Formulating a Linear Programming Problem (lecture – 1)

The LPP model

Formulating a maximization LPP

Formulating a minimization LPP

Introduction

- A large number of managerial decision problems faced by business managers involve allocation resources to various activities, with the objective of maximizing profit or minimizing time or cost.
- When plenty of resources are available – decision making is not a problem.
- But such situations are rare.
- Normally, there are several activities to be performed and resources have to allocated such as to obtain the objective.

Introduction

- This is called optimization of allocation of resources.
- Mathematical model making technique – Linear Programming is one such technique that helps decision making in such situations.

Linear programming

- It is a technique for choosing the best alternative from a set of feasible alternatives – in situations in which the objective function as well as the constraints can be expressed as linear mathematical functions.
- Requirements for using LPP to solve problems:
  - There should be an objective function which should be clearly identifiable and measurable in quantitative terms.
  - The activities to be included should be distinctly identifiable and measurable in quantitative terms.

Cont…

- The resources to be allocated should be identifiable and measurable in quantitative terms. They must be limited in supply.
- The relationships representing the objective function, and the resource constraints, must be linear in nature.
- There should be a series of feasible alternative courses of action available to the decision maker which are determined by the resource constraints.

Maximization case

An example of formulation of LPP

The decision problem

A firm is engaged in producing two products, A and B. each unit of A requires 2 kg of raw material and 4 hours of labour. Each unit of B requires 3 kg of raw material and 3 hours of labour. Every week, the firm has an availability of 60 kg of raw material and 96 labour hours. One unit of product A sold yields Rs 40 and every unit of Product B sold yields Rs 35 as profit.

Formulate this as a LPP to determine how many units of each of the products should be produced each week so that the firm can earn the maximum profit. Assume that there is no marketing constraint and that all that is produced can be sold.

Steps to Formulating a LPP

- Step 1 – identify the decision variables.
- Step 2 – write the objective function in terms of the decision variables
- Step 3 – Identify and write the constraints in terms of the decision variables.
- Step 4 – Identify and define whether the decision variables can take negative values or not.

The decision variables

- It is clear that the decision is regarding the number of units to be produced of products A and B.

- Let the number of units of product A to be produced in the week = x
- Let the number of units of product B to be produced in the week = y

The objective function

- The goal here is maximization of profit, which would be obtained by producing (& selling) x and y number of units.

- Objective Function:

$$\text{Maximize } Z = 40\,x + 35\,y$$

Since profit per unit of x and y is Rs 40 and Rs 35 respectively.

Constraints

- Resources are in limited supply.
- The mathematical relationships which is used to explain the limitation is inequality.
- The limitations itself is known as a constraint.
  - Each unit of product A requires 2 kg of raw material and each unit of product B requires 3 kg.
  - The total availability per week is 60 kg.
  - This can be expressed as: $\mathbf{2x + 3y \leq 60}$
  - Similarly, each unit of product A needs 4 hours of labor and each unit of B needs 3 hours.
  - The total availability is 96 hours
  - This can be expressed as: $\mathbf{4x + 3y \leq 96}$

Variables can take negative values or not?

- Obviously since decision variables are number of products to be produced – they cannot have negative values.
- Thus both the variables can assume values only *greater-than-or equal-to-zero.*
- This is called the non-negativity condition.
- It is expressed as: $x \geq 0, y \geq 0$

The completed LPP

- Let the number of units of product A to be produced in the week = x
- Let the number of units of product B to be produced in the week = y

Objective function

$$\text{Max } Z = 40x + 35y$$

Subject to:

raw material $2x + 3y \le 60$

labor hours $4x + 3y \le 96$

NNC $x \ge 0,\ y \ge 0$

Minimization case

An example of formulation of LPP

The problem

- The Agriculture Research Institute has advised a farmer to spread out at least 4800 kg of a special phosphate fertiliser and not less than 7200 kg of a special nitrogen fertiliser to raise productivity of crops in his fields.
- There are two sources for obtaining these – mixtures A and B. Both of these are available in bags weighing 100 kg each and they cost Rs 40 and Rs 24 respectively. Mixture A contains phosphate and nitrogen equivalent of 20 kg and 80 kg respectively, while mixture B contains these ingredients equivalent of 50 kg each.
- Write this as a LPP to determine how many bags of each type should the farmer but in order to obtain the required fertilizer at minimum cost.

The decision variables

- A combination of mixture A and B is to be bought such that total cost is minimized and the fertilizer requirement is satisfied.

- Let the number of bags of mixture A to be bought to satisfy the requirement = x
- Let the number of bags of mixture B to be bought to satisfy the requirement = y

The objective function

- The goal here is minimization of cost, which would be incurred by purchasing x and y number of bags of mixture A and B.

- Objective Function:

$$\text{Minimize } Z = 40x + 24y$$

Since cost per bag of x and y is Rs 40 and Rs 24 respectively.

CONSTRAINTS

- In this problem, there are two constraints, a minimum of 4800 kg of phosphate and 7200 of nitrogen ingredient is required.
- Each bag of mixture A contains 20 kg of Ph and 80 kg of N ingredient.
- Each bag of mixture B contains 50 kg of Ph and 50 kg of N ingredient.
- The phosphate requirement can be expressed as:

$$20\,x + 50\,y \geq 4800$$

- The Nitrogen ingredient can be expressed as:

$$80\,x + 50\,y \geq 7200$$

- The sign used here is $\geq$ since 4800 and 7200 are minimum requirements. Therefore the total must be at least *equal-to-or-greater-than*, but cannot be less than the required amount.

Variables can take negative values or not?

- Obviously since decision variables are number of bags to be purchased – they cannot have negative values.
- Thus both x and y can assume values only *greater-than-or equal-to-zero.*
- This is called the non-negativity condition.
- It is expressed as: $x \geq 0, y \geq 0$

Complete LPP

Let the number of bags of mixture A to be bought to satisfy the requirement = x
Let the number of bags of mixture B to be bought to satisfy the requirement = y

Objective function

$$\text{Min } Z = 40x + 24y$$

Subject to:

$$20x + 50y \geq 4800 \quad \text{(Ph requirement)}$$

$$80x + 50y \geq 7200 \quad \text{(N requirement)}$$

$$x \geq 0,\ y \geq 0$$

- In the next lecture we will learn how to solve the LPP graphically

Graphical solution of an LPP

Solving a maximization LPP

Solving a minimization LPP

Steps to solving an LPP graphically

- Step 1: Modify each constraint by replacing the inequality sign (≤ ≥) with the equal to sign (=)
- Step 2: Solve each equation to obtain two points such that the equation can be plotted on the graph paper.
  - This can be done by finding the value of ‘y’ when x = 0 and then the value of ‘x’ when y = 0.
  - A minimum of 2 points are needed to draw a line.
- Step 3: Taking an appropriate scale for the ‘x’ & ‘y’ axis, draw the constraint equations on the graph paper.

Steps to solving an LPP graphically

- Step 4: Identify the solution region for each constraint equation.
  - If the original constraint was $\leq$ type, the solution to the constraint lies between its line and the origin. This means the solution includes the origin (0,0).
  - If the original constraint was $\geq$ type, the solution to the constraint lies between its line and infinity. This means the solution does not include the origin (0,0).
  - If the original constraint was = type, all points on the line are a part of its solution.
- Step 5: Identify the *common solution region* that satisfies all given constraints.

Steps to solving an LPP graphically

- Step 6: Name the *corner points of the feasible region* and read their values from the graph.
- Step 7: Now find the *value of the objective function* at each of these points.
- Step 8: The *optimum solution* is the one that gives largest (for Max Z) and least (for Min Z) value of the objective function.
- Step 9: The values of the identified corner point is the *optimum value of the decision variable* of the LPP.

Maximization case

An example of graphical solution of LPP

The completed LPP *(from lecture 1)*

Objective function

$$\text{Max } Z = 40x + 35y$$

Subject to:

raw material $2x + 3y \leq 60$

labor hours $4x + 3y \leq 96$

NNC $x \geq 0,\ y \geq 0$

Solve to obtain 2 points per constraint

Material constraint $2x + 3y = 60$

When $x=0$, $y = 60/3 = 20$ hence the point is $(0, 20)$

When $y=0$, $x=60/2 = 30$ hence the point is $(30, 0)$

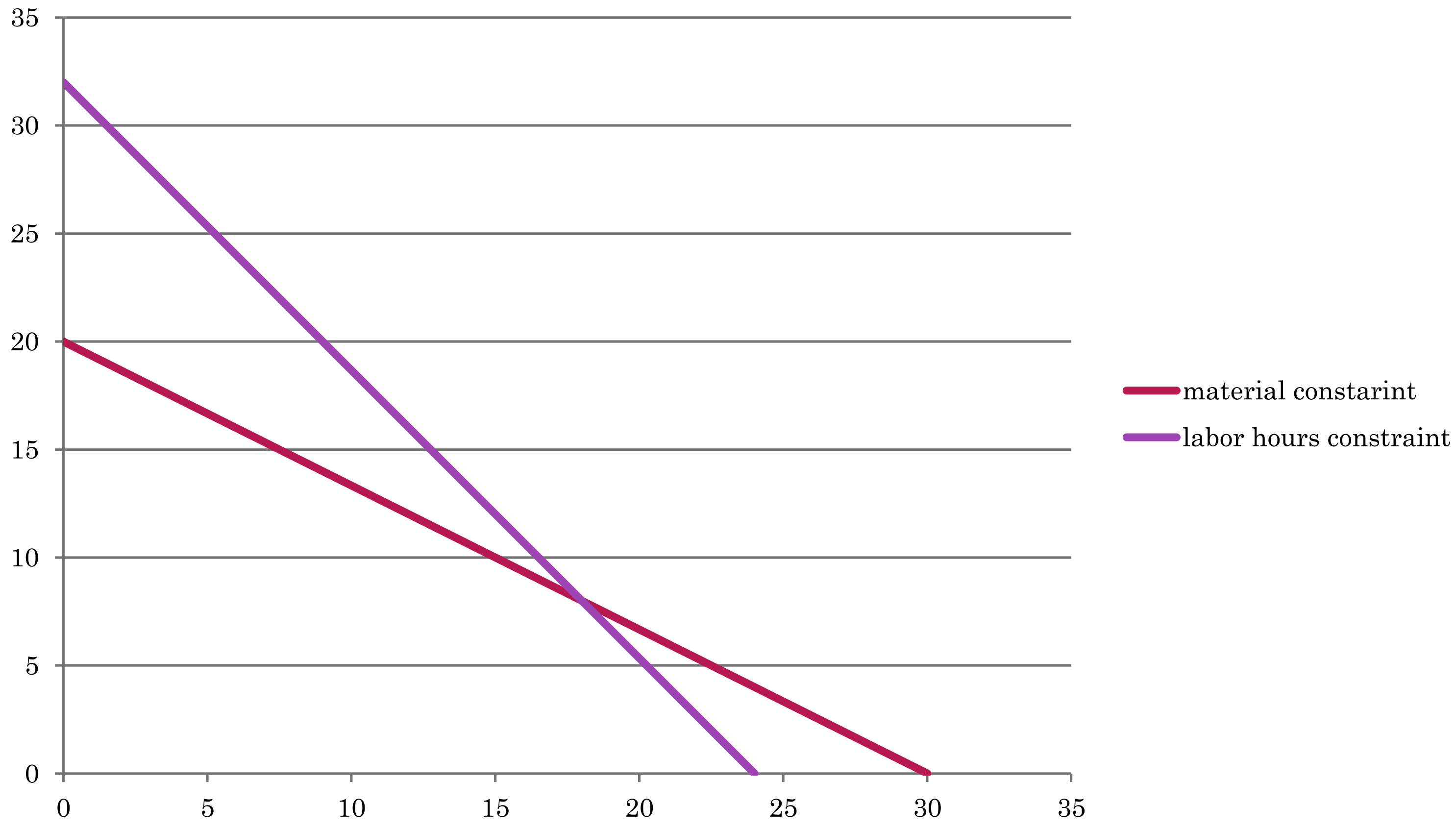
labor hours constraint $4x + 3y = 96$

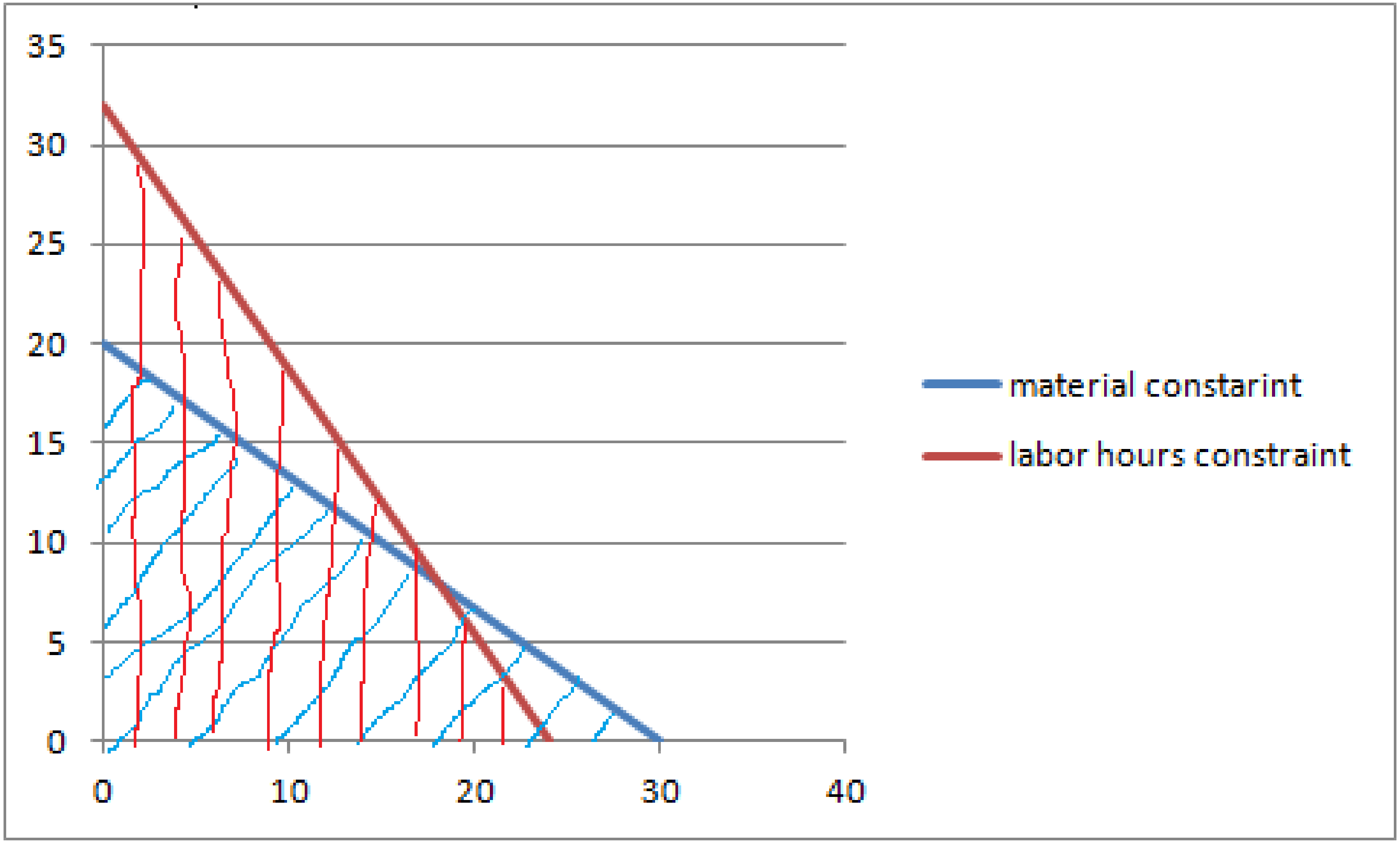
When $x=0$, $y = 96/3 = 32$ hence the point is $(0, 32)$

When $y=0$, $x= 96/4 = 24$ hence the point is $(24, 0)$

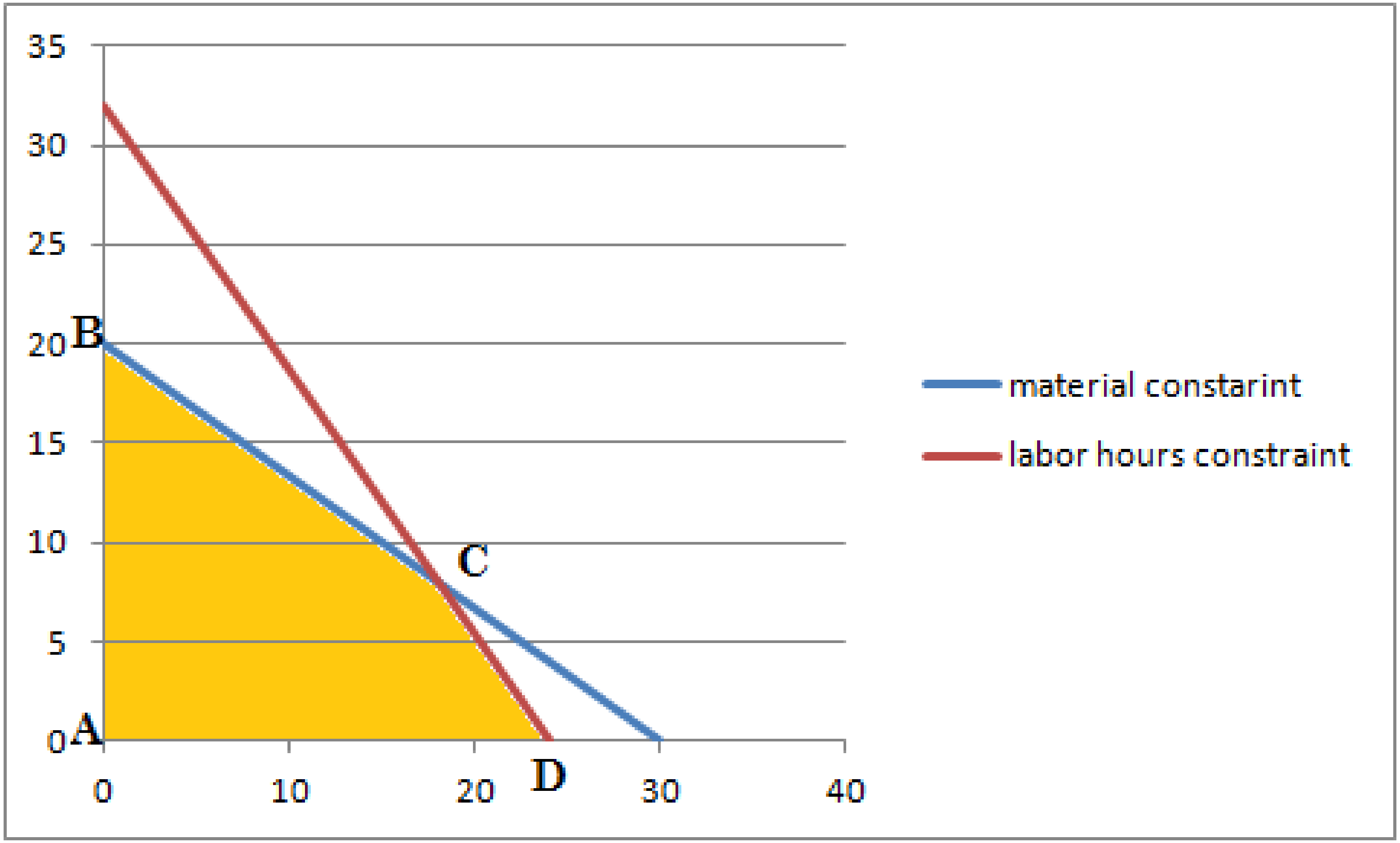
Drawing these on the graph paper will give us the following graph:

The graph of LPP





The feasible region of each equation



The feasible solution region- ABCD

The corner points of the solution region – A-B-C-D

- A (0, 0)
- B (0, 20)
- C (18, 8)
- D (24, 0)

Now calculate the value of the objective function at each of these points:

The objective function is

Max $Z = 40x + 35y$

- A (0, 0) = 40 (0) + 35 (0) = 0
- B (0, 20) = 40 (0) + 35 (20) = 700
- C (18, 8) = 40 (18) + 35 (8) = 1000
- D (24, 0) = 40 (24) + 35 (0) = 960

as can be observed, the largest value of objective function is obtained at point C (18, 8).

SOLUTION:

The optimum solution of the LPP:

The firm should manufacture18 units of product A and 8 units of product B at a profit of Rs 1000.

Minimization case

An example of formulation of LPP

Complete LPP *(from lecture 1)*

Objective function

$$\text{Min } Z = 40\,x + 24\,y$$

Subject to:

$$20\,x + 50\,y \geq 4800 \quad \text{(Ph requirement)}$$

$$80\,x + 50\,y \geq 7200 \quad \text{(N requirement)}$$

$$x \geq 0,\; y \geq 0$$

Solve to obtain 2 points per constraint

(Ph constraint) $20x + 50y = 4800$

When $x=0$, $y = 4800/50 = 96$ hence the point is (0, 96)

When $y=0$, $x=4800/20 = 240$ hence the point is (240,0)

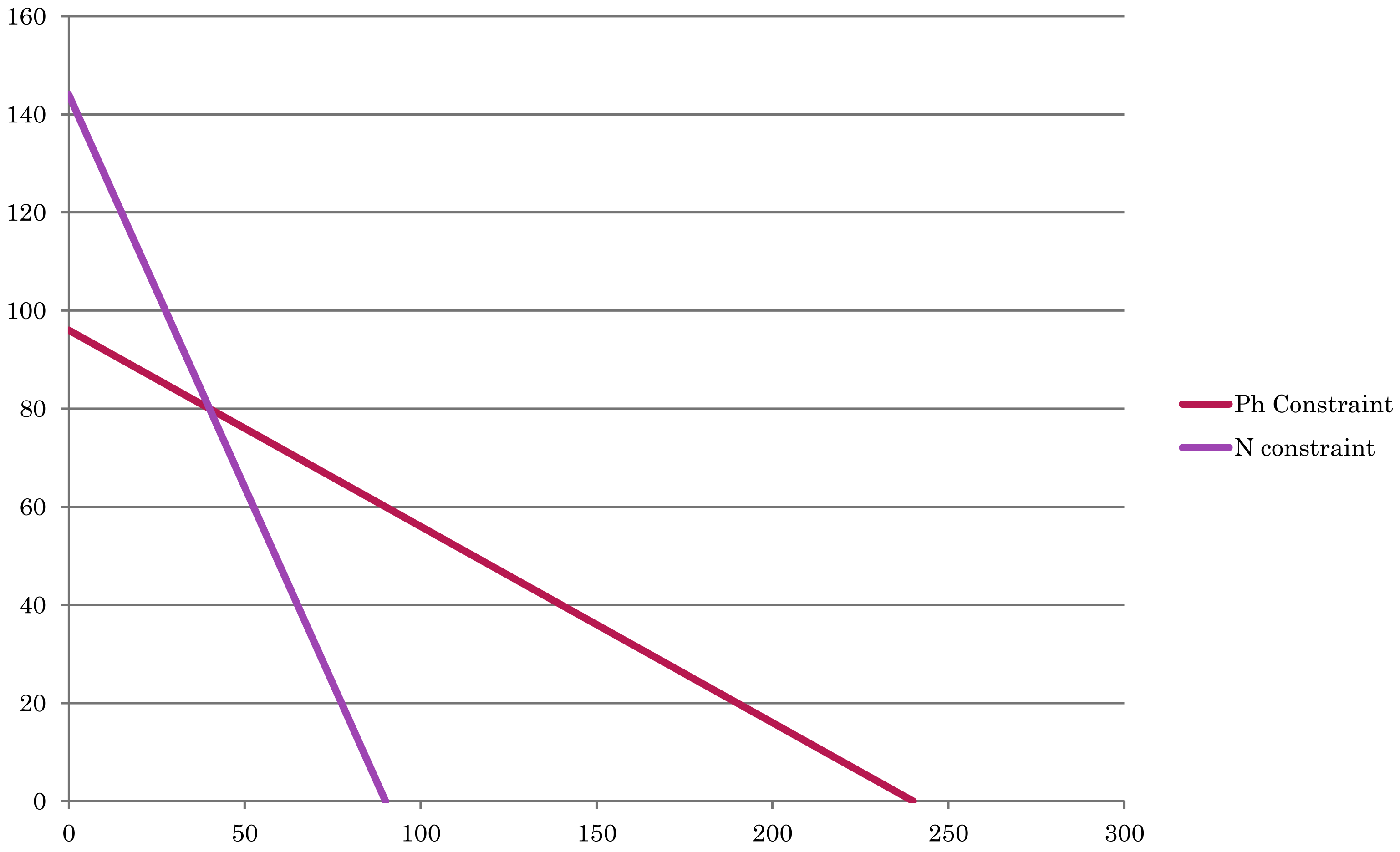
(N constraint) $80x + 50y = 7200$

When $x=0$, $y=7200/50 = 144$ hence the point is (0,144)

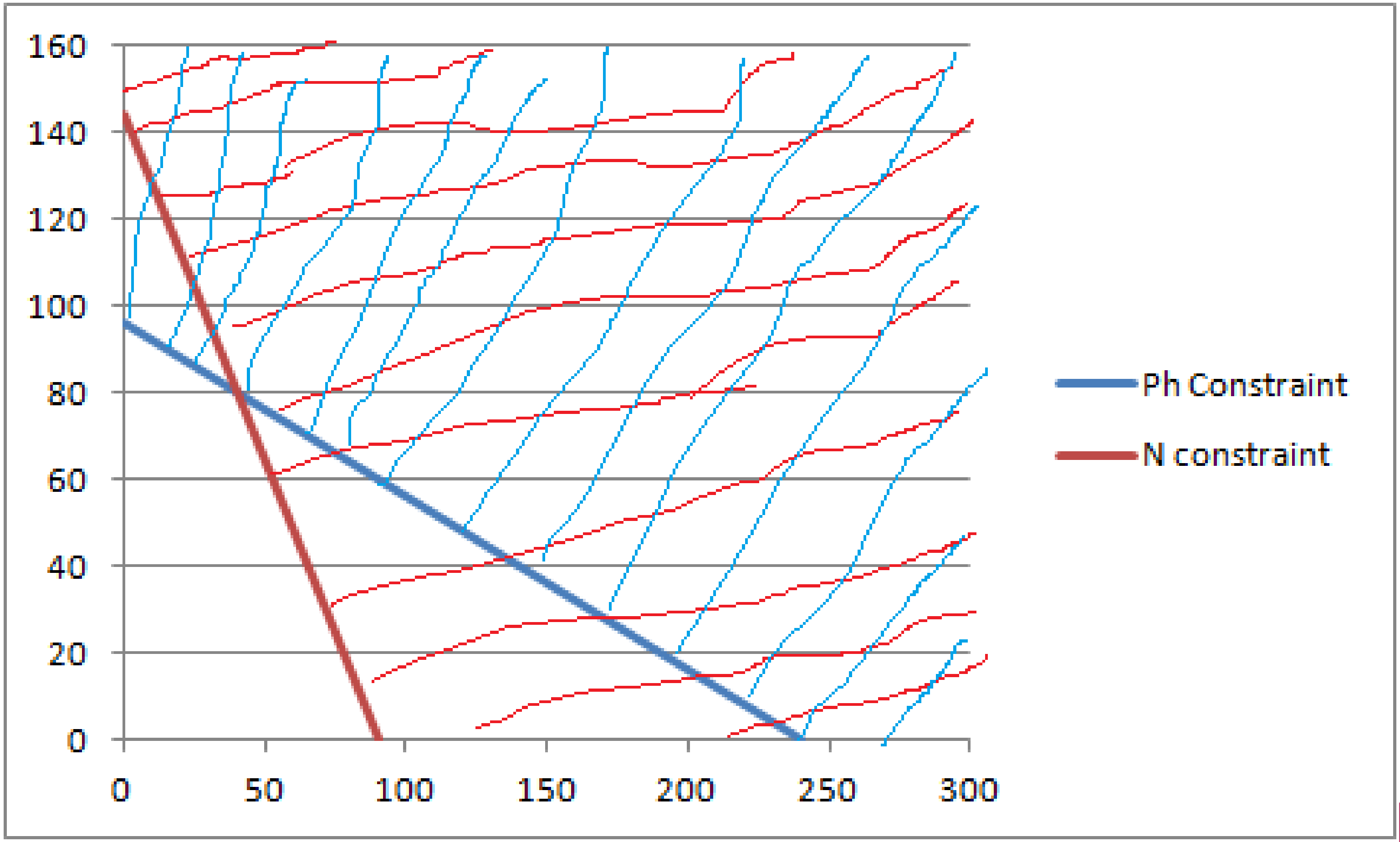
When $y=0$, $x= 7200/80 = 90$ hence the point is (90, 0)

Drawing these on the graph paper will give us the following graph:

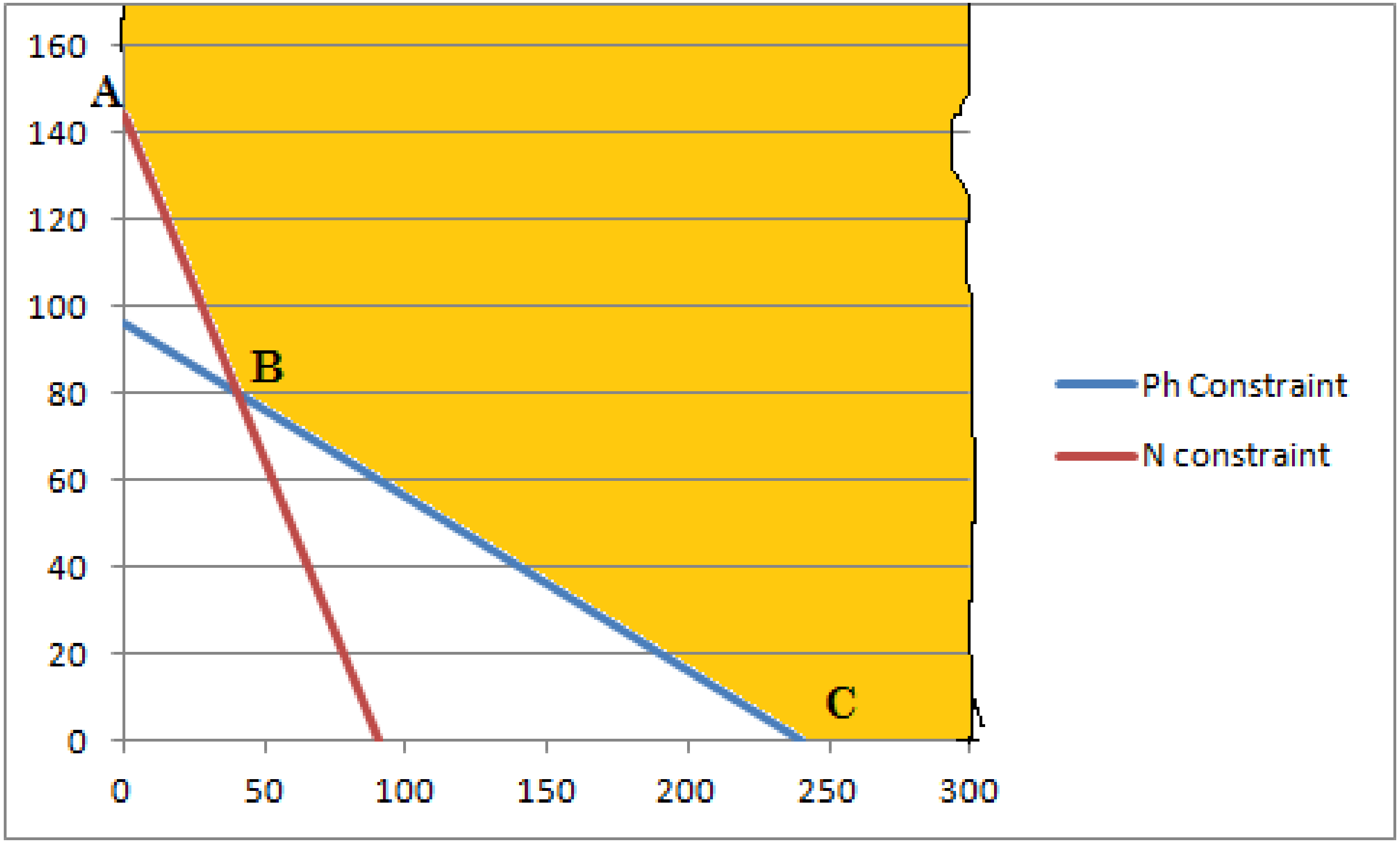
THE GRAPH OF LPP



IDENTIFY THE FEASIBLE REGION OF EACH CONSTRAINT



THE FEASIBLE AREA AND ITS CORNER POINTS – A-B-C



The corner points of the solution region – A-B-C

- A (0,144)
- B (40,80)
- C (240,0)

Now calculate the value of the objective function at each of these points:

Objective function
Min Z = 40 x + 24 y

- A (0,144) = 40 (0) + 24 (144) = 3456
- B (40,80) = 40 (40) + 24 (80) = 3520
- C (240,0) = 40 (240) + 24 (0) = 9600

as can be observed, the smallest value of objective function is obtained at point A (0, 144).

SOLUTION:

The optimum solution of the LPP:

The farmer should purchase 0 bags of mixture A and 144 bags of mixture B at a cost of Rs 3456.

OPTIMIZATION TECHNIQUES

UNIT-1

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- The ever-increasing demand on engineers to lower production costs to withstand global competition has prompted engineers to look for rigorous methods of decision making, such as optimization methods, to design and produce products and systems both economically and efficiently.
- Optimization techniques, having reached a degree of maturity in recent years, are being used in a wide spectrum of industries, including aerospace, automotive, chemical, electrical, construction, and manufacturing industries.
- With rapidly advancing computer technology, computers are becoming more powerful, and correspondingly, the size and the complexity of the problems that can be solved using optimization techniques are also increasing.

INTRODUCTION

- **Optimization is the act of obtaining the best result under given circumstances**. In design, construction, and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages.
- **The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit**. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, ***optimization*** can be defined as the process of finding the conditions that give the maximum or minimum value of a function.
- There is **no single method available for solving all optimization problems** efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems. The optimum seeking methods are also known as ***mathematical programming techniques*** and are generally studied as a part of **Operations Research**.

SCOPE OF OT

O.R. has a wide scope in everyday life as it provides better solutions to various decision-making problems with great speed and competence. It finds applications in a wide range of areas including defence operations, planning, agriculture, industry (finance, marketing, personal management, production management), research and development. We now describe the applications briefly.

In Planning for Economic Development

Careful planning is necessary for economic development of any country. Operations Research is used to frame future economic and social policies.

In Agriculture

Agricultural output needs to be increased due to increasing needs for adequate quantity and quality of food for our increasing population. But there are a number of restrictions under which agricultural production is studied. Problems of agricultural production under various restrictions such as optimum allocation of land to various crops in accordance with the climatic conditions, optimum distribution of water from various resources for irrigation purposes can easily be solved by application of Operations Research techniques.

In Industry

Now-a-days, due to complexities of operations and huge sizes of industries, important decisions regarding various sections of the organisation, e.g., planning, procurement, marketing, finance, etc. have to be taken division wise. For example, the production department needs to minimise the cost of production, but maximise output; the finance department needs to optimise capital investment; the personnel department needs to appoint competent work force at minimum cost. Each department has to plan its own objectives which may be in conflict with the objectives of other departments and may not conform to the overall objectives of the organisation. For example, the sales department of an organisation may want to keep sufficient stocks in the inventory, whereas the finance department may want to have minimum investment. In that case, both departments would be in conflict with each other. The applications of O.R. techniques to such situations help in overcoming this difficulty by evolving an optimal strategy and serving efficiently the interest of the organisation as a whole.

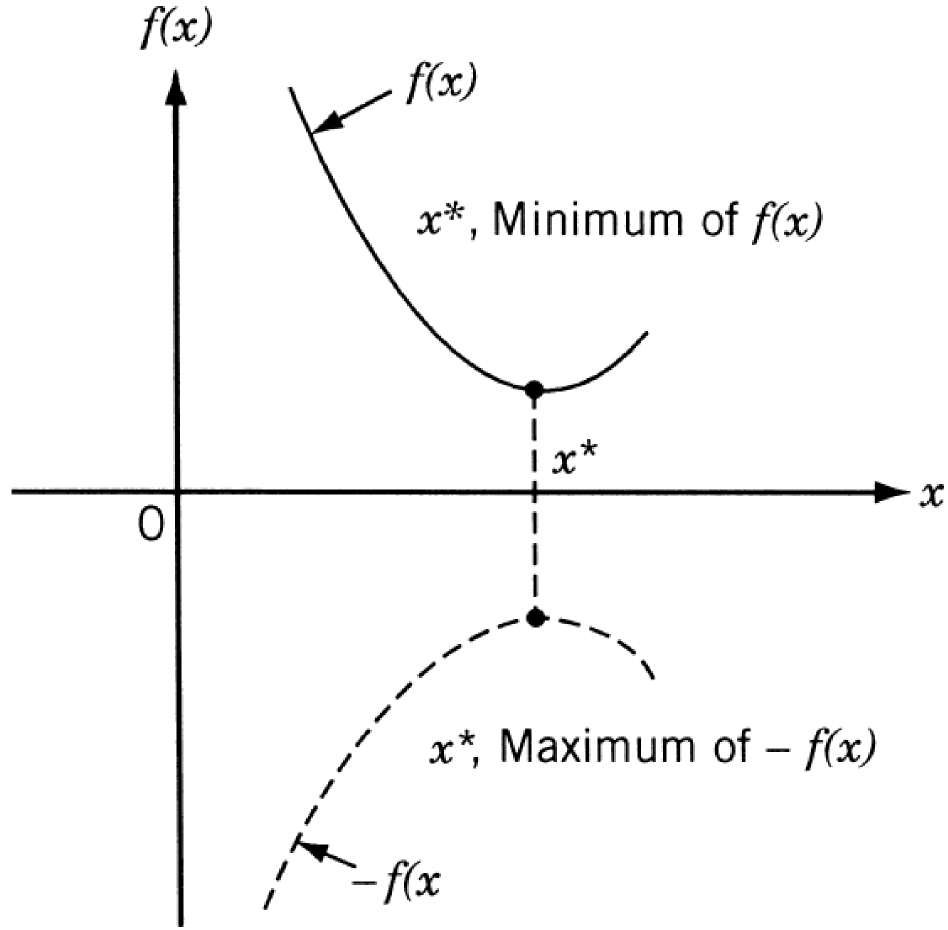
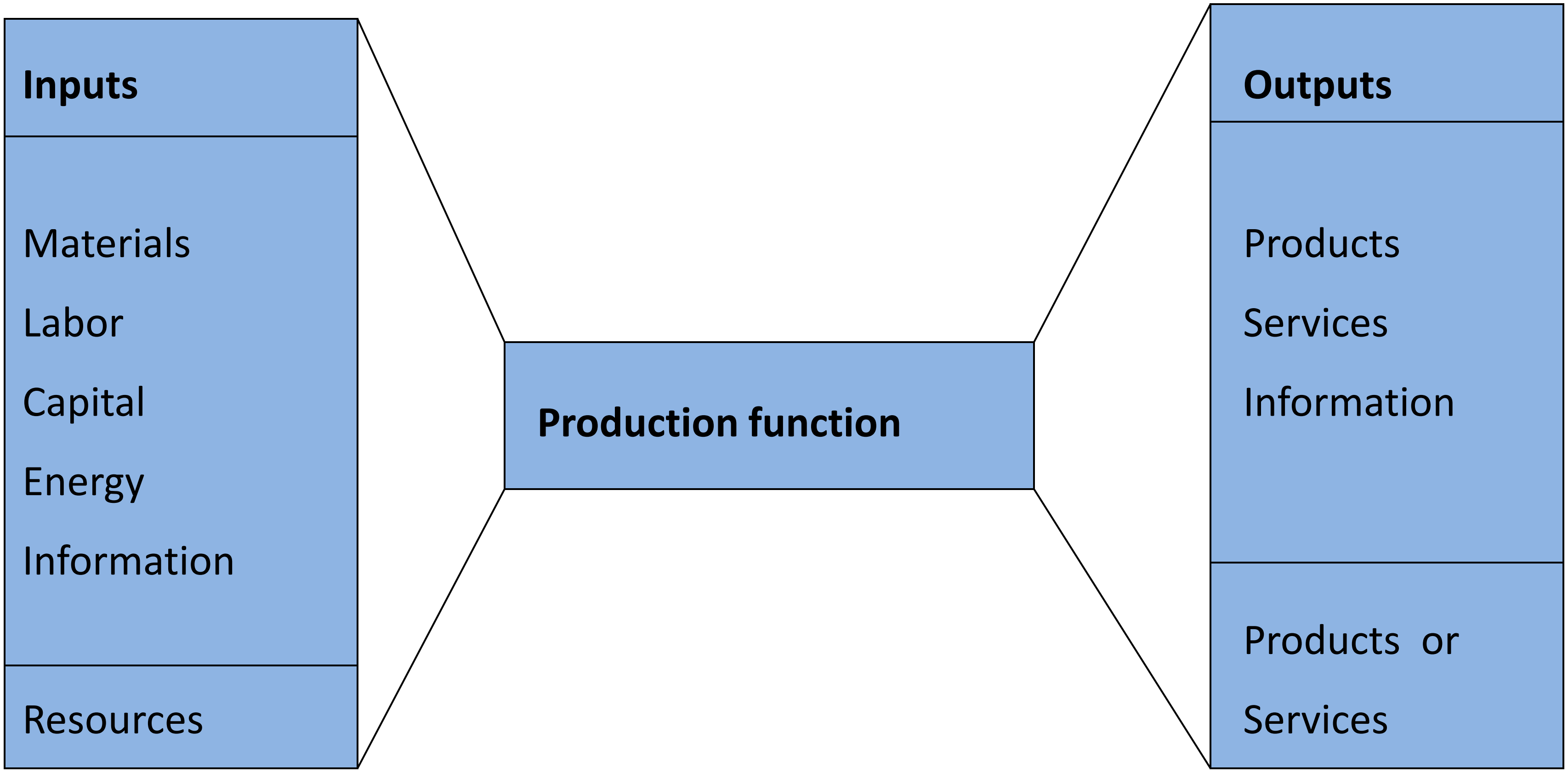


Figure 1.1 Minimum of $f(x)$ is same as maximum of $-f(x)$.


Inputs
Materials
Labor
Capital
Energy
Information
Resources
Production function
Outputs
Products
Services
Information
Products or
Services

ENGINEERING APPLICATIONS OF OPTIMIZATION

Some typical applications from different engineering disciplines include :

1. Design of aircraft and aerospace structures for minimum weight

2. Finding the optimal trajectories of space vehicles

3. Minimum-weight design of structures for earthquake, wind, other types of random loading

4. Selection of machining conditions in metal-cutting for minimum production cost

5. Shortest route taken by a salesperson visiting various cities during one tour

6. Optimal production planning, controlling, and scheduling

7. Design of optimum pipeline networks for process industries

8. Selection of a site for an industry

9. Planning of maintenance and replacement of equipment to reduce operating costs

10. Inventory control management

11. Allocation of resources among several activities to maximize the profit.

12. Controlling the waiting and idle times in production lines to reduce the costs

13. Optimum design of control systems in design of electronic appliances.

STATEMENT OF AN OPTIMIZATION PROBLEM

An optimization or a mathematical programming problem can be stated as follows.

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

subject to the constraints

$$\begin{aligned} g_j(\mathbf{X}) &\le 0, \qquad j = 1, 2, \ldots, m \\ l_j(\mathbf{X}) &= 0, \qquad j = 1, 2, \ldots, p \end{aligned} \tag{1.1}$$

where $\mathbf{X}$ is an n-dimensional vector called the *design vector*, $f(\mathbf{X})$ is termed the *objective function*, and $g_j(\mathbf{X})$ and $l_j(\mathbf{X})$ are known as *inequality* and *equality* constraints, respectively. The number of variables n and the number of constraints m and/or p need not be related in any way. The problem stated in Eq. (1.1) is called a *constrained optimization problem*.[†] Some optimization problems do not involve any constraints and can be stated as

OPTIMIZATION PROBLEM(2)

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

Such problems are called *unconstrained optimization problems*.

CONSTRAINTS IN OPTIMIZATION

- In many practical problems, the design variables cannot be chosen arbitrarily; rather, they have to satisfy certain specified functional and other requirements.
- The restrictions that must be satisfied to produce an acceptable design are collectively called *design constraints*.
- Constraints that represent limitations on the behavior or performance of the system are termed *behavior* or *functional constraints*.
- Constraints that represent physical limitations on design variables, such as availability, fabricability, and transportability, are known as *geometric* or *side constraints*.

CONSTRAINT SURFACE

For illustration, consider an optimization problem with only inequality constraints $g_j(\mathbf{X}) \le 0$. The set of values of $\mathbf{X}$ that satisfy the equation $g_j(\mathbf{X}) = 0$ forms a hypersurface in the design space and is called a *constraint surface*. Note that this is an $(n - 1)$-dimensional subspace, where n is the number of design variables. The constraint surface divides the design space into two regions: one in which $g_j(\mathbf{X}) < 0$ and the other in which $g_j(\mathbf{X}) > 0$. Thus the points lying on the hypersurface will satisfy the constraint $g_j(\mathbf{X})$ critically, whereas the points lying in the region where $g_j(\mathbf{X}) > 0$ are infeasible or unacceptable, and the points lying in the region where $g_j(\mathbf{X}) < 0$ are feasible or acceptable. The collection of all the constraint surfaces $g_j(\mathbf{X}) = 0$, $j = 1, 2, \ldots, m$, which separates the acceptable region is called the *composite constraint surface*.

CONSTRAINT SURFACE(2)

- Figure 1.4 shows a hypothetical two-dimensional design space where the infeasible region is indicated by hatched lines.
- A design point that lies on one or more than one constraint surface is called a *bound point* , and the associated constraint is called an *active constraint* .
- Design points that do not lie on any constraint surface are known as *free points*.
- Depending on whether a particular design point belongs to the acceptable or unacceptable region, it can be identified as one of the following four types:

1. Free and acceptable point

2. Free and unacceptable point

3. Bound and acceptable point

4. Bound and unacceptable point

All four types of points are shown in Fig. 1.4.

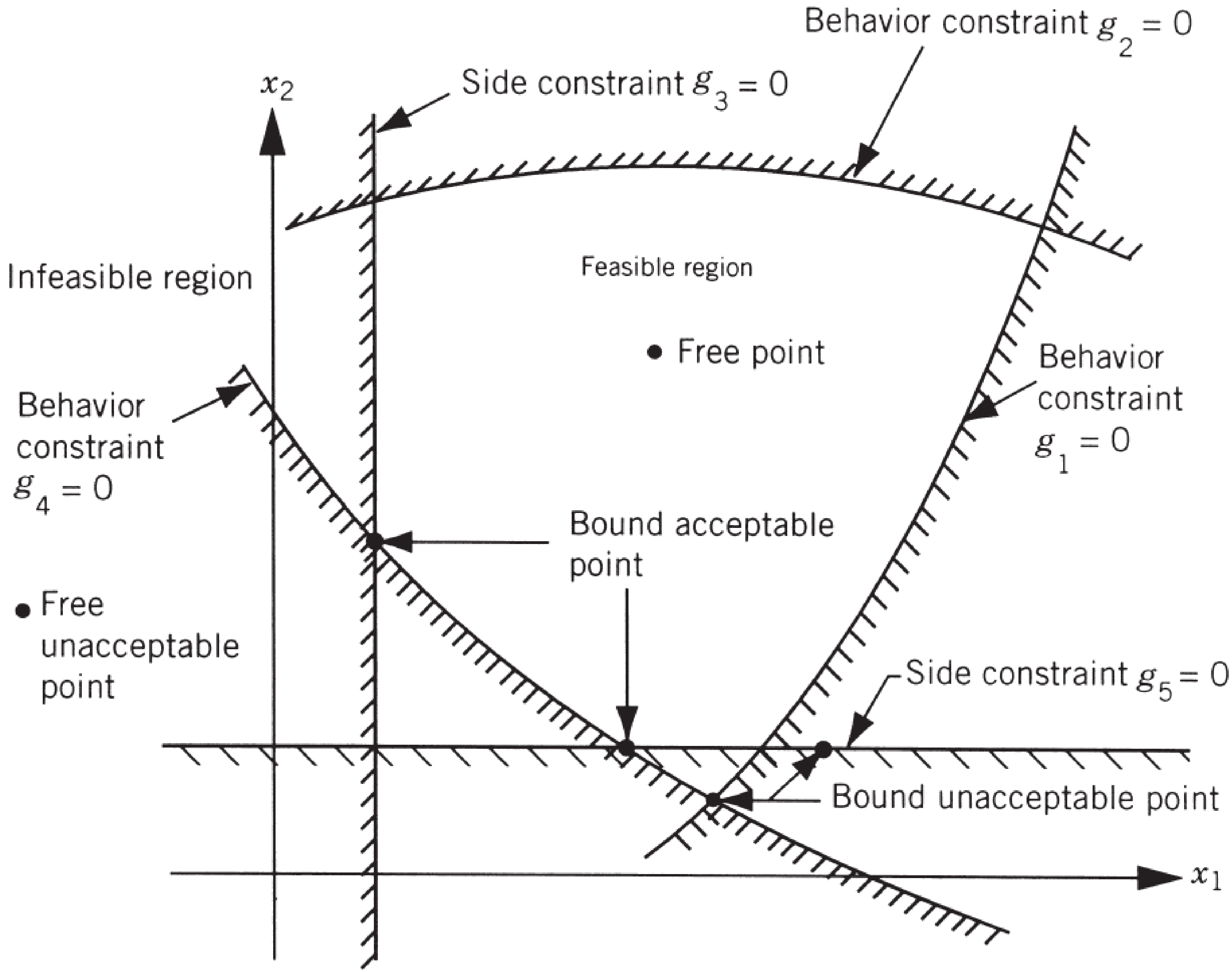


Figure 1.4 Constraint surfaces in a hypothetical two-dimensional design space.

CONSTRAINT SURFACE(2)

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1. Free and acceptable point

2. Free and unacceptable point

3. Bound and acceptable point

4. Bound and unacceptable point

All four types of points are shown in Fig. 1.4.

OBJECTIVE FUNCTION

- In general, there will be more than one acceptable design, and the purpose of optimization is to choose the best one of the many acceptable designs available.
- Thus a criterion has to be chosen for comparing the different alternative acceptable designs and for selecting the best one. The criterion with respect to which the design is optimized, when expressed as a function of the design variables, is known as the *criterion* or *merit* or *objective function*.
- The choice of objective function is governed by the nature of problem.
- The objective function for minimization is generally taken as weight in aircraft and aerospace structural design problems. In civil engineering structural designs, the objective is usually taken as the minimization of cost.
- In some situations, there may be more than one criterion to be satisfied simultaneously.
- For example, a gear pair may have to be designed for minimum weight and maximum efficiency while transmitting a specified horsepower.
- An optimization problem involving multiple objective functions is known as a *multiobjective programming problem*.

PROBLEM 1

- Reddy Mikks produces both interior and exterior paints from two raw materials M1 and M2. Following table provides the basic data of problem:
- A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.
- Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit? Find the optimal solution to this problem

	Tons of raw material per ton		
	Exterior paint	Interior paint	Max daily availability
Raw material, M1	6	4	24
Raw material, M2	1	2	6
profit per ton ($1000)	5	4	

SOLUTION

- The LP model, has three basic components:

1. Decision variables that we seek to determine.

2. Objective that we need to optimize (maximize or minimize).

3. Constraints that solution must satisfy.

The **decision variables** of model are defined as

x1=tons produced daily of exterior paint

x2=tons produced daily of interior paint

Objective function : The company wants to maximize (i.e., increase as much as possible) the total daily profit of both paints.

Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

Total profit from exterior paint=$5x_1$

Total profit from interior paint=$4x_2$

Let Z denote the total daily profit, then objective of the company is

Maximize $\mathbf{Z = 5x_1 + 4x_2}$

CONSTRAINTS : To construct the constraints that restrict raw material usage and product demand. The raw material restrictions are expressed verbally as

(usage of a raw material by both paints) ≤ (max. raw material availability)

Daily usage of raw material M1 is 6 tons of exterior paint and 4 tons of interior paint.

So, Usage of raw material M1 by exterior paint = $6x_1$ tons/day , and
Usage of raw material M1 by interior paint = $4x_2$ tons/day

Therefore, Usage of M1 by both paints = $6x_1 + 4x_2$ tons/day

By similar logic, Usage of M2 by both paints = $x_1 + 2x_2$ tons/day

Because the daily availabilities of raw materials M1 and M2 are limited to 24 and 6 tons respectively, the associated restrictions are given as

$6x_1 + 4x_2 \le 24$ **(Raw material M1)**

$x1 + 2x2 \le 6$ **(Raw material M2)**

Demand restriction stipulates that the excess of daily production of interior over exterior paint, x2-x1, should not exceed 1 ton, which translates to $x_2 - x_1 \le 1$ **(market limit)**

Second demand restriction- the max. daily demand of interior paint is limited to 2 tons, which translates to $x_2 \le 2$ **(demand limit)**.

An implicit (or "understood- to –be") restriction is that variables x_1 and x_2 cannot assume negative values. So $x_1 \ge 0$, $x_2 \ge 0$ **(nonnegativity restrictions)**

The complete Reddy Mikks Model is

$$\textbf{Maximize} \quad Z = 5x_1 + 4x_2$$

Subject to:

$$6x_1 + 4x_2 \leq 24 \qquad (1)$$

$$x_1 + 2x_2 \leq 6 \qquad (2)$$

$$x_2 - x_1 \leq 1 \qquad (3)$$

$$x_2 \leq 2 \qquad (4)$$

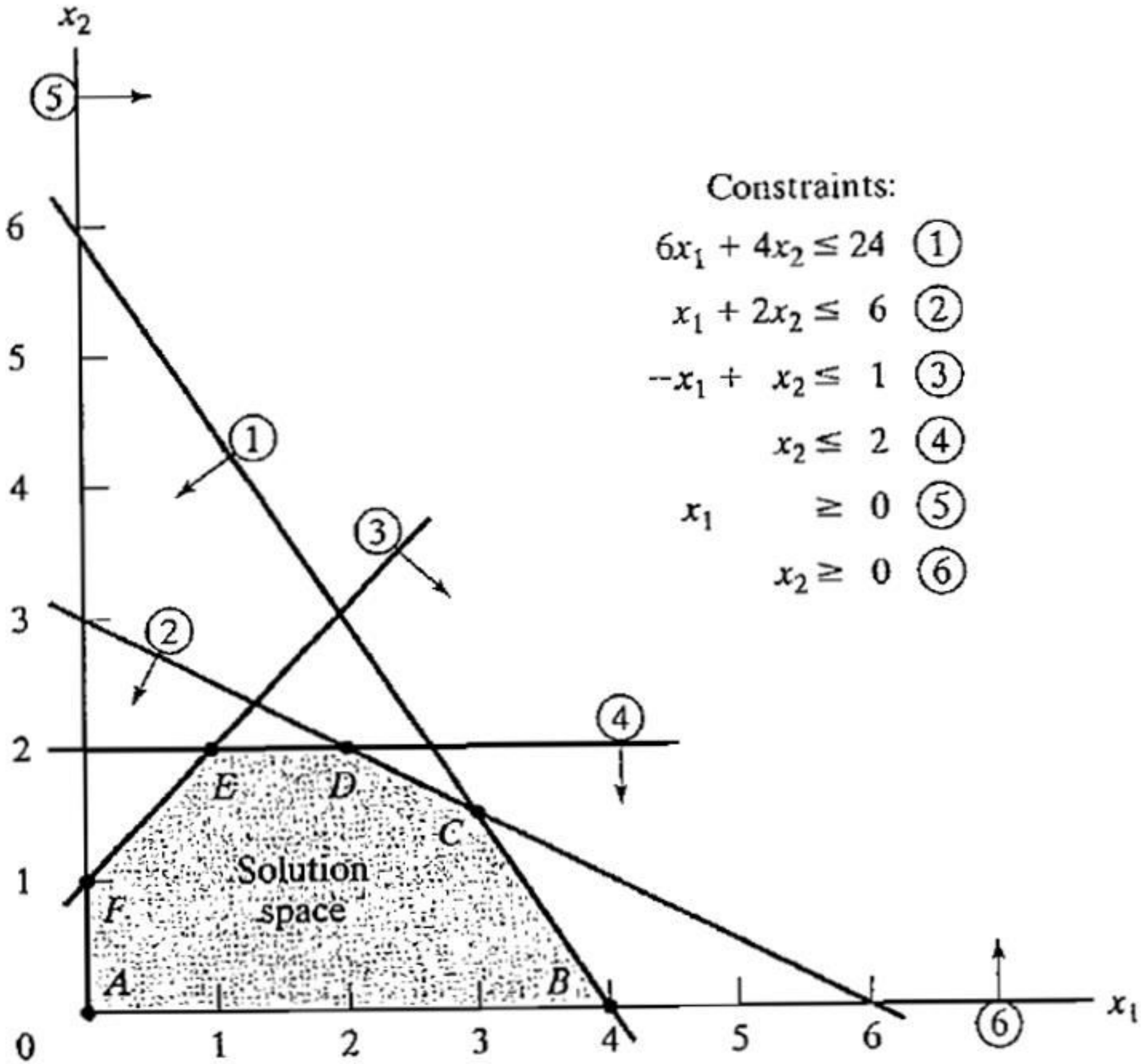
$$x_1, x_2 \geq 0 \qquad (5)$$

Any values of x_1 and x_2 that satisfy all five constraints constitute a feasible solution. Otherwise, the solution is infeasible.

For example, the solution, $x_1 = 3$ tons per day and $x_2 = 1$ ton per day, is feasible because it does not violate any of the constraints.

The goal of the problem is to find the best feasible solution, or the optimum, that maximizes the total profit. Before we can do that, we need to know how many feasible solutions the Reddy Mikks problem has. The answer, as we will see from the the graphical solution is an infinite number, which makes it impossible the solve the problem by enumeration.

GRAPHICAL SOLUTION OF LPP



The graphical procedure includes 2 steps:

1. Determination of the feasible solution space.

2. Determination of the optimum solution from among all the feasible points in the solution space

Determination of Optimum Solution

- The **feasible space** in figure is delineated by the line segments joining the points A, B, C, D, E, and F. **Any point within or on the boundary of the space ABCDEF is feasible.** Because the feasible space ABCDEF consists of an infinite number of points, we need a systematic procedure to identify the optimum solution.
- **An important characteristic of the optimum LP solution is that it is always associated with a corner point of the solution space** (where two lines intersect). This is true even if the objective function happens to be parallel to a constraint. For example, if the objective function is $z = 6x_1 + 4x_2$, which is parallel to constraint 1, we can always say that the optimum occurs at either corner point B or comer point C.
- The observation that the LP optimum is always associated with a corner point means that **the optimum solution can be found simply by enumerating all the corner points** as the following table shows:

Determination of Optimum Solution(2)

- As the number of constraints and variables increases, the number of corner points also increases.
- Nevertheless, the idea shows that, from the standpoint of determining the LP optimum, the solution space ABCDEF with its infinite number of solutions can, in fact, be replaced with a finite number of promising solution points-namely, the corner points, A, B, C, D, E, and F.
- The optimum solution is $x_1 = 3$ and $x_2 = 1.5$ with $Z = (5 * 3) + (4 * 1.5) = 21$. **This calls for a daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint. The associated daily profit is $21,000.**

Corner point	(x_1, x_2)	z
A	(0, 0)	0
B	(4, 0)	20
C	**(3, 1.5)**	**21 (OPTIMUM)**
D	(2, 2)	18
E	(1, 2)	13
F	(0, 1)	4

PRACTICE PROBLEM 1

A company produces two products, A and B. The sales volume for A is at least 80% of the total sales of both A and B.

However, the company cannot sell more than 100 units of A per day. Both products use one raw material, of which the maximum daily availability is 240 lb. The usage rates of the raw material are 2 lb per unit of A, and 4 lb per unit of B.

The profit units for A and B are \$20 and \$50, respectively.

Formulate the LPP for the same.

SOLUTION TO PRACTICE PROBLEM

If we let A = units of product A and B =units of product B, then we'll

$$\text{maximize } z = 20A + 50B$$

subject to

$$\begin{aligned} 2A + 4B &\leq 240 && \text{(raw material availability)} \\ A &\leq 100 && \text{(sales limit of A)} \\ -0.2A + 0.8B &\leq 0 && \text{(sales of A at least 80\%)} \\ A, B &\geq 0 && \text{(sign restrictions)} \end{aligned}$$

The sales volume for A is at least 80% of the total sales of both A and B. So, $A \geq 0.8\,(A + B)$ which gives us $0 \geq -0.2\,A + 0.8\,B$

PRACTICE PROBLEM 2

A company produces two types of items P and Q that require gold and silver.

Each unit of type P requires 4g silver and 1g gold while that of type Q requires 1g silver and 3g gold. The company can produce 8g silver and 9g gold.

If each unit of type P brings a profit of Rs 44 and that of type Q Rs 55, determine the number of units of each type that the company should produce to maximise the profit. What is the maximum profit?

Note: Q. 1 is compulsory.

Q1		(2.5*4)	CO
	(a) Explain Optimization, and its applications in Engineering		CO1
	(b) Depending on whether a particular point belongs to the acceptable or unacceptable region, it can be identified as one of the four types. Define and explain these types?		CO1
	(c) Explain the merits and limitations of the graphical method?		CO2
	(d) Explain decision variables, objective function, and constraints. Write an LPP to illustrate these terms?		CO2

Q2	(Attempt any Two Parts) UNIT-1 (CO1)	(5,5)
	(a) A soft drink manufacturing company has 300 ml and 150 ml canned cola as its products with profit margin of Rs. 4 and Rs. 2 per unit respectively. Both the products have to undergo process in three types of machine. The following data indicates the time required on each machine and the available machine-hours per week. Formulate the optimization problem as an LPP to maximize the total profit considering the limited resources.	

Requirement	Cola 300 ml	Cola 150 ml	Available machine hours per week
Machine 1	3	2	300
Machine 2	2	4	480
Machine 3	5	7	560

	(b) Discuss briefly about multiple and unbounded optimization Linear Programming Problems. Use appropriate example to justify your answer.	
	(C) Explain in detail the steps involved in formulating problems as mathematical programming problems? Explain the process, including the translation of design objectives and constraints into mathematical formulations	

Q3	(Attempt any Two Parts) UNIT-2 (CO2)	(5,5)
	(a) An airplane can carry a maximum of 200 passengers. A profit of Rs 1000 is made on each executive class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine using a graphical method how many tickets of each type must be sold in order to maximize the profit for the airline. What is the maximum profit?	
	(b) Use Simplex method to solve the following LP problem Maximize Z = 50x + 60y subject to: 2x + y <=300 ; 3x + 4y <=509; 4x + 7y <=812; x, y >=0.	
	(C) Write the algorithm to solve LPP using the simplex method OR explain the Integer Programming Problems in Optimization with an example.	

<u>**OTDM SOLUTION**</u>

SOLUTION OF 2a)

Let x_1 be the number of units of 300 ml cola and x_2 be the number of units of 150 ml cola to be produced respectively. Formulating the given problem, we get

$$Max.Z = 4x_1 + 2x_2$$

Subject to:

$$3x + 2x \leq 300$$
$$2x_1 + 4x_2 \leq 480$$
$$5x_1 + 7x_2 \leq 560$$
$$x_1, x_2 \geq 0$$

SOLUTION OF 3a)

Let x, y denote the number of executive class tickets and economy class tickets sold resp.

Since, aeroplane can carry maximum 200 passengers.

$\therefore x + y \leq 200$ ...(1)

Since, at least 20 tickets is reserved for executive class.

$\therefore x \geq 20$ ...(2)

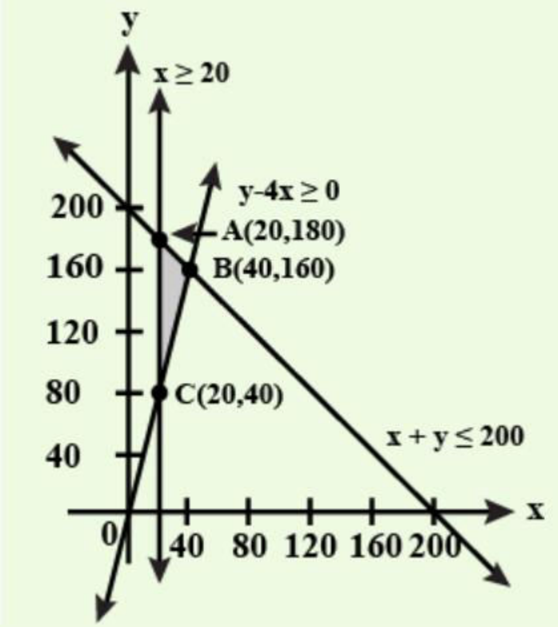
Since the number of tickets for economy class should be at least 4 times the executive class.

$\therefore y \geq 4x \Rightarrow y - 4x \geq 0$ ...(3)

Also, the number of tickets can't be negative. So, $x, y \geq 0$ ...(4)

Profit on an executive class ticket is 1000 Rs and profit on an economy class ticket is 600 Rs

So, Objective function is Maximize Z=1000x + 600y



Corner points	Value of Z = 1000X + 600Y
A (20, 180)	128000
B (40, 160)	136000 (Maximum)
C (20,80)	68000

We have to maximize the total profit. After plotting all the constraints given by equation (1), (2), (3) and (4), we get the feasible region as shown in the image above.

Hence, **Maximum profit will be 136000 Rs**, when number of executive class and economy class tickets sold will be 40 and 160 respectively.

Q3 b)
Use Simplex method to solve the following LP problem
Maximize Z = 50x + 60y subject to: 2x + y <=300 ; 3x + 4y <=509; 4x + 7y <=812; x, y >=0.

To solve this LPP using Simplex, let's first rewrite the problem in standard form:

Maximize Z = 50x + 60y Subject to:

1. 2x + y + s1 = 300
2. 3x + 4y + s2 = 509
3. 4x + 7y + s3 = 812
4. x, y, s1, s2, s3 >= 0

Basic Variables	x	y	s1	s2	s3	RHS
s1	2	1	1	0	0	300
s2	3	4	0	1	0	509
s3	4	7	0	0	1	812
Z-row	-50	-60	0	0	0	0

After the first iteration:

Basic Variables	x	y	s1	s2	s3	RHS
y	1	2	1/2	0	0	150
s2	0	1	-3/2	1	0	259
s3	0	1	-2	0	1	412
Z-row	0	10	0	0	0	3000

Now, since all coefficients in the Z-row are non-negative, we have reached the optimal solution.

Solution: x = 150 y = 259 , Z = 3000

Simplex method

The Simplex method is an approach to solving LPP involving 2 or more decision variables by hand using slack variables, tableaus, and pivot variables as a means to finding the optimal solution. Following steps are necessary:

- Standard form
- Introducing slack variables
- Creating the tableau
- Pivot variables
- Creating a new tableau
- Checking for optimality
- Identify optimal values

Basic variables:

Are the variables which coefficients One in the equations and Zero in the other equations.

Non-Basic variables:

Are the variables which coefficients are taking any of the values, whether positive or negative or zero.

Slack, surplus & artificial variables:

a) If the inequality be (less than or equal, then we add a slack variable + S to change to =.

b) If the inequality be (greater than or equal, then we subtract a surplus variable - S to change to =.

c) If we have = we use artificial variables.

ALGORITHM OF SIMPLEX METHOD:

Step 1:
Determine a starting basic feasible solution.

Step 2:
Select an entering variable using the optimality condition. Stop if there is no entering variable.

Step 3:
Select a leaving variable using the feasibility condition.

Optimality condition:

The entering variable in a maximization (minimization) problem is the non-basic variable having the most negative (positive) coefficient in the Z-row.
The optimum solution is reached at the iteration where all the Z-row coefficient of the non-basic variables are non-negative (non-positive).

Feasibility condition:

For both maximization and minimization problems the leaving variable is the basic associated with the smallest non-negative ratio (with strictly positive denominator).

Pivot row:

a) Replace the leaving variable in the basic column with the entering variable.
b) New pivot row equal to current pivot row divided by pivot element.
c) All other rows:
   New row=current row - (pivot column coefficient) *new pivot row

LIMITATION OF GRAPHICAL METHOD

The main limitation of graphical method for solving LPPs are that it is only applicable to problems with two decision variables and becomes awkward and impractical as the number of variables or constraints increases.
Additionally, it is not suitable for problems with three or more variables, yields approximate rather than precise results, and can be challenging to interpret if the feasible region is not convex or the objective function isn't linear.

1. **Limited to two variables:**

The graphical method relies on plotting constraints on a 2D plane (x-y axes), making it impossible to visualize problems with more than two decision variables.

2. **Impractical for larger problems:**

Even for two-variable problems, the method becomes tedious and difficult to manage as the number of constraints increases.

3. **Results can be approximate:**

The solution obtained from a graph is often an estimate, and precision depends on the accuracy of the drawing.

4. **Not applicable to problems with three or more variables:**

For LPPs with three or more decision variables, more advanced methods like the Simplex method are required.

5. **Difficulty with non-convex or non-linear scenarios:**

While the graphical method is for linear programming, it's difficult to find the optimal solution if the feasible region is not convex or the objective function is not strictly linear.

6. **May not have a feasible solution:**

In some cases, the constraints can be contradictory, resulting in no overlapping area on the graph and thus no feasible solution for the problem.

7. **Multiple optimal solutions:**

If the objective function line is parallel to one of the boundaries of the feasible region, there can be multiple optimal solutions, which the graphical method can identify but may be difficult to work with in complex cases.

OPTIMIZATION TECHNIQUES for DECISION MAKING

UNIT-1

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Need of Optimization Techniques

- The ever-increasing demand on engineers to lower production costs to withstand global competition has prompted engineers to look for rigorous methods of decision making, such as optimization methods, to design and produce products and systems both economically and efficiently.
- Optimization techniques, having reached a degree of maturity in recent years, are being used in a wide spectrum of industries, including aerospace, automotive, chemical, electrical, construction, and manufacturing industries.
- With rapidly advancing computer technology, computers are becoming more powerful, and correspondingly, the size and the complexity of the problems that can be solved using optimization techniques are also increasing.

INTRODUCTION

- **Optimization is the act of obtaining the best result under given circumstances**. In design, construction, and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages.
- **The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit**. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, ***optimization*** can be defined as the process of finding the conditions that give the maximum or minimum value of a function.
- There is **no single method available for solving all optimization problems** efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems. The optimum seeking methods are also known as ***mathematical programming techniques*** and are generally studied as a part of **Operations Research**.

SCOPE OF OT / OR

O.R. has a wide scope in everyday life as it provides better solutions to various decision-making problems with great speed and competence. It finds applications in a wide range of areas including defence operations, planning, agriculture, industry (finance, marketing, personal management, production management), research and development. We now describe the applications briefly.

Areas where Optimization is applied are:

1. **Science**
2. **Engineering**
3. **Management**
4. **Finance**
5. **Business**

In Planning for Economic Development

Careful planning is necessary for economic development of any country. Operations Research is used to frame future economic and social policies.

In Agriculture

Agricultural output needs to be increased due to increasing needs for adequate quantity and quality of food for our increasing population. But there are a number of restrictions under which agricultural production is studied. Problems of agricultural production under various restrictions such as optimum allocation of land to various crops in accordance with the climatic conditions, optimum distribution of water from various resources for irrigation purposes can easily be solved by application of Operations Research techniques.

In Industry

Now-a-days, due to complexities of operations and huge sizes of industries, important decisions regarding various sections of the organisation, e.g., planning, procurement, marketing, finance, etc. have to be taken division wise. For example, the production department needs to minimise the cost of production, but maximise output; the finance department needs to optimise capital investment; the personnel department needs to appoint competent work force at minimum cost. Each department has to plan its own objectives which may be in conflict with the objectives of other departments and may not conform to the overall objectives of the organisation. For example, the sales department of an organisation may want to keep sufficient stocks in the inventory, whereas the finance department may want to have minimum investment. In that case, both departments would be in conflict with each other. The applications of O.R. techniques to such situations help in overcoming this difficulty by evolving an optimal strategy and serving efficiently the interest of the organisation as a whole.

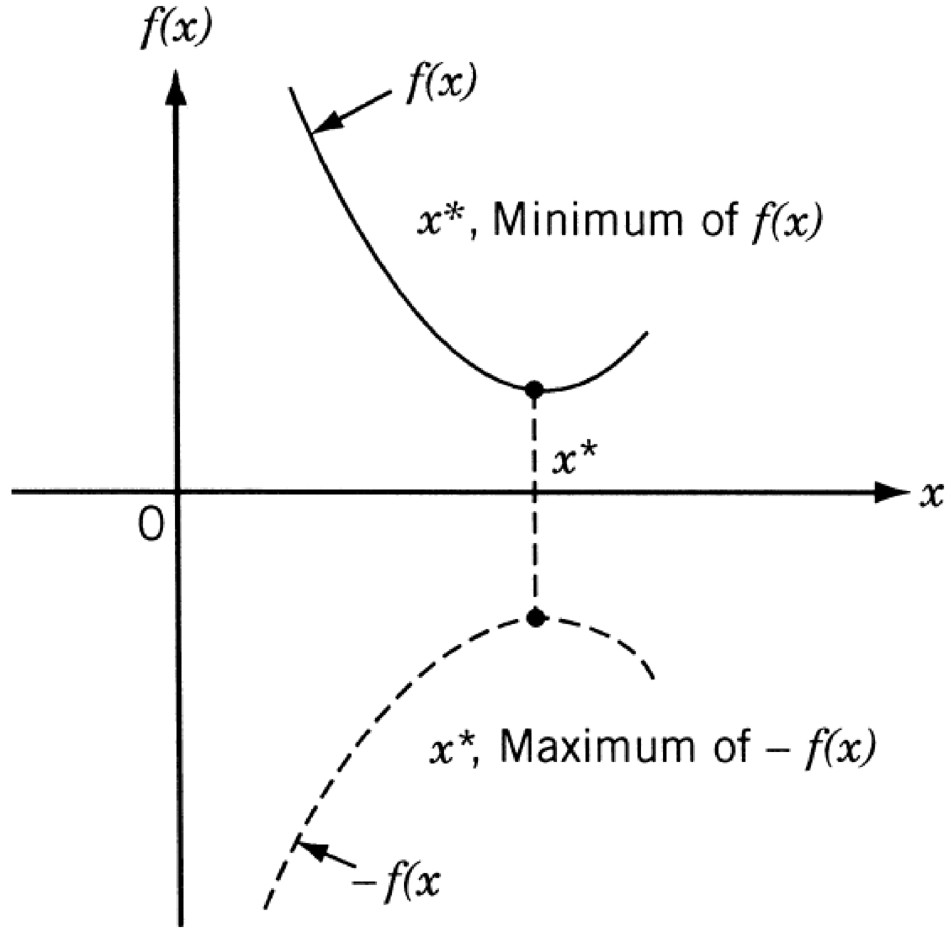

Figure 1.1 Minimum of $f(x)$ is same as maximum of $-f(x)$.

ENGINEERING APPLICATIONS OF OPTIMIZATION

Some typical applications from different engineering disciplines include :

1. Design of aircraft and aerospace structures for minimum weight

2. Finding the optimal trajectories of space vehicles

3. Minimum-weight design of structures for earthquake, wind, other types of random loading

4. Selection of machining conditions in metal-cutting for minimum production cost

5. Shortest route taken by a salesperson visiting various cities during one tour

6. Optimal production planning, controlling, and scheduling

7. Design of optimum pipeline networks for process industries

8. Selection of a site for an industry

9. Planning of maintenance and replacement of equipment to reduce operating costs

10. Inventory control management

11. Allocation of resources among several activities to maximize the profit.

12. Controlling the waiting and idle times in production lines to reduce the costs

13. Optimum design of control systems in design of electronic appliances.

GENERAL OPTIMIZATION PROBLEM

Minimize (Maximize) $f(X)$ objective function

where $f: R^n \rightarrow R$, $X = (x_1, x_2, x_3 \ldots x_n)$ decision values parameters

s.t. $X \in S \subseteq R^n$ where S is defined by

$g_k(X) \geq 0$, $k=1,2, \ldots m$ → inequality constraints

$h_j(X) = 0$, $j=1,2, \ldots l$ → equality constraints

$a_i \leq x_i \leq b_i$ → lower & upper bounds

COMPONENTS OF AN OPTIMIZATION MODEL

Decision variables

Objective function

Constraints

CLASSIFICATION

Linear Programming Problems (LPP)

Nonlinear Programming Problems (NLPP)

Unconstrained Optimization Problems

Constrained Optimization Problems

LINEAR PROGRAMMING PROBLEM

A **Linear Programming Problem** is an optimization problem where the **objective function** and all the **constraints** are linear in nature.

Objective function → A linear function of decision variables to be **maximized or minimized** (e.g., profit, cost, revenue).

Constraints → Linear equations or inequalities representing resource limitations.

Decision variables → Unknowns to be determined (e.g., number of products, allocation of resources).

The General form of LPP is as follows:

Maximize or Minimize:

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

...

$$x_i \geq 0 \quad (i = 1, 2, ..., n)$$

Where:

- Z = Objective function
- $x_1, x_2, ..., x_n$ = Decision variables
- c_i = Coefficients of objective function
- a_{ij} = Coefficients of constraints
- b_j = Resource availability

Problem:

A company produces two products A and B.

Profit from A = ₹3 per unit, profit from B = ₹2 per unit.

Constraints:

- Each unit of A requires 1 hour on machine 1 and 2 hours on machine 2.
- Each unit of B requires 1 hour on machine 1 and 1 hour on machine 2.
- Machine 1 is available for 8 hours, Machine 2 is available for 10 hours.

Formulation:

$$\text{Maximize } Z = 3x_1 + 2x_2$$

Subject to:

$$x_1 + x_2 \leq 8 \quad (Machine 1)$$

$$2x_1 + x_2 \leq 10 \quad (Machine 2)$$

$$x_1, x_2 \geq 0$$

Solution (Graphical):

- Plot the inequalities, find feasible region.
- Corner points = (0,0), (5,0), (2,6), (0,8).
- Evaluate Z:
  - (0,0) → 0
  - (5,0) → 15
  - (2,6) → 18
  - (0,8) → 16

Optimal Solution: Produce 2 units of A and 6 units of B → **Maximum Profit = ₹18**

NON-LINEAR PROGRAMMING

Non-Linear Programming involves optimization of a **non-linear objective function** and/or **non-linear constraints**.

General Formulation

$$\text{Maximize/Minimize } Z = f(x_1, x_2, \ldots, x_n)$$

Subject to:

$$g_j(x_1, x_2, \ldots, x_n) \leq b_j \quad \text{for } j = 1, 2, \ldots, m$$

$$x_i \geq 0$$

Where:

- $f(x)$ = non-linear objective function
- $g_j(x)$ = non-linear constraint functions

Methods of Solving NLPP

1. **Lagrange Multiplier Method** (for equality constraints)
2. **Kuhn-Tucker (KKT) Conditions** (for inequality constraints)
3. **Gradient Descent / Newton's Method** (iterative numerical methods)

CLASSIFICATION BASED ON TYPE OF DECISION VARIABLES

Dynamic Programming

Geometric Programming

Integer Programming

Quadratic Programming

Separable Programming

STATEMENT OF AN OPTIMIZATION PROBLEM

An optimization or a mathematical programming problem can be stated as follows.

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

subject to the constraints

$$\begin{aligned} g_j(\mathbf{X}) &\le 0, \qquad j = 1, 2, \ldots, m \\ l_j(\mathbf{X}) &= 0, \qquad j = 1, 2, \ldots, p \end{aligned} \tag{1.1}$$

where $\mathbf{X}$ is an n-dimensional vector called the *design vector*, $f(\mathbf{X})$ is termed the *objective function*, and $g_j(\mathbf{X})$ and $l_j(\mathbf{X})$ are known as *inequality* and *equality* constraints, respectively. The number of variables n and the number of constraints m and/or p need not be related in any way. The problem stated in Eq. (1.1) is called a *constrained optimization problem*.† Some optimization problems do not involve any constraints and can be stated as

OPTIMIZATION PROBLEM(2)

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

Such problems are called *unconstrained optimization problems*.

CONSTRAINTS IN OPTIMIZATION

- In many practical problems, the design variables cannot be chosen arbitrarily; rather, they have to satisfy certain specified functional and other requirements.
- The restrictions that must be satisfied to produce an acceptable design are collectively called *design constraints*.
- Constraints that represent limitations on the behavior or performance of the system are termed *behavior* or *functional constraints*.
- Constraints that represent physical limitations on design variables, such as availability, fabricability, and transportability, are known as *geometric* or *side constraints*.

CONSTRAINT SURFACE

For illustration, consider an optimization problem with only inequality constraints $g_j(\mathbf{X}) \le 0$. The set of values of $\mathbf{X}$ that satisfy the equation $g_j(\mathbf{X}) = 0$ forms a hypersurface in the design space and is called a *constraint surface*. Note that this is an $(n - 1)$-dimensional subspace, where n is the number of design variables. The constraint surface divides the design space into two regions: one in which $g_j(\mathbf{X}) < 0$ and the other in which $g_j(\mathbf{X}) > 0$. Thus the points lying on the hypersurface will satisfy the constraint $g_j(\mathbf{X})$ critically, whereas the points lying in the region where $g_j(\mathbf{X}) > 0$ are infeasible or unacceptable, and the points lying in the region where $g_j(\mathbf{X}) < 0$ are feasible or acceptable. The collection of all the constraint surfaces $g_j(\mathbf{X}) = 0,\ j = 1, 2, \ldots, m$, which separates the acceptable region is called the *composite constraint surface*.

CONSTRAINT SURFACE(2)

- Figure 1.4 shows a hypothetical two-dimensional design space where the infeasible region is indicated by hatched lines.
- A design point that lies on one or more than one constraint surface is called a *bound point* , and the associated constraint is called an *active constraint* .
- Design points that do not lie on any constraint surface are known as *free points*.
- Depending on whether a particular design point belongs to the acceptable or unacceptable region, it can be identified as one of the following four types:

1. Free and acceptable point

2. Free and unacceptable point

3. Bound and acceptable point

4. Bound and unacceptable point

All four types of points are shown in Fig. 1.4.

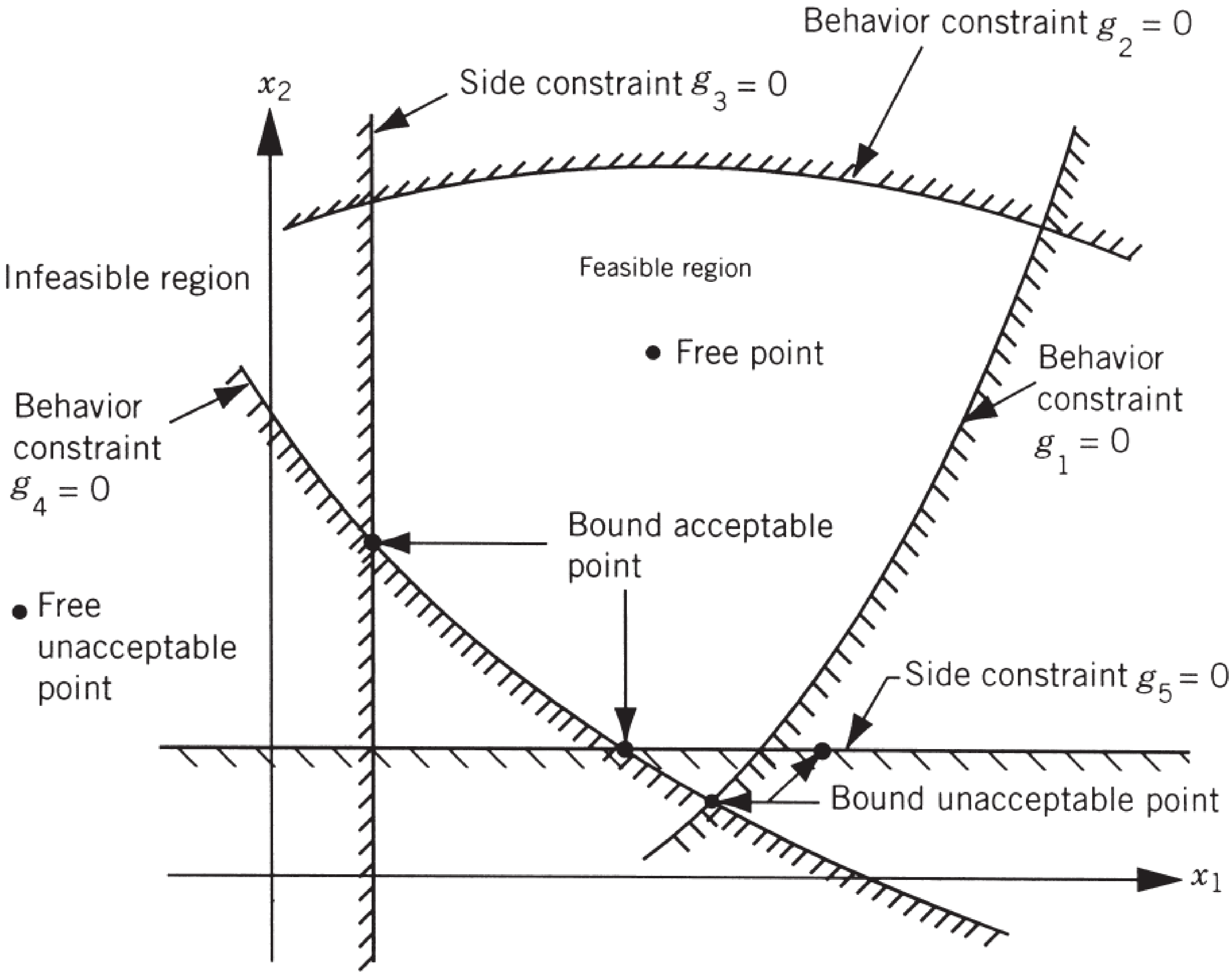

Figure 1.4 Constraint surfaces in a hypothetical two-dimensional design space.

CONSTRAINT SURFACE(2)

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1. Free and acceptable point

2. Free and unacceptable point

3. Bound and acceptable point

4. Bound and unacceptable point

All four types of points are shown in Fig. 1.4.

OBJECTIVE FUNCTION

- In general, there will be more than one acceptable design, and the purpose of optimization is to choose the best one of the many acceptable designs available.
- Thus a criterion has to be chosen for comparing the different alternative acceptable designs and for selecting the best one. The criterion with respect to which the design is optimized, when expressed as a function of the design variables, is known as the *criterion* or *merit* or *objective function*.
- The choice of objective function is governed by the nature of problem.
- The objective function for minimization is generally taken as weight in aircraft and aerospace structural design problems. In civil engineering structural designs, the objective is usually taken as the minimization of cost.
- In some situations, there may be more than one criterion to be satisfied simultaneously.
- For example, a gear pair may have to be designed for minimum weight and maximum efficiency while transmitting a specified horsepower.
- An optimization problem involving multiple objective functions is known as a *multiobjective programming problem*.

PROBLEM 1

1. Reddy Mikks produces both interior and exterior paints from two raw materials M1 and M2. Following table provides the basic data of problem:
2. A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton.
3. Also, the maximum daily demand for interior paint is 2 tons.
4. Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit?
5. Find the optimal solution to this problem

Tons of raw material per ton			
	Exterior paint	Interior paint	Max daily availability
Raw material, M1	6	4	24
Raw material, M2	1	2	6
profit per ton ($1000)	5	4	

SOLUTION

- The LP model, has three basic components:

1. Decision variables that we seek to determine.

2. Objective that we need to optimize (maximize or minimize).

3. Constraints that solution must satisfy.

The **decision variables** of model are defined as

$x1$= tons produced daily of exterior paint

$x2$= tons produced daily of interior paint

Objective function : The company wants to maximize (i.e., increase as much as possible) the total daily profit of both paints.

Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

Total profit from exterior paint = $5x_1$

Total profit from interior paint = $4x_2$

Let Z denote the total daily profit, then objective of the company is

Maximize $\mathbf{Z = 5x_1 + 4x_2}$

CONSTRAINTS : To construct the constraints that restrict raw material usage and product demand.

The raw material restrictions are expressed verbally as

(usage of a raw material by both paints) ≤ (max. raw material availability)

Daily usage of raw material M1 is 6 tons of exterior paint and 4 tons of interior paint.

So, Usage of raw material M1 by exterior paint = $6x_1$ tons/day ,

Usage of raw material M1 by interior paint = $4x_2$ tons/day

Therefore, Usage of M1 by both paints = $6x_1 + 4x_2$ tons/day

By similar logic,

Usage of M2 by both paints = $1x_1 + 2x_2$ tons/day

Because the daily availabilities of raw materials M1 and M2 are limited to 24 and 6 tons respectively, the associated restrictions are given as

$$6x_1 + 4x_2 \leq 24 \qquad \textbf{(Raw material M1)}$$

$$x_1 + 2x_2 \leq 6 \qquad \textbf{(Raw material M2)}$$

Demand restriction stipulates that the excess of daily production of interior over exterior paint, x_2-x_1, should not exceed 1 ton, which translates to $x_2 - x_1 \leq 1$ **(market limit)**

Second demand restriction- the max. daily demand of interior paint is limited to 2 tons, which translates to $x_2 \leq 2$ **(demand limit)**.

An implicit (or "understood- to –be") restriction is that variables x_1 and x_2 cannot assume negative values. So $x_1 \geq 0$, $x_2 \geq 0$ **(nonnegativity restrictions)**

The complete Reddy Mikks Model is

Maximize $\mathbf{Z = 5x_1 + 4x_2}$

Subject to:

$$6x_1 + 4x_2 \le 24 \qquad (1)$$

$$x_1 + 2x_2 \le 6 \qquad (2)$$

$$x_2 - x_1 \le 1 \qquad (3)$$

$$x_2 \le 2 \qquad (4)$$

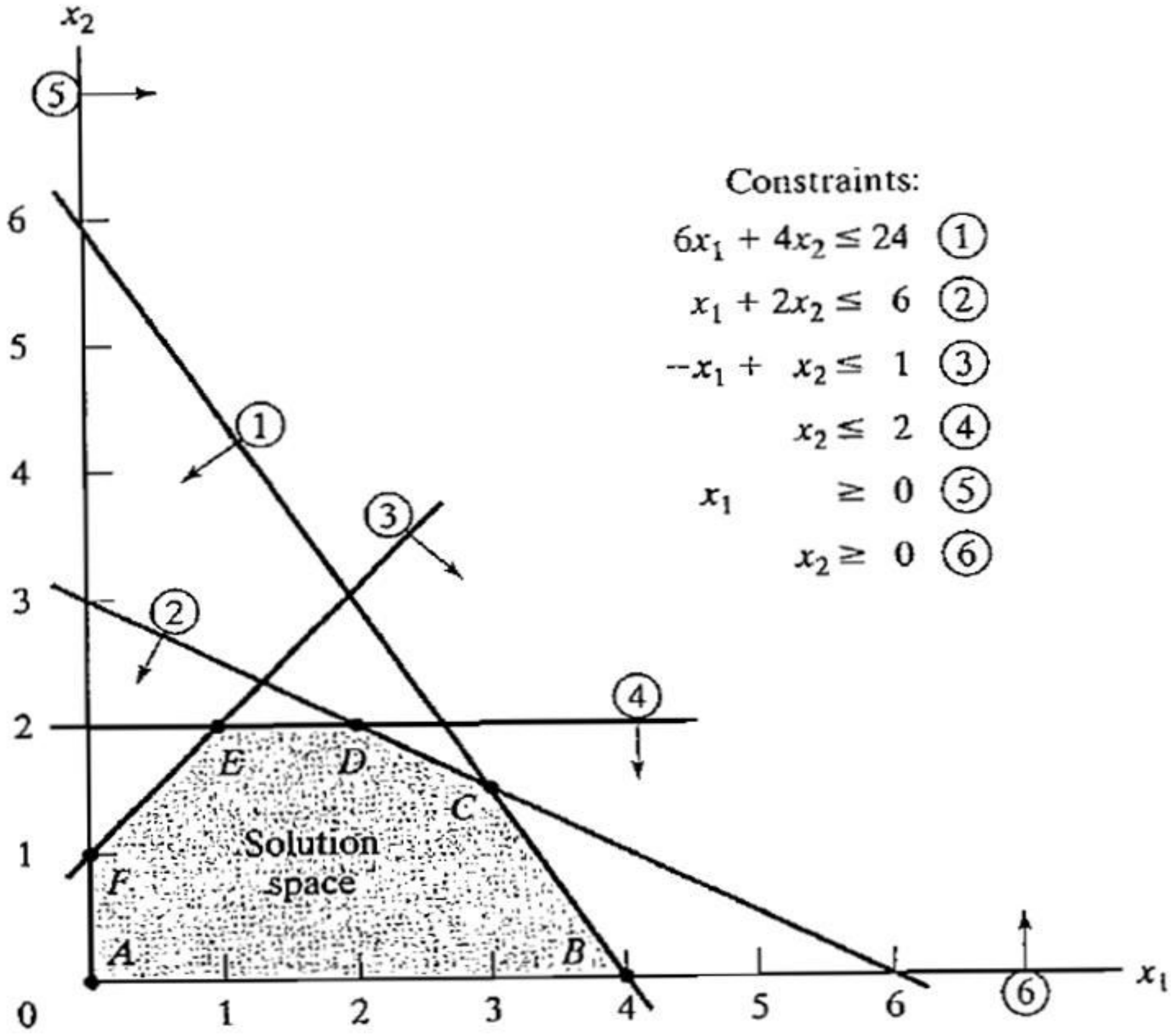
$$x_1, x_2 \ge 0 \qquad (5)$$

Any values of x_1 and x_2 that satisfy all five constraints constitute a feasible solution. Otherwise, the solution is infeasible.

For example, the solution, $x_1 = 3$ tons per day and $x_2 = 1$ ton per day, is feasible because it does not violate any of the constraints.

The goal of the problem is to find the best feasible solution, or the optimum, that maximizes the total profit. Before we can do that, we need to know how many feasible solutions the Reddy Mikks problem has. The answer, as we will see from the the graphical solution is an infinite number, which makes it impossible the solve the problem by enumeration.

GRAPHICAL SOLUTION OF LPP



The graphical procedure includes 2 steps:

1. Determination of the feasible solution space.

2. Determination of the optimum solution from among all the feasible points in the solution space

Determination of Optimum Solution

- The **feasible space** in figure is delineated by the line segments joining the points A, B, C, D, E, and F. **Any point within or on the boundary of the space ABCDEF is feasible.** Because the feasible space ABCDEF consists of an infinite number of points, we need a systematic procedure to identify the optimum solution.
- **An important characteristic of the optimum LP solution is that it is always associated with a corner point of the solution space** (where two lines intersect). This is true even if the objective function happens to be parallel to a constraint. For example, if the objective function is $z = 6x_1 + 4x_2$, which is parallel to constraint 1, we can always say that the optimum occurs at either corner point B or comer point C.
- The observation that the LP optimum is always associated with a corner point means that **the optimum solution can be found simply by enumerating all the corner points** as the following table shows:

Determination of Optimum Solution(2)

- As the number of constraints and variables increases, the number of corner points also increases.
- Nevertheless, the idea shows that, from the standpoint of determining the LP optimum, the solution space ABCDEF with its infinite number of solutions can, in fact, be replaced with a finite number of promising solution points-namely, the corner points, A, B, C, D, E, and F.
- The optimum solution is $x_1 = 3$ and $x_2 = 1.5$ with $Z = (5 * 3) + (4 * 1.5) = 21$. **This calls for a daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint. The associated daily profit is $21,000.**

Corner point	(x_1, x_2)	z
A	(0, 0)	0
B	(4, 0)	20
C	**(3, 1.5)**	**21 (OPTIMUM)**
D	(2, 2)	18
E	(1, 2)	13
F	(0, 1)	4

PRACTICE PROBLEM 1

A company produces two products, A and B. The sales volume for A is at least 80% of the total sales of both A and B.

However, the company cannot sell more than 100 units of A per day. Both products use one raw material, of which the maximum daily availability is 240 lb. The usage rates of the raw material are 2 lb per unit of A, and 4 lb per unit of B.

The profit units for A and B are $20 and $50, respectively.

Formulate the LPP for the same.

SOLUTION TO PRACTICE PROBLEM

If we let A = units of product A and B =units of product B, then we'll

$$\text{maximize } z = 20A + 50B$$

subject to

$$\begin{aligned} 2A + 4B &\leq 240 && \text{(raw material availability)} \\ A &\leq 100 && \text{(sales limit of A)} \\ -0.2A + 0.8B &\leq 0 && \text{(sales of A at least 80\%)} \\ A, B &\geq 0 && \text{(sign restrictions)} \end{aligned}$$

The sales volume for A is at least 80% of the total sales of both A and B. So, $A \geq 0.8\,(A + B)$ which gives us $0 \geq -0.2\,A + 0.8\,B$

PRACTICE PROBLEM 2

A company produces two types of items P and Q that require gold and silver.

Each unit of type P requires 4g silver and 1g gold while that of type Q requires 1g silver and 3g gold. The company can produce 8g silver and 9g gold.

If each unit of type P brings a profit of `44 and that of type Q `55, determine the number of units of each type that the company should produce to maximise the profit. What is the maximum profit?

Degeneracy in Linear Programming

- Degeneracy in LPP occurs when a basic feasible solution has one or more basic variables equal to zero, which can lead to the Simplex method cycling or failing to reach an optimal solution in a finite number of steps.
- It manifests when there is a tie in the minimum ratio test while selecting an outgoing variable in the Simplex table (**Ties in Replacement Ratios)**. Also degeneracy can arise if a**t least one of the constraints has a zero value on the right-hand side (Zero-Valued Basic Variables).** Degeneracy can also arise from **redundant constraints** that overly restrict the solution space.
- In the graphical method for LPPs, degeneracy occurs when **a basic feasible solution has at least one basic variable equal to zero,** , or when **multiple constraints intersect at the same corner point**, resulting in a degenerate corner point. Graphically, it means that more than one constraint line passes through a single corner of the feasible region.
- **Degeneracy can lead to cycling or difficulty in identifying unique optimal solutions.**

Consequences of Degeneracy

1. The primary consequences of degeneracy in LPP are the potential for the simplex algorithm to experience **cycling** (repeatedly visiting the same set of basic feasible solutions without improving the objective function) or **stalling** (failing to make progress toward the optimal solution).

2. **Increased iterations**: Even if it doesn't lead to cycling, degeneracy can significantly increase the number of iterations required for the simplex algorithm to converge to the optimal solution.

3. Degeneracy typically occurs when a pivot operation results in **no improvement to the objective function value**, often due to a tie in determining the outgoing variable in a simplex tableau, making it difficult (potentially requiring more iterations) to reach the optimal solution.

- Degeneracy does not affect the **existence of an optimal solution**.
- **Feasibility:** Degeneracy does not make a basic feasible solution infeasible.

Duality in LPP

- In LPPs, **duality is the concept that every LPP, called the Primal, has an associated LPP called the Dual**, derived from the same data and sharing the same solution.

Relationship between Primal and Dual :

- **Variables and Constraints:** The variables in the primal problem become the constraints in the dual problem, and vice-versa.
- **Objective Functions:** The objective function coefficients of primal become the RHS constants of the dual's constraints, and the RHS constants of the primal constraints become the objective function coefficients of the dual.
- **Optimization Direction:** If the primal is a maximization problem, its dual will be a minimization problem, and vice versa.

Duality Theorem: This theorem states that if the primal problem has an optimal solution, the dual problem also has an optimal solution, and the optimal values of their objective functions are equal.

Observations in Duality

- The number of constraints in the primal problem is equal to the number of variables in the dual problem.
- Similarly, the number of variables in the primal problem corresponds to the number of constraints in the dual problem.
- When primal is in maximization form, the dual is in minimization form.
- The coefficients in the objective function of the primal problem become the right-hand side(RHS) of the constraints in the dual problem.
- The right-hand side of the primal problem becomes the coefficients in the objective function of the dual problem.
- The coefficients of the variables in the constraints of the primal problem are transposed to form the coefficients of the variables in the constraints of the dual problem.

Primal-Dual Relationship

PRIMAL	CONVERSION	DUAL
Maximization Problem	↔	Minimization Problem
Minimization Problem	↔	Maximization Problem
Objective Coefficients	↔	Right Hand Side (RHS) values
Right Hand Side (RHS) values	↔	Objective Coefficients
Number of Variables	↔	Number of Constraints
Number of Constraints	↔	Number of Variables
Variables are in terms of X n	↔	Variables are in terms of Y n

Primal Problem (LPP)

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 4x_2 \\ \text{subject to} \quad & \frac{1}{2}x_1 + 2x_2 \leq 30 \\ & 3x_1 + x_2 \leq 25 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Dual LPP

$$\begin{aligned} \text{Minimize} \quad & Z = 30y_1 + 25y_2 \\ \text{subject to} \quad & \frac{1}{2}y_1 + 3y_2 \geq 3 \\ & 2y_1 + y_2 \geq 4 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Primal-Dual Relationship

Normal Primal Problem

$$\begin{aligned} \text{Maximize} \quad & Z = \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Normal Dual Problem

$$\begin{aligned} \text{Minimize} \quad & W = \mathbf{b}^\top \mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^\top \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned}$$

$$
\begin{aligned}
\text{Minimize} \quad & Z = 2x_1 + 3x_2 + 4x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + x_3 \geq 5 \\
& 3x_1 + x_2 + 2x_3 \geq 8 \\
& -3x_1 - x_2 - 2x_3 \geq -8 \\
& -x_1 - 4x_2 - 3x_3 \geq -10 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Minimize} \quad & Z = 2x_1 + 3x_2 + 4x_3 \\
\text{subject to} \quad & x_1y_1 + 2x_2y_1 + x_3y_1 \geq 5y_1 \\
& 3x_1y_2 + x_2y_2 + 2x_3y_2 \geq 8y_2 \\
& -3x_1y_3 - x_2y_3 - 2x_3y_3 \geq -8y_3 \\
& -x_1y_4 - 4x_2y_4 - 3x_3y_4 \geq -10y_4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Maximize} \quad & \boxed{Z = 5y_1 + 8y_2 - 8y_3 - 10y_4} \\
\text{subject to} \quad & \boxed{y_1 + 3y_2 - 3y_3 - y_4} \leq 2 \\
& \boxed{2y_1 + y_2 - y_3 - 4y_4} \leq 3 \\
& \boxed{y_1 + 2y_2 - 2y_3 - 3y_4} \leq 4 \\
& y_1, y_2, y_3, y_4 \geq 0
\end{aligned}
$$

Primal Problem (or Dual Problem)	**Dual Problem (or Primal Problem)**
Maximize Z (or W)	Minimize W (or Z)
Constraint i:	Variable y_i (or x_i):
$\leq$ form ⟷	$y_i \geq 0$
$=$ form ⟷	Unconstrained
$\geq$ form ⟷	$y_i' \leq 0$
Variable x_j (or y_j):	Constraint j:
$x_j \geq 0$ ⟷	$\geq$ form
Unconstrained ⟷	$=$ form
$x_j' \leq 0$ ⟷	$\leq$ form

Benefits of Duality in LPP

- **Alternative Formulations:** Provides another way to view and solve the same problem.
- **Solution Bounds:** Helps in establishing upper or lower bounds for the optimal solution of the primal problem.
- **Sensitivity Analysis:** Facilitates the calculation of shadow prices, which indicate the value of additional units of a resource.
- **Feasibility and Optimality:** Helps in evaluating whether a solution is feasible or optimal.

Fundamental Theorem of Linear Programming

If a linear programming problem (LPP) has an optimal solution, then at least one optimal solution occurs at a **corner point (vertex)** of the feasible region.

- **Implications:**
  - Search for optimal solutions can be restricted to corner points of the feasible region.
  - There may be:
    - **Unique solution** at one vertex.
    - **Multiple optimal solutions** if the objective function is parallel to a constraint.
    - **Unbounded solution** if feasible region is open in the direction of optimization.
    - **Infeasible problem** if feasible region is empty.

Degenerate Solutions in LPP

A solution is **degenerate** if one or more basic variables take the value zero at a basic feasible solution (BFS).

- **Causes:**
  - Redundant constraints.
  - Intersection of more than ‘m’ constraints at a BFS (where *m = number of constraints*).
- **Implications:**
  - May lead to **stalling** in the simplex method.
  - Could cause **cycling** (repetition of same BFS).

Simplex-Based Methods

- **Purpose:** Solve LPPs by moving from one BFS to another, improving the objective function until optimality.
- **Key Components:**
  - **Initial Basic Feasible Solution (IBFS):** Obtained using slack/surplus/artificial variables.
  - **Pivot Operations:** Exchange of basic and non-basic variables.
  - **Optimality Test:** When all reduced costs are ≥ 0 (for maximization).
  - **Unboundedness Check:** If entering variable has no positive ratio for leaving variable test.

Cycling in Simplex

- **Problem:** Simplex method may revisit the same set of BFS repeatedly due to degeneracy.
- **Result:** Infinite loop, no progress toward optimality.
- **Prevention Techniques:**
    - **Bland's Rule:** Always choose entering and leaving variables with smallest index.
    - **Perturbation Technique:** Slightly adjust constraints to remove degeneracy.

OPTIMIZATION TECHNIQUES for DECISION MAKING

UNIT-1

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Need of Optimization Techniques

- The ever-increasing demand on engineers to lower production costs to withstand global competition has prompted engineers to look for rigorous methods of decision making, such as optimization methods, to design and produce products and systems both economically and efficiently.
- Optimization techniques, having reached a degree of maturity in recent years, are being used in a wide spectrum of industries, including aerospace, automotive, chemical, electrical, construction, and manufacturing industries.
- With rapidly advancing computer technology, computers are becoming more powerful, and correspondingly, the size and the complexity of the problems that can be solved using optimization techniques are also increasing.

INTRODUCTION

- **Optimization is the act of obtaining the best result under given circumstances**. In design, construction, and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages.
- **The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit**. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, ***optimization*** can be defined as the process of finding the conditions that give the maximum or minimum value of a function.
- There is **no single method available for solving all optimization problems** efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems. The optimum seeking methods are also known as ***mathematical programming techniques*** and are generally studied as a part of **Operations Research**.

SCOPE OF OT / OR

O.R. has a wide scope in everyday life as it provides better solutions to various decision-making problems with great speed and competence. It finds applications in a wide range of areas including defence operations, planning, agriculture, industry (finance, marketing, personal management, production management), research and development. We now describe the applications briefly.

Areas where Optimization is applied are:

1. **Science**
2. **Engineering**
3. **Management**
4. **Finance**
5. **Business**

In Planning for Economic Development

Careful planning is necessary for economic development of any country. Operations Research is used to frame future economic and social policies.

In Agriculture

Agricultural output needs to be increased due to increasing needs for adequate quantity and quality of food for our increasing population. But there are a number of restrictions under which agricultural production is studied. Problems of agricultural production under various restrictions such as optimum allocation of land to various crops in accordance with the climatic conditions, optimum distribution of water from various resources for irrigation purposes can easily be solved by application of Operations Research techniques.

In Industry

Now-a-days, due to complexities of operations and huge sizes of industries, important decisions regarding various sections of the organisation, e.g., planning, procurement, marketing, finance, etc. have to be taken division wise. For example, the production department needs to minimise the cost of production, but maximise output; the finance department needs to optimise capital investment; the personnel department needs to appoint competent work force at minimum cost. Each department has to plan its own objectives which may be in conflict with the objectives of other departments and may not conform to the overall objectives of the organisation. For example, the sales department of an organisation may want to keep sufficient stocks in the inventory, whereas the finance department may want to have minimum investment. In that case, both departments would be in conflict with each other. The applications of O.R. techniques to such situations help in overcoming this difficulty by evolving an optimal strategy and serving efficiently the interest of the organisation as a whole.

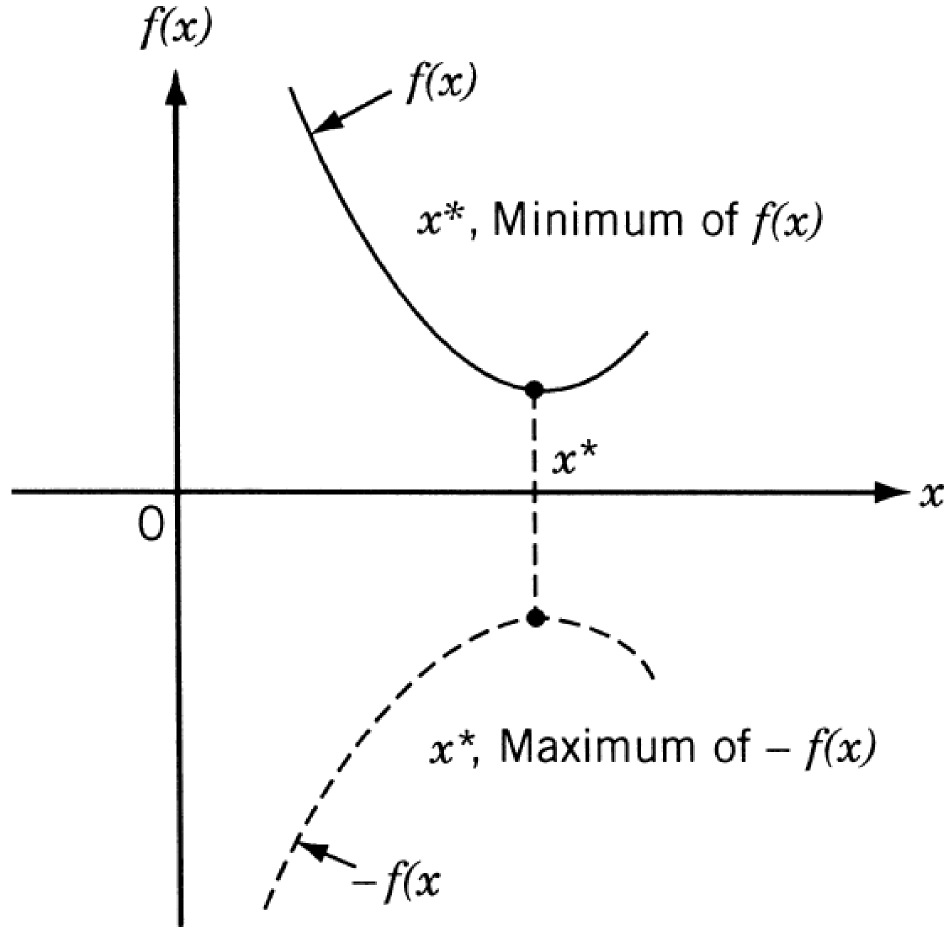

Figure 1.1 Minimum of $f(x)$ is same as maximum of $-f(x)$.

ENGINEERING APPLICATIONS OF OPTIMIZATION

Some typical applications from different engineering disciplines include :

1. Design of aircraft and aerospace structures for minimum weight

2. Finding the optimal trajectories of space vehicles

3. Minimum-weight design of structures for earthquake, wind, other types of random loading

4. Selection of machining conditions in metal-cutting for minimum production cost

5. Shortest route taken by a salesperson visiting various cities during one tour

6. Optimal production planning, controlling, and scheduling

7. Design of optimum pipeline networks for process industries

8. Selection of a site for an industry

9. Planning of maintenance and replacement of equipment to reduce operating costs

10. Inventory control management

11. Allocation of resources among several activities to maximize the profit.

12. Controlling the waiting and idle times in production lines to reduce the costs

13. Optimum design of control systems in design of electronic appliances.

GENERAL OPTIMIZATION PROBLEM

Minimize (Maximize) f (X) objective function

where f: $R^n \rightarrow R$, $X = (x_1, x_2, x_3 \ldots x_n)$ decision values parameters

s.t. $X \in S \subseteq R^n$ where S is defined by

$g_k(X) \geq 0$, k=1,2, … m → inequality constraints

$h_j(X) = 0$, j=1,2, … l → equality constraints

$a_i \leq x_i \leq b_i$ → lower & upper bounds

COMPONENTS OF AN OPTIMIZATION MODEL

Decision variables

Objective function

Constraints

CLASSIFICATION

Linear Programming Problems (LPP)

Nonlinear Programming Problems (NLPP)

Unconstrained Optimization Problems

Constrained Optimization Problems

LINEAR PROGRAMMING PROBLEM

A **Linear Programming Problem** is an optimization problem where the **objective function** and all the **constraints** are linear in nature.

Objective function → A linear function of decision variables to be **maximized or minimized** (e.g., profit, cost, revenue).

Constraints → Linear equations or inequalities representing resource limitations.

Decision variables → Unknowns to be determined (e.g., number of products, allocation of resources).

The General form of LPP is as follows:

Maximize or Minimize:

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

...

$$x_i \geq 0 \quad (i = 1, 2, ..., n)$$

Where:

- Z = Objective function
- $x_1, x_2, ..., x_n$ = Decision variables
- c_i = Coefficients of objective function
- a_{ij} = Coefficients of constraints
- b_j = Resource availability

Problem:

A company produces two products A and B.

Profit from A = ₹3 per unit, profit from B = ₹2 per unit.

Constraints:

- Each unit of A requires 1 hour on machine 1 and 2 hours on machine 2.
- Each unit of B requires 1 hour on machine 1 and 1 hour on machine 2.
- Machine 1 is available for 8 hours, Machine 2 is available for 10 hours.

Formulation:

$$\text{Maximize } Z = 3x_1 + 2x_2$$

Subject to:

$$x_1 + x_2 \leq 8 \quad (Machine1)$$

$$2x_1 + x_2 \leq 10 \quad (Machine2)$$

$$x_1, x_2 \geq 0$$

Solution (Graphical):

- Plot the inequalities, find feasible region.
- Corner points = (0,0), (5,0), (2,6), (0,8).
- Evaluate Z:
  - (0,0) → 0
  - (5,0) → 15
  - (2,6) → 18
  - (0,8) → 16

Optimal Solution: Produce 2 units of A and 6 units of B → **Maximum Profit = ₹18**

NON-LINEAR PROGRAMMING

Non-Linear Programming involves optimization of a **non-linear objective function** and/or **non-linear constraints**.

General Formulation

$$\text{Maximize/Minimize } Z = f(x_1, x_2, \ldots, x_n)$$

Subject to:

$$g_j(x_1, x_2, \ldots, x_n) \leq b_j \quad \text{for } j = 1, 2, \ldots, m$$

$$x_i \geq 0$$

Where:

- $f(x)$ = non-linear objective function
- $g_j(x)$ = non-linear constraint functions

Methods of Solving NLPP

1. **Lagrange Multiplier Method** (for equality constraints)
2. **Kuhn-Tucker (KKT) Conditions** (for inequality constraints)
3. **Gradient Descent / Newton's Method** (iterative numerical methods)

CLASSIFICATION BASED ON TYPE OF DECISION VARIABLES

Dynamic Programming

Geometric Programming

Integer Programming

Quadratic Programming

Separable Programming

STATEMENT OF AN OPTIMIZATION PROBLEM

An optimization or a mathematical programming problem can be stated as follows.

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

subject to the constraints

$$\begin{aligned} g_j(\mathbf{X}) &\le 0, \qquad j = 1, 2, \ldots, m \\ l_j(\mathbf{X}) &= 0, \qquad j = 1, 2, \ldots, p \end{aligned} \tag{1.1}$$

where $\mathbf{X}$ is an n-dimensional vector called the *design vector*, $f(\mathbf{X})$ is termed the *objective function*, and $g_j(\mathbf{X})$ and $l_j(\mathbf{X})$ are known as *inequality* and *equality* constraints, respectively. The number of variables n and the number of constraints m and/or p need not be related in any way. The problem stated in Eq. (1.1) is called a *constrained optimization problem*.† Some optimization problems do not involve any constraints and can be stated as

OPTIMIZATION PROBLEM(2)

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

Such problems are called *unconstrained optimization problems*.

CONSTRAINTS IN OPTIMIZATION

- In many practical problems, the design variables cannot be chosen arbitrarily; rather, they have to satisfy certain specified functional and other requirements.
- The restrictions that must be satisfied to produce an acceptable design are collectively called *design constraints*.
- Constraints that represent limitations on the behavior or performance of the system are termed *behavior* or *functional constraints*.
- Constraints that represent physical limitations on design variables, such as availability, fabricability, and transportability, are known as *geometric* or *side constraints*.

CONSTRAINT SURFACE

For illustration, consider an optimization problem with only inequality constraints $g_j(\mathbf{X}) \leq 0$. The set of values of $\mathbf{X}$ that satisfy the equation $g_j(\mathbf{X}) = 0$ forms a hypersurface in the design space and is called a *constraint surface*. Note that this is an $(n-1)$-dimensional subspace, where n is the number of design variables. The constraint surface divides the design space into two regions: one in which $g_j(\mathbf{X}) < 0$ and the other in which $g_j(\mathbf{X}) > 0$. Thus the points lying on the hypersurface will satisfy the constraint $g_j(\mathbf{X})$ critically, whereas the points lying in the region where $g_j(\mathbf{X}) > 0$ are infeasible or unacceptable, and the points lying in the region where $g_j(\mathbf{X}) < 0$ are feasible or acceptable. The collection of all the constraint surfaces $g_j(\mathbf{X}) = 0$, $j = 1, 2, \ldots, m$, which separates the acceptable region is called the *composite constraint surface*.

CONSTRAINT SURFACE(2)

- Figure 1.4 shows a hypothetical two-dimensional design space where the infeasible region is indicated by hatched lines.
- A design point that lies on one or more than one constraint surface is called a *bound point* , and the associated constraint is called an *active constraint* .
- Design points that do not lie on any constraint surface are known as *free points*.
- Depending on whether a particular design point belongs to the acceptable or unacceptable region, it can be identified as one of the following four types:

1. Free and acceptable point

2. Free and unacceptable point

3. Bound and acceptable point

4. Bound and unacceptable point

All four types of points are shown in Fig. 1.4.

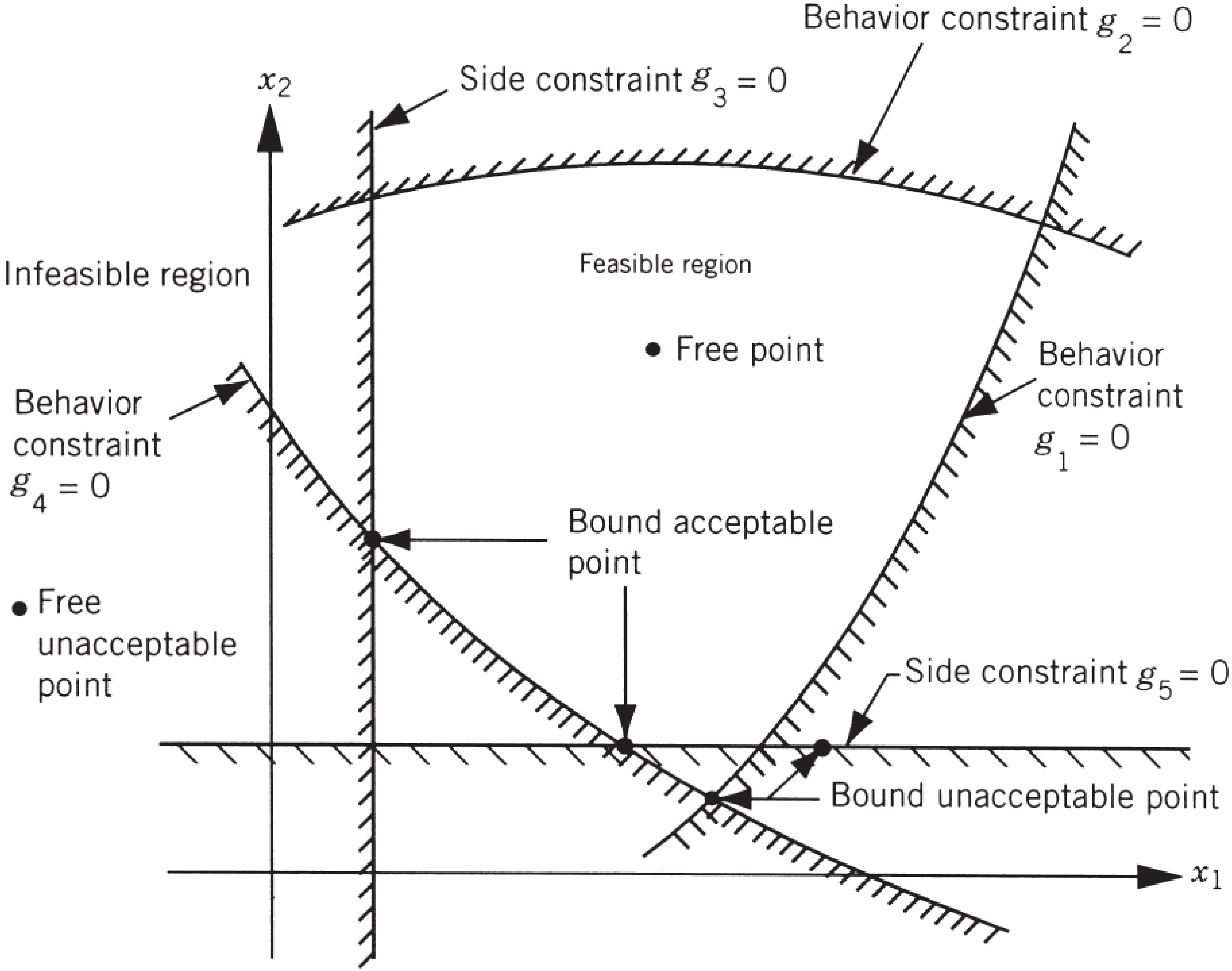

Figure 1.4 Constraint surfaces in a hypothetical two-dimensional design space.

CONSTRAINT SURFACE(2)

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1. Free and acceptable point

2. Free and unacceptable point

3. Bound and acceptable point

4. Bound and unacceptable point

All four types of points are shown in Fig. 1.4.

OBJECTIVE FUNCTION

- In general, there will be more than one acceptable design, and the purpose of optimization is to choose the best one of the many acceptable designs available.
- Thus a criterion has to be chosen for comparing the different alternative acceptable designs and for selecting the best one. The criterion with respect to which the design is optimized, when expressed as a function of the design variables, is known as the *criterion* or *merit* or *objective function*.
- The choice of objective function is governed by the nature of problem.
- The objective function for minimization is generally taken as weight in aircraft and aerospace structural design problems. In civil engineering structural designs, the objective is usually taken as the minimization of cost.
- In some situations, there may be more than one criterion to be satisfied simultaneously.
- For example, a gear pair may have to be designed for minimum weight and maximum efficiency while transmitting a specified horsepower.
- An optimization problem involving multiple objective functions is known as a *multiobjective programming problem*.

PROBLEM 1

1. Reddy Mikks produces both interior and exterior paints from two raw materials M1 and M2. Following table provides the basic data of problem:
2. A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton.
3. Also, the maximum daily demand for interior paint is 2 tons.
4. Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit?
5. Find the optimal solution to this problem

Tons of raw material per ton			
	Exterior paint	Interior paint	Max daily availability
Raw material, M1	6	4	24
Raw material, M2	1	2	6
profit per ton ($1000)	5	4	

SOLUTION

- The LP model, has three basic components:

1. Decision variables that we seek to determine.

2. Objective that we need to optimize (maximize or minimize).

3. Constraints that solution must satisfy.

The **decision variables** of model are defined as

$x1$= tons produced daily of exterior paint

$x2$= tons produced daily of interior paint

Objective function : The company wants to maximize (i.e., increase as much as possible) the total daily profit of both paints.

Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

Total profit from exterior paint = $5x_1$

Total profit from interior paint = $4x_2$

Let Z denote the total daily profit, then objective of the company is

Maximize $\quad Z = 5x_1 + 4x_2$

CONSTRAINTS : To construct the constraints that restrict raw material usage and product demand.

The raw material restrictions are expressed verbally as

(usage of a raw material by both paints) ≤ (max. raw material availability)

Daily usage of raw material M1 is 6 tons of exterior paint and 4 tons of interior paint.

So, Usage of raw material M1 by exterior paint = $6x_1$ tons/day ,

Usage of raw material M1 by interior paint = $4x_2$ tons/day

Therefore, Usage of M1 by both paints = $6x_1 + 4x_2$ tons/day

By similar logic,

Usage of M2 by both paints = $1x_1 + 2x_2$ tons/day

Because the daily availabilities of raw materials M1 and M2 are limited to 24 and 6 tons respectively, the associated restrictions are given as

$$6x_1 + 4x_2 \le 24 \quad \text{(Raw material M1)}$$

$$x_1 + 2x_2 \le 6 \quad \text{(Raw material M2)}$$

Demand restriction stipulates that the excess of daily production of interior over exterior paint, x_2-x_1, should not exceed 1 ton, which translates to $x_2 - x_1 \le 1$ **(market limit)**

Second demand restriction- the max. daily demand of interior paint is limited to 2 tons, which translates to $x_2 \le 2$ **(demand limit)**.

An implicit (or "understood- to –be") restriction is that variables x_1 and x_2 cannot assume negative values. So $x_1 \ge 0$, $x_2 \ge 0$ **(nonnegativity restrictions)**

The complete Reddy Mikks Model is

Maximize $\mathbf{Z = 5x_1 + 4x_2}$

Subject to:

$$6x_1 + 4x_2 \leq 24 \qquad (1)$$

$$x_1 + 2x_2 \leq 6 \qquad (2)$$

$$x_2 - x_1 \leq 1 \qquad (3)$$

$$x_2 \leq 2 \qquad (4)$$

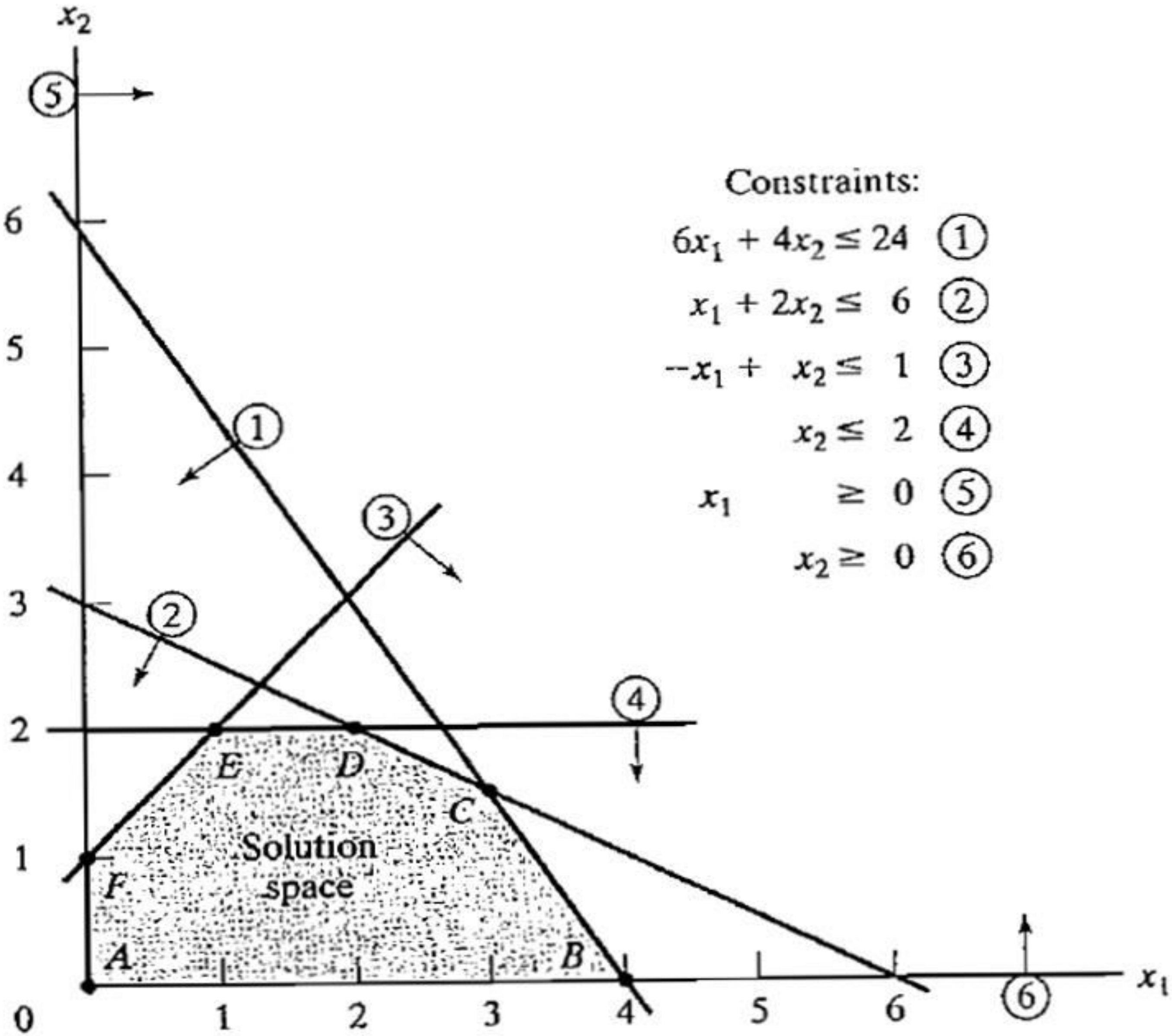
$$x_1, x_2 \geq 0 \qquad (5)$$

Any values of x_1 and x_2 that satisfy all five constraints constitute a feasible solution. Otherwise, the solution is infeasible.

For example, the solution, $x_1 = 3$ tons per day and $x_2 = 1$ ton per day, is feasible because it does not violate any of the constraints.

The goal of the problem is to find the best feasible solution, or the optimum, that maximizes the total profit. Before we can do that, we need to know how many feasible solutions the Reddy Mikks problem has. The answer, as we will see from the the graphical solution is an infinite number, which makes it impossible the solve the problem by enumeration.

GRAPHICAL SOLUTION OF LPP



The graphical procedure includes 2 steps:

1. Determination of the feasible solution space.

2. Determination of the optimum solution from among all the feasible points in the solution space

Determination of Optimum Solution

- The **feasible space** in figure is delineated by the line segments joining the points A, B, C, D, E, and F. **Any point within or on the boundary of the space ABCDEF is feasible.** Because the feasible space ABCDEF consists of an infinite number of points, we need a systematic procedure to identify the optimum solution.
- **An important characteristic of the optimum LP solution is that it is always associated with a corner point of the solution space** (where two lines intersect). This is true even if the objective function happens to be parallel to a constraint. For example, if the objective function is $z = 6x_1 + 4x_2$, which is parallel to constraint 1, we can always say that the optimum occurs at either corner point B or comer point C.
- The observation that the LP optimum is always associated with a corner point means that **the optimum solution can be found simply by enumerating all the corner points** as the following table shows:

Determination of Optimum Solution(2)

- As the number of constraints and variables increases, the number of corner points also increases.
- Nevertheless, the idea shows that, from the standpoint of determining the LP optimum, the solution space ABCDEF with its infinite number of solutions can, in fact, be replaced with a finite number of promising solution points-namely, the corner points, A, B, C, D, E, and F.
- The optimum solution is $x_1 = 3$ and $x_2 = 1.5$ with $Z = (5 * 3) + (4 * 1.5) = 21$. **This calls for a daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint. The associated daily profit is $21,000.**

Corner point	(x_1, x_2)	z
A	(0, 0)	0
B	(4, 0)	20
C	**(3, 1.5)**	**21 (OPTIMUM)**
D	(2, 2)	18
E	(1, 2)	13
F	(0, 1)	4

PRACTICE PROBLEM 1

A company produces two products, A and B. The sales volume for A is at least 80% of the total sales of both A and B.

However, the company cannot sell more than 100 units of A per day. Both products use one raw material, of which the maximum daily availability is 240 lb. The usage rates of the raw material are 2 lb per unit of A, and 4 lb per unit of B.

The profit units for A and B are \$20 and \$50, respectively.

Formulate the LPP for the same.

SOLUTION TO PRACTICE PROBLEM

If we let A = units of product A and B =units of product B, then we'll

$$\text{maximize } z = 20A + 50B$$

subject to

$$\begin{aligned} 2A + 4B &\le 240 && \text{(raw material availability)} \\ A &\le 100 && \text{(sales limit of A)} \\ -0.2A + 0.8B &\le 0 && \text{(sales of A at least 80\%)} \\ A, B &\ge 0 && \text{(sign restrictions)} \end{aligned}$$

The sales volume for A is at least 80% of the total sales of both A and B. So, $A \ge 0.8\,(A + B)$ which gives us $0 \ge -0.2\,A + 0.8\,B$

PRACTICE PROBLEM 2

A company produces two types of items P and Q that require gold and silver.

Each unit of type P requires 4g silver and 1g gold while that of type Q requires 1g silver and 3g gold. The company can produce 8g silver and 9g gold.

If each unit of type P brings a profit of `44 and that of type Q `55, determine the number of units of each type that the company should produce to maximise the profit. What is the maximum profit?

Degeneracy in Linear Programming

- Degeneracy in LPP occurs when a basic feasible solution has one or more basic variables equal to zero, which can lead to the Simplex method cycling or failing to reach an optimal solution in a finite number of steps.
- It manifests when there is a tie in the minimum ratio test while selecting an outgoing variable in the Simplex table (**Ties in Replacement Ratios)**. Also degeneracy can arise if a**t least one of the constraints has a zero value on the right-hand side** **(Zero-Valued Basic Variables).** Degeneracy can also arise from **redundant constraints** that overly restrict the solution space.
- In the graphical method for LPPs, degeneracy occurs when **a basic feasible solution has at least one basic variable equal to zero,** , or when **multiple constraints intersect at the same corner point**, resulting in a degenerate corner point. Graphically, it means that more than one constraint line passes through a single corner of the feasible region.
- **Degeneracy can lead to cycling or difficulty in identifying unique optimal solutions.**

Consequences of Degeneracy

1. The primary consequences of degeneracy in LPP are the potential for the simplex algorithm to experience **cycling** (repeatedly visiting the same set of basic feasible solutions without improving the objective function) or **stalling** (failing to make progress toward the optimal solution).

2. **Increased iterations**: Even if it doesn't lead to cycling, degeneracy can significantly increase the number of iterations required for the simplex algorithm to converge to the optimal solution.

3. Degeneracy typically occurs when a pivot operation results in **no improvement to the objective function value**, often due to a tie in determining the outgoing variable in a simplex tableau, making it difficult (potentially requiring more iterations) to reach the optimal solution.

- Degeneracy does not affect the **existence of an optimal solution**.
- **Feasibility:** Degeneracy does not make a basic feasible solution infeasible.

Duality in LPP

- In LPPs, **duality is the concept that every LPP, called the Primal, has an associated LPP called the Dual**, derived from the same data and sharing the same solution. **Dual LP problem** provides useful economic information about worth of resources to be used.

Relationship between Primal and Dual :

- **Variables and Constraints:** The variables in the primal problem become the constraints in the dual problem, and vice-versa.
- **Objective Functions:** The objective function coefficients of primal become the RHS constants of the dual's constraints, and the RHS constants of the primal constraints become the objective function coefficients of the dual.
- **Optimization Direction:** If the primal is a maximization problem, its dual will be a minimization problem, and vice versa.

Duality Theorem: This theorem states that if the primal problem has an optimal solution, the dual problem also has an optimal solution, and the optimal values of their objective functions are equal.

Observations in Duality

- The number of constraints in the primal problem is equal to the number of variables in the dual problem.
- Similarly, the number of variables in the primal problem corresponds to the number of constraints in the dual problem.
- When primal is in maximization form, the dual is in minimization form.
- The coefficients in the objective function of the primal problem become the right-hand side(RHS) of the constraints in the dual problem.
- The right-hand side of the primal problem becomes the coefficients in the objective function of the dual problem.
- The coefficients of the variables in the constraints of the primal problem are transposed to form the coefficients of the variables in the constraints of the dual problem.

Primal-Dual Relationship

If Primal	*Then Dual*
(i) Objective is to maximize	(i) Objective is to minimize
(ii) jth primal variable, x_j	(ii) jth dual constraint
(iii) ith primal constraint	(iii) ith dual variable, y_i
(iv) Primal variable x_j unrestricted in sign	(iv) Dual constraint j is = type
(v) Primal constraint i is = type	(v) Dual variable y_i is unrestricted in sign
(vi) Primal constraints $\leq$ type	(vi) Dual constraints $\geq$ type

Primal Problem (LPP)

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 4x_2 \\ \text{subject to} \quad & \frac{1}{2}x_1 + 2x_2 \leq 30 \\ & 3x_1 + x_2 \leq 25 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Dual LPP

$$\begin{aligned} \text{Minimize} \quad & Z = 30y_1 + 25y_2 \\ \text{subject to} \quad & \frac{1}{2}y_1 + 3y_2 \geq 3 \\ & 2y_1 + y_2 \geq 4 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Primal-Dual Relationship

Normal Primal Problem	Normal Dual Problem
$\begin{aligned} \text{Maximize} \quad & Z = \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$	$\begin{aligned} \text{Minimize} \quad & W = \mathbf{b}^\top \mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^\top \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned}$

$$
\begin{aligned}
\text{Minimize} \quad & Z = 2x_1 + 3x_2 + 4x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + x_3 \geq 5 \\
& 3x_1 + x_2 + 2x_3 \geq 8 \\
& -3x_1 - x_2 - 2x_3 \geq -8 \\
& -x_1 - 4x_2 - 3x_3 \geq -10 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Minimize} \quad & Z = 2x_1 + 3x_2 + 4x_3 \\
\text{subject to} \quad & x_1y_1 + 2x_2y_1 + x_3y_1 \geq 5y_1 \\
& 3x_1y_2 + x_2y_2 + 2x_3y_2 \geq 8y_2 \\
& -3x_1y_3 - x_2y_3 - 2x_3y_3 \geq -8y_3 \\
& -x_1y_4 - 4x_2y_4 - 3x_3y_4 \geq -10y_4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Maximize} \quad & \boxed{Z = 5y_1 + 8y_2 - 8y_3 - 10y_4} \\
\text{subject to} \quad & \boxed{y_1 + 3y_2 - 3y_3 - y_4} \leq 2 \\
& \boxed{2y_1 + y_2 - y_3 - 4y_4} \leq 3 \\
& \boxed{y_1 + 2y_2 - 2y_3 - 3y_4} \leq 4 \\
& y_1, y_2, y_3, y_4 \geq 0
\end{aligned}
$$

Primal Problem (or Dual Problem)		**Dual Problem (or Primal Problem)**
Maximize Z (or W)		Minimize W (or Z)
Constraint i:		Variable y_i (or x_i):
$\leq$ form	$\longleftrightarrow$	$y_i \geq 0$
$=$ form	$\longleftrightarrow$	Unconstrained
$\geq$ form	$\longleftrightarrow$	$y_i' \leq 0$
Variable x_j (or y_j):		Constraint j:
$x_j \geq 0$	$\longleftrightarrow$	$\geq$ form
Unconstrained	$\longleftrightarrow$	$=$ form
$x_j' \leq 0$	$\longleftrightarrow$	$\leq$ form

Obtain the dual of the following LPP:

Maximize $Z_x = x_1 - 2x_2 + 3x_3$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 = 2$, (ii) $2x_1 + 3x_2 + 4x_3 = 1$

and $x_1, x_2, x_3 \geq 0$

Solution Since both the primal constraints are of the equality type, the corresponding dual variables y_1 and y_2, will be unrestricted in sign. Following the rules of duality formulation, the dual of the given primal LP problem is

Minimize $Z_y = 2y_1 + y_2$

subject to the constraints

(i) $-2y_1 + 2y_2 \geq 1$, (ii) $y_1 + 3y_2 \geq -2$, (iii) $3y_1 + 4y_2 \geq 3$

and y_1, y_2 unrestricted in sign.

Benefits of Duality in LPP

- Study of duality helps to identify only an increase (or decrease) in the value of objective function due to per unit variation in the amount of resources available.

1. **Alternative Formulations:** Provides another way to view and solve the same problem.
2. **Solution Bounds:** Helps in establishing upper or lower bounds for the optimal solution of the primal problem.
3. **Sensitivity Analysis:** Facilitates the calculation of shadow prices, which indicate the value of additional units of a resource.
4. **Feasibility and Optimality:** Helps in evaluating whether a solution is feasible or optimal.

Fundamental Theorem of Linear Programming

If a linear programming problem (LPP) has an optimal solution, then at least one optimal solution occurs at a **corner point (vertex)** of the feasible region.

•**Implications:**

- Search for optimal solutions can be restricted to corner points of the feasible region.
- There may be:
  - **Unique solution** at one vertex.
  - **Multiple optimal solutions** if the objective function is parallel to a constraint.
  - **Unbounded solution** if feasible region is open in the direction of optimization.
  - **Infeasible problem** if feasible region is empty.

Degenerate Solutions in LPP

A solution is **degenerate** if one or more basic variables take the value zero at a basic feasible solution (BFS).

- **Causes:**
  - Redundant constraints.
  - Intersection of more than ‘m’ constraints at a BFS (where *m = number of constraints*).
- **Implications:**
  - May lead to **stalling** in the simplex method.
  - Could cause **cycling** (repetition of same BFS).

Simplex-Based Methods

•**Purpose:** Solve LPPs by moving from one BFS to another, improving the objective function until optimality.

•**Key Components:**

- **Initial Basic Feasible Solution (IBFS):** Obtained using slack/surplus/artificial variables.
- **Pivot Operations:** Exchange of basic and non-basic variables.
- **Optimality Test:** When all reduced costs are ≥ 0 (for maximization).
- **Unboundedness Check:** If entering variable has no positive ratio for leaving variable test.

Cycling in Simplex

- **Problem:** Simplex method may revisit the same set of BFS repeatedly due to degeneracy.
- **Result:** Infinite loop, no progress toward optimality.
- **Prevention Techniques:**
    - **Bland's Rule:** Always choose entering and leaving variables with smallest index.
    - **Perturbation Technique:** Slightly adjust constraints to remove degeneracy.

SENSITIVITY ANALYSIS in LPP

- **Alternative Formulations:** Provides another way to view and solve the same problem.
- **Solution Bounds:** Helps in establishing upper or lower bounds for the optimal solution of the primal problem.
- **Sensitivity Analysis:** Facilitates the calculation of shadow prices, which indicate the value of additional units of a resource.
- **Feasibility and Optimality:** Helps in evaluating whether a solution is feasible or optimal.

SENSITIVITY ANALYSIS in LPP

- The process of modifying an OR model to observe the effect upon its outputs is called **Sensitivity Analysis**. Purpose is to evaluate the effect on the optimal solution of an LP problem due to variations in the input coefficients (also called parameters), one at a time.
- In an LP model, the coefficients (also known as parameters) such as: (i) profit (cost) contribution (c_j) per unit of a decision variable, x_j (ii) availability of a resources (b_i), and (iii) consumption of resource per unit of decision variables (a_{ij}), are assumed to be constant and known with certainty.
- *Sensitivity analysis determines the sensitivity range (both lower and upper limit) within which the LP model parameters can vary (one at a time) without affecting the optimality of the current optional solution.*
- This analysis reveals the magnitude of change in the optimal solution of an LP model due to discrete variations (changes) in its parameters. The possible change in the parameter values, can range from zero to a substantial change.
- Thus, aim of sensitivity analysis is to determine the range within which the LP model parameters can change without affecting the current optimal solution.

SENSITIVITY ANALYSIS (2)

- The sensitivity analysis is also referred to as *post-optimality analysis* because it does not begin until the optimal solution to the given LP model has been obtained.
- Different parametric changes in an LPP are:

1. Profit (or cost) per unit (c_j) associated with both basic and non-basic decision variables (i.e., coefficients in the objective function).
2. Availability of resources (i.e., right-hand side constants, b_i in constraints).
3. Consumption of resources per unit of decision variables x_j (i.e., coefficients of decision variables in the constraints, a_{ij}).
4. Addition of a new variable to the existing list of variables in LP problem.
5. Addition of a new constraint to the existing list of constraints in the LP problem.

- *So, what happens to the optimal solution value when we have a* **change in the Objective Function Coefficient (c_j)**? Analysing this is termed as **SENSITIVITY ANALYSIS**.

OPTIMIZATION TECHNIQUES for DECISION MAKING

UNIT-1

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Need of Optimization Techniques

- The ever-increasing demand on engineers to lower production costs to withstand global competition has prompted engineers to look for rigorous methods of decision making, such as optimization methods, to design and produce products and systems both economically and efficiently.
- Optimization techniques, having reached a degree of maturity in recent years, are being used in a wide spectrum of industries, including aerospace, automotive, chemical, electrical, construction, and manufacturing industries.
- With rapidly advancing computer technology, computers are becoming more powerful, and correspondingly, the size and the complexity of the problems that can be solved using optimization techniques are also increasing.

INTRODUCTION

- **Optimization is the act of obtaining the best result under given circumstances**. In design, construction, and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages.
- **The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit**. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, ***optimization*** can be defined as the process of finding the conditions that give the maximum or minimum value of a function.
- There is **no single method available for solving all optimization problems** efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems. The optimum seeking methods are also known as ***mathematical programming techniques*** and are generally studied as a part of **Operations Research**.

SCOPE OF OT / OR

O.R. has a wide scope in everyday life as it provides better solutions to various decision-making problems with great speed and competence. It finds applications in a wide range of areas including defence operations, planning, agriculture, industry (finance, marketing, personal management, production management), research and development. We now describe the applications briefly.

Areas where Optimization is applied are:

1. **Science**
2. **Engineering**
3. **Management**
4. **Finance**
5. **Business**

In Planning for Economic Development

Careful planning is necessary for economic development of any country. Operations Research is used to frame future economic and social policies.

In Agriculture

Agricultural output needs to be increased due to increasing needs for adequate quantity and quality of food for our increasing population. But there are a number of restrictions under which agricultural production is studied. Problems of agricultural production under various restrictions such as optimum allocation of land to various crops in accordance with the climatic conditions, optimum distribution of water from various resources for irrigation purposes can easily be solved by application of Operations Research techniques.

In Industry

Now-a-days, due to complexities of operations and huge sizes of industries, important decisions regarding various sections of the organisation, e.g., planning, procurement, marketing, finance, etc. have to be taken division wise. For example, the production department needs to minimise the cost of production, but maximise output; the finance department needs to optimise capital investment; the personnel department needs to appoint competent work force at minimum cost. Each department has to plan its own objectives which may be in conflict with the objectives of other departments and may not conform to the overall objectives of the organisation. For example, the sales department of an organisation may want to keep sufficient stocks in the inventory, whereas the finance department may want to have minimum investment. In that case, both departments would be in conflict with each other. The applications of O.R. techniques to such situations help in overcoming this difficulty by evolving an optimal strategy and serving efficiently the interest of the organisation as a whole.

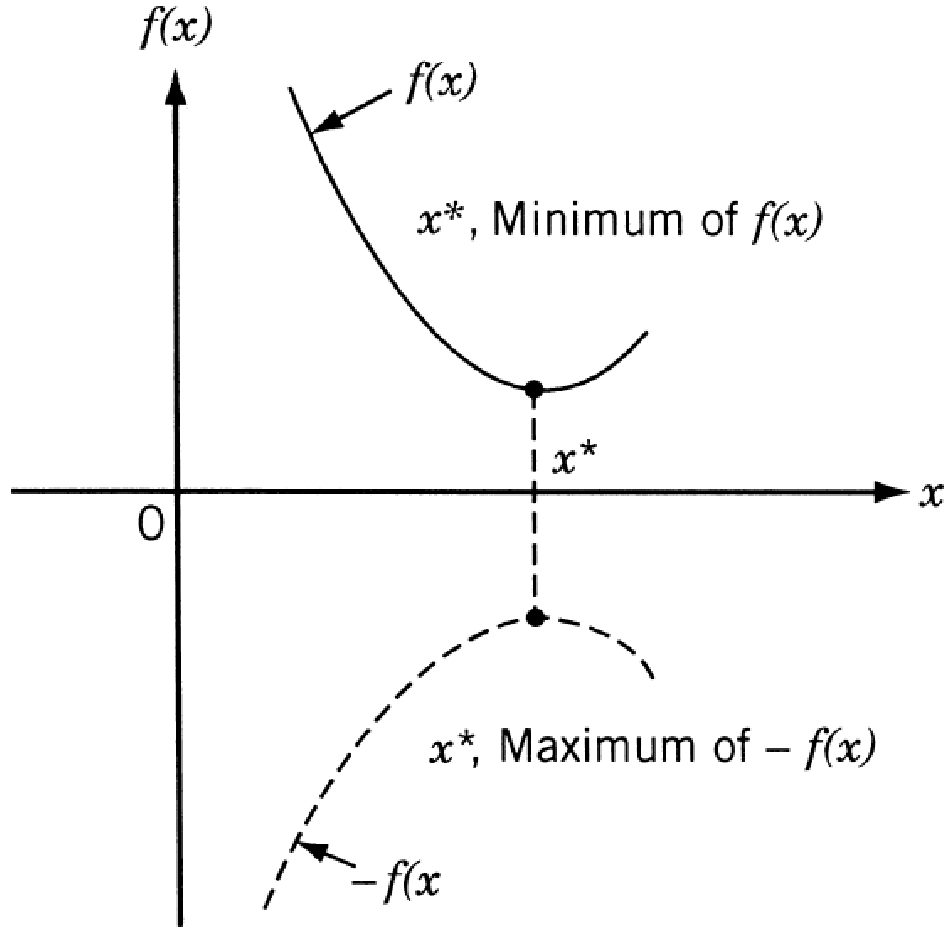

Figure 1.1 Minimum of $f(x)$ is same as maximum of $-f(x)$.

ENGINEERING APPLICATIONS OF OPTIMIZATION

Some typical applications from different engineering disciplines include :

1. Design of aircraft and aerospace structures for minimum weight

2. Finding the optimal trajectories of space vehicles

3. Minimum-weight design of structures for earthquake, wind, other types of random loading

4. Selection of machining conditions in metal-cutting for minimum production cost

5. Shortest route taken by a salesperson visiting various cities during one tour

6. Optimal production planning, controlling, and scheduling

7. Design of optimum pipeline networks for process industries

8. Selection of a site for an industry

9. Planning of maintenance and replacement of equipment to reduce operating costs

10. Inventory control management

11. Allocation of resources among several activities to maximize the profit.

12. Controlling the waiting and idle times in production lines to reduce the costs

13. Optimum design of control systems in design of electronic appliances.

GENERAL OPTIMIZATION PROBLEM

Minimize (Maximize) f (X) → objective function

where $f: R^n \rightarrow R$, $X = (x_1, x_2, x_3 \ldots x_n)$ → decision values parameters

s.t. $X \in S \subseteq R^n$ where S is defined by

$g_k (X) \geq 0$, k=1,2, … m → inequality constraints

$h_j (X) = 0$, j=1,2, … l → equality constraints

$a_i \leq x_i \leq b_i$ → lower & upper bounds

COMPONENTS OF AN OPTIMIZATION MODEL

Decision variables

Objective function

Constraints

CLASSIFICATION

Linear Programming Problems (LPP)

Nonlinear Programming Problems (NLPP)

Unconstrained Optimization Problems

Constrained Optimization Problems

LINEAR PROGRAMMING PROBLEM

A **Linear Programming Problem** is an optimization problem where the **objective function** and all the **constraints** are linear in nature.

Objective function → A linear function of decision variables to be **maximized or minimized** (e.g., profit, cost, revenue).

Constraints → Linear equations or inequalities representing resource limitations.

Decision variables → Unknowns to be determined (e.g., number of products, allocation of resources).

The General form of LPP is as follows:

Maximize or Minimize:

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

...

$$x_i \geq 0 \quad (i = 1, 2, ..., n)$$

Where:

- Z = Objective function
- $x_1, x_2, ..., x_n$ = Decision variables
- c_i = Coefficients of objective function
- a_{ij} = Coefficients of constraints
- b_j = Resource availability

Problem:

A company produces two products A and B.
Profit from A = ₹3 per unit, profit from B = ₹2 per unit.

Constraints:

- Each unit of A requires 1 hour on machine 1 and 2 hours on machine 2.
- Each unit of B requires 1 hour on machine 1 and 1 hour on machine 2.
- Machine 1 is available for 8 hours, Machine 2 is available for 10 hours.

Formulation:

$$\text{Maximize } Z = 3x_1 + 2x_2$$

Subject to:

$$x_1 + x_2 \leq 8 \quad (Machine1)$$

$$2x_1 + x_2 \leq 10 \quad (Machine2)$$

$$x_1, x_2 \geq 0$$

Solution (Graphical):

- Plot the inequalities, find feasible region.
- Corner points = (0,0), (5,0), (2,6), (0,8).
- Evaluate Z:
  - (0,0) → 0
  - (5,0) → 15
  - (2,6) → 18
  - (0,8) → 16

Optimal Solution: Produce 2 units of A and 6 units of B → **Maximum Profit = ₹18**

NON-LINEAR PROGRAMMING

Non-Linear Programming involves optimization of a **non-linear objective function** and/or **non-linear constraints**.

General Formulation

$$\text{Maximize/Minimize } Z = f(x_1, x_2, \ldots, x_n)$$

Subject to:

$$g_j(x_1, x_2, \ldots, x_n) \leq b_j \quad \text{for } j = 1, 2, \ldots, m$$

$$x_i \geq 0$$

Where:

- $f(x)$ = non-linear objective function
- $g_j(x)$ = non-linear constraint functions

Methods of Solving NLPP

1. **Lagrange Multiplier Method** (for equality constraints)
2. **Kuhn-Tucker (KKT) Conditions** (for inequality constraints)
3. **Gradient Descent / Newton's Method** (iterative numerical methods)

CLASSIFICATION BASED ON TYPE OF DECISION VARIABLES

Dynamic Programming

Geometric Programming

Integer Programming

Quadratic Programming

Separable Programming

STATEMENT OF AN OPTIMIZATION PROBLEM

An optimization or a mathematical programming problem can be stated as follows.

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

subject to the constraints

$$\begin{aligned} g_j(\mathbf{X}) &\le 0, \qquad j = 1, 2, \ldots, m \\ l_j(\mathbf{X}) &= 0, \qquad j = 1, 2, \ldots, p \end{aligned} \tag{1.1}$$

where $\mathbf{X}$ is an n-dimensional vector called the *design vector*, $f(\mathbf{X})$ is termed the *objective function*, and $g_j(\mathbf{X})$ and $l_j(\mathbf{X})$ are known as *inequality* and *equality* constraints, respectively. The number of variables n and the number of constraints m and/or p need not be related in any way. The problem stated in Eq. (1.1) is called a *constrained optimization problem*.[†] Some optimization problems do not involve any constraints and can be stated as

OPTIMIZATION PROBLEM(2)

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

Such problems are called *unconstrained optimization problems*.

CONSTRAINTS IN OPTIMIZATION

- In many practical problems, the design variables cannot be chosen arbitrarily; rather, they have to satisfy certain specified functional and other requirements.
- The restrictions that must be satisfied to produce an acceptable design are collectively called *design constraints*.
- Constraints that represent limitations on the behavior or performance of the system are termed *behavior* or *functional constraints*.
- Constraints that represent physical limitations on design variables, such as availability, fabricability, and transportability, are known as *geometric* or *side constraints*.

CONSTRAINT SURFACE

For illustration, consider an optimization problem with only inequality constraints $g_j(\mathbf{X}) \leq 0$. The set of values of $\mathbf{X}$ that satisfy the equation $g_j(\mathbf{X}) = 0$ forms a hypersurface in the design space and is called a *constraint surface*. Note that this is an $(n - 1)$-dimensional subspace, where n is the number of design variables. The constraint surface divides the design space into two regions: one in which $g_j(\mathbf{X}) < 0$ and the other in which $g_j(\mathbf{X}) > 0$. Thus the points lying on the hypersurface will satisfy the constraint $g_j(\mathbf{X})$ critically, whereas the points lying in the region where $g_j(\mathbf{X}) > 0$ are infeasible or unacceptable, and the points lying in the region where $g_j(\mathbf{X}) < 0$ are feasible or acceptable. The collection of all the constraint surfaces $g_j(\mathbf{X}) = 0,\ j = 1, 2, \ldots, m$, which separates the acceptable region is called the *composite constraint surface*.

CONSTRAINT SURFACE(2)

- Figure 1.4 shows a hypothetical two-dimensional design space where the infeasible region is indicated by hatched lines.
- A design point that lies on one or more than one constraint surface is called a *bound point* , and the associated constraint is called an *active constraint* .
- Design points that do not lie on any constraint surface are known as *free points*.
- Depending on whether a particular design point belongs to the acceptable or unacceptable region, it can be identified as one of the following four types:

1. Free and acceptable point

2. Free and unacceptable point

3. Bound and acceptable point

4. Bound and unacceptable point

All four types of points are shown in Fig. 1.4.

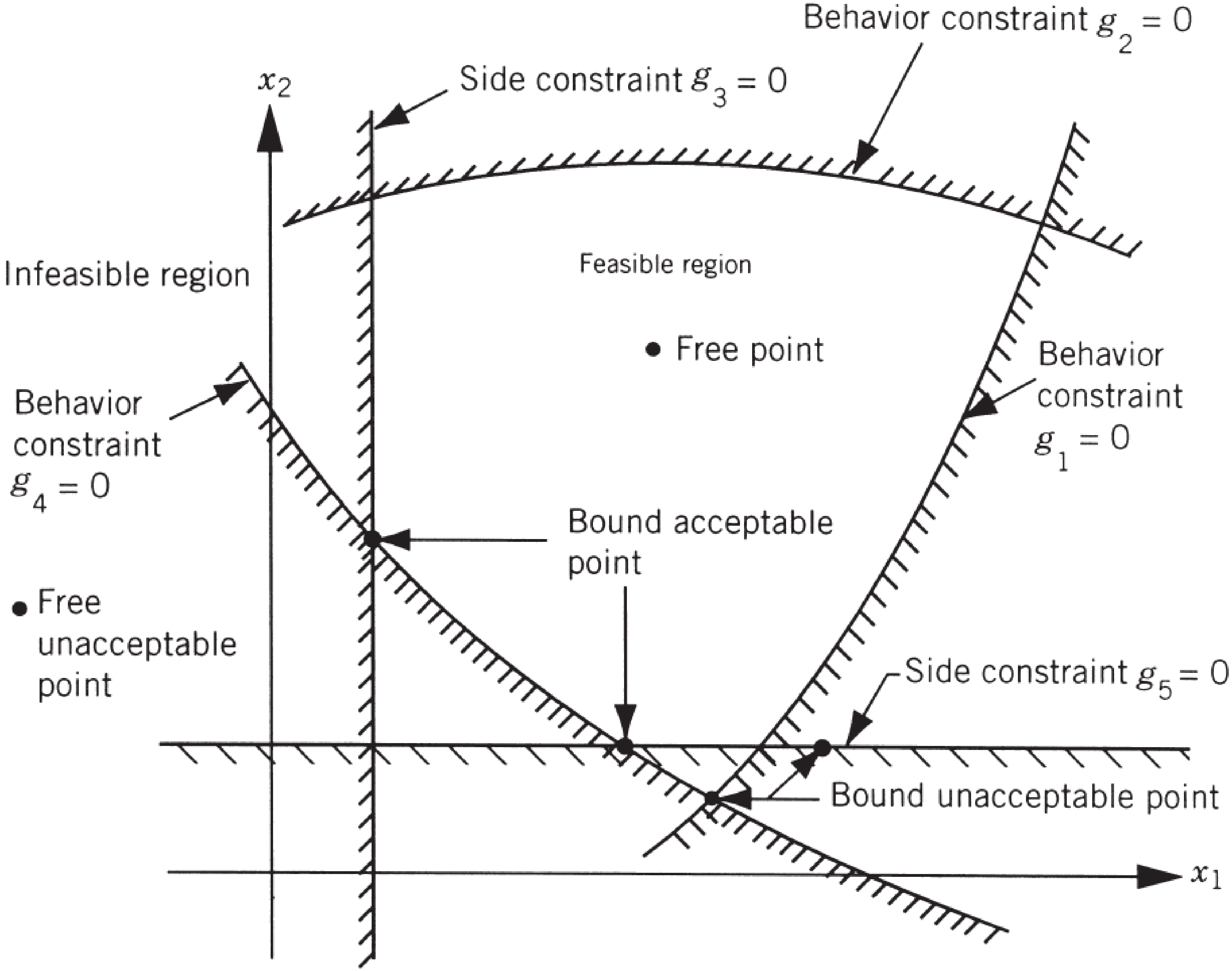

Figure 1.4 Constraint surfaces in a hypothetical two-dimensional design space.

CONSTRAINT SURFACE(2)

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2. Free and unacceptable point

3. Bound and acceptable point

4. Bound and unacceptable point

All four types of points are shown in Fig. 1.4.

OBJECTIVE FUNCTION

- In general, there will be more than one acceptable design, and the purpose of optimization is to choose the best one of the many acceptable designs available.
- Thus a criterion has to be chosen for comparing the different alternative acceptable designs and for selecting the best one. The criterion with respect to which the design is optimized, when expressed as a function of the design variables, is known as the *criterion* or *merit* or *objective function*.
- The choice of objective function is governed by the nature of problem.
- The objective function for minimization is generally taken as weight in aircraft and aerospace structural design problems. In civil engineering structural designs, the objective is usually taken as the minimization of cost.
- In some situations, there may be more than one criterion to be satisfied simultaneously.
- For example, a gear pair may have to be designed for minimum weight and maximum efficiency while transmitting a specified horsepower.
- An optimization problem involving multiple objective functions is known as a *multiobjective programming problem*.

PROBLEM 1

1. Reddy Mikks produces both interior and exterior paints from two raw materials M1 and M2. Following table provides the basic data of problem:
2. A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton.
3. Also, the maximum daily demand for interior paint is 2 tons.
4. Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit?
5. Find the optimal solution to this problem

Tons of raw material per ton			
	Exterior paint	Interior paint	Max daily availability
Raw material, M1	6	4	24
Raw material, M2	1	2	6
profit per ton ($1000)	5	4	

SOLUTION

- The LP model, has three basic components:

1. Decision variables that we seek to determine.

2. Objective that we need to optimize (maximize or minimize).

3. Constraints that solution must satisfy.

The **decision variables** of model are defined as

$x1$= tons produced daily of exterior paint

$x2$= tons produced daily of interior paint

Objective function : The company wants to maximize (i.e., increase as much as possible) the total daily profit of both paints.

Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

Total profit from exterior paint = $5x_1$

Total profit from interior paint = $4x_2$

Let Z denote the total daily profit, then objective of the company is

Maximize $Z = 5x_1 + 4x_2$

CONSTRAINTS : To construct the constraints that restrict raw material usage and product demand.

The raw material restrictions are expressed verbally as

(usage of a raw material by both paints) ≤ (max. raw material availability)

Daily usage of raw material M1 is 6 tons of exterior paint and 4 tons of interior paint.

So, Usage of raw material M1 by exterior paint = $6x_1$ tons/day ,

Usage of raw material M1 by interior paint = $4x_2$ tons/day

Therefore, Usage of M1 by both paints = $6x_1 + 4x_2$ tons/day

By similar logic,

Usage of M2 by both paints = $1x_1 + 2x_2$ tons/day

Because the daily availabilities of raw materials M1 and M2 are limited to 24 and 6 tons respectively, the associated restrictions are given as

$$\mathbf{6x_1 + 4x_2 \le 24} \quad \textbf{(Raw material M1)}$$

$$\mathbf{x_1 + 2x_2 \le 6} \quad \textbf{(Raw material M2)}$$

Demand restriction stipulates that the excess of daily production of interior over exterior paint, x_2-x_1, should not exceed 1 ton, which translates to $\mathbf{x_2 - x_1 \le 1}$ **(market limit)**

Second demand restriction- the max. daily demand of interior paint is limited to 2 tons, which translates to $\mathbf{x_2 \le 2}$ **(demand limit)**.

An implicit (or "understood- to –be") restriction is that variables x_1 and x_2 cannot assume negative values. So $\mathbf{x_1 \ge 0}$ **,** $\mathbf{x_2 \ge 0}$ **(nonnegativity restrictions)**

The complete Reddy Mikks Model is

Maximize $\mathbf{Z = 5x_1 + 4x_2}$

Subject to:

$$6x_1 + 4x_2 \le 24 \qquad (1)$$

$$x_1 + 2x_2 \le 6 \qquad (2)$$

$$x_2 - x_1 \le 1 \qquad (3)$$

$$x_2 \le 2 \qquad (4)$$

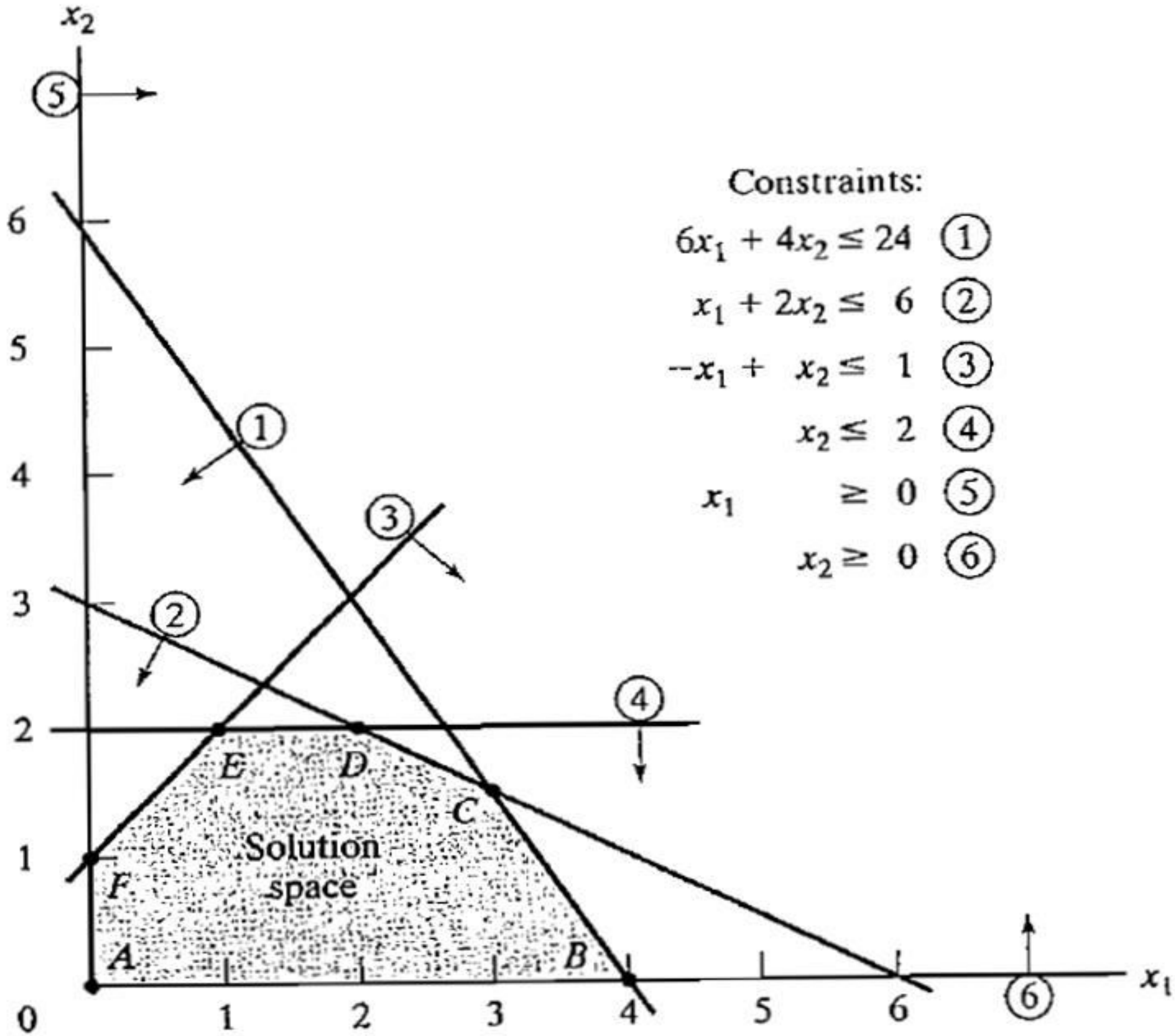
$$x_1, x_2 \ge 0 \qquad (5)$$

Any values of x_1 and x_2 that satisfy all five constraints constitute a feasible solution. Otherwise, the solution is infeasible.

For example, the solution, $x_1 = 3$ tons per day and $x_2 = 1$ ton per day, is feasible because it does not violate any of the constraints.

The goal of the problem is to find the best feasible solution, or the optimum, that maximizes the total profit. Before we can do that, we need to know how many feasible solutions the Reddy Mikks problem has. The answer, as we will see from the the graphical solution is an infinite number, which makes it impossible the solve the problem by enumeration.

GRAPHICAL SOLUTION OF LPP



The graphical procedure includes 2 steps:

1. Determination of the feasible solution space.

2. Determination of the optimum solution from among all the feasible points in the solution space

Determination of Optimum Solution

- The **feasible space** in figure is delineated by the line segments joining the points A, B, C, D, E, and F. **Any point within or on the boundary of the space ABCDEF is feasible.** Because the feasible space ABCDEF consists of an infinite number of points, we need a systematic procedure to identify the optimum solution.
- **An important characteristic of the optimum LP solution is that it is always associated with a corner point of the solution space** (where two lines intersect). This is true even if the objective function happens to be parallel to a constraint. For example, if the objective function is $z = 6x_1 + 4x_2$, which is parallel to constraint 1, we can always say that the optimum occurs at either corner point B or comer point C.
- The observation that the LP optimum is always associated with a corner point means that **the optimum solution can be found simply by enumerating all the corner points** as the following table shows:

Determination of Optimum Solution(2)

- As the number of constraints and variables increases, the number of corner points also increases.
- Nevertheless, the idea shows that, from the standpoint of determining the LP optimum, the solution space ABCDEF with its infinite number of solutions can, in fact, be replaced with a finite number of promising solution points-namely, the corner points, A, B, C, D, E, and F.
- The optimum solution is $x_1 = 3$ and $x_2 = 1.5$ with $Z = (5 * 3) + (4 * 1.5) = 21$. **This calls for a daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint. The associated daily profit is $21,000.**

Corner point	(x_1, x_2)	z
A	(0, 0)	0
B	(4, 0)	20
C	**(3, 1.5)**	**21 (OPTIMUM)**
D	(2, 2)	18
E	(1, 2)	13
F	(0, 1)	4

PRACTICE PROBLEM 1

A company produces two products, A and B. The sales volume for A is at least 80% of the total sales of both A and B.

However, the company cannot sell more than 100 units of A per day. Both products use one raw material, of which the maximum daily availability is 240 lb. The usage rates of the raw material are 2 lb per unit of A, and 4 lb per unit of B.

The profit units for A and B are $20 and $50, respectively.

Formulate the LPP for the same.

SOLUTION TO PRACTICE PROBLEM

If we let A = units of product A and B =units of product B, then we'll

$$\text{maximize } z = 20A + 50B$$

subject to

$$\begin{aligned} 2A + 4B &\le 240 && \text{(raw material availability)} \\ A &\le 100 && \text{(sales limit of A)} \\ -0.2A + 0.8B &\le 0 && \text{(sales of A at least 80\%)} \\ A, B &\ge 0 && \text{(sign restrictions)} \end{aligned}$$

The sales volume for A is at least 80% of the total sales of both A and B. So, $A \geq 0.8\,(A + B)$ which gives us $0 \geq -0.2\,A + 0.8\,B$

PRACTICE PROBLEM 2

A company produces two types of items P and Q that require gold and silver.

Each unit of type P requires 4g silver and 1g gold while that of type Q requires 1g silver and 3g gold. The company can produce 8g silver and 9g gold.

If each unit of type P brings a profit of `44 and that of type Q `55, determine the number of units of each type that the company should produce to maximise the profit. What is the maximum profit?

Degeneracy in Linear Programming

- Degeneracy in LPP occurs when a basic feasible solution has one or more basic variables equal to zero, which can lead to the Simplex method cycling or failing to reach an optimal solution in a finite number of steps.
- It manifests when there is a tie in the minimum ratio test while selecting an outgoing variable in the Simplex table (**Ties in Replacement Ratios)**. Also degeneracy can arise if a**t least one of the constraints has a zero value on the right-hand side (Zero-Valued Basic Variables).** Degeneracy can also arise from **redundant constraints** that overly restrict the solution space.
- In the graphical method for LPPs, degeneracy occurs when **a basic feasible solution has at least one basic variable equal to zero,** , or when **multiple constraints intersect at the same corner point**, resulting in a degenerate corner point. Graphically, it means that more than one constraint line passes through a single corner of the feasible region.
- **Degeneracy can lead to cycling or difficulty in identifying unique optimal solutions.**

Consequences of Degeneracy

1. The primary consequences of degeneracy in LPP are the potential for the simplex algorithm to experience **cycling** (repeatedly visiting the same set of basic feasible solutions without improving the objective function) or **stalling** (failing to make progress toward the optimal solution).

2. **Increased iterations**: Even if it doesn't lead to cycling, degeneracy can significantly increase the number of iterations required for the simplex algorithm to converge to the optimal solution.

3. Degeneracy typically occurs when a pivot operation results in **no improvement to the objective function value**, often due to a tie in determining the outgoing variable in a simplex tableau, making it difficult (potentially requiring more iterations) to reach the optimal solution.

- Degeneracy does not affect the **existence of an optimal solution**.
- **Feasibility:** Degeneracy does not make a basic feasible solution infeasible.

Duality in LPP

- In LPPs, **duality is the concept that every LPP, called the Primal, has an associated LPP called the Dual**, derived from the same data and sharing the same solution. **Dual LP problem** provides useful economic information about worth of resources to be used.

Relationship between Primal and Dual :

- **Variables and Constraints:** The variables in the primal problem become the constraints in the dual problem, and vice-versa.
- **Objective Functions:** The objective function coefficients of primal become the RHS constants of the dual's constraints, and the RHS constants of the primal constraints become the objective function coefficients of the dual.
- **Optimization Direction:** If the primal is a maximization problem, its dual will be a minimization problem, and vice versa.

Duality Theorem: This theorem states that if the primal problem has an optimal solution, the dual problem also has an optimal solution, and the optimal values of their objective functions are equal.

Observations in Duality

- The number of constraints in the primal problem is equal to the number of variables in the dual problem.
- Similarly, the number of variables in the primal problem corresponds to the number of constraints in the dual problem.
- When primal is in maximization form, the dual is in minimization form.
- The coefficients in the objective function of the primal problem become the right-hand side(RHS) of the constraints in the dual problem.
- The right-hand side of the primal problem becomes the coefficients in the objective function of the dual problem.
- The coefficients of the variables in the constraints of the primal problem are transposed to form the coefficients of the variables in the constraints of the dual problem.

Primal-Dual Relationship

If Primal	*Then Dual*
(i) Objective is to maximize	(i) Objective is to minimize
(ii) jth primal variable, x_j	(ii) jth dual constraint
(iii) ith primal constraint	(iii) ith dual variable, y_i
(iv) Primal variable x_j unrestricted in sign	(iv) Dual constraint j is = type
(v) Primal constraint i is = type	(v) Dual variable y_i is unrestricted in sign
(vi) Primal constraints $\leq$ type	(vi) Dual constraints $\geq$ type

Primal Problem (LPP)

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 4x_2 \\ \text{subject to} \quad & \frac{1}{2}x_1 + 2x_2 \leq 30 \\ & 3x_1 + x_2 \leq 25 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Dual LPP

$$\begin{aligned} \text{Minimize} \quad & Z = 30y_1 + 25y_2 \\ \text{subject to} \quad & \frac{1}{2}y_1 + 3y_2 \geq 3 \\ & 2y_1 + y_2 \geq 4 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Primal-Dual Relationship

Normal Primal Problem

$$\begin{aligned} \text{Maximize} \quad & Z = \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Normal Dual Problem

$$\begin{aligned} \text{Minimize} \quad & W = \mathbf{b}^\top \mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^\top \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned}$$

$$
\begin{aligned}
\text{Minimize} \quad & Z = 2x_1 + 3x_2 + 4x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + x_3 \geq 5 \\
& 3x_1 + x_2 + 2x_3 \geq 8 \\
& -3x_1 - x_2 - 2x_3 \geq -8 \\
& -x_1 - 4x_2 - 3x_3 \geq -10 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Minimize} \quad & Z = 2x_1 + 3x_2 + 4x_3 \\
\text{subject to} \quad & x_1y_1 + 2x_2y_1 + x_3y_1 \geq 5y_1 \\
& 3x_1y_2 + x_2y_2 + 2x_3y_2 \geq 8y_2 \\
& -3x_1y_3 - x_2y_3 - 2x_3y_3 \geq -8y_3 \\
& -x_1y_4 - 4x_2y_4 - 3x_3y_4 \geq -10y_4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Maximize} \quad & \boxed{Z = 5y_1 + 8y_2 - 8y_3 - 10y_4} \\
\text{subject to} \quad & \boxed{y_1 + 3y_2 - 3y_3 - y_4} \leq 2 \\
& \boxed{2y_1 + y_2 - y_3 - 4y_4} \leq 3 \\
& \boxed{y_1 + 2y_2 - 2y_3 - 3y_4} \leq 4 \\
& y_1, y_2, y_3, y_4 \geq 0
\end{aligned}
$$

Primal Problem (or Dual Problem)	**Dual Problem (or Primal Problem)**
Maximize Z (or W)	Minimize W (or Z)
Constraint i:	Variable y_i (or x_i):
$\leq$ form $\longleftrightarrow$	$y_i \geq 0$
$=$ form $\longleftrightarrow$	Unconstrained
$\geq$ form $\longleftrightarrow$	$y_i' \leq 0$
Variable x_j (or y_j):	Constraint j:
$x_j \geq 0$ $\longleftrightarrow$	$\geq$ form
Unconstrained $\longleftrightarrow$	$=$ form
$x_j' \leq 0$ $\longleftrightarrow$	$\leq$ form

Obtain the dual of the following LPP:

Maximize $Z_x = x_1 - 2x_2 + 3x_3$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 = 2$, (ii) $2x_1 + 3x_2 + 4x_3 = 1$

and $x_1, x_2, x_3 \geq 0$

Solution Since both the primal constraints are of the equality type, the corresponding dual variables y_1 and y_2, will be unrestricted in sign. Following the rules of duality formulation, the dual of the given primal LP problem is

Minimize $Z_y = 2y_1 + y_2$

subject to the constraints

(i) $-2y_1 + 2y_2 \geq 1$, (ii) $y_1 + 3y_2 \geq -2$, (iii) $3y_1 + 4y_2 \geq 3$

and y_1, y_2 unrestricted in sign.

Benefits of Duality in LPP

- Study of duality helps to identify only an increase (or decrease) in the value of objective function due to per unit variation in the amount of resources available.

1. **Alternative Formulations:** Provides another way to view and solve the same problem.
2. **Solution Bounds:** Helps in establishing upper or lower bounds for the optimal solution of the primal problem.
3. **Sensitivity Analysis:** Facilitates the calculation of shadow prices, which indicate the value of additional units of a resource.
4. **Feasibility and Optimality:** Helps in evaluating whether a solution is feasible or optimal.

Fundamental Theorem of Linear Programming

If a linear programming problem (LPP) has an optimal solution, then at least one optimal solution occurs at a **corner point (vertex)** of the feasible region.

- **Implications:**
  - Search for optimal solutions can be restricted to corner points of the feasible region.
  - There may be:
    - **Unique solution** at one vertex.
    - **Multiple optimal solutions** if the objective function is parallel to a constraint.
    - **Unbounded solution** if feasible region is open in the direction of optimization.
    - **Infeasible problem** if feasible region is empty.

Degenerate Solutions in LPP

A solution is **degenerate** if one or more basic variables take the value zero at a basic feasible solution (BFS).

- **Causes:**
  - Redundant constraints.
  - Intersection of more than ‘m’ constraints at a BFS (where *m = number of constraints*).
- **Implications:**
  - May lead to **stalling** in the simplex method.
  - Could cause **cycling** (repetition of same BFS).

Simplex-Based Methods

- **Purpose:** Solve LPPs by moving from one BFS to another, improving the objective function until optimality.
- **Key Components:**
  - **Initial Basic Feasible Solution (IBFS):** Obtained using slack/surplus/artificial variables.
  - **Pivot Operations:** Exchange of basic and non-basic variables.
  - **Optimality Test:** When all reduced costs are ≥ 0 (for maximization).
  - **Unboundedness Check:** If entering variable has no positive ratio for leaving variable test.

Cycling in Simplex

- **Problem:** Simplex method may revisit the same set of BFS repeatedly due to degeneracy.
- **Result:** Infinite loop, no progress toward optimality.
- **Prevention Techniques:**
  - **Bland’s Rule:** Always choose entering and leaving variables with smallest index.
  - **Perturbation Technique:** Slightly adjust constraints to remove degeneracy.

SENSITIVITY ANALYSIS in LPP

- The process of modifying an OR model to observe the effect upon its outputs is called **Sensitivity Analysis**. Purpose is to evaluate the effect on the optimal solution of an LP problem due to variations in the input coefficients (also called parameters), one at a time.
- In an LP model, the coefficients (also known as parameters) such as: (i) profit (cost) contribution (c_j) per unit of a decision variable, x_j (ii) availability of a resources (b_i), and (iii) consumption of resource per unit of decision variables (a_{ij}), are assumed to be constant and known with certainty.
- *Sensitivity analysis determines the sensitivity range (both lower and upper limit) within which the LP model parameters can vary (one at a time) without affecting the optimality of the current optional solution.*
- This analysis reveals the magnitude of change in the optimal solution of an LP model due to discrete variations (changes) in its parameters. The possible change in the parameter values, can range from zero to a substantial change.
- Thus, aim of sensitivity analysis is to determine the range within which the LP model parameters can change without affecting the current optimal solution.

SENSITIVITY ANALYSIS (2)

- The sensitivity analysis is also referred to as *post-optimality analysis* because it does not begin until the optimal solution to the given LP model has been obtained.
- Different parametric changes in an LPP are:

1. Profit (or cost) per unit (c_j) associated with both basic and non-basic decision variables (i.e., coefficients in the objective function).
2. Availability of resources (i.e., right-hand side constants, b_i in constraints).
3. Consumption of resources per unit of decision variables x_j (i.e., coefficients of decision variables in the constraints, a_{ij}).
4. Addition of a new variable to the existing list of variables in LP problem.
5. Addition of a new constraint to the existing list of constraints in the LP problem.

- *So, what happens to the optimal solution value when we have a* **change in the Objective Function Coefficient** (c_j)? Analysing this is termed as **SENSITIVITY ANALYSIS**.

Line Search Methods in Optimization

Definition: Techniques to determine optimal step size α_k along a search direction.

- Basic Idea:
  - Start at x_k.
  - Choose search direction d_k.
  - Update:

$$x_{k+1} = x_k + \alpha_k d_k$$

where α_k is step size.

Stationarity of Limit Points in Steepest Descent

- If:
    1. The objective function $f(x)$ is continuously differentiable.
    2. Step sizes α_k are chosen by exact or inexact line search (satisfying descent conditions).

Then:

- Any **accumulation point** of the sequence $\{x_k\}$ generated by steepest descent is a **stationary point** (i.e., satisfies $\nabla f(x^*) = 0$).
- In practice: steepest descent may take many iterations to reach acceptable accuracy.

Backtracking is simple and widely used.

Exact line search (finding optimal α analytically) is rare in practice because it may require solving another optimization problem.

Line Search Method Concept

- Need to find good step size α.
- Steepest descent converges to stationary points.
- Backtracking reduces α until sufficient decrease.

Problem 1: Exact Line Search

Minimize $f(x) = (x-2)^2$ using steepest descent, start $x_0 = 0$.

Solution:

- Gradient: $2(x-2)$. Direction: $d = -\nabla f(x_0) = +4$.
- Line search: $f(0+\alpha\cdot 4) = (4\alpha-2)^2$.
- Derivative wrt α: $2(4\alpha-2)(4) = 0 \Rightarrow \alpha = 0.5$.
- Update: $x_1 = 0 + 0.5\cdot 4 = 2$.
- Reached optimum in 1 step.

$$x_{k+1} = x_k + \alpha_k d_k$$

Successive Step-Size Reduction Algorithms

- Instead of fixing step size, start large and **reduce until progress is adequate.**
- Common rules:

(a) Backtracking Line Search

1. Choose initial $\alpha = 1$.
2. While:

$$f(x_k + \alpha d_k) > f(x_k) + c\alpha \nabla f(x_k)^T d_k$$

(Armijo condition not satisfied), reduce α (e.g., $\alpha \leftarrow \beta\alpha$, with $\beta \in (0, 1)$).

3. Accept the reduced α.

- Guarantees sufficient decrease in each step.

❑ **Backtracking is simple and widely used method in Line Search Optimization problems.**

❑ **Exact line search (finding optimal α analytically) is rare in practice because it may require solving another optimization problem.**

❑ **Another method is Wolfe condition**

(a) Backtracking Line Search

1. Choose initial $\alpha = 1$.
2. While:

$$f(x_k + \alpha d_k) > f(x_k) + c\alpha \nabla f(x_k)^T d_k$$

(Armijo condition not satisfied), reduce α (e.g., $\alpha \leftarrow \beta\alpha$, with $\beta \in (0,1)$).

Problem 2: Backtracking Line Search

Minimize

$$f(x) = x^2$$

starting from $x_0 = 1$, direction $d = -\nabla f(x_0) = -2$, initial $\alpha = 1$, $\beta = 0.5$.

- Gradient at $x_0 = 1$: 2. Direction = -2.
- Trial step: $x = 1 - 2(1) = -1$. $f(-1) = 1$.
- Armijo: $f(-1) \leq f(1) + c\alpha\nabla f(1)d$. (With $c = 0.5$).
- RHS = $1 + 0.5(1)(2)(-2) = -1$. LHS = 1 > -1 → not satisfied.
- Reduce step: $\alpha = 0.5$.
- New point: $x = 0$. $f(0) = 0$. Condition satisfied.
- Accept $\alpha = 0.5$.

$$x_{k+1} = x_k + \alpha_k d_k$$

Problem: Minimize $f(x) = x^2$ starting from $x_0 = 1$, direction $d = -\nabla f(x_0) = -2$. Initial step $\alpha_0 = 1$, shrink factor $\beta = 0.5$. Use the Armijo (sufficient decrease) condition in the backtracking loop.

1. compute gradient and direction

$$f(x) = x^2, \qquad \nabla f(x) = 2x.$$

At $x_0 = 1$:

$$\nabla f(1) = 2, \qquad d = -\nabla f(1) = -2.$$

The trial point for step α is

$$x(\alpha) = x_0 + \alpha d = 1 + \alpha(-2) = 1 - 2\alpha.$$

The value along the line is

$$\phi(\alpha) \equiv f(x(\alpha)) = (1 - 2\alpha)^2.$$

2. Armijo (sufficient decrease) condition

Armijo condition requires

$$\phi(\alpha) \le \phi(0) + c\,\alpha\,\nabla f(x_0)^\top d,$$

where $\phi(0) = f(x_0) = 1$ and $\nabla f(x_0)^\top d = (2) \cdot (-2) = -4$. So the right-hand side is

$$\phi(0) + c\alpha \nabla f(x_0)^\top d = 1 + c\alpha(-4) = 1 - 4c\alpha.$$

Thus Armijo inequality becomes

$$(1 - 2\alpha)^2 \le 1 - 4c\alpha.$$

3. Test $\alpha = \alpha_0 = 1$

Left-hand side:

$$\phi(1) = (1 - 2 \cdot 1)^2 = (-1)^2 = 1.$$

Right-hand side:

$$1 - 4c \cdot 1 = 1 - 4c.$$

Inequality is $1 \le 1 - 4c$, i.e. $0 \le -4c$. That is **false** for any $c > 0$.

So $\alpha = 1$ **fails** Armijo and we reduce α.

4. Reduce α: $\alpha \leftarrow \beta\alpha = 0.5$

Now $\alpha = 0.5$.

Left-hand side:

$$\phi(0.5) = (1 - 2 \cdot 0.5)^2 = (1 - 1)^2 = 0.$$

Right-hand side:

$$1 - 4c \cdot 0.5 = 1 - 2c.$$

Inequality is $0 \leq 1 - 2c$, i.e. $c \leq 0.5$. For any typical Armijo constant c (e.g. $c = 10^{-4}$ up to $c = 0.1$), this holds. So $\alpha = 0.5$ **satisfies** Armijo and is accepted.

5. Numerical check with $c = 0.1$ (concrete)

- For $\alpha = 1$: LHS $= 1$, RHS $= 1 - 4(0.1) = 0.6$. $1 \leq 0.6$ → fails.
- For $\alpha = 0.5$: LHS $= 0$, RHS $= 1 - 2(0.1) = 0.8$. $0 \leq 0.8$ → holds.

So backtracking accepts $\alpha = 0.5$.

---

6. Update iterate

$$x_1 = x_0 + \alpha d = 1 + 0.5 \cdot (-2) = 0.$$

Evaluate objective: $f(x_1) = 0$, which is the global minimum for $f(x) = x^2$. (In fact, an exact line search on this quadratic also gives $\alpha = 0.5$.)

final answer

- First trial $\alpha = 1$ fails Armijo.
- After one shrink ($\beta = 0.5$) we get $\alpha = 0.5$ which satisfies Armijo (for any common choice of $c \leq 0.5$, in particular $c = 0.1$).
- Accepted step $\alpha = 0.5$ → new iterate $x_1 = 0$.
- The method reached the exact minimizer in one accepted backtracking step.

Problem 3: Stationarity of Limit Points

Show that steepest descent for

$$f(x) = (x-1)^2$$

leads to a stationary point.

Solution:

- Gradient: $2(x-1)$.
- Update: $x_{k+1} = x_k - \alpha(2(x_k - 1))$.
- As $k \to \infty$, $x_k \to 1$.
- At $x = 1$, gradient = 0 → stationary point.

Problem 3: Backtracking Line Search

$f(x) = x^2$, start at $x_0 = 1$, direction $d = -\nabla f(1) = -2$.
Parameters: $c = 0.1, \beta = 0.5$.

We try $\alpha = 1$:
Check Armijo: $f(1-2) \leq f(1) + c\alpha\nabla f(1)d$.
LHS $= f(-1) = 1$. RHS $= 1 + 0.1(1)(-2)(-2) = 1 + 0.4 = 1.4$.
Condition holds → accept $\alpha = 1$.

Update: $x_1 = -1$.

$$x_{k+1} = x_k + \alpha_k d_k$$

OPTIMIZATION TECHNIQUES for DECISION MAKING

UNIT-1

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Need of Optimization Techniques

- The ever-increasing demand on engineers to lower production costs to withstand global competition has prompted engineers to look for rigorous methods of decision making, such as optimization methods, to design and produce products and systems both economically and efficiently.
- Optimization techniques, having reached a degree of maturity in recent years, are being used in a wide spectrum of industries, including aerospace, automotive, chemical, electrical, construction, and manufacturing industries.
- With rapidly advancing computer technology, computers are becoming more powerful, and correspondingly, the size and the complexity of the problems that can be solved using optimization techniques are also increasing.

INTRODUCTION

- **Optimization is the act of obtaining the best result under given circumstances**. In design, construction, and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages.
- **The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit**. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, ***optimization*** can be defined as the process of finding the conditions that give the maximum or minimum value of a function.
- There is **no single method available for solving all optimization problems** efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems. The optimum seeking methods are also known as ***mathematical programming techniques*** and are generally studied as a part of **Operations Research**.

SCOPE OF OT / OR

O.R. has a wide scope in everyday life as it provides better solutions to various decision-making problems with great speed and competence. It finds applications in a wide range of areas including defence operations, planning, agriculture, industry (finance, marketing, personal management, production management), research and development. We now describe the applications briefly.

Areas where Optimization is applied are:

1. **Science**
2. **Engineering**
3. **Management**
4. **Finance**
5. **Business**

In Planning for Economic Development

Careful planning is necessary for economic development of any country. Operations Research is used to frame future economic and social policies.

In Agriculture

Agricultural output needs to be increased due to increasing needs for adequate quantity and quality of food for our increasing population. But there are a number of restrictions under which agricultural production is studied. Problems of agricultural production under various restrictions such as optimum allocation of land to various crops in accordance with the climatic conditions, optimum distribution of water from various resources for irrigation purposes can easily be solved by application of Operations Research techniques.

In Industry

Now-a-days, due to complexities of operations and huge sizes of industries, important decisions regarding various sections of the organisation, e.g., planning, procurement, marketing, finance, etc. have to be taken division wise. For example, the production department needs to minimise the cost of production, but maximise output; the finance department needs to optimise capital investment; the personnel department needs to appoint competent work force at minimum cost. Each department has to plan its own objectives which may be in conflict with the objectives of other departments and may not conform to the overall objectives of the organisation. For example, the sales department of an organisation may want to keep sufficient stocks in the inventory, whereas the finance department may want to have minimum investment. In that case, both departments would be in conflict with each other. The applications of O.R. techniques to such situations help in overcoming this difficulty by evolving an optimal strategy and serving efficiently the interest of the organisation as a whole.

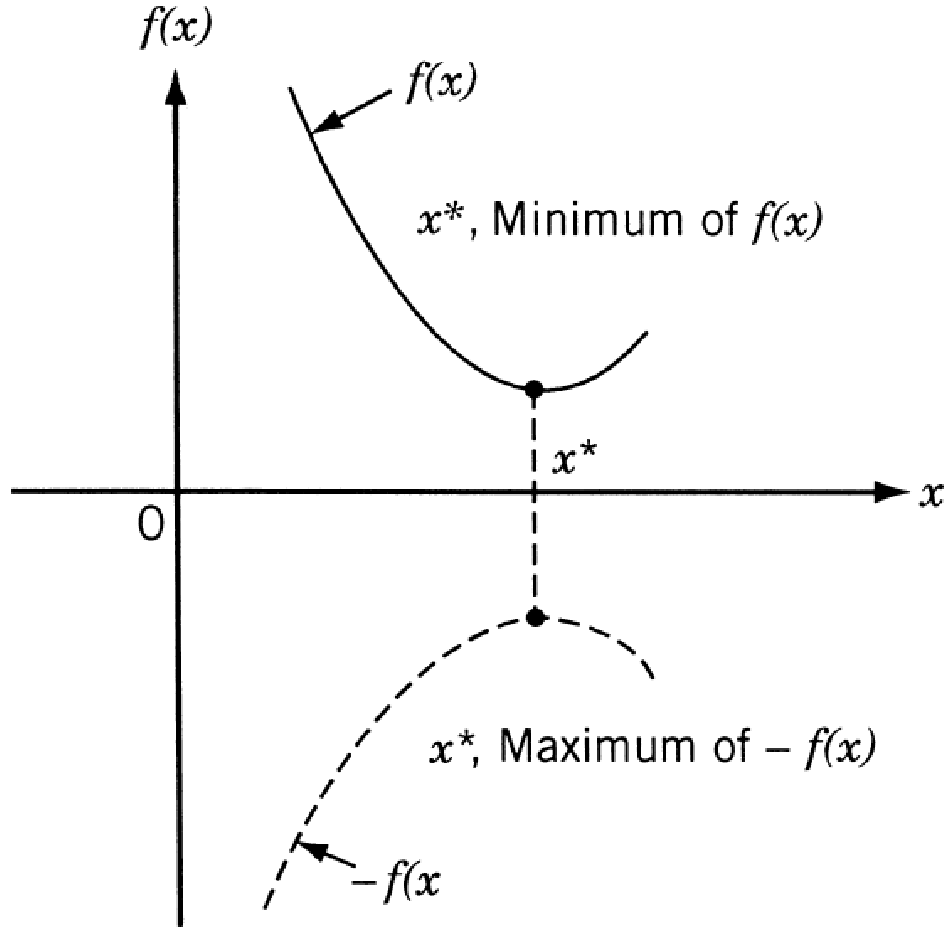

Figure 1.1 Minimum of $f(x)$ is same as maximum of $-f(x)$.

ENGINEERING APPLICATIONS OF OPTIMIZATION

Some typical applications from different engineering disciplines include :

1. Design of aircraft and aerospace structures for minimum weight

2. Finding the optimal trajectories of space vehicles

3. Minimum-weight design of structures for earthquake, wind, other types of random loading

4. Selection of machining conditions in metal-cutting for minimum production cost

5. Shortest route taken by a salesperson visiting various cities during one tour

6. Optimal production planning, controlling, and scheduling

7. Design of optimum pipeline networks for process industries

8. Selection of a site for an industry

9. Planning of maintenance and replacement of equipment to reduce operating costs

10. Inventory control management

11. Allocation of resources among several activities to maximize the profit.

12. Controlling the waiting and idle times in production lines to reduce the costs

13. Optimum design of control systems in design of electronic appliances.

GENERAL OPTIMIZATION PROBLEM

Minimize (Maximize) $f(X)$ → objective function

where $f: R^n \rightarrow R$, $X = (x_1, x_2, x_3 \ldots x_n)$ → decision values parameters

s.t. $X \in S \subseteq R^n$ where S is defined by

$g_k(X) \geq 0$, $k=1,2, \ldots m$ → inequality constraints

$h_j(X) = 0$, $j=1,2, \ldots l$ → equality constraints

$a_i \leq x_i \leq b_i$ → lower & upper bounds

COMPONENTS OF AN OPTIMIZATION MODEL

Decision variables

Objective function

Constraints

CLASSIFICATION

Linear Programming Problems (LPP)

Nonlinear Programming Problems (NLPP)

Unconstrained Optimization Problems

Constrained Optimization Problems

LINEAR PROGRAMMING PROBLEM

A **Linear Programming Problem** is an optimization problem where the **objective function** and all the **constraints** are linear in nature.

Objective function → A linear function of decision variables to be **maximized or minimized** (e.g., profit, cost, revenue).

Constraints → Linear equations or inequalities representing resource limitations.

Decision variables → Unknowns to be determined (e.g., number of products, allocation of resources).

The General form of LPP is as follows:

Maximize or Minimize:

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

...

$$x_i \geq 0 \quad (i = 1, 2, ..., n)$$

Where:

- Z = Objective function
- $x_1, x_2, ..., x_n$ = Decision variables
- c_i = Coefficients of objective function
- a_{ij} = Coefficients of constraints
- b_j = Resource availability

Problem:

A company produces two products A and B.
Profit from A = ₹3 per unit, profit from B = ₹2 per unit.

Constraints:

- Each unit of A requires 1 hour on machine 1 and 2 hours on machine 2.
- Each unit of B requires 1 hour on machine 1 and 1 hour on machine 2.
- Machine 1 is available for 8 hours, Machine 2 is available for 10 hours.

Formulation:

$$\text{Maximize } Z = 3x_1 + 2x_2$$

Subject to:

$$x_1 + x_2 \leq 8 \quad (Machine 1)$$

$$2x_1 + x_2 \leq 10 \quad (Machine 2)$$

$$x_1, x_2 \geq 0$$

Solution (Graphical):

- Plot the inequalities, find feasible region.
- Corner points = (0,0), (5,0), (2,6), (0,8).
- Evaluate Z:
  - (0,0) → 0
  - (5,0) → 15
  - (2,6) → 18
  - (0,8) → 16

Optimal Solution: Produce 2 units of A and 6 units of B → **Maximum Profit = ₹18**

NON-LINEAR PROGRAMMING

Non-Linear Programming involves optimization of a **non-linear objective function** and/or **non-linear constraints**.

General Formulation

$$\text{Maximize/Minimize } Z = f(x_1, x_2, \ldots, x_n)$$

Subject to:

$$g_j(x_1, x_2, \ldots, x_n) \leq b_j \quad \text{for } j = 1, 2, \ldots, m$$

$$x_i \geq 0$$

Where:

- $f(x)$ = non-linear objective function
- $g_j(x)$ = non-linear constraint functions

Methods of Solving NLPP

1.Lagrange Multiplier Method (for equality constraints)

2.Kuhn-Tucker (KKT) Conditions (for inequality constraints)

3.Gradient Descent / Newton's Method (iterative numerical methods)

CLASSIFICATION BASED ON TYPE OF DECISION VARIABLES

Dynamic Programming

Geometric Programming

Integer Programming

Quadratic Programming

Separable Programming

STATEMENT OF AN OPTIMIZATION PROBLEM

An optimization or a mathematical programming problem can be stated as follows.

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

subject to the constraints

$$\begin{aligned} g_j(\mathbf{X}) &\le 0, \qquad j = 1, 2, \ldots, m \\ l_j(\mathbf{X}) &= 0, \qquad j = 1, 2, \ldots, p \end{aligned} \tag{1.1}$$

where $\mathbf{X}$ is an n-dimensional vector called the *design vector*, $f(\mathbf{X})$ is termed the *objective function*, and $g_j(\mathbf{X})$ and $l_j(\mathbf{X})$ are known as *inequality* and *equality* constraints, respectively. The number of variables n and the number of constraints m and/or p need not be related in any way. The problem stated in Eq. (1.1) is called a *constrained optimization problem*.† Some optimization problems do not involve any constraints and can be stated as

OPTIMIZATION PROBLEM(2)

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

Such problems are called *unconstrained optimization problems*.

CONSTRAINTS IN OPTIMIZATION

- In many practical problems, the design variables cannot be chosen arbitrarily; rather, they have to satisfy certain specified functional and other requirements.
- The restrictions that must be satisfied to produce an acceptable design are collectively called *design constraints*.
- Constraints that represent limitations on the behavior or performance of the system are termed *behavior* or *functional constraints*.
- Constraints that represent physical limitations on design variables, such as availability, fabricability, and transportability, are known as *geometric* or *side constraints*.

CONSTRAINT SURFACE

For illustration, consider an optimization problem with only inequality constraints $g_j(\mathbf{X}) \le 0$. The set of values of $\mathbf{X}$ that satisfy the equation $g_j(\mathbf{X}) = 0$ forms a hypersurface in the design space and is called a *constraint surface*. Note that this is an $(n-1)$-dimensional subspace, where n is the number of design variables. The constraint surface divides the design space into two regions: one in which $g_j(\mathbf{X}) < 0$ and the other in which $g_j(\mathbf{X}) > 0$. Thus the points lying on the hypersurface will satisfy the constraint $g_j(\mathbf{X})$ critically, whereas the points lying in the region where $g_j(\mathbf{X}) > 0$ are infeasible or unacceptable, and the points lying in the region where $g_j(\mathbf{X}) < 0$ are feasible or acceptable. The collection of all the constraint surfaces $g_j(\mathbf{X}) = 0,\ j = 1, 2, \ldots, m$, which separates the acceptable region is called the *composite constraint surface*.

CONSTRAINT SURFACE(2)

- Figure 1.4 shows a hypothetical two-dimensional design space where the infeasible region is indicated by hatched lines.
- A design point that lies on one or more than one constraint surface is called a *bound point* , and the associated constraint is called an *active constraint* .
- Design points that do not lie on any constraint surface are known as *free points*.
- Depending on whether a particular design point belongs to the acceptable or unacceptable region, it can be identified as one of the following four types:

1. Free and acceptable point

2. Free and unacceptable point

3. Bound and acceptable point

4. Bound and unacceptable point

All four types of points are shown in Fig. 1.4.

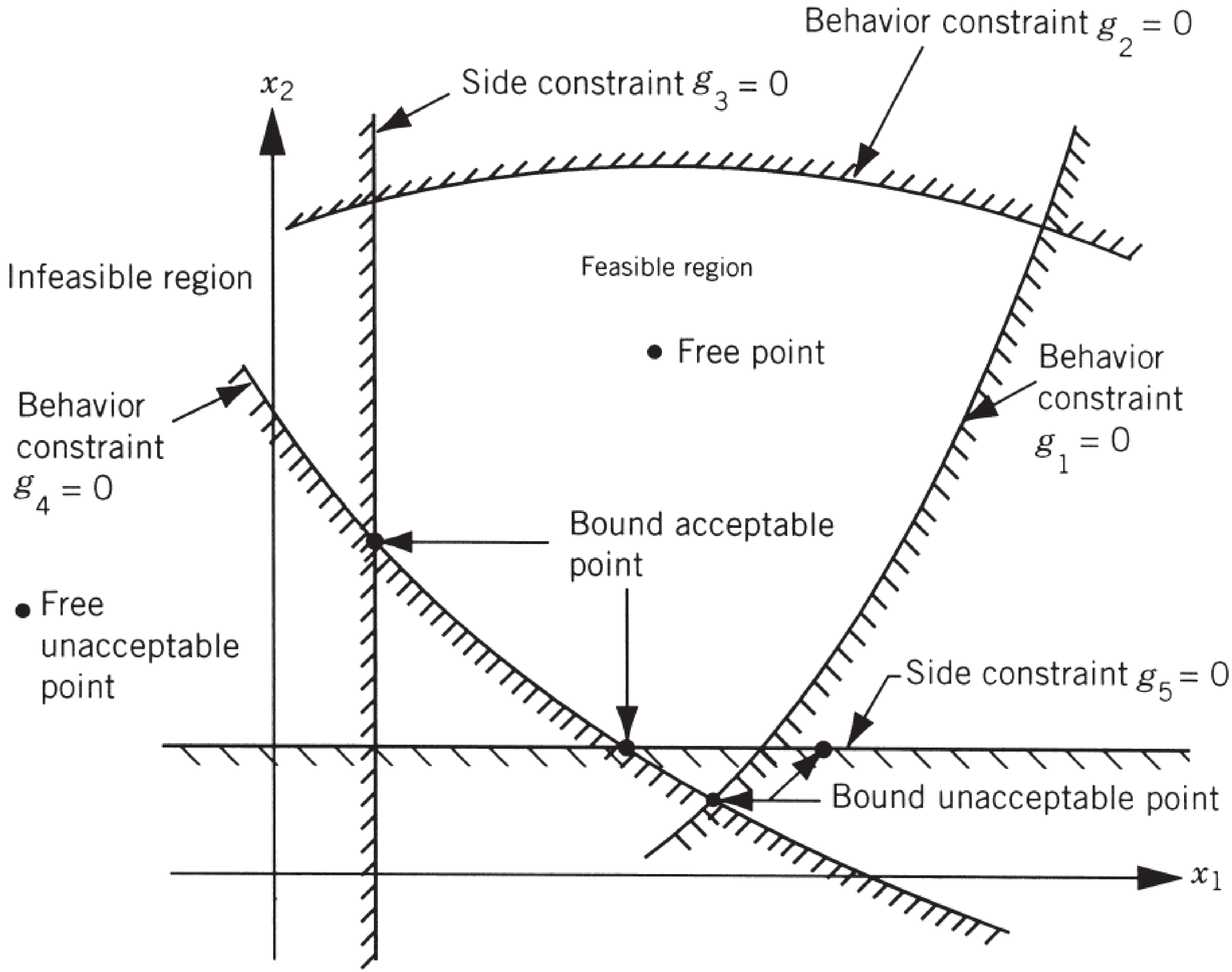

Figure 1.4 Constraint surfaces in a hypothetical two-dimensional design space.

CONSTRAINT SURFACE(2)

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2. Free and unacceptable point

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4. Bound and unacceptable point

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OBJECTIVE FUNCTION

- In general, there will be more than one acceptable design, and the purpose of optimization is to choose the best one of the many acceptable designs available.
- Thus a criterion has to be chosen for comparing the different alternative acceptable designs and for selecting the best one. The criterion with respect to which the design is optimized, when expressed as a function of the design variables, is known as the *criterion* or *merit* or *objective function*.
- The choice of objective function is governed by the nature of problem.
- The objective function for minimization is generally taken as weight in aircraft and aerospace structural design problems. In civil engineering structural designs, the objective is usually taken as the minimization of cost.
- In some situations, there may be more than one criterion to be satisfied simultaneously.
- For example, a gear pair may have to be designed for minimum weight and maximum efficiency while transmitting a specified horsepower.
- An optimization problem involving multiple objective functions is known as a *multiobjective programming problem*.

PROBLEM 1

1. Reddy Mikks produces both interior and exterior paints from two raw materials M1 and M2. Following table provides the basic data of problem:
2. A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton.
3. Also, the maximum daily demand for interior paint is 2 tons.
4. Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit?
5. Find the optimal solution to this problem

Tons of raw material per ton			
	Exterior paint	Interior paint	Max daily availability
Raw material, M1	6	4	24
Raw material, M2	1	2	6
profit per ton ($1000)	5	4	

SOLUTION

- The LP model, has three basic components:

1. Decision variables that we seek to determine.

2. Objective that we need to optimize (maximize or minimize).

3. Constraints that solution must satisfy.

The **decision variables** of model are defined as

$x1$= tons produced daily of exterior paint

$x2$= tons produced daily of interior paint

Objective function : The company wants to maximize (i.e., increase as much as possible) the total daily profit of both paints.

Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

Total profit from exterior paint = $5x_1$

Total profit from interior paint = $4x_2$

Let Z denote the total daily profit, then objective of the company is

Maximize $\mathbf{Z = 5x_1 + 4x_2}$

CONSTRAINTS : To construct the constraints that restrict raw material usage and product demand.

The raw material restrictions are expressed verbally as

(usage of a raw material by both paints) ≤ (max. raw material availability)

Daily usage of raw material M1 is 6 tons of exterior paint and 4 tons of interior paint.

So, Usage of raw material M1 by exterior paint = $6x_1$ tons/day ,

Usage of raw material M1 by interior paint = $4x_2$ tons/day

Therefore, Usage of M1 by both paints = $6x_1 + 4x_2$ tons/day

By similar logic,

Usage of M2 by both paints = $1x_1 + 2x_2$ tons/day

Because the daily availabilities of raw materials M1 and M2 are limited to 24 and 6 tons respectively, the associated restrictions are given as

$$6x_1 + 4x_2 \le 24 \quad \textbf{(Raw material M1)}$$

$$x_1 + 2x_2 \le 6 \quad \textbf{(Raw material M2)}$$

Demand restriction stipulates that the excess of daily production of interior over exterior paint, x_2-x_1, should not exceed 1 ton, which translates to $x_2 - x_1 \le 1$ **(market limit)**

Second demand restriction- the max. daily demand of interior paint is limited to 2 tons, which translates to $x_2 \le 2$ **(demand limit)**.

An implicit (or "understood- to –be") restriction is that variables x_1 and x_2 cannot assume negative values. So $x_1 \ge 0$, $x_2 \ge 0$ **(nonnegativity restrictions)**

The complete Reddy Mikks Model is

$$\textbf{Maximize} \quad \mathbf{Z = 5x_1 + 4x_2}$$

Subject to:

$$6x_1 + 4x_2 \le 24 \qquad (1)$$

$$x_1 + 2x_2 \le 6 \qquad (2)$$

$$x_2 - x_1 \le 1 \qquad (3)$$

$$x_2 \le 2 \qquad (4)$$

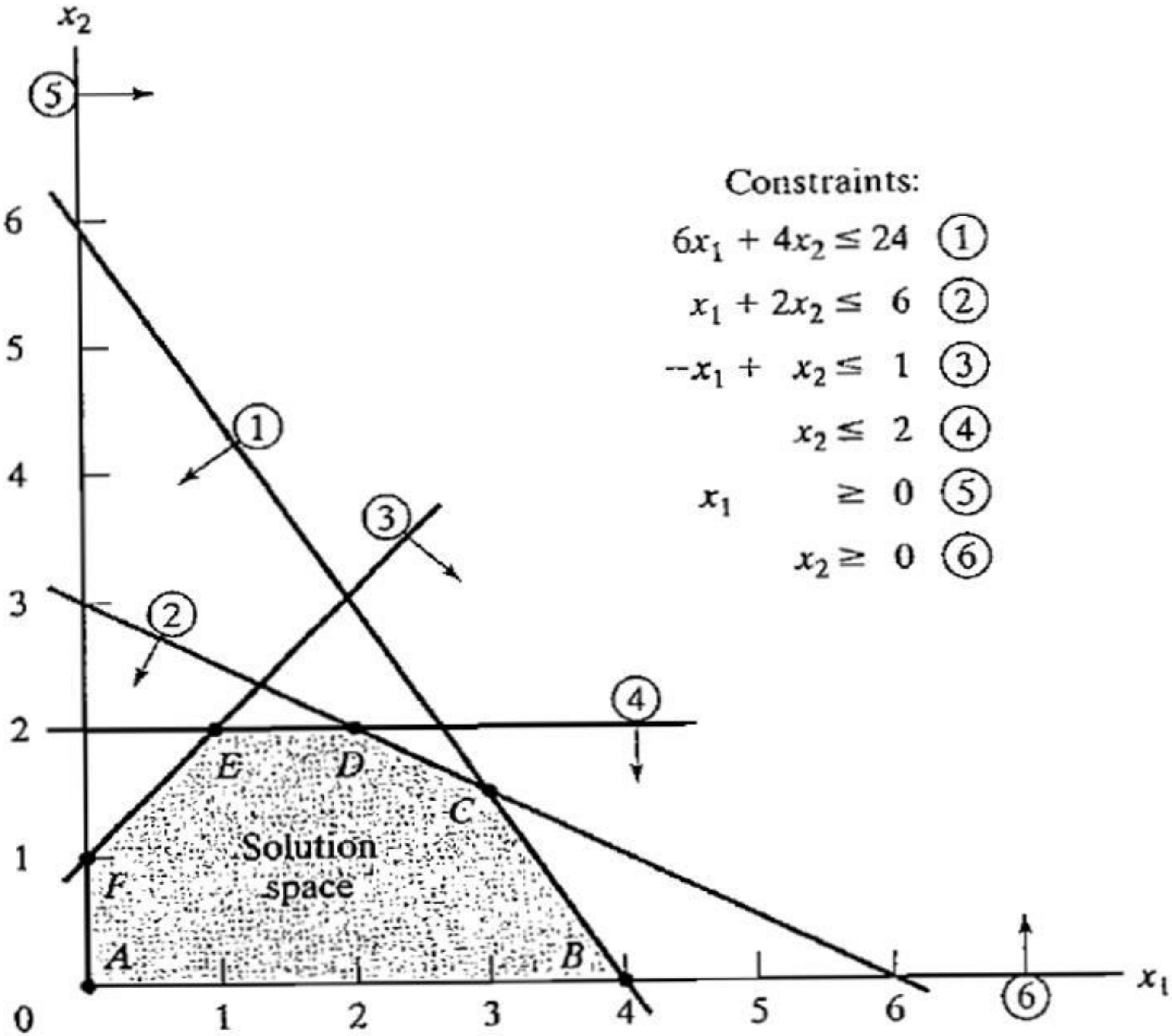
$$x_1, x_2 \ge 0 \qquad (5)$$

Any values of x_1 and x_2 that satisfy all five constraints constitute a feasible solution. Otherwise, the solution is infeasible.

For example, the solution, $x_1 = 3$ tons per day and $x_2 = 1$ ton per day, is feasible because it does not violate any of the constraints.

The goal of the problem is to find the best feasible solution, or the optimum, that maximizes the total profit. Before we can do that, we need to know how many feasible solutions the Reddy Mikks problem has. The answer, as we will see from the the graphical solution is an infinite number, which makes it impossible the solve the problem by enumeration.

GRAPHICAL SOLUTION OF LPP



The graphical procedure includes 2 steps:

1. Determination of the feasible solution space.

2. Determination of the optimum solution from among all the feasible points in the solution space

Determination of Optimum Solution

- The **feasible space** in figure is delineated by the line segments joining the points A, B, C, D, E, and F. **Any point within or on the boundary of the space ABCDEF is feasible.** Because the feasible space ABCDEF consists of an infinite number of points, we need a systematic procedure to identify the optimum solution.
- **An important characteristic of the optimum LP solution is that it is always associated with a corner point of the solution space** (where two lines intersect). This is true even if the objective function happens to be parallel to a constraint. For example, if the objective function is $z = 6x_1 + 4x_2$, which is parallel to constraint 1, we can always say that the optimum occurs at either corner point B or comer point C.
- The observation that the LP optimum is always associated with a corner point means that **the optimum solution can be found simply by enumerating all the corner points** as the following table shows:

Determination of Optimum Solution(2)

- As the number of constraints and variables increases, the number of corner points also increases.
- Nevertheless, the idea shows that, from the standpoint of determining the LP optimum, the solution space ABCDEF with its infinite number of solutions can, in fact, be replaced with a finite number of promising solution points-namely, the corner points, A, B, C, D, E, and F.
- The optimum solution is $x_1 = 3$ and $x_2 = 1.5$ with $Z = (5 * 3) + (4 * 1.5) = 21$. **This calls for a daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint. The associated daily profit is $21,000.**

Corner point	(x_1, x_2)	z
A	(0, 0)	0
B	(4, 0)	20
C	**(3, 1.5)**	**21 (OPTIMUM)**
D	(2, 2)	18
E	(1, 2)	13
F	(0, 1)	4

PRACTICE PROBLEM 1

A company produces two products, A and B. The sales volume for A is at least 80% of the total sales of both A and B.

However, the company cannot sell more than 100 units of A per day. Both products use one raw material, of which the maximum daily availability is 240 lb. The usage rates of the raw material are 2 lb per unit of A, and 4 lb per unit of B.

The profit units for A and B are $20 and $50, respectively.

Formulate the LPP for the same.

SOLUTION TO PRACTICE PROBLEM

If we let A = units of product A and B =units of product B, then we'll

$$\text{maximize } z = 20A + 50B$$

subject to

$$\begin{aligned} 2A + 4B &\leq 240 && \text{(raw material availability)} \\ A &\leq 100 && \text{(sales limit of A)} \\ -0.2A + 0.8B &\leq 0 && \text{(sales of A at least 80\%)} \\ A, B &\geq 0 && \text{(sign restrictions)} \end{aligned}$$

The sales volume for A is at least 80% of the total sales of both A and B. So, $A \geq 0.8\,(A + B)$ which gives us $0 \geq -0.2\,A + 0.8\,B$

PRACTICE PROBLEM 2

A company produces two types of items P and Q that require gold and silver.

Each unit of type P requires 4g silver and 1g gold while that of type Q requires 1g silver and 3g gold. The company can produce 8g silver and 9g gold.

If each unit of type P brings a profit of `44 and that of type Q `55, determine the number of units of each type that the company should produce to maximise the profit. What is the maximum profit?

Degeneracy in Linear Programming

- Degeneracy in LPP occurs when a basic feasible solution has one or more basic variables equal to zero, which can lead to the Simplex method cycling or failing to reach an optimal solution in a finite number of steps.
- It manifests when there is a tie in the minimum ratio test while selecting an outgoing variable in the Simplex table (**Ties in Replacement Ratios)**. Also degeneracy can arise if a**t least one of the constraints has a zero value on the right-hand side (Zero-Valued Basic Variables).** Degeneracy can also arise from **redundant constraints** that overly restrict the solution space.
- In the graphical method for LPPs, degeneracy occurs when **a basic feasible solution has at least one basic variable equal to zero,** , or when **multiple constraints intersect at the same corner point**, resulting in a degenerate corner point. Graphically, it means that more than one constraint line passes through a single corner of the feasible region.
- **Degeneracy can lead to cycling or difficulty in identifying unique optimal solutions.**

Consequences of Degeneracy

1. The primary consequences of degeneracy in LPP are the potential for the simplex algorithm to experience **cycling** (repeatedly visiting the same set of basic feasible solutions without improving the objective function) or **stalling** (failing to make progress toward the optimal solution).

2. **Increased iterations**: Even if it doesn't lead to cycling, degeneracy can significantly increase the number of iterations required for the simplex algorithm to converge to the optimal solution.

3. Degeneracy typically occurs when a pivot operation results in **no improvement to the objective function value**, often due to a tie in determining the outgoing variable in a simplex tableau, making it difficult (potentially requiring more iterations) to reach the optimal solution.

- Degeneracy does not affect the **existence of an optimal solution**.
- **Feasibility:** Degeneracy does not make a basic feasible solution infeasible.

Duality in LPP

- In LPPs, **duality is the concept that every LPP, called the Primal, has an associated LPP called the Dual**, derived from the same data and sharing the same solution. **Dual LP problem** provides useful economic information about worth of resources to be used.

Relationship between Primal and Dual :

- **Variables and Constraints:** The variables in the primal problem become the constraints in the dual problem, and vice-versa.
- **Objective Functions:** The objective function coefficients of primal become the RHS constants of the dual's constraints, and the RHS constants of the primal constraints become the objective function coefficients of the dual.
- **Optimization Direction:** If the primal is a maximization problem, its dual will be a minimization problem, and vice versa.

Duality Theorem: This theorem states that if the primal problem has an optimal solution, the dual problem also has an optimal solution, and the optimal values of their objective functions are equal.

Observations in Duality

- The number of constraints in the primal problem is equal to the number of variables in the dual problem.
- Similarly, the number of variables in the primal problem corresponds to the number of constraints in the dual problem.
- When primal is in maximization form, the dual is in minimization form.
- The coefficients in the objective function of the primal problem become the right-hand side(RHS) of the constraints in the dual problem.
- The right-hand side of the primal problem becomes the coefficients in the objective function of the dual problem.
- The coefficients of the variables in the constraints of the primal problem are transposed to form the coefficients of the variables in the constraints of the dual problem.

Primal-Dual Relationship

	If Primal		*Then Dual*
(i)	Objective is to maximize	(i)	Objective is to minimize
(ii)	jth primal variable, x_j	(ii)	jth dual constraint
(iii)	ith primal constraint	(iii)	ith dual variable, y_i
(iv)	Primal variable x_j unrestricted in sign	(iv)	Dual constraint j is = type
(v)	Primal constraint i is = type	(v)	Dual variable y_i is unrestricted in sign
(vi)	Primal constraints $\leq$ type	(vi)	Dual constraints $\geq$ type

Primal Problem (LPP)

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 4x_2 \\ \text{subject to} \quad & \frac{1}{2}x_1 + 2x_2 \le 30 \\ & 3x_1 + x_2 \le 25 \\ & x_1, x_2 \ge 0. \end{aligned}$$

Dual LPP

$$\begin{aligned} \text{Minimize} \quad & Z = 30y_1 + 25y_2 \\ \text{subject to} \quad & \frac{1}{2}y_1 + 3y_2 \ge 3 \\ & 2y_1 + y_2 \ge 4 \\ & y_1, y_2 \ge 0 \end{aligned}$$

Primal-Dual Relationship

Normal Primal Problem	Normal Dual Problem
Maximize $Z = \mathbf{c}^\top \mathbf{x}$	Minimize $W = \mathbf{b}^\top \mathbf{y}$
subject to $\mathbf{A}\mathbf{x} \le \mathbf{b}$	subject to $\mathbf{A}^\top \mathbf{y} \ge \mathbf{c}$
$\mathbf{x} \ge \mathbf{0}$	$\mathbf{y} \ge \mathbf{0}$

$$
\begin{aligned}
\text{Minimize} \quad & Z = 2x_1 + 3x_2 + 4x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + x_3 \geq 5 \\
& 3x_1 + x_2 + 2x_3 \geq 8 \\
& -3x_1 - x_2 - 2x_3 \geq -8 \\
& -x_1 - 4x_2 - 3x_3 \geq -10 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Minimize} \quad & Z = 2x_1 + 3x_2 + 4x_3 \\
\text{subject to} \quad & x_1y_1 + 2x_2y_1 + x_3y_1 \geq 5y_1 \\
& 3x_1y_2 + x_2y_2 + 2x_3y_2 \geq 8y_2 \\
& -3x_1y_3 - x_2y_3 - 2x_3y_3 \geq -8y_3 \\
& -x_1y_4 - 4x_2y_4 - 3x_3y_4 \geq -10y_4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Maximize} \quad & \boxed{Z = 5y_1 + 8y_2 - 8y_3 - 10y_4} \\
\text{subject to} \quad & \boxed{y_1 + 3y_2 - 3y_3 - y_4} \leq 2 \\
& \boxed{2y_1 + y_2 - y_3 - 4y_4} \leq 3 \\
& \boxed{y_1 + 2y_2 - 2y_3 - 3y_4} \leq 4 \\
& y_1, y_2, y_3, y_4 \geq 0
\end{aligned}
$$

Primal Problem (or Dual Problem)	**Dual Problem (or Primal Problem)**
Maximize Z (or W)	Minimize W (or Z)
Constraint i:	Variable y_i (or x_i):
$\le$ form $\longleftrightarrow$	$y_i \ge 0$
$=$ form $\longleftrightarrow$	Unconstrained
$\ge$ form $\longleftrightarrow$	$y_i' \le 0$
Variable x_j (or y_j):	Constraint j:
$x_j \ge 0$ $\longleftrightarrow$	$\ge$ form
Unconstrained $\longleftrightarrow$	$=$ form
$x_j' \le 0$ $\longleftrightarrow$	$\le$ form

Obtain the dual of the following LPP:

Maximize $Z_x = x_1 - 2x_2 + 3x_3$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 = 2$, (ii) $2x_1 + 3x_2 + 4x_3 = 1$

and $x_1, x_2, x_3 \geq 0$

Solution Since both the primal constraints are of the equality type, the corresponding dual variables y_1 and y_2, will be unrestricted in sign. Following the rules of duality formulation, the dual of the given primal LP problem is

Minimize $Z_y = 2y_1 + y_2$

subject to the constraints

(i) $-2y_1 + 2y_2 \geq 1$, (ii) $y_1 + 3y_2 \geq -2$, (iii) $3y_1 + 4y_2 \geq 3$

and y_1, y_2 unrestricted in sign.

Benefits of Duality in LPP

- Study of duality helps to identify only an increase (or decrease) in the value of objective function due to per unit variation in the amount of resources available.

1. **Alternative Formulations:** Provides another way to view and solve the same problem.
2. **Solution Bounds:** Helps in establishing upper or lower bounds for the optimal solution of the primal problem.
3. **Sensitivity Analysis:** Facilitates the calculation of shadow prices, which indicate the value of additional units of a resource.
4. **Feasibility and Optimality:** Helps in evaluating whether a solution is feasible or optimal.

Fundamental Theorem of Linear Programming

If a linear programming problem (LPP) has an optimal solution, then at least one optimal solution occurs at a **corner point (vertex)** of the feasible region.

•Implications:

- Search for optimal solutions can be restricted to corner points of the feasible region.
- There may be:
  - **Unique solution** at one vertex.
  - **Multiple optimal solutions** if the objective function is parallel to a constraint.
  - **Unbounded solution** if feasible region is open in the direction of optimization.
  - **Infeasible problem** if feasible region is empty.

Degenerate Solutions in LPP

A solution is **degenerate** if one or more basic variables take the value zero at a basic feasible solution (BFS).

- **Causes:**
  - Redundant constraints.
  - Intersection of more than 'm' constraints at a BFS (where *m = number of constraints*).
- **Implications:**
  - May lead to **stalling** in the simplex method.
  - Could cause **cycling** (repetition of same BFS).

Simplex-Based Methods

- **Purpose:** Solve LPPs by moving from one BFS to another, improving the objective function until optimality.
- **Key Components:**
  - **Initial Basic Feasible Solution (IBFS):** Obtained using slack/surplus/artificial variables.
  - **Pivot Operations:** Exchange of basic and non-basic variables.
  - **Optimality Test:** When all reduced costs are ≥ 0 (for maximization).
  - **Unboundedness Check:** If entering variable has no positive ratio for leaving variable test.

Cycling in Simplex

- **Problem:** Simplex method may revisit the same set of BFS repeatedly due to degeneracy.
- **Result:** Infinite loop, no progress toward optimality.
- **Prevention Techniques:**
  - **Bland’s Rule:** Always choose entering and leaving variables with smallest index.
  - **Perturbation Technique:** Slightly adjust constraints to remove degeneracy.

SENSITIVITY ANALYSIS in LPP

- The process of modifying an OR model to observe the effect upon its outputs is called **Sensitivity Analysis**. Purpose is to evaluate the effect on the optimal solution of an LP problem due to variations in the input coefficients (also called parameters), one at a time.
- In an LP model, the coefficients (also known as parameters) such as: (i) profit (cost) contribution (c_j) per unit of a decision variable, x_j (ii) availability of a resources (b_i), and (iii) consumption of resource per unit of decision variables (a_{ij}), are assumed to be constant and known with certainty.
- *Sensitivity analysis determines the sensitivity range (both lower and upper limit) within which the LP model parameters can vary (one at a time) without affecting the optimality of the current optional solution.*
- This analysis reveals the magnitude of change in the optimal solution of an LP model due to discrete variations (changes) in its parameters. The possible change in the parameter values, can range from zero to a substantial change.
- Thus, aim of sensitivity analysis is to determine the range within which the LP model parameters can change without affecting the current optimal solution.

SENSITIVITY ANALYSIS (2)

- The sensitivity analysis is also referred to as *post-optimality analysis* because it does not begin until the optimal solution to the given LP model has been obtained.
- Different parametric changes in an LPP are:

1. Profit (or cost) per unit (c_j) associated with both basic and non-basic decision variables (i.e., coefficients in the objective function).
2. Availability of resources (i.e., right-hand side constants, b_i in constraints).
3. Consumption of resources per unit of decision variables x_j (i.e., coefficients of decision variables in the constraints, a_{ij}).
4. Addition of a new variable to the existing list of variables in LP problem.
5. Addition of a new constraint to the existing list of constraints in the LP problem.

- *So, what happens to the optimal solution value when we have a* ***change in the Objective Function Coefficient*** (***c_j***)? Analysing this is termed as **SENSITIVITY ANALYSIS**.

Line Search Methods in Optimization

Definition: Techniques to determine optimal step size α_k along a search direction.

- **Basic Idea:**
  - Start at x_k.
  - Choose search direction d_k.
  - Update:

$$x_{k+1} = x_k + \alpha_k d_k$$

where α_k is step size.

Stationarity of Limit Points in Steepest Descent

- If:
  1. The objective function $f(x)$ is continuously differentiable.
  2. Step sizes α_k are chosen by exact or inexact line search (satisfying descent conditions).

Then:

- Any **accumulation point** of the sequence $\{x_k\}$ generated by steepest descent is a **stationary point** (i.e., satisfies $\nabla f(x^*) = 0$).
- In practice: steepest descent may take many iterations to reach acceptable accuracy.

Backtracking is simple and widely used.

Exact line search (finding optimal α analytically) is rare in practice because it may require solving another optimization problem.

Exact Line Search Method

- Find the best α that minimizes along the direction d_k.
- Usually involves solving a 1-variable optimization problem

Line Search Method Concept

- Need to find good step size α.
- Steepest descent converges to stationary points.
- Backtracking reduces α until sufficient decrease.

Problem 1: Exact Line Search

Minimize $f(x) = (x-2)^2$ using steepest descent, start $x_0 = 0$.

Solution:

- Gradient: $2(x-2)$. Direction: $d = -\nabla f(x_0) = +4$.
- Line search: $f(0 + \alpha \cdot 4) = (4\alpha - 2)^2$.
- Derivative wrt α: $2(4\alpha - 2)(4) = 0 \Rightarrow \alpha = 0.5$.
- Update: $x_1 = 0 + 0.5 \cdot 4 = 2$.
- Reached optimum in 1 step.

$$x_{k+1} = x_k + \alpha_k d_k$$

Successive Step-Size Reduction Algorithms

- Instead of fixing step size, start large and **reduce until progress is adequate.**
- Common rules:

(a) Backtracking Line Search

1. Choose initial $\alpha = 1$.
2. While:

$$f(x_k + \alpha d_k) > f(x_k) + c\alpha \nabla f(x_k)^T d_k$$

   (Armijo condition not satisfied), reduce α (e.g., $\alpha \leftarrow \beta\alpha$, with $\beta \in (0, 1)$).
3. Accept the reduced α.

- Guarantees sufficient decrease in each step.

- ❑ **Backtracking is simple and widely used method in Line Search Optimization problems.**
- ❑ **Exact line search (finding optimal α analytically) is rare in practice because it may require solving another optimization problem.**
- ❑ **Another method is Wolfe condition**

(a) Backtracking Line Search

1. Choose initial $\alpha = 1$.
2. While:

$$f(x_k + \alpha d_k) > f(x_k) + c\alpha \nabla f(x_k)^T d_k$$

(Armijo condition not satisfied), reduce α (e.g., $\alpha \leftarrow \beta\alpha$, with $\beta \in (0, 1)$).

Problem 2: Backtracking Line Search

Minimize

$$f(x) = x^2$$

starting from $x_0 = 1$, direction $d = -\nabla f(x_0) = -2$, initial $\alpha = 1$, $\beta = 0.5$.

- Gradient at $x_0 = 1$: 2. Direction = -2.
- Trial step: $x = 1 - 2(1) = -1$. $f(-1) = 1$.
- Armijo: $f(-1) \leq f(1) + c\alpha \nabla f(1) d$. (With $c = 0.5$).
- RHS = $1 + 0.5(1)(2)(-2) = -1$. LHS = 1 > -1 → not satisfied.
- Reduce step: $\alpha = 0.5$.
- New point: $x = 0$. $f(0) = 0$. Condition satisfied.
- Accept $\alpha = 0.5$.

$$x_{k+1} = x_k + \alpha_k d_k$$

Problem: Minimize $f(x) = x^2$ starting from $x_0 = 1$, direction $d = -\nabla f(x_0) = -2$. Initial step $\alpha_0 = 1$, shrink factor $\beta = 0.5$. Use the Armijo (sufficient decrease) condition in the backtracking loop.

1. compute gradient and direction

$$f(x) = x^2, \qquad \nabla f(x) = 2x.$$

At $x_0 = 1$:

$$\nabla f(1) = 2, \qquad d = -\nabla f(1) = -2.$$

The trial point for step α is

$$x(\alpha) = x_0 + \alpha d = 1 + \alpha(-2) = 1 - 2\alpha.$$

The value along the line is

$$\phi(\alpha) \equiv f(x(\alpha)) = (1 - 2\alpha)^2.$$

2. Armijo (sufficient decrease) condition

Armijo condition requires

$$\phi(\alpha) \le \phi(0) + c\,\alpha\,\nabla f(x_0)^\top d,$$

where $\phi(0) = f(x_0) = 1$ and $\nabla f(x_0)^\top d = (2) \cdot (-2) = -4$. So the right-hand side is

$$\phi(0) + c\alpha \nabla f(x_0)^\top d = 1 + c\alpha(-4) = 1 - 4c\alpha.$$

Thus Armijo inequality becomes

$$(1 - 2\alpha)^2 \le 1 - 4c\alpha.$$

3. Test $\alpha = \alpha_0 = 1$

Left-hand side:

$$\phi(1) = (1 - 2 \cdot 1)^2 = (-1)^2 = 1.$$

Right-hand side:

$$1 - 4c \cdot 1 = 1 - 4c.$$

Inequality is $1 \leq 1 - 4c$, i.e. $0 \leq -4c$. That is **false** for any $c > 0$.

So $\alpha = 1$ **fails** Armijo and we reduce α.

4. Reduce α: $\alpha \leftarrow \beta\alpha = 0.5$

Now $\alpha = 0.5$.

Left-hand side:

$$\phi(0.5) = (1 - 2 \cdot 0.5)^2 = (1 - 1)^2 = 0.$$

Right-hand side:

$$1 - 4c \cdot 0.5 = 1 - 2c.$$

Inequality is $0 \leq 1 - 2c$, i.e. $c \leq 0.5$. For any typical Armijo constant c (e.g. $c = 10^{-4}$ up to $c = 0.1$), this holds. So $\alpha = 0.5$ **satisfies** Armijo and is accepted.

5. Numerical check with $c = 0.1$ (concrete)

- For $\alpha = 1$: LHS $= 1$, RHS $= 1 - 4(0.1) = 0.6$. $1 \leq 0.6$ → fails.
- For $\alpha = 0.5$: LHS $= 0$, RHS $= 1 - 2(0.1) = 0.8$. $0 \leq 0.8$ → holds.

So backtracking accepts $\alpha = 0.5$.

---

6. Update iterate

$$x_1 = x_0 + \alpha d = 1 + 0.5 \cdot (-2) = 0.$$

Evaluate objective: $f(x_1) = 0$, which is the global minimum for $f(x) = x^2$. (In fact, an exact line search on this quadratic also gives $\alpha = 0.5$.)

final answer

- First trial $\alpha = 1$ fails Armijo.
- After one shrink ($\beta = 0.5$) we get $\alpha = 0.5$ which satisfies Armijo (for any common choice of $c \leq 0.5$, in particular $c = 0.1$).
- Accepted step $\alpha = 0.5$ → new iterate $x_1 = 0$.
- The method reached the exact minimizer in one accepted backtracking step.

Problem 3: Stationarity of Limit Points

Show that steepest descent for

$$f(x) = (x-1)^2$$

leads to a stationary point.

Solution:

- Gradient: $2(x-1)$.
- Update: $x_{k+1} = x_k - \alpha(2(x_k - 1))$.
- As $k \to \infty$, $x_k \to 1$.
- At $x = 1$, gradient = 0 → stationary point.

Problem 3: Backtracking Line Search

$f(x) = x^2$, start at $x_0 = 1$, direction $d = -\nabla f(1) = -2$.

Parameters: $c = 0.1, \beta = 0.5$.

We try $\alpha = 1$:

Check Armijo: $f(1-2) \leq f(1) + c\alpha\nabla f(1)d$.

LHS $= f(-1) = 1$. RHS $= 1 + 0.1(1)(-2)(-2) = 1 + 0.4 = 1.4$.

Condition holds → accept $\alpha = 1$.

Update: $x_1 = -1$.

$$x_{k+1} = x_k + \alpha_k d_k$$

Unconstrained Optimization

- Unconstrained optimization is a mathematical process that finds the minimum or maximum value of an objective function without any restrictions on the decision variables
- Unconstrained optimization plays a crucial role in the training of neural networks.
- Unlike constrained optimization, where the solution must satisfy certain constraints, unconstrained optimization seeks to minimize (or maximize) an objective function **without any restrictions on the variable values**.
- **Common Unconstrained Optimization Techniques**
  - **Gradient descent method (Steepest Descent Method) :** move in the direction of the negative gradient.
  - **Newton's method:** use both gradient and Hessian for faster convergence

1. What is Unconstrained Optimization?

An **unconstrained optimization problem** is one where we want to **minimize (or maximize) an objective function** without any restrictions (constraints) on the variables. Formally:

$$\min_{x \in \mathbb{R}^n} f(x)$$

where:

- $f(x)$ is the objective function,
- $x = (x_1, x_2, \dots, x_n)$ is the decision vector,
- No equality or inequality constraints are imposed.

2. First-Order Condition (Stationary Point)

A necessary condition for x^* to be an **optimal point** (minimum or maximum) is:

$$\nabla f(x^*) = 0$$

This means the **gradient** of $f(x)$ must vanish at x^*. Such a point is called a **stationary point**.

3. Second-Order Condition

To classify a stationary point x^* ,we examine the **Hessian matrix** $H(x) = \nabla^2 f(x)$:

- If $H(x^*)$ is **positive definite** → x^* is a **local minimum.**
- If $H(x^*)$ is **negative definite** → x^* is a **local maximum.**
- If $H(x^*)$ is **indefinite** (both positive and negative eigenvalues) → x^* is a **saddle point** (neither min nor max).

4. Methods for Unconstrained Optimization

Since there are no constraints, algorithms focus directly on improving the objective function:

(a) Analytical methods

- Solve $\nabla f(x) = 0$ directly if possible.

(b) Iterative numerical methods

- **Steepest Descent Method**: move in the direction of the negative gradient.
- **Newton's Method**: use both gradient and Hessian for faster convergence.

Example 1 (1D Function)

$$f(x) = x^2 - 4x + 5$$

Step 1: First derivative (gradient in 1D).

$$f'(x) = 2x - 4$$

Set equal to zero:

$$2x - 4 = 0 \;\Rightarrow\; x^* = 2$$

Step 2: Second derivative.

$$f''(x) = 2 > 0$$

So $x^* = 2$is a **local (and global) minimum**.

Function value:

$$f(2) = 2^2 - 4(2) + 5 = 1$$

Minimum value of $f(x)$ is 1 at x = 2

6. Example 2 (2D Function)

$$f(x, y) = x^2 + y^2 - 2x - 4y + 5$$

Step 1: Gradient.

$$\nabla f(x, y) = \begin{bmatrix} 2x - 2 \\ 2y - 4 \end{bmatrix}$$

Set $\nabla f = 0$:

$$2x - 2 = 0 \Rightarrow x^* = 1$$

$$2y - 4 = 0 \Rightarrow y^* = 2$$

Step 2: Hessian.

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

This is positive definite (all eigenvalues = 2 > 0).

So $(1, 2)$is a **local minimum**.

Function value:

$$f(1, 2) = 1^2 + 2^2 - 2(1) - 4(2) + 5 = 2$$

Important Points :

- Unconstrained optimization = **no restrictions** on variables.
- Stationary points are found by setting $\nabla f(x)(\text{derivative}) = 0$.
- Second-order test (Hessian) classifies stationary points.
- Numerical iterative methods are used when analytical solutions are difficult.

Problem 2 (1D Cubic)

$$f(x) = x^3 - 3x^2 + 2$$

Solution:

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Stationary points: $x = 0, 2$.

Second derivative:

$$f''(x) = 6x - 6$$

- At $x = 0$: $f''(0) = -6 < 0 \Rightarrow$ local **maximum**.
- At $x = 2$: $f''(2) = 6 > 0 \Rightarrow$ local **minimum**.

Function values:

$$f(0) = 2 \quad \text{(local max)}, \quad f(2) = 8 - 12 + 2 = -2 \quad \text{(local min)}$$

Problem 3: 2D Quadratic with Cross Term

$$f(x, y) = (x - 1)^2 + 2(y + 2)^2 + xy$$

(a) Find the stationary points by solving $\nabla f = 0$.

(b) Compute the Hessian and classify the stationary point(s).

(c) State whether the stationary point is global and why.

Step 1: Expand the function (for clarity)

$$f(x, y) = (x^2 - 2x + 1) + 2(y^2 + 4y + 4) + xy$$

Simplify:

$$f(x, y) = x^2 + 2y^2 + xy - 2x + 8y + 9$$

Step 2: Compute the first derivatives

$$\frac{\partial f}{\partial x} = 2x + y - 2$$

$$\frac{\partial f}{\partial y} = 4y + x + 8$$

Step 3: Stationary point condition ($\nabla f = 0$)

Set both partial derivatives to zero:

$$\begin{cases} 2x + y - 2 = 0 \\ x + 4y + 8 = 0 \end{cases}$$

Step 4: Solve the linear system

From the first equation:

$$y = 2 - 2x$$

Substitute into the second:

$$x + 4(2 - 2x) + 8 = 0$$

Simplify:

$$x + 8 - 8x + 8 = 0$$

$$-7x + 16 = 0$$

$$x = \frac{16}{7}$$

Now substitute back into $y = 2 - 2x$:

$$y = 2 - 2\left(\frac{16}{7}\right) = 2 - \frac{32}{7} = \frac{14 - 32}{7} = -\frac{18}{7}$$

✅ Stationary point:

$$(x^*, y^*) = \left(\frac{16}{7}, -\frac{18}{7}\right)$$

Step 5: Compute the Hessian matrix

Second derivatives:

$$f_{xx} = 2, \quad f_{yy} = 4, \quad f_{xy} = f_{yx} = 1$$

So,

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

Step 6: Test positive definiteness of Hessian

For a 2×2 symmetric matrix

$$H = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

it is **positive definite** if:

1. $a > 0$
2. Determinant $ac - b^2 > 0$

Compute:

$$a = 2 > 0$$

$$\det(H) = (2)(4) - (1)^2 = 8 - 1 = 7 > 0$$

Hence, H is **positive definite**.

Step 7: Classify the stationary point

Since H is positive definite,

$$\Rightarrow (x^*, y^*) = \left(\frac{16}{7}, -\frac{18}{7}\right)$$

is a **local minimum.**

Step 8: Compute the minimum value

Substitute $x = \frac{16}{7}$, $y = -\frac{18}{7}$ into $f(x, y) = x^2 + 2y^2 + xy - 2x + 8y + 9$:

$$f = \left(\frac{16}{7}\right)^2 + 2\left(\frac{-18}{7}\right)^2 + \left(\frac{16}{7}\right)\left(\frac{-18}{7}\right) - 2\left(\frac{16}{7}\right) + 8\left(\frac{-18}{7}\right) + 9$$

Simplify step-by-step:

$$f = \frac{256}{49} + 2\left(\frac{324}{49}\right) - \frac{288}{49} - \frac{32}{7} - \frac{144}{7} + 9$$

Convert everything to denominator 49:

$$f = \frac{256 + 648 - 288}{49} - \frac{176 \times 7}{49} + \frac{9 \times 49}{49}$$

$$f = \frac{616}{49} - \frac{1232}{49} + \frac{441}{49} = \frac{-175}{49} = -\frac{25}{7}$$

Minimum value of f is : - 25/7 = - 3.57

Step 9: Check if global

Since $f(x, y)$ is a **quadratic function** with a **positive definite Hessian**, it is **strictly convex**.

→ The stationary point is the **unique global minimum**.

Quantity	Result
Function	$f(x,y) = (x-1)^2 + 2(y+2)^2 + xy$
Stationary point	$(x^*, y^*) = \left(\frac{16}{7}, -\frac{18}{7}\right)$
Hessian	$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$, Positive definite
Classification	Local **and** global minimum
Minimum value	$f_{min} = -\frac{25}{7} \approx -3.571$

Problem 4 — 2D Non-Convex Polynomial

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

Find all stationary points, use the Hessian to classify them, and identify which are minima, maxima, or saddle points.

Step 1: Compute the gradient (first-order partial derivatives)

$$\frac{\partial f}{\partial x} = 4x^3 - 4y$$

$$\frac{\partial f}{\partial y} = 4y^3 - 4x$$

Stationary points satisfy $\nabla f = 0$:

$$4x^3 - 4y = 0, \qquad 4y^3 - 4x = 0$$

or equivalently,

$$x^3 = y, \qquad y^3 = x$$

Step 2: Solve for stationary points

Substitute $y = x^3$ into $y^3 = x$:

$$(x^3)^3 = x \quad \Rightarrow \quad x^9 = x$$

$$x(x^8 - 1) = 0$$

Hence

$$x = 0 \quad \text{or} \quad x^8 = 1$$

For real x:

$$x = 0, \quad x = \pm 1$$

Now obtain corresponding y values from $y = x^3$:

x	$y = x^3$
0	0
1	1
−1	−1

Stationary points: (0,0), (1,1), (−1,−1)

Step 3: Compute the Hessian matrix

$$f_{xx} = 12x^2, \quad f_{yy} = 12y^2, \quad f_{xy} = f_{yx} = -4$$

Thus

$$H(x,y) = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

Step 4: Evaluate Hessian at each stationary point

(a) At (0,0):

$$H = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

Determinant $D = (0)(0) - (-4)^2 = -16 < 0.$

→ **Indefinite Hessian ⇒ Saddle point.**

(b) At (1,1):

$$H = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

Compute principal minors:

$$a_{11} = 12 > 0, \quad D = (12)(12) - (-4)^2 = 144 - 16 = 128 > 0$$

→ **Positive definite ⇒ Local minimum.**

Function value:

$$f(1,1) = 1 + 1 - 4(1)(1) + 1 = -1$$

Local minimum at (1,1) with $f = -1$.

(c) At (−1,−1):

$$H = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

(same structure as above)

→ **Positive definite ⇒ Local minimum.**

Function value:

$$f(-1,-1) = (-1)^4 + (-1)^4 - 4(-1)(-1) + 1 = 1 + 1 - 4 + 1 = -1$$

Local minimum at (-1, -1) with $f = -1$.

Step 5: Summary of classification

Point	Hessian	Determinant	Type	$f(x, y)$
(0, 0)	$\begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$	−16	Saddle point	1
(1, 1)	$\begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$	128	Local min	−1
(−1, −1)	same as (1,1)	128	Local min	−1

Step 6: Global behavior

For large $|x|, |y|$, the quartic terms $x^4 + y^4$ dominate $-4xy$.
Since $x^4 + y^4 \geq 0$ and grows to infinity, $f(x, y) \to \infty$ as $|(x, y)| \to \infty$.

Hence both local minima at $(\pm 1, \pm 1)$ are also **global minima**.

Step 7: Visual intuition

- The function has **two symmetric global minima** at (1, 1) and (−1, −1).
- The origin (0, 0) is a **saddle point** (a “mountain pass” between two valleys).
- The surface is non-convex overall because the Hessian is not positive definite everywhere.

Final Results

Quantity	Value / Interpretation
Stationary points	(0, 0), (1, 1), (−1, −1)
Classification	(0, 0): Saddle; (±1, ±1): Local & Global Minima
Minimum function value	$f_{min} = -1$
Non-convex?	Yes — Hessian not positive definite for all (x,y)

Unbounded Solution in a Linear Programming Problem

In Linear Programming, we aim to:

$$\text{Maximize (or Minimize)} \quad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$Ax \leq b, \quad x \geq 0$$

The **feasible region** is the set of all points x satisfying the constraints.

An **LPP has an unbounded solution** if the objective function can increase (for maximization) or decrease (for minimization) **indefinitely** without violating any constraint, i.e.

$$Z \to +\infty \text{ (for maximization)} \quad \text{or} \quad Z \to -\infty \text{ (for minimization)}$$

while still remaining inside the feasible region.

Geometric Explanation

In two dimensions:

•The feasible region (intersection of constraints) may **open infinitely** in one or more directions.

•If the **objective function line** (isoprofit or isocost line) can move indefinitely in that open direction without leaving the feasible region, the problem is **unbounded**.

Example sketch idea:

•If you maximize $Z = 3x + 2y$

•and feasible region extends infinitely in direction of increasing x, y, then Zhas no finite maximum — it is **unbounded**.

Algebraic (Simplex Method) Condition for Unboundedness

During the **Simplex algorithm**, we work with the **tableau** and repeatedly select:

1. **Entering variable:**

   The non-basic variable with the **most negative coefficient** (for maximization) in the Z-row.

2. **Leaving variable:**

   Determined using the **minimum ratio test:**

$$\text{Ratio} = \frac{\text{RHS}}{\text{positive entry in pivot column}}$$

- Only **positive** pivot column entries** are considered.
- The **smallest positive ratio** decides which variable leaves the basis.

Condition for Unbounded Solution (in Simplex) : If in the selected **pivot (entering) column, all entries are ≤ 0**, then the LPP is **unbounded**.

Reason:

- The entering variable can **increase indefinitely**
- Since all entries in that column are ≤ 0, increasing that variable will **not violate any constraint** (the RHS values won't become negative).
- Therefore, Z can be increased (in maximization) or decreased (in minimization) infinitely — i.e., no finite optimum exists.

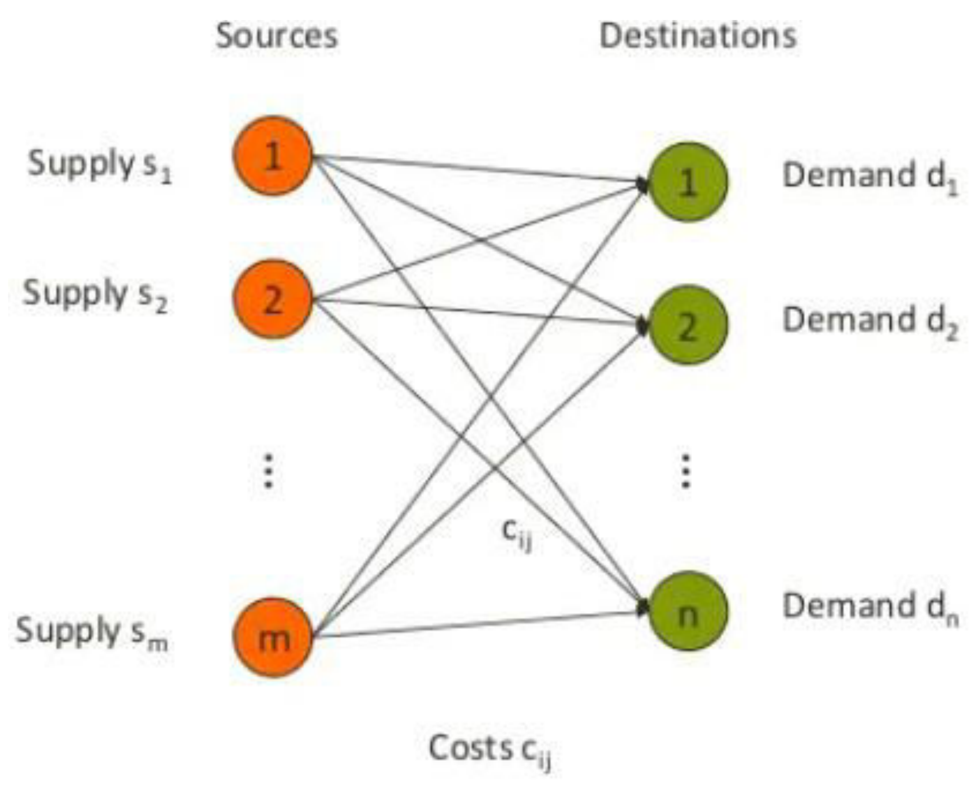

Chicago
Transportation Models

TRANSPORTATION PROBLEM IN SCM

- **How much should be shipped from several sources to several destinations**
  - Sources: Factories, warehouses, etc.
  - Destinations: Warehouses, stores, etc.
- **Transportation models**
  - Find lowest cost shipping arrangement
  - Used primarily for existing distribution systems
- **A Transportation Model Requires**
  - The origin points, and the capacity or supply per period at each
  - The destination points and the demand per period at each
  - The cost of shipping one unit from each origin to each destination

Transportation Model

- The transportation model is a special class of LPPs that deals with transporting(=shipping) a commodity from **sources** (e.g. factories) to **destinations** (e.g. warehouses).
- The objective is to determine the **optimum transportation schedule** that minimizes the **total transportation(shipping)cost** while satisfying supply and demand limits.
- We assume that the shipping cost is proportional to the number of units shipped on a given route.
- The total transportation cost, distribution cost or shipping cost and production costs are to be minimized by applying the model



Mathematical Formulation of Transportation Problem

The transportation problem applies to situations where a single commodity is to be transported from various sources of supply **(origins)** to various demands **(destinations).**
Let there be *m* sources of supply S1, S2, …...............Sm having ai (i = 1,2,......m) units of supplies respectively to be transported among *n* destinations D1, D2 ………Dn with bj (j = 1,2…..n) units of requirements respectively.
Let Cij be the cost for shipping one unit of the commodity from source i, to destination j for each route.
If xij represents the units shipped per route from source i, to destination j, then
the problem is to determine the transportation schedule which minimizes the total transportation cost of satisfying supply and demand conditions.
The transportation problem can be stated mathematically as a LPP as below:

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to constraints,

$$\sum_{j=1}^{n} x_{ij} = a_{i,} \qquad i = 1,2,\ldots..m \text{ (supply constraints)}$$

$$\sum_{i=1}^{n} x_{ij} = b_{j,} \qquad j = 1,2,\ldots..m \text{ (demand constraints)}$$

$$\text{and } x_{ij} \geq 0 \text{ for all } \quad i = 1,2,\ldots..m \text{ and,}$$

$$j = 1,2,\ldots..m$$

- We assume that there are m sources 1,2, …, m and n destinations 1,2, …, n.
- The cost of shipping one unit from Source i to Destination j is Cij.
- We assume that the availability at source i is ai (i=1, 2, …, m) and the demand at the destination j is bj (j=1, 2, …, n).
- Let xij be the amount of commodity to be shipped from the source i to the destination j
- **We make an important assumption: the problem is a balanced one. That is,**

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

That is, total availability equals total demand.

Balanced Transportation

Source	Destination				
	1	2	3	4	Supply
1	10	16	9	12	200
2	12	12	13	5	300
3	14	8	13	4	300
4	0	0	0	0	200
Demand	100	200	450	250	1000/1000

- We can always meet this condition by introducing a dummy source (if the total demand is more than the total supply) or a dummy destination (if the total supply is more than the total demand).

Destination

Source

	1	2	.	.	n	Supply
1	c_{11}	c_{12}			c_{1n}	a_1
2	c_{21}	c_{22}			c_{2n}	a_2
.						
.						
m	c_{m1}	c_{m2}			c_{mn}	a_m
Demand	b_1	b_2			b_n	

NETWORK REPRESENTATION OF TRANSPORTATION MODEL

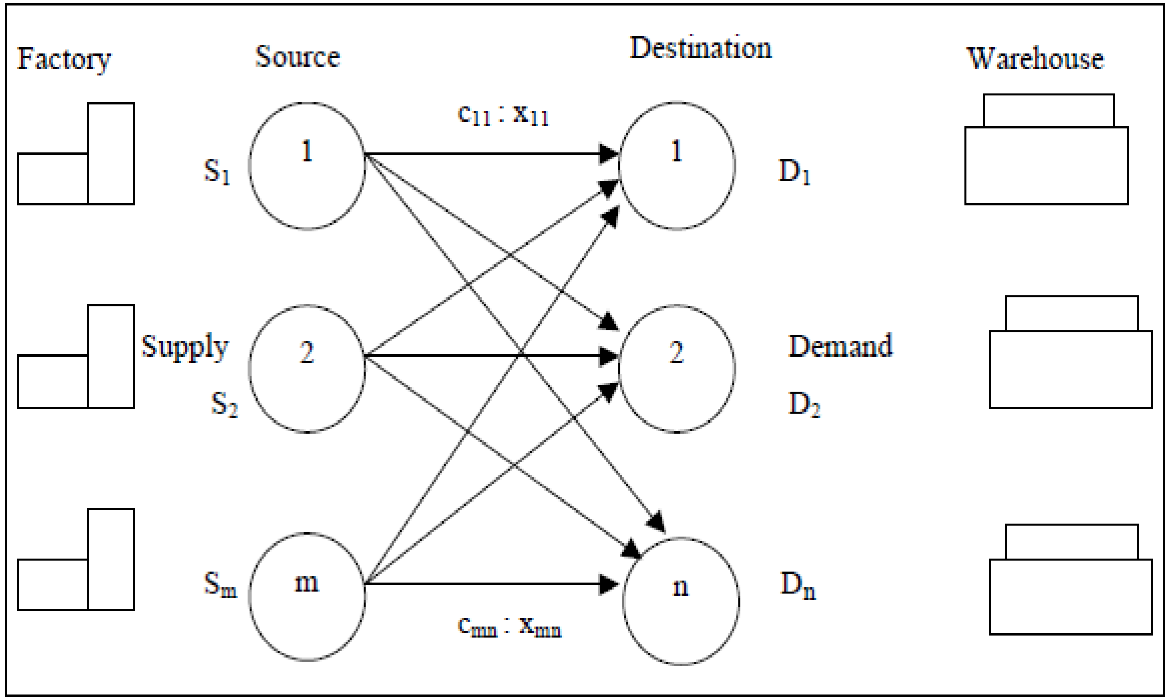


Figure: Network Transportation Model

where,
m be the number of sources,
n be the number of destinations,
S_m be the supply at source m,
D_n be the demand at destination n,
c_{ij} be the cost of transportation from source i to destination j, and
x_{ij} be the number of units to be shipped from source i to destination j.

The objective is to minimize the total transportation cost by determining the unknowns x_{ij}, i.e., the number of units to be shipped from the sources and the destinations while satisfying all the supply and demand requirements.

GENERAL REPRESENTATION OF TRANSPORTATION MODEL

The Transportation problem can also be represented in a tabular form as shown in Table

- Let C_{ij} be the cost of transporting a unit of the product from i_{th} origin to j_{th} destination.
- a_i be the quantity of the commodity available at source i,
- b_j be the quantity of the commodity needed at destination j, and
- x_{ij} be the quantity transported from i_{th} source to j_{th} destination

To / From	D_1	D_2	...	D_n	Supply A_i
S_1	C_{11} x_{11}	C_{12} x_{12}	...	C_{1n}	A_1
S_2	C_{21} x_{21}	C_{22} x_{22}	...	C_{2n}	A_2
⋮	⋮	⋮	⋮	⋮	⋮
S_m	C_{m1} x_{m1}	C_{m2} x_{m2}	...	C_{mn}	A_m
B_j	B_1	B_2	...	B_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

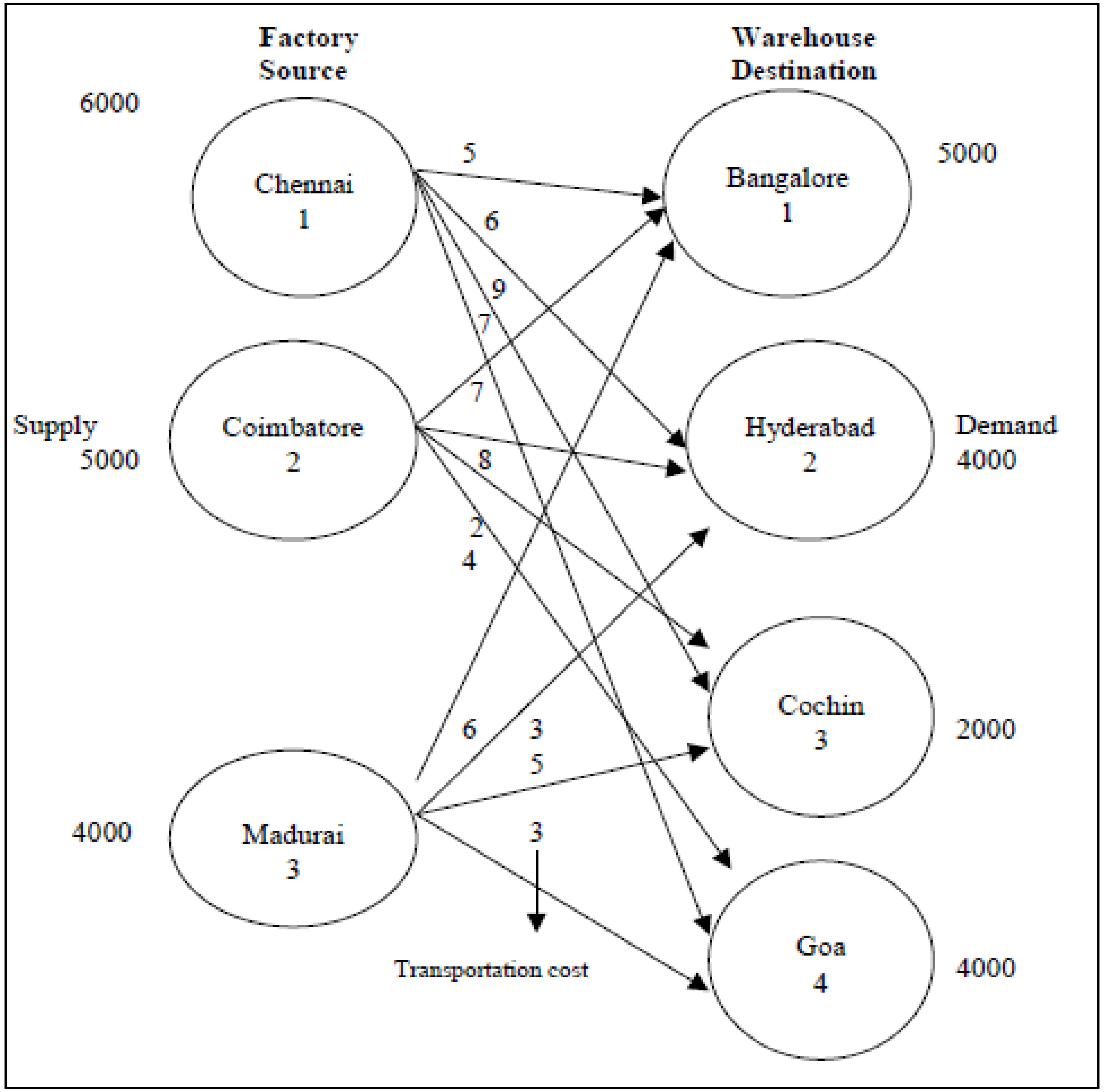
$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

If the total supply is equal to total demand, then the given transportation problem is a balanced one.

Tabular Representation of Transportation Model

- **Feasible solution**- any set of non negative allocations which satisfies row and column requirement
- **Basic feasible solution**-a feasible solution is called basic feasible solution if the number of nonnegative allocations is equal to m+n-1, where m is the no of rows and n is the number of columns
- **Steps involved in solution of transportation problem are :**
  - To find an initial basic feasible solution(IBFS)
  - To check the above solution for optimality
  - To revise the solution
- **Methods to determine IBFS are as follows :**
  - North West Corner Method (NWCM)
  - Least Cost Method (LCM)
  - Vogel's Approximation Method (VAM)

USE OF LPP TO SOLVE TRANSPORTATION PROBLEM



- The network diagram on the left represents the transportation model of **M/s GM Textiles** units located at Chennai, Coimbatore and Madurai.
- GM Textiles produces ready-made garments at these locations with capacities 6000, 5000 and 4000 units per week at Chennai, Coimbatore and Madurai respectively.
- The textile unit distributes its ready-made garments through four of its wholesale distributors situated at four locations Bangalore, Hyderabad, Cochin and Goa.
- The weekly demand of the distributors are
- 5000, 4000, 2000 and 4000 units for Bangalore, Hyderabad, Cochin and Goa respectively.
- The cost of transportation per unit varies between different supply points and destination
- The management would like to determine the number of units to be shipped from each textile unit to satisfy the demand of each wholesale distributor.

Supply	Textile Unit	Weekly Production (Units)
1	Chennai	6000
2	Coimbatore	5000
3	Madurai	4000

Table 1: Production Capacities

Destination	Wholesale Distributor	Weekly Demand (Units)
1	Bangalore	5000
2	Hyderabad	4000
3	Cochin	2000
4	Goa	4000

Table 2: Demand Requirements

Supply	Destination			
	B'lore	Hyderabad	Cochin	Goa
Chennai	5	6	9	7
Coimbatore	7	8	2	4
Madurai	6	3	5	3

Table 3: Transportation Cost per Unit

Let, X_{11} be number of units shipped from source1 (Chennai) to destination 1 (B'lore)
X_{12} be number of units shipped from source1 (Chennai) to destination 2 (Hyderabad)
X_{13} number of units shipped from source 1 (Chennai) to destination 3 (Cochin)
X_{14} number of units shipped from source 1 (Chennai) to destination 4 (Goa) and so on.
X_{ij} = number of units shipped from source i to destination j, where i = 1,2,……..m and, j = 1,2,………n.

Objective function: The objective is to minimize the total transportation cost.

Using the cost data table, the following equation can be arrived at:

Transportation cost for units shipped from Chennai = $5x_{11}+6x_{12}+9x_{13}+7x_{14}$
Transportation cost for units shipped from Coimbatore = $7x_{21}+8x_{22}+2x_{23}+4x_{24}$
Transportation cost for units shipped from Madurai = $6x_{31}+3x_{32}+5x_{33}+3x_{34}$

Combining the transportation cost for all the units shipped from each supply point with the objective to minimize the transportation cost, **the objective function will be,**
Minimize $Z = 5x_{11}+6x_{12}+9x_{13}+7x_{14}+7x_{21}+8x_{22}+2x_{23}+4x_{24}+6x_{31}+3x_{32}+5x_{33}+3x_{34}$

Constraints:

In transportation problems, there are supply constraints for each source, and demand constraints for each destination.

Supply constraints:

For Chennai, $x_{11}+ x_{12}+ x_{13}+ x_{14} \leq 6000$
For Coimbatore, $x_{21}+ x_{22}+ x_{23}+ x_{24} \leq 5000$
For Madurai, $x_{31}+ x_{32}+ x_{33}+ x_{34} \leq 4000$

Demand constraints:

For B'lore, $x_{11}+ x_{21}+ x_{31}= 5000$
For Hyderabad, $x_{12}+ x_{22}+ x_{32}= 4000$
For Cochin, $x_{13}+ x_{23}+ x_{33}= 2000$
For Goa, $x_{14}+ x_{24}+ x_{34}= 4000$

The linear programming model for GM Textiles will be:

Minimize $Z = 5x_{11} + 6x_{12} + 9x_{13} + 7x_{14} + 7x_{21} + 8x_{22} + 2x_{23} + 4x_{24} + 6x_{31} + 3x_{32} + 5x_{33} + 3x_{34}$

Subject to constraints,

$x_{11}+x_{12}+x_{13}+x_{14} \leq 6000$ ....(i)

$x_{21}+x_{22}+x_{23}+x_{24} \leq 5000$ ....(ii)

$x_{31}+x_{32}+x_{33}+x_{34} \leq 4000$ ....(iii)

$x_{11}+x_{21}+x_{31} = 5000$ ....(iv)

$x_{12}+x_{22}+x_{32} = 4000$ ....(v)

$x_{13}+x_{23}+x_{33} = 2000$ ....(vi)

$x_{14}+x_{24}+x_{34} = 4000$ ....(vii)

where, $x_{ij} \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$.

In the above LP problem, there are $m \times n = 3 \times 4 = 12$ decision variables and $m + n = 3+4 = 7$ constraints

PROCEDURE TO SOLVE TRANSPORTATION PROBLEM

Step 1: Formulate the problem.

Formulate the given problem and set up in a matrix form. Check whether the problem is a balanced or unbalanced transportation problem. If unbalanced, add dummy source (row) or dummy destination (column) as required.

Step 2: Obtain the initial feasible solution.

The initial feasible solution can be obtained by any of the following three methods:

i. Northwest Corner Method (NWCM)

ii. Least Cost Method (LCM)

iii. Vogel's Approximation Method (VAM)

The transportation cost of the initial basic feasible solution through Vogel's approximation method, VAM will be the least when compared to the other two methods which gives the value nearer to the optimal solution or optimal solution itself. Algorithms for all the three methods to find the initial basic feasible solution are given.

Remarks: The initial solution obtained by any of the three methods must satisfy the following conditions:

(a) The solution must be feasible, i.e., the supply and demand constraints must be satisfied (also known as rim conditions).

(b) The number of positive allocations, N must be equal to m+n-1, where m is the number of rows and n is the number of columns.

Procedure for NWCM

i. Select the North-west (i.e., upper left) corner cell of the table and allocate the maximum possible units between the supply and demand requirements. During allocation, the transportation cost is completely discarded (not taken into consideration).

ii. Delete that row or column which has no values (fully exhausted) for supply or demand.

iii. Now, with the new reduced table, again select the North-west corner cell and allocate the available values.

iv. Repeat steps (ii) and (iii) until all the supply and demand values are zero.

v. Obtain the initial basic feasible solution.

Example 1

The cost of transportation per unit from three sources and four destinations are given below. Obtain the initial basic feasible solutions using the following methods :

a) NWCM b) LCM c) VAM

Source	Destination				Supply
	1	2	3	4	
1	4	2	7	3	250
2	3	7	5	8	450
3	9	4	3	1	500
Demand	200	400	300	300	1200

Solution: The problem given in Table above is a **balanced** one as the total sum of supply is equal to the total sum of demand.
The problem can be solved by all the three methods.

Solution using NWCM

Source	Destination				Supply
	1	2	3	4	
1	4	2	7	3	250
2	3	7	5	8	450
3	9	4	3	1	500
Demand	200	400	300	300	1200

In the given matrix, select the North-West corner cell.

The North-West corner cell is (1,1) and the supply and demand values corresponding to cell (1,1) are 250 and 200 respectively. Allocate the maximum possible value to satisfy the demand from the supply. Here the demand and supply are 200 and 250 respectively. Hence allocate 200 to the cell (1,1) as shown below.

Destination

		1	2	3	4	Supply
Source	1	4 200	2	7	3	~~250~~ 50
	2	3	7	5	8	450
	3	9	4	3	1	500
Demand		~~200~~ 0	400	300	300	

Allocated 200 to the Cell (1, 1)

Now, delete the exhausted column 1 which gives a new reduced table as shown below in Tables Tables 2, 3, 4, 5. Again repeat the steps.

Table 2 : Exhausted Column 1 Deleted

Destination

Source		2	3	4	Supply
	1	2 50	7	3	~~50~~ 0
	2	7	5	8	450
	3	4	3	1	500
Demand		~~400~~ 350	300	350	

Table after deleting Row 1 is

Destination

Source		2	3	4	Supply
	2	7 350	5	8	~~450~~ 100
	3	4	3	1	500
Demand		~~350~~ 0	300	300	

Table 3: Exhausted Row 1 Deleted

Table after deleting Column 2 is

Destination

Source		3	4	Supply
	2	cost 5, ~~100~~	cost 8	~~100~~ 0
	3	cost 3	cost 1	500
Demand		~~300~~ 200	300	

Table 4: Exhausted Column 2 Deleted

Finally, after deleting Row 2, we have

Destination

Source		3	4	Supply
	3	cost 3, 300	cost 1, 200	~~500~~
Demand		~~300~~ 0	~~200~~ 0	

Table 5: Exhausted Row 2 Deleted

Now only source 3 is left.
Allocating to destinations 3 and 4 satisfies the supply of 500.

The initial basic feasible solution using NWCM method is shown in Table 6 below

Source	Destination				Supply
	1	2	3	4	
1	4	2	7	3	250
2	3	7	5	8	450
3	9	4	3	1	500
Demand	200	400	300	300	1200

Transportation cost = (4 × 200) + (2 × 50) + (7 × 350) + (5 × 100) + (2 × 300) + (1 × 300)
= 800 + 100 + 2450 + 500 + 600 + 300
= **Rs. 4,750.00**

Procedure for Least Cost Method

(i) Select the smallest transportation cost cell available in the entire table and allocate the supply and demand.

(ii) Delete that row/column which has exhausted. The deleted row/column must not be considered for further allocation.

(iii) Again select the smallest cost cell in the existing table and allocate.

(Note: In case, if there are more than one smallest costs, select the cells where maximum allocation can be made)

(iv) Obtain the initial basic feasible solution.

Least Cost Method

Select the minimum cost cell from the entire Table 1, the least cell is (3,4). The corresponding supply and demand values are 500 and 300 respectively. Allocate the maximum possible units.

The allocation is shown in Table 1 below

		Destination				
		1	2	3	4	Supply
	1	4	2	7	3	250
Source	2	3	7	5	8	450
	3	9	4	3	1 300	~~500~~ 200
Demand		200	400	300	~~300~~ 0	

Table 1: Allocation of Maximum Possible Units

From the supply value of 500, the demand value of 300 is satisfied. Subtract 300 from the supply value of 500 and subtract 300 from the demand value of 300. The demand of destination 4 is fully satisfied. Hence, delete the column 4; as a result we get, the table as shown in Table 2.

Destination

Source	1	2	3	Supply
1	4	2 250	7	~~250~~
2	3	7	5	450
3	9	4	3	200
Demand	200	~~400~~ 150	300	

Table 2: Exhausted Column 4 deleted

Now, again take the minimum cost value available in the existing table and allocate it with a value of 250 in the cell (1,2).
The reduced matrix is shown in table 3 below:

Desitnation

Source	1	2	3	Supply
2	3 200	7	5	~~450~~ 250
3	9	4	3	450
Demand	~~200~~	150	300	

Table 3: Exhausted Row 1 deleted

In the reduced Table 3, the minimum value 3 exists in cell (2,1) and (3,3), which is a tie. If there is a tie, it is preferable to select a cell where maximum allocation can be made.
In this case, the maximum allocation is 200 in both the cells.
Choose a cell arbitrarily and allocate.
The cell allocated in (2,1) is shown in Table 3. The reduced matrix is shown in Table 4.

Destination

		2	3	Supply
	2	7	5	250
Source	3	4	3 200	~~200~~ 0
Demand		~~150~~	~~300~~ 100	

Table 4: Reduced Matrix

Now, deleting the exhausted demand row 3, we get the matrix as shown in Table 5 below:

Destination

		2	3	Supply
Source	2	7 150	5 100	~~250~~ 0
Demand		~~150~~ 0	~~100~~ 0	

Table 5: Exhausted Row 3 deleted

The initial basic feasible solution using LCM is shown in a single Table as shown:

		Destination				
		1	2	3	4	Supply
	1	4	2 250	7	3	250
Source	2	3 200	7 150	5 100	8	450
	3	9	4	3 200	1 300	500
Demand		200	400	300	300	

Table 6: Initial basic feasible solution Using LCM Method

Transportation Cost using LCM =

$(2 \times 250) + (3 \times 200) + (7 \times 150) + (5 \times 100) + (3 \times 200) + (1 \times 300)$

$= 500 + 600 + 1050 + 500 + 600 + 300 =$ **Rs. 3550**

Procedure for Vogel's Approximation Method

(i) Calculate penalties for each row and column by taking the difference between the smallest cost and next highest cost available in that row/column. If there are two smallest costs, then the penalty is zero.

(ii) Select the row/column, which has the largest penalty and make allocation in the cell having the least cost in the selected row/column. If two or more equal penalties exist, select one where a row/column contains minimum unit cost. If there is again a tie, select one where maximum allocation can be made.

(iii) Delete the row/column, which has satisfied the supply and demand.

(iv) Repeat steps (i) and (ii) until the entire supply and demands are satisfied.

(v) Obtain the initial basic feasible solution.

The penalties for each row and column are calculated (steps given on previous slide). Choose the row/column, which has the maximum value for allocation.

In this case there are five penalties, which have the maximum value 2.

The cell with least cost (cost = 1) is Row 3 and hence select cell (3,4) for allocation.

The supply and demand are 500 and 300 respectively and hence allocate 300 in cell (3,4) as shown

Penalty Calculation for each Row and Column

Destination

		1	2	3	4	Supply	Penalty
	1	4	2	7	3	250	(1)
Source	2	3	7	5	8	450	(2)
	3	9	4	3	1 300	~~500~~ 200	← (2)
Demand		200	400	300	~~300~~ 0		
		(1)	(2)	(2)	(2)		

Since the demand is satisfied for destination 4, delete column 4 . Now again calculate the penalties for the remaining rows and columns.

Table 2: Exhausted Column 4 Deleted

Source		Destination 1	Destination 2	Destination 3	Supply	Penalty
	1	4	2 250	7	~~250~~ 0	(2)
	2	3	7	5	450	(2)
	3	9	4	3	200	(1)
Demand		200	~~400~~ 150	300		
		(1)	(2)	(2)		

In the Table 3, there are four maximum penalties of values which is 2.
Selecting the least cost cell, (1,2) which has the least unit transportation cost 2.
The cell (1, 2) is selected for allocation as shown in Table 2. Table 3 shows the reduced table after deleting row l.

Table 3: Row 1 Deleted

Destination

Source	1	2	3	**Supply**	**Penalty**
2	3 200	7	5	~~450~~ 250	(2)
3	9	4	3	200	(1)
Demand	~~200~~	150	300		
	(6) ↑	(3)	(2)		

After deleting column 1 we get the table as shown in the Table 4 shown alongside.

Table 4 : Column 1 Deleted

Destination

Source	**2**	**3**	**Supply**	**Penalty**
2	7	5	250	(2)
3	4 150	3	~~200~~ 50	(1)
Demand	~~150~~ 0	300		
	(3) ↑	(2)		

Finally we get the reduced table as shown on the left.

The initial basic feasible solution is shown in Table below now:

Destination

Source	3	Supply
2	5 [250]	~~250~~ 0
3	3 [50]	~~50~~ 0
Demand	~~300~~ 0	

Destination

	W_1	W_2	W_3	W_4	Supply
F_1	17 [140]	5	9	65	140 (4) (4) (8) (48) (48)
F_2	20 [50]	10	12	65 [210]	260 (2) (2) (8) (45) (45)
F_3	15 [10]	0 [100]	5 [250]	65	360 (5) (5) (10) (50) __
F_4	13	1 [220]	10	65	220 (9) __ __ __ __ __
Demand	200	320	250	210	
	(2)	(1)	(4)	(0)	
	(2)	(5)	(4)	(0)	
	(2)	__	__	(0)	
	(2)	__	__	(0)	
	(3)	__	__	(0)	

Transportation cost = (2 × 250) + (3 × 200) + (5 × 250) + (4 × 150) + (3 × 50) + (1 × 300)
= 500 + 600 + 1250 + 600 + 150 + 300 = Rs. 3,400.00

Find the initial basic solution for the transportation problem and hence solve it.

		Destination 1	2	3	4	Supply
Source	1	4	2	7	3	250
	2	3	7	5	8	450
	3	9	4	3	1	500
Demand		200	400	300	300	

Find the initial basic solution for the transportation problem using VAM and hence solve it.

		Destination				
		1	2	3	4	Supply
	1	4	2	7	3	250
Source	2	3	7	5	8	450
	3	9	4	3	1	500
Demand		200	400	300	300	

TRAVELLING SALESMAN PROBLEM (TSP)
OR
VEHICLE ROUTING PROBLEM(VRP)

INTRODUCTION

- The Traveling Salesman Problem (TSP) is a problem in combinatorial optimization studied in operations research (O R).
- A map of cities is given to the salesman and he has to visit all the cities only once to complete a tour such that the length of the tour is the shortest among all possible tours for this map.
- Consider an example related with travelling salesman problem and explain in detail how to find optimal solution.

- Problem Statement
  - If there are n cities and cost of traveling from any city to any other city is given.
  - Then we have to obtain the cheapest round-trip such that each city is visited exactly ones returning to starting city, completes the tour.
  - Typically travelling salesman problem is represent by weighted graph.

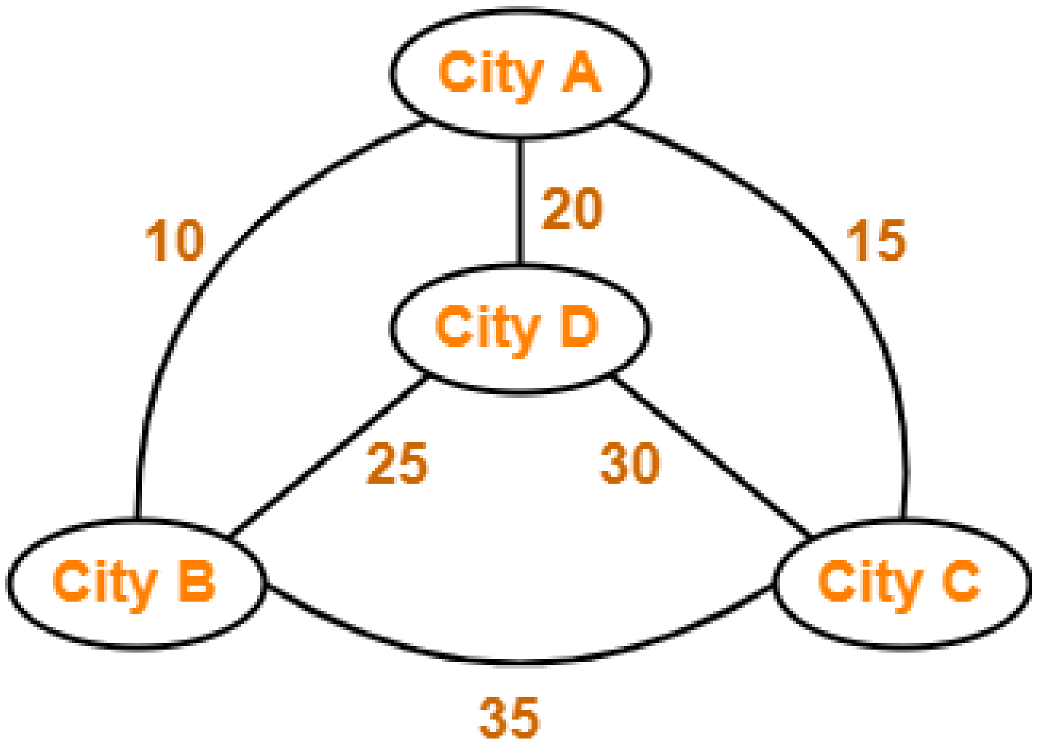
Travelling Salesman Problem-

You are given-

- A set of some cities
- Distance between every pair of cities

Travelling Salesman Problem states-

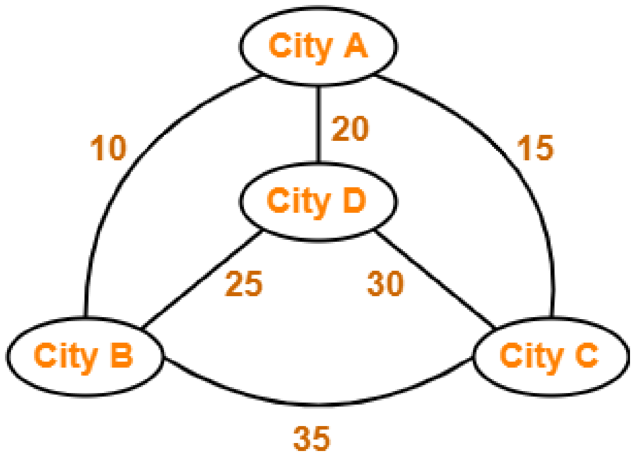
- A salesman has to visit every city exactly once.
- He has to come back to the city from where he starts his journey.
- What is the shortest possible route that the salesman must follow to complete his tour?



Travelling Salesman Problem

EXAMPLE OF TSP

The following graph shows a set of cities and distance between every pair of cities-



If salesman starting city is A, then a TSP tour in the graph is-

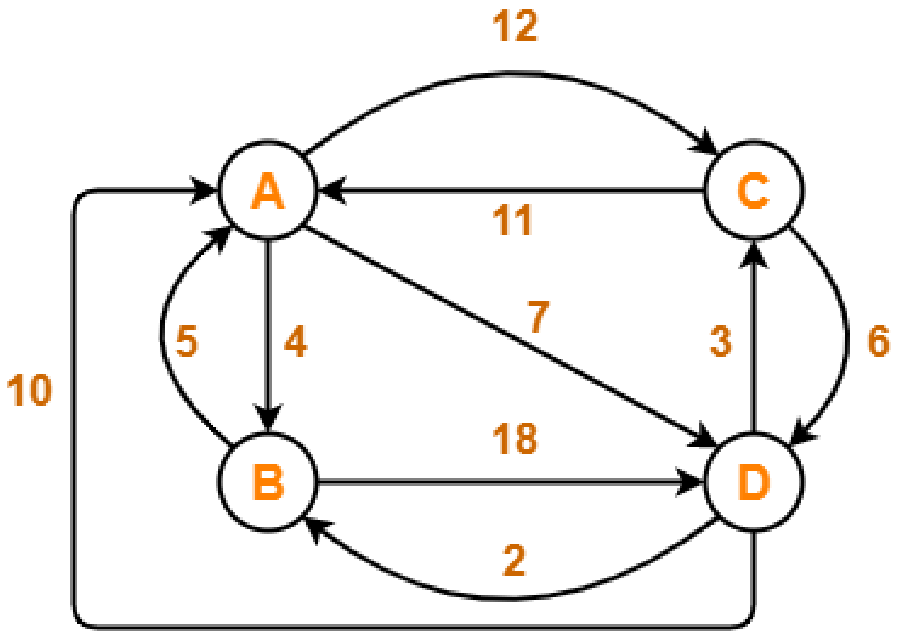
$$A \rightarrow B \rightarrow D \rightarrow C \rightarrow A$$

Cost of the tour

= 10 + 25 + 30 + 15

= 80 units

Solve Travelling Salesman Problem using Branch and Bound Algorithm in the following graph-



$$
\begin{array}{c|c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} - & 10 & 3 & 6 & 9 \\ 5 & - & 5 & 4 & 2 \\ 4 & 9 & - & 7 & 8 \\ 7 & 1 & 3 & - & 4 \\ 3 & 2 & 6 & 5 & - \end{bmatrix}
\end{array}
$$

- **1 2 3 3 7**
- $4 \to 2 \to 5 \to 1 \to 3 \to 4$

Non-Linear Programming (NLP)

Optimization problems where either the objective function or the constraints (or both) are non-linear fall into the category of Non-Linear Programming.

1. First-Order and Second-Order Conditions in NLP

Consider a general unconstrained optimization problem:

$$\min f(x), \quad x \in \mathbb{R}^n$$

where $f(x)$ is continuously differentiable.

1.1 First-Order Necessary Condition (FONC)

- At a **local minimum** x^*:

$$\nabla f(x^*) = 0$$

  That is, the **gradient (vector of first derivatives)** vanishes.

- Geometric interpretation:
  At the minimum, the tangent plane is flat → no slope in any direction.
- Note: This is a **necessary** condition, not sufficient — because it also holds at local maxima and saddle points.

1.2 Second-Order Conditions (SOC)

- Let $H(x) = \nabla^2 f(x)$ denote the **Hessian matrix** (matrix of second derivatives).

At a candidate point x^* where $\nabla f(x^*) = 0$:

- If $H(x^*)$ is **positive definite**, then x^* is a **strict local minimum**.
- If $H(x^*)$ is **positive semi-definite**, then x^* may be a local minimum (not strict).
- If $H(x^*)$ is **negative definite**, then x^* is a **strict local maximum**.
- If $H(x^*)$ is **indefinite** (some eigenvalues positive, some negative), then x^* is a **saddle point**.

1.3 Constrained NLP

- With constraints:

$$\min f(x), \quad g_i(x) \leq 0, \quad h_j(x) = 0$$

Karush-Kuhn-Tucker (KKT) conditions are the generalization of first-order conditions.

- KKT introduces **Lagrange multipliers** to handle constraints.
  (This belongs to later lectures, but good to keep in mind.)

2. Iterative Methods in NLP

- Analytical solutions (closed-form) are rare in NLP. Instead, we use **iterative methods**:

$$x_{k+1} = x_k + \alpha_k d_k$$

where:

- x_k: current iterate,
- d_k: search direction,
- α_k: step size (learning rate).

2.1 Gradient Descent (Steepest Descent)

- $d_k = -\nabla f(x_k)$.
- Moves in the direction of steepest decrease.
- Convergence can be **very slow** in ill-conditioned problems.

2.2 Newton's Method

- Uses second-order (Hessian) information:

$$d_k = -H(x_k)^{-1} \nabla f(x_k)$$

- Quadratic convergence near the optimum (very fast).
- Issues: computing Hessian is costly, may not be positive definite.

2.3 Quasi-Newton Methods

- Approximate Hessian instead of computing exactly.
- Example: **BFGS algorithm** (widely used in optimization libraries).

2.4 Issues with Iterative Methods

1. **Convergence to Local vs. Global Minimum**
   - Non-convex functions may trap algorithms in local minima.
2. **Choice of Initial Guess x_0**
   - Strongly affects performance and final result.
3. **Step Size (Learning Rate) Selection**
   - Too large → divergence.
   - Too small → very slow convergence.
4. **Ill-conditioning**
   - If level curves are elongated (like a narrow valley), gradient descent zig-zags and converges slowly.
   - Preconditioning or using Newton-type methods helps.

3. Line Search Methods

Line search methods focus on choosing **optimal step size** α_k in each iteration.

General iteration:

$$x_{k+1} = x_k + \alpha_k d_k$$

3.1 Stationarity of Limit Points in Steepest Descent

- If:
  1. The objective function $f(x)$ is continuously differentiable.
  2. Step sizes α_k are chosen by exact or inexact line search (satisfying descent conditions).

Then:

- Any **accumulation point** of the sequence $\{x_k\}$ generated by steepest descent is a **stationary point** (i.e., satisfies $\nabla f(x^*) = 0$).
- In practice: steepest descent may take many iterations to reach acceptable accuracy.

3.2 Successive Step-Size Reduction Algorithms

- Instead of fixing step size, start large and **reduce until progress is adequate**.
- Common rules:

(a) Backtracking Line Search

1. Choose initial $\alpha = 1$.
2. While:

$$f(x_k + \alpha d_k) > f(x_k) + c\alpha \nabla f(x_k)^T d_k$$

   (Armijo condition not satisfied), reduce α (e.g., $\alpha \leftarrow \beta\alpha$, with $\beta \in (0, 1)$).
3. Accept the reduced α.

- Guarantees sufficient decrease in each step.

(b) Wolfe Conditions

- Stronger conditions to balance **sufficient decrease** and **curvature condition**.
- Ensures both progress and stability.

Remarks on Line Search

- Backtracking is simple and widely used.
- Exact line search (finding optimal α\alphaα analytically) is rare in practice because it may require solving another optimization problem.
- In machine learning and data analytics, **fixed learning rate with occasional reduction** is common (a practical variant of step-size reduction).

First-order condition: Gradient must vanish at local optima.

Second-order condition: Hessian determines nature of stationary point (min/max/saddle).

Iterative methods: Gradient descent, Newton's, Quasi-Newton are main techniques.

Issues: Step-size selection, local minima, ill-conditioning, sensitivity to starting point.

Line search: Essential for efficient convergence; backtracking and Wolfe conditions widely used.

Steepest descent: Limit points are stationary, but convergence may be slow.

First and Second Order Conditions

Concept Recap

- At local optimum, gradient must vanish: $\nabla f(x^*) = 0$.
- Nature of point determined by Hessian matrix.

Problem 1:

Find the stationary points of

$$f(x) = x^2 - 4x + 5$$

and determine their nature.

Solution:

- Gradient:
  $\frac{df}{dx} = 2x - 4$.
- Set = 0: $2x - 4 = 0 \Rightarrow x = 2$.
- Hessian: $f''(x) = 2 > 0$.
- **Conclusion:** Minimum at $x = 2$, value = $f(2) = 1$.

Problem 2:

For the function

$$f(x, y) = x^2 + y^2 - 2x - 4y + 5$$

find the minimum point.

Solution:

- Gradient:
  $\nabla f(x, y) = (2x - 2,\ 2y - 4)$.
- Stationary point: $2x - 2 = 0 \Rightarrow x = 1$, $2y - 4 = 0 \Rightarrow y = 2$.
- Hessian:
  $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (positive definite).
- **Conclusion:** Minimum at $(1, 2)$, value = $f(1, 2) = 0$.

Problem 3:

Check the nature of stationary point for:

$$f(x, y) = x^2 - y^2$$

Solution:

- Gradient: $(2x, -2y)$.
- Stationary point: $(0, 0)$.
- Hessian:
  $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$.
- Eigenvalues: $+2, -2$ → indefinite.
- **Conclusion:** $(0, 0)$ is a **saddle point**.

Remarks:

First-order → stationarity.

Second-order → min/max/saddle classification.

Iterative Methods in NLP

Concept

- Use iterative updates:

$$x_{k+1} = x_k + \alpha_k d_k$$

- Gradient descent: $d_k = -\nabla f(x_k)$.
- Newton's method: $d_k = -H^{-1}(x_k)\nabla f(x_k)$.

Problem 1: Gradient Descent in 1D

Minimize

$$f(x) = (x - 3)^2$$

using gradient descent, starting from $x_0 = 0$, with $\alpha = 0.1$.

Solution:

- Gradient: $\nabla f(x) = 2(x - 3)$.
- Iteration: $x_{k+1} = x_k - 0.1(2(x_k - 3))$.
- Step 1: $x_1 = 0 - 0.1(-6) = 0.6$.
- Step 2: $x_2 = 0.6 - 0.1(-4.8) = 1.08$.
- Step 3: $x_3 = 1.08 - 0.1(-3.84) = 1.464$.
- Converges towards $x = 3$.

Problem 2: Newton's Method in 1D

Minimize

$$f(x) = x^2 + 4x + 4$$

(start from $x_0 = 2$).

Solution:

- Gradient: $f'(x) = 2x + 4$.
- Hessian: $f''(x) = 2$.
- Newton update: $x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$.
- Step 1: $x_1 = 2 - (8/2) = -2$.
- Gradient at -2 is 0 → convergence.
- **Minimum at $x = -2$.**

Problem 3: Gradient Descent in 2D

Minimize

$$f(x, y) = (x - 1)^2 + (y - 2)^2$$

starting from $(0, 0)$, $\alpha = 0.1$.

Solution:

- Gradient: $\nabla f(x, y) = (2(x - 1), 2(y - 2))$.
- Start: $(0, 0)$. Gradient = $(-2, -4)$.
- Update: $(x, y) = (0, 0) - 0.1(-2, -4) = (0.2, 0.4)$.
- Next step: Gradient = $(-1.6, -3.2)$.
- Update: $(0.36, 0.72)$.
- Converges towards $(1, 2)$.

Remarks:

- Gradient descent = slow but general.
- Newton = fast but needs Hessian.