UNIT 5 UNCONSTRAINED OPTIMISATION

Structure

5.0 OBJECTIVES

After going through this unit we should able to:

- solve simple unconstrained optimisation problems; and
- solve simple economic problems dealing with unconstrained optimisation.

5.1 INTRODUCTION

In this unit, our objective is to find and explain the necessary and sufficient conditions for any unconstrained optimisation problem. Optimisation in mathematical sense (and therefore for our purpose) means either maximisation or, minimisation of different mathematical functions. Optimisation is broadly distinguishable in two respects viz., Unconstrained Optimisation (which is our current focus) and Constrained Optimisation (which we discuss in the next unit). Unconstrained optimisation deals with those problems whose domain is not compressed by any constrained. We are going to use simple calculus to solve these kinds of optimisation problems. First, let us we find and discuss the necessary first order conditions for unconstrained optimisation before moving on to the sufficient condition or, the second order condition. In the next section, we will focus on some simple economic applications that deal with unconstrained optimisation.

5.2 UNCONSTRAINED OPTIMISATION

Under optimisation exercise we either maximise or, minimise some mathematical functions consists of either one variable or, several variables. To optimize a mathematical function, we need to find necessary first order conditions from which the optimum values of the variables can be found. But to find whether the value is maximum or, minimum we have to check the sufficient second order condition.

5.2.1 First Order Condition

Consider the function $y = f(x)$, which can be plotted as shown below. The function has the maximum point A and the minimum point B. Note that at these two points the value of y must be stationary. We can say that it is a necessary conditions for an extremum of y that $dy = 0$ instantaneously as x varies. This condition is known as the differential version of the first order condition for an extremum and we have already discussed it in the preceding unit. As we have said earlier, this condition is necessary but not sufficient for either a maximum or, a minimum. Therefore, it is clear that the necessary first order condition for both maxima and minima are the same.

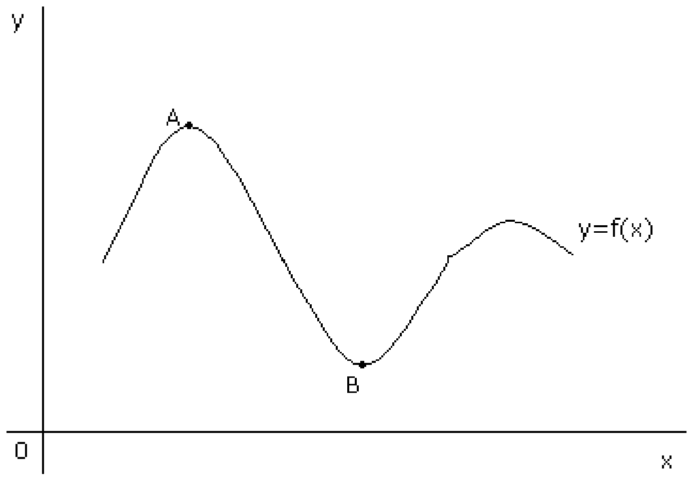

Fig 5.1: First order condition for optimum

We will see that the above condition is equivalent to the derivative version of the first order condition $\frac{dy}{dx} = 0$ or, $f^{'}(x) = 0$. We note that when there is no change in x (dx = 0), dy will automatically be zero. But this, of course, is not what the first order condition is all about. What the first order condition requires is that dy be zero as x is varied, that is, as arbitrary (positive or, negative, but not zero) infinitesimal changes of x occur. In such a context, with $dx \neq 0$, dy can be zero if and only if $f^{'}(x) = 0$. Thus, the derivative condition $f^{'}(x) = 0$ and the differential condition dy = 0 "for arbitrary nonzero values of dx" are indeed equivalent.

Now suppose $y = f(x_1, x_2, ..., x_n)$ then in this case, the first order condition for optimisation is given by the n-equation viz., $\frac{\partial y}{\partial x_1} = f^1(x) = 0$, $\frac{\partial y}{\partial x_2} = f^2(x) = 0$, …, $\frac{\partial y}{\partial x_n} = f^n(x) = 0$. From these n equations, we can find $x_1^*, x_2^*, ..., x_n^*$.

Example: Find the optimal vales of $y = x^3 - x$ in the domain $-\infty < x < +\infty$.

Solution: The necessary first order condition gives $\frac{dy}{dx} = 0$, which can be transformed to $3x^2 - 1 = 0$ - (I). From equation (I) we get at optimum either $x^* = +\frac{1}{\sqrt{3}}$ or, $x^* = -\frac{1}{\sqrt{3}}$. Putting these values of x, we get optimum y i.e., y^*.

Example: Find the optimal values of $z = y^2 - xy + x^2$ on the domain $-\infty < x < +\infty$ and $-\infty < y < +\infty$.

Solution: The necessary first order condition gives $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$, from which we can get $-y + 2x = 0$ and $2y - x = 0$. We solve this and get $x^* = 0$ and $y^* = 0$. Finally putting these optimum values into the original equation we get $z^* = 0$.

5.2.2 Second Order Condition

For the time being, let us assume the function $z = f(x, y)$ can give rise two first order partial derivatives, $f_x \equiv \frac{\partial z}{\partial x}$ and $f_y \equiv \frac{\partial z}{\partial y}$. Since f_x is itself a function of x (as well as y), we can measure the rate of change of f_x with respect to x, while y remains fixed, by a particular second order (or, second) partial derivative denoted by either f_{xx} or, $\frac{\partial^2 z}{\partial x^2}$.

$f_{xx} \equiv \frac{\partial}{\partial x}(f_x)$ Or, $\frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$. The notation f_{xx} has a double subscript signifying that the primitive function f has been differentiated partially with respect to x twice, whereas the notation $\frac{\partial^2 z}{\partial x^2}$ resembles that of $\frac{d^2 z}{dx^2}$ except for the use of the partial symbol. In a perfectly analogues manner, we can use the second partial derivative $f_{yy} \equiv \frac{\partial}{\partial y}(f_y)$ or, $\frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$ to denote the rate of change of f_y with respect to y, while x is held constant.

Recall however, that f_x is also a function of x. Hence, there can be written two more partial derivatives: $f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$ and $f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$.

These are called cross (or, mixed) partial derivatives because each measures the rate of change of one first order partial derivative with respect to the "other" variable. We can also note that using total differential equation we get $d^2 z \equiv d(dz) \equiv \frac{\partial(\partial z)}{\partial x} dx + \frac{\partial(\partial z)}{\partial y} dy$.

Using the concept of $d^2 z$, we can state the second order sufficient condition for a maximum of $z = f(x, y)$ as follows: $d^2 z < 0$ for arbitrary values of dx and dy, not both zero – (a). The rationale behind is very similar to that of the $d^2 z$ condition for the one variable case and it can be explained by means of Fig 5.2.1, which depicts the bird's-eye view of a surface. Let point A on the surface – the point lying directly above the point (x_0, y_0) in the domain – satisfy the first order condition. Then the point A is a prospective candidate for a maximum. Whether it in fact qualifies depends on the surface configuration in the neighborhood of A. If an infinitesimal movement away from A in any direction along the surface invariable results in a decrease in z – that is, if dz < 0 for arbitrary values of dx and dy, not both zero – A is a pick of a dome. Given that dz = 0 at point A, however, the condition dz < 0 at point in the neighborhood of A amounts to the stipulation that dz is decreasing, that is, $d^2 z \equiv d(dz) < 0$, for arbitrary values of dx and dy, not both zero. Thus (a) constitutes a sufficient condition for identifying a stationary value as a maximum of z. Analogous reasoning would show that a counterpart second order sufficient condition for identifying value as a minimum of z = f(x, y) is as follows: $d^2 z > 0$ for arbitrary values of dx and dy, not both zero – (b).

The reason why (a) and (b) are sufficient conditions but not necessary conditions, is that it is again possible for d^2z to take a zero value at a maximum or, a minimum. For this reason, second order necessary conditions must be stated with weak inequalities as follows: For maximum of z: $d^2 z \le 0$ for arbitrary values of dx and dy, not both zero. For minimum of z: $d^2 z \ge 0$ for arbitrary values of dx and dy, not both zero – (c). In the following, however, we pay more attention to the second order sufficient conditions.

For operational convenience, second order differential conditions can be translated into equivalent conditions on second order derivatives. In the two-variable case, this would entail restrictions on the signs of the second partial derivatives f_{xx}, f_{yy} and f_{xy}. The actual translation would require knowledge of quadratic forms. But we may first introduce the main results here: For any values of dx and dy, not both zero, we have

$d^2 z < 0$ if and only if $f_{xx} < 0$, $f_{yy} < 0$, and $f_{xx} f_{yy} > f_{xy}^2$

$d^2 z > 0$ if and only if $f_{xx} > 0$, $f_{yy} > 0$, and $f_{xx} f_{yy} > f_{xy}^2$.

Note that the sign of d^2z hinges not only on f_{xx} and f_{yy}, which have to do with the surface configuration around point A, but also on the cross partial derivative f_{xy}. The role played by this latter partial derivative is to ensure that the surface in question will yield (two-dimensional) cross sections with the same type of configuration in all possible directions.

Table 5. 1: Conditions for Relative Extremum: z=f(x, y)

Condition	**Maximum**	**Minimum**
First order necessary condition	$f_x = f_y = 0$	$f_x = f_y = 0$
Second order sufficient condition	$f_{xx}, f_{yy} < 0$	$f_{xx}, f_{yy} > 0$
	and $f_{xx} f_{yy} > f_{xy}^2$	and $f_{xx} f_{yy} > f_{xy}^2$

The above result, together with the first order condition, enables us to construct Table 5.1. It should be understood that all the second partial derivatives therein are to be evaluated at the stationary point where $f_x = f_y = 0$. It should also be stressed that the second order sufficient condition is not necessary for an extremum. In particular, if a stationary value is characterised by $f_{xx} f_{yy} = f_{xy}^2$ in violation of that condition, that stationary value may nevertheless turn out to be an extremum. On the other hand, in the case of another type of violation, with a stationary point characterised by $f_{xx} f_{yy} < f_{xy}^2$, we can identify that point as a saddle point, because the sign of d^2z will in that case be indefinite (positive for same values of dx and dy, but negative for others).

For n-variable case we can generalise the above second order condition. The satisfaction of the first order condition earmarks certain values of z as the stationary values of the objective function. If at a stationary value of z we find that d^2z is positive definite, this will suffice to establish that value of z as a minimum. Analogously, the negative definiteness of d^2z is a sufficient condition for the stationary value to be maximum.

When there are n choice variables, the objective function may be expressed as $z = f(x_1, x_2, ..., x_n)$.

The total differential will then be $dz = f_1 dx_1 + f_2 dx_2 + ... + f_n dx_n$ so that the necessary condition for extremum (dz = 0 for arbitrary dx_i) means that all the n first order partial derivatives are required to be zero.

The second order differential d^2z will again be a quadratic form, which can be expressed as an $n \times n$ array. The coefficients of that array, properly arranged, will now give the (symmetric) Hessian

$$|H| = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

with principle minors $|H_1| = f_{11}$, $|H_2| = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \ldots,$ $|H_n| = |H|$. The second order sufficient condition for extremum is, as before, that all the n principal minors be positive (for a minimum in z) and that they duly alternate in sign (for a maximum in z), the first one being negative.

In summary, then – if we concentrate on the determinantal test – we have the criteria as listed in Table 5.2, which is valid for an objective function of any number of choice variables. As special cases, we can have n = 1 or, n = 2. When n = 1, the objective function is z = f(x), and the condition for maximization, $f_1 = 0$ and $|H_1| < 0$, reduce to $f'(x) = 0$ and $f''(x) < 0$, exactly as we learned above. Similarly, when n = 2, the objective function is $z = f(x_1, x_2)$, so that the first order condition for maximum is $f_1 = f_2 = 0$, whereas the second order sufficient condition becomes $f_{11} < 0$ and $\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = f_{11} f_{22} - f_{12}^2 > 0$, which is merely a restatement of the information given above.

Table 5. 2: Determinantal Test for Relative Optimum: $z = f(x_1, x_2, ..., x_n)$

Condition	**Maximum**	**Minimum**
First order necessary condition	$f_1 = f_2 = ... = f_n$	$f_1 = f_2 = ... = f_n$
Second order sufficient condition	$(-1)^i \, \lvert H_i \rvert > 0 \; \forall$ i =1,…,n	$\lvert H_i \rvert > 0 \; \forall$ i=1,…,n

Example: Find the extreme value(s) of $z = x^2 + xy + 2y^2 + 3$ and determine whether they are maximum or, minimum.

Solution: The relevant first order conditions are the following:

$f_x = 2x + y = 0$ and $f_y = x + 4y = 0$. Solving these two equations we get $x^* = 0$ and $y^* = 0$. Note that $f_{xx} = 2 > 0$ and $f_{yy} = 4 > 0$ and as $f_{xy} = 1$ we have $f_{xx} f_{yy} > f_{xy}^2$. So at $x^* = 0$ and $y^* = 0$ the function z is minimized.

Example: Find the extreme value(s) of $z = -x^2 + xy - y^2 + 2x + y$ and determine whether they are maximum or, minimum.

Solution: The relevant first order conditions are the following:

$f_x = -2x + y + 2 = 0$ and $f_y = x - 2y + 1 = 0$. Solving these two equations we get $x^* = 5/3$ and $y^* = 4/3$. Now note that $f_{xx} = -2 < 0$ and $f_{yy} = -2 < 0$ and as $f_{xy} = 1$ we have $f_{xx}f_{yy} > f_{xy}^2$. So at $x^* = 5/3$ and $y^* = 4/3$ the function z is maximised.

Check Your Progress 1

1) Find the extreme value(s) of $z = -x^2 + xy - y^2 + x + 5y$ and determine whether they are maximum or, minimum.

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2) Find the extreme value(s) of $y = x^2 - 6$ and determine whether they are maximum or, minimum.

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3) Find the extreme value(s) of $y = x^3 - 2x^2 + x - 6$ and determine whether they are maximum or, minimum.

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4) Find the extreme value(s) of $z = x^2 - 2x - y^2$ and determine whether they are maximum or, minimum.

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5.3 SOME ECONOMIC PROBLEMS

In this section, we will give some economic application concerning unconstrained optimisation. Our first example is concerning the multi-product firm who takes the prices as given i.e., she produces under competitive structure. The second example is also regarding multi-product firm but here the firm acts as a monopolist where she can choose how much to produce and at what price. In both the cases the firm's objective is to maximise her own profit where profit is defined as net revenue minus net cost. These objectives we take for simplicity i.e., one can think of other objective of the firm viz., revenue maximisation, but that is not going to be discussed in this unit. For simplicity again, we take cost function to be the same in both example.

Example 1: This example is regarding the problem of a multi-product firm. Let us first postulate a two-product firm under circumstances of pure competition. Since with pure competition the prices of both commodities must be taken as exogenous, these will be given by $p_1 = 5$ and $p_2 = 3$, respectively. Accordingly, the firm's revenue function will be $R = p_1q_1 + p_2q_2 = 5q_1 + 3q_2$, where q_i represents the output level of the i^{th} product per unit of time. The firm's cost function is assumed to be $C = 2q_1^2 + 2q_2^2 + q_1q_2$.

Note that $\frac{\partial C}{\partial q_1} = 4q_1 + q_2$ (the marginal cost of the first product) is a function not only of q_1 but also of q_2. Similarly, the marginal cost of the second product also depends, in part, on the output level of the first product. Thus, according to the assumed cost function, the two commodities are seen to be technically related in production.

The profit function of this hypothetical firm can now be written readily as $\pi = R - C = 5q_1 + 3q_2 - 2q_1^2 - 2q_2^2 - q_1q_2$, a function of two choice variables (q_1 and q_2) and two price parameters. It is our task to find the levels of q_1 and q_2, which in combination, will maximise π. For this purpose, we find the first order partial derivatives of the profit function:

$$\pi_1\left(\equiv \frac{\partial \pi}{\partial q_1}\right) = 5 - 4q_1 - q_2 \text{ and } \pi_2\left(\equiv \frac{\partial \pi}{\partial q_2}\right) = 3 - 4q_2 - q_1 \quad \text{- (a)}$$

Setting these both equal to zero, to satisfy the necessary condition for maximum, we get the two simultaneous equations $4q_1 + q_2 = 5$ and $q_1 + 4q_2 = 3$, which yield the unique solution $q_1^* = \frac{4p_1 - p_2}{15} = \frac{17}{15}$ and $q_2^* = \frac{4p_2 - p_1}{15} = \frac{7}{15}$. Thus, the maximum profit that the firm will get is $\pi^* = 5.51$ (approx).

To be sure that this does represent a maximum profit, let us check the second order condition. The second partial derivatives, obtainable by partial differentiation of equation (a), give us the following Hessian:

$$|H| = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix}$$

Since $|H_1| = -4 < 0$ and $|H_2| = 15 > 0$, the Hessian matrix (or, d^2z) is negative definite, and the solution does maximise the profit. In fact, since the signs of the principal minors do not depends on where they are evaluated, d^2z is in this case everywhere negative definite. Thus, the objective function must be strictly concave, and the maximum profit found above is actually a unique absolute maximum.

Example 2: This example again regarding a multi-product firm but with the modification that in earlier case the firm face given price i.e., multi-product firm under competitive structure, in this example the multi-product firm is a monopoly in both the product i.e., the firm can decide how much quantity to be produced and at the same time she can choose any price for her product. Now by virtue of this new market-structure assumption, the revenue function must be modified to reflect the fact that the prices of the two products will now vary with their output levels (which are assumed to be identical with their sales levels, no inventory accumulation being contemplated in the model). The exact manner in which prices will vary with output levels is, of course, to be found in the demand functions for the firm's two products.

Suppose that the demand facing the monopolistic firm, are as follows:

$$q_1 = 40 - 2p_1 + p_2 \text{ and } q_2 = 15 + p_1 - p_2 \quad \text{- (a)}$$

These equations reveal that the two commodities are related in consumption; specifically, they are substitute goods, because an increase in the price of one will raise the demand for the other. As given, equation (a) expresses the quantities demanded q_1 and q_2 as functions of prices, but for our present purposes, it will be more convenient to have prices p_1 and p_2 expressed in terms of the sales volumes q_1 and q_2, that is, to have average revenue functions for the two products. Since equation (a) can be rewritten as

$-2p_1 + p_2 = q_1 - 40$ and $p_1 - p_2 = q_2 - 15$, we may (considering q_1 and q_2 as parameters) apply Cramer's rule to solve for p_1 and p_2 as follows:

$$p_1 = 55 - q_1 - q_2 \text{ and } p_2 = 70 - q_1 - 2q_2 \quad \text{- (b)}$$

These constitute the desired average-revenue functions, since $p_1 \equiv AR_1$ and $p_2 \equiv AR_2$. Consequently, the firm's total-revenue function can be written as:

$$R = p_1q_1 + p_2q_2$$
$$= (55 - q_1 - q_2)q_1 + (70 - q_1 - 2q_2)q_2$$
$$= 55q_1 + 70q_2 - 2q_1q_2 - q_1^2 - 2q_2^2$$

If we again assume the total cost function to be $C = q_1^2 + q_1q_2 + q_2^2$, then the profit function will be:

$$\pi = R - C$$
$$= 55q_1 + 70q_2 - 3q_1q_2 - 2q_1^2 - 3q_2^2 \text{ - (c),}$$

which is an objective function with two choice variables. Once the profit maximizing output levels q_1^* and q_2^* are found, however, the optimal prices p_1^* and p_2^* are easy enough to find from equation (b).

The objective function yields the following first and second partial derivatives:

$$\pi_1 = 55 - 3q_1 - 4q_2 \qquad \pi_2 = 70 - 3q_1 - 6q_2$$
$$\pi_{11} = -4 \qquad \pi_{12} = \pi_{21} = -3 \qquad \pi_{22} = -6$$

To satisfy the first order condition for a maximum of π, we must have $\pi_1 = \pi_2 = 0$; that is $4q_1 + 3q_2 = 55$ and $3q_1 + 6q_2 = 70$, thus the solution output levels (per unit of time) are $(q_1^*, q_2^*) = (8, 7\frac{2}{3})$.

Upon substitution of this result into equation (b) and (c), respectively, we find that $p_1^* = 39\frac{1}{3}$ $p_2^* = 46\frac{2}{3}$ and $\pi^* = 488\frac{1}{3}$ (per unit of time)

Inasmuch as the Hessian is $\begin{bmatrix} -4 & -3 \\ -3 & -6 \end{bmatrix}$, we have $|H_1| = -4 < 0$ and $|H_2| = 15 > 0$, so that the value of π^* does represent the maximum profit. Here, the signs of the principle minors are again independent of where they are evaluated. Thus, the Hessian matrix is everywhere negative definite, implying that the objective function is strictly concave and that it has a unique absolute maximum.

5.4 LET US SUM UP

In this unit we emphasise an important tool, which has enormous application in economic viz., unconstrained optimisation. Unconstrained optimisation problem deals with optimisation (i.e., either minimisation or, maximisation) problems where the domain of objective function is not limited by any constrained. There are many real life economic problems for which unconstrained optimisation technique is necessary. Any problem regarding profit maximisation, we use the technique of unconstrained optimisation. In this unit, we emphasise the technique, how to solve unconstrained optimisation problems. For that we first introduced the first order necessary conditions for optimisation. We saw that if there are n choice variables we must have n number of equality as first order condition to solve the optimum value of each variable. We also saw that the first order conditions are same for both maximisation and minimisation problem, and therefore we can’t get whether by putting the optimum values of variables we get maximum or, minimum value of the objective function. To solve this kind of problem we therefore introduce second order sufficient condition for optimisation. We saw second order condition is different for maximisation and minimisation exercise. We also saw that for a given objective function and for given optimum values of choice variables, if the corresponding Hessian matrix is negative definite at optimum then the optimum values give the maximum value of the objective function. Analogously, if the Hessian matrix is positive definite at optimum then the optimum values give the minimum value of the objective function. Therefore, from the first order condition (supported by second order condition), we can determine the maximum or, the minimum value of the objective function. There is a notion of necessary second order condition which tells that the necessary second order condition for maximization is that, the Hessian matrix should be negative semi-definite and the necessary second order condition for minimization is that, the Hessian matrix should be positive semi-definite.

Finally, we give some interesting economic applications that clear the techniques of unconstrained optimisation problem. For that our first example is a problem regarding multi-product firm under competitive structure. Then we give an example again concerning multi-product firm but now the firm acting as monopolist in all markets.

5.5 KEY WORDS

Positive Definite Matrix: A symmetric matrix A is positive definite if it is true that for every vector x, the real number which results from the multiplication x'Ax > 0 for all nonzero x. Such a matrix has real positive eigenvalues.

Positive Semi-definite Matrix: A symmetric matrix A is positive definite if it is true that for every vector x, the real number which results from the multiplication x'Ax ³ 0 for all nonzero x.

Negative Definite Matrix: A symmetric matrix A is negative definite if it is true that for every vector x, the real number which results from the multiplication x'Ax ≤ 0 for all nonzero x. Such a matrix has real negative eigenvalues.

Negative Semi-definite Matrix: A symmetric matrix A is negative semi-definite if it is true that for every vector x, the real number which results from the multiplication x'Ax ≤ 0 for all nonzero x.

Stationary Point: In calculus, a **stationary point** is referred to a point on a graph where the tangent to the graph is parallel to the *x*-axis or, equivalently, where the derivative of the function equals zero.

Necessary and Sufficient Condition for Optimum: Given a function $z = f(x_1, x_2, ..., x_n)$, the conditions are obtained as the following:

Condition	Maximum	Minimum
First order necessary condition	$f_1 = f_2 = ... = f_n$	$f_1 = f_2 = ... = f_n$
Second order sufficient condition	$(-1)^i \mid H_i \mid > 0 \ \forall$ i =1,…,n	$\mid H_i \mid > 0 \ \forall$ i=1,…,n

5.6 SOME USEFUL BOOKS

Henderson, James M. and Quandt, Richard E. (1980), *Microeconomic Theory*, McGrow-Hill Book Company, New York.

Allen, R.G.D. (1938), *Mathematical Analysis for Economists*, St. Martin's Press, New York.

Varian, Hal (1992), *Microeconomic Analysis*, W.W. Norton & Company, Inc., Yew York.

5.7 ANSWERS OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See the example given in section 5.2.2.

2) The objective function is $y = x^2 - 6$. Therefore, from the first order condition for optimisation, we get $2x = 0$. Solving this equation we have x = 0. The second order condition gives that $d^2y = 2 > 0$, which is not a function of x. Therefore, we have $d^2y = 2$ at x = 0. So at x = 0, we reach the minima of the objective function.

3) The objective function is $y = x^3 - 2x^2 + x - 6$. Therefore, we get the first order condition $3x^2 - 4x + 1 = 0$ or, either x = $\frac{1}{3}$ or, x = 1. Note that $d^2y = 6x - 4$. So, $d^2y > 0$ at x = 1 and $d^2y < 0$ at x = $\frac{1}{3}$. Therefore, at x = 1, we have the minima and at x = $\frac{1}{3}$.

4) The objective function is $z = x^2 - 2x - y^2$. The necessary first order condition yields $2x - 2 = 0$ and 2y = 0. From these two first order conditions we get x = 1 and y = 0. The corresponding Hessian matrix is $|H| = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, so we get that $|H_1| = 2 > 0$ and $|H_2| = 4 > 0$. Therefore, we have the minimum value of z at x =1 and y = 0.