^2)^{1/2}}$$

$$S = \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial}{\partial x}\left[8x\left\{\frac{(a^2-x^2-2y^2)}{(a^2-x^2-y^2)}\right\}1/2\right]$$

$$= 8\left[x\left\{\frac{(a^2-x^2-y^2)^{1/2}(-2x)-(a^2-x^2-2y^2)\frac{1}{2}(a^2-x^2-y^2)^{1/2}(-2x)}{(a^2-x^2-y^2)}\right\}\right.$$

$$\left.+\left\{\frac{a^2-x^2-2y^2}{(a^2-x^2-y^2)^{1/2}}\right\}\right]$$

$$8 = \left[x^2\left\{\frac{(a^2-x^2-y^2)-2(a^2-x^2-y^2)}{(a^2-x^2-y^2)^{3/2}}\right\}+\left\{\frac{a^2-x^2-y^2}{(a^2-x^2-y^2)^{1/2}}\right\}\right]$$

From (5), we have

$$S = \frac{\partial^2 V}{\partial x \partial y} = \frac{-16x^2}{(a^2-x^2-y^2)^{1/2}}$$

Now we have
$$rt - s^2 = \left\{\frac{-32xy}{(a^2-x^2-y^2)^{1/2}}\right\}\left\{\frac{-32xy}{(a^2-x^2-y^2)^{1/2}}\right\} + \left\{\frac{-16x^2}{(a^2-x^2-y^2)^{1/2}}\right\}^2$$

$$= \frac{256x^2(x^2+4y^2)}{(a^2-x^2-y^2)}$$

At $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$, $rt - s^2$ = (five) and $r < 0$

i.e., V is maximum.

The maximum value of $V = 8xyz$

$$= 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{a}{\sqrt{3}}\right)\left(\frac{a}{\sqrt{3}}\right) = \frac{8a^3}{3\sqrt{3}}$$

Example 4. Find the dimensions of a cylindrical tin (with top and bottom) made up of sheet metal to maximize its volume such that the total surface area is equal to $A = 24\ \pi$.

Solution: Let x and y be the radius of the base and length of the tin respectively. The problem is

Maximize $$f(x,\ y) = \pi r^2 y \tag{1}$$

subject to constraint

$$2\pi x^2 + 2\pi xy = A = 24\pi$$

we have $$g(x,\ y) = 2\pi x^2 + 2\pi xy - 24\pi = 0 \tag{2}$$

Construct the Lagrangian function L is

$$L\ (x,\ y;\ \lambda) = \pi x^2 y + \lambda\ (2\pi x^2 + 2\pi xy - A) \tag{3}$$

The necessary condition for extreme of L are

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2\pi xy + 4\lambda\pi x + 2\pi\lambda y = 0 \\ \frac{\partial L}{\partial y} &= \pi x^2 + 2\pi\lambda x = 0 \\ \text{and} \quad \frac{\partial L}{\partial \lambda} &= 2\pi x^2 + 2\pi xy - A = 0 \end{aligned}\right\} \tag{4}$$

Solving these, we get $$x = \sqrt{\frac{A}{6\pi}}\ ,\ y = \sqrt{\frac{2A}{3A}} \text{ and } \lambda = \sqrt{\frac{A}{24\pi}}\ .$$

for $A = 24\pi$, we have $x = 2$, $y = 4$, $\lambda = -1$.

Differentiating partially equation (4) and we get

$$\frac{\partial^2 L}{\partial x^2} = 2\pi y + 4\pi\lambda = 4\pi \text{ at } (2,\ 4;\ -1)$$

$$\frac{\partial^2 L}{\partial x \partial y} = 2\pi x + 2\pi\lambda = 2\pi \text{ at } (2,\ 4;\ -1)$$

$$\frac{\partial^2 L}{\partial y \partial x} = 2\pi x + 2\pi\lambda = 2\pi \text{ at } (2,\ 4;\ -1)$$

$$\frac{\partial^2 L}{\partial y^2} = 0 \text{ at } (2,\ 4;\ -1)$$

$$\frac{\partial g}{\partial x} = 4\pi x + 2\pi y = 16\pi \text{ at } (2,\ 4;\ -1)$$

$$\frac{\partial g}{\partial y} = 2\pi x = 4\pi \text{ at } (2,\ 4;\ -1)$$

The sufficient condition for extreme point is

$$H = \begin{vmatrix} \dfrac{\partial^2 L}{\partial x^2} - k & \dfrac{\partial^2 L}{\partial x \partial y} & \dfrac{\partial g}{\partial x} \\ \dfrac{\partial^2 L}{\partial y \partial x} & \dfrac{\partial^2 L}{\partial y^2} - k & \dfrac{\partial g}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & 0 \end{vmatrix} = 0$$

i.e.
$$H = \begin{vmatrix} 4\pi - k & 2\pi & 16\pi \\ 2\pi & 0 - k & 4\pi \\ 16\pi & 4\pi & 0 \end{vmatrix} = 0$$

or $(4\pi - k)(0 - 16\pi^2) - 2\pi(0 - 64\pi^2) + 16\pi(8\pi^2 + 16\pi k) = 0$

or $-64\pi^3 + 16\pi^2 k + 128\pi^3 + 144\pi^3 + 256\pi^2 k = 0$

or $208\pi^3 + 272\pi^2 k = 0$

$\Rightarrow k$ is negative

so f is maximum at $(2, 4; -1)$

The maximum value of $f = 16\pi$.

Example 5. Find the extreme point for the function

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

Subject to constraint $x_1^2 + x_2^2 + x_3^2 = 1$

Determine whether the extreme points are maximum or minimum.

Solution: Given that

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3 \tag{1}$$

Subject to constraint

$$g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 1 = 0 \text{ (2)} \tag{2}$$

Construct the Lagrangian function L is

$$L(x_1, x_2, x_3; \lambda) = x_1 + x_2 + x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 1) \tag{3}$$

The necessary conditions for extreme of L are

$$\left.\begin{aligned}\frac{\partial L}{\partial x_1} &= 1 + 2\lambda x_1 = 0\\ \frac{\partial L}{\partial x_2} &= 1 + 2\lambda x_2 = 0\\ \frac{\partial L}{\partial x_3} &= 1 + 2\lambda x_3 = 0\\ \frac{\partial L}{\partial \lambda} &= x_1^2 + x_2^2 + x_3^2 - 1 = 0\end{aligned}\right\} \quad (4)$$

Solving these, we get $x_1 = x_2 = x_3 = -\dfrac{1}{2\lambda}$, using equation (2), we have $\lambda = \pm\ \sqrt{3}/2$.

when $\lambda = +\ \sqrt{3}/2$ then point is $\left(-\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}\right)$ and

when $\lambda = -\ -\sqrt{3}/2$ then point is $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$.

Differentiating partially equation (4) again and we get

$$\frac{\partial^2 L}{\partial x_1^2} = 2\lambda,\ \frac{\partial^2 L}{\partial x_1 \partial x_2} = 0,\ \frac{\partial^2 L}{\partial x_1 \partial x_3} = 0,\ \frac{\partial^2 L}{\partial x_2 \partial x_1} = 0,\ \frac{\partial^2 L}{\partial x_2^2} = 2\lambda,$$

$$\frac{\partial^2 L}{\partial x_2 \partial x_3} = 0,\ \frac{\partial^2 L}{\partial x_3 \partial x_1} = 0,\ \frac{\partial^2 L}{\partial x_3 \partial x_2} = 0,\ \frac{\partial^2 L}{\partial x_3^2} = 2\lambda,\ \frac{\partial g}{\partial x_1} = 2x_1,\ \frac{\partial g}{\partial x_2} = 2x_2,\ \frac{\partial g}{\partial x_3} = 2x_3.$$

The sufficient condition for extreme point is

$$H = \begin{vmatrix} 2\lambda - k & 0 & 0 & 2x_1 \\ 0 & 2\lambda - k & 0 & 2x_2 \\ 0 & 0 & 2\lambda - k & 2x_3 \\ 2x_1 & 2x_2 & 2x_3 & 0 \end{vmatrix} = 0$$

or $(2\lambda - k)\begin{vmatrix} 2\lambda - k & 0 & 2x_2 \\ 0 & 2\lambda - k & 2x_3 \\ 2x_2 & 2x_3 & 0 \end{vmatrix} - 2x_1 \begin{vmatrix} 0 & 2\lambda - k & 0 \\ 0 & 0 & 2\lambda - k \\ 2x_1 & 2x_2 & 2x_2 \end{vmatrix} = 0$

or $$(2\lambda - k)^2\ [-4x_3^2 - 4x_2^2 - 4x_1^2] = 0$$

or $$-4\ (2\lambda - k)^2 = 0$$

or $$k = 2\lambda,\ 2\lambda$$

Here both the values of k are of same sign

$$\text{It } \lambda = \frac{\sqrt{3}}{2},\ k \text{ is time, so } \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \text{ is the minimum point}$$

Then the minimum value of f is $-\sqrt{3}$

$$\text{It } \lambda = -\frac{\sqrt{3}}{2},\ k \text{ is –ve, so } \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ is the maximum point.}$$

Then the maximum value of f is $\sqrt{3}$.

Example 6. Find the axtreme points of $U = 2x + y + 10$ subject to constraint $g(x, y) = x + 2y^2 - 3 = 0$.

Solution: Given that
$$U = 2x + y + 10 \tag{1}$$
subject to constraint
$$g(x, y) = x + 2y^2 - 3 = 0 \tag{2}$$

Construct the Lagrangian function L is
$$L\ (x, y; \lambda) = 2x + y + 10 + \lambda\ (x + 2y^2 - 3) \tag{3}$$

The necessary condition for extreme of L are
$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2 + \lambda = 0 \\ \frac{\partial L}{\partial y} &= 1 + 4\lambda y = 0 \\ \frac{\partial L}{\partial \lambda} &= x + 2y^2 - 3 = 0 \end{aligned}\right\} \tag{4}$$

Solving these, we get $x = \frac{95}{32}$, $y = \frac{1}{8}$ and $\lambda = -2$.

Differentiating partailly equation 4 again
$$\frac{\partial^2 L}{\partial x_2} = 0,\ \frac{\partial^2 L}{\partial x \partial y} = 0,\ \frac{\partial^2 L}{\partial y \partial x} = 0,\ \frac{\partial^2 L}{\partial y^2} = 4\lambda = -8,$$
$$\frac{\partial g}{\partial x} = 1,\ \frac{\partial g}{\partial y} = 4y.$$

The sufficient condition for extreme point is

$$H = \begin{vmatrix} 0-k & 0 & 1 \\ 0 & -8-k & 4y \\ 1 & 4y & 0 \end{vmatrix} = 0$$

$\Rightarrow \qquad -k\,(0 - 16y^2) + (0 + 8 + k) = 0$

$\Rightarrow \qquad 16ky^2 + 8 + k = 0$

At $\left(\dfrac{95}{32}, \dfrac{1}{8}\right)$ the value of k is

$$\frac{k}{4} + k + 8 = 0$$

$\Rightarrow \qquad 5k/4 = -8$

$\Rightarrow \qquad k = -32/5.$

Here only value of k which is negative so U is maximum

at $\left(\dfrac{95}{32}, \dfrac{1}{8}\right)$

The maximum value of U is

$$U = 2.\left(\frac{95}{32}\right) + \frac{1}{8} + 10$$

$$= \frac{257}{16}.$$

Example 7. Minimize $f(x) = \dfrac{1}{2}(x_1^2 + x_2^2 + x_3^2)$
subject to constraints

$$g_1(x) = x_1 - x_2 = 0$$

and $g_2(x) = x_1 + x_2 + x_3 - 1 = 0$

by Lagrange's multiplier method.

Solution: Given that minimize $f(x) = \dfrac{1}{2}(x_1^2 + x_2^2 + x_3^2) \quad (1)$
subject to constraints

$$g_1(x) = x_1 - x_2 = 0 \quad (2)$$

and $g_2(x) = x_1 + x_2 + x_3 - 1 = 0 \quad (3)$

Construct the Lagrangian function L is

$$L\,(x_1, x_2, x_3; \lambda_1, \lambda_2) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \lambda_1(x_1 - x_2) + \lambda_2(x_1 + x_2 + x_3 - 1)$$

The necessary condition for extreme of L are

$$\left.\begin{aligned}
\frac{\partial L}{\partial x_1} &= x_1 + \lambda_1 + \lambda_2 = 0\\
\frac{\partial L}{\partial x_2} &= x_2 - \lambda_1 + \lambda_2 = 0\\
\frac{\partial L}{\partial x_3} &= x_3 + \lambda_2 = 0\\
\frac{\partial L}{\partial \lambda_1} &= x_1 - x_2 = 0\\
\frac{\partial L}{\partial \lambda_2} &= x_1 + x_2 + x_3 - 1 = 0
\end{aligned}\right\} \tag{4}$$

Solving these, we get $\quad x_1 = \frac{1}{3}, x_2 = x_3 = \frac{1}{3}; \quad \lambda_1 = 0, \lambda_2 = -\frac{1}{3}.$

Differentiating partially equation (4) again and we get

$$\frac{\partial^2 L}{\partial x_1^2} = 1, \frac{\partial^2 L}{\partial x_1 \partial x_2} = 0, \frac{\partial^2 L}{\partial x_1 \partial x_3} = 0, \frac{\partial^2 L}{\partial x_2 \partial x_1} = 0, \frac{\partial^2 L}{\partial x_2^2} = 1,$$

$$\frac{\partial^2 L}{\partial x_2 \partial x_3} = 0, \frac{\partial^2 L}{\partial x_3 \partial x_1} = 0, \frac{\partial^2 L}{\partial x_3 \partial x_2} = 0, \frac{\partial^2 L}{\partial x_3^2} = 1, \frac{\partial g_1}{\partial x_1} = 1,$$

$$\frac{\partial g_1}{\partial x_2} = -1, \frac{\partial g_2}{\partial x_1} = 1, \frac{\partial g_2}{\partial x_2} = 1, \frac{\partial g_2}{\partial x_3} = 1.$$

The sufficient condition for extreme point is

$$H = \begin{vmatrix} 1-k & 0 & 0 & 1 & 1 \\ 0 & 1-k & 0 & -1 & 1 \\ 0 & 0 & 1-k & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{vmatrix} = 0$$

$\Rightarrow k = 1, 1, 1$; here all the values of k are of same sign and positive, i.e. f is minimum at

$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; 0, -\frac{1}{3}\right).$

The minimum value of $$f = \frac{1}{2}\left(\frac{1}{9}+\frac{1}{9}+\frac{1}{9}\right)$$

$$= \frac{1}{6}.$$

Example 8. Find the optimal solution of the following problem optimize $f(x_1, x_2, x_3) = x_1 x_2 x_3$ subject to constraint $x_1 + x_2 + x_3 - 1 = 0$

Solution: Given that $$f(x_1, x_2, x_3) = x_1 x_2 x_3 \tag{1}$$
subject to constraint
$$g(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 1 = 0 \tag{2}$$

Construct the Lagrangian function L is

$$L\,(x_1, x_2, x_3; \lambda) = x_1 x_2 x_3 + \lambda\,(x_1 + x_2 + x_3 - 1) \tag{3}$$

The necessary condition for extreme of L are

$$\left.\begin{aligned}\frac{\partial L}{\partial x_1} &= x_2 x_3 + \lambda = 0\\ \frac{\partial L}{\partial x_2} &= x_1 x_3 + \lambda = 0\\ \frac{\partial L}{\partial x_3} &= x_1 x_2 + \lambda = 0\\ \frac{\partial L}{\partial \lambda} &= x_1 + x_2 + x_3 - 1 = 0\end{aligned}\right\} \tag{4}$$

Solving these, we get $x_1 = x_2 = x_3$ (by symmetry)

using $x_1 + x_2 + x_3 = 1$

we have $3x_1 = 1$ or

$$x_1 = x_2 = x_3 = \frac{1}{3} \text{ and } \lambda = -\frac{1}{9}$$

Differentiating partially equation 4 again and we get

$$\frac{\partial^2 L}{\partial x_1^2} = 0,\ \frac{\partial^2 L}{\partial x_1 \partial x_2} = x_3,\ \frac{\partial^2 L}{\partial x_1 \partial x_3} = x_2,\ \frac{\partial^2 L}{\partial x_2 \partial x_1} = x_3,\ \frac{\partial^2 L}{\partial x_2^2} = 0,$$

$$\frac{\partial^2 L}{\partial x_2 \partial x_3} = x_1,\ \frac{\partial^2 L}{\partial x_3 \partial x_1} = x_2,\ \frac{\partial^2 L}{\partial x_3 \partial x_2} = x_1,\ \frac{\partial^2 L}{\partial x_3^2} = 0,\ \frac{\partial g}{\partial x_1} = 1,$$

$$\frac{\partial g}{\partial x_2} = 1,\ \frac{\partial g}{\partial x_3} = 1.$$

The sufficient condition for extreme point is

$$H = \begin{vmatrix} 0-k & x_3 & x_2 & 1 \\ x_3 & 0-k & x_1 & 1 \\ x_2 & x_1 & 0-k & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0$$

Solving these Haessian matrix, we get values of k are $-\frac{1}{3}, -\frac{1}{3}$ (both are same and nagative). So the point is maxima.

The maximum value of f is $\frac{1}{27}$.

Example 9. Find the point on the plane $x + 2y + 3z = 1$ which is nearest to the point (–1, 0, 1).

Solution: Suppose

$$U = (x + 1)^2 + (y - 0)^2 + (z - 1)^2$$

$$= (x + 1)^2 + y^2 + (z - 1)^2 \tag{1}$$

and subject to constraint

$$g(x, y, z,) = x + 2y + 3z - 1 = 0 \tag{2}$$

Construct the Lagrangian function L as

$$L\,(x, y, z; \lambda) = (x + 1)^2 + y^2 + (z-1)^2 + \lambda(x + 2y + 3z - 1) \tag{3}$$

The necessary conditions for extreme of L are

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2\,(x + 1) + \lambda = 0 \\ \frac{\partial L}{\partial y} &= 2y + 2\lambda = 0 \\ \frac{\partial L}{\partial z} &= 2\,(z - 1) + 3\lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= x + 2y + 3z - 1 = 0 \end{aligned}\right\} \tag{4}$$

Solving these, we get $x = -\frac{15}{14}, y = -\frac{2}{14}, z = \frac{11}{14}$ and $\lambda = \frac{1}{7}$

Differentiating partially equation (4) again and we get

$$\frac{\partial^2 L}{\partial x_2} = 2,\ \frac{\partial^2 L}{\partial x \partial y} = 0,\ \frac{\partial^2 L}{\partial x \partial z} = 0,\ \frac{\partial^2 L}{\partial y \partial x} = 0$$

$$\frac{\partial^2 L}{\partial y^2} = 2,\ \frac{\partial^2 L}{\partial y \partial z} = 0,\ \frac{\partial^2 L}{\partial z \partial x} = 0,\ \frac{\partial^2 L}{\partial z \partial y} = 0,$$

$$\frac{\partial^2 L}{\partial z^2} = 2,\ \frac{\partial g}{\partial x} = 1,\ \frac{\partial g}{\partial y} = 2,\ \frac{\partial g}{\partial z} = 3.$$

The sufficient condition for extreme point is

$$H = \begin{vmatrix} \frac{\partial^2 L}{\partial x^2} - k & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial x \partial z} & \frac{\partial g}{\partial x} \\ \frac{\partial^2 L}{\partial y \partial x} & \frac{\partial^2 L}{\partial y^2} - k & \frac{\partial^2 L}{\partial y \partial z} & \frac{\partial g}{\partial y} \\ \frac{\partial^2 L}{\partial z \partial x} & \frac{\partial^2 L}{\partial z \partial y} & \frac{\partial^2 z}{\partial z^2} - k & \frac{\partial g}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} & 0 \end{vmatrix} = 0$$

or
$$H = \begin{vmatrix} 2-k & 0 & 0 & 1 \\ 0 & 2-k & 0 & 2 \\ 0 & 0 & 2-k & 3 \\ 1 & 2 & 3 & 0 \end{vmatrix} = 0$$

or
$$(2 - \lambda) \begin{vmatrix} 2-k & 0 & 2 \\ 0 & 2-k & 3 \\ 2 & 3 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 2-k & 0 \\ 0 & 0 & 2-k \\ 1 & 2 & 3 \end{vmatrix} = 0$$

or
$$(2 - \lambda)\ [-9\ (2 - k) - 4\ (2 - k)] - (2 - k)^2 = 0$$

or
$$(2 - k)^2 = 0$$

or
$$k = 2, 2$$

Here values of k are of same sign and positive. So point $\left(-\frac{15}{14}, -\frac{2}{14}, \frac{11}{14}\right)$ is minima.

The minimum value of U is

$$U = \left(-\frac{15}{14}+1\right)^2 + \left(-\frac{2}{14}\right)^2 + \left(\frac{11}{14}-1\right)^2$$

$$= \frac{1}{196} + \frac{4}{196} + \frac{9}{146}$$

$$= \frac{1}{14}$$

Example 10. Consider the following problem

Minimize $f(x) = x_1^2 + x_2^2 + x_3^2$

subject to constraints

$$x_1 + x_2 + x_3 \geq 5$$

$$x_2 x_3 - 2 \geq 0$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Determine whether Kuhn-Tucker conditions are satisfied at the following point:

(i) $\left(\frac{3}{2}, \frac{3}{2}, 2\right)$ (ii) $\left(\frac{4}{3}, \frac{2}{3}, 3\right)$ (iii) (2, 1, 2).

Solution: Given that

Minimize $f(x) = x_1^2 + x_2^2 + x_3^2$ (1)

subject to constraints

$$\left.\begin{aligned} g_1(x) &= x_1 + x_2 + x_3 \geq 5 \\ g_2(x) &= x_2 x_3 - 2 \geq 0 \\ g_3(x) &= x_1 \geq 0 \\ g_4(x) &= x_2 \geq 0 \\ g_5(x) &= x_3 \geq 2 \end{aligned}\right\} \quad (2)$$

Now define a Lagrangian function

$L\ (x_1, x_2, x_3; \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5; y_1, y_2, y_3, y_4, y_5)$

$$= f(x) + \sum_{j=1}^{5} \lambda_j \ [g_j(x) - y_j^2] \quad (3)$$

Using (1), (2) and (3), we have

$$L = x_1^2 + x_2^2 + x_3^2 + \lambda(x_1 + x_2 + x_3 - 5 - y_1^2) + \lambda_2(x_2 x_3 - 2 - y_2^2)$$
$$+ \lambda_3 (x_1 - y_3^2) + \lambda_4(x_2 - y_4^2) + \lambda_5 (x_3 - 2 - y_5^2) \quad (4)$$

The Kuhn-Tucker necessary condition for minimization of L (with $g_j(x) \geq 0$) are

$$\frac{\partial L}{\partial x_i} = 0; \quad i = 1, 2, 3$$

$$\lambda_j \, g_j = 0; \quad j = 1, 2, \ldots, 5$$

and $$\lambda_j \leq 0; \quad j = 1, 2 \ldots, 5$$

Differentiating partially equation (4) and we get

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= 2x_1 + \lambda_1 + \lambda_3 = 0 \\ \frac{\partial L}{\partial x_2} &= 2x_2 + \lambda_1 + \lambda_2 x_3 + \lambda_4 = 0 \\ \frac{\partial L}{\partial x_3} &= 2x_3 + \lambda_1 + \lambda_2 x_2 + \lambda_5 = 0 \end{aligned}\right\} \quad (5)$$

$$\left.\begin{aligned} \lambda_1 g_1 &= \lambda_1 (x_1 + x_2 + x_3 - 5) = 0 \\ \lambda_2 g_2 &= \lambda_2 (x_2 x_3 - 2) = 0 \\ \lambda_3 g_3 &= \lambda_3 \, x_1 = 0 \\ \lambda_4 g_4 &= \lambda_4 x_2 = 0 \\ \lambda_5 g_5 &= \lambda_5 (x_3 - 2) = 0 \end{aligned}\right\} \quad (6)$$

$$\lambda_j \leq 0; \quad j = 1, 2, \ldots, 5 \quad (7)$$

(i) Taking $x_1 = \frac{3}{2}, \; x_2 = \frac{3}{2}, \; x_3 = 2$

From (6), we have

$$g_1 = 0 \Rightarrow \lambda_1 \neq 0$$

$$g_2 = 1 \Rightarrow \lambda_2 = 0$$

$$g_3 = x_1 = \frac{3}{2} \Rightarrow \lambda_3 = 0$$

$$g_4 = x_2 = \frac{3}{2} \Rightarrow \lambda_4 = 0$$

$$g_5 = x_{3-2} = 0 \Rightarrow \lambda_5 \neq 0$$

Now from (5), we have

$$2x_1 + \lambda_1 + \lambda_3 = 0 \Rightarrow \lambda_1 = -3$$

$$2x_2 + \lambda_1 + \lambda_2 x_3 + \lambda_4 = 0 \Rightarrow \lambda_1 = -3$$

$$2x_3 + \lambda_1 + \lambda_2 x_2 + \lambda_5 = 0 \Rightarrow \lambda_5 = -1$$

Hence, equation (7) ($\lambda_j \leq 0$; $j = 1, 2, \ldots, 5$) is satisfied at $\left(\frac{3}{2}, \frac{3}{2}, 2\right)$.

(ii) Taking $$x_1 = \frac{4}{3}, x_2 = \frac{2}{3}, x_3 = 3$$

From (6), we have

$$g_1 = 0 \Rightarrow \lambda_1 \neq 0$$

$$g_2 = 0 \Rightarrow \lambda_2 \neq 0$$

$$g_3 = x_1 = \frac{4}{3} \Rightarrow \lambda_3 = 0$$

$$g_4 = x_2 = \frac{2}{3} \Rightarrow \lambda_4 = 0$$

$$g_5 = x_{3-2} = 0 \Rightarrow \lambda_5 \neq 0$$

Now from (5), we have

$$2x_1 + \lambda_1 + \lambda_3 = 0 \Rightarrow \lambda_1 = -\frac{8}{3}$$

$$2x_2 + \lambda_1 + \lambda_2 x_3 + \lambda_4 = 0 \Rightarrow \lambda_2 = \frac{4}{9}$$

Hence, equation (7) ($\lambda_j \leq 0$; $j = 1, 2, \ldots, 5$) is not satisfied at $\left(\frac{4}{3}, \frac{2}{3}, 3\right)$.

(iii) Taking $$x_1 = 2, x_2 = 1, x_3 = 2$$

From (6), we have

$$g_1 = 0 \Rightarrow \lambda_1 \neq 0$$

$$g_2 = 0 \Rightarrow \lambda_2 \neq 0$$

$$g_3 = x_1 = 2 \Rightarrow \lambda_3 = 0$$

$$g_4 = x_2 = 1 \Rightarrow \lambda_4 = 0$$

$$g_5 = 0 \Rightarrow \lambda_5 \neq 0$$

Now from (5), we have

$$2x_1 + \lambda_1 + \lambda_3 = 0 \Rightarrow \lambda_1 = -4$$

$$2x_2 + \lambda_1 + \lambda_2 x_3 + \lambda_4 = 0 \Rightarrow \lambda_2 = 0$$

$$2x_3 + \lambda_1 + \lambda_2 x_2 + \lambda_5 = 0 \Rightarrow \lambda_5 = 0$$

Hence, equation 7 ($\lambda_j \leq 0$; $j = 1, 2, \ldots, 5$) is satisfied at (2, 1, 2).

Example 11. State Kuhn-Tucker conditions. Use them to minimize

$$f(x, y, z) = x^2 + y^2 + z^2 + 20x + 10y$$

subject to constraints

$$x \geq 40,$$

$$x + y \geq 80,$$

$$x + y + z \geq 120,$$

Solution: Consider the optimization problem

Minimize $$Z = f(X) \quad (1)$$

subject to constraints

$$g_j(x) \geq 0; \quad j = 1, 2, 3 \quad (2)$$

where $$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

From (2), we have

$$G_j(X, Y) = g_j(X) - y_j^2 = 0; \quad j = 1, 2, 3 \quad (3)$$

Now define a Lagrangian function

$$L(x, y, z; \lambda_1, \lambda_2, \lambda_3; y_1, y_2, y_3) = f(X) + \sum_{j=1}^{3} \lambda_j [g_j(X) - y_j^2] \quad (4)$$

Given that minimize $f(x, y, z) = x^2 + y^2 + z^2 + 20x + 10y$ (5)

subject to constraints

$$x \geq 40$$

$$x + y \geq 80$$

$$x + y + z \geq 120$$

$$g_1 = x - 40, \; g_2 = x + y - 80, \; g_3 = x + y + z - 120$$

Using (4) , (5) and (6), we have

$$L = x^2 + y^2 + z^2 + 20x + 10y + \lambda_1 (x - 40 - y_1^2) + \lambda_2 (x + y - 80 - y_2^2)$$
$$+ \lambda_3(x + y + z - 120 - y_3^2) \quad (7)$$

The Kuhn-Tucker necessary condition for minimization of L (with $g_j(X) \geq 0$)

are $\dfrac{\partial L}{\partial x} = 0, \; \dfrac{\partial L}{\partial y} = 0 \; \dfrac{\partial L}{\partial z} = 0$

$$\lambda_j \, g_j = 0; \quad j = 1, 2, 3$$

$$\lambda_j \leq 0; \quad j = 1, 2, 3$$

Differentiating partially equation 7 with respect to x, y and z, both sides respectively, we get

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2x + 20 + \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \frac{\partial L}{\partial y} &= 2y + 10 + \lambda_2 + \lambda_3 = 0 \\ \frac{\partial L}{\partial z} &= 2z + \lambda_3 = 0 \end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned} \lambda_1 g_1 &= \lambda_1 (x - 40) = 0 \\ \lambda_2 g_2 &= \lambda_2 (x + y - 80) = 0 \\ \lambda_3 g_3 &= \lambda_3 (x + y + z - 120) = 0 \end{aligned}\right\} \quad (9)$$

$$\lambda_1, \lambda_2, \lambda_3 \leq 0 \quad (10)$$

If $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$ from (9), we have

$$\left.\begin{aligned} x - 40 &= 0 \\ x + y - 80 &= 0 \\ x + y + z - 120 &= 0 \end{aligned}\right\} \quad (11)$$

Solving these, we get $x = 40, y = 40, z = 40$

using values of x, y and z.

From (6), we get $\lambda_1 = -10, \lambda_2 = -10, \lambda_3 = -80$

Hence, the condition $\lambda_j \leq 0$ of equation 10 are satisfied.

Hence, the optimum solution is

$$x = y = z = 40$$

And minimize $f = (40)^2 + (40)^2 + (40)^2 + 20 (40) + 10(40)$

$$= 6000.$$

Example 12. Solve the following problem:

Minimize $f(X) = x_1 + x_2^2 + x_3^2$

subject to constraints

$$g_1(X) = 2x_1 + x_2 - 5 \leq 0$$
$$g_2(X) = x_1 + x_3 - 2 \leq 0$$
$$g_3(X) = 1 - x_1 \leq 0$$
$$g_4(X) = 2 - x_2 \leq 0$$
$$g_5(X) = -x_3 \leq 0$$

Solution: Given that minimize $f(X) = x_1^2 + x_2^2 + x_3^2$ (1)

subject to constraints

$$\left.\begin{aligned} g_1(X) &= 2x_1 + x_2 - 5 \leq 0 \\ g_2(X) &= x_1 + x_3 - 2 \leq 0 \\ g_3(X) &= 1 - x_1 \leq 0 \\ g_4(X) &= 2 - x_2 \leq 0 \\ g_5(X) &= -x_3 \leq 0 \end{aligned}\right\} \quad (2)$$

Now define a Lagrangian function by introducing slack variables y_j^2, we have

$$L(x_1, x_2, x_3; \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5; y_1, y_2, y_3, y_4, y_5) = f(X) + \sum_{j=1}^{5} \lambda_j [g_j(X) - y_j^2] \quad (3)$$

From equations (1), (2) and (3), we have

$$L = x_1^2 + x_2^2 + x_3^2 + \lambda_1, (2x_1 + x_2 - 5 + y_1^2) + \lambda_2 (x_1 + x_3 - 2 + y_2^2) + \lambda_3(1 - x_1 + y_3^2) + \lambda_4(2 - x_2 + y_4^2) + \lambda_5(-x_3 + y_5^2) \quad (4)$$

The Kuhn-Tucker necessary condition for minimization of L (with $g_j(X) \leq 0$)

are
$$\frac{\partial L}{\partial x_i} = 0; \quad i = 1, 2, 3$$
$$\lambda_j g_j = 0 \text{ and } \lambda_j \geq 0; \quad j = 1, 2, 3, 4, 5.$$

Differentiating partially equation (4) with respect to x_1 x_2 and x_3 respectively, we get

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= 2x_1 + 2\lambda_1 + \lambda_2 - \lambda_3 = 0 \\ \frac{\partial L}{\partial x_2} &= 2x_2 + \lambda_1 - \lambda_4 = 0 \\ \frac{\partial L}{\partial x_3} &= 2x_3 + \lambda_2 - \lambda_5 = 0 \end{aligned}\right\} \quad (5)$$

$$\left.\begin{aligned} \lambda_1 (2x_1 + x_2 - 5) &= 0 \\ \lambda_2 (x_1 + x_3 - 2) &= 0 \\ \lambda_3 (1 - x_1) &= 0 \\ \lambda_4 (2 - x_2) &= 0 \\ \lambda_5 (-x_3) &= 0 \end{aligned}\right\} \quad (6)$$

$$\lambda_j \geq 0; \quad j = 1, 2, 3, 4, 5 \quad (7)$$

Suppose $\lambda_3 \neq 0$ and $\lambda_4 \neq 0$, we have

$$x_1 = 1, x_2 = 2, x_3 = 0 \text{ or } \lambda_5 = 0$$

If we take $x_3 = 0$, $x_1 = 1$, $x_2 = 2$ the equation (6) is satisfied.

Now from (6), we have

$$2x_1 + x_2 - 5 \neq 0 \text{ and } x_1 + x_3 - 2 \neq 0$$

So $\lambda_1 = 0$ and $\lambda_2 = 0$

From equation (5), we have

$$2 - \lambda_3 = 0 \Rightarrow \lambda_3 = 2$$

$$4 - \lambda_4 = 0 \Rightarrow \lambda_4 = 4$$

$$\lambda_2 - \lambda_5 = 0 \Rightarrow \lambda_2 = \lambda_5$$

Hence, the optimum solution is

$$x_1 = 1, x_2 = 2, x_3 = 0,$$

$$\lambda_1 = \lambda_2 = \lambda_5 = 0, \lambda_3 = 2, \lambda_4 = 4.$$

and minimize

$$f = (1)^2 + (2)^2 + (0)^2 = 5.$$

EXERCISE 9.2

1. Find the extreme value of $x^p y^q z^r$ subject to the constraint $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$.

   (**Ans:** $\frac{a^p b^q c^r}{p^p q^q r^r} (p + q + r)^{p+q+r}$)

2. Find the minimum value of $x^2 + y^2 + z^2$ when $ax + by + cz = p$.

   (**Ans:** $\frac{p^2}{a^2 + b^2 + c^2}$)

3. Find the maxima and minima of $x^2 + y^2 + z^2$ subject to the constraints ax^2 + by^2 + cz^2 =1 and $lx + my + nz = 0$.

$$\left(\textbf{Ans: } \frac{l^2}{au-1} + \frac{m^2}{au-1} + \frac{n^2}{cu-1} = 0\right)$$

4. Find the maximum and minimum values of $U = x + y + z$ subject to the constraint $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$.

$$\left(\textbf{Ans: } (\sqrt{a} + \sqrt{b} + \sqrt{c})^2\right)$$

5. Find the maximum and minimum value of $\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ where $lx + my + nz = 0$ and $\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = 1$.

$$\left(\textbf{Ans: } \frac{l^2a^4}{a^2u-1} + \frac{m^2b^4}{b^2u-1} + \frac{n^2c^4}{c^2u-1} = 0\right)$$

6. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\left(\textbf{Ans: } \frac{8abc}{3\sqrt{3}}\right)$$

7. Using Lagrange multiplier method solve the following problems:

   Minimize $f(x_1, x_2) = 3x_1^2 + x_2^2 + 2x_1x_2 + 6x_1 + 2x_2$

   subject to constraint

$$2x_1 - x_2 = 4.$$

   (**Ans:** (1 – 2), $f = 5$)

8. Maximize $f(x_1, x_2) = 6x_1 + 8x_2 - x_1^2 - x_2^2$

   subject to constraint

$$4x_1 + 3x_2 = 16$$

$$\left(\textbf{Ans: } \left(\frac{35}{11}, \frac{12}{11}\right), f = 281\right)$$

   and $3x_1 + 5x_2 = 15$ by Lagrange's multiplier method.

9. Minimize $f = 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1 x_2 - 8x_1, - 6x_2 - 4x_3 + 2x_1x_2 + 9$

   subject to constraint

$$x_1 + x_2 + 2x_3 = 3$$

by (i) direct method and (ii) Lagrange multiplier method.

(**Ans:** $\left(\frac{4}{3}, \frac{7}{9}, \frac{4}{9}\right)$)

10. Using Lagrange multiplier method

optimize $f(x_1, x_2) = 6x_1x_2$

subject to constraint

$$2x_1 + x_2 = 10$$

Also state whether the stationary point is a maxima or minima.

(**Ans:** $\left(\frac{5}{2}, \frac{5}{2}\right)$, $f = 75$)

11. Find the solution of the following problem using the Lagrange's multiplier method.

Minimize $f(x, y) = kx^{-1}y^{-2}$

subject to constraint

$$g(x, y) = x^2 + y^2 - a^2 = 0.$$

(**Ans:** $\left(\frac{a}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}a\right)$)

12. Minimize $f(x_1, x_2) = 6x_1^2 + 5x_2^2$

subject to constraint

$$x_1 + 5x_2 = 3$$

by Lagrange's multiplier method.

(**Ans:** $\left(\frac{3}{31}, \frac{18}{31}\right)$, $f = \frac{54}{31}$)

13. Minimize $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 200$

subject to constraint

$$x_1 + x_2 + x_3 = 11$$

by (i) direct method and (ii) Lagrange's multiplier method.

(**Ans:** (6, 2, 3))

14. Maximize $f(X) = -x_1^2 - x_2^2 + x_1x_2 + 8x_1 + 4x_2$

subject to constraints

$$2x_1 + 3x_2 \le 24$$

$$-5x_1 + 12x_2 \le 24$$

$$x_2 \le 5$$

by using Kuhn-Tucker conditions.

(**Ans:** No solution)

15. Using the Kuhn-Tucker conditions, solve the following problem.

(i) Min $f(x_1, x_2) = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$

subject to constraints

$$x_1 + 3x_2 \le 6$$

$$5x_1 + 2x_2 \le 10$$

and $x_1, x_2 \ge 0$ (**Ans:** $\left(1, \frac{1}{3}\right)$, $f = 17/8$

(ii) Min $f(x_1, x_2) = 2x_1^2 - 7x_2^2 + 12x_1x_2$

subject to constraints

$$2x_1 + 5x_2 \le 98$$

and $x_1, x_2 \ge 0$

(**Ans:** (4, 4, 2), $f = 4900$)

(iii) Max $f(x_1, x_2) = 8x_1^2 + 2x_2^2$

subject to constraint

$$x_1^2 + x_2^2 \le 9$$

and $x_1, x_2 \ge 0$

(**Ans:** $(2, \sqrt{5})$, $f = 42$)

(iv) Max $f(x_1, x_2) = -x_1^2 + x_2 + 2x_1$

subject to constraints

$$2x_1 + 3x_2 \le 6$$

$$2x_1 + x_2 \le 4$$

and $x_1, x_2 \ge 0$

(**Ans:** $\left(\frac{2}{3}, \frac{14}{9}\right)$, $f = \frac{22}{9}$.)

16. A firm manufacturing refrigerators has made a contract to supply 50 units on the last day of the month for three successive months. The production cost of x refrigerators Rs. x^2. If the production is more than the requirement in any month, it is carried over to the next month at a further additional cost of Rs. 20 per unit as holding cost. Assuming no initial inventory, determine the number of units produced in each months so as to minimize the total cost.

(**Ans:** $x_1 = x_2 = x_3 = 50$)

17. A company producing small water heaters contracted to supply 50 water heaters at the end of the first month, 50 at the end of the second month and 50 at the end of the third month. The cost of producing x water heaters in any month is given by Rs. $(x^2 + 1000)$. To carry a water heater from one month to the next costs Rs. 20 per unit. Assuming that there is no

intial stock, determine the number of water heaters to be produced in each month to minimize the total cost.

(**Ans:** $x_1 = x_2 = x_3 = 50$)

18. Maximize $f(x_1, x_2) = -x_1^2 - x_2^2 + 8x_1 + 10x_2$

subject to constraints

$$3x_1 + 2x_2 \le 6$$

$$x_1 \ge 0$$

and $$x_2 \ge 0$$

using by Kuhn-Tucker conditions.

(**Ans:** $\left(\frac{4}{13}, \frac{33}{13}\right)$, $f = 213$)

Chapter 10

Non-linear Programming

10.1 INTRODUCTION

All programming problems are not linear. As we know that if either of the objective function or constraint is non-linear, the programming problem is a non-linear programming problem (NLPP).

A method of solving an NLPP, namely, separable programming method in which an NLPP is approximated by an LPP will be discussed in article 8.9.

We shall first be discussing one more method of solving an NLPP, that too a special class, here. We shall not be rigorous in our discussion.

10.2 QUADRATIC FORM

A real valued function of n-variables is called of a quadratic form if it is in the following form

$$\begin{aligned} f(X) &= \alpha_{11}x_1^2 + \alpha_{22}x_2^2 + \ldots + \alpha_{nn}x_n^2 + 2\alpha_{12}x_1x_2 + 2\alpha_{13}x_1x_3 \\ &\quad + \ldots + 2\alpha_{1n}x_1x_n + 2\alpha_{23}x_2x_3 + \ldots + 2\alpha_{2n}x_2x_n \\ &\quad + 2\alpha_{34}x_3x_4 + 2\alpha_{35}x_3x_5 + \ldots + 2\alpha_{3n}x_3x_n \\ &\quad + 2\alpha_{(n-1)n}x_{n-1}x_n, \end{aligned}$$

Where, $X = (x_1, x_2, \ldots, x_n)^T$.

In matrix form, it can be expressed as

$$f(X) = X^TAX, = (x_1, x_2, \ldots x_n)\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \ldots & \alpha_{2n} \\ \vdots & & & & \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \ldots & \alpha_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Where $X = (x_1, x_2, \ldots, x_n)^T$, and

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1n} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} & \dots & \alpha_{2n} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \dots & \alpha_{3n} \\ & & & & \\ \alpha_{1n} & \alpha_{2n} & \alpha_{3n} & \dots & \alpha_{nn} \end{bmatrix}$$

Which is a symmetrix matrix.

Definition:

(i) A quadratic form X^TAX is said to be *Positive Definite* if $X^TAX > 0 \;\; \forall \; X \; (\neq 0)$.

(ii) A quadratic form X^TAX is said to be *Positive Semi-Definite* if $X^TAX \geq 0 \;\; \forall \; X \neq 0$ and $\exists$ $a \; X \; (\neq 0) \; \ni \; X^TAX = 0$.

(iii) A quadratic form X^TAX is said to be *Negative Definite* if $X^TAX < 0 \;\; \forall \; X \; (\neq 0)$.

(iv) A quadratic form X^TAX is said to be *Negative Semi-Definite* if $X^TAX \leq 0 \;\; \forall \; X \; (\neq 0)$ and $\exists \; a \; X \; (\neq 0) \; \ni \; X^TAX \; \leq 0$.

(v) If $f(X)$ is none of the above, it is called indefinite.

Remark 1: If a quadratic form $f(X) = X^TAX$ is positive (negative) definite, then $-f(X) = -X^TAX$ is negative (positive) definite.

Remark 2: If a quadratic form $f(X) = X^TAX$ is positive (negative) semi-definite, then $-f(X) = -X^TAX$ is negative (positive) semi-definite.

Example 1: Find whether the following

$f(X) = x_1^2 + 2x_2^2 + 3x_3^2$ is positive definite or not.

Solution: It can be expressed as

$$f(X) = [x_1 \; x_2 \; x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

It is positive definite as it is the sum of squares and, therefore, $f(X) > 0 \;\; \forall \; X \; (\neq 0)$.

Example 2: Show that $f(X) = x_1^2 + 4x_2^2 + 3x_3^2 - 4x_1x_2$ is positive semi-definite.

Solution: Let

$$f(x) = (x_1^2 + 4x_2^2 + 3x_3^2 - 4x_1x_2)$$

$$= [x_1 \; x_2 \; x_3] \begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Since, $f(x) = (x_1 - 2x_2)^2 + 3x_3^2$, it is $\geq 0 \;\; \forall \; X$ and $\exists$ (2, 1, 0) at which $f(x) = 0$. Hence, it is positive semi-definite.

10.3 METHODS OF TESTING OF A QUADRATIC FORM

We shall give two methods for testing (without proof) whether $f(X)$ is any of the definite or indefinite.

(a) ***First Method:*** Let $f(X) = X^TAX$ be a quadratic form and D denotes det A. Let $D_1, D_2, \ldots, D_n$, principal minors be defined by

$$D_1 = |\alpha_{11}|,\ D_2 = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{vmatrix},\ D_3 = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{32} & \alpha_{33} \end{vmatrix},\ \ldots\ D_n = D = |A|$$

(i) $f(X)$ is *Positive Definite* if all principal minors are > 0, i.e., $D_1 > 0$, $D_2 > 0$, $D_3 > 0$, $\ldots$, $D_n > 0$.

(ii) $f(X)$ is *Positive Semi-definite* if first principal minor D_1 is positive and others are non-negative, i.e., $D_1 > 0$ and $D_i \geq 0$, $i = 2, 3, \ldots, n$.

(iii) $f(X)$ is *Negative Definite* if $(-1)^i\, D_i > 0$, $i = 1, 2, \ldots, n$ i.e., alternate principal minors are negative and positive with first as negative.

(iv) $f(X)$ is *Negative Semi-definite* if $D_1 < 0$ and $D_2 \geq 0$, $D_3 \leq 0$, $\ldots$, $(-1)^i D_i \geq 0$, $i = 2, 3, \ldots, n$.

(v) If none of the above four, it is indefinite.

(b) ***Second Method:*** Let $f(X) = X^TAX$ be a quadratic form and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of the symmetric matrix A (hence, λ_i are real) then

(i) $f(X)$ is positive definite *iff* $\lambda_i > 0\ \ \forall\ \ i = 1, 2, \ldots, n$

(ii) $f(X)$ is positive semi-definite *iff* $\lambda_i \geq 0\ \ \forall\ \ i = 1, 2, \ldots, n$ and at least one $\lambda_i = 0$.

(iii) $f(X)$ is negative definite *iff* $\lambda_i < 0\ \ \forall\ \ i = 1, 2, \ldots, n$

(iv) $f(X)$ is negative semi-definite *iff* $\lambda_i \leq 0\ \ \forall\ \ i = 1, 2, \ldots, n$ and at least one $\lambda_i = 0$

(v) $f(X)$ is indefinite *iff* at least one $\lambda_i > 0$, and at least one $\lambda_i < 0$.

Example 1: Find whether the $f(X) = 25x_1^2 + 34x_2^2 + 41x_3^2 - 24x_2x_3$ is a positive definite or not.

Solution:

$$f(X) = 25x_1^2 + 34x_2^2 + 41x_3^2 - 24x_2x_3$$

$$= [x_1,\ x_2,\ x_3] \begin{bmatrix} 25 & 0 & 0 \\ 0 & 34 & -12 \\ 0 & -12 & 41 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Here, $$D = \det A = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 34 & -12 \\ 0 & -12 & 41 \end{bmatrix}$$

$$D_1 = 25 > 0,\ D_2 = 850 > 0,\ D_3 = 31250 > 0.$$

Hence, by first method, it is positive definite.

Example 2: Show that $f(X) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2$ is a positive semi-definite.

Solution:

$$f(X) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2$$

$$= [x_1\ x_2\ x_3]\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

Here, $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and eigenvalues of A are 0, 1, 2

$$[A - \lambda I] = 0$$

Hence, by second method $f(X)$ is positive semi-definite.

10.4 CONVENTIONAL METHODS OF OPTIMISATION

Let us first recall certain methods of optimisation i.e., finding maxima and/or minima based on calculus.

(a) ***Functions of one variable:*** We have done a method of finding maxima/minima of a function of one variable, namely, if $f(x)$ is such a function then points of optima are given by the points, where $f'(x) = 0$ and criteria of maxima/minima are given by the signs of $f''(x)$ or $f'''(x)$, etc.

(b) ***Functions of more than one variable:*** If $f(x)$ is a function of n-variable, then we know by Taylor's series.

$$f(x_1 + h_1, \ldots, x_n + h_n) = f(h_1, h_2, \ldots, h_n) + \left(h_1 \frac{\partial}{\partial x_1} + \ldots + h_n \frac{\partial}{\partial x_n}\right) f + \left(h_1 \frac{\partial}{\partial x_1} + \ldots + h_n \frac{\partial}{\partial x_n}\right)^2 f + \ldots$$

and points of optima are given by

$$\frac{\partial f}{\partial x_1} = 0, \ \frac{\partial f}{\partial x_2} = 0, \ \ldots, \ \frac{\partial f}{\partial x_n} = 0$$

If $X^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is a solution of the above equation i.e., a point of optima, then the sign of

$$A = \left(h_1 \frac{\partial}{\partial x_1} + \ldots + h_n \frac{\partial}{\partial x_n}\right)^2 f(X^*)$$

gives the maxima or minima. It is maximum if $A < 0$, minimum if $A > 0$ and a saddle point if for some h_i, $A < 0$ and for some $A > 0$.

We have,

$$A = \left(h_1 \frac{\partial}{\partial x_1} + \ldots + h_n \frac{\partial}{\partial x_n}\right)^2 f(X^*)$$

$$= h_1^2 \frac{\partial^2 f}{\partial x_1^2} + \ldots + h_n^2 \frac{\partial^2 f}{\partial x_n^2} + 2h_1 h_2 \frac{\partial^2 f}{\partial x_1 \partial x_2}$$

$$+ 2h_1 h_3 \frac{\partial^2 f}{\partial x_1 \partial x_3} + \ldots + 2h_1 h_n \frac{\partial^2 f}{\partial x_1 \partial x_n}$$

$$+ 2h_2 h_3 \frac{\partial^2 f}{\partial x_2 \partial x_3} + \ldots + 2h_2 h_n \frac{\partial^2 f}{\partial x_2 \partial x_n} + \ldots + 2h_{n-1} h_n \frac{\partial^2 f}{\partial x_{n-1} \partial x_n}.$$

Which is a quadratic form. It can be written as

$$A = [h_1\ h_2\ \ldots\ h_n] \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \cdots & f_{2n} \\ f_{31} & f_{32} & f_{33} & \cdots & f_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{n1} & f_{n2} & f_{n3} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$$

Where, $f_{ij} = f_{ji} = \dfrac{\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\partial^2 f}{\partial x_j \partial x_i}$. $(h_1, h_2, \ldots, h_n)^T$ is change in X^*. So we write it as ΔX^*.

The matrix

$$H(X^*) = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \cdots & f_{2n} \\ f_{31} & f_{32} & f_{33} & \cdots & f_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{n1} & f_{n2} & f_{n3} & \cdots & f_{nn} \end{bmatrix}$$

is called **Hessian Matrix** of f.

Thus, $$A = (\Delta X^*)^T\ (H(X^*))\ (\Delta X^*)$$

and it is maximum if A is negative definite, minimum if A is positive definite and saddle point if A is indefinite.

In case of function of many variables, there is another method, known as method of **Lagrange's Multiplier method** which is used when optimum is required under certain constraints.

Let the problem be

Opt. $f(X)$, $X = (x_1, x_2, \ldots, x_n)$

Subject to $\quad g_i(X) = 0 \;\; i = 1, 2, \ldots, m$

To do this, one method is, we eliminate m variables with the help of constraints and get the objective function of $n–m$ variables and solve by Taylor's series as above. As a second method, we write the Auxiliary function

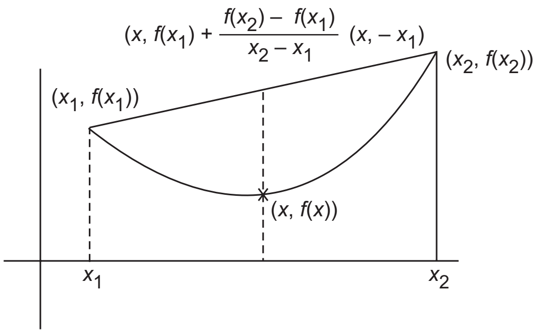
$$\phi(X, \lambda) = f(X) + \lambda_1 g_1(X) + \ldots + \lambda_m (g)_m(X)$$

Where, λ_i's are known as Lagrange's multiplier and the points of optima are obtained from the equations

$$\phi_{x_1} = \phi_{x_2} = \ldots = \phi(x_n) = \phi_{\lambda_1} = \phi_{\lambda_2} = \ldots = \phi_{\lambda_m} = 0$$

10.5 CONVEX FUNCTIONS

A function $f(x)$ of one variable is said to be convex, if its graph is of the form



i.e., $$f(x) \le f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) \;\; \forall \; x \in [x_1, x_2]$$

$$x \in [x_1, x_2] \Rightarrow x = (1 - \lambda)x_1 + \lambda x_2, \; 0 \le \lambda \le 1$$

Therefore,

$$f((1 - \lambda)x_1 + \lambda x_2) \le f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(-\lambda x_1 + \lambda x_2)$$

$$= f(x_1) + \lambda(f(x_2) - f(x_1))$$

or, $f((1 - \lambda)x_1 + \lambda x_2) \le (1 - \lambda)f(x_1) + \lambda f(x_2) \;\; \forall \; x_1, x_2 \in D_f$

Here, D_f is an interval, a convex set.

Keeping this in mind and taking a thread from it, we define in E_n,

(i) A function $f(X) = f(x_1, x_2, \ldots, x_n)$ defined over a convex set S, is said to be a **convex function**, if

$$f((1 - \lambda)X_1 + \lambda X_2) \le (1 - \lambda)\, f(X_1) + \lambda f(X_2) \;\; \forall \;\; X_1, X_2 \in S.$$

(ii) $f(X)$, defined over a convex set S, is called **Strictly convex function,** if

$$f((1-\lambda)X_1 + \lambda X_2) < (1-\lambda)f(X_1) + \lambda f(X_2) \;\; \forall \;\; X_1, X_2 \in S.$$

(iii) $f(X)$, defined over a convex set S, is called a **Concave function,** if

$$f((1-\lambda)X_1 + \lambda X_2) \geq (1-\lambda)f(X_1) + \lambda f(X_2) \;\; \forall \;\; X_1, X_2 \in S.$$

(iv) $f(X)$, defined over a convex set S, is called a **Strictly concave function,** if

$$f((1-\lambda)X_1 + \lambda X_2) > (1-\lambda)f(X_1) + \lambda f(X_2) \;\; \forall \;\; X_1, X_2 \in S.$$

We prove below certain theorems on convex functions.

Theorem 1: Prove that $f(X) = C^TX - \alpha$ is a convex function.

Proof: Let the domain be a convex set S. Let $X_1, X_2 \in S$.

Then,

$$\begin{aligned} f((1-\lambda)X_1 + \lambda X_2) &= C^T((1-\lambda)X_1 + \lambda X_2) - \alpha \\ &= (1-\lambda)C^TX_1 + \lambda\, C^TX_2 - (1-\lambda)\alpha - \lambda\alpha \\ &= (1-\lambda)\,(C^TX_1 - \alpha) + \lambda(C^TX_2 - \alpha) \\ &= (1-\lambda)f(X_1) + \lambda f(X_2) \end{aligned}$$

Hence, proved.

Theorem 2: Let $f(X) = X^TAX$ be a quadratic form defined over a convex set S. If $f(X)$ is positive semi-definite then $f(X)$ is convex.

Proof: Let $X_1, X_2 \in S$. Then

$$\begin{aligned} f((1-\lambda)X_1 + \lambda X_2) &= ((1-\lambda)X_1 + \lambda X_2)^T A\, ((1-\lambda)X_1 + \lambda X_2) \\ &= (1-\lambda)^2 X_1^TAX_1 + \lambda^2 X_2^TAX_2 + \lambda(1-\lambda)\, X_2^TAX_1 \\ &\qquad + \lambda(1-\lambda)X_1^TAX_2 \end{aligned}$$

Since $X_2^TAX_1$ is a real number, $X_1^TAX_2 = (X_1^TAX_2)^T = X_2^TA^TX_i = X_2^TAX_1$

$$\therefore \quad f((1-\lambda)X_1 + \lambda X_2) = (1-\lambda)^2\, X_1^TAX_1 + \lambda^2\, X_2^T\, AX_2 + 2\lambda(1-\lambda)X_1^TAX_2$$

Also $\quad (1-\lambda)f(X_1) + \lambda f(X_2) = (1-\lambda)\, X_1^TAX_1 + \lambda\, X_2^T\, AX_2$

$$\begin{aligned} \therefore \quad & f((1-\lambda)X_1 + \lambda X_2) - [(1-\lambda)f(X_1) + \lambda f(X_2)] \\ &= [(1-\lambda)^2 - (1-\lambda)]\, X_1^TAX_1 + (\lambda^2 - \lambda)\, X_2^T\, AX_2 + 2\lambda(1-\lambda)\, X_1^TAX_2 \\ &= -\lambda(1-\lambda)\, X_1^TAX_1 + \lambda(\lambda - 1)\, X_2^TAX_2 + 2\lambda(1-\lambda)\, X_1^TAX_2 \\ &= -\lambda(1-\lambda)\, [X_1^TAX_1 - 2X_1^TAX_2 + X_2^TAX_2] \\ &= -\lambda(1-\lambda)\, [X_1^T(AX_1 - AX_2) - (X_1^T - X_2^T)AX_2] \\ &= -\lambda(1-\lambda)\, [X_1^T(AX_1 - AX_2) - (X_1^T - X_2^T)AX_2] \\ &= -\lambda(1-\lambda)\, [(X_1^T - X_2^T)\,(AX_1 - AX_2) - (X_1^T - X_2^T)\, AX_2 \\ &\qquad + X_2^T(AX_1 - AX_2)] \\ &= -\lambda(1-\lambda)\, [(X_1^T - X_2^T)\, A(X_1 - X_2) + X_2^TAX_1 - X_1^TAX_2] \end{aligned}$$

But $\quad X_2^TAX_1 = X_1^TAX_2.$

Therefore,

$$\begin{aligned} f((1-\lambda)X_1 + \lambda X_2) &- [(1-\lambda)f(X_1) + \lambda f(X_2)] \\ &= -\lambda(1-\lambda)\, (X_1 - X_2)^T\, A(X_1 - X_2) \end{aligned}$$

$$= -\lambda(1 - \lambda)f(X_1 - X_2)$$
$$\leq 0.$$

as $f(X_1 - X_2)$ is positive semi-definite. Hence,

$$f((1 - \lambda)X_1 + \lambda X_2) \leq (1 - \lambda)f(X_1) + \lambda f(X_2)$$

Hence, the theorem.

Theorem 3: Let $f(X)$ and $g(X)$ be convex functions defined over a convex set S. Then the sum $f(X) + g(X)$ is a convex function.

Proof: Let $X_1, X_2 \in S$. Then,

$$\begin{aligned}(f + g)((1 - \lambda)X_1 + \lambda X_2) &= f((1 - \lambda)X_1 + \lambda X_2) + g((1 - \lambda)X_1 + \lambda X_2)\\ &\leq (1 - \lambda)f(X_1) + \lambda f(X_2) + (1 - \lambda)g(X_1) + \lambda g(X_2)\\ &\leq (1 - \lambda)(f(X_1) + g(X_1)) + \lambda(f(X_2) + g(X_2))\\ &\leq (1 - \lambda)(f + g)(X_1) + \lambda(f + g)(X_2))\end{aligned}$$

Hence, $f + g$ is convex.

From the above three results, it is now clear that a quadratic form is convex, a linear function is a convex function. Also sum of a quadratic form and linear function is convex.

Example 1: Find whether the function

$$f(X) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2$$

is convex or not.

Solution:

$$f(X) = [x_1\ x_2\ x_3]\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Since, eigenvalues of A are 0, 1, 2, $f(X)$ is positive semi-definite. Hence, $f(X)$ is convex.

We now give a theorem, which gives an easy way to determine whether a given function is convex or not. We shall not give its proof.

Theorem 4: Let $f(X) \in C^{(2)}(S)$ be a function defined over a convex set S. The $f(X)$ is convex over S *iff* Hessian Matrix $H(X)$ of $f(X)$ is positive semi-definite over S.

Example 2: Let $f(X) = x_1^3 + 2x_2^2$. Find whether or not it is convex.

Solution:

$$H(x) = \begin{bmatrix} 6x_1 & 0 \\ 0 & +4 \end{bmatrix}$$

$$D_1 = 6x_1 \quad \text{and} \quad D_2 = +24x_1$$

Thus, $\qquad D_1 > 0\ \forall\ x_1 > 0$ and $D_2 \geq 0\ \forall\ x_1 \geq 0$.

Thus, in the right half plane $f(X)$ is convex.

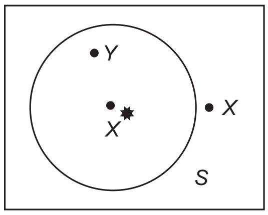
We know that a local minimum is in a neighbourhood of a point. It may not be a global (absolute) minimum. We shall now prove a result which says that a local minimum would be a global minimum if $f(X)$ is a convex function.

Theorem 5: Let $f(X)$ be a convex function over a convex set S. Then a local minima of $f(X)$ is a global minima.

Proof: Let $f(X)$ has a local minima at $X^* \in S$.

Then, $\quad f(X^*) \le f(Y) \;\forall\; Y \in \delta$-neighbourhood of X^*.

Let X be any point in S. $\exists\; Y_1$ in δ-neighbourhood of X^* such that $Y_1 = (1 - \lambda)X + \lambda X^*$, $0 < \lambda < 1$



Thus,

$$f(X^*) \le f(Y_1) = f((1 - \lambda)X + \lambda X^*)$$
$$\le (1 - \lambda)\, f(X) + \lambda f(X^*)$$

as f is convex. Hence,

$$(1 - \lambda)\, f(X^*) \le (1 - \lambda)\, f(X)$$

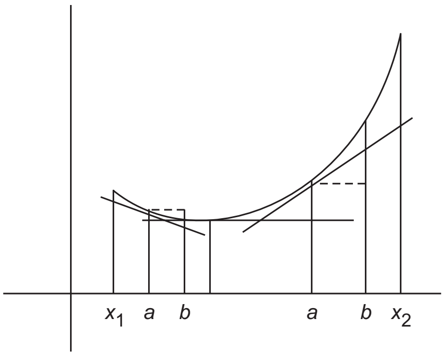
But $\lambda \ne 1$, we get

$$f(X^*) \le f(X)$$

Since, $X \in S$ is arbitrary, we get the result.

Remark: If we look at the steps in the proof, we find that if $f(X)$ is strictly convex, then $f(X^*) < f(X) \;\forall\; X \in S$, i.e., global minimum is unique.

In case of function of one variable, we know that if $f(X)$ is convex over the interval $[x_1, x_2]$, then for $a, b \in [x_1, x_2]$



Such that $b > a$, we have

$$f(b) - f(a) \ge (b - a)\, f'(a)$$

or, $\quad f(a) - f(b) \le (a - b)\, f'(a)$ if $a > b$.

This property also holds for a convex function $f(X)$ over a convex set S of E_n. Thus, we have

$$f(x_2) - f(x_1) \ge (x_2 - x_1)^T\, \nabla f(x_1)$$

and $\quad f(x_1) - f(x_2) \ge (x_1 - x_2)^T\, \nabla f(x_2) \;\forall\; x_1, x_2 \in S.$

Thus, if X_1 is a global minimum, then $\forall\; X_2 \in S$, we have $f(X_1) \le f(X_2)$, i.e.,

$$(X_2 - X_1)^T \, \nabla f(X_2) \geq 0$$

or, $\quad (X - X^*)^T \, \nabla f(X^*) \geq 0 \;\; \forall \;\; X \in S.$

10.6 CONVEX NON-LINEAR PROGRAMMING PROBLEM (CNLPP)

Now we define a special class of NLPP, namely, CNLPP i.e., convex Non-Linear Programming Problem.

An NLPP is called a CNLPP, if it is of the form

Min $f(X)$

Subject to $\quad g_i(X) \leq 0 \;\; \forall \;\; i = 1, 2, \ldots, m$

and, $\quad X \geq 0$

Where, $f(X)$ and $g_i(X)$ are all convex functions in E_n.

It should be noted that a CNLPP is always taken as minimisation problem.

We first show that the set of BFS of CNLPP is a convex set.

Theorem 1: The set S_F of BFS of a CNLPP is convex.

Proof: Let CNLPP be

Min $f(X)$

Subject to $\quad g_i(X) \leq 0 \;\; i = 1, 2, \ldots, m$

$\quad X \geq 0$

Let $X_1, X_2 \in S_F$ and $X = (1 - \lambda)X_1 + \lambda X_2$, $0 \leq \lambda \leq 1$.

Then,
$$\begin{aligned} g_i(X) &= g_i((1 - \lambda)X_1 + \lambda X_2) \\ &\leq (1 - \lambda)\, g_i(X_1) + \lambda g_i(X_2) \end{aligned}$$

because $g_i(X)$ are convex functions.

$$g_i(X) \leq 0$$

as $\quad g_i(X_1),\; g_i(X_2) \leq 0 \quad$ and $\quad \lambda,\, (1 - \lambda) \geq 0.$

Hence, X satisfies the constraints. Since $X_1, X_2 \geq 0$ and λ, $(1 - \lambda) \geq 0$, we have $X \geq 0$. Therefore, $X \in S_F$. Hence, S_F is convex.

If all the constraints of CNLPP are strictly equality then we can employ Lagrange's Multiplier method to solve it. If one or more constraints are inequalities, then the Lagrange's multiplier method cannot be used.

In this case we give another method, which is **Kuhn-Tucker Method** and is an extension of Lagrange's Multiplier Method.

We construct the function

$$\begin{aligned} L(X, \lambda) &= f(X) + \lambda_1 g_1(X) + \lambda_2 g_2(X) + \ldots + \lambda_m g_m(X) \\ &= f(X) + \lambda^T G(X) \end{aligned}$$

Where, $\quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$

and, $\quad G(X) = (g_1(x), g_2(x), \ldots, g_m(x))^T$

As usual λ_i's are called Lagrange's Multipliers.

We now first define a point which is going to play an important role, namely, saddle point and then prove a result showing the relation between a saddle point and the optimal solution of CNLPP.

Definition: The point (X^*, λ^*) is called a non-negative saddle point of $L(X, \lambda)$, if

$$L(X^*, \lambda) \leq L(X^*, \lambda^*) \leq L(X, \lambda^*) \;\; \forall \;\; X, \lambda \geq 0$$

i.e., it is a non-negative minimum point with respect to X and non-negative maximum point with respect to λ of $L(X, \lambda^*)$ and $L(X^*, \lambda)$ respectively. Clearly $X^*, \lambda^* \geq 0$.

Theorem 2: If (X^*, λ^*) is the non-negative saddle point of $L(X, \lambda)$ then X^* is the optimal solution of the CNLPP whose Lagrangian function is $L(X, \lambda)$.

Proof: Let the CNLPP be

$$\text{Min } f(X)$$

Subject to $\qquad g_i(X) \leq 0, \; X \geq 0$

We need to show that X^* satisfies the constraints, i.e.,

$$g_i(X^*) \leq 0, \; X^* \geq 0$$

and, $\qquad f(X^*) \leq f(X) \;\; \forall \;\; X \in S$, set of feasible solutions.

Given that (X^*, λ^*) is a non-negative saddle point of $L(X, \lambda)$. Therefore,

$$L(X^*, \lambda) \leq L(X^*, \lambda^*) \leq L(X, \lambda^*) \;\; \forall \;\; X, \lambda \geq 0.$$

i.e., $\qquad f(X^*) + \lambda^T G(X^*) \leq f(X^*) + \lambda^{*T} G(X^*) \leq f(X) + \lambda^{*T} G(X)$

Which gives from the left inequality.

$$\lambda^T G(X^*) \leq \lambda^{*T} G(X^*) \;\; \forall \;\; \lambda \geq 0$$

or, $\qquad (\lambda - \lambda^*)^T \, G(X^*) \leq 0 \;\; \forall \;\; \lambda \geq 0$

Which means $G(X^*) \leq 0$, i.e., $g_i(X^*) \leq 0 \;\; \forall \;\; i = 1, 2, \ldots, m$ because it is Already done that $X^* \geq 0$. Hence, X^* satisfies the constraints.

From the right hand inequality, we have

$$f(X^*) + \lambda^{*T} G(X^*) \leq f(X) + \lambda^{*T} G(X).$$

Now $X \in S$, therefore $G(X) \leq 0$, $X \geq 0$. Also $\lambda^* \geq 0$.

$\therefore \qquad \lambda^{*T} G(X) \leq 0$

Since, $X^* \in S$, we get $\lambda^{*T} G(X^*) \leq 0$

But, we have seen above

$$\lambda^T G(X^*) \leq \lambda^{*T} G(X^*) \;\; \forall \;\; \lambda \geq 0$$

So, it holds even for $\lambda = 0$. Therefore, $\lambda^{*T} G(X^*) \geq 0$. Thus,

We get $\lambda^{*T} G(X^*) = 0$. Hence,

$$f(X^*) \leq f(X) + \lambda^{*T} G(X) \leq f(X)$$

Since, $\lambda^{*T} G(X) \leq 0$. Hence, the theorem.

The above theorem gives the sufficient conditions for the existence of an optimal solution of CNLPP. Since we have not used the condition of convexity of functions, it holds for any LPP.

Now we take up the necessary conditions. In proving the necessary conditions, we want to be sure that while moving from a point X in any direction, we do not go out of the feasible region. For this Kuhn-Tucker introduced a condition known as Constraint Qualification (CQ) which guarantees that we will not go out of feasible region. This Constraint Qualification is given below.

10.7 CONSTRAINT QUALIFICATION (CQ)

Let X_0 be any feasible solution of NLPP (i.e., $f(X)$ and $g_i(X)$ need not be convex) and h be any direction at X_0 ∋

$$\nabla g_i(X_0)^T h \le 0 \;\; i = 1, 2, \ldots, m \text{ and } g_i(X) \in C^{(1)}.$$

Then it is said that Constraint Qualification (CQ) holds at X_0 provided there exists a vector function

$F(t) = (f_1(t), f_2(t), \ldots, f_n(t))$, $t \ge 0$, $f_i(t)$ real valued function

Such that

(i) $F(0) = X_0$

(ii) $F(t) = X \in S.\ \forall\ 0 \le t \le T, T > 0$ and

(iii) $\lim\limits_{t \to 0} \dfrac{F(t) - F(0)}{t} = h.$

With this, we state necessary conditions.

If X^* is the optimal solution of NLPP, then there exists a $\lambda^* \ge 0$ such that (X^*, λ^*) is a non-negative saddle point of the Lagrange's function $L(X, \lambda)$ is also true under the restriction that $f(X) \in C^{(1)}$ and all $g_i(X) \in C^{(1)}$ and that at X^* (CQ) Conditions hold.

It is to be pointed out here without proof that *CQ* is satisfied for an NLPP having convex feasible region.

Now we prove a theorem which derives conditions for (X^*, λ^*) to be a non-negative saddle point of $L(X, \lambda)$. These are known as **Kuhn-Tucker Conditions** (K-T Conditions)

Theorem 1: Let the CNLPP be

$$\text{Min } f(X)$$

Subject to $\quad g_i(X) \le 0,\ i = 1, 2, \ldots, m$

$$X \ge 0,$$

Where, $f(X)$, $g_i(X)$ are convex functions belonging to $C^{(1)}$. Then the necessary and sufficient conditions for X^* to be the solution of the above problem are, provided *CQ* conditions are satisfied at X^*,

$$\left[\begin{array}{l} \dfrac{\partial L}{\partial x_j}(X^*, \lambda^*) \ge 0;\ x_j^* \dfrac{\partial L(X^*, \lambda^*)}{\partial x_j} = 0;\ j = 1, 2, \ldots, n \\ \dfrac{\partial L}{\partial \lambda_i}(X^*, \lambda^*) = g_i(X^*) \le 0;\ \lambda_i^* \dfrac{\partial L(X^*, \lambda^*)}{\partial \lambda_i} = \lambda_i^* g_i(X^*) = 0;\ i = 1, 2, \ldots, m \\ x_j^* \ge 0,\ \lambda_i^* \ge 0;\ j = 1, 2, \ldots, n;\ i = 1, 2, \ldots, m. \end{array}\right.$$

Conditions are known as Kuhan-Tucker Conditions

The proof of this theorem is omitted.

Remark: A constraint of the '≥' type is to be first converted into a '≤' type, or it can be done by replacing the Corresponding Kuhn-Tucker condition by '≥' type and corresponding $\lambda \ge 0$ by $\lambda \le 0$. For if $g_k(X) \ge 0$ then we write it as

$$-g_k(X) \le 0$$

and in $L(X, \lambda)$, it would be $-\lambda_k\, g_k(X)$.

Therefore, $$\frac{\partial L(X^*, \lambda^*)}{\partial \lambda_k} = -g_K(X^*) \le 0, \text{ i.e., } g_k(X^*) \ge 0$$

and λ_k would be $(-\lambda_k) \ge 0$ i.e., $\lambda_k \le 0$.

Example 1: Solve the following using Kuhn-Tucker conditions.

$$\text{Max } \phi(X) = -x_1^2 - x_2^2$$

Subject to $$-x_1 - x_2 \ge -1$$
$$-2x_1 + 2x_2 \le 1,\ x_1\ x_2 \ge 0$$

Solution: It can be written in standard CNLPP form as

$$\text{Min } F(X) = x_1^2 + x_2^2$$
$$x_1 + x_2 \le 1$$
$$-2x_1 + 2x_2 \le 1,\ x_1, x_2 \ge 0$$
$$L(X, \lambda) = (x_1^2 + x_2^2) + \lambda_1(x_1 + x_2 - 1) + \lambda_2(-2x_1 + 2x_2 - 1)$$

Thus, K-T conditions give

$$2x_1 + \lambda_1 - 2\lambda_2 \ge 0 \text{ (i)} \qquad x_1(2x_1 + \lambda_1 - 2\lambda_2) = 0 \text{ (v)}$$
$$2x_2 + \lambda_1 + 2\lambda_2 \ge 0 \text{ (ii)} \qquad x_2(2x_2 + \lambda_1 + 2\lambda_2) = 0 \text{ (vi)}$$
$$x_1 + x_2 \le 1 \text{ (iii)} \qquad \lambda_1(x_1 + x_2 - 1) = 0 \text{ (vii)}$$
$$-2x_1 + 2x_2 \le 1 \text{ (iv)} \qquad \lambda_2(-2x_1 + 2x_2 - 1) = 0 \text{ (viii)}$$

Solving (v) to (viii), we get

from either $x_1 = 0$ or $2x_1 + \lambda_1 - 2\lambda_2 = 0$

if $x_1 = 0$, we get

$$\left.\begin{aligned} \lambda_1 x_2 + 2\lambda_2 x_2 &= 0 \\ \lambda_1 x_2 - \lambda_1 &= 0 \\ 2\lambda_2 x_2 - \lambda_2 &= 0 \end{aligned}\right| \Rightarrow \quad \begin{aligned} \lambda_1 + \lambda_2 &= 0 \\ \text{i.e., } \lambda_1 &= -\lambda_2 \\ \text{or, } -\lambda_2 x_2 + 2\lambda_2 x_2 &= 0 \end{aligned}$$
$$\lambda_2 x_2 = 0 = \lambda_2 = \lambda_1$$

and x_2 is arbitrary.

But (i), (ii), (iii), (iv) give $0 \le x_2 \le \dfrac{1}{2}$.

If $2x_1 + \lambda_1 - 2\lambda_2 = 0$, then we get

$$x_2(4\lambda_2) = 0 \text{ or } x_2\lambda_2 = 0$$

i.e., $$\text{either } x_2 = 0 \text{ or } \lambda_2 = 0$$
$$x_2 = 0 \quad \Rightarrow \quad \lambda_1(x_1 - 1) = 0$$
$$\lambda_2(-2x_1 - 1) = 0$$

$\Rightarrow$ $$\text{either } \lambda_1 = 0 \text{ or } x_1 = 1,\ \lambda_2 = 0$$

$\Rightarrow$ $$x_1 = \frac{-1}{2} \text{ or } \lambda_2 = 0\ \lambda_1 = -2$$

$$\lambda_2 = \frac{-1}{2} = 0$$

$\therefore$ (a) $x_1 = 1$, $x_2 = 0$, $\lambda_2 = 0$, $\lambda_1 = -2$; or (b) $x_1 = -\frac{1}{2}$, $x_2 = 0$, $\lambda_1 = 0$, $\lambda_2 = -\frac{1}{2}$, (c) $x_1 = 0$, $x_2 = 0$, $\lambda_1 = 0$, $\lambda_2 = 0$

(a), (b) do not satisfy (i) to (iv)

Hence, the solutions are

(i) $x_1 = 0$, $x_2 = 0$, $\lambda_1 = 0$, $\lambda_2 = 0$

(ii) $x_1 = 0$, $0 \le x_2 \le \frac{1}{2}$, $\lambda_1 = 0$, $\lambda_2 = 0$

only (i) gives the maximum value. Hence, it is the solution.

10.8 QUADRATIC PROGRAMMING

If all constraints are linear and objective function a quadratic function, then the problem is known as QLPP. We take a special case of it, **Convex Quadratic Programming problem**. It is of the form

$$\text{Min } f(X) = X^T CX + P^T X$$

Subject to $\quad AX \le b$, $X \ge 0$.

Where clearly all constraints are linear, $X^T CX$, C a symmetric positive definite or positive semi-definite matrix, is a quadratic form, hence, Convex, $P^T X$, a linear function. Thus, all constraints and $f(X)$ are convex functions. Hence, it is a CNLPP.

or, $\quad$ If Min $f(X) = X^T CX + P^T X$

Subject to $\quad AX \le b$, $X \ge 0$

will be a convex quadratic form. If $C = (C_{ij})$ is a $n \times n$ positive definite or positive semi-definite matrix.

$$\begin{aligned} P &= (p_1, p_2, \ldots, p_n)^T \text{ is an column vector.} \\ A &= (\alpha_{ij})_{m\times n} \\ X &= (x_1, x_2, \ldots, x_n)^T \\ b &= (b_1, b_2, \ldots, b_m)^T \end{aligned}$$

We now give a method, known as **Wolfe's method** to solve this convex QLPP.

It is done as follows:

Let Lagrange's function be

$$\begin{aligned} L(X, \lambda) &= f(X) + \lambda^T G(X) \\ &= X^T CX + P^T X + \lambda^T (AX - b) \end{aligned}$$

Where, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$ are Lagrangian multipliers.

or, $\quad L(X, \lambda) = C_{11}x_1^2 + 2C_{12}x_1x_2 + 2C_{13}x_1x_3 + \ldots + 2C_{1n}x_1x_n$

$$+ C_{22}x_2^2 + 2C_{23}x_2x_3 + \ldots + 2C_{2n}x_2x_n + C_{33}x_3^2$$
$$+ 2C_{34}x_3x_4 + \ldots + 2C_{3n}x_3x_n + \ldots + 2C_{n-1,n}\, x_{n-1}\, x_n + C_{nn}x_n^2$$
$$+ p_1x_1 + p_2x_2 + \ldots + p_nx_n + \lambda_1(\alpha_{11}x_1 + \alpha_{12}x_2 + \ldots + \alpha_{1n}x_n - b_1)$$
$$+ \lambda_2(\alpha_{21}x_1 + \alpha_{22}x_2 + \ldots + \alpha_{2n}x_n - b_2) + \ldots + \lambda_m(\alpha_{m1}x_1 + \alpha_{m2}x_2$$
$$+ \ldots + \alpha_{mn}\, x_n - b_m),$$
$$c_{ij} = c_{ji}$$

Hence, Kuhn-Tucker Conditions are: $\dfrac{\partial L}{\partial x_j} \geq 0,\ \dfrac{\partial L}{\partial \lambda_i} \leq 0$

and, $$x_j\left(\frac{\partial L}{\partial x_j}\right) = 0,\ \lambda_i\left(\frac{\partial L}{\partial \lambda_i}\right) = 0$$

i.e.,

$$\left.\begin{array}{l}
2C_{11}x_1 + 2C_{12}x_2 + \ldots + 2C_{1n}x_n + p_1 + \lambda_1\alpha_{11} + \ldots + \lambda_m\alpha_{m1} \geq 0 \\
\text{---} \\
2C_{n1}x_1 + 2C_{n2}x_2 + \ldots + 2C_{nn}x_n + p_n + \lambda_1\alpha_{1n} + \ldots + \lambda_m\alpha_{mn} \geq 0 \\
\alpha_{11}x_1 + \alpha_{12}x_2 + \ldots + \alpha_{1n}x_n - b_1 \leq 0 \\
\text{--------------------------} \\
\alpha_{m1}x_1 + \alpha_{m2}x_2 + \ldots + \alpha_{mn}x_n - b_m \leq 0
\end{array}\right]\ \text{A}$$

and,

$$\left.\begin{array}{l}
x_1(2C_{11}x_1 + 2C_{12}x_2 + \ldots + 2C_{1n}x_n + \alpha_{11}\lambda_1 + \ldots + \alpha_{m1}\lambda_m + p_1) = 0 \\
\text{---} \\
x_n(2C_{n1}x_1 + 2C_{n2}x_2 + \ldots + 2C_{nn}x_n + \alpha_{1n}\lambda_1 + \ldots + \alpha_{mn}\lambda_m + p_n) = 0 \\
\lambda_1(\alpha_{11}x_1 + \ldots + \alpha_{1n}x_n - b_1) = 0 \\
\text{----------------------} \\
\lambda_m(\alpha_{m1}x_1 + \ldots + \alpha_{mn}x_n - b_m) = 0
\end{array}\right]\ \text{B}$$

On adding slack/surplus variables in A, we obtain

$$\left.\begin{array}{l}
2C_{11}x_1 + 2C_{12}x_2 + \ldots + 2C_{1n}x_n + \alpha_{11}\lambda_1 + \ldots + \alpha_{m1}\lambda_m - \mu_1 = -p_1 \\
\text{---} \\
2C_{n1}x_1 + 2C_{n2}x_2 + \ldots + 2C_{nn}x_n + \alpha_{1n}\lambda_1 + \ldots + \alpha_{mn}\lambda_m - \mu_n = -p_n \\
\alpha_{11}x_1 + \alpha_{12}x_2 + \ldots + \alpha_{1n}x_n + S_1 = b_1 \\
\text{----------------------} \\
\alpha_{m1}x_1 + \alpha_{m2}x_2 + \ldots + \alpha_{mn}x_n + S_m = b_m
\end{array}\right]\ \text{C}$$

and writing the values of μ_i, S_j in B, we get

$$\left.\begin{array}{r} x_1\mu_1 = 0 \\ x_2\mu_2 = 0 \\ ------- \\ x_n\mu_n = 0 \\ \lambda_1 S_1 = 0 \\ \lambda_2 S_2 = 0 \\ \vdots \\ \lambda_m S_m = 0 \end{array}\right] \text{D}$$

The minimisation is already by K-T conditions. So, now the problem is to obtain solution of A and B, i.e., of C and D.

So, we add artificial variables to C and solve C to get a solution with artificial variable 0. So, we solve C by Phase I under the conditions D. Condition D would be satisfied if either both x_i, μ_i are non-basic variables or at least one of them is a non-basic variable. Similarly, either both or at least one of λ_i, S_i is a non-basic variable. Both of them can be basic variable, if at least one of them has value 0. Thus, in simplex iteration, we would take care of it. Therefore, we term D as restricted basis conditions.

We now illustrate it by example.

Example 1: Solve the following:

$$\text{Max } \phi(X) = 6x_1 + 3x_2 - 4x_1x_2 - 2x_1^2 - 3x_2^2$$

$$\text{Subject to } x_1 + x_2 \leq 1$$

$$2x_1 + 3x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solution: The above problem can be written as

$$\text{Min } F(X) = 2x_1^2 + 4x_1x_2 + 3x_2^2 - 6x_1 - 3x_2$$

$$\text{Subject to } x_1 + x_2 - 1 \leq 0$$

$$2x_1 + 3x_2 - 4 \leq 0 \quad x_1, x_2 \geq 0.$$

$$L(X, \lambda) = (2x_1^2 + 4x_1x_2 + 3x_2^2 - 6x_1 - 3x_2) + \lambda_1(x_1 + x_2 - 1) + \lambda_2(2x_1 + 3x_2 - 4)$$

KT conditions give

$$4x_1 + 4x_2 - 6 + \lambda_1 + 2\lambda_2 \geq 0, \quad x_1(4x_1 + 4x_2 - 6 + \lambda_1 + 2\lambda_2) = 0$$

$$4x_1 + 6x_2 - 3 + \lambda_1 + 3\lambda_2 \geq 0, \quad x_2(4x_1 + 6x_2 - 3 + \lambda_1 + 3\lambda_2) = 0$$

$$x_1 + x_2 - 1 \leq 0, \quad \lambda_1(x_1 + x_2 - 1) = 0$$

$$2x_1 + 3x_2 - 4 \leq 0, \quad \lambda_2(2x_1 + 3x_2 - 4) = 0$$

$$x_1, x_2, \lambda_1, \lambda_2 \geq 0$$

or
$$4x_1 + 4x_2 + \lambda_1 + 2\lambda_2 - \mu_1 = 6, \quad x_1\mu_1 = 0$$
$$4x_1 + 6x_2 + \lambda_1 + 3\lambda_2 - \mu_2 = 3, \quad x_2\mu_2 = 0$$
$$x_1 + x_2 + S_1 = 1, \quad \lambda_1 S_1 = 0$$
$$2x_1 + 3x_2 + S_2 = 4, \quad \lambda_2 S_2 = 0$$
$$x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2, S_1, S_2 \geq 0$$

Thus, we solve Min $R = R_1 + R_2$
$$4x_1 + 4x_2 + \lambda_1 + 2\lambda_2 - \mu_1 + R_1 = 6$$
$$4x_1 + 6x_2 + \lambda_1 + 3\lambda_2 - \mu_2 + R_2 = 3$$
$$x_1 + x_2 + S_1 = 1$$
$$2x_1 + 3x_2 + S_2 = 4$$
$$x_i, \lambda_i, \mu_i, R_i, S_i, \geq 0$$

and restricted basis conditions.
$$x_1\mu_1 = 0,\ x_2\mu_2 = 0,\ \lambda_1 S_1 = 0,\ \lambda_2 S_2 = 0$$

Initial Simplex table.

$B.V$	x_1	x_2	λ_1	λ_2	μ_1	μ_2	R_1	R_2	S_1	S_2	Solution	
R	0	0	0	0	0	0	–1	–1	0	0	0	
R_1	4	4	1	2	–1	0	1	0	0	0	6	
R_2	4	6	1	3	0	–1	0	1	0	0	3	
S_1	1	1	0	0	0	0	0	0	1	0	1	
S_2	2	3	0	0	0	0	0	0	0	1	4	Initial table
R	8	10 ↓	2	5	–1	–1	0	0	0	0	9	
R_1	4	4	1	2	–1	0	1	0	0	0	6	
$\leftarrow R_2$	4	$\boxed{6}$	1	3	0	–1	0	1	0	0	3	Starting table
S_1	1	1	0	0	0	0	0	0	1	0	1	
S_2	2	3	0	0	0	0	0	0	0	1	4	
R	4/3 ↓	0	2/6	0	–1	4/6	0	–10/6	0	0	4	
R_1	4/3	0	2/6	0	–1	4/6	1	–4/6	0	0	4	
$\leftarrow x_2$	$\boxed{2/3}$	1	1/6	1/2	0	–1/6	0	1/6	0	0	1/2	
S_1	1/3	0	–1/6	–1/2	0	1/6	0	–1/6	1	0	1/2	
S_2	0	0	–1/2	–3/2	0	1/2	0	–1/2	0	1	5/2	

x_2 can enter as μ_2 is a non-basic variable.

B.V	x_1	x_2	λ_1	λ_2	μ_1	μ_2	R_1	R_2	S_1	S_2	Sol.	
R	0	−2	0	−1	−1	1	0	−2	0	0	3	
R_1	0	−2	0	−1	−1	6/6	1	−1	0	0	3	
x_1	1	3/2	1/4	3/4	0	−1/4	0	1/4	0	0	3/4	
$\leftarrow S_1$	0	−1/2	−1/4	−3/4	0	$\boxed{1/4}$	0	−1/4	1	0	1/4	
S_2	0	0	−1/2	−3/2	0	+1/2	0	−1/2	0	1	5/2	
R	0	0	1	2	−1	0	0	−1	−4	0	2	
$\leftarrow R_1$	0	0	$\boxed{1}$	2	−1	0	1	0	−4	0	2	Here though suitable enter is λ_2 we take λ_1 as λ_2 cannot enter
x_1	1	1	0	0	0	0	0	0	1	0	1	
μ_2	0	−2	−1	−3	0	1	0	−1	4	0	1	
S_2	0	1	0	0	0	0	0	0	−2	1	2	
R	0	0	0	0	0	0	−1	−1	0	0	0	
λ_1	0	0	1	2	−1	0	1	0	−4	0	2	
x_1	1	1	0	0	0	0	0	0	1	0	1	
μ_2	0	−2	0	−1	−1	1	1	−1	0	0	3	
S_2	0	1	0	0	0	0	0	0	−2	1	2	

Thus, solution exists and the solution is

$$x_1 = 1,\ x_2 = 0,\ \lambda_1 = 2,\ \lambda_2 = 0,\ \mu_1 = 0,\ \mu_2 = 3,\ S_1 = 0,\ S_2 = 2$$

and the Minimum value is −4.

and the Maximum value is 4.

EXERCISE 10.1

1. Which of the following matrix is positive definite or positive semi-definite? (Justify your answer).

   (a) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 25 & 0 & 0 \\ 0 & 34 & -12 \\ 0 & -12 & 41 \end{pmatrix}$

   (**Ans:** (a) Positive semi definite, (b) Positive-definite)

2. Show that Min $Z = \dfrac{1}{2}x_1^2 - x_1x_2 + x_2^2 - x_1 - x_2$

Subject to

$$+x_1^2 + x_2^2 \le 4$$
$$-x_1 - x_2 \ge -5/2$$
$$x_1, x_2 \ge 0$$

is a convex non-linear programming problem.

3. Show that the following are convex non-linear programming problems.

(a) Min $Z = -3x_1 - 4x_2$

Subject to

$$-2x_1 - 5x_2 \ge -10$$
$$-3x_1 - 4x_2 \le -12$$
$$x_1, x_2, \ge 0$$

(b) Min $Z = x_1^2 - 2x_1 + x_2 + 1$

Subject to

$$-x_1 \ge 2$$
$$x_2 \le 2$$
$$x_1, x_2 \ge 0$$

4. Use the Kuhn-Tucker Conditions to solve the following problem

Max $Z = 2x_1 + x_2$

Subject to

$$-x_1 + x_2 \ge 0$$
$$x_1^2 + x_2^2 \le 4$$
$$x_1, x_2 \ge 0$$

(**Ans:** $x_1 = \sqrt{2}$, $x_2 = \sqrt{2}$)

5. Solve the following using Kuhu-Tucker Conditions.

Max $Z = -2x_1 - 6x_2$

Subject to

$$-x_1 + x_2 \ge 0$$
$$x_1^2 + x_2^2 \ge 4$$
$$x_1, x_2 \ge 0$$

(**Ans:** $x_1 = 0$, $x_2 = 2$ and $x_1 = \sqrt{2} = x_2$ are two solutions of the problem)

6. Show that $C^T X - 4$ is a convex function.

7. Let $f(X)$ and $g(X)$ be convex functions over convex sets S contained in E_n. Prove that 4 $f(X)$ + 6 $g(X)$ is also a convex function over S.

8. Solve the following quadratic programming problem.

Max $f(X) = -2x_1^2 - 2x_2^2 + 4x_1 + 4x_2$

Subject to

$$-2x_1 - 3x_2 \ge -6$$
$$x_1, x_2 \ge 0$$

(**Ans:** $x_1 = 1 = x_2$, Max $f(X) = 4$)

9. Solve the following quadratic programming problem by Wolfe's method.
   Min $Z = -2x_1 - x_2 + x_1^2$
   Subject to
   $$-2x_1 - 3x_2 \geq -6$$
   $$-2x_1 - x_2 \geq -4$$
   $$x_1, x_2 \geq 0$$
   $$\left(\textbf{Ans: } x_1 = \frac{2}{3}, x_2 = \frac{14}{9}, Z = \frac{22}{9}\right)$$
10. Use Wolfe's method to solve the following quadratic programming problem:
    Max $Z = -x_1^2 + x_1 + x_2$
    Subject to
    $$-x_1 - x_2 \geq -1$$
    $$x_1, x_2 \geq 0$$
    (**Ans:** $x_1 = 0$, $x_2 = 1$, $Z = 1$)

10.9 SEPARABLE PROGRAMMING

10.9.1 Introduction

A large class of Non-linear Programming Problems (NLPP) can be approximated by LPP which on solving by simplex method yield a solution of LPP. This solution can be regarded as a solution of NLPP. Obviously this solution of LPP would be a approximate solution of NLPP.

A method to handle NLPP as mentioned above is separable programming which is being discussed below.

10.9.2 Separable Programming

This method is applicable only to those classes of NLPP in which each objective function and constraint which is non-linear can be expressed as sum of functions, each of which is a function of one variable, i.e., variables are separable.

Thus, let the NLPP be

Opt. $Z = f(x_1, x_2, \ldots, x_n) = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n)$

Subject to the conditions

$$g^1(X) = g_1^1(x_1) + g_2^1(x_2) + \ldots + g_n^1(x_n) \lesseqgtr b_1$$
$$g^2(X) = g_1^2(x_1) + g_2^2(x_2) + \ldots + g_n^2(x_n) \gtreqless b_2$$

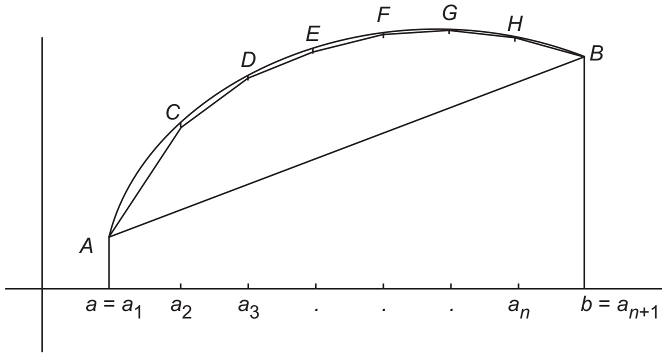
$$g^m(X) = g_1^m(x_1) + g_2^m(x_2) + \ldots + g_n^m(x_n) \lesseqgtr b_m$$
$$X = (x_1, x_2, \ldots, x_n) \geq 0,$$

Where each $f_i(x_i)$, $g_i^k(x_i)$, $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, m$ is a function of a single variable x_i.

We shall first take the method of approximating a function of one variable by a linear function of that variable.

10.9.3 Linearisation of a Non-linear Function

Let $y = f(x)$ be a non-linear function. Its graph is a curve, in the interval $[a, b]$.



Approximating it by a linear function means approximating it by lines. We can approximate the curve AB by the chord AB, but this is too much and error would be large. So, we divide the interval $[a, b]$ into subintervals $[a_1, a_2]$, $[a_2, a_3]$, …, $[a_n, a_{n+1}]$ and in each subinterval, we approximate it by chords AC, AD, etc., we approximate the curve AB by piecewise linear functions. This would decrease the error substantially. If the number of subintervals goes on increasing, error goes on decreasing.

In the interval $[a_1, a_2]$, function $f(x)$ is approximated by the function $\hat{f}(x)$ by,

$$\hat{f}(x) = f(a_1) + \frac{f(a_2) - f(a_1)}{a_2 - a_1}(x - a_1),\ a_1 \le x \le a_2.$$

Since, $x \in [a_1, a_2]$, we have $x = (1 - \lambda)a_1 + \lambda a_2$, $0 \le \lambda \le 1$

or, $$x = \lambda_1 a_1 + \lambda_2 a_2,\ \lambda_1, \lambda_2 \ge 0;\ \lambda_1 + \lambda_2 = 1.$$

$$\hat{f}(x) = f(a_1) + \frac{f(a_2) - f(a_1)}{a_2 - a_1}(\lambda_1 a_1 + \lambda_2 a_2 - a_1)$$

$$= f(a_1) + \frac{f(a_2) - f(a_1)}{a_2 - a_1}(\lambda_1 a_2 - a_1(1 - \lambda_1))$$

$$= f(a_1) + \frac{f(a_2) - f(a_1)}{a_2 - a_1}\lambda_2(a_2 - a_1)$$

or, $$\hat{f}(x) = f(a_1) + \lambda_2(f(a_2)) - \lambda_2(f(a_1))$$

$$\hat{f}(x) = \lambda_1 f(a_1) + \lambda_2 f(a_2);\ \lambda_1, \lambda_2 \ge 0;\ \lambda_1 + \lambda_2 = 1.$$

Similarly in $[a_2, a_3]$, we have

$$\hat{f}(x) = \lambda_2 f(a_2) + \lambda_3 f(a_3);\ \lambda_2, \lambda_3 \ge 0;\ \lambda_2 + \lambda_3 = 1.$$

Where, we have taken same λ_2 as earlier, because

in $[a_1, a_2]$, $x = \lambda_1 a_1 + \lambda_2 a_2$ $0 \le \lambda_1, \lambda_2, \lambda_1 + \lambda_2 = 1$

and, in $[a_2, a_3]$, $x = \lambda_2 a_2 + \lambda_3 a_3$, $\lambda_2, \lambda_3 \geq 0$, $\lambda_2 + \lambda_3 = 1$

Thus, same value of λ_2 i.e., $\lambda_2 = 1$ ($\lambda_1 = \lambda_3 = 0$) gives the same point $x = a_2$.

Continuing as above, we get in $[a_n, a_{n+1}]$,

$$\hat{f}(x) = \lambda_n f(a_n) + \lambda_{n+1} f(a_{n+1});\ \lambda_n, \lambda_{n+1} \geq 0,\ \lambda_n + \lambda_{n+1} = 1.$$

Combining this, we obtain the approximation piecewise in $[a_1 = a, a_{n+1} = b]$. Thus,

$$\hat{f}(x) = \lambda_1 f(a_1) + \lambda_2 f(a_2) + \ldots + \lambda_{n+1} f(a_{n+1})$$

and, $$x = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_{n+1} a_{n+1},$$

Where, $\lambda_1, \lambda_2, \ldots, \lambda_{n+1} \geq 0$; $\lambda_1 + \lambda_2 + \ldots + \lambda_{n+1} = 1$ with the restriction that

(i) at the most two λ_i's are non-zero, i.e., positive and (ii) two adjacent λ_i's are allowed to be non-zero. This would guarantee that at a point of time x lies only in one subinterval.

Using the above technique, after deciding the range of each variable and the number of subintervals we would like to have, we approximate each non-linear $f_i(x_i)$, $g_i^j(x_i)$ and substitute in NLPP and thus obtain an LPP which we solve by simplex method and get an approximate solution of the NLPP.

We now take illustrative examples.

Example 1: Approximate the following NLPP by an LPP.

$$\text{Max } Z = x_1 + x_2^4$$
$$3x_1 + 2x_2^2 \leq 9$$
$$x_1, x_2 \geq 0$$

Solution: It is clear that $f_1(x_1) = x_1$, $f_2(x_2) = x_2^4$

$$g_1(x_1) = 3x_1,\ g_2(x_2) = 2x_2^2$$

$f_1(x_1)$ and $g_1(x_1)$ are already linear and $f_2(x_2)$ and $g_2(x_2)$ need to be linearised.

We find that $0 \leq x_1 \leq 3$ and $0 \leq x_2 \leq 3$.

Let us divide x_2 in 3 subintervals [0, 1], [1, 2], [2, 3]

k	a_2^k	$f_2(a_2^k)$	$g_2(a_2^k)$
1	0	0	0
2	1	1	2
3	2	16	8
4	3	81	18

Thus, $$\hat{f}_2(x_2) = \lambda_2^1 \cdot 0 + \lambda_2^2 \cdot 1 + \lambda_2^3 \cdot 16 + \lambda_2^4 \cdot 81$$

$$= \lambda_2^2 + 16\lambda_2^3 + 81\lambda_2^4$$

$$\hat{g}_2(x_2) = 2\lambda_2^2 + 8\lambda_2^3 + 18\lambda_2^4$$

Use the same weights λ's. Also, we have used lower suffix 2 to signify that these are for the variable x_2.

Thus, the approximated LPP is

$$\text{Max } Z = x_1 + \lambda_2^2 + 16\lambda_2^3 + 81\lambda_2^4$$

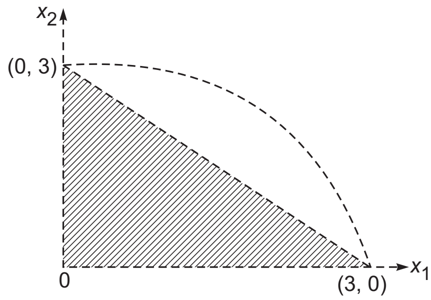
Subject to $3x_1 + 2\lambda_2^2 + 8\lambda_2^3 + 18\lambda_2^4$

$$\lambda_2^1 + \lambda_2^2 + \lambda_2^3 + \lambda_2^4 = 1$$

$$x_1, \lambda_2^i \geq 0, i = 1, 2, 3, 4.$$

Under the natural restriction that at the most two adjacent $\lambda_i's$ are non-zero.

Example 2: Solve the following NLPP by separable programming.



$$\text{Max } Z = 3x_1^2 + 2x_2^2$$

Subject to

$$x_1^2 + x_2^2 \leq 9$$
$$x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Solution: Here $f_1(x_1)$, $f_2(x_2)$, $g_1^1(x_1)$, $g_2^1(x_2)$ are non-linear. $g_1^2(x_1)$, $g_2^2(x_2)$ are already linear.

Graphically, $0 \leq x_1 \leq 3;\ 0 \leq x_2 \leq 3.$

We take 3 subintervals of both x_1, x_2.

k	$x_1^{(k)}$	$f_1(x_1^{(k)})$	$g_1^1(x_1^{(k)})$	k	$x_2^{(k)}$	$f_2(x_2^{(k)})$	$g_2^1(x_2^{(k)})$
1	0	8	0	1	0	0	0
2	1	3	1	2	1	2	1
3	2	12	4	3	2	8	4
4	3	27	9	4	3	18	9

Thus,

$$\hat{f}_2(x_1) = 3\lambda_1^2 + 12\lambda_1^3 + 27\lambda_1^4$$
$$\hat{f}_2(x_2) = 2\lambda_2^2 + 8\lambda_2^3 + 18\lambda_2^4$$
$$\hat{g}_1(x_1) = \lambda_1^2 + 4\lambda_1^3 + 9\lambda_1^4$$
$$\hat{g}_2(x_2) = \lambda_2^2 + 4\lambda_2^3 + 9\lambda_2^4$$
$$x_1 = \lambda_1^2 + 2\lambda_1^3 + 3\lambda_1^4$$
$$x_2 = \lambda_2^2 + 2\lambda_2^3 + 3\lambda_2^4$$

Substituting in the above NLPP, we obtain

$$\text{Max } Z = 3\lambda_1^2 + 12\lambda_1^3 + 27\lambda_1^4 + 2\lambda_2^2 + 8\lambda_2^3 + 18\lambda_2^4$$
$$\lambda_1^2 + 4\lambda_1^3 + 9\lambda_1^4 + \lambda_2^2 + 4\lambda_2^3 + 9\lambda_2^4 \leq 9$$
$$\lambda_1^2 + 2\lambda_1^3 + 3\lambda_1^4 + \lambda_2^2 + 2\lambda_2^3 + 3\lambda_2^4 \leq 3$$
$$\lambda_i^j \geq 0\ i = 1, 2, \quad j = 1, 2, 3, 4$$

$$\lambda_1^1 + \lambda_1^2 + \lambda_1^3 + \lambda_1^4 = 1$$
$$\lambda_2^1 + \lambda_2^2 + \lambda_2^3 + \lambda_2^4 = 1$$

With the natural restriction. We now apply simplex iteration.

Since, λ_2^1, λ_1^1 do not appear in 1st and 2nd constraint, we would get identity matrix even with artificial variable in the last 2 constraints. Thus, the table and simplex iteration is as follows:

BV	λ_1^2	λ_1^3	λ_1^4	λ_2^2	λ_2^3	λ_2^4	S_1	S_2	λ_1^1	λ_2^1	Solution
Z	–3	–12	–27 ↓	–2	–8	–18	0	0	0	0	0
S_1	1	4	9	1	4	9	1	0	0	0	9
S_2	1	2	3	1	2	3	0	1	0	0	3
$\leftarrow \lambda_1^1$	1	1	$\boxed{1}$	0	0	0	0	0	1	0	1
λ_2^1	0	0	0	1	1	1	0	0	0	1	1
Z	24	15	0	–2 ↓	–8	–18	0	0	27	0	27
S_1	–8	–5	0	1	4	9	1	0	–9	0	0
$\leftarrow S_2$	–2	–1	0	$\boxed{1}$	2	3	0	1	–3	0	0
λ_1^4	1	1	1	0	0	0	0	0	1	0	1
λ_2^1	0	0	0	1	1	1	0	0	0	1	1
Z	20	13	0	0	–4	–12 ↓	0	2	21	0	27
S_1	–6	–4	0	0	2	6	1	–1	–6	0	0
$\leftarrow \lambda_2^2$	–2	–1	0	1	2	$\boxed{3}$	0	1	–3	0	0
λ_1^4	1	1	1	0	0	0	0	0	1	0	1
λ_2^1	2	1	0	0	–1	–2	0	–1	3	1	1
Z	12	9	0	4	4	0	0	6	9	0	27
S_1	–2	–2	0	–2	–2	0	1	–3	0	0	0
λ_2^4	–2/3	–1/3	0	1/3	2/3	1	0	1/3	–1	0	0
λ_1^4	1	1	1	0	0	0	0	0	1	0	1
λ_2^1	2/3	1/3	0	2/3	1/3	0	0	–1/3	1	1	1

It is an optimal table. We have permitted λ_2^4 to enter basis even when λ_2^1 was present because one of them is zero. Thus, the solution is

$\lambda_1^1 = \lambda_1^2 = \lambda_1^3 = 0;\ \lambda_1^4 = 1;\ \lambda_2^1 = 1,\ \lambda_2^2 = \lambda_2^3 = \lambda_2^4 = 0,\ S_1 = S_2 = 0$

Thus, $x_1 = 3$; $x_2 = 0$; Optimal value = 27.

EXERCISE 10.2

1. Solve the following by separable programming.

$$\text{Max } Z = 2x_1 + 3x_2^4 + 4$$

Subject to

$$4x_1 + 2x_2^2 \leq 16$$
$$x_1, x_2 \geq 0$$

$$\left(\textbf{Ans: } x_1 = 0, x_2 = \frac{14}{15}, Z = 188\right)$$

2. Using separable programming to solve the following problem.

$$\text{Max } Z = 2x_1 + 3x_2^4 + 4$$

Subject to

$$3x_1^2 + 4x_2^2 \leq 36$$
$$x_1, x_2 \geq 0$$

(**Ans:** $x_1 = 0$, $x_2 = 3$, $Z = 247$)

3. Solve the following by separable programming.

$$\text{Minimize } Z = 2x_1 - 3x_2$$

Subject to

$$4x_1^2 + 9x_2^2 \leq 36$$
$$x_1, x_2 \geq 0$$

(**Ans:** $x_1 = 0$, $x_2 = 2$, $Z = -6$)

4. Min $Z = 4(x_1 - 6)^2 + (x_2 - 2)^2$

Subject to

$$3(x_1 + 1)^2 + 6x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

using separable programming.

(**Ans:** $x_1 = 1$, $x_2 = 0$, $Z = 104$)

Chapter 11

Networking

11.1 INTRODUCTION

A project is a job that has a definite starting and end point. In order to complete a job one has to perform a set of activities. These activities are performed in some set sequence.

For example, take a project of building a house. Starting point could be taken as the decision of building a house and end point as completing the house. The activities could be in broad sense (i) layout map, (ii) cleaning of the ground, (iii) digging of the foundation, (iv) filling the foundation, (v) erecting the walls, (vi) laying the roof, (vii) plastering, (viii) electric fitting, (ix) plumbering, (x) woodwork, (xi) finishing (xii) furnishing.

If we notice then we find that some of these activities are themselves big projects and consist of various activities like arranging for man, material, etc.

The problem here is to manage the large projects. The technique of scheduling is to identify the trouble spots, minimise them so as to complete the job in scheduled time or with minimum delays, cost, etc.

In managing a project three basic steps are involved, namely, **planning, scheduling, & controlling**.

In planning, we identify various activities, subactivities, that constitute the project, their completion time normally called duration of activity, requirement of resources, etc. and also the sequence in which the activities can follow.

In scheduling, we identify the time schedule of the project, which determines the start and finish time of each activity, relationship of each activity with other activities, critical activities which require special attention, if the job is to be completed in scheduled time.

In controlling, we control and monitor the project so that the project is completed in scheduled time. As we know that there is a difference between the actual performance in an activity and the time schedule. Here we control and minimise this difference.

11.2 NETWORK

Network diagram plays an important role in managing a project. This network depicts the details of a project. It is made in the following manner.

We first identify the activities and their interrelations i.e., it can follow on the completion of which activities and what activities can follow after its completion and also its time duration. An activity is denoted by an arrow with time duration over it, as

$$\xrightarrow{6}$$

The start and end point of an activity is denoted by a circle and is called node. These nodes can be regarded as a specific point of time. Thus,

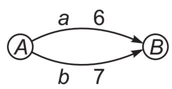
$$A \xrightarrow{\text{'}a\text{'}\ 6} B$$

Node A denotes a specific point of time at this time an activity 'a' of duration 6 can start and should end by a specific point of time denoted by node B.

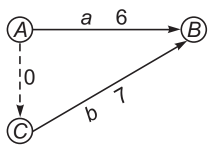
A network is a graphical representation of projects management and comprises arrows and nodes.

A network has the following conditions:

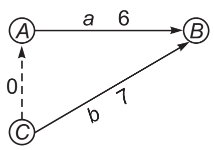
(i) Each activity is represented by one and only one arrow.
(ii) Each activity must start from a node and must end at a node.
(iii) More than one activity can start from a single node.
(iv) More than one activity may end at a single node.
(v) No two distinct activities should have same starting node as well as end node, i.e.,



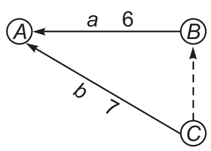
is not permissible. In case of the above situation, we introduce a dummy activity of '0' duration either in the beginning or at the end, i.e., we correct the above situation by adopting any of the following four.



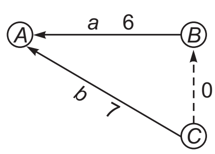
or,



or,

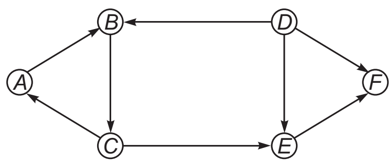


or,



Note that dummy activity is denoted by a broken arrow.

(vi) Arrows should not form a closed loop, i.e.,



is not permissible as (A)→(B)→(C)→(D) is a closed loop

(vii) Starting and end nodes should be unique.

Notations

(i) We shall write numerals inside nodes and call them as node number.

(ii) We shall use capital alphabets for activities.

(iii) We shall use ordered pair (i, j) to represent an activity from node 'i' to node 'j'.

Example 1: A project consists of activities A, B, C, D, E, F, G, H. The details are as follows:

(i) A, B, C start immediately.

(ii) D, E, F can start when A is completed.

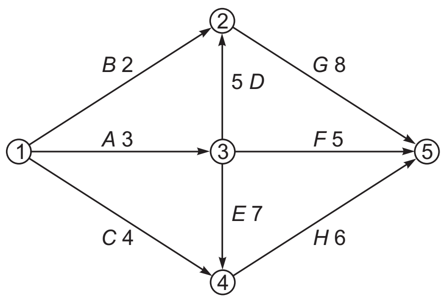
(iii) G can start after B and D are completed.

(iv) H can start after C and E are completed.

(v) F, G, H are terminal activities.

Construct a network if duration of activities are respectively 3, 2, 4, 5, 7, 5, 8, 6.

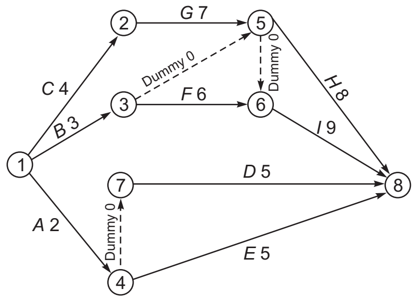
Solution:



Example 2: A project consists of tasks $A, B, C, D, E, F, G, H, I$ with durations 2, 3, 4, 5, 5, 6, 7, 8, 9, respectively. The details are as follows:

(i) A, B, C start simultaneously, (ii) D, E cannot start unless A is completed, (iii) F cannot start unless B is completed, (iv) G cannot start unless C is completed, (v) H cannot start unless B and G are completed, (vi) I cannot start unless F and G are completed, (vii) D, E, H, I are terminal activities. Construct the network.

Solution:



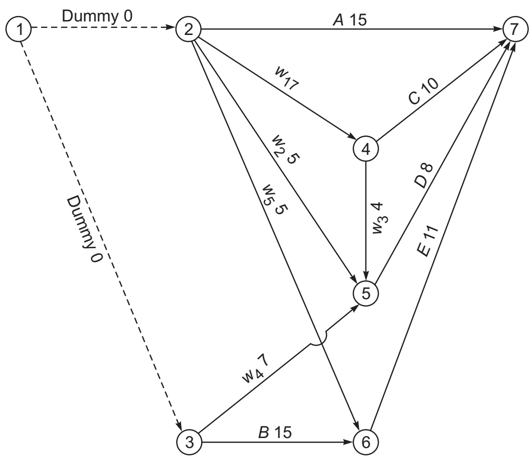
Example 3: A task consists of activities *A*, *B*, *C*, *D*, *E* of durations 15, 15, 10, 8, 11 days, respectively. The details are as follows:

(i) *A* and *B* can start simultaneously, (ii) *C* can start after 7 days and *D* after 5 days of starting *A*, (iii) *D* can start after 4 days of starting *C* and 7 days of starting *B* (iv) *E* can start after *A* is $\frac{1}{3}$ finished and *B* is completely finished, (v) *A*, *C*, *D*, *E* are terminal activities.

Construct the network.

Solution: This problem has a special requirement that one has to wait for a period of time to start an activity, so we consider waiting also an activity and denoted it by ω_i.

There are waiting activities ω_1(7 days) for *C*, ω_2(5 days) for *D*, ω_3(4 days), ω_4(7 days) for *D*, ω_5(5 days) for *E*. Thus, we have the following network.



11.3 CRITICAL PATH METHOD (CPM)

Many techniques exist for managing a project and to determine the time schedule for a project. Some of them are:

(i) Critical Path Method (CPM)
(ii) Programming, Evaluation and Review Technique (PERT)

(iii) Review Analysis of Multiple Project (RAMP), etc.

We shall take up only first two, namely, CPM and PERT. These methods use the network of the project. Here we first discuss about CPM.

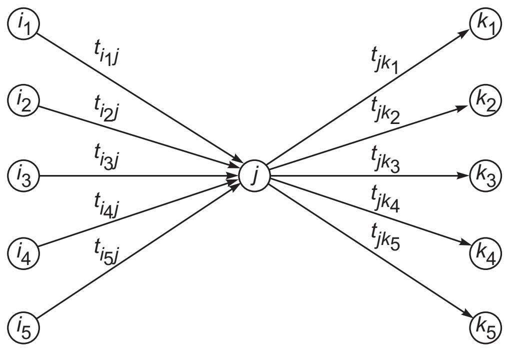
In a project the time estimate of the duration of activities may be deterministic or probabilistic. For example, time duration of erecting a wall is deterministic while time duration of conducting an experiment in a research lab is probabilistic. We normally use CPM in deterministic situations.

In order to determine the time duration of a project, it is wrong to consider it as the sum of time duration of all activities, but it is the sum of time duration of certain activities which are termed critical activities. Delay or earlier execution of these critical activities alone can affect the time duration of the project. The path formed by these critical activities from starting point to end point is called critical path.

Once the critical path is determined one can answer certain natural questions like (i) In how much time the project would be completed? (ii) If an activity is delayed, will it affect the time duration of the project, (iii) how economically the time duration be reduced, etc.

For all this, the first task is to determine the critical activities, hence critical path.

In order to do this, we proceed as follows. Let t_{ij} denote the time duration of the activity (i, j); ES_j denote the Earlier Start time at the node j (the point of time j) and LC_j, the latest completion time at the node j (the point of time j).



ES_j, the earliest start time, at node j, means the earliest time at which activities at this node can start. Naturally, it is only when all the activies ending at this node are over.

Therefore,

$$ES_j = \operatorname*{Max}_i \{ES_i + t_{ij}\}, \text{ and } ES_1 = 0$$

for all incoming activities (i, j), $i = i_1, i_2, i_3, \ldots$

This is denoted and written in a rectangle near the node j.

LC_j, the latest completion time, at node j, means the latest time by which all the activities ending this node must be completed. Obviously, it is only when all the activities starting at this node are ready to start.

Therefore, starting from the last node l, we have

$$ES_l = LC_l$$

$$LC_j = \operatorname*{Min}_k \{-t_{jk} + L_{Ck}\}$$

for all outgoing activities (j, k), $k = k_1, k_2, \ldots$. This is written in a triangle near the node j.

Obviously, at each node $LC_j \geq ES_j$. Whenever, $LC_j = ES_j$ this means as soon as all activities ending at node j are over activities starting at this node can start.

If it happens at both starting and end node of an activity and also, if the difference is equal to the time duration of this activity then this activity is known as critical activity which clearly means that the moment this activity is over, activities following it will start.

Thus, an activity (i, j) is critical, if

$$ES_i = LC_i$$
$$ES_j = LC_j$$
$$ES_j - ES_i = LC_j - LC_i = t_{ij}$$

Path determined by critical activities is the critical path. By definition $ES_l = LC_l$ and this is the completion time of the total project. A critical activity is denoted by a thick or double arrow.

Now we will define some more terms which will be used later for optimal scheduling of the project.

Total float of activity (i, j) TF_{ij} is the largest time schedule to complete the activity minus the time required to complete the activity i.e.

$$TF_{ij} = (LC_j - ES_i) - t_{ij}$$

Free float of activity FF_{ij} is the excess of available time over the time required to complete the activity (assuming that all activities start as early as possible) i.e.,

$$FF_{ij} = (ES_j - ES_i) - t_{ij}$$

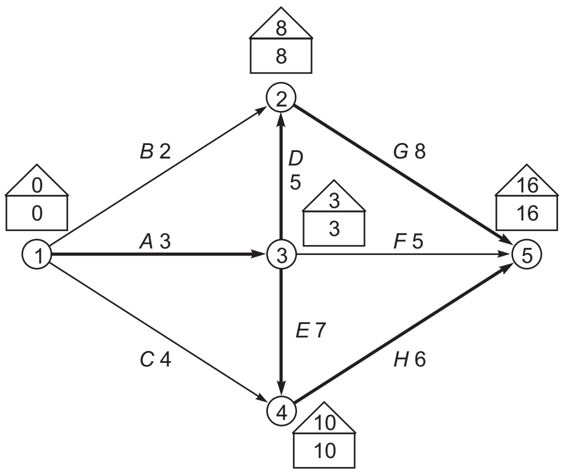
An activity is said to critical activity *iff*

$$FF_{ij} = TF_{ij} = 0$$

But non-critical activity may also have a zero free float.

We shall now illustrate the determination of critical path by example.

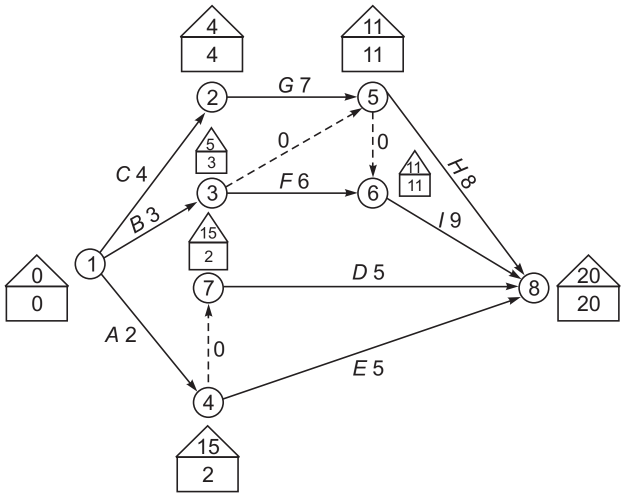
Example 1: Take the network of the project in example 1 and find the critical activities and path.



Here (1, 3), (3, 2), (3, 4), (2, 5), (4, 5) are critical activities which give two critical paths 1, 3, 2, 5 and 1, 3, 4, 5.

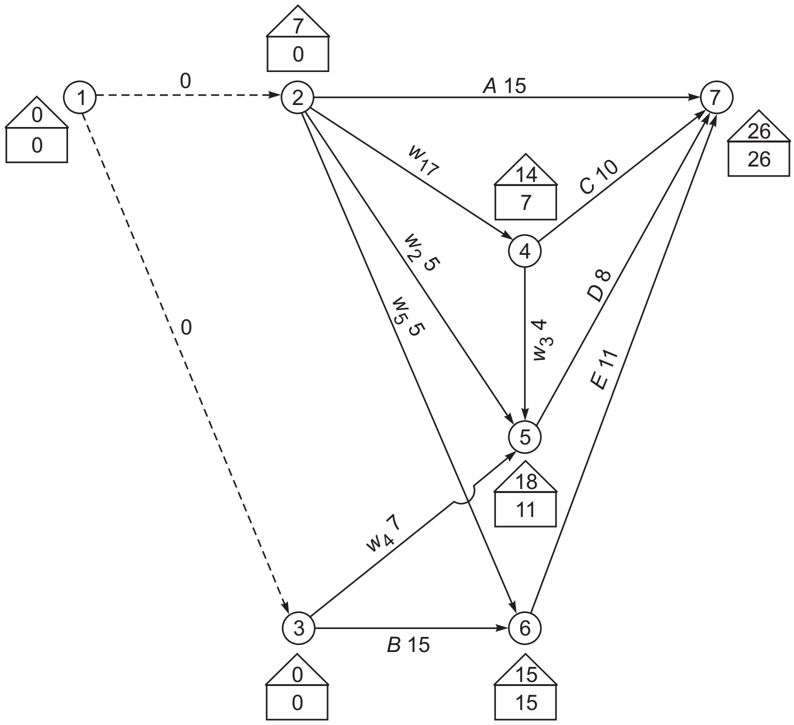
Activities (1, 2), (1, 4) and (3, 5) are not critical activities. Duration of the project is 16.

Example 2: Find the critical activities and path for the project in example 2.



Critical Activities (1, 2), (2, 5), (5, 6), (6, 8).
Critical Path (1, 2, 5, 6, 8).
Duration of Project 20.

Example 3: Find the critical activities, critical path, time duration of the project in example 3.



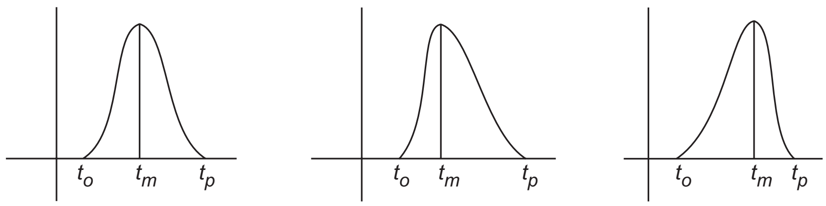
Critical Activities (1, 3) (3, 6), (6, 7).
Critical Path (1, 3, 6, 7).
Duration of Project 26 days.

11.4 PROGRAMMING EVALUATION AND REVIEW TECHNIQUE (PERT)

If the time duration of an activity is probabilistic, we shall use PERT. In this case time duration is taken to be a random variable and we formulate the probabilistic model of the project. Let T_{ij} be the time duration of the activity (i, j).

Let t_o denote the optimistic time, i.e., time required if execution goes well; t_p denote the pessimistic time, i.e., time required if execution goes badly; and t_m denote the most likely time which is determined by most likelihood function.

By experiments, it has been found that the probability density function of the random variable T_{ij} has a graph of either form, given in figure below.



In general, normal distribution function is used to approximate the probability density function. But here, since end points are finite and non-negative and also the shapes are not symmetrical, we cannot use normal distribution, but Beta distribution fulfils the requirements. So we use Beta distribution. The probability density function of Beta distribution is

$$f(t) = \begin{cases} kt^{p-1}(1-t)^{q-1} & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Where, $k = \dfrac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)}$, p and q have their standard meaning.

By adjusting p and q, $f(t)$ can be made to take any of the shapes. Assuming that the mid-point $\dfrac{t_o + t_p}{2}$ weighs half as much as most likely point t_m, the expected value of T_{ij} is

$$E(T_{ij}) = \frac{\dfrac{t_o + t_p}{2} + 2t_m}{3} = \frac{t_o + 4t_m + t_p}{6}$$

To ensure that 90% data falls in the interval $[t_o, t_p]$, in view of Chebyshev's inequality, we take the variance V_{ij}^2 of T_{ij}, given by

$$V_{ij}^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

Thus, in PERT, we first calculate $E(T_{ij})$ for all activities, with this, then we determine Critical Path as in CPM. Let T denote the Shortest Completion time. Then T is defined by

$$T = E(T) = \Sigma E(T_{ij})$$

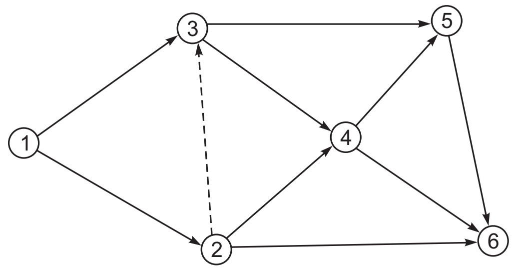
for Critical Activities (i, j)

and, $$V_T^2 = \Sigma V_{ij}^2$$

for Critical Activities (i, j)

We now illustrate the determination of Critical Path by PERT.

Example 1: A project is represented by the following network and has the data given by the following table



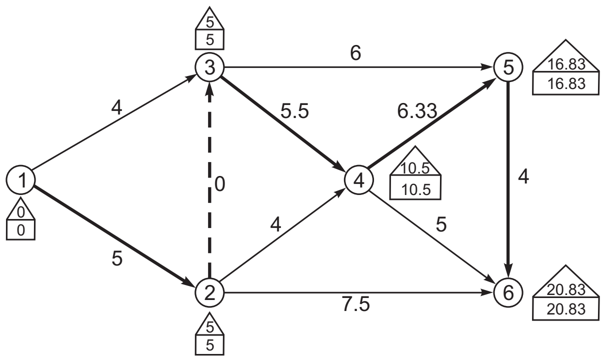
Activity	t_o	t_p	t_m
(1, 2)	2	8	5
(1, 3)	1	7	4
(2, 3)	0	0	0
(3, 4)	3	10	5
(3, 5)	3	9	6
(2, 4)	2	6	4
(2, 6)	5	12	7
(4, 5)	4	10	6
(4, 6)	2	8	5
(5, 6)	2	6	4

Find the critical path and expected time duration and also variance of the time duration.

Solution: We first Calculate $E(T_{ij})$ and V_{ij}^2 for each activity.

Activity	t_o	t_p	t_m	$E(T_{ij})$	V_{ij}^2
(1, 2)	2	8	5	5	1
(1, 3)	1	7	4	4	1
(2, 3)	0	0	0	0	0
(3, 4)	3	10	5	5.5	49/36
(3, 5)	3	9	6	6	1
(2, 4)	2	6	4	4	16/36
(2, 6)	5	12	7	7.5	49/36
(4, 5)	4	10	6	6.33	1
(4, 6)	2	8	5	5	1
(5, 6)	2	6	4	4	16/36

Taking $E(T_{ij})$ as duration of activity (i, j) we determine Critical Path

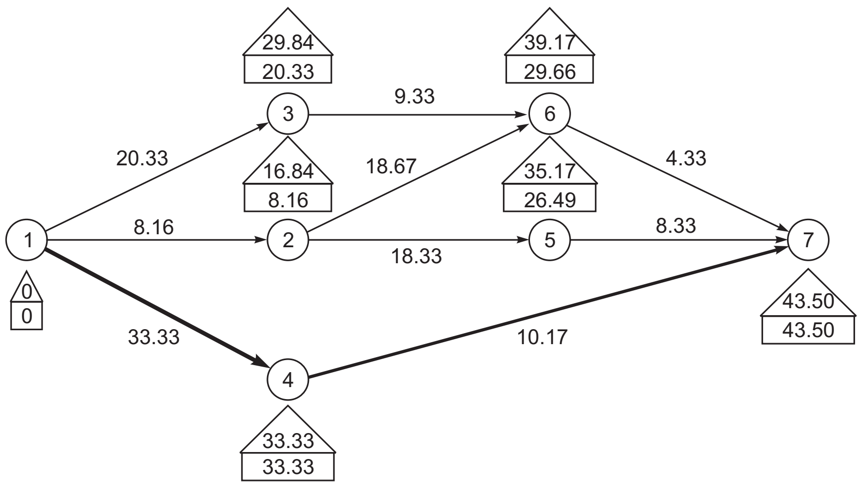


Therefore, critical path is (1, 2, 3, 4, 5, 6)

$$E(T) = 20.83$$

$$V_T^2 = 137/36$$

Example 2: Consider a project whose details are given below. Make the network. Find the critical path, variance of expected duration of project.



Activity	t_o	t_p	t_m	$E(T_{ij})$	V_{ij}^2
(1, 2)	6	11	8	8.16	25/36
(1, 3)	19	23	20	20.33	16/36
(1, 4)	27	41	33	33.33	196/36
(2, 5)	17	21	18	18.33	16/36
(2, 6)	16	26	20	18.67	100/36
(3, 6)	7	13	9	9.33	1
(4, 7)	8	13	10	10.17	25/36
(5, 7)	8	10	8	8.33	4/36
(6, 7)	4	6	4	4.33	4/36

Solution: $E(T_{ij})$ and V_{ij}^2 for all activities are calculated and written in the given table based on which critical path is determined and also shown in the given network diagramme above.

Critical Path (1, 4, 7)

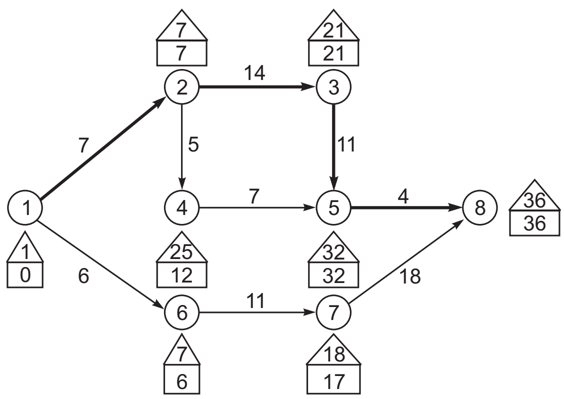
Expected Time Schedule = 43.50

Variance of Time Schedule = $\dfrac{221}{36}$ = 6.14

Example 3: Consider a project whose details are given below. (a) Make the network and (b) Find the critical path, variance of expected duration of project.

Activity	t_o	t_p	t_m	$E(T_{ij})$	V_{ij}^2
(1, 2)	3	15	6	7	4
(1, 6)	2	14	5	6	4
(2, 3)	6	30	12	14	16
(2, 4)	2	8	5	5	1
(3, 5)	5	17	11	11	4
(4, 5)	3	15	6	7	4
(6, 7)	3	27	9	11	16
(5, 8)	1	7	4	4	1
(7, 8)	4	28	19	18	16

Solution: The $E(T_{ij})$ and V_{ij}^2 are calculated and written above in the right hand side of the given table.



Critical Path: 1, 2, 3, 5, 8

Expected Time duration: 36

Variance of random variable $T = 25$

We can also answer many naturally arising questions. One of them is: Can we finish all the activities ending at the node i in a given number of days?

This question we shall answer in terms of the probability of finishing the task till the node i. To do this we proceed as follows:

The earliest occurrence time of event i, i.e., the earliest start time at the node i be denoted by Z_i which is a random variable. Therefore, $E(Z_i) = ES_i$. Let $V_{Z_i}^2$ be the variance of Z_i which is the sum of variances of activities leading to node i. If there are more than one path leading to node i, then $E(Z_i)$ is taken one having largest value. In case of a tie we take the one having largest $V_{Z_i}^2$ as it shows greater uncertainty.

The random variable Z_i is the sum of many independent variables T_{ij}. Hence, by central limit theorem, the random variable

$$Z = \frac{Z_i - E(Z_i)}{V_{Z_i}}$$

has the standard normal distribution $N(0, 1)$. Let ST_i be the scheduled time for earliest occurrence time at node i, i.e., event i. Let ST_i be given, say α. Then the earliest occurrence time of i-th event is

$$P(Z_i \leq \alpha) = P\left(\frac{Z_i - E(Z_i)}{V_{Z_i}} \leq \frac{\alpha - E(Z_i)}{V_{Z_i}}\right)$$

$$= P\left(Z \leq \frac{\alpha - E(Z_i)}{V_{Z_i}}\right)$$

Since $E(Z_i)$, V_{Z_i} are known, we know the value of $\dfrac{\alpha - E(Z_i)}{V_{Z_i}}$. Hence, by normal distribution table, the probability can be calculated.

Example 4: Find the probability of completing the job (i) at the node 4 in 12 days (ii) at the node 5 in 15 days, (iii) complete job in 23 days in example 7.

Solution:

(i) $$P(Z_4 \leq 12) = P\left(Z \leq \frac{12 - E(Z_4)}{V_{Z_4}}\right)$$

$$= P\left(Z \leq \frac{12 - 10.5}{\sqrt{85}/6}\right) = P(Z \leq .98)$$

$$= .8365$$

(ii) $$P(Z_5 \leq 15) = P\left(Z \leq \frac{15 - 16.83}{11/6}\right) = P(Z \leq -.999)$$

$= .1587$

(iii) $P(Z_6 = T \leq 23) = P\left(Z \leq \dfrac{23 - 20.83}{\sqrt{137/6}}\right) = P(Z \leq 1.11)$

$= .8665$

Example 5: Find the probability of completing the project in example 8 in 42 days.

Solution:

$$P(T \leq 42) = P\left(Z \leq \frac{42 - E(T)}{V_T}\right)$$
$$= P\left(Z \leq \frac{42 - 43.50}{\sqrt{221/6}}\right)$$
$$= P(Z \leq -.606)$$
$$= .2733$$

Example 6: Find the probability of completing the job of example 9 in 37.5 days.

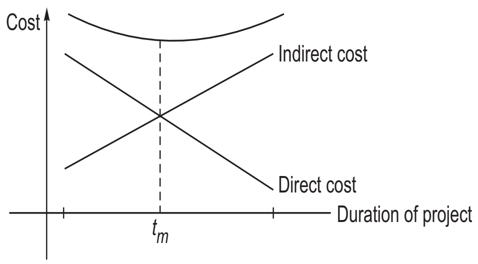
Solution:

$$P(T \leq 37.5) = P\left(T \leq \frac{37.5 - 36}{5}\right)$$
$$= P(T \leq .3)$$
$$= .6179$$

11.5 OPTIMUM SCHEDULING BY CPM

We now proceed to answer another question. How economically one can reduce the project duration?

Two types of costs are involved in a project. One is direct cost, like manpower, material and other resources and the other is indirect cost like supervision, etc. The direct cost goes on increasing as the time reduces and indirect cost decreases as time decreases. The total cost is the sum of these two costs.



We shall deal with the direct costs only as indirect costs can be computed as it would be dependent only on the duration. These direct costs would now be referred to as cost.

If duration is reduced, that can be done by employing more resources, the cost would increase. But there is a limit to this reduction. Duration which cannot be further reduced would be called "Crash duration" and the other which is normally used would be termed "Normal Duration". The cost of completing an activity in Normal duration would be called "Normal Cost" and in crash duration, it would be called "Crash Cost" and would be denoted by T_C, T_N, C_N, C_C, respectively.

Thus, in order to get an optimum scheduling, we should know about T_C, T_N, C_N, C_C of each activity.

Our procedure would follow in the following manner.

We first define the following:

Let (i, j) be an activity.

(i) Let T_N, T_C, C_N, C_C be Normal time, Crash Time, Normal Cost and Crash Cost, respectively. The ratio $\dfrac{C_C - C_N}{T_n - T_c}$ is called the slope of the activity. It is the cost per unit time. If time is reduced by one unit, the cost would go up by this amount (slope).

(ii) Total Float of activity (i, j), denoted by TF_{ij}, is defined by

$$TF_{ij} = (LC_j - ES_i) - t_{ij}$$

i.e., largest time available to complete this activity minus time required to complete it, and,

(iii) Free Float of activity (i, j), denoted by FF_{ij}, is defined by

$$FF_{ij} = (ES_j - ES_i) - t_{ij}$$

i.e., excess of time available over time required, assuming that all activities start as early as possible.

For a critical activity $LC_j = ES_j$, therefore $TF_{ij} = FF_{ij}$.

Also, for a critical activity $ES_j - ES_j = t_{ij}$. So $FF_{ij} = TF_{ij} = 0$, it is obvious that $TF_{ij} = 0 \Leftrightarrow (i, j)$ is a critical activity. But $FF_{ij} = 0 \nRightarrow (i, j)$ is a critical activity, and (i, j) is a critical activity $\Rightarrow FF_{ij} = 0$.

Proof:

(i) (i, j) is a critical activity. So

$$LC_j = ES_j$$
$$LC_i = ES_i$$
$$LC_j - LC_i = ES_j - ES_i = t_{ij}$$

Hence,

$$TF_{ij} = 0 \quad \text{and} \quad FF_{ij} = 0$$

(ii) $FF_{ij} = 0 \Rightarrow ES_j - ES_i = t_{ij}$

but $LC_j \neq ES_j$ and or $ES_i \neq LC_i$.

(iii) $TF_{ij} = 0 \Rightarrow LC_j = ES_i + t_{ij}$

But $$ES_j = \operatorname*{Max}_i \{ES_i + t_{ij}\}$$

or, $$ES_i + t_{ij} \leq ES_j \;\; \forall \;\; i.$$

or, $\quad LC_j \le ES_j.$

But $\quad ES_j \le LC_j$

$\therefore \quad LC_j = ES_j$

Now, similarly $\quad ES_i = LC_j - t_{ij} \ge LC_i$

as $\quad LC_i = \underset{j}{\text{Min}} \ (LC_j - t_{ij})$

Also $\quad ES_i \le LC_i$

$\therefore \quad LC_i = ES_i$

Now $\quad LC_j - LC_i = ES_j - ES_i$

$$= ES_i + t_{ij} - ES_i = t_{ij}$$

Hence, (i, j) is a critical activity.

We know that duration of the project is the sum of the duration of critical activities. Thus, in order to reduce the duration of the project we have to reduce the duration of critical activities along a critical path. If there are more than one critical paths, then duration of critical activities along critical paths should be reduced by the same amount.

Which critical activity should be reduced? We select that critical activity which has least slope, as it would increase the cost the least. Of course the activity should not be at its crash duration.

By what amount, we can reduce the activity at one stroke? There are two methods to do this.

(i) Stepping step procedure

(ii) *FF* limit process.

(i) *Stepping Step Procedure:* In this method, we reduce the duration of activities by 1 unit.

We reduce the duration of activity by 1 unit and recalculate *LC*, *ES* and critical path, and increase the cost by its slope(s), and mark the new critical paths, if developed. Note that earlier critical paths would remain critical.

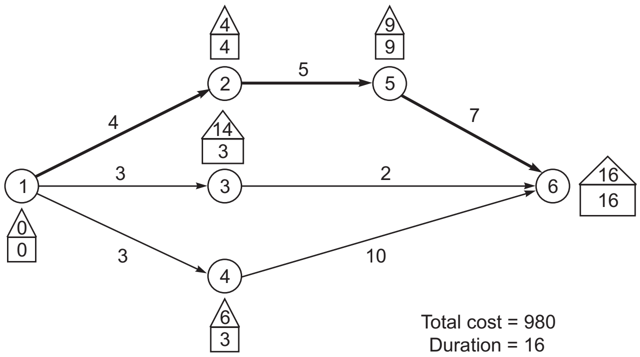
The steps would then be repeated till, either required duration is reached or along a critical path all critical activities have reached to their crash duration.

We illustrate this concept by examples.

Example 1: The crash and normal duration and cost are given below. Calculate the different minimum cost schedule by compressing the project duration by one unit at a time.

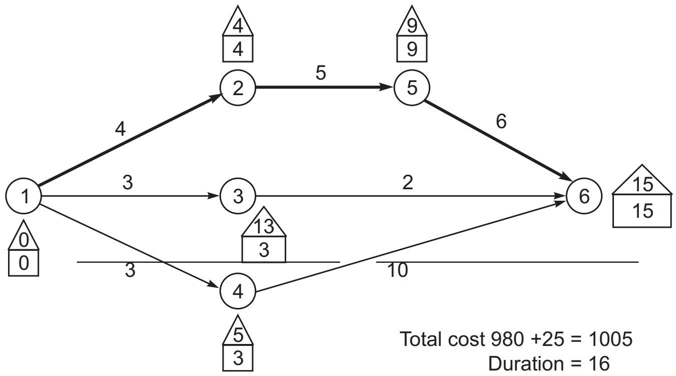
Activity	*Normal*		*Crash*		*Slope*
	Duration	*Cost*	*Duration*	*Cost*	
(1, 2)	4	140	1	230	30
(1, 3)	3	140	1	160	10
(1, 4)	3	200	1	240	20
(2, 5)	5	100	2	200	33.33
(3, 6)	2	50	1	80	30
(4, 6)	10	150	9	180	30
(5, 6)	7	200	5	250	25

Solution: Slope for each activity is calculated and written in the given Table

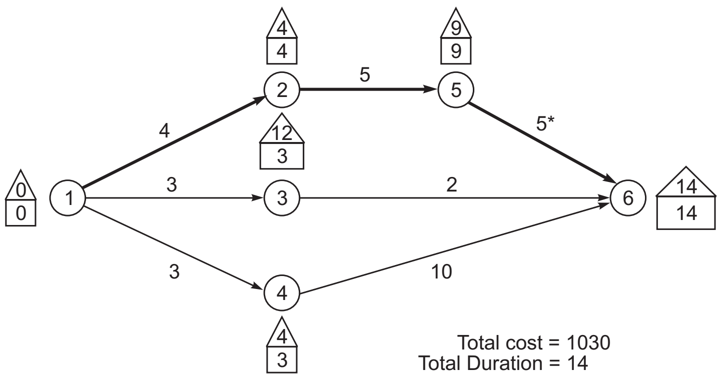


Here we have to reduce the duration of the activities (1, 2), (2, 5), (5, 6).

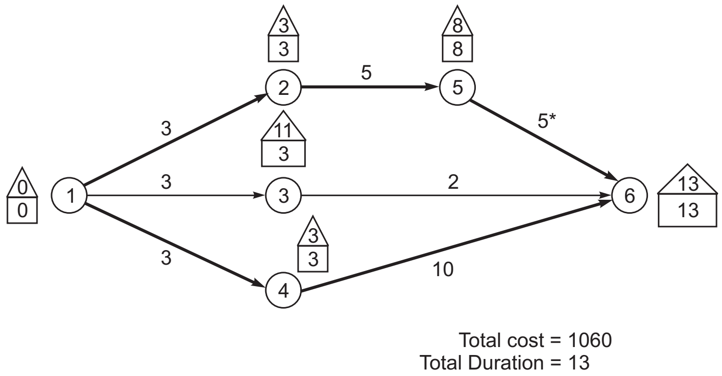
We choose (5, 6) as its slope 25 is minimum. We compress the activity (5, 6) by 1 unit and again solve it. We get



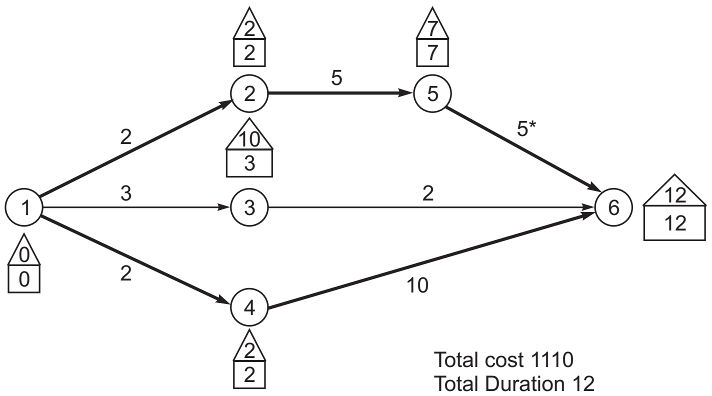
Again, no more critical path. We compress (5, 6) by 1 unit. We get



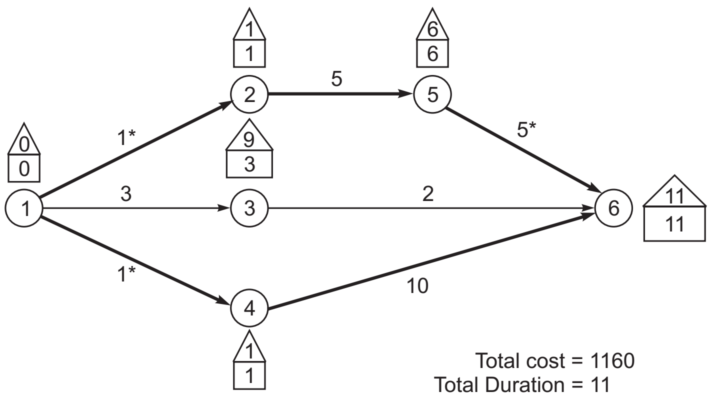
Now activity (5, 6) has reached to its crash value, so we put * over duration, signifying that it cannot be further reduced. No more critical path. We now compress (1, 2) as its slope 30 is less than the slope of (2, 5). We compress it by 1 unit. We get.



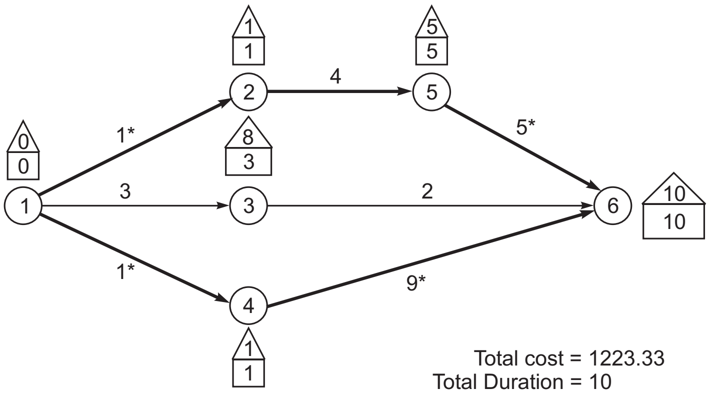
Now one more critical path has developed, so we have to compress activities on both paths by same amount. On path economical activity is (1, 2) while on the new path, it is (1, 4) so we compress (1, 2) and (1, 4) by 1 unit each. We get



No more new path. Again we compress both (1, 2) and (1, 4) by 1 unit each. These two activities would reach to their crash values. We get



We have no choice now we compress (2, 5) and (4, 6) by 1 unit each and get



Here activity (4, 6) has also reached to its crash limit. So on the path 1, 4, 6 all activities have reached to their crash limits. So no more compression is possible. Therefore, the cost schedule is

Duration	16	15	14	13	12	11	10
Cost	980	1005	1030	1060	1110	1150	1223.33

Now one knows the cost schedule. On looking his needs and resources, can decide the duration which he can afford.

This is known as stepping step procedure wherein we compress activities by 1 unit at a time. We now take up another method.

(ii) *FF limit procedure:* In this method, procedure is same as in stepping step method except that we compress activities by more than 1 unit at a time.

How to decide by how much amount an activity is to be compressed?

It is clear that we cannot compress an activity less than its crash duration. For this we coin a word

$$\text{Crash limit} = \text{Present duration} - \text{Crash duration} = C$$

Thus, we cannot compress an activity by an amount less than C, i.e., its Crash limit. Secondly we have to take care that in between no new critical path is developed. For this purpose, we use FF, the free float as follows:

We know that FF of critical activities is zero and of non-critical activities it is greater than or equal to zero. So we calculate FF for each activity or at least of non-critical activity.

We then see by compressing the said activity by a unit, which FF gets changed. This can be done by two methods. One by compressing the activity by 1 unit and doing the calculation and then noting or even mentally. But in a large network, it is not possible to do mentally, so we compress by 1 unit and note it.

Therefore, the compression cannot be done by an amount which is less than the minimum of these FF's which is called Free Float Limit, which is defined as:

$$\text{Free Float limit} = F = \underset{(i,j)}{\text{Min}} \; \{FF_{ij} \,|\, (i, j) \text{ is an activity whose } FF \text{ is affected.}\}$$

Thus, the compression has a limit at one step known as compression limit, defined as

$$\text{Compression limit} = \text{Min}\ (F, C)$$

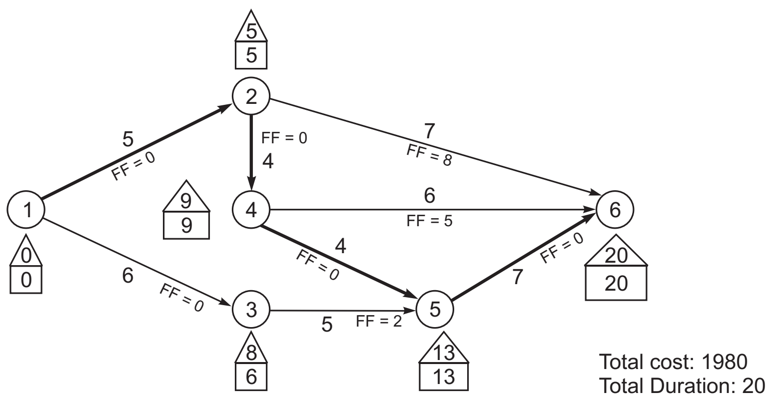
Therefore, we can compress an activity at one step by an amount less than or equal to the compression limit. If FF of each activity has reached to zero and further compression is possible because $C \neq 0$ then we can compress it by an amount C. Rest is as in stepping step procedure.

We shall now illustrate it by examples.

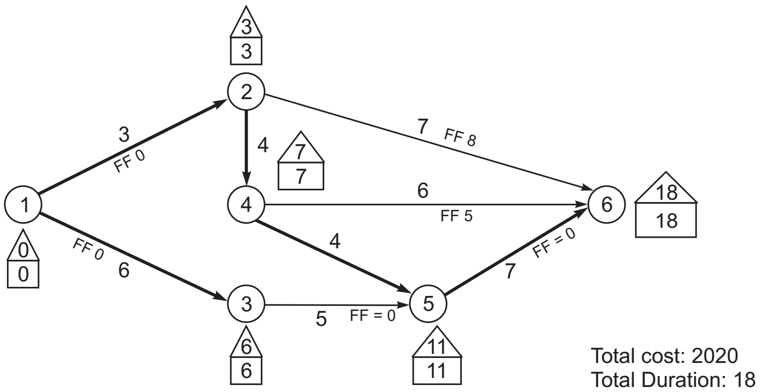
Example 2: Find by FF limit method, the different minimum cost schedule of the project whose details are given below. Also find the minimum total cost schedule (including indirect costs) if the indirect cost is Rs. 40 per day.

Activity	*Normal*		*Crash*		*Slope*
	Duration	*Cost*	*Duration*	*Cost*	
(1, 2)	5	200	2	260	20
(1, 3)	6	220	3	310	30
(2, 4)	4	310	2	390	40
(2, 6)	7	250	4	400	50
(3, 5)	5	350	3	390	20
(4, 5)	4	150	2	230	40
(4, 6)	6	300	3	420	40
(5, 6)	7	200	4	290	30

Solution: Slope for each activity calculated and written in the given table above.

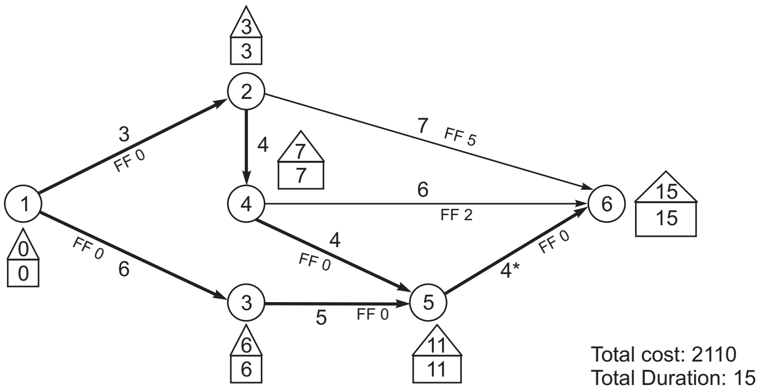


Out of the activities (1, 2), (2, 4), (4, 5) and (5, 6), we decide to compress the activity (1, 2) as it has the least slope 20. We notice that by this compression ES_2, ES_4, ES_5, ES_6 get changed so no effect on $FF_{2,6}$, $FF_{4,6}$, $FF_{1,3}$. But $FF_{3,5}$ gets affected as ES_3 has not changed. So $F = 2$ and $C = 3$. Therefore, compression limit is 2. So we compress (1, 2) by 2 units and obtain

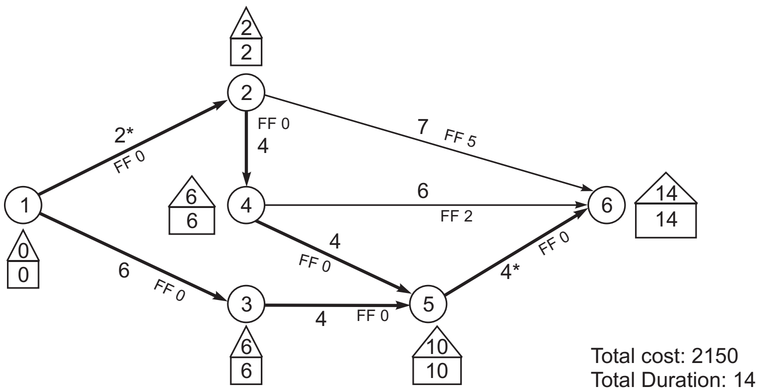


A new path 1, 3, 5, 6 has also developed. On the earlier path most economic activity to compress is (1, 2) with slope 20 and new path (3, 5) with slope 20. Thus, it would raise the cost by 40 per unit time. But activity (5, 6) common to both paths has slope 30. So it is economical. So we decide to compress it.

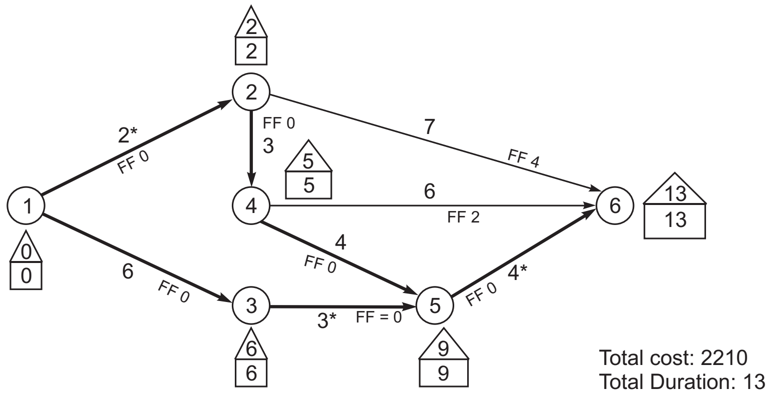
Here, $C = 3$ and both $FF_{4,6}$, $F_{2,6}$ get effected so $F = 5$. Hence, compression limit is 3.



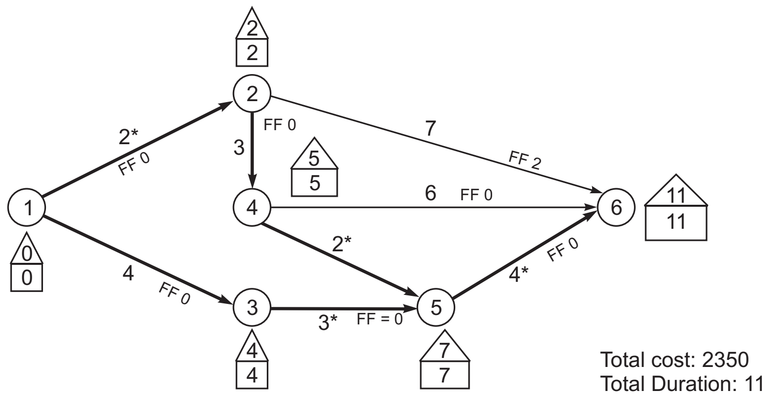
No new path has developed. Most economical would be to compress (1, 2) and (3, 5). It does not affect any non-zero *FF*. So we compress it minimum of the two crash limits, i.e., by 1, we obtain



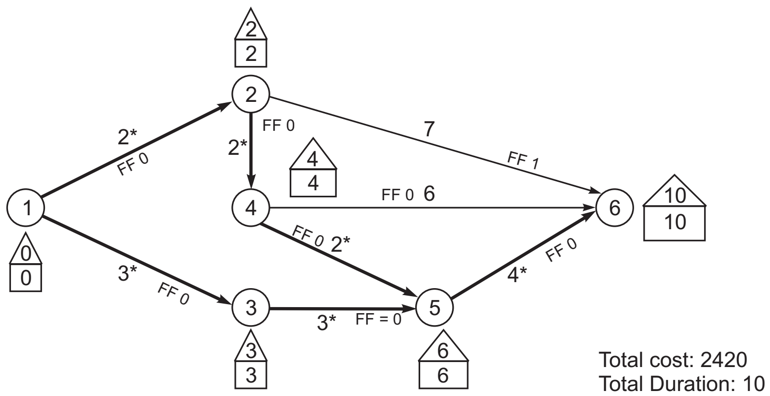
Most economical would be (2, 4) (or (4, 5)) and (3, 5). Since (3, 5) can be compressed only by 1 unit. So we do not go for further calculation, compress them by 1 unit. We get



Most economical would be to compress (1, 3) and (4, 5). Here crash limit is 2 and F = 2. So Compression limit. Hence, we compress by 2 units.



One more path has developed, namely, (1, 2, 4, 6). Most economical and also no choice compressing (1, 3) by 1 and (2, 4) by 1 as it would take care of both. We get



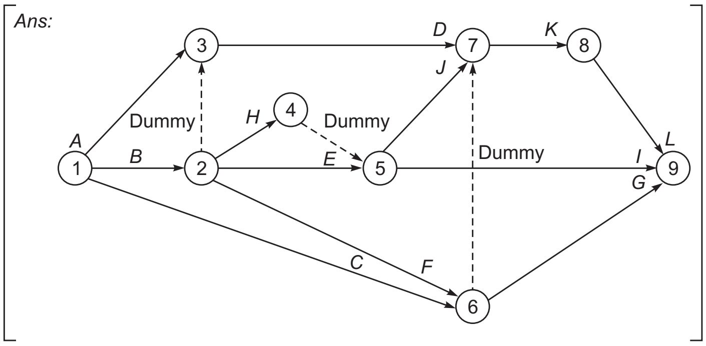
No more compression is possible. Along two critical paths all activities have reached to their crash values. Cost schedule is

Duration	20	18	15	14	13	11	10
Direct Cost	1980	2020	2110	2150	2210	2350	2420
Indirect Cost	800	720	600	560	520	440	400
Total	2780	2740	2710	2710	2730	2790	2820

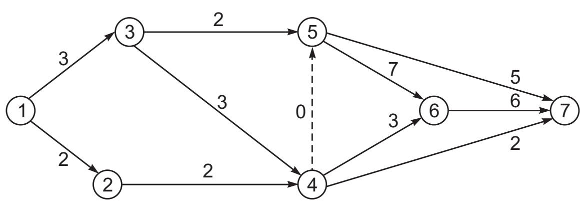
From the table we find that most economical schedule would be of either 15 or 14 days and the cost would be 2710.

EXERCISE 11.1

1. A project consists activities *A* to *L* with the following details.
   (a) *A*, *B* and *C*, the first activities of the project, can start simultaneously.
   (b) *A* and *B* precede *D*.
   (c) *B* precedes *E*, *F* and *H*.
   (d) *F* and *C* precede *G*.
   (e) *E* and *H* precede *I* and *J*.
   (f) *C*, *D*, *F* and *J* precede *K*.
   (g) *K* precedes *L*.
   (h) *I*, *G* and *L* are the terminal activities.
   Construct the network.



2. Determine the critical path for the following network as per details given (duration of the activities are given in days).



(Critical path: 1 → 3 → 4 → 5 → 6 → 7 with 19 days)

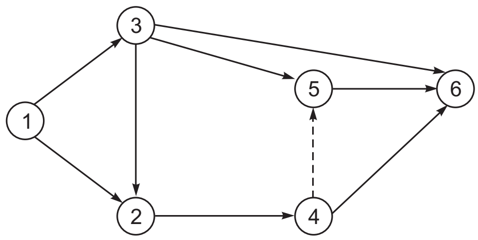
3. Consider the following project as per details given below:

Activity	*Duration (in days)*		
	t_o	t_m	t_p
(1, 2)	3	5	7
(1, 3)	4	6	8
(2, 3)	1	3	5
(2, 4)	5	8	11
(3, 5)	1	2	3
(3, 6)	9	11	13
(4, 5)	0	0	0
(4, 6)	1	1	1
(5, 6)	10	12	14

(a) Draw the network.
(b) Find the critical path with project duration.
(c) Find the probability that the project will be completed in 26 days.

Ans.

(a) Network:



(b) Critical path: $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$
Project duration: 25 days

(c) 0.7693

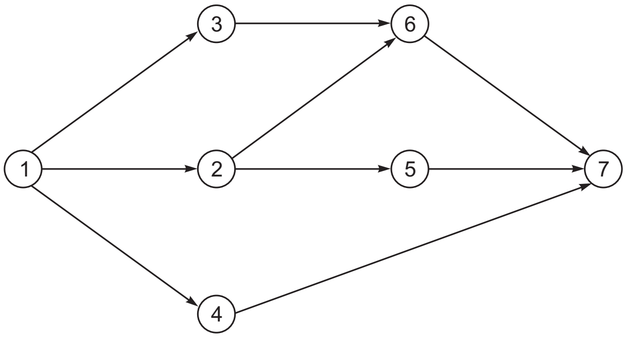
4. Consider the following project as per details given below:

Activity	*(Duration in day)*		
	t_o	t_p	t_m
(1, 2)	2	8	5
(1, 3)	16	20	18
(1, 4)	25	35	30
(2, 5)	19	21	20
(2, 6)	16	24	20
(3, 6)	9	11	7
(4, 7)	10	12	8
(5, 7)	8	12	10
(6, 7)	4	8	6

(a) Draw the network.
(b) Find the critical path of the project.
(c) Find the probability of completing the project in 39 days.

Ans:

(a) Network



(b) Critical path : 1 → 4 → 7
(c) 0.50

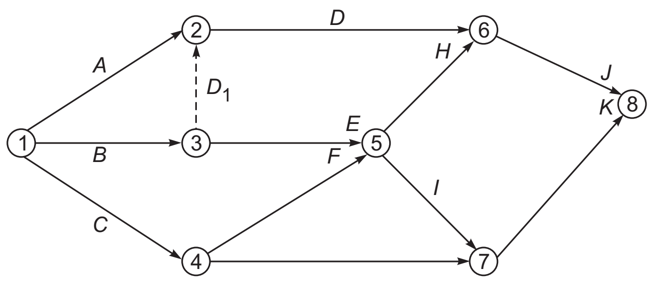
5. Consider the project whose details are given below:

Activity	*Predecessor(s)*	*Duration (weeks)*		
		t_o	t_m	t_p
A	–	6	7	8
B	–	1	2	9
C	–	1	4	7
D	A	1	2	3
E	A, B	1	2	9
F	C	1	5	9
G	C	2	2	8
H	E, F	4	4	4
I	E, F	4	4	10
J	D, H	2	5	14
K	I, G	2	2	8

(a) Construct the project network.
(b) Find the expected duration and variance for each activity.
(c) Find the critical path and the expected project completion time.
(d) What is the probability that the project will be completed in maximum 25 weeks?
(e) If the probability of completing the project is 0.84, find the expected project completion time.

Ans:

(a) Network.



(b)

Activity	*Mean duration*	*Variance*
A	7	1/9
B	3	16/9
C	4	1.00
D	2	1/9
E	3	16/9
F	5	16/9
G	3	1.00
H	4	0.00
I	5	1.00
J	6	4.00
K	3	1.00

(c) Critical path $A \to D_1 \to E \to H \to$ I
Project Completion time = 20 weeks.

(d) $P(T \leq 25) = 0.9803$

(e) 23 weeks.

6. The activities involved in Shyam Garments Ltd. are listed with their time estimates in the following table.

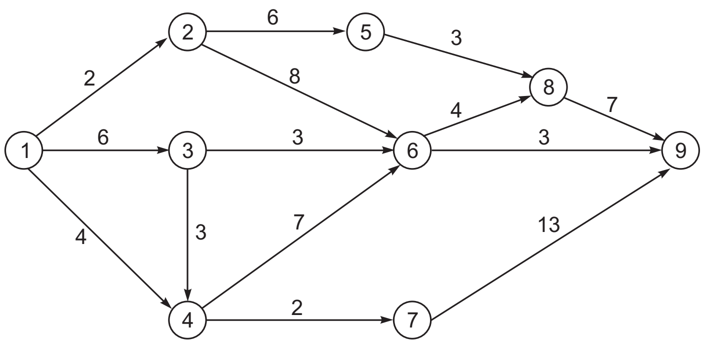
Activity	*Description*	*Immediate Predecessor (s)*	*Duration (days)*
A	Forecast sales volume	–	12
B	Study competitive market	–	9
C	Design item and facilities	A	7
D	Prepare production plan	C	5
E	Estimate Cost of production	D	4
F	Set sales price	B, E	3
G	Prepare Budget	F	16

(a) Construct the network.
(b) Find the critical path.
(c) What is the project completion time?

Ans:

(b) Critical path $A \to C \to D \to E \to F \to G$
(c) 47 days.

7. Consider the network whose details are given below. (Duration of each activity is in months.)



(a) Determine the critical path and project completion time.
(b) Calculate the total floats and free floats for the non-critical activities.

Ans:

(a) Critical path: $1 \to 3 \to 4 \to 6 \to 8 \to 9$
Project completion time: 27 months.

(b)

Activity	(1,2)	(1,3)	(1,4)	(2,5)	(2,6)	(3,4)	(3,6)	(4,6)	(4,7)	(5,8)	(6,8)	(6,9)	(7,9)	(8,9)
Total floats	6	0	5	7	6	0	7	0	3	7	0	8	3	0
Free floats	0	0	5	0	6	0	7	0	0	7	0	8	3	0

8. Consider the data of a project as per details given below:

Activity	*Normal*		*Crash*	
	Time (weeks)	*Cost (Rs.)*	*Time (weeks)*	*Cost (Rs.)*
(1, 2)	13	700	9	900
(1, 3)	5	400	4	460
(1, 4)	7	600	4	810
(2, 5)	12	800	11	865
(3, 2)	6	900	4	1130
(3, 4)	5	1000	3	1180
(4, 5)	9	1500	6	1800

Find the optimal crashed project completion time, if the indirect cost per week is Rs. 160. (**Ans:** Crashed project completion time is 21 weeks with total cost of Rs. 9,535, Critical paths are $1 \to 2 \to 5$ & $1 \to 3 \to 2 \to 5$)

9. Find the optimal crashed project completion time of a project whose details are given below:

Activity	*Normal*		*Crash*	
	Time (in Weeks)	*Cost (Rs.)*	*Time (Weeks)*	*Cost (Rs.)*
(1, 2)	15	1000	13	1500
(2, 3)	10	700	4	2500
(2, 4)	12	1500	8	3500
(3, 5)	5	400	1	1000
(4, 6)	16	1200	12	1600
(5, 6)	8	1000	6	1400

(**Ans:** Crashed Completion Time: 33 days with cost Rs. 9150. Critical paths are $1 \to 2 \to 3 \to 5 \to 6$ and $1 \to 2 \to 4 \to 6$.)

10. Solve exercise 9 by *FF* limit.
11. Consider the project whose details are given below:

Activity	*Normal*		*Crash*	
	Time (days)	*Cost (Rs.)*	*Time (days)*	*Cost (Rs.)*
(1, 2)	5	200	2	260
(1, 3)	6	220	3	310
(2, 4)	4	310	2	390
(2, 6)	7	250	4	400
(3, 6)	5	350	3	390
(4, 5)	4	150	2	230
(4, 6)	6	300	3	420
(5, 6)	7	200	4	290

Find the optimal schedule by Stepping step procedure if indirect costs are Rs. 40 per day.

12. Consider exercise 11.
    (a) Find by *FF* limit method the different minimum cost schedules between the normal and crash point.
    (b) Use *FF* limit method to find the optimal schedule if the indirect costs are Rs. 40 per days.

Ans:

(a)

1980	2020	2110	2150	2210	2350	2420
20	18	15	14	13	11	10

Crash Critical paths are: $1 \to 2 \to 4 \to 5 \to 6$ and $1 \to 3 \to 5 \to 6$.

(b) Duration Time: 14 days: Cost Rs. 2710

13. Consider the data of a project as given below:

Activity	*Normal*		*Crash*	
	Time (weeks)	*Cost (Rs.)*	*Time (weeks)*	*Cost (Rs.)*
(1, 2)	8	800	5	950
(1, 3)	5	500	3	700
(1, 4)	9	600	6	1050
(2, 5)	10	900	8	1300
(3, 5)	5	700	3	1100
(3, 6)	6	1200	5	1500
(4, 6)	7	1300	5	1400
(5, 7)	2	400	1	500
(6, 7)	4	500	2	900

If the indirect cost per week is Rs. 300, find the optimal crashed project completion time by

(i) Stopping step procedure.

(ii) *FF* limit method.

Ans:

(a) Crash Critical paths: $1 \rightarrow 2 \rightarrow 5 \rightarrow 7$ and $1 \rightarrow 4 \rightarrow 6 \rightarrow 7$
Project Completion Limit = 16 weeks and total Cost = Rs. 12,350

Chapter 12

Linear Goal Programming

12.1 INTRODUCTION

So far we have considered problems involving a single objective function. Now we shall take multi-objective function problems. If each objective function and constraints are linear, then these would be termed linear problems.

Sometimes in optimising, we must be having certain goals in our mind. For example, if we are optimising a profit we must have set a goal to achieve the target of profit A and we have to optimise the profit so as to reach as close to A as possible and in other words the deviation of the optimal profit obtained and A should be as less as possible. This deviation could be positive or negative.

If for each objective function a goal is fixed, we call such a problem goal programming problem. We shall consider now linear goal programming problem.

12.2 LINEAR GOAL PROGRAMMING PROBLEM

We shall try to understand the problem and derive a method to solve a goal programming problem with the help of an illustrative example. It would help us to understand the problem better in conceptual manner.

Example 1: Let a manufacturer produces two types of vegetable oils A and B which require to pass through machines $M1$ and $M2$. The following table gives the time (in hours) for each litre of A and B on each machine $M1$ and $M2$. The total time available on each machine per week is also given. Also given the net profit and gross profit per litre of oils A and B.

	Oil A	*Oil B*	*Time available/Week*
Machine $M1$	0.4	0.3	50
Machine $M2$	0.2	0.5	60
Net Profit per litre	4	5	
Gross Profit per litre	1	5	
Cash Profit per litre	2	1	

If x_1 litre and x_2 litre of oils are produced, per week then the cash position is estimated to be $2x_1 + x_2$. The decision-maker has set the goals for net profit, gross profit and cash position as 800, 700 and 300, respectively.

The problem is to maximise the net profit, gross profit and cash position keeping the goal in mind.

Solution:

Formulation

Objective functions: We have to maximise net profit i.e., $4x_1 + 5x_2$ with the goal of 800, i.e., we have to maximise $4x_1 + 5x_2$ so as to reach near to 800. This we write as

$$\text{Max } 4x_1 + 5x_2 \approx 800 \text{ (Net Profit, Goal 1)},$$

'≈' stands for 'approximately equal to'.

Thus, we have three objective functions:

$$\text{Max } Z_1 = 4x_1 + 5x_2 \approx 800 \text{ (Net Profit, Goal 1)}$$
$$\text{Max } Z_2 = x_1 + 5x_2 \approx 700 \text{ (Gross Profit, Goal 2)}$$
$$\text{Max } Z_3 = 2x_1 + x_2 \approx 300 \text{ (Cash position, Goal 3)}$$

Constraints:

$$.4x_1 + .3x_2 \leq 50$$
$$.2x_1 + .5x_2 \leq 60$$

Non-negativity Condition:

$$x_1, x_2 \geq 0$$

Thus, the problem becomes

$$\text{Max } Z_1 = 4x_1 + 5x_2 \approx 800 \text{ (Goal 1)}$$
$$\text{Max } Z_2 = x_1 + 5x_2 \approx 700 \text{ (Goal 2)}$$
$$\text{Max } Z_3 = 2x_1 + x_2 \approx 300 \text{ (Goal 3)}$$

Subject to

$$.4x_1 + .3x_2 \leq 50$$
$$.2x_1 + .5x_2 \leq 60$$
$$x_1, x_2 \geq 0$$

Writing it in standard form and assuming objective functions also as constraints with near about availability, we obtain

$$4x_1 + 5x_2 + S_1 = 800 \quad \dots (1)$$
$$x_1 + 5x_2 + S_2 = 700 \quad \dots (2)$$
$$2x_1 + x_2 + S_3 = 300 \quad \dots (3)$$
$$.4x_1 + .3x_2 + S_4 = 50 \quad \dots (4)$$
$$.2x_1 + .5x_2 + S_5 = 60 \quad \dots (5)$$
$$x_1, x_2, S_4, S_5 \geq 0; \; S_1, S_2, S_3 \text{ unrestricted}$$

as goals could be over or under achieved. But these achievements over or under should be as small as possible in order to be near about goals. Therefore, the objective function should be

Min $S_1 + S_2 + S_3$.

Now a question that arises is whether minimisation of $S_1 + S_2 + S_3$ guarantee the minimisation of S_1, S_2 and S_3 separately as they are unrestricted in sign? Does it ensure the optimal solution of multi-objective goal oriented problem?

To discuss it further, we solve the following 4 problems by simplex method.

1st Problem	*2nd Problem*	*3rd Problem*
Min S_1	Min S_2	Min S_3
Subject to 1, 4, 5	Subject to 2, 4, 5	Subject to 3, 4, 5
$S_1, x_1, x_2, S_4, S_5 \geq 0$	$S_2, x_1, x_2, S_4, S_5 \geq 0$	$S_3, x_1, x_2, S_4, S_5 \geq 0$

4th Problem

Min $S_1 + S_2 + S_3$

Subject to 1, 2, 3, 4, 5

$x_1, x_2, S_4, S_5 \geq 0; S_1, S_2, S_3 \geq 0$

	Problem 1	*Problem 2*	*Problem 3*	*Problem 4*
	Min S_1 $4x_1 + 5x_2 + S_1 = 800$ $.4x_1 + .3x_2 + S_4 = 50$ $.2x_1 + .5x_2 + S_5 = 60$ $x_1, x_2, S_1, S_4, S_5 \geq 0$	Min S_2 $x_1 + 5x_2 + S_2 = 700$ $.4x_1 + .3x_2 + S_4 = 50$ $.2x_1 + .5x_2 + S_5 = 60$ $x_1, x_2, S_2, S_4, S_5 \geq 0$	Min S_3 $2x_1 + x_2 + S_3 = 300$ $.4x_1 + .3x_2 + S_4 = 50$ $.2x_1 + .5x_2 + S_5 = 60$ $x_1, x_2, S_3, S_4, S_5 \geq 0$	Min $S_1 + S_2 + S_3$ $4x_1 + 5x_2 + S_1 = 800$ $x_1 + 5x_2 + S_2 = 700$ $2x_1 + x_2 + S_3 = 300$ $.4x_1 + .3x_2 + S_4 = 50$ $.2x + .5x_2 + S_5 = 60$ $x_1, x_2, S_1, S_2, S_3, S_4, S_5 \geq 0$
x_1	50	0	125	50
x_2	100	120	0	100
S_1	100	–	–	100
S_2	–	100	–	150
S_3	–	–	50	100
S_4	0	14	0	0
S_5	0	0	35	0
Z_1	700	–	–	700
Z_2	–	600	–	50
Z_3	–	–	250	200

Looking at the solutions of the above problems, with all $S_i \geq 0$, we find that the values of decision variables change and also that the deviations from goals are too much. This means that it won't work. So, we have to think of attaching some weights to deviations i.e., S_1, S_2, S_3, etc.

In a nutshell, it can be visualised that any solution may not be able to minimise S_1, S_2, S_3, etc., i.e., any solution may not be able to desired goals of all the objective functions. Thus, we have to fix our priorities which will be denoted by P_1, P_2, etc. Thus, we fix our priorities to these objective functions which are goal oriented. By fixing priorities, we mean that an upper priority goal must be much more near to its goal in comparison to a lower priority goal, i.e., deviation in an upper priority goal must be lower than a lower priority goal. Considering these priorities P_1, P_2, etc., as big positive numbers such that

$$P_1 \gg P_2 \gg P_3 \gg P_4 \gg \ldots \gg P_n$$

We attach them as weights to the respective deviation(s).

Secondly, in order to get optimal solution it may also be needed that one may have to go out of S_F (feasible region) determined by actual constraints. So, the decision-maker has to decide which constraint he can relax and which one he cannot.

A constraint which cannot be relaxed would be termed ***'Resource constraint'*** and the one which can be relaxed would be termed ***Goal constraint***. The Resource constraints are not be touched as they cannot be relaxed. We have to fix priorities to Goal constraints only. These priorities to Goal constraints could be after the priorities of objective functions or could be above the priorities of some or all the objective functions. Actually these Goal constraints could be treated as Goal oriented objective functions. In other words, we assign priorities to objective functions and Goal constraints together.

Let, in the above problem, us decide that both of the constraints are Goal constraints. So we assign priorities to all the five. Let the priorities be $P_1, P_2, \ldots, P_5$ to them in the order they are written, i.e.,

$$
\begin{aligned}
4x_1 + 5x_2 + S_1 &= 800 \ \ldots \ P_1 \\
x_1 + 5x_2 + S_2 &= 700 \ \ldots \ P_2 \\
2x_1 + x_2 + S_3 &= 300 \ \ldots \ P_3 \\
.4x_1 + .3x_2 + S_4 &= 50 \ \ldots \ P_4 \\
.2x_1 + .5x_2 + S_5 &= 60 \ \ldots \ P_5
\end{aligned}
$$

$$x_1, x_2 \geq 0;\ S_1, S_2, S_3, S_4, S_5 \text{ unrestricted in sign.}$$

Since, S_i's are unrestricted in sign, we write them as $S_i^- - S_i^+$, S_i^- and $S_i^+ \geq 0$. Thus, we get

$$
\begin{aligned}
\text{Min } Z = P_1 S_1^+ + P_1 S_1^- + P_2 S_2^+ + P_2 S_2^- + P_3 S_3^+ \\
+ P_3 S_3^- + P_4 S_4^+ + P_5 S_5^+
\end{aligned}
$$

$$
\begin{aligned}
\text{Subjec to } 4x_1 + 5x_2 + S_1^- - S_1^+ &= 800 \\
x_1 + 5x_2 + S_2^- - S_2^+ &= 700 \\
2x_1 + x_2 + S_3^- - S_3^+ &= 300 \\
.4x_1 + .3x_2 + S_4^- - S_4^+ &= 50 \\
.2x_1 + .5x_2 + S_5^- - S_5^+ &= 60
\end{aligned}
$$

The objective function which is written above is written as follows.

Both the variables S_i^+ and S_i^-, $i = 1, 2, 3$ would appear in objective function as these are Goal oriented objective functions and these are to be multiplied by P_i which is priority weight to them. But both S_i^+ and S_i^-, $i = 4, 5$ will not appear. To decide which one would appear we proceed as follows:

If the constraint $i \leq$ type, S_i^- (which is with positive sign) has to be there and we have to minimise S_i^+. If the constraint is $\geq$ type, S_i^+ (which is with negative sign) has to be there and we have to minimise S_i^-. Therefore, we have to include that which is to be minimised. We include them after multiplying them by their priorities. In our problem, both constraints are '$\leq$' type so we include $P_4 S_4^+$ and $P_5 S_5^+$. After deciding this, we take all and add and objective function becomes minimise. Hence, the objective function is as written above.

If suppose the last constraint is a resource constraint then we do not write S_5 as $S_5^- - S_5^+$ but write it as S_5 and it will not appear in objective function.

Then solve it by simplex iteration.

Now we take an example and solve it.

Example 2:

Max $\quad Z = x_1 - 2x_2 \approx 6 \qquad P_1$

Subject to $\quad 2x_1 - x_2 \geq 2 \qquad P_2$

$x_1 + x_2 \leq 4 \qquad P_3$

$x_1, x_2 \geq 0$

Without considering it a Goal programming problem, by simplex method, we obtain its value as 4.

Now we consider it a Goal problem. Let both the constraints be Goal constraints and fix the priorities as above. Thus, the problem becomes

$$x_1 - 2x_2 + S_1 = 6$$
$$2x_1 - x_2 - S_2 = 2$$
$$x_1 + x_2 + S_3 = 4$$
$$x_1, x_2 \geq 0, S_1, S_2, S_3 \text{ unrestricted.}$$

or, $\quad$ Min $P_1(S_1^+ + S_1^-) + P_2\, S_2^- + P_3\, S_3^+$

$$x_1 - 2x_2 + S_1^- - S_1^+ = 6$$
$$2x_1 - x_2 - S_2^- + S_2^+ = 2$$
$$x_1 + x_2 + S_3^- - S_3^+ = 4$$
$$x_1, x_2, S_1^-, S_1^+, S_2^-, S_2^+, S_3^-, S_3^+ \geq 0.$$

and solve it by simplex method. We have to keep the following in mind.

(i) P's are to be treated as big positive numbers with the condition that $P_i \gg P_j$ if $i < j$.

(ii) Both S_i^- and S_i^+ do not appear in basis because of obvious reasons.

(iii) We apply simplex method like big M method.

It should be noted that $Z_i - C_i$ below each variable would appear as $\alpha P_1 + \beta P_2 + \gamma P_3$ also. In order to make table legible and non-confusing, we write this also in tabular form as

	…	…	x_i	…
P_1			α	
P_2			β	
P_3			γ	

Thus, our initial table of the problem is given below. It is not starting table.

BV	x_1	x_2	S_1^+	S_2^-	S_3^+	S_1^-	S_2^+	S_3^-	*Soln.*	
P_1	0	0	–1	0	0	–1	0	0	0	
P_2	0	0	0	0	0	0	–1	0	0	
P_3	0	0	0	0	–1	0	0	0	0	
S_1^-	1	–2	–1	0	0	1	0	0	6	'I.T.'
S_2^+	2	–1	0	–1	0	0	1	0	2	(Initial
S_3^-	1	1	0	0	–1	0	0	1	4	table)
P_1	↓ 1	–2	–2	0	0	0	0	0	6	'S.T.'
P_2	2	–1	0	–1	0	0	0	0	2	(Starting
P_3	0	0	0	0	–1	0	0	0	0	table)
S_1^-	1	–2	–1	0	0	1	0	0	6	
← S_2^+	$\boxed{2}$	–1	0	–1	0	0	1	0	2	
S_3^-	1	1	0	0	–1	0	0	1	4	

Since, it is a minimisation problem, x_1^- enters S_2^+ leaves. On applying simplex method, we obtain

BV	x_1	x_2	S_1^+	S_2^-	S_3^+	S_1^-	S_2^+	S_3^-	*Soln.*
P_1	0	–3/2	–2	1/2	0	0	–1/2	0	5
P_2	0	0	0	0	0	0	–1	0	0
P_3	0	0	0	0	–1	0	0	0	0
S_1^-	0	–3/2	–1	1/2	0	1	–1/2	0	5
x_1	1	–1/2	0	–1/2	0	0	1/2	0	1
S_3^-	0	3/2	0	1/2	–1	0	–1/2	1	3
P_1	0	–3	–2	0	1	0	0	–1	2
P_2	0	0	0	0	0	0	–1	0	0
P_3	0	0	0	0	–1	0	0	0	0
S_1^-	0	–3	–1	0	1	1	0	–1	2
x_1	1	1	0	0	–1	0	0	1	4
S_2^-	0	3	0	1	–2	0	–1	2	6
P_1	0	0	–1	0	0	–1	0	0	0
P_2	0	0	0	0	0	0	–1	0	0
P_3	0	–3	–1	0	0	1	0	–1	2
S_3^+	0	–3	–1	0	1	1	0	–1	2
x_1	1	–2	–1	0	0	1	0	0	6
S_2^-	0	–3	–2	1	0	2	–1	0	10

Since, $P_1 \gg P_3$, it is an optimal table. Hence, the solution

$$x_1 = 6,\ x_2 = 0,\ S_1^- = 0,\ S_1^+ = 0,\ S_2^- = 10,\ S_2^+ = 0\ S_3^- = 0,\ S_3^+ = 2$$

This gives that objective function

$x_1 - 2x_2$ approaches the goal value 6 as deviation from goal $S_1^- - S_1^+ = 0$

The goal constraint

$2x_1 - x_2$ (= 12) ≥ 2 is satisfied and remains as resource constraint as deviation $S_2^- - S_2^+ \geq 0$

The goal constraint

$$x_1 + x_2 \text{ (= 6)} \leq 4 \text{ requires deviation}$$

If decision can provide this much relaxation, the above is the required optimal solution.
It gives the optimal solution which satisfies our target (goal) exactly.
Now we will discuss partition algorithm to find solution of a Linear Goal Programming problem.

12.3 PARTITION ALGORITHM FOR LINEAR GOAL PROGRAMMING

Problem: It is discussed in the following steps:

Step 1: Bring the Linear goal programming problem (LGPP) into the following standard form.

$$\text{Min } Z = \sum_{i=1}^{m} P_i(W_i^+ S_i^+ + W_i^- S_i^-)$$

Subject to

$$\sum_{j=1}^{n} \alpha_{ij} x_j + S_i^+ - S_i^- = b_i \text{ for all } i = 1 \text{ to } m$$

$$x_j,\ S_i^-,\ S_i^+ \geq 0 \text{ for all } j = 1 \text{ to } n \text{ and } \forall\ i = 1 \text{ to } m.$$

Where, P_i is the priority attached to i^{th} constraint and W_i^+ & W_i^- are respectively the weights attached to S_i^+ and S_i^- of i^{th} constraint. The W_i^+ & W_i^- are 1 or 0 according to S_i^+, S_i^- to be included in objective function or not.

In the standard form of LGPP defined above the main features are two.

(i) Each Constraint contains both surplus and slack variables.

(ii) The surplus and slack variable S_i^+ & S_i^- of i^{th} constraint are not present in j^{th} constraint for all $j \neq i$.

Step 2: Solve the first subproblem which consists of the objective function involving first priority P_1 and the Constraint and Constraints with P_1. Problem is solved by simplex method because slack variables give the identity matrix and get the optimal solution. There arise two cases (a) There exists an alternate solution (b) The optimal solution is unique. In case of (a) go to step 3 and in case of (b) go to step 4.

Step 3: If alternate optimal solution exists then it is possible to negotiate with the constraint (constraints) of next higher priority. For this we define the following steps.

3 (a) In the optimal table of Step 2 no $Z_j - C_j$ will be positive as it is an optimal solution of minimisation problem. If in this table the relative cost of slack or surplus variables is negative, then this variable is dropped because these variables are not involved in any of the remaining constraints and cannot give a better solution, then step 2 if entered at a later stage.

3 (b) Delete the objective function row.

3 (c) Add the constraint (or constraints) with priority 2 by sensitivity analysis. This will not create any problem because both $+S^-$ and $-S^+$ are present. While adjusting the columns corresponding to basic variables to proper format, one of these will give the required identity column.

3 (d) Add the part of the objective function for corresponding to priority P_2 by using sensitivity analysis. (i.e., make the relative cost of basic variable zero).

3 (e) Use simplex method to get the optimal solution of the second subproblem.

3 (f) Go to step 2 with above problem as first subproblem.

Step 4: In case there exists no alternate optimal solution, the optimal table of step 2 gives the optimal solution for the given LGPP with respect to all of the priorities. This is because P_1 is the priority of highest order and this subproblem has one and only one optimal solution. So there is no scope to accommodate the constraints with subsequent lower priorities. The values of other decision variables are obtained by substituting the values of variables given by the optimal table of step 2 in the goal constraints of the lower priorities. This completes the algorithm. STOP.

Example 1: Solve the following

$$\text{Max } 4x_1 + 5x_2 \simeq 800 \text{ with } P_1 \text{ (Goal 1)}$$
$$\text{Max } x_1 + 5x_2 \simeq 700 \text{ with } P_2 \text{ (Goal 2)}$$

subject to

$$.4x_1 + .3x_2 \le 50 \text{ with } P_3$$
$$.2x_1 + .5x_2 \le 60 \text{ with } P_4$$
$$x_1, x_2 \ge 0$$

Solve:

LGPP is

$$\text{Min } P_1(S_1^- + S_2^+) + P_2(S_2^- + S_2^+) + P_3 S_3^+ + P_4 S_4^+$$

subject to

$$4x_1 + 5x_2 + S_1^- - S_1^+ = 800 \text{ with } P_1$$
$$x_1 + 5x_2 + S_2^- - S_2^+ = 700 \text{ with } P_2$$
$$.4x_1 + .3x_2 + S_3^- - S_3^+ = 50 \text{ with } P_3$$
$$.2x_1 + .5x_2 + S_4^- - S_4^+ = 60 \text{ with } P_4$$

all variable ≥ 0 and $P_1 >> P_2 >> P_3 >> P_4$

Step 2: The first subproblem

Min $S_i^- + S_i^+$

subject to

$$4x_1 + 5x_2 + S_1^- - S_1^+ = 800$$
$$\text{all var.} \ge 0$$

Solution by simplex method.

	BV	Z	x_1	x_2	S_1^+	S_1^-	Sol.	
	Z	1	0	0	–1	–1	0	
	S_1^-	0	4	5	–1	1	800	Initial table
	Z	1	4	5↓	–2	0	800	
←	S_1^-	0	4	$\boxed{5}$	–1	1	800	Starting table
	Z	1	0	0	–1	–1	0	
	x_2	0	4/5	1	–1/5	1/5	160	Optimal table

$Z_1 - C_1 = 0$, alternate optimal solution exists.

drop. S_1^- and S_1^+ and add the constraint of next higher priority. Add $x_1 + 5x_2 + S_2^- - S_2^+ = 700$

	BV	Z	x_1	x_2	S_2^+	S_2^-	Sol.	
	Z	1						
	x_2 S_2^-	0 0	4/5 1	1 5	0 –1	0 1	160 700	x_2 is basic variable adjust the column
	Z	1						
	x_2 S_2^-	0 0	4/5 –3	1 0	0 –1	0 1	160 –100	here feasibility disturbed Change S_2^- with S_2^+
	Z	1						
	x_2 S_2^+	0 0	4/5 3	1 0	0 1	0 –1 $S_2^- + S_2^+$	160 100	Now table is feasible. Now add Min
	Z	1	0	0	–1	–1	0	
	x_2 S_2^+	0 0	4/5 3	1 0	0 1	0 –1	160 100	Initial table
	Z	1	3	0	0	–2	100	
←	x_2	0	$\boxed{415}$ ↓	1	0	0	160	Starting table
	S_2^+	0	3	0	1	–1	100	

BV	Z	x_1	x_2	S_2^+	S_2^-	Sol.	
Z	1	0	0	–1	–1	0	
x_2	0	0	1	–4/15	4/15	400/3	
x_1	0	1	0	1/3	–1/3	100/3	Optimal table

Optimal table of II Subproblem. No alternate solution
So algorithm terminates. So the optimal solution is

$$x_1 = 100/3, x_2 = 400/3$$

Substituting in III constraints, we get

$$0.4 \times \frac{100}{3} + .3 \times \frac{400}{3} + S_3^- - S_3^+ = 50$$

$$= \frac{40}{3} + \frac{120}{3} + S_3^- - S_3^+ = 50$$

$$S_3^- - S_3^+ = 50 - \frac{160}{3} = \frac{-10}{3}$$

$$S_3^- = 0, S_3^+ = \frac{10}{3}$$

$\therefore$ both S_3^- and S_3^+ cannot be positive $\Rightarrow$ that third Constraint is to be relaxed.

Similarly, by 4th constraint

$$S_4^- = 0, S_4^+ = 40/3$$

Example 2: Consider the following LGPP

$$2x_1 + x_2 \simeq 12 \text{ with } P_1$$
$$x_1 + x_2 \simeq 10 \text{ with } P_1$$
$$x_1 \geq 7 \text{ with } P_2$$
$$x_1 + 4x_2 \geq 4 \text{ with } P_3$$

Solve the above as LGPP.

Solution: LGPP is

Min $P_1(S_1^- + S_2^+) + P_1(S_2^- + S_2^+) + P_2 S_3^- + P_3 S_4^-$

subject to

$$2x_1 + x_2 + S_1^- - S_1^+ = 12$$
$$x_1 + x_2 + S_2^- - S_2^+ = 10$$
$$x_1 + S_3^- - S_3^+ = 7$$
$$x_1 + 4x_2 + S_4^- - S_4^+ = 7$$

$P_1 \gg P_2 \gg P_3$

all variable ≥ 0 and 7.

Solve the first subproblem

Min $S_1^- + S_1^+ + S_2^- + S_2^+$

subject to

$$2x_1 + x_2 + S_1^- - S_1^+ = 12$$
$$x_1 + x_2 + S_2^- - S_2^+ = 10$$

all variable ≥ 0

	BV	Z	x_1	x_2	S_1^+	S_2^+	S_1^-	S_2^-	$Sol.$	
	Z	1	0	0	–1	–1	–1	–1	0	
	S_1^-	0	2	1	–1	0	1	0	12	
	S_2^-	0	1	1	0	–1	0	1	10	Initial table
	Z	1	3	2	–2	–2	0	0	22	
←	S_1^-	0	$\boxed{2}$ ↓	1	–1	0	1	0	12	
	S_2^-	0	1	1	0	–1	0	1	10	Starting table
	Z	1	0	1/2	–1/2	–2	–3/2	0	4	
	x_1	0	1	1/2↓	–1/2	0	1/2	0	6	
←	S_2^-	0	0	$\boxed{1/2}$	1/2	–1	–1/2	1	4	
	Z	1	0	0	–1	–1	–1	–1	0	
	x_1	0	1	0	–1	1	1	–1	2	
	x_2	0	0	1	1	–2	–1	2	8	Optimal table

No Alternate Solution

$$\Rightarrow \qquad x_1 = 2, \quad x_2 = 8$$

but 3rd constraint is not satisfied if we want to satisfy 3rd and 4th constraints, give first priority to them because they are resource constraints, so problem becomes

$$\begin{aligned} 2x_1 + x_2 &= 12 \text{ with } P_2 \\ x_1 + x_2 &= 10 \text{ with } P_3 \\ x_1 &\geq 7 \text{ with } P_1 \\ x_1 + 4x_2 &\geq 8 \text{ with } P_1 \\ \text{all variables} &\geq 0 \end{aligned}$$

So LGPP is

Min $P_1(S_3^-) + P_1 S_4^- + P_2(S_1^- + S_1^+) + P_3(S_2^- + S_2^-)$

subject to

$$\begin{aligned} 2x_1 + x_2 + S_1^- - S_1^+ &= 12 \\ x_1 + x_2 + S_2^- - S_2^+ &= 10 \\ x_1 + S_3^- - S_3^+ &= 7 \\ x_1 + 4x_2 + S_4^- - S_4^+ &= 4 \\ \text{all variables} \geq 0 \text{ and } P_1 \gg P_2 \gg P_3 \end{aligned}$$

Solution of first subproblem

Min $S_3^- + S_4^-$, subject to $x_1 + S_3^- - S_3^+ = 7$, $x_1 + 4x_2 + S_4^- - S_4^+ = 4$

all variables ≥ 0

	BV	Z	x_1	x_2	S_3^+	S_4^+	S_3^-	S_4^-	$Sol.$	
	Z	1	0	0	0	0	–1	–1	0	
	S_3^-	0	1	0	–1	0	1	0	7	
	S_4^-	0	1	4	0	–1	0	1	4	Initial table
	Z	1	2	4	–1	–1	0	0	11	
	S_3^-	0	1	0↓	–1	0	1	0	7	
←	S_4^-	0	1	$\boxed{4}$	0	–1	0	1	4	
	Z	1	1	0	0	0	–1	–2	7	
	S_3^-	0	1↓	0	–1	0↓	1	0	7	
←	x_2	0	$\boxed{1/4}$	1	0	–1/4	0	1/4	1	
	Z	1	0	–4	0	1	–1	–3	3	
←	$\boxed{S_3}$	0	0	–4	–1	$\boxed{1}$↓	1	–1	3	
	x_1	0	1	4	0	–1	0	1	4	
	Z	1	0	0	0	0	–1	–1	0	
	S_4^+	0	0	–4	–1	1	1	–1	3	
	x_1	0	1	0	–1	0	1	0	7	Optimal table

Drop S_3^- and S_4^- as $Z_j - C_j < 0$ Add the first constraint which is with priority P_2 then adjust the column of basic variables.

Z	x_1	x_2	S_3^+	S_4^+	S_1^-	S_1^+	Sol.	
S_4^+	0	–4	–1	1	0	0	3	
x_1	1	0	–1	0	0	0	7	
S_1^-	2	1	0	0	1	–1	12	
Z								
S_4^+	0	–4	–1	1	0	0	3	
x_1	1	0	–1	0	0	0	7	
S_1^-	0	1	2	0	1	–1	–2	Interchange S_1^- with S_1^+
Z	0	–1	–2	0	–2	0	2	
		0	0		–1	–1		
S_4^+	0	–4	–1	1	0	0	3	
x_1	1	0	–1	0	0	0	7	
S_1^+	0	–1	–2	0	–1	1	2	Add objective function of P_2

This is optimal table with no alternate solution, hence process terminates. The optimal solution of the LGPP is

$$x_1 = 7,\ x_2 = 0,\ S_4^+ = 3,\ S_1^+ = 2,\ S_4^- = 0,\ S_1^- = 0\ S_3^+ = 0,\ S_3^- = 0$$

Putting these values in 4th constraint

We get

$$7 + 4(0) + S_4^- - S_4^+ = 4$$
$$7 + S_4^- - 3 = 4,\ S_4^- = 0$$

resource constraint is satisfied

EXERCISE 12.1

1. Consider

   Maximize $2x_1 - 4x_2 \simeq 12$ (Goal 1)

   Subject to

   $$4x_1 - 2x_2 \geq 4 \text{ (Goal 2)}$$
   $$2x_1 + 2x_2 \leq 8 \text{ (Goal 3)}$$
   $$x_1, x_2 \geq 0$$

   Formulate the above as LGPP.

   Ans: Min $P_1(S_1^+ + S_1^-) + P_2\, S_2^- + P_3\, S_3^+$

   Subject to

   $$2x_1 - 4x_2 - S_1^+ + S_1^- = 12 \text{ (Goal 1)}$$
   $$4x_1 - 2x_2 - S_2^+ + S_2^- = 4 \text{ (Goal 2)}$$
   $$2x_1 + 2x_2 - S_3^+ + S_3^- = 8 \text{ (Goal 3)}$$

   All variables ≥ 0

   and $P_1 \gg P_2 \gg P_3$.

2. Each metre of woollen and cotton clothes manufactured by Shiva Mills Ltd. require operations M_1 and M_2. The following table gives the time (in hours) each metre of woollen and cotton clothes requires in M_1 and M_2. The available time (in hours) and profit (in Rs.) is also shown below.

	Woollen	*Cotton*	*Time Available/Week.*
M_1	0.5	0.4	60
M_2	0.3	0.6	70
Gross profit/metre	50	10	
Net profit/metre	10	2	

   Let x_1 metre of woollen and x_2 metre of cotton cloth is manufactured. It is estimated that the cash position is $3x_1 + 2x_2$. The decision-makers have goals for net profit, gross profit and cash positions of Rs. 500000, Rs. 1000000 and Rs. 300000, respectively. Formulate the above as LGPP.

   Ans:

   LPP is Max $Z_1 = 50x_1 + 10x_2 \simeq 1000000$ (Goal 1)

   Max $Z_2 = 10x_1 + 2x_2 \simeq 500000$ (Goal 2)

   Max $Z_3 = 3x_1 + 2x_2 \simeq 300000$ (Goal 3)

Subject to

$$0.5x_1 + 0.4x_2 \leq 60$$
$$0.3x_1 + 0.6x_2 \leq 70$$
$$x_1, x_2 \geq 0$$

GLPP is

Min $S_1 + S_2 + S_3$

Subject to

$$50x_1 + 10x_2 + S_1 = 1000000 \text{ (Goal 1)}$$
$$10x_1 + 2x_2 + S_2 = 500000 \text{ (Goal 2)}$$
$$3x_1 + 2x_2 + S_3 = 300000 \text{ (Goal 3)}$$
$$0.5x_1 + 0.4x_2 + S_4 = 60$$
$$0.3x_1 + 0.6x_2 + S_5 = 70$$
$$\text{all variables} \geq 0.$$

3. Consider the exercise 2 with a modification that P_1, P_2, P_3 and P_4 priorities are given to objective functions 1 and 2, and the first and second constraint respectively i.e., third objective function is dropped. Formulate this as LGPP.

Ans:

Min $P_1(S_1^- + S_1^+) + P_2(S_2^- + S_2^+) + P_3\, S_3^+ + P_4\, S_4^+$

Subject to

$$50x_1 + 10x_2 + S_1^- - S_1^+ = 1000000 \text{ (Goal 1)}$$
$$10x_1 + 2x_2 + S_2^- - S_2^+ = 500000 \text{ (Goal 2)}$$
$$0.5x_1 + 0.4x_2 + S_3^- - S_3^+ = 60 \text{ (Goal 3)}$$
$$0.3x_1 + 0.6x_2 + S_4^- - S_4^+ = 70 \text{ (Goal 4)}$$
$$\text{all variables} \geq 0$$

and, $\qquad P_1 \gg P_2 \gg P_3 \gg P_4$

4. Solve the following LGPP by partition algorithm.

$$2x_1 + x_2 \simeq 12 \text{ with priority } P_2$$
$$x_1 + x_2 \simeq 10 \text{ with priority } P_3$$
$$x_1 \geq 7 \text{ with priority } P_1$$
$$x_1 + 4x_2 \geq 4 \text{ with priority } P_1$$
$$\text{all variables} \geq 0$$

(**Ans:** $x_1 = 7$, $x_2 = 0$, $S_4^+ = 3$, $S_4^- = 0$, $S_1^+ = 2$, $S_1^- = 0$, $S_3^+ = S_3^- = 0$, $S_2^- = 3$, $S_2^+ = 0$)

5. Solve the following LGPP

$$3x_1 + 6x_2 \leq 60 \text{ with priority } 2P_2$$
$$4x_1 + 2x_2 \leq 20 \text{ with priority } P_2$$
$$20x_1 + 24x_2 \simeq 300 \text{ with priority } P_1$$
$$\text{all variables} \geq 0$$

$\left(\textbf{Ans: } x_1 = \dfrac{15}{2}, x_2 = \dfrac{25}{4}, S_2^+ = 21/2 \text{ all other variables are equal to zero.}\right)$

Chapter 13

Sequencing

13.1 INTRODUCTION

Let there be n jobs and m machines and each of n jobs is to be processed on some or all of m different machines. For any given sequence of jobs at each machine, the effectiveness in terms of time, cost, etc. are to be measured and we have to select that sequence for which the measure of effectiveness is optimum. Theoretically, there are $(n!)^m$ possible sequences in which n jobs can be processed through m machines. Theoretically, it is always possible to compare all the sequences but practically it is not possible even $n = 4$ and $m = 4$ because when $n = 4$ and $m = 4$ the possible number of sequences will be $(4!)^4$ which is equal to 331776. So to select the best sequence we have to compare 331776 sequences which is not a simple task rather impossible. Then what to do? To get best sequence we need an easier method by which we can select best sequence.

Before proceeding to actual discussion, first we shall explain what is a sequencing problem? This may be defined as follows:

Definition: Let there be n jobs ($J_1, J_2, \ldots, J_n$) and m machines ($M_1, M_2, \ldots, M_m$) each of n jobs is to be processed one at a time at each of m machines. The order of processing each job through machines is given. The time that each job must require on machine is also known. The problem is to find a sequence among $(n!)^m$ number of all possible sequences (or order) (or combination) for processing the jobs so that the total elapsed time for all n jobs is minimum.

Mathematically, let

T_{1j} = Time for job j on machine M_1

T_{2j} = Time for job j on machine M_2

$\vdots$

T_{mj} = Time for job j on machine M_m or in general

T_{ij} = Time for job j on machine M_i ($i = 1, 2, \ldots, m$)

T = Time from start of first job to completion of the last job.

Now, the problem is to determine for each machine a sequence of jobs for which T is minimum.

Now before we discuss the method to find the optimal sequence, we shall discuss different terminology and notations used.

13.2 TERMINOLOGY AND NOTATIONS

Below are some terminology and notations which will be used in this chapter.

(i) ***Number of Machines:*** Number of machines means the service facilities through which a job passes before that job is completed. For example, a house to be built has to be processed purchasing land, making map, filling foundation, erecting walls, flooring, roofing, plumbering, electrifying, furnituring and finishing, etc. In this example the house constitutes the job and different processes constitute the number of machines.

(ii) ***Processing Order:*** It is the order in which machines are required to complete the job.

(iii) ***Processing Time:*** It is time which is required by each job on each machine. T_{ij} will denote the time of processing job j (j = 1 to n) on i^{th} machine (i = 1 to m).

(iv) ***Idle time on a machine:*** The time for which a machine remains idle during the total elapsed time is called idle time. I_{ij} will be used to denote the idle time of machine i between the $(j - 1)^{th}$ job and the start of j^{th} job.

(v) ***Total Elapsed time:*** The time between starting the first job and completing the last job is called total elapsed time. This time also includes idle time of a machine, if exits and it will be denoted by T.

(vi) ***No Passing Rule:*** This rule means that no passing is allowed i.e., the same order of jobs is maintained over each machine. If each of the n-jobs is to be processed through two machines M_1 and M_2 in the order M_1 M_2, then this rule means that each job will go to machine M_1 first and then machine M_2.

13.3 MAIN ASSUMPTIONS IN A SEQUENCING PROBLEM

Following are the main assumptions which are to be followed in each sequencing problem:

(i) No machine can process more than one job at a time i.e., each machine is allowed to process one and only one job at time.

(ii) Once a job is started it should be completed i.e., no job should be left incompleted before starting a new job.

(iii) Each job must be completed before any other job, which it must precede, can begin.

(iv) A job is an entity i.e., even though the job represents a lot of individual parts, no lot should be processed by more than one at a time.

(v) Time intervals for processing jobs are independent of the order in which jobs are performed.

(vi) Each machine is of different type i.e., there is only one machine of each type.

(vii) A job is processed without delay subject to ordering requirements.

(viii) All jobs are known and are ready to start processing before the period under consideration starts.

(ix) The time of transferring a job from one machine to another machine is negligible.

13.4 SOLUTION OF SEQUENCING PROBLEM

At present following cases of sequencing problem are available:

1. n jobs and two machines M_1 and M_2, all jobs are processed in the order M_1 M_2.
2. n jobs and three machines M_1, M_2, and M_3, all jobs are processed in the order M_1 M_2 M_3. Other conditions are given in section 11.6 when this case will be discussed.

3. m machines and two jobs. Each job is to be processed through the machines in a prescribed order.
4. m-machines and n jobs.

Only first two cases will be explained in the following sections one by one without proof. We will not be discussing case 3 and 4.

13.5 PROCESSING n JOBS THROUGH TWO MACHINES

This problem can be described as:

(i) Only two machines M_1 and M_2 are involved.

(ii) Each job is processed in the order M_1 M_2 i.e., first on machine M_1 and then on machine M_2.

(iii) The exact or expected times T_{ij} (i = 1, 2; j = 1, 2,…, n) are known and shown in the following table:

Machines	*Jobs*					
	1	2	…	j	…	n
M_1	T_{11}	T_{12}	…	T_{ij}	…	T_{1n}
M_2	T_{21}	T_{22}	…	T_{2j}	…	T_{2n}

The problem here is to sequence the jobs so that the total elapsed time is minimum. The procedure for solving the problem is given as follows:

Procedure:

Step I: Select the least processing time occurring in the list T_{ij} (i = 1, 2; j = 1, 2,…, n). In case of tie select any of the smallest processing time.

Step II: If the least processing time is T_{1p} select p^{th} job first. If it is T_{2q}, do the q^{th} job last (because order is given M_1 M_2).

Step III: There are now (n – 1) jobs left to be ordered. Repeat step I and II for the reduced set of processing times obtained by deleting processing times for both the machines corresponding to the job already assigned.

Continue this till all jobs have been ordered. This will minimize the total elapsed time T.

Example 1: There are 5 jobs, each of which must go through the two machines M_1 and M_2 in the order M_1 M_2. Processing times are given in the table below:

Processing Time (in hours)

Maching↓/Job→	1	2	3	4	5
M_1	6	2	10	4	11
M_2	3	7	8	9	5

Find a sequence for five jobs which minimizes the elapsed time T. Also calculate the total idle time for both the machines in this period.

Solution: Apply step I and step II of the procedure. We see that the smallest processing time is 2 for job 2 for the machine M_1. So write the job 2 at the first place in the following manner.

2				

After this the reduced list of processing times becomes as follows.

Job	M_1	M_2
1	6	3
3	10	8
4	4	9
5	11	5

Now the smallest processing time is 3 which for job 1 on the machine M_2. So as per procedure write the job 1 in the last.

2				1

Now next reduced list of processing times is as following:

Job	M_1	M_2
3	10	8
4	4	9
5	11	5

In the above table the smallest processing time is 4 which is for job 4 on machine M_1. Write this job 4 next to 2 as below.

2	4			1

Reduced list is

Job	M_1	M_2
3	10	8
5	11	5

The smallest processing time is 5 for job 5 on machine M_2. Write this 5 from the last as below.

2	4		5	1

Finally, the optimal sequence becomes

2	4	3	5	1

After finding the optimal sequence, we are in a position to find the minimum elapsed time for this sequence using the given processing time for each job on each machine. The details are shown in the following table.

Job Sequence	*Machine M_1*		*Machine M_2*	
	Time in	*Time out*	*Time in*	*Time out*
2	0	2	2	9
4	2	6	9	18
3	6	16	18	26
5	16	27	27	32
1	27	33	33	36

Therefore, the minimum time i.e., the time for starting of job 2 to completion of last job 1 is 36 hours only. During this time machine M_1 is idle for three hours i.e., (33 to 36 hours) and the machine M_2 is idle for 4 hours only i.e., (from 0 to 2 hours, 26 to 27 hours and 32 to 33 hours).

Example 2: In a machine shop, 8 different products are being manufactured each requiring time on two machines M_1 and M_2 as given below:

Product	*Time (in minutes) on machine M_1*	*Time (in minutes) on M_2*
1	30	20
2	45	30
3	15	50
4	20	35
5	80	36
6	120	40
7	65	50
8	10	20

Determine the optimum sequence of processing of different products in order to minimize the total manufacturing time for all the products. Also find total idle time on both the machines M_1 and M_2.

Solution: Using the steps given in the procedure to solve, we find that the smallest processing time is 10 for product 8 on machine M_1. Write 8 at first place. After this get the reduced list of processing time removing processing times for product 8 on both the machine and find the smallest one, we see that it is 15 for product 3 on M_1, write it after 8. Continue this till we get the optimal sequence of products which is given as below:

8	3	4	7	6	5	2	1

Sequence is $8 \rightarrow 3 \rightarrow 4 \rightarrow 7 \rightarrow 6 \rightarrow 5 \rightarrow 2 \rightarrow 1$

Total elapsed time and idle times for each machine are shown in the following table.

Product Sequence	*Machine M_1*		*Machine M_2*	
	Time in	*Time out*	*Time in*	*Time out*
8	0	10	10	30
3	10	25	30	80
4	25	45	80	115
7	45	110	115	165
6	110	230	230	270
5	230	310	310	346
2	310	355	355	385
1	355	385	385	405

Minimum elapsed time = 405 minutes
Idle time for machine M_1 = 385 to 405 = 20 minutes
Idle time for machine M_2 =

(i) 0 to 10 = 10 minutes
(ii) 165 to 230 = 65 minutes
(iii) 270 to 310 = 40 minutes
(iv) 346 to 355 = 9 minutes

Total idle time for machine M_2 = 10 + 65 + 40 + 9 = 124 minutes
Hence, total idle time for both the machines
M_1 & M_2 = 20 + 124 = 144 minutes

Example 3: Following table shows the processing time (in hours) for 5 jobs to be processed on two machines M_1 and M_2.

Job	1	2	3	4	5
Machine M_1	3	7	4	5	7
Machine M_2	6	2	7	3	4

Passing is not allowed. Find the optimal sequence in which jobs should be processed. Also find minimum elapsed time for optimal sequence and idle time for machine M_1 and M_2, respectively.

Solution: Using the steps described in the procedure to find the optimal sequence, we get the following sequence.

1	3	5	4	2

Optimal sequence is $1 \rightarrow 3 \rightarrow 5 \rightarrow 4 \rightarrow 2$

Elapsed time and idle times are shown in the following table:

Job	*Machine M_1*		*Machine M_2*	
	Time in	*Time out*	*Time in*	*Time out*
1	0	3	3	9
3	3	7	9	16
5	7	14	16	20
4	14	19	20	23
2	19	26	26	28

Minimum elapsed time = 28 hours
Idle time for machine M_1 = 2 hours (i.e., from 26 to 28 hours)
Idle time for machine M_2 = 6 hours i.e., (0 to 3 hours and 23 to 26 hours)

13.6 PROCESSING *n* JOBS THROUGH THREE MACHINES

We can describe the problem as follows:

(i) Only three machines M_1, M_2 and M_3 are involved in processing.
(ii) Each job is processed on the every machine in the order of M_1 M_2 M_3.
(iii) Transfer of job is not allowed.
(iv) Exact or expected processing times for each job on each machine are shown in the following table:

Machine	*Job*					
	1	2	…	j	…	n
M_1	T_{11}	T_{12}	…	T_{ij}	…	T_{1n}
M_2	T_{21}	T_{22}	…	T_{2j}	…	T_{2n}
M_3	T_{31}	T_{32}	…	T_{3j}	…	T_{3n}

When T_{ij} = the processing time of j^{th} job on machine i (i = 1, 2, 3; j = 1, 2,…, n)

Procedure to find optimal solution: So far no general solution procedure is known for getting an optimal sequence in this case. However, the earlier method used to solve optimal sequence in the case of n jobs and 2 machines can be extended to find the optimal solution as special cases where either one or both of the following conditions are satisfied.

1. The minimum processing time on machine $M_1 \geq$ the maximum processing time on machine M_2.
2. The minimum processing time on machine $M_3 \geq$ the maximum processing time on machine M_2.

The procedure which is explained here (without proof) is to replace the given problem with an equivalent problem involving n jobs and two fictitious machines denoted by M_4 and M_5 and corresponding processing times T_{4j} and T_{5j} (j = 1, 2,…, n), which are defined as follows:

$$T_{4j} = T_{1j} + T_{2j}$$
$$T_{5j} = T_{2j} + T_{3j}$$

Now solve this problem with prescribed order M_4 M_5. The resulting optimal sequence will also be optimal for the original problem.

Example 1: There are 5 jobs each of which must go through three machines M_1, M_2 and M_3 in the order M_1 M_2 M_3. Processing times are given in the following table.

Job	*Processing Times*		
	M_1	M_2	M_3
1	7	4	3
2	9	5	8
3	5	1	7
4	6	2	5
5	10	3	4

Determine a sequence for five jobs that will minimize the elapsed time T

Solution: Here, Minimum T_{1j} = 5, Maximum T_{2j} = 5 and Minimum T_{3j} = 3

Since, one of two conditions Minimum T_{1j} = Maximum T_{2j} so process adopted in example 1, 2 and 3 can be followed.

The equivalent problem, involving five jobs and two fictitious machines M_4 and M_5 becomes.

Job	*Processing Times*	
	$T_{4j} = T_{1j} + T_{2j}$	$T_{5j} = T_{2j} + T_{3j}$
1	11	7
2	14	13
3	6	8
4	8	7
5	13	7

This new problem can be solved by the procedure described earlier. Because of ties possible optimal sequences are:

(i) | 3 | 2 | 1 | 4 | 5 |

(ii) | 3 | 2 | 4 | 5 | 1 |

(iii) | 3 | 2 | 1 | 5 | 4 |

(iv) | 3 | 2 | 4 | 1 | 5 |

(v) | 3 | 2 | 5 | 4 | 1 |

(vi) | 3 | 2 | 5 | 1 | 4 |

The elapsed time can be calculated for the first sequence as follows:

Job	*Machine M_1*		*Machine M_2*		*Machine M_3*	
	Time in	*Time out*	*Time in*	*Time out*	*Time in*	*Time out*
3	0	5	5	6	6	13
2	5	14	14	19	19	27
1	14	21	21	25	27	30
4	21	27	27	29	30	35
5	27	37	37	40	40	44

Minimum elapsed time T = 44 hours
Idle time for machine M_1 = 7 hours (37 to 44)
Idle time for machine M_2 = 25 hours

(i) 0 to 5 (iii) 19 to 21
(iv) 25 to 27 (ii) 6 to 14
(v) 29 to 37

Idle time for machine M_3 = 17 hours

(i) 0 to 6 (ii) 13 to 19
(iii) 35 to 40

Similarly, we can calculate for other alternatives.

Remark:

1. If conditions minimum $T_{1j} \geq$ maximum T_{2j} and/or minimum $T_{3j} \geq$ maximum T_{2j} do not bold, then there is no general procedure available till today by which we can find optimal sequence.
2. The idle times for individual machines may be different for alternative optimal sequences. Although, the minimum elapsed time will be same for all alternative optimal sequences.

Example 2: We have 4-jobs each of which has to go through the machines M_i, i = 1, 2, 3, …, 6 in the order of M_1 M_2 M_3 M_4 M_5 M_6. The processing time (in hours) is given below:

Machine \ Job	*A*	*B*	*C*	*D*
Machine M_1	18	17	11	20
Machine M_2	8	6	5	4
Machine M_3	7	9	8	3
Machine M_4	2	6	5	4
Machine M_5	10	8	7	8
Machine M_6	25	19	15	12

Determine a sequence of these four jobs that minimizes the total elapsed time.

Solution: Here, minimum M_{1j} = 11, minimum M_{6j} = 12 and maximum of M_{2j}, M_{3j}, M_{4j} and M_{5j} are 8, 9, 6 and 10 respectively. Since the conditions

Minimum $M_{1j} \geq$ Maximum M_{ij} (i = 2, 3, 4, 5)

and Minimum $M_{6j} \geq$ Maximum M_{ij} (i = 2, 3, 4, 5)

are satisfied. So, the given sequencing problem can be converted into a 4-jobs and 2-machines. The two factitious machines denoted by G and H and the corresponding processing times are shown in the following table:

Machine \ Job	A	B	C	D
Machine G	45	46	36	39
Machine H	52	48	40	31

Where $G_j \sum_{i=1}^{5} = M_{ij}$ and $H_j = \sum_{i=2}^{6} M_{ij}$ for j = 1, 2, 3, 4.

Now using the n-jobs and 2-machines method solved this new problem. We get the following optimal sequence

C	A	B	D

For total elapsed time, we have

Job	Machine					
	M_1	M_2	M_3	M_4	M_5	M_6
C	0-11	11-16	16-24	24-29	29-36	36-51
A	11-29	29-37	37-44	44-46	46-56	56-81
B	29-46	46-52	52-61	61-67	67-75	81-100
D	46-66	66-70	70-73	73-77	77-85	100-112

The total minimum elapsed time (T) is 112 hours.

Example 3: Given the following data: (i)

Machine \ Job	J_1	J_2	J_3	J_4	J_5	J_6
Machine M_1	12	10	9	14	7	9
Machine M_2	7	6	6	5	4	4
Machine M_3	6	5	6	4	2	4

(ii) The order of processing job on machines is M_1 M_3 M_2.

(iii) Sequence suggested: Jobs J_5, J_3, J_6, J_2, J_1, J_4.

(a) Determine the total elapsed time for the sequence suggested.

(b) Is the given sequence optimal?

(c) If your answer to (b) is ‘No’, determine the optimal sequence and the total elapsed time associated with it.

Solution: Here, minimum $M_{1j} = 7$, minimum $M_{2j} = 4$ and maximum $M_{3j} = 6$. Since the condition

$$\text{Minimum } M_{1j} \geq \text{Maximum } M_{3j} \quad (i = 2, 3, 4, 5)$$

is satisfied. So, the given sequencing problem can be converted into a 6-jobs and 2-machines. The two factitious machines denoted by G and H and the corresponding processing times are shown in the following table:

Machine \ Job	J_1	J_2	J_3	J_4	J_5	J_6
Machine G	18	15	15	18	9	13
Machine H	13	11	12	9	6	8

Where $G_j = M_{1j} + M_{3j}$ and $H_j = \sum_{i=2}^{3} M_{ij}$ for $j = 1, 2, \ldots, 6$.

Now using the n-jobs and 2-machines method solved this new problem. We get the following optimal sequence

J_1	J_3	J_2	J_4	J_6	J_5

For total elapsed time, we have

Job	Machines		
	M_1	M_2	M_3
J_1	0-12	12-18	18-25
J_3	12-21	21-27	27-33
J_2	21-31	31-36	36-42
J_4	31-45	45-49	49-54
J_6	45-54	54-58	58-62
J_5	54-61	61-63	63-67

The total minimum elapsed time (T) is 67 hours.

Given sequence of jobs are J_5, J_3, J_6, J_2, J_1, J_4.

Now for total elapsed time according to given sequence J_5-J_3-J_6-J_2-J_1-J_4. we have

Job	Machines		
	M_1	M_2	M_3
J_5	0–7	7–9	9–13
J_3	7–16	16–22	22–28
J_6	16–25	25–29	29–31
J_2	25–35	35–40	40–46
j_1	35–47	47–53	53–60
j_4	47–61	61–65	65–70

The total minimum elapsed time (T) is 70 hours.

(ii) Given sequence is not optimal.

(iii) Hence, the total minimum elapsed time is 67 hours.

13.7 PROCESSING 2-JOBS THROUGH K-MACHINES

This type of sequencing problem can be described as:-

(i) There are two jobs J_1 and J_2.

(ii) There are k-machines say M_1, M_2, M_3, …, M_k.

(iii) The technological ordering of each of two jobs on through k-machines is known in advance and ordering may not be the same for both the jobs J_1 and J_2.

(iv) The exact or expected time M_{ij} (i = 1, 2, 3, …, k; j = 1, 2) are known and shown as in the following table:

Job	Machine						
Job J_1	M_1	M_2	M_3	… …	M_i	… …	M_k
Job J_2	M_3	M_1	M_4	… …	M_i	… …	M_k

and

Job / Machine	Machine						
	M_1	M_1	M_1	…	M_i	… …	M_k
Job J_1	M_{11}	M_{21}	M_{31}	…	M_{ii}	… …	M_{k1}
Job J_2	M_{12}	M_{22}	M_{32}	…	M_{i2}	… …	M_{k2}

The optimal sequnce in this type of sequencing problem can be obtained by graphical method to make graph. The solution procedure of this problem is given as follows:

Solution Procedure:

Step 1: First we draw two perpendicular lines, x-axis representing the processing time for job J_1 on different k-machines while job J_2 remains idle and y-axis representing the processing time for job J_2 on different k-machines while job J_1 remains idle.

Step 2: Mark the processing time for job J_1 on x-axis and job J_2 on y-axis according to the technological order of k-machines.

Step 3: Construct the various block diagrams starting from the origin by pairing the same machine until the end point.

Step 4: After constructing blocks, we draw a line from origin included at an angle of 45° with the horizontal, will represent the work on the both jobs simultaneaously. The horizontal segment of this line means that job J_1 is processing while job J_2 is idle and vertically segment (moving upwards) means that job J_2 is processing while job J_1 is idle and diagonal segment (moving diagonally) means that the both jobs j_1 and J_2 are processing simultaneously.

Step 5: An optimum path is that which minimizes the idle time for job J_1 and J_2 i.e., a path in which diagonal movement is maximum.

Step 6: The total time is obtained by adding the idle time for either job to the processing time for that job i.e.,
Total elapsed time = Processing time of job J_1 + Idle time job J_1.
or Total elapsed time = Processing time of job J_2 + Idle time for job J_2.

Example 1: A machine shop has five machines A, B, C, D and E. Two jobs must be processed through each of these machines. The time (in hours) taken on each of the machines and the necessary sequence of jobs through the shop are given below:

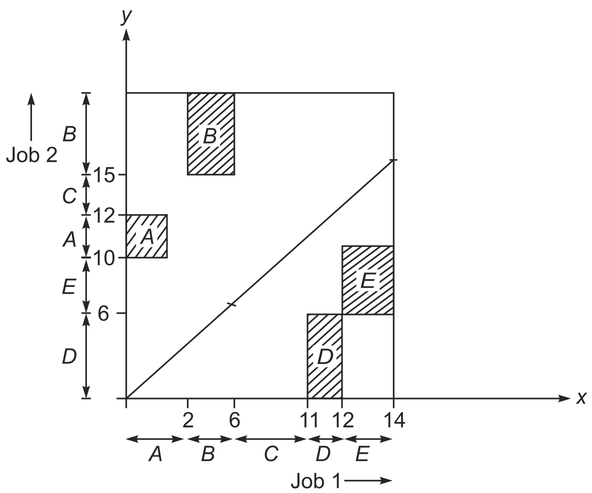
Job 1

	Machines				
Sequence:	*A*	*B*	*C*	*D*	*E*
Time:	2	4	5	1	2

Job 2

	Machines				
Sequence:	*D*	*E*	*A*	*C*	*B*
Time:	6	4	2	3	6

Use graphical method to determine the total minimum elapsed time.
Solution:



(i) First we mark the processing times for job 1 and job 2 on the x-axis and y-axis respectively according to the technological order of five machines.

(ii) Now construct the rectangular blocks by pairing the same machines.

(iii) Mark a path from origin to the end point of moving along the 45° line at much as possible.

Hence, the total elapsed time = Processing time of job J_1 + Idle time for job J_1.
= 14 + 7 = 21 hours.

or Total elapsed time = Processing time of job J_2 + Idle time for job J_2.
= 21 hours.

Example 2: Use graphical method to minimize the time added to process the following jobs on the machines shown, i.e., for each machines find the job which should be done first. Also calculate the total time elapsed to complete both the jobs:

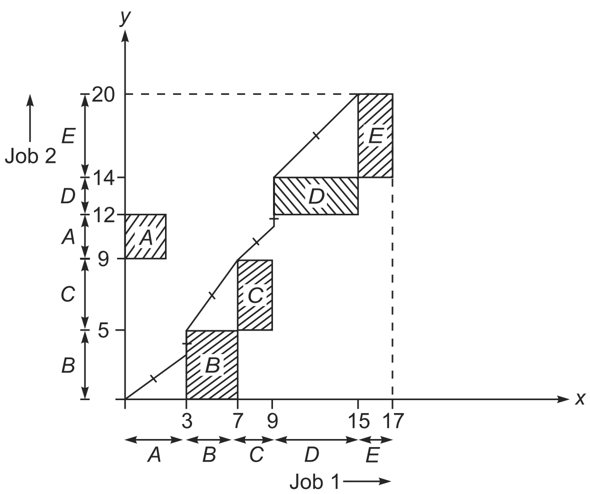
Job 1

	Machines				
Sequence:	*A*	*B*	*C*	*D*	*E*
Time:	3	4	2	6	2

Job 2

	Machines				
Sequence:	*B*	*C*	*A*	*D*	*E*
Time:	5	4	3	2	6

Solution:



(i) First we mark the processing times for job 1 and job 2 on the x-axis and y-axis respectively according to the technological order of four machines.

(ii) Now construct the rectangular blocks by pairing the same machines.

(iii) Mark a path from origin to the end point of moving along the 45° line at much as possible.

Hence, the total elapsed time = Processing time of job J_1 + Idle time for job J_1.

= 17 + (2 + 3) = 22 hours.

or Total elapsed time = Processing time of job J_2 + Idle time for job J_2.

= 20 + 2 = 22 hours.

EXERCISE 13.1

1. A publisher has one printing press, one binding machine and the manuscripts of 6 different books. The time required to perform the printing and binding operations for each book are shown below. Determine the order in which books should be processed, so that the total elapsed time is minimum.

Books	1	2	3	4	5	6
Printing time (in hours)	30	120	50	20	90	110
Binding time (in hours)	80	100	90	60	30	10

Also calculate idle time for printing and binding machines, respectively.
(**Ans:** optimum sequence: 4 → 1 → 3 → 2 → 5 → 6
Minimum elapsed time = 430 hours.
Idle time on printing machine = 10 hours
Idle time on binding machine = 60 hours)

2. Six jobs processed first on machine M_1 and then on machine M_2. The following table gives the processing times (in hours) for six jobs on two machines M_1 and M_2

Job	1	2	3	4	5	6
Machine M_1	5	9	4	7	8	6
Machine M_2	7	4	8	3	9	5

Find the sequence of jobs that minimizes the total elapsed time to complete the jobs.
(**Ans:** 3 → 1 → 5 → 6 → 2 → 4 Total Time = 42 hours)

3. Find the sequence that minimizes the total elapsed time required to complete the following jobs:

Processing Time in hours

No. of Jobs	1	2	3	4	5	6
Machine M_1	4	8	3	6	7	5
Machine M_2	6	3	7	2	8	4

(**Ans:** 3 → 1→ 5 → 6 → 2 → 4 Minimum total elapsed time = 35 hours)

4. Seven jobs which first processed on machine M_1 and then on machine M_2 in order M_1 M_2. The processing times (in hours) are given as follows:

Job	1	2	3	4	5	6	7
Machine M_1	3	12	15	6	10	11	9
Machine M_2	8	10	10	6	12	1	3

Determine a sequence of these jobs which minimizes the total elapsed time T. Is there any alternate optimal sequence?

(**Ans:** (i) $1 \to 4 \to 5 \to 3 \to 2 \to 7 \to 6$ (ii) $1 \to 4 \to 5 \to 2 \to 3 \to 7 \to 6$, $T = 67$ hours)

5. A company has six jobs A, B, C, D, E and F and all of these six jobs are to processed on machines M_1 and M_2, respectively. The time required for each job on each machine (in hours) is given below:

Job	A	B	C	D	E	F
Machine M_1	3	12	18	9	15	6
Machine M_2	9	18	24	24	3	15

Determine a sequence of jobs which minimizes the total elapsed time. Also calculate idle time for machine M_1 and M_2.

(**Ans:** optimal sequence: $A \to F \to D \to B \to C \to E$

Minimum elapsed time = 96 hours.

Idle time on M_1 = 33 hours

Idle time on M_2 = 3 hours)

6. A company has six jobs to be processed on three machines I, II and III in the order I II III. The processing time (in hours) for each job on each machine is given as follows:

Jobs	1	2	3	4	5	6
Machine I	18	12	29	36	43	37
Machine II	7	12	11	2	6	12
Machine III	19	12	23	47	28	36

What is the sequence that minimizes the total elapsed time? Also calculate idle times for machines I, II and III.

(**Ans:** optimal sequence: $2 \to 1 \to 4 \to 6 \to 5 \to 3$

Minimum elapsed time = 209 hours

Idle time for machine I = 34 hours

Idle time for machine II = 159 hours

Idle time for machine III = 44 hours)

7. Five jobs are to be processed on three machines *A*, *B* and *C* in the *ABC* order. The processing times (in hours) is given in the following table:

Processing Times (in hours)

Job No. →	1	2	3	4	5
Machine *A*	5	7	6	9	5
Machine *B*	2	1	4	5	3
Machine *C*	3	7	5	6	7

Determine a sequence for the jobs that minimizes the total elapsed time and idle time for each machine.

(**Ans:** optimal sequences: (i) 2 → 5 → 4 → 3 → 1
(ii) 5 → 4 → 3 → 2 → 1
(iii) 5 → 2 → 4 → 3 → 1

Minimum elapsed time = 40 hours

Idle time for each machine respectively are 8 hours for *A*, 25 hours for *B* and 12 hours for *C*.)

8. We have 5 jobs, each of which must be processed on machines 1, 2 and 3 in the order of 123. Processing times (in hours) on each machine for each job is given below. Determine a sequence which minimizes the elapsed time. Also find idle time on each machine.

Machine↓/Job→	I	II	III	IV	V
1	8	5	4	6	5
2	6	2	9	7	4
3	10	13	11	10	12

(**Ans:** optimal sequence: II → V → IV → III → I

Minimum elapsed time: 63 hours

Idle time on machine 1 = 35 hours

Idle time on machine 2 = 35 hours

Idle time on machine 3 = 7 hours)

9. Find the optimal sequence for the following sequencing problem.

Machine	Job			
	1	2	3	4
M_1	3	7	8	10
M_2	4	5	7	15
M_3	10	8	5	6

(**Ans:** optimal sequence: 1 → 2 → 4 → 3).

10. What is no passing rule in a sequencing problem? Explain the main assumptions made while dealing with sequencing problems.
11. Use graphical method to find the minimum elapsed total time sequence of 2-jobs and 5-machines, when we are given the following information:

Job 1

	Machines				
Sequence:	*A*	*B*	*C*	*D*	*E*
Time:	2	3	4	6	2

Job 2

	Machines				
Sequence:	*C*	*A*	*D*	*E*	*B*
Time:	4	5	3	2	6

(**Ans:** total elapsed time = 20 hours)

12. A machine shop has six machines *A*, *B*, *C*, *D*, *E* and *F*. Two jobs must be processed through each of machines. The time on machines and the necessary sequence of the jobs through the shop are given below:

Job 1

	Machines					
Sequence:	*A*	*C*	*D*	*B*	*E*	*F*
Time:	20	10	10	30	25	16

Job 2

	Machines					
Sequence:	*A*	*C*	*B*	*D*	*F*	*E*
Time:	10	30	15	10	15	20

Determine the optimum sequence for the job in order to minimize the total time necessary to finish the jobs.

(**Ans:** total elapsed time = 150 hours)

13. Two jobs are to be processed on four machines *A*, *B*, *C* and *D*. The technological order for these jobs on machines is as follows:

Job 1	*A*	*B*	*C*	*D*
Job 2	*D*	*B*	*A*	*C*

The processing times are given in the following table:

Job	Machine			
	A	*B*	*C*	*D*
1	4	6	7	3
2	4	7	5	8

Find the optimal sequence of jobs on each of the machines.

(**Ans:** total elapsed time = 24 hours)

14. Using graphical method, calculate the minimum time needed to process jobs 1 and 2 on five machines *A*, *B*, *C*, *D* and *E* i.e., for each machine find the job which should be done first. Also calculate the total needed to calculate both jobs.

Job 1

	Machines				
Sequence:	*A*	*B*	*C*	*D*	*E*
Time (hours):	6	8	4	12	4

Job 2

	Machines				
Sequence:	*B*	*C*	*A*	*D*	*E*
Time (hours):	10	8	6	4	12

(**Ans:** total elapsed time = 44 hours)

15. Use the graphical method to minimize the time needed to process the following jobs on the machines shown below:

Job 1

	Machines					
Sequence:	*A*	*B*	*C*	*D*	*E*	*F*
Time (hours):	8	10	2	6	12	10

Job 2

	Machines					
Sequence:	*B*	*A*	*C*	*F*	*D*	*E*
Time (hours):	12	6	4	8	6	10

(**Ans:** total elapsed time = 26 hours)

16. Explain the optimal sequencing procedure for solving the sequencing problem of processing 2-jobs through *k*-machines.

Index

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