VERY SHORT ANSWER QUESTIONS (2 Marks)

STRAIGHT LINES

1. Prove that the points (1,11), (2,15) and (-3, -5) are collinear and find the equation of the stright line containing them.

Sol. Let A = (1, 11); B = (2, 15) ; C = (-3, -5)

slope of BC (m) = $\frac{-5-15}{-3-2} = \frac{-20}{-5} = 4$ $\boxed{\because \text{slope of the line passing through}(x_1,y_1)\&(x_2,y_2) \text{ is } \frac{y_2-y_1}{x_2-x_1}}$

Equation of BC is $y - y_1 = m(x - x_1)$

$$\Rightarrow y+5 = 4(x+3) \Rightarrow 4x - y + 7 = 0 \text{ ....}(1)$$

put A(1,11) in equation (1) $\Rightarrow 4(1)-11+7=0$

$\therefore$ A(1,11) satisfies equation (1) $\Rightarrow$ the points A,B,C are collinear and the equation of the stright line is 4x-y+7=0

2. Find the condition for the points (a, 0), (h, k) and (0, b) where $ab \neq 0$ to be collinear. (Mar-10)

Sol. Let A = (a, 0); B = (h, k) ; C = (0, b)

Given that A, B, C are collinear.

$\therefore$ **slope of AB = slope of BC**

$\Rightarrow \frac{k-0}{h-a} = \frac{b-k}{0-h}$ $\boxed{\because \text{slope of the line passing through}(x_1,y_1)\&(x_2,y_2) \text{ is } \frac{y_2-y_1}{x_2-x_1}}$

$\Rightarrow \frac{k}{h-a} = \frac{b-k}{-h}$ $\Rightarrow -hk = bh - hk - ab + ak \Rightarrow bh + ak = ab$

$\Rightarrow \frac{bh}{ab} + \frac{ak}{ab} = 1 \Rightarrow \frac{h}{a} + \frac{k}{b} = 1$

3. Transform the equations into normal form $(i)\, x + y + 1 = 0$ **(May-10)**

$(ii)\, x + y - 2 = 0$ **(Mar-12)**

Sol. **(i)** Given that $x + y + 1 = 0 \Rightarrow x + y = -1 \Rightarrow -x - y = 1$

Divide with $\sqrt{2}$ $\left[\because \sqrt{a^2+b^2} = \sqrt{1+1} = \sqrt{2}\right]$

$$\left(\frac{-1}{\sqrt{2}}\right)x + \left(\frac{-1}{\sqrt{2}}\right)y = \frac{1}{\sqrt{2}}$$

This is in the form of $x\cos\alpha + y\sin\alpha = p$ **(normal form)**

where $\cos\alpha = \frac{-1}{\sqrt{2}} < 0, \sin\alpha = \frac{-1}{\sqrt{2}} < 0, p = \frac{1}{\sqrt{2}}$

$$\alpha \in Q_3 \Rightarrow \alpha = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

$\therefore$ The equation of st.line in normal form is $x\cos\frac{5\pi}{4} + y\sin\frac{5\pi}{4} = \frac{1}{\sqrt{2}}$

(ii) $x + y - 2 = 0 \Rightarrow x + y = 2$

Divide with $\sqrt{2}$ $\left[\because \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2}\right]$

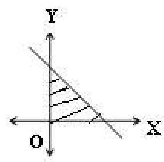
$\left(\frac{1}{\sqrt{2}}\right)x + \left(\frac{1}{\sqrt{2}}\right)y = \sqrt{2} \Rightarrow x\cos\frac{\pi}{4} + y\sin\frac{\pi}{4} = \sqrt{2}$

This is in the form of $\mathbf{x\cos\alpha + y\sin\alpha = p}$ (Normal form)

where $\cos\alpha = \frac{1}{\sqrt{2}}$; $\sin\alpha = \frac{1}{\sqrt{2}}$ and $p = \sqrt{2}$, $\alpha = \frac{\pi}{4}$

4. **If the area of the triangle formed by the straight lines x = 0, y = 0 and 3x+4y=a (>o) is 6.Find the value of a. (May-07, Mar-09)**

Sol. Given that $3x + 4y = a \Rightarrow 3x + 4y - a = 0 ------(1)$



Also, Given that

The area of the triangle formed with (1) and the co-ordinate axes = 6

$\Rightarrow \frac{c^2}{2|ab|} = 6$

$\boxed{\because \textit{area of the traingle formed by the line } ax + by + c = 0 \textit{ with coordinate axes is } \frac{c^2}{2|ab|}}$

$\Rightarrow \frac{a^2}{2|3.4|} = 6 \Rightarrow a^2 = 144 \Rightarrow a = 12 [\because a > 0]$

5. **If the product of the intercepts made by the straight line**

$x\tan\alpha + y\sec\alpha = 1\left(0 \le \alpha < \frac{\pi}{2}\right)$ **on the co-ordinate axes is equal to** $\sin\alpha$, **find** α **.**

Sol. Given that $x\tan\alpha + y\sec\alpha = 1 \Rightarrow \frac{x}{\cot\alpha} + \frac{y}{\cos\alpha} = 1$

This is in the form of $\mathbf{\frac{x}{a} + \frac{y}{b} = 1}$

where x - intercept, a = $\cot\alpha$; y - intercept, b = $\cos\alpha$

Given that, the product of intercepts $= \sin\alpha$

$\Rightarrow \cot\alpha.\cos\alpha = \sin\alpha \Rightarrow \frac{\cos\alpha}{\sin\alpha}.\cos\alpha = \sin\alpha \Rightarrow \frac{\cos^2\alpha}{\sin^2\alpha} = 1 \Rightarrow \cot^2\alpha = 1$

$\Rightarrow \cot\alpha = 1\left[\because 0 \le \alpha < \frac{\pi}{2}\right] \Rightarrow \cot\alpha = \cot 45^0 \Rightarrow \alpha = 45^0$

6. Find the area of the triangle formed by the straight line $x-4y+2=0$ with the co-ordinate axes.

Sol. Given that $x-4y+2=0$

Area of the triangle formed by the line $ax+by+c=0$ with co-ordinate axes $= \dfrac{c^2}{2|ab|}$

$$= \frac{4}{2|1(-4)|} = \frac{1}{2} sq.units$$

7. Find the equation of the straight line passing through (-4, 5)and cutting off equal non zero intercepts on the co-ordinate axes. (May-10, Mar 2007, May 2008)

Sol. Let the equation of a striaght line is $\frac{x}{a}+\frac{y}{b}=1------(1)$

Given that intercepts are equal

$\therefore a=b$

then **the equation of straight line is** $\frac{x}{a}+\frac{y}{a}=1$

$\Rightarrow x+y=a$ --------(2)

it passes through (-4, 5)

$\therefore\ -4+5=a \quad \Rightarrow a=1$

$\therefore$ Required equation of the straight line is $x+y=1$ $\left[\because from(2)\right]$

8. Find the equation of the straight line passing through the point (3, -4) and making X and Y- intercepts which are in the ratio 2:3. (Mar-08)

Sol. Let **the equation of the straight line be** $\frac{x}{a}+\frac{y}{b}=1$------(1)

Given that, a : b = 2 : 3 $\Rightarrow \frac{a}{b}=\frac{2}{3} \Rightarrow a=\frac{2b}{3}------(2)$

from (1) and (2), $\dfrac{x}{\left(\frac{2b}{3}\right)}+\dfrac{y}{b}=1$

$\Rightarrow \frac{3x}{2b}+\frac{y}{b}=1 \quad \Rightarrow \frac{3x+2y}{2b}=1 \Rightarrow 3x+2y=2b$ ------------(3)

it passes through (3, -4)

9 - 8 = 2b $\Rightarrow 2b=1 \Rightarrow b=\frac{1}{2}$

$\therefore$ The required equation of the straight line is $3x+2y=1$ $\left[\because from(3)\right]$

9. Find the equation of the straight line passing through the points $\left(at_1^2, 2at_1\right)$ and $\left(at_2^2, 2at_2\right)$ (Mar-04)

Sol. Let $A=\left(at_1^2, 2at_1\right)$; $B=\left(at_2^2, 2at_2\right)$

slope of AB, $m=\dfrac{2at_2-2at_1}{at_2^2-at_1^2}$ $\boxed{\because \textit{If } A(x_1,y_1) \textit{ and } B(x_2,y_2) \text{ are two points, then } \textit{slope of } \overline{AB}=\dfrac{y_2-y_1}{x_2-x_1}}$

$$=\frac{2a\left(t_2-t_1\right)}{a\left(t_2^2-t_1^2\right)}=\frac{2\left(t_2-t_1\right)}{\left(t_2-t_1\right)\left(t_2+t_1\right)}=\frac{2}{\left(t_2+t_1\right)} \quad \boxed{\because a^2-b^2=(a-b)(a+b)}$$

The equation of line passing through (x_1,y_1) with slope m is $y-y_1=m(x-x_1)$

$$\Rightarrow y-2at_1=\frac{2}{t_1+t_2}\left(x-at_1^2\right)$$

$$\Rightarrow \left(t_1+t_2\right)y-2at_1\left(t_1+t_2\right)=2x-2at_1^2 \Rightarrow 2x-\left(t_1+t_2\right)y=2at_1^2-2at_1\left(t_1+t_2\right)$$

$$\Rightarrow 2x-\left(t_1+t_2\right)y=2at_1\left(t_1-t_1-t_2\right) \Rightarrow 2x-\left(t_1+t_2\right)y=-2at_1t_2$$

$$\Rightarrow 2x-\left(t_1+t_2\right)y+2at_1t_2=0$$

10. Find the length of the perpendicular drawn from the point (-2, -3) to the straight line $5x-2y+4=0$ (Mar-09)

Sol. Given that $5x-2y+4=0$ ------ (1)

given point $\left(x_1,y_1\right)=\left(-2,-3\right)$

The length of the perpendicular drawn from $\left(-2,-3\right)$ to (1)

$$=\frac{\left|5(-2)-2(-3)+4\right|}{\sqrt{25+4}} \quad \boxed{\because \text{The } \perp^{lr} \text{ distance from } P(x_1,y_1) \text{ to the line } ax+by+c=0 \text{ is } \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}}$$

$$=\frac{\left|-10+6+4\right|}{\sqrt{29}}=0$$

11. Find the distance between the parallel lines $5x-3y-4=0; 10x-6y-9=0$ (Mar-09)

Sol. Given that $5x-3y-4=0 \Rightarrow 10x-6y-8=0$ ------ (1)

$10x-6y-9=0$ ------ (2)

Distance between the parallel lines (1) and (2)

$$=\frac{\left|-8+9\right|}{\sqrt{100+36}} \quad \boxed{\because \text{The distance between the parallel lines } ax+by+c_1=0 \ \& \ ax+by+c_2=0 \text{ is } \frac{|c_1-c_2|}{\sqrt{a^2+b^2}}}$$

$$=\frac{1}{\sqrt{136}}=\frac{1}{2\sqrt{34}} \textit{ units}$$

12. Find the equation of straight line passing through the point (5, 4) and parallel to the line 2x + 3y+ 7 = 0. (March-2013)

Sol. The equation of straight line parallel to the line ax + by + c = 0 is ax + by = k.

The equation of straight line parallel to the line 2x + 3y + 7 = 0 is 2x + 3y =k -------- (1)

(5, 4) passes through the equation (1)

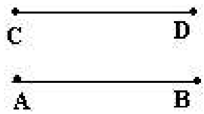
$2(5) + 3(4) = k \quad \Rightarrow k = 22$

substitute k = 22 in equation (1)

$\therefore\ 2x + 3y = 22$

$2x + 3y - 22 = 0$

13. Find the value of y, if the line joining (3, y) and (2, 7) is parallel to the line joining the points (-1, 4) and (0, 6). (March-2014 SAQ) (Mar-08)



Sol. Let A = (3, y); B = (2, 7)

C = (-1, 4); D = (0, 6)

Given that $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$

$\therefore$ **slope of AB = slope of CD**

$$\frac{7-y}{2-3} = \frac{6-4}{0+1} \qquad \boxed{\because \text{ slope of the line AB}(m) = \frac{y_2 - y_1}{x_2 - x_1}}$$

$$\Rightarrow \frac{7-y}{-1} = 2 \Rightarrow 7 - y = -2 \Rightarrow y = 9$$

14. Find the value of k, if the straight lines $6x - 10y + 3 = 0$ and $kx - 5y + 8 = 0$ are parallel.

Sol. Given that $6x - 10y + 3 = 0 \ldots\ldots\ldots(1), \quad kx - 5y + 8 = 0 \ldots\ldots\ldots(2)$

slope of (1), $m_1 = \frac{-6}{-10} = \frac{3}{5}$ $\qquad \boxed{\because \textit{slope of the line } ax + by + c = 0 \textit{ is } m = \frac{-a}{b}}$

slope of (2), $m_2 = \frac{-k}{-5} = \frac{k}{5}$

Given that (1) and (2) are parallel

$\Rightarrow \mathbf{m_1 = m_2} \quad \Rightarrow \frac{3}{5} = \frac{k}{5} \quad \Rightarrow k=3$

15. Find the value of p, if the straight lines $3x + 7y - 1 = 0$ and $7x - py + 3 = 0$ are mutually perpendicular.

Sol. Given that $3x + 7y - 1 = 0$ ------(1), $\quad 7x - py + 3 = 0$ ------(2)

slope of (1), $m_1 = \frac{-3}{7}$ $\qquad \boxed{\because \textit{slope of the line } ax + by + c = 0 \textit{ is } m = \frac{-a}{b}}$

slope of (2), $m_2 = \frac{-7}{-p} = \frac{7}{p}$

Given that (1) and (2) are perpendicular

$\therefore m_1 m_2 = -1$ $\boxed{\because \textit{If } m_1, m_2 \textit{ be the slopes of two non vertical lines are } \perp^{lr} \textit{ then } m_1 m_2 = -1}$

$\Rightarrow \left(\frac{-3}{7}\right)\left(\frac{7}{p}\right) = -1 \quad \Rightarrow p = 3$

16. Find the equation of the straight line passing through (2, 3) and making non-zero intercepts whose sum is zero. (Mar-12)

Sol. Let **the equation of the straight line be** $\frac{x}{a} + \frac{y}{b} = 1 ------(1)$

Given that $a + b = 0 \Rightarrow b = -a$ ------(2)

from (1) and (2)

$\frac{x}{a} + \frac{y}{-a} = 1 \Rightarrow x - y = a ------(3)$

it passes through (2, 3)

2 - 3 = a $\Rightarrow a = -1$

$\therefore$ Required equation of the straight line is $x - y = -1 \left[\because from(3)\right] \Rightarrow x - y + 1 = 0$

17. Find the equation of the straight line passing through (-2, 4) and making non zero intercepts whose sum is zero. (May-09)

Sol. Let **the equation of the straight line be** $\frac{x}{a} + \frac{y}{b} = 1 ------(1)$

Given that a + b = 0 $\Rightarrow b = -a$ ------(2)

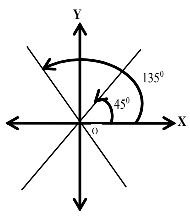
from (1) and (2); $\frac{x}{a} + \frac{y}{-a} = 1 \Rightarrow x - y = a ------(3)$

it passes through (-2, 4); $-2 - 4 = a \Rightarrow a = -6$

$\therefore$ Required equation of the straight line is $x - y = -6 \left[\because from(3)\right] \Rightarrow x - y + 6 = 0$

18. Find the equations of the straight lines passing through the origin and making equal angles with the co-ordinate axes. (May-05)

Sol.



The equation of the straight line which passes through the origin is y = mx ....(1)

case(i) Inclination of the line $\theta = 45^0 \Rightarrow m = \tan 45^0 = 1$ $\boxed{\because m = \tan\theta}$

$\therefore$ Equation of straight line is $y = 1.x \quad \left[\because from(1)\right]$

$\Rightarrow y = x \Rightarrow x - y = 0$

case(ii) Inclination of the other line $\theta = 135^0 \Rightarrow m = \tan 135^0 = -1$

$\therefore$ Equation of straight line is $y = -1.x \left[\because from(1)\right]$

$\Rightarrow y = -x \Rightarrow x + y = 0$

19. Find the value of P, if the straight lines x + p = 0, y + 2 = 0, 3x + 2y + 5 = 0 are concurrent. (March-13)

Sol. Given straight lines x + p = 0, y + 2 = 0, 3x + 2y + 5 = 0 are concurrent then

$$\begin{vmatrix} 1 & 0 & P \\ 0 & 1 & 2 \\ 3 & 2 & 5 \end{vmatrix} = 0 \quad \boxed{\because \textit{If three lines are concurent then the } \det \textit{ volue is } 0}$$

$$\Rightarrow 1(5-4) - 0 + P(0-3) = 0 \quad \Rightarrow 1 - 3P = 0 \quad \Rightarrow P = \frac{1}{3}.$$

20. If $2x - 3y - 5 = 0$ is the perpendicular bisector of the line segment joining (3, -4) and (α, β) then find $\alpha + \beta$. (Mar-11)

Sol. Given that $2x - 3y - 5 = 0$ ----------(1)

(α, β) be the image of (3, -4) w.r.to (1)

We know that (h,k) is the image of (x_1, y_1) w.r.t the line $ax + by + c = 0$ then

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$

$$\therefore \frac{\alpha - 3}{2} = \frac{\beta + 4}{-3} = \frac{-2(6 + 12 - 5)}{4 + 9} \Rightarrow \frac{\alpha - 3}{2} = \frac{\beta + 4}{-3} = \frac{-2(13)}{13} \Rightarrow \frac{\alpha - 3}{2} = \frac{\beta + 4}{-3} = -2$$

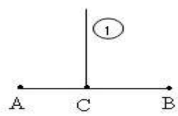
$$\Rightarrow \alpha - 3 = -4 \; ; \; \beta + 4 = 6 \Rightarrow \alpha = -1 \; ; \; \beta = 2$$

$$\therefore \alpha + \beta = 1$$

Alternate Method:

Given that

$$2x - 3y - 5 = 0 \text{ ------}(1)$$



Let A = (3, -4) ; B = (α, β)

Let C be the midpoint of AB

$$\therefore C = \left(\frac{3+\alpha}{2}, \frac{-4+\beta}{2}\right) \quad \boxed{\because \text{If } A(x_1, y_1)\, B(x_2, y_2) \text{ are two points, then mid point of AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)}$$

C lies on (1)

$$2\left(\frac{3+\alpha}{2}\right) - 3\left(\frac{-4+\beta}{2}\right) - 5 = 0 \Rightarrow \frac{6 + 2\alpha + 12 - 3\beta - 10}{2} = 0 \Rightarrow 2\alpha - 3\beta + 8 = 0 \text{ ------}(2)$$

Also, $(1) \perp AB$

slope of (1) $\times$ slope of AB = -1

$$\left(\frac{2}{3}\right)\left(\frac{\beta+4}{\alpha-3}\right)=-1 \Rightarrow 2\beta+8=-3\alpha+9 \Rightarrow 3\alpha+2\beta-1=0 \text{ ------(3)}$$

Solving (2) and (3)

	$\underline{\alpha}$	$\underline{\beta}$	1
-3	8	2	-3
2	-1	3	2

$$\frac{\alpha}{3-16}=\frac{\beta}{24+2}=\frac{1}{4+9} \Rightarrow \frac{\alpha}{-13}=\frac{\beta}{26}=\frac{1}{13} \Rightarrow \alpha=\frac{-13}{13}=-1; \beta=\frac{26}{13}=2$$

$\therefore \alpha+\beta=-1+2=1$

21. Find the value of k, if the straight lines $y-3kx+4=0$ and $(2k-1)x-(8k-1)y-6=0$ are perpendicular. (Mar-10).

Sol. Given that $y-3kx+4=0 \Rightarrow -3kx+y+4=0$ ------(1)

$(2k-1)x-(8k-1)y-6=0$ ------(2)

slope of (1), $m_1=\frac{-(-3k)}{1}=3k$

slope of (2), $m_2=\frac{-(2k-1)}{-(8k-1)}=\frac{2k-1}{8k-1}$

Given that (1) and (2) are perpendicular

$\therefore m_1m_2=-1$ $\boxed{\because \textit{If } m_1, m_2 \textit{ are slopes of two} \perp \textit{ler lines then } m_1m_2=-1}$

$$\Rightarrow 3k\left(\frac{2k-1}{8k-1}\right)=-1 \Rightarrow 6k^2-3k=-8k+1 \Rightarrow 6k^2+5k-1=0$$

$$\Rightarrow 6k^2+6k-k-1=0 \Rightarrow 6k(k+1)-1(k+1)=0 \Rightarrow (k+1)(6k-1)=0$$

$$\Rightarrow k+1=0 (or) 6k-1=0 \Rightarrow k=-1 (or) k=\frac{1}{6}$$

22. If θ is the angle between the lines $\frac{x}{a}+\frac{y}{b}=1, \frac{x}{b}+\frac{y}{a}=1$ then find the value of $\sin\theta(a>b)$. (May-09)

Sol. Given that $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow bx+ay-ab=0$ ------(1)

$\frac{x}{b}+\frac{y}{a}=1 \Rightarrow ax+by-ab=0$ ------(2)

Let θ be the actu angle between (1) and (2)

$$\sin\theta = \frac{|b(b)-a(a)|}{\sqrt{b^2+a^2}\sqrt{a^2+b^2}}$$

$\boxed{\because \textit{If } \theta \textit{ is the angle between the lines } a_1x+b_1y+c_1=0 \textit{ and } a_2x+b_2y+c_2=0 \textit{ then } \sin\theta = \frac{|a_1b_2-a_2b_1|}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}}$

$$= \frac{|b^2-a^2|}{\sqrt{b^2+a^2}\sqrt{a^2+b^2}} \quad = \frac{a^2-b^2}{a^2+b^2}[\because a>b]$$

23. Transform the equation $(2+5k)x-3(1+2k)y+(2-k)=0$ into the form $L_1+\lambda L_2=0$ and find the point of concurrency of the family of straight lines.

Sol. Given that $(2+5k)x-3(1+2k)y+(2-k)=0$

$\Rightarrow 2x+5kx-3y-6ky+2-k=0$

$\Rightarrow (2x-3y+2)+k(5x-6y-1)=0$

It represents a family of concurrent straight lines are $2x-3y+2=0$------(1)

$5x-6y-1=0$---(2)

$\boxed{\because \textbf{The equation of family of lines passing through the point of intersection of } L_1=0 \textit{ and } L_2=0 \textit{ is } L_1+\lambda L_2=0}$

Solving (1) and (2)

	$\underline{x}$	$\underline{y}$	$\underline{1}$	
-3	2	2	-3	
-6	-1	5	-6	

$$\frac{x}{3+12} = \frac{y}{10+2} = \frac{1}{-12+15} \quad \Rightarrow \frac{x}{15} = \frac{y}{12} = \frac{1}{3}$$

$\therefore x=5, y=4$

$\therefore$ Point of concurrency = (5, 4) **(1M)**

24. Find the ratio in which the straight line $2x+3y-20=0$ divides the join of the points (2, 3) and (2, 10).

Sol. Let the given straight line is $L \equiv 2x+3y-20=0$

$(x_1,y_1)=(2,3);(x_2,y_2)=(2,10)$

We know that the ratio in which the line $ax+by+c=0$ divides the line joining of (x_1,y_1) and $(x_2,y_2) = -L_{11}:L_{22}$

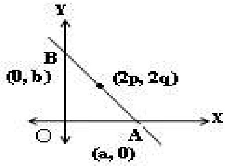
$= -(4+9-20):(4+30-20)$

$= -(-7):14$

$= 7:14 = 1:2$

25. If a portion of a straight line intercepted between the axes of coordinates is bisected at $(2p, 2q)$ write the equation of the straight line.

Sol.



Let the equation of the straight line is $\frac{x}{a}+\frac{y}{b}=1$

$\therefore$ A =(a, 0) ; B =(0, b)

Given that (2p, 2q) bisects $\overline{AB}$

$\therefore$ (2p, 2q) = midpoint of AB

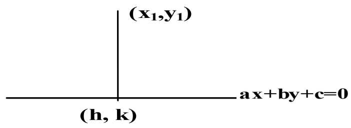
$\Rightarrow (2p, 2q)=\left(\frac{a}{2}, \frac{b}{2}\right)$ $\boxed{\because \text{If } A(x_1, y_1)\, B(x_2, y_2) \text{ are two points, then mid point of AB} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)}$

$\Rightarrow a=4p, b=4q$

$\therefore$ The equation of the straight line is $\frac{x}{4p}+\frac{y}{4q}=1 \quad \Rightarrow \frac{qx+py}{4pq}=1 \Rightarrow qx+py=4pq$

26. Find the foot of the perpendicular drawn from(4, 1) upon the straight line $3x-4y+12=0$.

Sol. Given that $3x-4y+12=0$ ------(1)



Let $(x_1, y_1)=(4,1)$

Let (h, k) be the foot of the perpendicular from (4, 1) on (1)

We know that (h,k) is the foot of the $\perp$ ler from $P(x_1, y_1)$ w.r.t the line $ax+by+c=0$ then $\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-(ax_1+by_1+c)}{a^2+b^2}$

$$\Rightarrow \frac{h-4}{3}=\frac{k-1}{-4}=\frac{-(12-4+12)}{9+16}$$

$$\Rightarrow \frac{h-4}{3}=\frac{k-1}{-4}=\frac{-20}{25} \qquad \Rightarrow \frac{h-4}{3}=\frac{k-1}{-4}=\frac{-4}{5}$$

$$\frac{h-4}{3}=\frac{-4}{5} \quad ; \quad \frac{k-1}{-4}=\frac{-4}{5}$$

$$h-4=\frac{-12}{5} \quad ; \quad k-1=\frac{16}{5}$$

$h = \dfrac{-12}{5} + 4$; $k = \dfrac{16}{5} + 1$

$h = \dfrac{8}{5}$; $k = \dfrac{21}{5}$

$\therefore$ Foot of the perpendicular $= \left(\dfrac{8}{5}, \dfrac{21}{5}\right)$

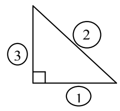
27. Find the orthocentre of the triangle whose sides are given by
$4x - 7y + 10 = 0, x + y = 5 \; and \; 7x + 4y = 15$.

Sol. Given that $4x - 7y + 10 = 0$ ------(1) , $x + y - 5 = 0$ ------(2), $7x + 4y - 15 = 0$ ------(3)

Slope of eq(1) $= \dfrac{4}{7}$ $\boxed{\because \textit{slope of a line } ax + by + c = 0 \textit{ is } \dfrac{-a}{b}}$

Slope of eq(3) $= \dfrac{-7}{4}$

$$slope\ of\ (1) \times slope\ of\ (3) = \left(\frac{4}{7}\right)\left(\frac{-7}{4}\right) = -1$$



$\therefore$ Eq(1) and Eq(3) are perpendicular
$\therefore$ (1),(2),(3) forms a right angled triangle
In right angled triangle, the orthocentre is the point of intersection of perpendicular sides.
Solving (1) and (3),

	x	y	1
-7	10	4	-7
4	-15	7	4

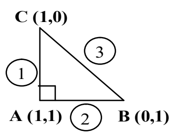
$$\frac{x}{105-40} = \frac{y}{70+60} = \frac{1}{16+49} \Rightarrow \frac{x}{65} = \frac{y}{130} = \frac{1}{65} \Rightarrow x = 1, y = 2$$

$\therefore$ Orthocentre $= (1, 2)$

28. Find the incentre of the triangle whose sides are $x = 1, y = 1, x + y = 1$

Sol. Given that $x = 1$ ------(1), $y = 1$ ------(2), $x + y = 1$ ------(3)

Let the point of intersection of (1) & (2) is A (1, 1)
The point of intersection of (2) and (3) is B (0,1)
The point of intersection of (3) and (1) is C (1,0)



$a = BC = \sqrt{2}$; $b = CA = 1$; $c = AB = 1$

$$\therefore \textbf{Incentre} = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

$$= \left(\frac{\sqrt{2}(1) + 1(0) + 1(1)}{\sqrt{2}+1+1}, \frac{\sqrt{2}(1) + 1(1) + 1(0)}{\sqrt{2}+1+1}\right) = \left(\frac{\sqrt{2}+1}{\sqrt{2}+2}, \frac{\sqrt{2}+1}{\sqrt{2}+2}\right)$$

$$=\left(\frac{\sqrt{2}+1}{\sqrt{2}\left(1+\sqrt{2}\right)},\frac{\sqrt{2}+1}{\sqrt{2}\left(1+\sqrt{2}\right)}\right)=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$

29. If a,b,c are in A.P. then show that $ax+by+c=0$ represents a family of concurrent lines and find the point of concurrency.

Sol. Given that **a,b,c are in A.P.** $\Rightarrow 2b=a+c \Rightarrow c=2b-a$ ------(1)

Also, $ax+by+c=0$ ------(2)

from (1) and (2), $ax+by+2b-a=0 \Rightarrow a(x-1)+b(y+2)=0$

$\therefore$ It represents a family of concurrent lines.It represents the straight lines

$\because$ ***The equation of family of lines passing through the point of intersection of $L_1=0$ and $L_2=0$ is $L_1+\lambda L_2=0$***

$x-1=0$; $y+2=0 \Rightarrow x=1$; $y=-2$

$\therefore$ Point of concurrency = (1, -2)

30. Find the ratio in which the straight line $2x+3y=5$ divides the join of the points (0, 0) and (-2, 1). (Mar-2014)

Sol: Let the given straight line is $L \equiv 2x+3y-5=0$

$(x_1,y_1)=(0,0);(x_2,y_2)=(-2,1)$

We know that the ratio in which the line $ax+by+c=0$ divides the line joining of (x_1,y_1) and $(x_2,y_2)=-L_{11}:L_{22}$

$$=-(0+0-5):(-4+3-5)=-5:6$$

3D-GEOMETRY

31) Find the centroid of the triangle whose vertices are (5,4,6),(1,-1,3) and (4,3,2) (Mar 04)

Sol: Given vertices of triangle $A(x_1,y_1,z_1)=(5,4,6);B(x_2,y_2,z_2)=(1,-1,3);C(x_3,y_3,z_3)=(4,3,2)$

We know that the centroid of a triangle whose vertices are $A(x_1,y_1,z_1),B(x_2,y_2,z_2)$, $C(x_3,y_3,z_3)$ is $G(x,y,z)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$

$$G=\left(\frac{5+1+4}{3},\frac{4-1+3}{3},\frac{6+3+2}{3}\right)=\left(\frac{10}{3},2,\frac{11}{3}\right)$$

32) Find the coordinates of the vertex'c'of ΔABC if its centroid is the orgin and the verices A,B are (1,1,1)and (-2,4,1) respectively

SOL Given two vertices of triangle A (1,1,1) B (-2,4,1)

Let the third vertex be C(x, y, z)

Given Centriod G= (0,0,0)

We know that the centroid of a triangle whose vertices are $A(x_1,y_1,z_1),B(x_2,y_2,z_2)$, $C(x_3,y_3,z_3)$ is $G(x,y,z)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$

$$\Rightarrow (0,0,0)=\left(\frac{1-2+x}{3},\frac{1+4+y}{3},\frac{1+1+z}{3}\right)$$

$$\Rightarrow \frac{x-1}{3}=0,\frac{y+5}{3}=0,\frac{z+2}{3}=0 \Rightarrow x=1, y=-5, z=-2$$

Third vertex C = (1,-5,-2)

33) Find the centroid of the tetrahedron whose vertices are (2,3,-4),(-3,3,-2),(-1,4,2),(3,5,1)

SOL: Given vertices of tetrahedron are

$$A(x_1,y_1,z_1)=(2,3,-4), B(x_2,y_2,z_2)=(-3,3,-2), C(x_3,y_3,z_3)=(-1,4,2), D(x_4y_4z_4)=(3,5,1)$$

We know that the centroid of a tetrahedron whose vertices are $A(x_1,y_1,z_1), B(x_2,y_2,z_2)$, $C(x_3,y_3,z_3)$, $D(x_4,y_4,z_4)$ is $G(x,y,z)=\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4},\frac{z_1+z_2+z_3+z_4}{4}\right)$

$$G=\left(\frac{2-3-1+3}{4},\frac{3+3+4+5}{4},\frac{-4-2+2+1}{4}\right)=\left(\frac{1}{4},\frac{15}{4},\frac{-3}{4}\right)$$

34) If (3,2,-1),(4,1,1) and (6,2,5) are three vertices and (4,2,2) is the centroid of a tetrahedron, find the fourth vertex. (Mar-09)

SOL: given vertices of tetrahedron are A(3,2,-1), B(4,1,1), C(6,2,5)

Let the fourth vertex be D(x,y,z)

Centroid of tetrahedron G = (4,2,2)

We know that the centroid of a tetrahedron whose vertices are $A(x_1,y_1,z_1), B(x_2,y_2,z_2)$, $C(x_3,y_3,z_3)$, $D(x_4,y_4,z_4)$ is $G(x,y,z)=\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4},\frac{z_1+z_2+z_3+z_4}{4}\right)$

$$\Rightarrow (4,2,2)=\left(\frac{3+4+6+x}{4},\frac{2+1+2+y}{4},\frac{-1+1+5+z}{4}\right)$$

$$\Rightarrow (4,2,2)=\left(\frac{13+x}{4},\frac{5+y}{4},\frac{5+z}{4}\right)$$

$$\frac{13+x}{4}=4,\frac{5+y}{4}=2,\frac{5+z}{4}=2 \Rightarrow 13+x=16, 5+y=8, 5+z=8$$

$$\Rightarrow x=3, y=3, z=3$$

Fourth vertex D = (3,3,3)

35) Show that the point A(3,-2,4),B(1,1,1),C(-1,4,-2) are collinear.

Sol: Given points are A(3,-2,4) B(1,1,1) C(-1,4,-2)

$$AB=\sqrt{(3-1)^2+(-2-1)^2+(4-1)^2}$$

$\because$ The distance between two points $A(x_1,y_1,z_1), B(x_2,y_2,z_2)$ is $AB=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

$$=\sqrt{4+9+9}=\sqrt{22}$$

$$BC=\sqrt{(1+1)^2+(1-4)^2+(1+2)^2}=\sqrt{4+9+9}=\sqrt{22}$$

$$AC = \sqrt{(3+1)^2 + (-2-4)^2 + (4+2)^2} = \sqrt{16+36+36} = \sqrt{88} = 2\sqrt{22}$$

$\Rightarrow AB + BC = AC$

$\therefore$ Given points are Collinear. $\boxed{\because \text{If } A, B, C \text{ are collinear iff } AB + BC = CA \text{ or } BC + CA = AB \text{ or } CA + AB = BC}$

36. Show that the points $A(1,2,3)\, B(7,0,1)\, C(-2,3,4)$ are collinear (March 2013)

Sol: The given points are $A(1,2,3)\, B(7,0,1)\, C(-2,3,4)$

$$AB = \sqrt{(1-7)^2 + (2-0)^2 + (3-1)^2}$$

$\boxed{\because \text{The distance between two points } A(x_1, y_1, z_1), B(x_2, y_2, z_2) \text{ is } AB = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}}$

$$= \sqrt{36+4+4} = \sqrt{44} = 2\sqrt{11}$$

$$BC = \sqrt{(7+2)^2 + (0-3)^2 + (1-4)^2} = \sqrt{81+9+9} = \sqrt{99} = 3\sqrt{11}$$

$$CA = \sqrt{(-2-1)^2 + (3-2)^2 + (4-3)^2} = \sqrt{9+1+1} = \sqrt{11} = \sqrt{11}$$

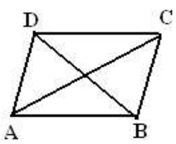
$\Rightarrow AB + CA = BC$

$\therefore$ Given points are collinear $\boxed{\because \text{If } A, B, C \text{ are collinear iff } AB + BC = CA \text{ or } BC + CA = AB \text{ or } CA + AB = BC}$

37) Find the fourth vertex of the parallelogram whose consecutive vertices are (2,4,-1),(3,6,-1)and (4,5,1) (june 2003, mar-11)

Sol: Given three vertices of parallelogram are A(2,4,-1) B(3,6,-1) and C(4,5,1)

Let the fourth vertex be D(x, y, z)



In Parallelogram,Midpoint of $\overline{AC}$ = midpoint of $\overline{BD}$

$$\left(\frac{2+4}{2}, \frac{4+5}{2}, \frac{-1+1}{2}\right) = \left(\frac{3+x}{2}, \frac{6+y}{2}, \frac{-1+z}{2}\right)$$

$\boxed{\because \text{Midpoint of } A(x_1, y_1, z_1) \,\&\, B(x_2, y_2, z_2) \text{ is } \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)}$

$\Rightarrow 3 + x = 6, 6 + y = 9, z - 1 = 0 \Rightarrow x = 3, y = 3, z = 1$

Fourth vertex D = (3,3,1)

38) Find the ratio in which YZ-plane divides the line joining A(2,4,5)andB(3,5,-4).Also find the point of intersection. (May-10)

Sol: Given points $A(x_1, y_1, z_1) = (2,4,5)$; $B(x_2, y_2, z_2) = (3,5,-4)$

YZ-plane divides the line joining $A(x_1, y_1, z_1)$ *and* $B(x_2, y_2, z_2)$ in the ratio $= -x_1 : x_2$

= -2 : 3

Let line AB meet plane at P,

P divides $\overline{AB}$ in the ratio $l : m$ = 2 : 3 externally

$\therefore$ point of intersection $=\left(\frac{2(3)-3(2)}{2-3},\frac{2(5)-3(4)}{2-3},\frac{2(-4)-3(5)}{2-3}\right)=(0,2,23)$

$\because$ If P divides the line segment joining $A(x_1,y_1,z_1)\,B(x_2,y_2,z_2)$ in the ratio $l:m$ externally then $P=\left(\frac{lx_2-mx_1}{l-m},\frac{ly_2-my_1}{l-m},\frac{lz_2-mz_1}{l-m}\right)$

39) Find x if the distance between (5,-1,7) and (x,5,1)is 9 units. (May-11)

Sol: Let point A(5,-1,7) B(x,5,1)

Given that AB = 9 $\Rightarrow AB^2=81$

$\because$ The distance between two points $A(x_1,y_1,z_1),B(x_2,y_2,z_2)$ is $AB=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

$\Rightarrow (x-5)^2+(5+1)^2+(1-7)^2=81$

$\Rightarrow (x-5)^2+36+36=81 \Rightarrow (x-5)^2=9$

$\Rightarrow x-5=\pm 3 \Rightarrow x=5+3=8, x=5-3=2 \Rightarrow x=8(or)x=2$

40) Show that the points(1,2,3),(2,3,1)and(3,1,2) from an equilateral triangle.

Sol: Let the points A (1,2,3) B(2,3,1) C (3,1,2)

$\because$ The distance between two points $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$ is $AB=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

$AB=\sqrt{(1-2)^2+(2-3)^2+(3-1)^2}=\sqrt{1+1+4}=\sqrt{6}$

$BC=\sqrt{(2-3)^2+(3-1)^2+(1-2)^2}=\sqrt{1+4+1}=\sqrt{6}$

$CA=\sqrt{(3-1)^2+(1-2)^2+(2-3)^2}=\sqrt{4+1+1}=\sqrt{6}$

Clearly AB=BC=CA

$\therefore$ Given points form an equilateral triangle.

41. If H,G,S and I respectively denotes orthocentre, centroid, circumcentre and in-centre of a triangle formed by the points $(1,2,3)(2,3,1)$ *and* $(3,1,2)$ then find H,G,,S,I

$\because$ The distance between two points $A(x_1,y_1,z_1),B(x_2,y_2,z_2)$ is $AB=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

Sol: $AB=\sqrt{(2-1)^2+(3-2)^2+(1-3)^2}=\sqrt{6}$

$BC=\sqrt{(3-2)^2+(1-3)^2+(2-1)^2}=\sqrt{6}$

$CA=\sqrt{(1-3)^2+(2-1)^2+(3-2)^2}=\sqrt{6}$

$\therefore AB=BC=CA$

$\therefore$ ABC is an equilateral triangle.

We know that orthocentre(H), centroid(G), circumcentre(S) and incentre(I) of an equiletral triangle are the same

centroid $G=\left(\frac{1+2+3}{3},\frac{2+3+1}{3},\frac{3+1+2}{3}\right)=(2,2,2)$

$\because$ The centroid of a triangle whose vertices are $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$, $C(x_3,y_3,z_3)$ is $G\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$

$\therefore H=(2,2,2),\ S=(2,2,2),\ I=(2,2,2)$

42. Show that the points A(-4,9,6) B(-1,6,6) and C(0,7,10) form a rightangled isosceles triangle.

Sol: Let the points $A(-4,9,6), B(-1,6,6), C(0,7,10)$

$AB=\sqrt{(-4+1)^2+(9-6)^2+(6-6)^2}$

$\because$ The distance between two points $A(x_1,y_1,z_1), B(x_2,y_2,z_2)$ is $AB=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

$=\sqrt{9+9}=3\sqrt{2}$

$BC=\sqrt{(-1-0)^2+(6-7)^2+(6-10)^2}=\sqrt{1+1+16}=3\sqrt{2}$

$CA=\sqrt{(0+4)^2+(7-9)^2+(10-6)^2}=\sqrt{16+4+16}=6$

Clearly $AB=BC$ and $AB^2+BC^2=CA^2$

$\therefore$ Given points form a rightangled isosceles triangle.

43. If the point (1,2,3) is changes to the point (2,3,1) through translation of axes. find the new origin.

Sol: Given $(x,y,z)=(1,2,3)$ *and* $(X,Y,Z)=(2,3,1)$

We know that the origin is shifted to $O'(h,k,l)$ by translation of axes the co-ordinates of (x,y,z) changed to (X,Y,Z) then $x=X+h, y=Y+k, z=Z+l$

$\Rightarrow (h,k,l)=(x-X,y-Y,z-Z)$

$(h,k,l)=(1-2,\ 2-3,\ 3-1)=(-1,-1,2)$

$\therefore O'=(-1,-1,2)$ is the new origin

44. Find the ratio in which the point P(5,4,-6) divides the line segment joining the points A(3,2,-4) and B(9,8,-10) . Also,find the harmonic conjugate of P.

Sol: Let the points $A(3,2,-4)B(9,8,-10)$ and $P(5,4,-6)$

$P(x,y,z)$ divides the line segment $A(x_1,y_1,z_1), B(x_2,y_2,z_2)$ in the ratio $x_1-x:x-x_2$

$=3-5:5-9=1:2$ internally

Let Q be the harmonic conjugate of P then Q divides AB in the ratio 1 : 2 enternally

$=\left(\frac{1(9)-2(3)}{1-2},\frac{1(8)-2(2)}{1-2},\frac{1(-10)-2(-4)}{1-2}\right)$

$\because$ If P divides the line segment joining $A(x_1,y_1,z_1)B(x_2,y_2,z_2)$ in the ratio $l:m$ externally then $P=\left(\frac{lx_2-mx_1}{l-m},\frac{ly_2-my_1}{l-m},\frac{lz_2-mz_1}{l-m}\right)$

$=(-3,-4,2)$

$\therefore Q(-3,-4,2)$ is the harmonic conjugate of $P(5,4,-6)$

THE PLANE

45. Find the angle between the planes x+2y+2z-5=0 and 3x+3y+2z-8=0. (M-09)

Sol Given equation of the planes are x+2y+2z-5=0 and 3x+3y+2z-8=0

Here $a_1 = 1, b_1 = 2, c_1 = 2$
$a_2 = 3, b_2 = 3, c_2 = 2$

We know that the acute angle between two plane $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ is θ then $\cos\theta = \dfrac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$

$$\Rightarrow \cos\theta = \frac{|3+6+4|}{\sqrt{(1+4+4)}\sqrt{(9+9+4)}} = \frac{13}{3\sqrt{22}}$$

$$\therefore \text{Angle between planes } \theta = Cos^{-1}\left(\frac{13}{3\sqrt{22}}\right)$$

46. Find the angle between the planes 2x-y+z=6and $x + y + 2z = 7$. (Mar-11)

Sol: Given equation of the planes are 2x-y+z=6 and $x + y + 2z = 7$

Here $a_1 = 2, b_1 = -1, c_1 = 1$
$a_2 = 1, b_2 = 1, c_2 = 2$

We know that the acute angle between two plane $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ is θ then $\cos\theta = \dfrac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$

$$\Rightarrow \cos\theta = \frac{|2-1+2|}{\sqrt{(4+1+1)}\sqrt{(1+1+4)}} = \frac{3}{6} = \frac{1}{2}$$

$$\Rightarrow Cos\theta = Cos\frac{\pi}{3}$$

$\therefore$ Angle between the planes $\theta = \dfrac{\pi}{3}$

47) Find the equation of the plane whose intercepts on x, y, z axes are 1,2,4 respectively. (Mar-10)

Sol: Given intercepts are a=1,b=2,c=4 on x,y,z axes respectively

We know that the equation of the plane whose x, y,z -intercepts a, b, c is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

$$\Rightarrow \frac{x}{1} + \frac{y}{2} + \frac{z}{4} = 1 \Rightarrow 4x + 2y + z = 4$$

48) Transform the Equation 4x-4y+2z+5=0 into intercept form. (Mar-12)

Sol Given equation of the plane 4x-4y+2z= -5

divide with -5 on both sides

$$\Rightarrow \frac{4x}{-5}-\frac{4y}{-5}+\frac{2z}{-5}=1 \Rightarrow \frac{x}{\left(\frac{-5}{4}\right)}+\frac{y}{\left(\frac{5}{4}\right)}+\frac{z}{\left(\frac{-5}{2}\right)}=1$$

Which is of the form $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

The Intercepts on x, y and z axes respectively are $a=\frac{-5}{4},\ b=\frac{5}{4},\ c=\frac{-5}{2}$

49. Find the intercepts of the plane 4x+3y-2z+2=0 on the coordinate axes. (Mar-11)

Sol Given equation of the plane 4x+3y-2z = -2

divide with -2 on both sides

$$\Rightarrow \frac{4x}{-2}+\frac{3y}{-2}-\frac{2z}{-2}=\frac{-2}{-2} \quad \Rightarrow -2x-\frac{3}{2}y+z=1 \quad \Rightarrow \frac{x}{\left(-\frac{1}{2}\right)}+\frac{y}{\left(-\frac{2}{3}\right)}+\frac{z}{(1)}=1$$

Which is of the from $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

X-intercept $(a)=\frac{-1}{2}$; Y- intercept $(b)=\frac{-2}{3}$; Z-intercept (c) =1

50) Find the direction cosines of the normal to the plane x+2y+2z-4=0. (Mar-13)

Sol Given equation of the plane x+2y+2z=4

Divide with $\sqrt{a^2+b^2+c^2}=\sqrt{1+4+4}=3$ on both sides

$$\frac{1}{3}x+\frac{2}{3}y+\frac{2}{3}z=\frac{4}{3}$$

Which is of the form *lx+my+nz=p* ; *where (l,m,n) are D.C.'s of the plane.*

$\therefore$ Dc's of normal to plane $(l,m,n)=\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$

51. Reduce the Equation x+2y-3z-6=0 of plane in to the normal form.

Sol: Given equation of plane x+2y-3z=6

Divided with $\sqrt{a^2+b^2+c^2}=\sqrt{1+4+9}=\sqrt{14}$ on both sides

$$\left(\frac{1}{\sqrt{14}}\right)x+\left(\frac{2}{\sqrt{14}}\right)y+\left(\frac{-3}{\sqrt{14}}\right)z=\frac{6}{\sqrt{14}}$$

Which is of the form *lx+my+nz = p*

Where (l,m,n) $=\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{-3}{\sqrt{14}}\right)$ are dc's of normal to plane

The perpendicular distance from origin to the given plane $p=\frac{6}{\sqrt{14}}$

52) Find the equation of the plane passing though the point (1,1,1) and parallel to the plane x+2y+3z-7=0. (May-09,10,11)

Sol: Given equation of the plane is x+2y+3z-7=0

here a=1,b=2,c=3 and point $(x_1, y_1, z_1) = (1,1,1)$

We know that the Equation of the plane parallel to plane $ax + by + cz + d = 0$ and passing through (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\Rightarrow 1(x-1)+2(y-1)+3(z-1)=0 \Rightarrow x+2y+3z=6$

53) Find the equation of the plane passing through the point(-2,1,3)and having(3,-5,4)as direction ratio of its normal.

Sol Dr's of normal to plane are (a, b, c)=(3,-5,4)

Given point is (x_1, y_1, z_1)=(-2,1,3)

We know that the Equation of the plane parallel to plane $ax + by + cz + d = 0$ and passing through (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\Rightarrow 3(x+2)-5(y-1)+4(z-3)=0 \Rightarrow 3x-5y+4z-1=0$

54) Find the Equation to the plane parallel to the ZX-plane and passing through (0,4,4).

Sol: **We know that the equation of the plane which is parallel to the ZX-plane is y=k**

It passes through (0,4,4)

$\therefore$ 4=k $\Rightarrow$ y=4

The required equation of the plane is $y - 4 = 0$

55. Find the midpoint of the line joining the points (1,2,3) and (-2, 4, 2) (May-12)

Ans. Given points are A(1,2,3) and B(-2, 4, 2)

Midpoint of line joining the points (x_1, y_1, z_1) *and* (x_2, y_2, z_2) $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

$= \left(\frac{1-2}{2}, \frac{2+4}{2}, \frac{3+2}{2}\right) = \left(\frac{-1}{2}, \frac{6}{2}, \frac{5}{2}\right) = \left(\frac{-1}{2}, 3, \frac{5}{2}\right)$

56. Find the equation of the plane passing through the points (2,0,1) and (3,-3,4) and perpendicular to x-2y+z=6

Sol: Let (a, b, c) are Dr's of normal to required plane.

Equation of plane passing through the point (x_1, y_1, z_1) and having (a,b,c) are Dr's normal is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

It passes through $(2,0,1)$ is $a(x-2)+b(y-0)+c(z-1)=0$ $\longrightarrow$ (1)

It also passes through $(3,-3,4)$ then $a(3-2)+b(-3)+c(4-1)=0$

$a-3b+3c=0$ $\longrightarrow$ (2)

Plane (1) is perpendicula to the plane $x-2y+z=6$

$\Rightarrow a-2b+c=0$ $\longrightarrow$ (3)

Solving (2) & (3) for a, b, c we get $\frac{a}{3} = \frac{b}{2} = \frac{c}{1}$

Substitute these values in equation (1)

$\therefore$ equation of required plane is $3x+2y+z=7$

57. Find the equation of the plane passing through (2,3,4) and perpendicular to X-axis.

Sol: Equation of a plane perpendiculr to x-axis is the plane parallel to YX - plane

Equation of the plane parallel to YZ plane is x = k.

Since it passes through $(2,3,4) \Rightarrow k = 2$

$\therefore$ Equation of required plane is $x = 2$

LIMITS

58. Find $\lim_{x\to0}\left(\frac{\sqrt{1+x}-1}{x}\right)$. **(MAR-04, 07, MAY-10)**

Sol : $\lim_{x\to0}\left(\frac{\sqrt{1+x}-1}{x}\right) = \lim_{x\to0}\left(\frac{\sqrt{1+x}-1}{x}\right)\left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}\right)$ $\left[\because rationlising\ with\ \sqrt{1+x}+1\right]$

$$=\lim_{x\to0}\frac{1+x-1}{x\left(\sqrt{1+x}+1\right)} \qquad \boxed{\because (a-b)(a+b)=a^2-b^2}$$

$$=\frac{1}{\left(\sqrt{1+0}+1\right)}=\frac{1}{2}$$

59. Find $\lim_{x\to0}\left(\frac{3^x-1}{\sqrt{1+x}-1}\right)$ **(MAR-05)**

Sol: $\lim_{x\to0}\left(\frac{3^x-1}{\sqrt{1+x}-1}\right)=\lim_{x\to0}\left(\frac{3^x-1}{\sqrt{1+x}-1}\right)\left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}\right)=\lim_{x\to0}\left(\frac{(3^x-1)\left(\sqrt{1+x}+1\right)}{1+x-1}\right)$

$$=\lim_{x\to0}\left(\frac{3^x-1}{x}\right)\lim_{x\to0}\left(\sqrt{1+x}+1\right)$$

$$=(\log 3)(2)=2\log 3=\log 9 \qquad \boxed{\because \lim_{x\to0}\frac{a^x-1}{x}=\log_e a} \ \boxed{\because \log a^m = m\log a}$$

60. Compute $\lim_{x\to0}\left(\frac{e^x-1}{\sqrt{1+x}-1}\right)$ **(MARCH 2009)**

Sol. $\lim_{x\to0}\left(\frac{e^x-1}{\sqrt{1+x}-1}\right)=\lim_{x\to0}\left(\frac{e^x-1}{\sqrt{1+x}-1}\right)\left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}\right)=\lim_{x\to0}\frac{(e^x-1)\left(\sqrt{1+x}+1\right)}{\left(\sqrt{1+x}\right)^2-(1)^2}$ $\boxed{\because (a-b)(a+b)=a^2-b^2}$

$$=\lim_{x\to0}\frac{(e^x-1)\left(\sqrt{1+x}+1\right)}{1+x-1}=\lim_{x\to0}\left(\frac{e^x-1}{x}\right)\lim_{x\to0}\left(\sqrt{1+x}+1\right)$$

$=(1)\left(\sqrt{1+0}+1\right)$ $\boxed{\because \lim\limits_{x\to 0}\frac{e^x-1}{x}=1}$

$=(1)(2)=2$

61. Compute $\lim\limits_{x\to 0}\frac{a^x-1}{b^x-1}$ $(a>0,\ b>0,\ b\neq 1)$. **(Mar 02,08,13, Jun-02)**

Sol. $\lim\limits_{x\to 0}\frac{a^x-1}{b^x-1}=\lim\limits_{x\to 0}\frac{\left(\frac{a^x-1}{x}\right)}{\left(\frac{b^x-1}{x}\right)}=\frac{\lim\limits_{x\to 0}\left(\frac{a^x-1}{x}\right)}{\lim\limits_{x\to 0}\left(\frac{b^x-1}{x}\right)}$

$=\frac{\log_e a}{\log_e b}$ $\boxed{\because \lim\limits_{x\to 0}\frac{a^x-1}{x}=\log_e a}$

$=\log_b a$

62. Find $\lim\limits_{x\to 0}\frac{\sin(a+bx)-\sin(a-bx)}{x}$ **(Mar-05,12, May 2009)**

Sol. $\lim\limits_{x\to 0}\frac{\sin(a+bx)-\sin(a-bx)}{x}=\lim\limits_{x\to 0}\frac{2\cos a\sin bx}{x}$ $\boxed{\because \sin(A+B)-\sin(A-B)=2\cos A\sin B}$

$=2\cos a\lim\limits_{bx\to 0}\frac{\sin bx}{bx}.b=2\cos a(b)$ $(as\ x\to 0\Rightarrow bx\to 0)$ $\boxed{\because \lim\limits_{x\to 0}\frac{\sin x}{x}=1}$

$=2b\cos a$

63. Find $\lim\limits_{x\to 0}\left(\frac{\cos ax-\cos bx}{x^2}\right)$ **(MARCH 2004, MARCH 2007)**

Sol: $\lim\limits_{x\to 0}\left(\frac{\cos ax-\cos bx}{x^2}\right)=\lim\limits_{x\to 0}\frac{-2\sin\left(\frac{ax+bx}{2}\right)\sin\left(\frac{ax-bx}{2}\right)}{x^2}$ $\boxed{\because \cos C-\cos D=-2\sin\frac{(C+D)}{2}\sin\frac{C-D}{2}}$

$=-2\lim\limits_{x\to 0}\frac{\sin\left(\frac{a+b}{2}\right)x}{x}\lim\limits_{x\to 0}\frac{\sin\left(\frac{a-b}{2}\right)x}{x}=-2\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right)$ $\boxed{\because \lim\limits_{x\to 0}\frac{\sin nx}{x}=n}$

$=-\left(\frac{a^2-b^2}{2}\right)=\left(\frac{b^2-a^2}{2}\right)$ $\boxed{\because (a+b)(a-b)=a^2-b^2}$

64. Find $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$. **(MARCH 2008, MARCH -09,10,11)**

Sol: $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim\limits_{x\to\infty}\frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\left(\sqrt{x^2+x}+x\right)}=\lim\limits_{x\to\infty}\frac{\left(\sqrt{x^2+x}\right)^2-x^2}{\left(\sqrt{x^2+x}+x\right)}$ $\boxed{\because (a-b)(a+b)=a^2-b^2}$

$$= \lim_{x \to \infty} \frac{x^2 + x - x^2}{\sqrt{x^2 + x} + x} = \lim_{x \to \infty} \frac{x}{x\left[\sqrt{1 + \frac{1}{x}} + 1\right]} \qquad \left(\because as\ x \to \infty \Rightarrow \frac{1}{x} \to 0\right)$$

$$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}$$

65. **Find** $\lim_{x \to \frac{\pi}{2}} \frac{Cosx}{\left(x - \frac{\pi}{2}\right)}$ **(JUNE 2005, MARCH 2008)**

Sol: $\lim_{x \to \frac{\pi}{2}} \frac{Cosx}{\left(x - \frac{\pi}{2}\right)} = \lim_{x \to \frac{\pi}{2}} \left(\frac{Sin\left(\frac{\pi}{2} - x\right)}{\left(x - \frac{\pi}{2}\right)} \right)$ $\boxed{\because \sin(90-\theta) = \cos\theta}$

$$= - \lim_{x - \frac{\pi}{2} \to 0} \left(\frac{\sin\left(x - \frac{\pi}{2}\right)}{\left(x - \frac{\pi}{2}\right)} \right) = -1 \qquad \boxed{\because \sin(-\theta) = -\sin\theta} \ \boxed{\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$

66. **Compute** $\lim_{x \to 0} \frac{1 - \cos 2mx}{\sin^2 nx}$ **(MARCH-2010)**

Sol: $\lim_{x \to 0} \frac{1 - \cos 2mx}{\sin^2 nx} = \lim_{x \to 0} \frac{2\sin^2 mx}{\sin^2 nx}$ $\boxed{\because 1 - \cos 2\theta = 2\sin^2\theta}$

$$= 2 \frac{\left(\lim_{mx \to 0} \frac{\sin mx}{mx}\right)^2 (mx)^2}{\left(\lim_{nx \to 0} \frac{\sin nx}{nx}\right)^2 (nx)^2} = 2\, \frac{1\,.m^2}{1\,.n^2} \qquad \boxed{\because \lim_{x \to 0} \frac{\sin x}{x} = 1}$$

$$= 2 \frac{m^2}{n^2}$$

67. **Find** $\lim_{x\to\infty} \dfrac{11x^3-3x+4}{13x^3-5x^2-7}$. **(MAY 2007)**

Sol $$\lim_{x\to\infty}\frac{x^3\left(11-\frac{3x}{x^3}+\frac{4}{x^3}\right)}{x^3\left(13-\frac{5x^2}{x^3}-\frac{7}{x^3}\right)}=\lim_{x\to\infty}\left(\frac{11-\frac{3}{x^2}+\frac{4}{x^3}}{13-\frac{5}{x}-\frac{7}{x^3}}\right)$$

$$=\frac{11-0+0}{13-0-0}=\frac{11}{13} \qquad \left(as\ x\to\infty\Rightarrow\frac{1}{x},\frac{1}{x^2},\frac{1}{x^3}\to 0\right)$$

68. **Evaluate** $\underset{x\to 1}{Lt}\dfrac{\sin(x-1)}{x^2-1}$ **(MAY 2006)**

Sol: $$\lim_{x\to 1}\frac{Sin(x-1)}{x^2-1}=\lim_{x\to 1}\frac{\sin(x-1)}{(x-1)(x+1)} \qquad \boxed{\because \mathbf{a^2-b^2=(a-b)(a+b)}}$$

$$=\left(\lim_{x-1\to 0}\frac{\sin(x-1)}{(x-1)}\right)\left(\lim_{x\to 1}\frac{1}{x+1}\right)$$

$$=1.\frac{1}{2}=\frac{1}{2} \qquad \boxed{\because \lim_{x\to 0}\frac{\sin x}{x}=1}$$

69. **Compute** $\lim_{x\to 0}\dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$ **(MAY 2006)**

Sol: $$\lim_{x\to 0}\frac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}=\lim_{x\to 0}\frac{\sqrt[3]{1+x}-1+1-\sqrt[3]{1-x}}{x} \quad (\because \text{ add and subtrack by 1})$$

$$=\lim_{x\to 0}\frac{\sqrt[3]{1+x}-1}{1+x-1}+\lim_{x\to 0}\frac{\sqrt[3]{1-x}-1}{1-x-1}$$

$$=\lim_{x\to 0}\frac{(1+x)^{\frac{1}{3}}-(1)^{\frac{1}{3}}}{(1+x)-1}+\lim_{x\to 0}\frac{(1-x)^{\frac{1}{3}}-(1)^{\frac{1}{3}}}{(1-x)-1} \quad =\frac{1}{3}(1)^{\frac{1}{3}-1}+\frac{1}{3}(1)^{\frac{1}{3}-1} \quad \boxed{\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=n.a^{n-1}}$$

$$=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$$

70. **Find** $\lim_{x\to a}\dfrac{\sin(x-a)\tan^2(x-a)}{(x^2-a^2)^2}$ **(MARCH 2006)**

Sol: $$\lim_{x\to a}\frac{\sin(x-a)Tan^2(x-a)}{(x^2-a^2)^2}$$

$$=\left(\lim_{x\to a}\frac{\sin(x-a)}{x-a}\right)(x-a).\lim_{x\to a}\frac{Tan^2(x-a)}{(x-a)^2(x+a)^2}\quad \boxed{\because (a-b)(a+b)=a^2-b^2}$$

$$=1.\lim_{x\to a}\frac{Tan^2(x-a)}{(x-a)^2}.\frac{(x-a)}{(x+a)^2}\quad \boxed{\because \lim_{x\to 0}\frac{\sin x}{x}=1}$$

$$=\left(\lim_{x\to a}\frac{Tan(x-a)}{(x-a)}\right)^2.\lim_{x\to a}\frac{(x-a)}{(x+a)^2}$$

$$=1.\frac{0}{(2a)^2}=0\quad \boxed{\because \lim_{x\to 0}\frac{\tan x}{x}=1}$$

71. **Compute** $\lim_{x\to a}\left(\frac{x\sin a-a\sin x}{x-a}\right)$ **(MARCH 2011)**

Sol: $\lim_{x\to a}\left(\frac{x\sin a-a\sin x}{x-a}\right)=\lim_{x\to a}\left(\frac{x\sin a-a\sin a+a\sin a-a\sin x}{x-a}\right)$ [$\because$ Add & subtract with '$a\sin a$']

$$=\lim_{x\to a}\left(\frac{\sin a(x-a)+a(\sin a-\sin x)}{(x-a)}\right)=\lim_{x\to a}\left(\sin a+a.\left(\frac{\sin a-\sin x}{x-a}\right)\right)$$

$$=\sin a-a.\lim_{x\to a}\left(\frac{\sin x-\sin a}{x-a}\right)=\sin a-a\lim_{x\to a}\left(\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a}\right)\quad \boxed{\because \sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}}$$

$$=\sin a-2a.\lim_{x\to a}\cos\left(\frac{x+a}{2}\right).\lim_{x\to a}\frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}.\frac{1}{2}$$

$$=\sin a-a.\cos\left(\frac{a+a}{2}\right).1\quad \boxed{\because \lim_{x\to 0}\frac{\sin x}{x}=1}$$

$$=\sin a-a\cos a$$

72. **Show that** $\lim_{x\to 2^-}\frac{|x-2|}{x-2}=-1$ **(JUNE 2004)**

Sol: $x\to 2^-\Rightarrow x<2\Rightarrow x-2<0$

$\therefore |x-2|=-(x-2)$

$$\lim_{x\to 2^-}\frac{|x-2|}{x-2}=\lim_{x\to 2}\frac{-(x-2)}{(x-2)}=-1$$

73. **Show that** $\lim_{x \to 0^+}\left(\frac{2|x|}{x}+x+1\right)=3$ **(MAY 2008)**

Sol: $\lim_{x\to 0^+}\left(\frac{2|x|}{x}+x+1\right)$

$x \to 0^+ \Rightarrow x > 0 \Rightarrow |x| = x$

$=\lim_{x\to 0^+}\left(\frac{2|x|}{x}+x+1\right)=\lim_{x\to 0}\left(\frac{2x}{x}+x+1\right)=2+0+1=3$

74. **Find** $\lim_{x \to \infty}\frac{8|x|+3x}{3|x|-2x}$. **(MAY 2009, MARCH 2010, MARCH 2012)**

Sol: As $x \to \infty \Rightarrow x > 0$

$\therefore |x| = x$

$\lim_{x\to\infty}\frac{8|x|+3x}{3|x|-2x}=\lim_{x\to\infty}\frac{8x+3x}{3x-2x}=\lim_{x\to\infty}\frac{11x}{x}=11$

75. **Compute** $\lim_{x\to 2}\frac{\left(2x^2-7x-4\right)}{(2x-1)\left(\sqrt{x}-2\right)}$. **(MAY 2007)**

$\lim_{x\to 2}\frac{\left(2x^2-7x-4\right)}{(2x-1)\left(\sqrt{x}-2\right)}=\lim_{x\to 2}\frac{2x^2-8x+x-4}{(2x-1)\left(\sqrt{x}-2\right)}=\lim_{x\to 2}\frac{(2x+1)(x-4)}{(2x-1)\left(\sqrt{x}-2\right)}.\frac{\left(\sqrt{x}+2\right)}{\left(\sqrt{x}+2\right)}$

$=\lim_{x\to 2}\frac{(2x+1)(x-4)\left(\sqrt{x}+2\right)}{(2x-1)(x-4)}$ $\boxed{\because (a-b)(a+b)=a^2-b^2}$

$=\lim_{x\to 2}\frac{(2x+1)\left(\sqrt{x}+2\right)}{(2x-1)}=\frac{5\left(2+\sqrt{2}\right)}{3}$

76. **Compute** $\underset{x\to 0}{lt}\frac{e^{3+x}-e^3}{x}$

Sol: $\underset{x\to 0}{lt}\frac{e^{3+x}-e^3}{x}=\underset{x\to 0}{lt}\frac{e^3.e^x-e^3}{x}$

$=\underset{x\to 0}{lt}\frac{e^3\left(e^x-1\right)}{x}=e^3\,\underset{x\to 0}{lt}\frac{e^x-1}{x}=e^3.1=e^3$ $\boxed{\because \lim_{x\to 0}\frac{e^x-1}{x}=1}$

77. Compute $\underset{x\to 0}{lt}\dfrac{\log_e(1+5x)}{x}$

Sol: $\underset{x\to 0}{lt}\dfrac{\log_e(1+5x)}{x}=\underset{x\to 0}{lt}\dfrac{\log(1+5x)}{5x}\times 5$

$x\to 0 \ as\ 5x\to 0$

$\underset{5x\to 0}{lt}\dfrac{\log(1+5x)}{5x}\times 5 \ =1\times 5=5$ $\boxed{\because \lim\limits_{x\to 0}\dfrac{\log_e(1+x)}{x}=1}$

78. Compute $\underset{x\to 0}{lt}\dfrac{e^{3x}-1}{x}$

Sol:

$\underset{x\to 0}{lt}\dfrac{e^{3x}-1}{x}=\underset{x\to 0}{lt}\dfrac{e^{3x}-1}{3x}\times 3$

$x\to 0 \ as\ 3x\to 0$

$\underset{3x\to 0}{lt}\dfrac{e^{3x}-1}{3x}\times 3=1\times 3=3$ $\boxed{\because \lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1}$

79. Compute $\lim\limits_{x\to 2+}([x]+x)$ **and** $\lim\limits_{x\to 2-}([x]+x)$**.**

Sol: $\lim\limits_{x\to 2+}([x]+x)=\lim\limits_{x\to 2}(2+x)=$ 2+2=4 $\left(\because x\to 2^+\Rightarrow [x]=2\right)$ $\boxed{\because (x-1)<[x]\le x}$

$\lim\limits_{x\to 2-}([x]+x)\ =\lim\limits_{x\to 2}(1+x)=$1 + 2= 3 $\left(\because x\to 2^-\Rightarrow [x]=1\right)$

80. Find $\lim\limits_{x\to -\infty}\left(\dfrac{2x+3}{\sqrt{x^2-1}}\right)$

Sol: $\lim\limits_{x\to -\infty}\left(\dfrac{2x+3}{\sqrt{x^2-1}}\right)=\lim\limits_{x\to -\infty}\dfrac{x\left(2+\dfrac{3}{x}\right)}{|x|\sqrt{1-\dfrac{1}{x^2}}}=\lim\limits_{x\to -\infty}\dfrac{x\left(2+\dfrac{3}{x}\right)}{-x\sqrt{1-\dfrac{1}{x^2}}}$ $\left(\because As\ x\to -\infty, |x|=-x\right)$

$=-\dfrac{2+0}{\sqrt{1-0}}=-2$

81. Compute $\lim\limits_{x\to 2}\dfrac{x-2}{x^3-8}$**.**

Sol: $\lim\limits_{x\to 2}\dfrac{x-2}{x^3-8}=\lim\limits_{x\to 2}\dfrac{(x-2)}{(x-2)(x^2+2x+4)}$ $\boxed{\because a^3-b^3=(a-b)(a^2+ab+b^2)}$

$=\lim\limits_{x\to 2}\dfrac{1}{x^2+2x+4}=\dfrac{1}{4+4+4}=\dfrac{1}{12}$

82. Compute $\lim\limits_{x\to 2}\left(\dfrac{1}{x-2}-\dfrac{4}{x^2-4}\right)$.

Sol: $\lim\limits_{x\to 2}\left(\dfrac{1}{x-2}-\dfrac{4}{x^2-4}\right)=\lim\limits_{x\to 2}\left(\dfrac{1}{(x-2)}-\dfrac{4}{(x+2)(x-2)}\right)$ $\boxed{\because a^2-b^2=(a-b)(a+b)}$

$$=\lim_{x\to 2}\frac{x+2-4}{(x+2)(x-2)}=\lim_{x\to 2}\frac{x-2}{(x+2)(x-2)}=\frac{1}{2+2}=\frac{1}{4}$$

83. Compute $\lim\limits_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)$.

Sol: $\lim\limits_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)=\lim\limits_{x\to\infty}\dfrac{\left(\sqrt{x+1}-\sqrt{x}\right)\times\left(\sqrt{x+1}+\sqrt{x}\right)}{\left(\sqrt{x+1}+\sqrt{x}\right)}$ [$\because$ rationalize with $\sqrt{x+1}+\sqrt{x}$]

$$=\lim_{x\to\infty}\frac{x+1-x}{\sqrt{x+1}+\sqrt{x}}$$ $\boxed{\because (a-b)(a+b)=a^2-b^2}$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{x}\left(\sqrt{1+\frac{1}{x}}+1\right)}=\lim_{\frac{1}{x}\to 0}\frac{\sqrt{\frac{1}{x}}}{\sqrt{1+\frac{1}{x}}+1}$$ $\boxed{\because \text{as } x\to\infty\Rightarrow\frac{1}{x}\to 0}$

$$=\frac{0}{\sqrt{1+0}+1}=\frac{0}{2}=0$$

84. Compute $\lim\limits_{x\to 0}\dfrac{1-\cos mx}{1-\cos nx}, n\neq 0$

Sol: $\lim\limits_{x\to 0}\dfrac{1-\cos mx}{1-\cos nx}$

$$=\lim_{x\to 0}\frac{2\sin^2\left(\frac{mx}{2}\right)}{2\sin^2\left(\frac{nx}{2}\right)}$$ $\boxed{\because 1-\cos\theta=2\sin^2\frac{\theta}{2}}$

$$=\frac{\left(\lim\limits_{\frac{mx}{2}\to 0}\dfrac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right)^2.\left(\frac{mx}{2}\right)^2}{\left(\lim\limits_{\frac{nx}{2}\to 0}\dfrac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right)^2\left(\frac{nx}{2}\right)^2}=\frac{1.\frac{m^2}{4}}{1.\frac{n^2}{4}}=\frac{m^2}{n^2}$$ $\boxed{\because \lim\limits_{x\to 0}\frac{\sin x}{x}=1}$

85. Show that $\lim_{x\to 3}\frac{x-3}{\sqrt{|x^2-9|}}=0$

Sol: Given $\lim_{x\to 3}\frac{x-3}{\sqrt{|x^2-9|}}$

$$For\ x^2\neq 9, \lim_{x\to 3}\left|\frac{x-3}{\sqrt{x^2-9}}\right| = \lim_{x\to 3}\left|\frac{\left(\sqrt{x-3}\right)\left(\sqrt{x-3}\right)}{\left(\sqrt{x-3}\right)\left(\sqrt{x+3}\right)}\right| \qquad \boxed{\because a^2-b^2=(a-b)(a+b)}$$

$$=\lim_{x\to 3}\sqrt{\frac{x-3}{x+3}}=\sqrt{\frac{3-3}{3+3}}=0$$

86. Compute $\lim_{x\to a}\frac{\tan(x-a)}{x^2-a^2}$

Sol: $$\lim_{x\to a}\frac{\tan(x-a)}{x^2-a^2}=\lim_{x\to a}\frac{\tan(x-a)}{(x-a)(x+a)} \qquad \boxed{\because a^2-b^2=(a-b)(a+b)}$$

$$=\lim_{(x-a)\to 0}\frac{\tan(x-a)}{(x-a)}\lim_{x\to a}\frac{1}{x+a}=\frac{1}{a+a}=\frac{1}{2a} \qquad \boxed{\because \lim_{x\to 0}\frac{\tan x}{x}=1}$$

87. Compute $\lim_{x\to\infty}\frac{x^2-\sin x}{x^2-2}$

Sol : $$\lim_{x\to\infty}\frac{x^2-\sin x}{x^2-2}=\lim_{x\to\infty}\frac{x^2\left(1-\frac{\sin x}{x^2}\right)}{x^2\left(1-\frac{2}{x^2}\right)} = \underset{\frac{1}{x}\to 0}{Lt}\left(\frac{1-\left(\frac{sinx}{x}\right)\left(\frac{1}{x}\right)}{1-2\left(\frac{1}{x}\right)^2}\right)$$

$$=\frac{1-0}{1-0}=\frac{1}{1}=1 \quad \left(as\ x\to\infty \Rightarrow \frac{1}{x},\frac{1}{x^2}\to 0\right) \qquad \boxed{\because \lim_{x\to 0}\frac{\sin x}{x}=1}$$

88. Show that $\lim_{x\to 0}\frac{|x|}{x}$ **does not exist**

Sol: We know that $|x|=\begin{cases} x\ if\ x\geq 0 \\ -x\ if\ x<0\end{cases}$

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0}\frac{x}{x}=\lim_{x\to 0}(1)=1$$

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0}\frac{-x}{x}=\lim_{x\to 0}(-1)=-1$$

$\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$

$\therefore \lim_{x\to 0}\frac{|x|}{x}$ does not exist

89. Compute $\lim_{x\to 0}\frac{\sin ax}{x\cos x}$

Sol: $\lim_{x\to 0}\frac{\sin ax}{x\cos x} = \lim_{x\to 0}\frac{\sin ax}{x}\lim_{x\to 0}\frac{1}{\cos x}$

$= \left(\lim_{ax\to 0}\frac{\sin ax}{ax}\right)(a)\frac{1}{\cos 0} = 1.a.1 = a$ $\boxed{\because \lim_{x\to 0}\frac{\sin x}{x} = 1}$

90. Compute $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$, $b \neq 0,\ a \neq b$

Sol: $\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\left(\frac{\sin ax}{ax}.\frac{bx}{\sin bx}.\frac{a}{b}\right)$

$= \frac{a}{b}\lim_{ax\to 0}\frac{\sin ax}{ax}.\lim_{bx\to 0}\frac{bx}{\sin bx}$ $\left(\because as\ x\to 0, ax\to 0\ and\ bx\to 0\right)$

$= \frac{a}{b}.1.1 = \frac{a}{b}$ $\boxed{\because \lim_{x\to 0}\frac{\sin x}{x} = 1}$

91. Evalute $\lim_{x\to 1}\frac{\log_e x}{x-1}$

Sol: Put $x-1 = y$ $as\ \ x\to 1 then\ x-1\to 0$

$\Rightarrow y\to 0$

$\lim_{x\to 1}\frac{\log_e x}{x-1} = \lim_{y\to 0}\frac{\log_e(1+y)}{y} = 1$ $\boxed{\because \lim_{x\to 0}\frac{\log_e(1+x)}{x} = 1}$

92. Compute $\underset{x\to 0}{Lt}\frac{e^x - \sin x - 1}{x}$ **(Mar-2013)**

Sol: $\underset{x\to 0}{Lt}\frac{e^x - \sin x - 1}{x} = \underset{x\to 0}{Lt}\frac{e^x - 1}{x} - \underset{x\to 0}{Lt}\frac{\sin x}{x}$

$= 1 - 1 = 0$ $\boxed{\because \lim_{x\to 0}\frac{e^x - 1}{x} = 1}$ $\boxed{\because \lim_{x\to 0}\frac{\sin x}{x} = 1}$

DIFFERENTIATION

93. **If** $y = ax^{n+1} + bx^{-n}$ $then$ $P.T$ $x^2 y'' = n(n+1)y$. **(MAR-06, MAY-10)**

Sol. Given $y = ax^{n+1} + bx^{-n}$ ....(1) .

Differentiating w.r.t "x" on both sides

$$\frac{dy}{dx} = (n+1)ax^{n+1-1} + (-n)bx^{-n-1}$$ $\boxed{\because \frac{d}{dx}(x^n) = nx^{n-1}}$

$$\frac{dy}{dx} = a(n+1)x^n - bnx^{-n-1}$$

Again differentiating w.r.t. on both sides

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = na(n+1)x^{n-1} + bn(n+1)x^{-n-1-1}$$ $\boxed{\because \frac{d}{dx}(x^n) = nx^{n-1}}$

$$\Rightarrow \frac{d^2y}{dx^2} = n(n+1)\left[ax^{n-1} + bx^{-n-2}\right]$$

Multiply with x^2 on both sides

$$\Rightarrow x^2\frac{d^2y}{dx^2} = n(n+1)\left[a.x^{n-1}x^2 + bx^{-n-2}.x^2\right]$$

$$\Rightarrow x^2 y^{11} = n(n+1)\left[ax^{n-1+2} + bx^{-n-2+2}\right] = n(n+1)\left[ax^{n+1} + bx^{-n}\right]$$

$$\therefore x^2 y^{11} = n(n+1)y \qquad \text{(from (1))}$$

94. **If** $y = \sec\left(\sqrt{\tan x}\right)$, **Find** $\frac{dy}{dx}$ **(MAY 2007)**

Sol. Given $y = \sec\left(\sqrt{\tan x}\right)$

Differentiating w.r.t "x" on both sides

$$\frac{d}{dx}(y) = \frac{d}{dx}\sec\left(\sqrt{\tan x}\right) = \sec\left(\sqrt{\tan x}\right).\tan\left(\sqrt{\tan x}\right)\frac{d}{dx}\sqrt{\tan x}$$ $\boxed{\because \frac{d}{dx}(\sec x) = \sec x \tan x}$

$$= \sec\sqrt{\tan x}.\tan\left(\sqrt{\tan x}\right)\frac{1}{2\sqrt{\tan x}}\frac{d}{dx}\tan x$$ $\boxed{\because \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}}$

$$= \sec\sqrt{\tan x}.\tan\left(\sqrt{\tan x}\right).\frac{1}{2\sqrt{\tan x}}\sec^2 x$$ $\boxed{\because \frac{d}{dx}(\tan x) = \sec^2 x}$

$$= \frac{\sec^2 x}{2\sqrt{\tan x}}.\sec\sqrt{\tan x}.\tan\sqrt{\tan x}$$

95. Find the derivative of the function $f(x)=a^x.e^{x^2}$ **(MAY 2008)**

Sol. Given $f(x)=a^x.e^{x^2}$

Diff. w.r.to. 'x' we get

$$f^{|}(x)=a^x\left(e^{x^2}\right)^{|}+e^{x^2}\left(a^x\right)^{|} \quad \boxed{\because (uv)^{|}=uv^{|}+vu^{|}}$$

$$f^{|}(x)=a^x\left(e^{x^2}\right)2x+e^{x^2}\left(a^x\right)\log a \quad \boxed{\because \frac{d}{dx}\left(e^x\right)=e^x} \boxed{\because \frac{d}{dx}\left(x^n\right)=nx^{n-1}} \boxed{\because \frac{d}{dx}\left(a^x\right)=a^x.\log a}$$

$$\Rightarrow f^{|}(x)=a^xe^{x^2}\left[2x+\log a\right]$$

96. If $f(x)=7^{x^3+3x}(x>0)$, then find $f^1(x)$. **(MAY 2005)**

Sol. Given $f(x)=7^{x^3+3x}$

$$\Rightarrow f^{|}(x)=7^{x^3+3x}\log 7\frac{d}{dx}\left(x^3+3x\right) \quad \boxed{\because \frac{d}{dx}\left(a^x\right)=a^x.\log a}$$

$$=7^{x^3+3x}\log 7\left(3x^2+3\right) \quad \boxed{\because \frac{d}{dx}\left(x^n\right)=nx^{n-1}}$$

$$=7^{x^3+3x}3\left(x^2+1\right)\log 7$$

97. If $x=\tan\left(e^{-y}\right)$, then show that $\frac{dy}{dx}=\frac{-e^y}{1+x^2}$. **(MARCH 2005)**

Sol. Given $x=\tan\left(e^{-y}\right)\Rightarrow \tan^{-1}(x)=e^{-y}$

Diff. w.r.to 'x' we get

$$\Rightarrow \frac{d}{dx}\left(\tan^{-1}(x)\right)=\frac{d}{dx}\left(e^{-y}\right)$$

$$\Rightarrow \frac{1}{1+x^2}=-e^{-y}.\frac{dy}{dx} \quad \boxed{\because \frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}} \boxed{\frac{d}{dx}\left(e^y\right)=e^y\frac{dy}{dx}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\frac{1}{1+x^2}}{-e^{-y}}=\frac{-e^y}{1+x^2}$$

98. If $f(x)=\log(\sec x+\tan x)$, find $f^1(x)$ **[March-2014, May -11]**

Sol. Given $f(x)=\log(\sec x+\tan x)$

Diff. w.r.to 'x' we get

$$f^1(x)=\frac{1}{\sec x+\tan x}\times\frac{d}{dx}\left(\sec x+\tan x\right) \quad \boxed{\because \frac{d}{dx}(\log x)=\frac{1}{x}}$$

$= \frac{1}{\sec x + \tan x} \times \sec x . \tan x + \sec^2 x$ $\boxed{\because \frac{d}{dx}(\sec x) = \sec x \tan x}$ $\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$

$= \frac{\sec(\tan x + \sec x)}{\sec x + \tan x} = \sec x$

99. **If $y = \left(Cot^{-1}x^3\right)^2$, find $\frac{dy}{dx}$.** **(MAY-09)**

Sol. Put $u = \cot^{-1} x^3$ so that $y = u^2$

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$;

$\frac{dy}{dx} = 2u \times \frac{-1}{\left(1 + x^6\right)} .3x^2$ $\boxed{\because \frac{d}{dx}(x^n) = nx^{n-1}}$ $\boxed{\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}}$

$\Rightarrow \frac{dy}{dx} = -6x^2 \frac{\cot^{-1}\left(x^3\right)}{1 + x^6}$

100. **If $y = \log\left(\sin^{-1}\left(e^x\right)\right)$ then find $\frac{dy}{dx}$.** **(MAR-10)**

Sol. Given $y = \log\left(\sin^{-1}\left(e^x\right)\right)$

Diff. w.r.to 'x' we get

$\frac{dy}{dx} = \frac{1}{\sin^{-1}\left(e^x\right)} . \left(\sin^{-1}\left(e^x\right)\right)^{|}$ $\boxed{\because \frac{d}{dx}(\log x) = \frac{1}{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin^{-1}\left(e^x\right)} . \frac{1}{\sqrt{1-\left(e^x\right)^2}} \left(e^x\right)^{|}$ $\boxed{\because \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin^{-1}\left(e^x\right) . \sqrt{1-\left(e^x\right)^2}} \left(e^x\right)$ $\boxed{\frac{d}{dx}(e^x) = e^x}$

$\Rightarrow \frac{dy}{dx} = \frac{\left(e^x\right)}{\sin^{-1}\left(e^x\right) . \sqrt{1 - e^{2x}}}$

101. **If $f(x) = x^2 2^x \log x (x > 0)$, then find $f^{|}(x)$** **(MAY-10)**

Sol. Given $f(x) = x^2 2^x \log x (x > 0)$

Diff. w.r.to. 'x' we get

$f^{|}(x) = x^2 2^x (\log x)^{|} + x^2 \log x \left(2^x\right)^{|} + 2^x \log x \left(x^2\right)^{|}$ $\boxed{\because (uvw)^{|} = uvw^{|} + uwv^{|} + vwu^{|}}$

$= x^2 2^x \frac{1}{x} + x^2 \log x \; 2^x \log 2 + 2^x \log x (2x)$ $\boxed{\because \frac{d}{dx}(\log x) = \frac{1}{x}}$ $\boxed{\frac{d}{dx}(a^x) = a^x . \log a}$ $\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$

$= x2^x + x^2 \log x\ 2^x \log 2 + 2^x \log x(2x)$

$f^{|}(x) = x2^x[1 + x\log x \log 2 + 2\log x]$

102. If $y = \cos[\log(\cot x)]$, then find $\frac{dy}{dx}$ (Mar-09)

Sol. Given $y = \cos[\log(\cot x)]$

Diff. w.r.to. 'x' we get

$\frac{dy}{dx} = -\sin[\log(\cot x)](\log(\cot x))^{|}$ $\boxed{\because \frac{d}{dx}(\cos x) = -\sin x}$

$= -\sin[\log(\cot x)]\frac{1}{\cot x}(\cot x)^{|}$ $\boxed{\because \frac{d}{dx}(\log x) = \frac{1}{x}}$

$= \frac{-\sin[\log(\cot x)]}{\cot x}(-\cos ec^2 x)$ $\boxed{\because \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$

$= \frac{\cos ec^2 x.\sin[\log(\cot x)]}{\cot x}$

103. If $y = \log(\cosh 2x)$ then find $\frac{dy}{dx}$ (Mar-12)

Sol. Given $y = \log(\cosh 2x)$

Diff w.r.to. 'x' we get

$\frac{dy}{dx} = \frac{1}{\cosh 2x}(\cosh 2x)^{|}$ $\boxed{\because \frac{d}{dx}(\log x) = \frac{1}{x}}$

$= \frac{1}{\cosh 2x}(\sinh 2x)(2)$ $\boxed{\because \frac{d}{dx}(\cosh x) = \sinh x}$

$= 2\frac{\sinh 2x}{\cosh 2x} = 2\tanh 2x$ $\boxed{\because \tanh\theta = \frac{\sinh\theta}{\cosh\theta}}$

104. Find $\frac{dy}{dx}$, if $y = \sin^{-1}\sqrt{x}$ (March-2013)

Sol: Given $y = \sin^{-1}\sqrt{x}$

Diff w.r.to. 'x' we get

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}}.\frac{d}{dx}(\sqrt{x})$ $\boxed{\because \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}}$

$= \frac{1}{\sqrt{1-x}}.\left(\frac{1}{2\sqrt{x}}\right)$ $\boxed{\because \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}}$

$= \frac{1}{2\sqrt{x(1-x)}} = \frac{1}{2\sqrt{x-x^2}}$

105. Find the derivative of $f(x)=\frac{ax+b}{cx+d}$ (Mar-12)

Sol: Diff w.r.to. 'x' we get

$$y=f(x)=\frac{ax+b}{cx+d}$$

$$\frac{dy}{dx}=\frac{(cx+d)a-(ax+b)c}{(cx+d)^2}$$ $\boxed{\because \left(\frac{u}{v}\right)^{|}=\frac{uv^{|}-vu^{|}}{v^2}}$ $\boxed{\frac{d}{dx}(k)=0}$ $\boxed{\frac{d}{dx}(x)=1}$

$$=\frac{acx+ad-acx-bc}{(cx+d)^2}=\frac{ad-bc}{(cx+d)^2}$$

106. $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ **find** $\frac{dy}{dx}$ **(March 2013)**

Sol: Given $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

put $x=\cos\theta$

$\Rightarrow \theta=\cos^{-1}(x)$

$\therefore y=\sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right)=\sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$ $\boxed{\because 2\cos^2 A-1=\cos 2A}$

$=\sec^{-1}(\sec 2\theta)=2\theta$ $\boxed{\because \sec^{-1}(\sec\theta)=\theta}$

$y=2.\cos^{-1}x$

Differentation w.r.to x

$\therefore \frac{dy}{dx}=\frac{2.(-1)}{\sqrt{1-x^2}}=\frac{-2}{\sqrt{1-x^2}}$ $\boxed{\because \frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}}$

107. If $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$**, then find** $\frac{dy}{dx}$

Sol. Given $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$\Rightarrow y=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)$ $\boxed{\because \sin 2A=\frac{2\tan A}{1+\tan^2 A}}$

$=2\theta=2\tan^{-1}x$ $\boxed{\because \sin^{-1}(\sin\theta)=\theta}$

Diff. w.r.to 'x' we get

$\frac{dy}{dx}=2.\frac{1}{1+x^2}=\frac{2}{1+x^2}$ $\boxed{\because \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}}$

108. **If** $x = a\cos^3 t,\ y = a\ \sin^3 t$, **find** $\frac{dy}{dx}$**.**

Sol: Given that $x = a\cos^3 t,\ y = a\sin^3 t$

Now $x = a(\cos t)^3$

Diff b.s. w.r.to t then , $\frac{d}{dt}x = a\frac{d}{dt}(\cos t)^3$

$\Rightarrow \frac{dx}{dt} = a.3(\cos t)^2.\frac{d}{dt}\cos t \Rightarrow \frac{dx}{dt} = 3a\cos^2 t(-\sin t)$ $\boxed{\because \frac{d}{dx}(x^n) = nx^{n-1}}$ $\boxed{\because \frac{d}{dx}(\cos x) = -\sin x}$

$\therefore \frac{dx}{dt} = -3a\cos^2 t\sin t\ -(1)$

Now $y = a(\sin t)^3$

Diff b.s. w.r.to t then

$\frac{d}{dt}y = a\frac{d}{dt}(\sin t)^3 \Rightarrow \frac{dy}{dt} = a.3(\sin t)^2.\frac{d}{dt}(\sin t)$ $\boxed{\because \frac{d}{dx}(x^n) = nx^{n-1}}$

$\therefore \frac{dy}{dt} = 3a\sin^2 t(\cos t)\ -(2)$ $\boxed{\because \frac{d}{dx}(\sin x) = \cos x}$

From (1) and (2), $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$;

$$\frac{dy}{dx} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t}$$

$\therefore \frac{dy}{dx} = -\tan t$ $\boxed{\because \tan\theta = \frac{\sin\theta}{\cos\theta}}$

109. $y = \log(\sin(\log x))$, **find** $\frac{dy}{dx}$

Sol Given $y = \log(\sin(\log x))$

Diff.w r.to 'x' we get

$\frac{dy}{dx} = \frac{1}{\sin(\log x)}(\sin(\log x))^{|}$ $\boxed{\because \frac{d}{dx}(\log x) = \frac{1}{x}}$

$= \frac{1}{\sin(\log x)}.\cos(\log x)(\log x)^{|}$ $\boxed{\because \frac{d}{dx}(\sin x) = \cos x}$

$= \frac{\cos(\log x)}{\sin(\log x)}.\frac{1}{x}$

$\frac{dy}{dx} = \frac{\cot(\log x)}{x}$ $\boxed{\because \cot\theta = \frac{\cos\theta}{\sin\theta}}$

110. If $y = x^x$ then find $\frac{dy}{dx}$.

Sol. given y = x^x

take 'log' on both sides we get

$\log y = \log\left(x^x\right) \Rightarrow \log y = x\log x$ $\boxed{\because \log x^n = n\log x}$

Diff. w.r.to 'x'

$\frac{1}{y}\frac{dy}{dx} = x\left(\log x\right)^{|} + \log x\left(x\right)^{|}$ $\boxed{\because \left(uv\right)^{|} = uv^{|} + vu^{|}}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \log x\left(1\right)$ $\boxed{\because \frac{d}{dx}(x) = 1}$ $\boxed{\because \frac{d}{dx}(\log x) = \frac{1}{x}}$

$\Rightarrow \frac{dy}{dx} = y\left[1 + \log x\right]$

$\Rightarrow \frac{dy}{dx} = x^x\left[1 + \log x\right]$

111. If $x^3 + y^3 - 3axy = 0$, find $\frac{dy}{dx}$. **(APR 2000)**

Sol. Given $x^3 + y^3 - 3axy = 0$

Differentiating w.r.to "x" on both sides

$3x^2 + 3y^2\frac{dy}{dx} - 3a\left[x\frac{dy}{dx} + y\right] = 0$ $\boxed{\because \left(uv\right)^{|} = uv^{|} + vu^{|}}$ $\boxed{\because \frac{d}{dx}\left(x^n\right) = nx^{n-1}}$ $\boxed{\because \frac{d}{dx}\left(y^n\right) = ny^{n-1}\frac{dy}{dx}}$

$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} - 3ay = 0$

$\Rightarrow 3\left[x^2 + y^2\frac{dy}{dx} - ax\frac{dy}{dx} - ay\right] = 0$

$\Rightarrow \left(x^2 - ay\right) + \frac{dy}{dx}\left(y^2 - ax\right) = 0 \Rightarrow \frac{dy}{dx}\left(y^2 - ax\right) = -\left(x^2 - ay\right) = \left(ay - x^2\right) \Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$

112. Find the derivative of the following functions w.r.to x.

i) $Cos^{-1}\left(4x^3 - 3x\right)$ **ii)** $Tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$ **(JUNE 2002)**

Sol. (i) Let y= $\cos^{-1}\left(4x^3 - 3x\right)$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$y = \cos^{-1}\left(4\cos^3\theta - 3\cos\theta\right) \Rightarrow y = \cos^{-1}\left(\cos 3\theta\right) \Rightarrow y = 3\theta \Rightarrow y = 3\cos^{-1}x$

Diff. w.r.to x on both sides $\frac{d}{dx}y = 3\frac{d}{dx}\cos^{-1}x$

$\Rightarrow \frac{dy}{dx} = 3.\frac{-1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}$ $\boxed{\because \frac{d}{dx}\left(\cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}}}$

(ii) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

Let $y = \tan^{-1}\left(\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}\right) = \tan^{-1}\left(\frac{\sqrt{2\sin^2 x/2}}{\sqrt{2\cos^2 x/2}}\right)$ $\boxed{\because 1-\cos A = 2\sin^2\frac{A}{2}}$ $\boxed{1+\cos A = 2\cos^2\frac{A}{2}}$

$= \tan^{-1}\left(\frac{\sin x/2}{\cos x/2}\right)$

$= \tan^{-1}\left(\tan\frac{x}{2}\right)$ $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

$\Rightarrow y = \frac{x}{2}$ $\boxed{\because \tan^{-1}(\tan\theta) = \theta}$

Diff. both sides w.r.to x then $\frac{d}{dx}(y) = \frac{d}{dx}\frac{1}{2}.x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}.\frac{d}{dx}x \Rightarrow \frac{1}{2}.1 = \frac{1}{2}$ $\boxed{\frac{d}{dx}(x) = 1}$

113. Differentiate $f(x)$ with respect to $g(x)$ if $f(x) = e^x,\ g(x) = \sqrt{x}$ (MARCH 2003)

Sol. Given $f(x) = e^x,\ g(x) = \sqrt{x}$

$f(x) = e^x \ -(1);\ g(x) = \sqrt{x}\ -(2)$

Diff (1) w.r.to x then $f^{|}(x) = e^x$ $\boxed{\because \frac{d}{dx}\left(e^x\right) = e^x}$

Diff (2) w.r.to x then $g^{|}(x) = \frac{1}{2\sqrt{x}}$ $\boxed{\because \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}}$

Now diff. f(x) w.r to ‘g(x)’

$$\frac{f^{|}(x)}{g^{|}(x)} = \frac{e^x}{\frac{1}{2\sqrt{x}}} = 2e^x\sqrt{x}$$

114. Find the derivative of the following functions w.r.to x.

i) $Tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ **(JUNE 2003)** ii) $\tan^{-1}\left(\frac{1+x}{1-x}\right)$ **(May-12).**

Sol. (i) Let y= $Tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$y=\tan^{-1}\left(\dfrac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)\Rightarrow y=\tan^{-1}\left(\dfrac{\sec\theta-1}{\tan\theta}\right)$ $\boxed{\because \sec\theta=\sqrt{1+\tan^2\theta}}$

$\Rightarrow y=\tan^{-1}\left(\dfrac{\sec\theta}{\tan\theta}-\dfrac{1}{\tan\theta}\right)$

$\Rightarrow y=\tan^{-1}\left(\dfrac{1}{\cos\theta}\times\dfrac{\cos\theta}{\sin\theta}-\dfrac{\cos\theta}{\sin\theta}\right)\Rightarrow y=\tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$ $\boxed{\because \sec\theta=\dfrac{1}{\cos\theta}}$ $\boxed{\tan\theta=\dfrac{\sin\theta}{\cos\theta}}$

$\Rightarrow \quad y=\tan^{-1}\left(\dfrac{2\sin^2\theta/2}{2\sin\theta/2\,\cos\theta/2}\right)$ $\boxed{\because 1-\cos A=2\sin^2\dfrac{A}{2}}$ $\boxed{\sin A=2\sin\dfrac{A}{2}\cos\dfrac{A}{2}}$

$\Rightarrow y=\tan^{-1}\left(\tan\dfrac{\theta}{2}\right)$

$\Rightarrow y=\dfrac{\theta}{2}\Rightarrow y=\dfrac{1}{2}\tan^{-1}x$ $\boxed{\because \tan^{-1}(\tan\theta)=\theta}$

Diff. both sides w.r.to x then $\dfrac{d}{dx}y=\dfrac{1}{2}\dfrac{d}{dx}\tan^{-1}x$

$\dfrac{dy}{dx}=\dfrac{1}{2}.\dfrac{1}{1+x^2}\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{2(1+x^2)}$ $\boxed{\because \dfrac{d}{dx}(\tan^{-1}x)=\dfrac{1}{1+x^2}}$

(ii) Let $y=\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

Put $x=\tan\theta \quad \Rightarrow\theta=\tan^{-1}(x)$

$\therefore\ y=\tan^{-1}\left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)$

$=\tan^{-1}\left(\tan\left(\dfrac{\pi}{4}+\theta\right)\right)$ $\boxed{\because \dfrac{1+\tan A}{1-\tan A}=\tan\left(\dfrac{\pi}{4}+A\right)}$

$=\dfrac{\pi}{4}+\theta=\dfrac{\pi}{4}+\tan^{-1}x$ $\boxed{\because \tan^{-1}(\tan\theta)=\theta}$

Diff. w.r.to x, we get $\dfrac{dy}{dx}=0+\dfrac{1}{1+x^2}=\dfrac{1}{1+x^2}$ $\boxed{\because \dfrac{d}{dx}(\tan^{-1}x)=\dfrac{1}{1+x^2}}$

115. If $f(x)=xe^x\sin x,$ *then find* $f^{1}(x)$

Sol. Given $f(x)=xe^x\sin x$

Here $u(x)=x,\ v(x)=e^x,\ w(x)=\sin x$

$f^{|}(x)=xe^x\dfrac{d}{dx}(\sin x)+x\sin x\dfrac{d}{dx}(e^x)+e^x\sin x\dfrac{d}{dx}(x)$ $\boxed{\because (uvw)^{|}=uvw^{|}+uwv^{|}+vwu^{|}}$

$f^1(x) = xe^x \cos x + xe^x \sin x + e^x \sin x$ $\boxed{\because \frac{d}{dx}(\sin x) = \cos x}$ $\boxed{\frac{d}{dx}(e^x) = e^x}$ $\boxed{\because \frac{d}{dx}(x) = 1}$

$= e^x (x\cos x + x\sin x + \sin x)$

116. If $y = ae^{nx} + be^{-nx}$ then prove that $y'' = n^2 y$.

Sol. Given that $y = ae^{nx} + be^{-nx}$ $-(1)$

Diff b.s. w.r.to x then $\frac{d}{dx}y = \frac{d}{dx}\left(ae^{nx} + be^{-nx}\right)$

$y^1 = a\frac{d}{dx}\left(e^{nx}\right) + b\frac{d}{dx}\left(e^{-nx}\right)$

$y^1 = a.e^{nx}.n + b.e^{-nx}(-n)$ $\boxed{\because \frac{d}{dx}\left(e^{ax}\right) = ae^{ax}}$

$\Rightarrow y^1 = ane^{nx} - bne^{-nx}$ $-(2)$

Again diff. b.s. w.r.to x then

$\frac{d}{dx}y^1 = an\frac{d}{dx}e^{nx} - bn\frac{d}{dx}e^{-nx}$

$\Rightarrow y^{11} = ane^{nx}.n - bne^{-nx}(-n)$ $\boxed{\because \frac{d}{dx}\left(e^{ax}\right) = ae^{ax}}$

$\Rightarrow y^{11} = an^2e^{nx} + bn^2e^{-nx}$

$\therefore y^{11} = n^2\left[ae^{nx} + be^{-nx}\right] \therefore y^{11} = n^2 y\left[\because \text{from } (1)\right]$

117. If $y = \sin(\log x)$, *find* $\frac{dy}{dx}$.

Sol. $y = \sin(\log x)$

Diff. both sides. w.r.to x

$\frac{d}{dx}y = \cos(\log x)\frac{d}{dx}(\log x)$ $\boxed{\because \frac{d}{dx}(\sin x) = \cos x}$

$\Rightarrow \frac{dy}{dx} = \cos(\log x).\frac{1}{x}$ $\boxed{\because \frac{d}{dx}(\log x) = \frac{1}{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x}\cos(\log x)$

118. If $f(x) = 1 + x + x^2 + ... + x^{100}$ then find $f^1(1)$.

Sol. Given $f(x) = 1 + x + x^2 + x^3 ..... + x^{100}$

Diff. both sides. w.r.to x.

$f^1(x) = 1 + 2x + 3x^2 + ... + 100x^{99}$ $\boxed{\because \frac{d}{dx}\left(x^n\right) = nx^{n-1}}$

$\Rightarrow f^1(1) = 1+2+3+..............+100$

$f^1(1) = \dfrac{n(n+1)}{2}$ here $n = 100$ $\boxed{\because 1+2+3+.........+n = \dfrac{n(n+1)}{2}}$

$\Rightarrow f^1(1) = \dfrac{100(100+1)}{2} \Rightarrow f^1(1) = 50(101) = 5050$

119. If $y = e^{a\sin^{-1}x}$ then prove that $\dfrac{dy}{dx} = \dfrac{ay}{\sqrt{1-x^2}}$

Sol. Given $y = e^{a\sin^{-1}x}$

Diff. w.r.to 'x' we get

$\dfrac{dy}{dx} = e^{a\sin^{-1}x} a\left(\sin^{-1}x\right)^{|}$ $\boxed{\because \dfrac{d}{dx}\left(e^{ax}\right) = ae^{ax}}$

$= a.e^{a\sin^{-1}x}.\dfrac{1}{\sqrt{1-x^2}}$ $\boxed{\dfrac{d}{dx}\left(\sin^{-1}x\right) = \dfrac{1}{\sqrt{1-x^2}}}$

$= \dfrac{a.e^{a\sin^{-1}x}}{\sqrt{1-x^2}} = \dfrac{ay}{\sqrt{1-x^2}}$

120. Find the derivative of $20^{\log(\tan x)}$

Sol: $y = f(x) = 20^{\log(\tan x)}$

$\dfrac{dy}{dx} = 20^{\log(\tan x)} \log 20.\dfrac{d}{dx}(\log(\tan x))$ $\boxed{\because \dfrac{d}{dx}\left(a^x\right) = a^x.\log a}$

$= 20^{\log(\tan x)} \log 20.\dfrac{1}{\tan x}\dfrac{d}{dx}(\tan x)$ $\boxed{\because \dfrac{d}{dx}(\log x) = \dfrac{1}{x}}$

$= 20^{\log(\tan x)} \log 20.\dfrac{\sec^2 x}{\tan x}$ $\boxed{\because \dfrac{d}{dx}(\tan x) = \sec^2 x}$

121. Find the derivative of $f(x) = e^x(x^2+1)$ w.r.t x

Sol. Given $f(x) = e^x\left(x^2+1\right)$

diff. w.r.to 'x' we get

$f^{|}(x) = e^x\dfrac{d}{dx}\left(x^2+1\right) + \left(x^2+1\right)\dfrac{d}{dx}\left(e^x\right)$ $\boxed{\because (uv)^{|} = uv^{|} + vu^{|}}$

$= e^x(2x+0) + \left(x^2+1\right)\left(e^x\right)$ $\boxed{\because \dfrac{d}{dx}\left(x^n\right) = nx^{n-1}}$ $\boxed{\dfrac{d}{dx}\left(e^x\right) = e^x}$ $\boxed{\dfrac{d}{dx}(k) = 0}$

$= e^x\left(2x+x^2+1\right) = e^x(x+1)^2$ $\boxed{\because (a+b)^2 = a^2+2ab+b^2}$

$\therefore f^{|}(x) = e^x(x+1)^2$

122. If $f(x)=\dfrac{a-x}{a+x}$ then find $f^{1}(x)$

Sol. Given $f(x)=\dfrac{a-x}{a+x}$

diff. w.r.to 'x' , we get

$$f^{|}(x)=\frac{(a+x)\frac{d}{dx}(a-x)-(a-x)\frac{d}{dx}(a+x)}{(a+x)^2}$$ $\boxed{\because \left(\frac{u}{v}\right)^{|}=\frac{vu^{|}-uv^{|}}{v^2}}$

$$=\frac{(a+x)(-1)-(a-x)(1)}{(a+x)^2}$$ $\boxed{\because \frac{d}{dx}(x)=1}$ $\boxed{\frac{d}{dx}(k)=0}$

$$=\frac{-a-x-a+x}{(a+x)^2}=\frac{-2a}{(a+x)^2}$$

123. If $y=\left(x^3+6x^2+12x-13\right)^{100}$ then find $\dfrac{dy}{dx}$

Sol. Given $y=\left(x^3+6x^2+12x-13\right)^{100}$

diff. w.r.to 'x', we get

$$\frac{dy}{dx}=100\left(x^3+6x^2+12x-13\right)^{99}.\frac{d}{dx}\left(x^3+6x^2+12x-13\right)$$ $\boxed{\because \frac{d}{dx}(x^n)=nx^{n-1}}$

$$=100\left(x^3+6x^2+12x-13\right)^{99}.\left(3x^2+12x+12\right)$$

$$=300\left(x^3+6x^2+12x-13\right)^{99}.\left(x^2+4x+4\right)$$

$$\therefore \frac{dy}{dx}=300\left(x^3+6x^2+12x-13\right)^{99}.(x+2)^2$$ $\boxed{\because a^2+2ab+b^2=(a+b)^2}$

124. If $f(x)=\log_7(\log x)$ then find $f^{1}(x)$

Sol. Given $f(x)=\log_7(\log x)$

$$f(x)=\frac{\log(\log x)}{\log 7}$$ $\boxed{\because \log_b a=\frac{\log a}{\log b}}$

diff. w.r.to 'x' , we get

$$f^{|}(x)=\frac{1}{\log 7}\frac{1}{\log x}\frac{d}{dx}(\log x)$$ $\boxed{\because \frac{d}{dx}(\log x)=\frac{1}{x}}$

$$=\frac{1}{\log 7}\frac{1}{\log x}\frac{1}{x}=\frac{1}{x\log 7\log x}$$

125. If $y=\frac{1}{ax^2+bx+c}$ then find $\frac{dy}{dx}$

Sol. Given $y=\frac{1}{ax^2+bx+c}$

$$\frac{dy}{dx}=\frac{-1}{\left(ax^2+bx+c\right)^2}\frac{d}{dx}\left(ax^2+bx+c\right) \quad \boxed{\because \frac{d}{dx}\left(\frac{1}{f(x)}\right)=\frac{-1}{\left(f(x)\right)^2}f^{|}(x)}$$

$$=\frac{-\left(2ax+b\right)}{\left(ax^2+bx+c\right)^2} \quad \boxed{\because \frac{d}{dx}\left(x^n\right)=nx^{n-1}} \boxed{\because \frac{d}{dx}(x)=1} \boxed{\frac{d}{dx}(k)=0}$$

126. If $y=\cos ec^{-1}(e^{2x+1})$, find $\frac{dy}{dx}$

Sol. Given $y=\cos ec^{-1}\left(e^{2x+1}\right)$

diff. w.r.to 'x', we get

$$\frac{dy}{dx}=\frac{-1}{e^{2x+1}\sqrt{\left(e^{2x+1}\right)^2-1}}\frac{d}{dx}\left(e^{2x+1}\right) \quad \boxed{\because \frac{d}{dx}\left(\text{cosec}^{-1}x\right)=\frac{-1}{x\sqrt{x^2-1}}}$$

$$=\frac{-1}{e^{2x+1}\sqrt{e^{4x+1}-1}}2\left(e^{2x+1}\right) \quad \boxed{\because \frac{d}{dx}\left(e^{ax}\right)=e^{ax}}$$

$$=\frac{-2}{\sqrt{e^{4x+1}-1}}$$

127. If $y=\frac{1-\cos 2x}{1+\cos 2x}$ then Find $\frac{dy}{dx}$

Sol: Given $y=\frac{1-\cos 2x}{1+\cos 2x}=\frac{2\sin^2 x}{2\cos^2 x}$ $\boxed{\because 1+\cos 2A=2\cos^2 A}$ $\boxed{1-\cos 2A=2\sin^2 A}$

$y=Tan^2 x$ $\boxed{\because \tan\theta=\frac{\sin\theta}{\cos\theta}}$

Diff. w.r.to 'x' we get

$$\frac{dy}{dx}=2\tan x\left(\sec^2 x\right) \quad \boxed{\because \frac{d}{dx}\left(x^n\right)=nx^{n-1}} \boxed{\frac{d}{dx}(\tan x)=\sec^2 x}$$

128. If $f(x)=\sinh^{-1}\left(\frac{3x}{4}\right)$ then find $f^{1}(x)$

Sol. Given $f(x)=\sinh^{-1}\left(\frac{3x}{4}\right)$

$$f^{|}(x)=\frac{1}{\sqrt{1+\left(\frac{3x}{4}\right)^2}}\frac{d}{dx}\left(\frac{3x}{4}\right) \qquad \boxed{\because \frac{d}{dx}\left(\sinh^{-1}x\right)=\frac{1}{\sqrt{1+x^2}}}$$

$$=\frac{1}{\sqrt{1+\frac{9x^2}{16}}}\left(\frac{3}{4}\right) \qquad \boxed{\because \frac{d}{dx}(kx)=k}$$

$$=\frac{1}{\frac{\sqrt{16+9x^2}}{4}}\left(\frac{3}{4}\right)=\frac{3}{\sqrt{16+9x^2}}$$

129. If $y=\sin^{-1}\left(3x-4x^3\right)$ then find $\frac{dy}{dx}$

Sol. Given $y=\sin^{-1}\left(3x-4x^3\right)$

put $x=\sin\theta \Rightarrow \theta=\sin^{-1}x$

$$y=\sin^{-1}\left(3\sin\theta-4\sin^3\theta\right)$$

$$=\sin^{-1}\left(\sin 3\theta\right)=3\theta \qquad \boxed{\because \sin 3A=3\sin A-4\sin^3 A} \quad \boxed{\sin^{-1}(\sin\theta)=\theta}$$

$$=3\sin^{-1}x$$

Diff. w.r.to ‘x’ we get

$$\frac{dy}{dx}=3.\frac{1}{\sqrt{1-x^2}}=\frac{3}{\sqrt{1-x^2}} \qquad \boxed{\because \frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}}$$

130. If $y=\frac{\cos x}{\sin x+\cos x}$ then find $\frac{dy}{dx}$

Sol. Given $y=\frac{\cos x}{\sin x+\cos x}$

Diff. w.r.to ‘x’ we get

$$\frac{dy}{dx}=\frac{(\sin x+\cos x)(\cos x)^{|}-(\cos x)(\sin x+\cos x)^{|}}{(\sin x+\cos x)^2} \qquad \boxed{\because \left(\frac{u}{v}\right)^{|}=\frac{vu^{|}-uv^{|}}{v^2}}$$

$$\frac{dy}{dx}=\frac{(\sin x+\cos x)(-\sin x)-(\cos x)(\cos x-\sin x)}{(\sin x+\cos x)^2} \qquad \boxed{\because \frac{d}{dx}(\sin x)=\cos x} \quad \boxed{\frac{d}{dx}(\cos x)=-\sin x}$$

$$=\frac{-\sin^2 x-\sin x\cos x-\cos^2 x+\sin x\cos x}{(\sin x+\cos x)^2}$$

$$=\frac{-(\sin^2 x+\cos^2 x)}{(\sin x+\cos x)^2}=\frac{-1}{(\sin x+\cos x)^2} \qquad \boxed{\because \cos^2\theta+\sin^2\theta=1}$$

131. If x=at², y= 2at find $\frac{dy}{dx}$

Sol. Given $x=at^2$, ; $y = 2at$

Diff. w.r.to 't' we get ; Diff. w.r.to 't' we get

$\Rightarrow \frac{dx}{dt}=a(2t)$ $\boxed{\because \frac{d}{dx}(x^n)=nx^{n-1}}$; $\Rightarrow \frac{dy}{dt}=2a(1)$ $\boxed{\because \frac{d}{dx}(x)=1}$

$\Rightarrow \frac{dx}{dt}=2at$; $\Rightarrow \frac{dy}{dt}=2a$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2a}{2at}=\frac{1}{t}$$

ERRORS AND APPROXIMATIONS

132. Find Δy and dy if $y=x^2+3x+6$. When x = 10, $\Delta x=0.01$. (MAR-2005,2011,2014)

Sol. Given $f(x)=x^2+3x+6$

x = 10 and $\Delta x=0.01$

(i) $\mathbf{\Delta y=f(x+\Delta x)-f(x)}$

$$=\left[(x+\Delta x)^2+3(x+\Delta x)+6\right]-(x^2+3x+6) \qquad \boxed{\because (a+b)^2=a^2+2ab+b^2}$$

$$=2x\Delta x+(\Delta x)^2+3\Delta x=(2x+\Delta x+3)\Delta x=(23.01)(0.01)=0.2301$$

(ii) $\mathbf{dy=f^{|}(x).\Delta x}$

$$=(2x+3)\Delta x \qquad \boxed{\because \frac{d}{dx}(x^n)=nx^{n-1}} \ \boxed{\frac{d}{dx}(k)=0}$$

$$=[2(10)+3](0.01)=(23)(0.01)=0.23$$

133. Find Δy and dy if $y = x^2 + x$, at x = 10, $\Delta x = 0.1$

Sol. Given $f(x) = x^2 + x$

x = 10, $\Delta x = 0.1$

(i) $\mathbf{\Delta y = f(x+\Delta x) - f(x)}$

$= \left[(x+\Delta x)^2 + (x+\Delta x)\right] - (x^2 + x)$ $\boxed{\because (\mathbf{a}+\mathbf{b})^2 = \mathbf{a}^2 + 2\mathbf{ab} + \mathbf{b}^2}$

$= 2x\Delta x + (\Delta x)^2 + \Delta x = (2x + \Delta x + 1)(\Delta x)$ = (21.1) (0.1)= 2.11

(ii) $\mathbf{dy = f^{|}(x).\Delta x}$

$= (2x+1)(\Delta x)$ = (21) (0.1)= 2.1 $\boxed{\because \frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{x}^{\mathbf{n}}) = \mathbf{nx}^{\mathbf{n-1}}}$ $\boxed{\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{k}) = \mathbf{0}}$

134. Find Δy and dy If $y = \frac{1}{x+2}$ when x = 8, $\Delta x = 0.02$

Sol. Given $f(x) = \frac{1}{x+2}$

x = 8; $\Delta x = 0.02$

(i) $\mathbf{\Delta y = f(x+\Delta x) - f(x)}$

$= \frac{1}{(x+\Delta x)+2} - \frac{1}{x+2} = \frac{-\Delta x}{(x+\Delta x+2)(x+2)} = \frac{-0.02}{(10.02)(10)}$ = - 0.0001996

(ii) $\mathbf{dy = f^{|}(x).\Delta x}$

$= \frac{-1}{(x+2)^2}\Delta x$ $\boxed{\because \frac{\mathbf{d}}{\mathbf{dx}}\left(\frac{\mathbf{1}}{\mathbf{x}}\right) = \frac{\mathbf{-1}}{\mathbf{x}^2}}$

$= \frac{-0.02}{100}$ = - 0.0002

135. Find Δy and dy for $y = e^x + x$, when x = 5, $\Delta x = 0.02$

Sol. Given $f(x) = e^x + x$

x = 5 and $\Delta x = 0.02$

(i) $\mathbf{\Delta y = f(x+\Delta x) - f(x)}$

$= \left[e^{x+\Delta x} + x + \Delta x\right] - (e^x + x)$

$= e^{5.02} + 5.02 - (e^5 + 5) = e^5(e^{0.02} - 1) + 0.02$

(ii) $\mathbf{dy = f^{|}(x).\Delta x}$

$= (e^x + 1)\Delta x = (e^5 + 1)(0.02)$ $\boxed{\because \frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{e}^{\mathbf{x}}) = \mathbf{e}^{\mathbf{x}}}$ $\boxed{\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{x}^{\mathbf{n}}) = \mathbf{nx}^{\mathbf{n-1}}}$

136. Find Δy and dy if $y = \cos x, x = 60^0$ and $\Delta x = 1^0$

Sol: Given y = f(x) = $\cos x$, $x = 60^0$ and $\Delta x = 1^0$

(i) $\mathbf{\Delta y = f(x+\Delta x) - f(x)}$

$= \cos\left(60^0 + 1^0\right) - \cos 60^0 \quad = \cos\left(61^0\right) - \cos 60^0$

$= 0.4848 - 0.5 = -0.0152$ $\boxed{\because \cos 61^0 = 0.4848}$ $\boxed{\cos 60^0 = 0.5}$

(ii) $\mathbf{dy = f^{|}(x).\Delta x}$

$= -\sin x.\Delta x = \left(-\sin 60^0\right)\left(1^0\right)$ $\boxed{\because \frac{d}{dx}(\cos x) = -\sin x}$

$= -(0.866)(0.0174) = -0.015$ $\boxed{\because 1^0 = 0.0174^c}$ $\boxed{\sin 60^0 = 0.866}$

137. Find the approximate value of $\sqrt{82}$ (March-2013, May-2009)

Sol. Let $f(x) = \sqrt{x}$

x = 81, and $\Delta x = 1$

approximate value of $\sqrt{82}$ = $\mathbf{f(x+\Delta x) \simeq f(x) + f^{|}(x)(\Delta x)}$

$\simeq x^{1/2} + \frac{1}{2}x^{-1/2}\Delta x$ $\boxed{\because \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}} = \frac{x^{\frac{-1}{2}}}{2}}$

$\simeq 81^{1/2} + \frac{1}{2}81^{-1/2}(1)$

$\simeq 9 + \frac{1}{2}(9)^{-1} \simeq 9 + \frac{1}{18} \simeq 9 + 0.056 \simeq 9.056$

138. Find the approximate value of $\cos\left(60^o 5^|\right)$ ($\therefore$ 1^0=0.0174 radians)

Sol. Let $f(x) = \cos x$

$x = 60^o$ and $\Delta x = 5^| = \left(\frac{5}{60}\right)^o = \frac{5}{60} \times 0.0174$ radians

approximate value of $\cos 60^0\, 5^|$ = $\mathbf{f(x+\Delta x) \simeq f(x) + f^{|}(x)(\Delta x)}$

$\simeq \cos x + (-\sin x)(\Delta x) \simeq \cos 60^o - \sin 60^o\left(\frac{5}{60}\right) \times 0.0174$ $\boxed{\because \frac{d}{dx}(\cos x) = -\sin x}$

$\simeq 0.5 - 0.8 \times \frac{1}{12} \times 0.0174 \simeq 0.5 - 0.00124 \simeq 0.4987$

139. Find the approximate value of $\sqrt[3]{65}$.

Sol. Let $f(x)=\sqrt[3]{x}$

x = 64, and $\Delta x = 1$

approximate value of $\sqrt[3]{65} = \mathbf{f(x+\Delta x) \simeq f(x)+f^{I}(x)(\Delta x)}$

$\simeq x^{1/3}+\frac{1}{3}x^{-2/3}\Delta x$ $\boxed{\because \frac{d}{dx}\left(\sqrt[3]{x}\right)=\frac{1}{3}x^{\frac{-2}{3}}}$

$\simeq (64)^{1/3}+\frac{1}{3}(64)^{-2/3}(1) \simeq 4+\frac{1}{48} \simeq 4+0.0208 \simeq 4.0208$

140. Find the approximate value of $\sqrt[3]{7.8}$

Sol. Let $f(x)=\sqrt[3]{x}$

x = 8 and $\Delta x = -0.2$

approximate value of $\sqrt[3]{7.8} = \mathbf{f(x+\Delta x) \simeq f(x)+f^{I}(x)(\Delta x)}$

$\simeq x^{1/3}+\frac{1}{3}x^{-2/3}(\Delta x)$ $\boxed{\because \frac{d}{dx}\left(\sqrt[3]{x}\right)=\frac{1}{3}x^{\frac{-2}{3}}}$

$\simeq 8^{1/3}+\frac{1}{3}8^{-2/3}(-0.2) \simeq 2+\frac{1}{3}\left(\frac{1}{4}\right)(-0.2) \simeq 2-\frac{1}{60} \simeq 2-0.016 \simeq 1.984$

141. (i) If the increase in the side of a square is 4% . Then find the approximate percentage of increase in the area of square.
(ii) If the increase in the side of a square is 2% . Then find the approximate percentage of increase in the area of square.

Sol. Let the side of the square be 'x' units.

Given $\frac{\Delta x}{x}\times 100 = 4$

Area of square = x^2 i.e. $A = x^2$

Take log on both sides

$\log A = \log x^2$

$\log A = 2\log x$ $\boxed{\because \log_e x^n = n\log_e x}$

$\frac{\Delta A}{A} = 2\frac{\Delta x}{x}$ $\boxed{\because \text{change in } \log x = \frac{1}{x}\Delta x}$

$\frac{\Delta A}{A}\times 100 = 2\frac{\Delta x}{x}\times 100 = 2 \text{ x } 4 = 8$

$\therefore$ percentage change in area of a square = 8

(ii) Let the side of the square be ‘x’ units.

Given $\frac{\Delta x}{x} \times 100 = 2$

Area of square = x^2 i.e. $A = x^2$

Take log on both sides

$\log A = \log x^2$

$\log A = 2\log x$ $\boxed{\because \log_e x^n = n\log_e x}$

$\frac{\Delta A}{A} = 2\frac{\Delta x}{x}$ $\boxed{\because \text{change in } \log x = \frac{1}{x}\Delta x}$

$\frac{\Delta A}{A} \times 100 = 2\frac{\Delta x}{x} \times 100 = 2 \text{ x } 2 = 4$

$\therefore$ percentage change in area of a square = 4

142. If the radius of a sphere is increased from 7cm to 7.02cm then find the approximate increase in the volume of the sphere

Sol. Given radius of sphere $r = 7\,cm$

and $\Delta r = 0.02cm$

Volume of sphere $\mathbf{(V) = \frac{4}{3}\pi r^3}$

The change in V is $\Delta V = \frac{4}{3}\pi 3r^2.\Delta r$ $\boxed{\because \text{change in } r^3 = 3r^2\Delta r}$

$= 4\pi(7)^2(0.02) = 4\left(\frac{22}{7}\right)(49)(0.02) = 12.32 \text{ cm}^3$

143. If $y = f(x) = kx^n$ show that the approximate relative error in y is n times the relative error in x where n and k are constants.

Sol: Given $y = kx^n$

Take log on both sides

$\log y = \log kx^n$

$\Rightarrow \log y = \log k + \log x^n$ $\boxed{\because \log(ab) = \log a + \log b}$

$\Rightarrow \log y = \log k + n\log x$ $\boxed{\because \log(x^n) = n\log x}$

$\Rightarrow \frac{1}{y}\Delta y = 0 + n.\frac{1}{x}\Delta x$

$\Rightarrow \frac{\Delta y}{y} = n.\frac{\Delta x}{x}$

Relative error in y = n.(relative error in x)

144. The diameter of sphere is measured to be 40cm. If an error of 0.02cm is made in it. Then Find approximate errors in volume and surface area of the sphere.

Sol. Given diameter of a sphere = 40cm = d

$$\Delta d = 0.02cm$$

Volume of sphere $(V) = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3$ $\boxed{\because \text{Diameter of sphere is } d = 2r \Rightarrow r = \frac{d}{2}}$

$$V = \frac{\pi}{6}d^3$$

$$\Delta V = \frac{\pi}{6}3d^2.\Delta d = \frac{\pi}{2}(40)^2(0.02) = 16\pi\, cm^3$$

Surface area of sphere $S = 4\pi r^2$

$$S = 4\pi\left(\frac{d}{2}\right)^2 = \pi d^2$$

$$\Delta S = \pi.2d.\Delta d = 2\pi(40)(0.02) = 1.6\pi\, cm^2$$

145. The time t, of a complete oscillation of a simple pendulum of length 'l' **is given by the equation** $t = 2\pi\sqrt{\frac{l}{g}}$ **where 'g' is gravational constant. Find approximate percentage error in 't' when the perecentage of error in 'l' is 1%.**

Sol. Given $\frac{\Delta l}{l}\times 100 = 1$ and $t = 2\pi\sqrt{\frac{l}{g}}$

Take log on both sides

$$\Rightarrow \log t = \log\left(2\pi\left(\frac{l}{g}\right)^{1/2}\right)$$

$$\Rightarrow \log t = \log 2\pi + \frac{1}{2}\log\left(\frac{l}{g}\right)$$ $\boxed{\because \log(ab) = \log a + \log b}$ $\boxed{\because \log(x^n) = n\log x}$

$$\Rightarrow \log t = \log 2\pi + \frac{1}{2}(\log l - \log g)$$ $\boxed{\because \log\left(\frac{a}{b}\right) = \log a - \log b}$

$$\Rightarrow \frac{\Delta t}{t} = 0 + \frac{1}{2}\left[\frac{\Delta l}{l} - 0\right]$$ $\boxed{\because g \text{ is a constant then change in } \log g = 0}$

$$\Rightarrow \frac{\Delta t}{t}\times 100 = \frac{1}{2}\times\frac{\Delta l}{l}\times 100 = \frac{1}{2}\times 1 = \frac{1}{2}$$

$\therefore$ The percentage error in time 't' is $\frac{1}{2}$

ROLLE'S THEOREM AND LAGRANGE'S MEAN VALUE THEOREM

146. State Rolle's theorem

Sol: If $f:[a,b]\to R$ be a function satisfying the following conditions

(i) f is continuous on $[a,b]$

(ii) f is differentiable on (a,b)

(iii) $f(a)=f(b)$ then there exists at least one $c\in(a,b)$ such that $f^{'}(c)=0$

147. State Lagrange's theorem

Sol: If $f:[a,b]\to R$ be a function satisfying the following conditions

(i) f is continuous on $[a,b]$

(ii) f is differentiable on (a,b) then there exists atleast one $c\in(a,b)$

such that $f^{|}(c)=\dfrac{f(b)-f(a)}{b-a}$.

148. $f(x)=(x-1)(x-2)(x-3)$ **prove that there is more than one 'c' in** $(1,3)$ **such that** $f^{'}(c)=0$ **(Mar-2013)**

Sol: $f(x)=(x-1)(x-2)(x-3)$ is polynomial which is continuous and differentiable on **R**

f is continuous on $[1,3]$

f is differentiable on $(1,3)$

$f(1)=(1-1)(1-2)(1-3)=0$

$f(3)=(3-1)(3-2)(3-3)=0$

$\therefore\ f(1)=f(3)=0$ $\boxed{\because f(a)=f(b)}$

$\therefore\ f(x)$ satisfies all the conditions of Rolle's theorem

$f^{'}(x)=(x-1)(x-2)+(x-1)(x-3)+(x-2)(x-3)$

$\boxed{\because (uvw)^{'}=u^{'}(vw)+u(v^{'})w+(uv)w^{'}}$ $\boxed{\dfrac{d}{dx}(x)=1}$

$=3x^2-12x+11$

By Rolle's theorem, $f^{|}(c)=0$

$\Rightarrow 3c^2-12c+11=0$

$\Rightarrow c=\dfrac{-(-12)\pm\sqrt{(-12)^2-4(3)(11)}}{2(3)}$ $\boxed{\because \text{if } ax^2+bx+c=0\Rightarrow x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}}$

$$\Rightarrow c = \frac{12 \pm \sqrt{144-132}}{6} \Rightarrow c = \frac{12 \pm 2\sqrt{3}}{6}$$

$$\therefore c = 2 \pm \frac{1}{\sqrt{3}} \in (1,3)$$

149. Find the value of c in Rolle's theorm for the function y=f(x) = $x^2 + 4$ on [-3,3]

Sol: $f(x) = x^2 + 4$ as $x^2 + 4$ is polynomial which is continuous and differentiable on **R**

f is continuous on [-3,3]

f is differentiable on (-3, 3)

$f(3) = (3)^2 + 4 = 13$ and $f(-3) = (-3)^2 + 4 = 13$

$\therefore f(-3) = f(3)$

$\therefore f(x)$ satisfies all the conditions of Rolle's theorem $\boxed{\because \mathbf{f(a) = f(b)}}$

$f'(x) = 2x$ $\boxed{\because \frac{d}{dx}(x^n) = nx^{n-1}}$

By Rolle's theorem , $f'(c) = 0$

$2c = 0 \Rightarrow c = 0 \in (-3, 3)$

Thus Rolle's theorm is verified.

150. Find the value of 'c' from Rolle's theorem for the function $f(x) = x^2 - 1$ on $[-1,1]$

(Mar-2014)

Sol: f is continuous on $[-1,1]$

f is differentiable on $(-1,1)$

$f(-1) = (-1)^2 - 1 = 0$ and $f(1) = (1)^2 - 1 = 0$

$\therefore f(-1) = f(1)$

$\therefore f(x)$ satisfies all the conditions of Rolle's theorem $\boxed{\because \mathbf{f(a) = f(b)}}$

$f'(x) = 2x$ $\boxed{\because \frac{d}{dx}(x^n) = nx^{n-1}}$

By Rolle's theorem , $f'(c) = 0$

$\Rightarrow 2c = 0$

$\Rightarrow c = 0 \in (-1,1)$

151. It is given that Rolle's theorm holds for the function f(x) = $x^3 + bx^2 + ax$ on [1, 3] with $c = 2+\frac{1}{\sqrt{3}}$. Find the values of a and b

Sol: $f(x) = x^3 + bx^2 + ax$ is polynomial which is continuous and differentiable on **R**

f is continuous on [1, 3]

f is differentiable on $(1,3)$

By Roll's theorem f(a) = f(b) $\Rightarrow f(1) = f(3)$

$\Rightarrow 1^3 + b(1)^2 + a(1) = 3^3 + b(3)^2 + a(3)$

$\Rightarrow a + 4b + 13 = 0$

$\Rightarrow a = -4b - 13$ ....(1)

$f^{|}(x) = 3x^2 + 2bx + a$ $\boxed{\because \frac{d}{dx}(x^n) = nx^{n-1}}$

By Rolle's theorem , $f^{|}(c) = 0$

$3c^2 + 2bc + a = 0$

$\Rightarrow 3\left(2+\frac{1}{\sqrt{3}}\right)^2 + 2b\left(2+\frac{1}{\sqrt{3}}\right) - 4b - 13 = 0$ (From (1))

$\Rightarrow 3\left(4+\frac{1}{3}+\frac{4}{\sqrt{3}}\right) + 4b + \frac{2b}{\sqrt{3}} - 4b - 13 = 0$ $\boxed{\because (a+b)^2 = a^2 + 2ab + b^2}$

$\Rightarrow 13 + \frac{12}{\sqrt{3}} + \frac{2b}{\sqrt{3}} - 13 = 0$

$\Rightarrow b = -6$

From (1) $\Rightarrow$ a=-4(-6)-13=11

152. Verify Rolle's theorem for function f(x) = sin x – sin 2x on [0,π]

Sol: Let f(x) = sin x – sin 2x is continuous and differentible on R

f is continuous on [0, π]

f is differentiable on $(0,\pi)$

$f(0) = \sin 0 - \sin 2(0) = 0$

$f(\pi) = \sin\pi - \sin 2(\pi) = 0$

$\therefore$ f (0) = f(π)

$\therefore f(x)$ satisfies all the conditions of Rolle's theorem $\boxed{\because f(a) = f(b)}$

$f^{|}(x) = \cos x - 2\cos 2x$ $\boxed{\because \frac{d}{dx}(\sin(ax)) = a\cos(ax)}$

By Rolle's theorem , $f^{|}(c) = 0$

$\Rightarrow$ cos c – 2 cos 2c = 0

$\Rightarrow$ cos c – 2 (2 cos^{2}c – 1) = 0 $\boxed{\because \cos 2A = 2\cos^2 A - 1}$

$\Rightarrow$ cos c – 4 cos^2 c + 2 = 0

$\Rightarrow$ 4cos^2 c – cos c – 2 = 0

$$\cos c = \frac{1 \pm \sqrt{1+32}}{2(4)} = \frac{1 \pm \sqrt{33}}{8} \qquad \boxed{\because \text{if } ax^2+bx+c=0 \Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

$$\Rightarrow c = Cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right) \in (0,\pi)$$

Thus Rolle's theorem is verified.

153. Verify Rolle's theorem for the function $(x^2 – 1)(x – 2)$ on [-1, 2]. Find the point in the interval where the derivate vanishes.

Sol: $f(x) = (x^2 – 1)(x – 2)$ is polynomial which is continuous and differentiable on **R**

f is continuous on [-1,2]

f is differentiable on (-1,2)

$f(x) = x^3 – 2x^2 – x + 2$

$f(-1) = (-1)^3 – 2(-1)^2 – (-1) + 2 = -1 -2 + 1 + 2 = 0$

$f(2) = (2)^3 – 2(2)^2 – 2 + 2 = 8 – 8 – 2 + 2 =0$

$f(-1)=f(2)$

$\therefore f(x)$ satisfies all the conditions of Rolle's theorem $\boxed{\because f(a)=f(b)}$

$f^{|}(x) = 3x^2 - 4x - 1$ $\boxed{\because \frac{d}{dx}(x^n) = nx^{n-1}}$

By Rolle's theorem , $f^{|}(c) = 0$

$\Rightarrow 3c^2 – 4c – 1 = 0$

$$\Rightarrow c = \frac{4 \pm \sqrt{16+12}}{6} \qquad \boxed{\because \text{if } ax^2+bx+c=0 \Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

$$= \frac{4 \pm \sqrt{28}}{6} = \frac{2 \pm \sqrt{7}}{3} \in (-1,2)$$

Thus Rolle's theorem is verified.

154. Verify Rolle's theroem for the function $f(x) = x(x+3)e^{-x/2}$ on $[-3,0]$

Sol $f(x) = x(x+3)e^{-x/2}$

f is continuous on [-3,0]

f is differentiable on (-3,0)

$$f(-3) = -3(-3+3)e^{\frac{3}{2}} = 0$$

$f(0)=0(0+3)e^{0}=0$

$\therefore f(-3)=f(0)$

$\therefore\ f(x)$ satisfies all the conditions of Rolle's theorem $\boxed{\because f(a)=f(b)}$

$f^{'}(x)=(1)(x+3)e^{-x/2}+x(1)e^{-x/2}+x(x+3)e^{-x/2}\left(\frac{-1}{2}\right)$

$\boxed{\because (uvw)^{'}=u^{'}(vw)+u(v^{'})w+(uv)w^{'}}$ $\boxed{\frac{d}{dx}(x)=1}$ $\boxed{\frac{d}{dx}(e^{ax})=ae^{ax}}$

$\Rightarrow f^{'}(x)=e^{-x/2}\left[\frac{2x+6+2x-x^2-3x}{2}\right]$

$\Rightarrow f^{'}(x)=e^{-x/2}\left[\frac{-x^2+x+6}{2}\right]$

By Rolle's theorem , $f^{|}(c)=0$

$\Rightarrow e^{-c/2}\left[\frac{-c^2+c+6}{2}\right]=0$

$\Rightarrow c^2-c-6=0 \Rightarrow c^2-3c+2c-6=0 \Rightarrow c(c-3)+2(c-3)=0$

$\Rightarrow (c+2)(c-3)=0$

$\Rightarrow c-3=0 \Rightarrow c=3 \notin (-3,0)$

$\Rightarrow c+2=0 \Rightarrow c=-2 \in (-3,0)$

Thus Rolle's theorem is verified.

155. Show that there is no real number 'k' for which the equation $x^2-3x+k=0$ has two distinct roots in [0, 1]

Sol: *Let* $f(x)=x^2-3x+k$ is polynomial which is continuous and differentiable on **R**

Let α,β are two distinct roots of $f(x)=0$, $0<\alpha<\beta<1$

$\therefore f(\alpha)=0, f(\beta)=0$

f is continuous on $[\alpha,\beta]$

f is differentiable on (α,β)

$\therefore f(\alpha)=f(\beta)$

$\therefore\ f(x)$ satisfies all the conditions of Rolle's theorem $\boxed{\because f(a)=f(b)}$

$f^{'}(x)=2x-3$ $\boxed{\because \frac{d}{dx}(x^n)=nx^{n-1}}$

By Rolle's theorem , $f^{|}(c) = 0$

$$\Rightarrow 2c - 3 = 0 \Rightarrow c = \frac{3}{2} \notin (\alpha, \beta)$$

$\therefore$ f(x) has two distinct roots in [0,1] for no real value of k

156. Find c so that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ in the following cases

i) $f(x) = x^2 - 3x - 1$, $a = \dfrac{-11}{7}$, $b = \dfrac{13}{7}$ ii) $f(x) = e^x$, $a = 0$, $b = 1$

Sol :i) $f(x) = x^2 - 3x - 1$, $a = \dfrac{-11}{7}$, $b = \dfrac{13}{7}$

$$f(b) = f\left(\frac{13}{7}\right) = \frac{169}{49} - \frac{39}{7} - 1 = \frac{169 - 273 - 49}{49} = \frac{-153}{49}$$

$$f(a) = f\left(\frac{-11}{7}\right) = \frac{121}{49} + \frac{33}{7} - 1 = \frac{121 + 231 - 49}{49} = \frac{303}{49}$$

$$f^{|}(x) = 2x - 3 \qquad \boxed{\because \frac{d}{dx}\left(x^n\right) = nx^{n-1}}$$

$$\mathbf{f^{|}(c) = \frac{f(b) - f(a)}{b - a}}$$

$$\Rightarrow 2c - 3 = \frac{\dfrac{-153}{49} - \dfrac{303}{49}}{\dfrac{13}{7} + \dfrac{11}{7}} = \frac{-456/49}{24/7} = \frac{-19}{7}$$

$$\Rightarrow 2c = \frac{-19}{7} + 3 = \frac{2}{7} = \frac{2}{7} \Rightarrow c = \frac{1}{7}$$

ii) $f(x) = e^x$, $a = 0$, $b = 1$

$f(b) = f(1) = e^1 = e$

$f(a) = f(0) = e^0 = 1$

$$f(x) = e^x \Rightarrow f^{|}(x) = e^x \qquad \boxed{\because \frac{d}{dx}\left(e^x\right) = e^x}$$

$$\mathbf{f^{|}(c) = \frac{f(b) - f(a)}{b - a}}$$

$$e^c = \frac{e - 1}{1 - 0} \Rightarrow e^c = e - 1 \Rightarrow c = \log_e(e - 1)$$

157. Verify the conditions of the Lagrange's mean value theorem for the following functions In each case find a point c in the interval as stated by the theorem.
i) $x^2 - 1$ on [2, 3] ii) sin x – sin 2x on [0, π] iii) log x on [1, 2]

Sol: i) $f(x) = x^2 - 1$ is polynomial which is continuous and differentiable on **R**

f is continuous on [2, 3]

f is differentiable on (2,3)

$f^{|}(x) = 2x$ $\boxed{\because \frac{d}{dx}\left(x^n\right) = nx^{n-1}}$

By Lagrange's mean value theorem $f^{|}(c) = \frac{f(b)-f(a)}{b-a}$

$$f^1(c) = \frac{f(3)-f(2)}{3-2} \Rightarrow 2c = \frac{8-3}{3-2} \Rightarrow 2c = 5 \Rightarrow c = \frac{5}{2} \in (2,3)$$

Thus Lagrange's mean value theorem is verified.

ii) $f(x) = \sin x - \sin 2x$

f is continuous on [0, π]

f is differentiable on (0, π)

$f^{|}(x) = \cos x - 2\cos 2x$ $\boxed{\because \frac{d}{dx}(\sin(ax)) = a\cos(ax)}$

By Lagrange's mean value theorem $f^{|}(c) = \frac{f(b)-f(a)}{b-a}$

$$f^{'}(c) = \frac{f(\pi)-f(0)}{\pi - 0}$$

$\Rightarrow \cos c - 2\cos 2c = 0$

$\Rightarrow \cos c - 2(2\cos^2 c - 1) = 0$ $\boxed{\because \cos 2A = 2\cos^2 A - 1}$

$\Rightarrow \cos c - 4\cos^2 + 2 = 0$.

$\Rightarrow 4\cos^2 c - \cos c - 2 = 0$

$\Rightarrow \cos c = \frac{1 \pm \sqrt{33}}{8}$ $\boxed{\because \text{if } ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

$\Rightarrow c = \cos^{-1} \frac{1 \pm \sqrt{33}}{8} \in (0, \pi)$

Thus Lagrange's mean value theorem is verified.

iii) $f(x) = \log x$

f is continuous on [1, 2]

f is differentiable on $(1, 2)$

$f^1(x) = \frac{1}{x}$ $\boxed{\because \frac{d}{dx}(\log x) = \frac{1}{x}}$

By Lagrange's mean value theorem $\mathbf{f^{|}(c) = \dfrac{f(b)-f(a)}{b-a}}$

$$\Rightarrow f^{|}(c) = \frac{f(2)-f(1)}{2-1}$$

$$\Rightarrow \frac{1}{c} = \frac{\log 2 - \log 1}{2-1}$$

$$\Rightarrow \frac{1}{c} = \log 2 \quad \Rightarrow c = \frac{1}{\log_e 2} = \log_2 e$$

Thus Lagrange's mean value theorem is verified.

158. Find a point on the graph of the curve $y = x^3$ where the tangent is parallel to chord joining the points (1, 1) and (3, 27).

Sol: $y = x^3$

$$\frac{dy}{dx} = 3x^2$$

$\boxed{\because \dfrac{d}{dx}(x^n) = nx^{n-1}}$

Let $A(x_1, y_1) = (1,1)$, $B(x_2, y_2) = (3,27)$

Slope of chord AB$= \dfrac{27-1}{3-1} = 13$

$\boxed{\because \text{slope} = \dfrac{y_2 - y_1}{x_2 - x_1}}$

Given the tangent is parallel to the chord joining the points A and B

i.e. **Slope of tangent = slope of chord**

$$\Rightarrow 3x^2 = 13$$

$$\Rightarrow x = \sqrt{\frac{13}{3}} = \sqrt{\frac{13(3)}{3(3)}} = \frac{\sqrt{39}}{3}$$

$$y = \left(\frac{\sqrt{39}}{3}\right)^3 = \frac{13\sqrt{39}}{9}$$

$$\therefore \text{ Point} = \left(\sqrt{\frac{13}{3}}, \frac{13\sqrt{39}}{9}\right)$$

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