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Kinetic Models in the Near-Equilibrium Regime

Modèles cinétiques au voisinage de l'équilibre

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Abstract

This thesis concentrates on two principal questions arising in the kinetic theory.

In the first part we study the connection between kinetic models with Fermi-Dirac statistics and macroscopic fluid dynamics. We obtain these macroscopic limits when the fluid is dense enough that particles undergo many collisions per unit of time. This situation is described via a small parameter ε, called the Knudsen number, that represents the ratio of mean free path of particles between collisions to some characteristic length of the flow. We derive formal limits; in order to do that, we introduce a scaling for standard kinetic equation of the form

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} C(F_\varepsilon).$$

Here F_ε is a non-negative function representing the density of particles with position x and velocity v in the single-particle phase space $\mathbb{R}^3_x \times \mathbb{R}^3_v$ at time t. The interaction of particles through collisions is given by the operator $C(F)$; this operator acts only on variable v and is non-linear in the general case.

We base the connection between kinetic and macroscopic dynamics on the conservation properties and entropy relations implying that the equilibria are Fermi-Dirac (i.e. of the form $1/(1+\exp(c_1 + c_2|v - u|^2))$ distributions.

In the first chapter we establish that moments and parameters of Fermi-Dirac distributions are related by a diffeomorphism. Also, for a very large class of collision operators we give the conditions that allow us to formally derive the generalised Euler equations from the Boltzmann equation. These conditions are related to the H-theorem and assume a formally consistent convergence for fluid dynamical moments and entropy of the kinetic equation. We also discuss the well-posedness of the obtained Euler equations by using Godunov's criterion of hyperbolicity.

In the second chapter we use the Chapman-Enskog expansion to study the connection between the solutions of kinetic equations with Fermi-Dirac statistics and the solutions of the compressible Navier-Stokes equations for a specific form of the collision operator C.

We establish the analytic properties of the linearised collision operator; in particular, we prove that under certain hypothesis on the collision kernel the linearised collision operator satisfies the Fredholm alternative. We describe a general approach allowing to reuse the existing results from Maxwellian case. We build approximate solutions of order two of the scaled kinetic equation by using the solution of the Navier-Stokes equations with a particular form of viscosity and heat flow.

In the third chapter we extend the results obtained in the previous two chapters by establishing the limiting form of the fluid dynamic equations in the incompressible case. We introduce several scalings for the kinetic equations with the Fermi-Dirac statistics with the same collision operator as in the second chapter. Under stronger assumptions and a formally consistent convergence for the fluid dynamical moments we can formally derive the limiting fluid dynamic equations with the help of the moment method expansion.

In the fourth chapter we consider the Boltzmann equation linearised about a global Maxwellian; global Maxwellian distribution function are local Maxwellian functions satisfying the free transport equation. We establish analytical properties on the linearised collision operator, most notably, sufficient conditions for its continuity. Then we prove the existence of solutions of the linearised Boltzmann equation for an initial value problem and for boundary value problems, and, moreover, prove the existence of limits for large time of these solutions. These results allow us to introduce the scattering operator, which can be understood as "evolution of the density function modulo free transport". The key result of this chapter is that this scattering operator is bounded and it has a spectral gap in a weighted Hilbert space.

Résumé

Cette thèse se concentre sur deux questions de la théorie de modèles cinétiques.

Dans la première partie on étudie la connexion entre les modèles cinétiques avec la statistique de Fermi-Dirac et la dynamique macroscopique de fluide. On obtient ces limites macroscopiques lorsque la fluide est suffisamment dense pour que les particules fassent beaucoup de collisions pour une unité de temps. Cette situation est décrite par un petit paramètre ε, appelé le nombre de Knudsen, qui représente la rapport du libre parcours moyen des particules entre les collisions et de la longueur caractéristique du flot.

On dérive les limites formelles ; pour ce faire, on introduit un scaling pour l'équation cinétique de la forme

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} C(F_\varepsilon).$$

Ici F_ε est une fonction non-négative représentante la densité des particules avec la position x et vitesse v dans l'espace de phase $\mathbb{R}^3_x \times \mathbb{R}^3_v$ au temps t. L'interaction des particules via des collisions est donnée par l'opérateur $C(F)$; cet opérateur agit seulement sur la variable v et il est non-linéaire dans le cas général.

On base cette connexion entre la dynamique cinétique et la dynamique macroscopique sur les lois de conservation et les relations d'entropie impliquant que les états d'équilibre sont des distributions de Fermi-Dirac (i.e. de la forme $1/(1 + \exp(c_1 + c_2|v - u|^2))$.

Dans le premier chapitre on établit que les moments et les paramètres des distributions de Fermi-Dirac sont liées par un difféomorphisme. En plus, pour une large classe des opérateurs de collisions on donne les conditions sous lesquelles on peut dériver formellement les équations de Euler généralisées à partir de l'équation de Boltzmann. Ces conditions sont liées au théorème H et elles supposent une convergence formellement consistante de moments dynamiques de fluide et d'entropie de l'équation cinétique. On discute également si ces équations d'Euler sont bien posées en utilisant le critère d'hyperbolicité de Godunov.

Dans le deuxième chapitre on utilise le développement de Chapman-Enskog pour étudier la relation entre les solutions des équations cinétiques avec la statistique de Fermi-Dirac et les solutions des équations de Navier-Stokes compressibles pour une forme spécifique de l'opérateur de collision C.

On établit les propriétés analytiques d'opérateur de collision linéarisé ; en particulier, on démontre que sous certaines hypothèses sur le noyau de collision l'opérateur de collision linéarisé satisfait l'alternative de Fredholm. On décrit l'approche générale permettant de réutiliser des résultat existants pour le cas Maxwellien. On construit les solutions approchées d'ordre deux de l'équation cinétique sur la base des solutions d'équations de Navier-Stokes avec une forme particulière de dissipation, viscosité et flot de chaleur.

Dans le troisième chapitre on étend les résultats obtenus dans les chapitres précédents par établir la forme limite des équations de dynamique de fluide incompressible. On introduit les différents échelles de temps et d'espace pour les équations cinétiques avec la statistique de Fermi-Dirac pour le même opérateur de collision que dans le deuxième chapitre. Sous les hypothèses plus fortes et la convergence formelles des moments de dynamique de fluide, à l'aide de méthode de développement des moments, on obtient comme la limite formelle les équations de dynamique de fluide incompressible.

Dans le quatrième chapitre on considère l'équation de Boltzmann linéarisée près d'une fonction Maxwellienne globale. Une Maxwellienne globale est une fonction Maxwellienne locale qui satisfait en même temps l'équation de transport libre. On établit les propriétés analytiques de l'opérateur de collision linéarisé, notamment les conditions suffisantes pour que cet opérateur soit borné. Puis on démontre l'existence de solutions de l'équation de Boltzmann linéarisée pour un problème à valeur initiale et pour les problèmes à valeur à bord. Pour ces solutions on montre l'existence de limites au temps grand. Ces résultats nous permettent d'introduire l'opérateur de scattering qu l'on peut comprendre comme "évolution de la fonction de densité modulo transport libre". Le résultat clé de ce chapitre est que cet opérateur de scattering est borné et qu'il possède un gap spectrale dans un espace de Hilbert à poids spécifique.

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Contents

Introduction

1 Kinetic theory

The purpose of kinetic theory is to study systems composed of a large number of particles. For such a system one can use different levels of description.

The microscopic approach studies the trajectories of each individual particle. This method has several drawbacks, most notably the large number of particles. The macroscopic approach consists in examination of the observable macroscopic quantities of the system, i.e. mass density, fluid velocity field and temperature. Kinetic theory sits at the intermediary level between these two approaches; it can be called the mesoscopic approach. This description is statistical — it studies the "typical" behaviour of particles. This enables us both to simplify the examination of particles trajectories and to have access to physical properties of the system. The scope of the kinetic theory also allows us to study the relaxation of dynamical systems to equilibrium and to predict the long time behavior of the system.

In this thesis we consider the mesoscopic description. The studied system consists of a large number of particles and therefore can be seen as a continuum. The state of the system is described by a density of particles $F(t,x,v)$ at the time $t \geq 0$, position $x \in \mathbb{R}^3$, and velocity $v \in \mathbb{R}^3$.

As we mentioned earlier, by to the mesoscopic description we have access to observable macroscopic quantities such as local density ρ, local average velocity u, and local local internal energy $\mathcal{E}$

$$
\begin{aligned}
\rho(t,x) &= \int_{\mathbb{R}^3} F(t,x,v)\,\mathrm{d}v,\\
u(t,x) &= \frac{1}{\rho(t,x)} \int_{\mathbb{R}^3} vF(t,x,v)\,\mathrm{d}v,\\
\mathcal{E}(t,x) &= \frac{1}{3} \int_{\mathbb{R}^3} |v-u(t,x)|^2 F(t,x,v)\,\mathrm{d}v.
\end{aligned}
\tag{1.1}
$$

Evolution of particle density

The goal is to study the evolution of the particle density F. By Newton's laws, the absence of external forces and the absence of interaction between particles leads to particles moving along straight lines at constant velocity:

$$v = \frac{dx}{dt}, \qquad \frac{dv}{dt} = 0.$$

Hence the density F is the solution of the free transport equation:

$$\partial_t F + v \cdot \nabla_x F = 0.$$

Now, in order to take into account particle interactions, we modify the right-hand side of this equation:

$$\partial_t F + v \cdot \nabla_x F = C(F).$$

We will specify the nature of the operator $C(F)$ for each model studied here.

The Boltzmann equation

Among all kinetic equations, the Boltzmann equation used to described rarefied gas dynamics, plays a central role. It is the oldest equation first derived formally by Boltzmann in [5] after Maxwell's seminal results in [33]. Moreover, it is one of the few equations which can be rigorously derived from microscopic dynamics. The Boltzmann-Grad [25] limit provides a scope allowing us to derive the Boltzmann equation from Newton's laws of motion applied to each particle. The rigorous derivation for this case was established in [29] and [16].

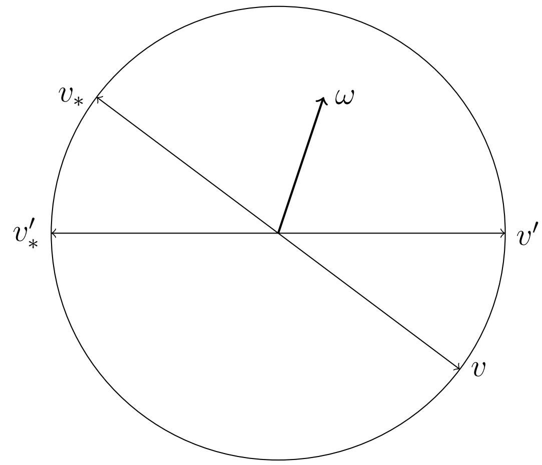
Let us review the assumptions made by Boltzmann:

1. The particles undergo only *binary collisions*, the process under which two sufficiently close particles change their velocities in a short amount of time. Boltzmann's theory implicitly assumes that the medium is rarefied enough so that there are no collisions involving three or more particles.

2. The collisions are *localized in time and space*: the scale of time and space of these collisions are negligibly small compared to the described typical length and time scales.

3. The collisions are *elastic*: we have the conservation of momentum and energy in the collision process. If we denote v and v_* (respectively, v' and v'_*) the velocities of two particles after (respectively, before) the collision, then we have

$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$



We obtain four (in space dimension 3) equations; the velocities v' and v'_* can be represented in terms of v and v_* as

$$v' = v - \omega(v - v_*, \omega), \quad v'_* = v_* + \omega(v - v_*, \omega)$$

with $\omega \in \mathbb{S}^2$.

4. The collisions are *micro-reversible*. From the statistical point of view, the probability that the velocities (v', v'_*) are changed to (v, v_*) is the same as the probability that the velocities (v, v_*) are changed to (v', v'_*).

5. The Boltzmann's "molecular chaos" hypothesis holds: the velocities of colliding particles are uncorrelated, and independent of position.

Under these hypotheses Boltzmann showed that collision integral has the form

$$C_B(F) = \iiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} b(v - v_*, \omega)(F'F'_* - FF_*)\,\mathrm{d}v_*\,\mathrm{d}\omega$$

with the standard notation

$$\begin{aligned} F &= F(t, x, v), & F_* &= F(t, x, v_*), \\ F' &= F(t, x, v'), & F'_* &= F(t, x, v'_*). \end{aligned}$$

The function b is called the collision kernel; it is a positive function depending only the relative velocity of two particles and the angle between these velocities. We will discuss the nature of this function in the presentation of the models studied here.

Quantum case

Boltzmann's kinetic theory can be applied to model the evolution of quantum particles. In this thesis, we shall exclusively deals with quantum particles following Fermi-Dirac statistics.

Fermi-Dirac statistics was first presented in 1926 in works by Fermi and Dirac ([12, 15]). It adds an additional hypothesis on the behaviour of particles: they obey the Pauli exclusion principle and therefore the velocities before and after the collision are no longer uncorrelated. However, at variance with the classical case, a similarly rigorous derivation of the Boltzmann equation is yet to be established. A phenomenological description implies that for the quantum case and under the Fermi-Dirac statistics the collision integral writes (cf., for example, [28])

$$C(F) = \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(v-v_*,\omega)(F'F'_*(1-F)(1-F_*) - FF_*(1-F')(1-F'_*))\,\mathrm{d}v_*\,\mathrm{d}\omega. \tag{1.2}$$

Fermi-Dirac distributions

Fermi-Dirac distributions play an essential role in the kinetic theory for the Fermi-Dirac statistics. They are introduced as functions of the form (see [28, 38])

$$F_{f,u,\theta}(t,x,v) = \frac{1}{1+\exp\left(\frac{|v-u(t,x)|^2}{2\theta(t,x)} - f(t,x)\right)},$$

where θ is the temperature, u the bulk velocity, and $\mu = f/\theta$ is the total chemical potential.

While it is useful to adopt the parametrization (f,u,θ) to study the analytical properties of Fermi-Dirac distributions, in order to study the hydrodynamic limits one needs the macroscopic observable quantities — in other words, the moments of the distribution — i.e. the quantities ρ and $\mathcal{E}$ defined in (1.1).

One of the key results in chapter I is the following theorem (see theorem I.2):

Theorem. *There exists a diffeomorphism expressing the parameters (f,u,θ) of a Fermi-Dirac distribution $F_{f,u,\theta}$ in terms of its moments $\int_{\mathbb{R}^3} F\,\mathrm{d}v$, $\int_{\mathbb{R}^3} vF\,\mathrm{d}v$, and $\int_{\mathbb{R}^3} |v|^2 F\,\mathrm{d}v$.*

One of the major corollaries of this theorem is that we can parametrize Fermi-Dirac distributions by their associated moments.

Conservation laws

Note the microscopic conservation properties of particle collisions render as macroscopic quantities: mass, momentum, and energy. One can formally show by changing variables that for each measurable test function g decaying fast enough at infinity, the operator C defined in (1.2) satisfies

$$\int_{\mathbb{R}^3} C(F)g(v)\,\mathrm{d}v = -\frac{1}{4}\iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \mathcal{N}(g'_* + g' - g - g_*)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega, \tag{1.3}$$

with

$$\mathcal{N} = F'F'_*(1-F)(1-F_*) - FF_*(1-F')(1-F'_*).$$

This allows us, in particular, to conclude

$$\int_{\mathbb{R}^3} C(F)g(v)\,\mathrm{d}v = 0 \quad \text{for} \quad g(v) = 1,\, v_1,\, v_2,\, v_3,\, |v|^2.$$

Entropy production

The identity (1.3) with $g = \ln\left(\frac{1-F}{F}\right)$ also leads to the following inequality:

$$D(F) = \int_{\mathbb{R}^3} C(F)\ln\left(\frac{1-F}{F}\right)\,\mathrm{d}v \geq 0.$$

If we define the entropy functional as

$$H(F) = \int_{\mathbb{R}^3\times\mathbb{R}^3} (F\ln F + (1-F)\ln(1-F))\,\mathrm{d}x\,\mathrm{d}v,$$

then the above inequality yields the relation

$$\frac{d}{dt}H(F) = -\int_{\mathbb{R}^3} D(F)\,\mathrm{d}x \leq 0.$$

Moreover, the functional D allows us to formulate a version of Boltzmann's H-theorem. Specifically, $D(F) = 0$ whenever F is a local Fermi-Dirac distribution, i.e. a local thermodynamic equilibrium.

Global Maxwellian functions

As was mentioned earlier, in the theory of kinetic equations the Boltzmann equations plays an essential role. The collision operator in the classical case writes

$$C_B(F) = \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(v-v_*,\omega)(F'F'_* - FF_*)\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

The local thermodynamic equilibria are functions such that $C_B(F) = 0$. In this case, they are local Maxwellian distributions, i.e. distribution functions of the form

$$\frac{\rho(t,x)}{(2\pi\theta(t,x))^{3/2}} \exp\left(-\frac{|v-u(t,x)|^2}{2\theta(t,x)}\right),$$

where ρ is the local density, u is the average velocity, and θ is the local temperature.

The global Maxwellian distribution functions, in addition to being local Maxwellian distribution functions, satisfy the free transport equation. It is easy to see that the function $e^{-|x-tv|^2}$ is a global Maxwellian function. One can prove that global Maxwellians of finite mass are the functions of the form (see [30])

$$\frac{m}{(2\pi)^d}\sqrt{\det(Q)} \exp\left(-\frac{1}{2}q(t,x,v)\right),$$

where

$$q(t,x,v) = a|x-tv|^2 + 2b(x-tv)\cdot v + c|v|^2 + 2v\cdot B(x-tv)\times v,$$

where $a > 0$, $c > 0$, $b \in \mathbb{R}$, $B = -B^T \in M_d(\mathbb{R})$, and the matrix $Q = (ac - b^2)I + B^2$ is positive definite.

2 General scope of the thesis

The goal of the first three chapters is to study the hydrodynamic limits of the Boltzmann equation for the Fermi-Dirac statistics. We obtain these macroscopic limits when the fluid is dense enough so that particles undergo many collisions per unit of time. As a parameter of the problem we use the Knudsen number Kn. This dimensionless number represents the relation of the mean free path of a particle to the characteristic length to the flow. The case $\mathrm{Kn} \to +\infty$ corresponds to the vacuum; we are interested in the case $\mathrm{Kn} \to 0$, i.e. in the situation when the fluid becomes dense.

The fourth chapter deals with in the Boltzmann equation linearised around a global Maxwellian function. The goal is to study the existence of its solutions, their behaviour for large time, and estimate these solutions. Moreover, we can introduce a scattering operator, "density evolution modulo free transport", and estimate its norm and spectral gap.

In order to simplify the notations, we will denote for a function $s(v) \in L^1(\mathbb{R}^3)$

$$\langle s \rangle = \int_{\mathbb{R}^3} s(v)\,\mathrm{d}v.$$

3 The models considered

Euler limit

In the first chapter we place ourselves in the most abstract setting. We assume that the collision operator $C(F)$ satisfies the conservation laws:

$$\langle C(F)\rangle = 0, \quad \langle vC(F)\rangle = 0, \quad \langle |v|^2 C(F)\rangle = 0,$$

that the entropy production rate is non-negative, i.e.

$$\left\langle C(F) \ln\left(\frac{1-F}{F}\right)\right\rangle \geq 0,$$

and that the analog of Boltzmann's H-theorem holds: in other words, the following assertions are equivalent:

- F is a Fermi-Dirac distribution,
- $C(F) = 0$,
- the entropy production rate is zero $\left\langle C(F)\ln\left(\frac{1-F}{F}\right)\right\rangle = 0$.

First, we show that such an operator exists; indeed, the operator defined in (1.2) satisfies these conditions.

Second, we suppose that the Knudsen number is of order ε. This yields the rescaled kinetic equation

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} C(F_\varepsilon). \tag{3.1}$$

Under formally consistent assumptions on the convergence of moments, entropy density, and entropy production rate of solutions F_ε of the scaled kinetic equation (3.1), we obtain the following theorem (see theorem I.5 in the main body of this thesis):

Theorem. *The limit $F = \lim_{\varepsilon\to 0} F_\varepsilon$ is a Fermi-Dirac distribution; moreover, there exists a parametrization $\vec{\rho}(t,x)$ of F such that $\vec{\rho}$ satisfies a hyperbolic Euler-type system of conservation laws.*

Compressible Navier-Stokes equations

In the second chapter we pursue the investigations described in chapter I. We take the collision operator of the form (1.2). This additional information allows us to obtain the Navier-Stokes equations as a correction of the previously obtained Euler equations via a Chapman-Enskog expansion.

We consider the first Fréchet derivative of $C(F)$ at a Fermi-Dirac distribution F:

$$L_F[g] = \frac{1}{F(1-F)} DC(F) \circ (F(1-F)g).$$

This operator is naturally defined as a linear, potentially unbounded operator on the space $L^2(F(1-F)\,\mathrm{d}v)$. We will assume that this operator is self-adjoint, nonpositive, and satisfies the Fredholm alternative. We provide sufficient conditions on the collision kernel b ensuring the aforementioned properties of the operator L_F (see theorems II.1, II.3).

Theorem. *If the collision kernel b has the separate form*

$$b(v - v_*, \omega) = |v - v_*| \hat{b}\left(\frac{(v - v_*)\cdot\omega}{|v - v_*|}\right)$$

and satisfies the weak cut-off condition

$$\hat{b}(\omega) \in L^1(\mathbb{S}^2),$$

then L_F is self-adjoint, nonpositive, and satisfies the Fredholm alternative in the space $L^2(F(1-F)\,\mathrm{d}v)$. Moreover its nullspace is spanned by the functions 1, v_1, v_2, v_3, and $|v|^2$.

These properties of the operator L_F together with the Chapman-Enskog expansion allow us to establish the following result (theorem II.5 of the main body):

Theorem. *There exist a Navier-Stokes system with a specific form of viscosity and thermal diffusivity on the variables $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ such that if the Fermi-Dirac distribution F_ε is parametrized by these variables, then, under formally consistent assumptions on the convergence, there exist functions g_ε and w_ε such that the function*

$$H_\varepsilon = F_\varepsilon + \varepsilon F_\varepsilon(1 - F_\varepsilon)(g_\varepsilon + \varepsilon w_\varepsilon)$$

is an approximate solution of order two of the scaled kinetic equation

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\varepsilon} C(F).$$

Incompressible Navier-Stokes limit

In this chapter, we merge the viscous correction with low Mach number limit. We obtain incompressible Navier-Stokes equations governing an absolute Fermi-Dirac distribution F. We assume that the Knudsen number is of order ε^q, the considered time compared to a typical time scale is of order ε, and the distance to the Fermi-Dirac distribution F is of order ε^r. The scaled kinetic equation then becomes

$$\varepsilon\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon^q} C(F_\varepsilon)$$

and we seek its solution in the form

$$F_\varepsilon = F + \varepsilon^r F(1-F) g_\varepsilon.$$

Note that only the case $r = q = 1$ is compatible with the usual incompressible Navier-Stokes equations.

The key result of this chapter is the following theorem (theorem III.1):

Theorem. *Under the formally consistent assumptions on the convergence of the function g_ε and its moments as $\varepsilon \to 0$, the limiting relative number density fluctuation $g = \lim_{\varepsilon\to 0} g_\varepsilon$ has the form*

$$g(t,x,v) = \rho(t,x) + u(t,x) \cdot v + \theta(t,x) \left(\frac{|v|^2}{2} - K_g \right),$$

where K_g is an appropriate constant, where the velocity field u is divergence-free

$$\nabla_x \cdot u = 0,$$

and where the density and temperature fluctuations satisfy the Boussinesq relation

$$\nabla_x(\rho + \theta) = 0.$$

Moreover, following different possible values of $r \geq 1$ and $q \geq 1$, the variables ρ, u, and θ are weak solutions of fluid dynamic equations and we can write these equations explicitly. For example, in the case $r = q = 1$ these equations are

$$\partial_t u + (u \cdot \nabla_x) u + \nabla_x p = \frac{\mu_*}{E_2} \Delta u,$$

$$\partial_t \theta + u \cdot \nabla_x \theta = \frac{k_*}{C_A} \Delta\theta,$$

for positive constants μ_*, k_*, E_2, and C_A defined in section 2 of chapter III.

Solutions of the Boltzmann equation linearised about a global Maxwellian

In chapter IV we consider the Boltzmann equation linearised about a global Maxwellian function M:

$$\partial_t g(t,x,v) + v \cdot \nabla_x g(t,x,v) = L_t[g](t,x,v), \tag{3.2}$$

where the operator L_t writes

$$L_t[g](t,x,v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} M_* b(v - v_*, \omega)(g' + g'_* - g - g_*) \,\mathrm{d}v_* \,\mathrm{d}\omega.$$

There are two competing mechanisms in the density evolution for this equation: on the one hand, the dissipation increases entropy and therefore the solution relaxes to a thermodynamic equilibrium state; on the other hand, dispersion rarefies the collisions between particles and hence diminishes the effect of dissipation. The particular choice of the function M allows us to find a balance between these two mechanisms. Given that the linearized collision operator L_t is naturally defined on the Hilbert space $\mathcal{X}_M(t) = L^2(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v)$, and taking into account that this operator is non-positive, it is logical to seek solutions of the linearized Boltzmann equation (3.2) in the space $\mathcal{Y}_M = L^\infty(\mathbb{R}, \mathcal{X}_M(t))$.

We provide the sufficient conditions such that the following statements hold:

Theorem. *The operator L_t is a linear bounded operator on the space $\mathcal{X}_M(t)$ and the integral of its norm is finite:*

$$\mu = \int_{\mathbb{R}} \|L_t\| \,\mathrm{d}t < +\infty.$$

See theorem IV.1 in the main body for more details.

Theorem. *If $\mu < 1$, then for all $g^{in} \in \mathcal{X}_M(0)$ there exists a unique mild solution $g(t,x,v)$ of the equation (3.2) such that $g(0,x,v) = g^{in}(x,v)$. Moreover, there exist unique functions $g^{-\infty} \in \mathcal{X}_M(0)$ and $g^{+\infty} \in \mathcal{X}_M(0)$ such that*

$$\lim_{t \to \pm\infty} \|g(t, x + tv, v) - g^{\pm\infty}(x,v)\|_{\mathcal{X}_M(0)} = 0.$$

These results are shown in theorems IV.4 and IV.6. Moreover, we also establish the following result (see theorem IV.5 in the main body):

Theorem. *If $\mu < 1$, then*

- *for all* $g^{+\infty} \in \mathcal{X}_M(0)$ *there exists a unique mild solution* $g(t,x,v) \in \mathcal{Y}_M$ *of the equation* (3.2) *such that*

$$\lim_{t\to+\infty} \|g(t,x+tv,v) - g^{+\infty}(x,v)\|_{\mathcal{X}_M(0)} = 0,$$

- *for all* $g^{-\infty} \in \mathcal{X}_M(0)$ *there exists a unique mild solution* $g(t,x,v) \in \mathcal{Y}_M$ *of the equation* (3.2) *such that*

$$\lim_{t\to-\infty} \|g(t,x+tv,v) - g^{-\infty}(x,v)\|_{\mathcal{X}_M(0)} = 0.$$

The above results allow us to define the scattering operator $\mathcal{S}$ acting on $\mathcal{X}_M(0)$ as "evolution of the density factored by free transport". More formally, if $g^{-\infty} \in \mathcal{X}_M(0)$, then the above theorem give the existence of a function $g \in \mathcal{Y}_M$ such that $\lim_{t\to-\infty} g = g^{-\infty}$ and g is a mild solution of the equation (3.2). Then for this function g we obtain a limit $g^{+\infty}$ as $t \to +\infty$ and then put $\mathcal{S}[g^{-\infty}] = g^{+\infty}$. The key property of this scattering operator is the following theorem (see theorem IV.9)

Theorem. *If* $\mu < 1$ *and if the function* $g_0 \in \mathcal{X}_M(0)$ *satisfies the identities*

$$\int_{\mathbb{R}^d} g_0 M \,\mathrm{d}x\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}^d} v g_0 M \,\mathrm{d}x\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}^d} x g_0 M \,\mathrm{d}x\,\mathrm{d}v = 0,$$
$$\int_{\mathbb{R}^d} |v|^2 g_0 M \,\mathrm{d}x\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}^d} |x|^2 g_0 M \,\mathrm{d}x\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}^d} (x\cdot v) g_0 M \,\mathrm{d}x\,\mathrm{d}v = 0,$$
$$\int_{\mathbb{R}^d} (x\times v) g_0 M \,\mathrm{d}x\,\mathrm{d}v = 0,$$

then

$$0 \le \|g_0\|^2_{\mathcal{X}_M(0)} - \|\mathcal{S}[g_0]\|^2_{\mathcal{X}_M(0)} \le \|g_0\|^2_{\mathcal{H}},$$

where $\mathcal{H} = L^2(\nu(v)M(0,x,v)\,\mathrm{d}x\,\mathrm{d}v)$ *for a known function* ν *vanishing at infinity.*

It is important to notice that under the condition of finite μ one can not avoid the function weight μ in the definition of the space $\mathcal{H}$, because the operator L_t does not have a spectral gap.

In [22], Golse considers the question on comparability of two entities: on the one side, the difference $H(f) - H(\mathcal{S}[f])$, where $\mathcal{S}$ is the scattering operator for nonlinear Boltzmann equation and H is the associated H-function for this equation, and $H(f) - H(M_f(0))$, where M_f is a global Maxwellian function admitting the same moments as the function f. It was shown that

$$0 \le H(f) - H(\mathcal{S}[f]) \le H(f) - H(M_f(0)).$$

The question is whether one can obtain an inequality of the form

$$c(H(f) - H(M_f(0)))^\alpha \leq H(f) - H(\mathcal{S}[f])$$

for some constants $\alpha > 0$ and $c > 0$, which is analogous to the Cercignani's conjecture on entropy production in the context of the Boltzmann equation over $\mathbb{R}^d$ in the scattering regime (see, for example, [39]).

Our result can be viewed as a negative answer to the same question but for the linearized Boltzmann equation.

On the other hand, this implies that thanks to the particular choice of the global Maxwellian function M the solutions of the equation (3.2) are not relaxing to a thermodynamic equilibrium.

4 List of the works constituting this thesis

The chapters of this thesis are based on the following works:

- Chapter I: article [45], published in *Asymptotic Analysis*.
- Chapter II: article [47], submitted to *Kinetic and Related Models*.
- Chapter III: article [46], submitted to *Bulletin des Sciences Mathématiques*
- Chapter IV: article [44], in preparation.

Chapter I

The Euler limit for kinetic models with Fermi-Dirac statistics

1 Introduction

In this work we establish the connection between kinetic theory for Fermi-Dirac statistics and macroscopic fluid dynamics. We derive formal limits; in order to do that, we introduce a scaling for standard kinetic equation (see, for example, [32]) of the form

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} C(F_\varepsilon). \tag{I.1.1}$$

Here F_ε is a non-negative function representing the density of particles with position x and velocity v in the single-particle phase space $\mathbb{R}^3_x \times \mathbb{R}^3_v$ at time t. The interaction of particles through collisions is given by the operator $C(F)$; this operator acts only on variable v and is non-linear in the general case. We will keep this operator abstract.

We base the connection between kinetic and macroscopic dynamics on the conservation properties and entropy relations implying that the equilibria are Fermi-Dirac (i.e. of the form $1/(1+\exp(c_1 + c_2|v-u|^2))$) distributions.

It is important to notice that our approach differs from Hilbert expansion employed in works of C. Cercignani. Our results highlight the role of entropy as it was done in [2]; the convergence assumptions are similar to those in [4]. We also adopt the formalism for moments of distributions proposed in [3].

In the section 4 we examine the moments of Fermi-Dirac distributions and establish that the parameters of such a distribution are related to the moments by a diffeomorphism. While in the case of Maxwellian distributions such a relation is rather evident, the case of Fermi-Dirac distributions requires an additional analysis.

We obtain the macroscopic limits when the fluid is dense enough that particles undergo many collisions per unit of time. In order to describe this situation, we introduce a small parameter ε, called the Knudsen number, that represents the ratio of mean free path of particles between collisions to some characteristic length of the flow.

Conservation properties are used to derive the compressible Euler equations from (I.1.1); we will do so in section 6, assuming a formally consistent convergence for fluid dynamical moments and entropy of the kinetic equation (I.1.1) (see theorem I.5).

2 Kinetic models with Fermi-Dirac statistics

Denote for an integrable function s its moment $\langle s(v) \rangle = \int\limits_{\mathbb{R}^3} s(v)dv$.

We assume that for all measurable functions rapidly decaying on infinity

$$F(t,x,v) : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \quad 0 \le F \le 1 \text{ a.e.} \tag{I.2.1}$$

the collision operator C satisfies the conservation properties

$$\langle C(F)\rangle = 0, \quad \langle C(F)v\rangle = 0, \quad \langle C(F)|v|^2\rangle = 0, \tag{I.2.2}$$

corresponding to the conservation of mass, momentum, and energy through the collision process. If we multiply the equation (I.1.1) by 1, v, $|v|^2$ and integrate with respect to v, we obtain the respective local conservation laws:

$$\begin{aligned} \partial_t\langle F_\varepsilon\rangle + \nabla_x \cdot \langle F_\varepsilon v\rangle =&0,\\ \partial_t\langle F_\varepsilon v\rangle + \nabla_x \cdot \langle F_\varepsilon v \otimes v\rangle =&0,\\ \partial_t\langle F_\varepsilon |v|^2\rangle + \nabla_x \cdot \langle F_\varepsilon |v|^2 v\rangle =&0. \end{aligned} \tag{I.2.3}$$

We also assume that for every measurable rapidly decaying function F satisfying (I.2.1) the non-negative quantity

$$\left\langle C(F)\ln\left(\frac{1-F}{F}\right)\right\rangle \geq 0 \tag{I.2.4}$$

is the entropy production rate for this collision process.

Observe that

$$\partial_t(F_\varepsilon \ln F_\varepsilon + (1-F_\varepsilon)\ln(1-F_\varepsilon)) = (\ln F_\varepsilon - \ln(1-F_\varepsilon))\partial_t F_\varepsilon,$$

$$\nabla_x(F_\varepsilon \ln F_\varepsilon + (1-F_\varepsilon)\ln(1-F_\varepsilon)) = (\ln F_\varepsilon - \ln(1-F_\varepsilon))\nabla_x F_\varepsilon,$$

therefore multiplying both parts of the equation (I.1.1) by $\ln\left(\frac{F_\varepsilon}{1-F_\varepsilon}\right)$ and integrating with respect to v yields

$$\left\langle \ln\left(\frac{1-F_\varepsilon}{F_\varepsilon}\right)\partial_t F_\varepsilon + \ln\left(\frac{1-F_\varepsilon}{F_\varepsilon}\right) v\cdot\nabla_x F_\varepsilon\right\rangle$$

$$\begin{aligned} = \langle \partial_t(F_\varepsilon \ln F_\varepsilon + (1-F_\varepsilon)\ln(1-F_\varepsilon)) &\\ + v\cdot\nabla_x(F_\varepsilon \ln F_\varepsilon + (1-F_\varepsilon)\ln(1-F_\varepsilon))\rangle& \end{aligned}$$

$$= -\left\langle C(F_\varepsilon)\ln\left(\frac{1-F_\varepsilon}{F_\varepsilon}\right)\right\rangle \leq 0,$$

which gives us a local entropy inequality

$$\begin{aligned} \partial_t \left\langle (F_\varepsilon \ln F_\varepsilon + (1-F_\varepsilon)\ln(1-F_\varepsilon))\right\rangle &\\ + \nabla_x \cdot \left\langle (vF_\varepsilon \ln F_\varepsilon + v(1-F_\varepsilon)\ln(1-F_\varepsilon))\right\rangle \leq 0.& \end{aligned}$$

Finally, the equilibria are assumed to be characterized by zero entropy production rate and are given by the class of Fermi-Dirac distributions (for more information see, for example, [28])

$$F(v) = \left(1 + \exp\left(\frac{|v-u|^2}{2\theta} - f\right)\right)^{-1}, \quad u \in \mathbb{R}^3, \quad f \in \mathbb{R}, \quad \theta > 0. \tag{I.2.5}$$

We assume the following analogy of the Boltzmann's H-theorem for the Fermi-Dirac statistics:

Theorem I.1. *For every measurable rapidly decaying function F satisfying* (I.2.1) *with at most polynomially increasing* $|\ln\left(\frac{1-F}{F}\right)|$ *the following properties are equivalent:*

$$\begin{aligned} &1) \quad C(F) = 0, \\ &2) \quad \left\langle C(F)\ln\left(\tfrac{1-F}{F}\right)\right\rangle = 0, \\ &3) \quad F \text{ is a Fermi-Dirac distribution of the form (I.2.5).} \end{aligned} \tag{I.2.6}$$

3 Example of a collision operator for Fermi-Dirac statistics

In this section we give an example of a collision operator satisfying the conservation properties (I.2.2) and with positive entropy production rate (I.2.4) together with H-theorem.

Consider the operator studied in [14]:

$$C(F) = \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b\left(F'F'_*(1-F)(1-F_*) - FF_*(1-F')(1-F'_*)\right) \mathrm{d}v_* \, \mathrm{d}\omega,$$

where for $\omega \in \mathbb{S}^2$

$$v' = v - \omega(v - v_*, \omega), \quad v'_* = v_* + \omega(v - v_*, \omega),$$

$$F' = F(t,x,v'), \quad F_* = F(t,x,v_*), \quad F'_* = F(t,x,v'_*)$$

and $b = b(|v-v_*|, |(\omega, v-v_*)|)$ — a collision kernel. Observe that the definition of vectors v' and v'_* implies the following relations:

$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2. \tag{I.3.1}$$

Observe also that

$$|v - v_*| = |v' - v'_*|, \quad |(v - v_*, \omega)| = |(v' - v'_*, \omega)|.$$

The above expression allows us to take the collision kernel in the form

$$\bar{b}_\omega(v, v_*, v', v'_*) = b\left(\frac{1}{2}|v - v_*| + \frac{1}{2}|v' - v'_*|, \frac{1}{2}|(\omega, v - v_*)| + \frac{1}{2}|(v' - v'_*, \omega)|\right).$$

In order to prove that the operator C indeed satisfies the relations (I.2.2) and (I.2.4), we first establish the following proposition:

Proposition 1. *Suppose that we have measurable functions $W : (\mathbb{R}^3)^4 \to \mathbb{C}$ and $h : \mathbb{R}^3 \to \mathbb{C}$ such that the integral*

$$I = \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} |W(v, v_*, v - (v - v_*, \omega)\omega, v_* + (v - v_*, \omega)\omega)h(v)| \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega$$

exists. Then the following identity holds:

$$I = \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} W(v_*, v, v_* + (v - v_*, \omega)\omega, v - (v - v_*, \omega)\omega)h(v_*) \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

$$= \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} W(v', v'_*, v, v_*)h(v') \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega$$

$$= \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} W(v'_*, v', v_*, v)h(v'_*) \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

We give the proof of this proposition in appendix B.

The following proposition shows that the operator C indeed satisfies the conservation properties (I.2.2):

Proposition 2.

$$\langle C(F)\rangle = 0, \quad \langle vC(F)\rangle = 0, \quad \langle |v|^2 C(F)\rangle = 0$$

for any measurable function F rapidly decaying on infinity satisfying (I.2.1).

Proof. Take any function $h : \mathbb{R}^3 \to \mathbb{R}$, $h(v) \in \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}$. Let $w_i \in \mathbb{R}^3$; introduce a function

$$\begin{aligned} W(w_1, w_2, w_3, w_4) =& \bar{b}(w_1, w_2, w_3, w_4)F(w_3)F(w_4)(1 - F(w_1))(1 - F(w_2)) \\ &- \bar{b}(w_1, w_2, w_3, w_4)F(w_1)F(w_2)(1 - F(w_3))(1 - F(w_4)) \end{aligned} \tag{I.3.2}$$

and observe that

$$\begin{aligned} W(w_1, w_2, w_3, w_4) &= W(w_2, w_1, w_3, w_4) \\ = W(w_1, w_2, w_4, w_3) &= -W(w_3, w_4, w_1, w_2) \end{aligned}$$

for all w_i. Applying proposition 1 to W defined above and h we can conclude that

$$\langle h(v)C(F)\rangle = -\frac{1}{4}\iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b\left(F'F'_*(1-F)(1-F_*) - FF_*(1-F')(1-F'_*)\right) \cdot \left(h(v') + h(v'_*) - h(v) - h(v_*)\right)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

Since $h(v) \in \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}$, applying the relation (I.3.1) yields

$$h(v') + h(v'_*) - h(v) - h(v_*) = 0$$

for all v, v_*, and ω. Therefore

$$\langle C(F)h(v)\rangle = 0$$

and the proposition holds. □

Proposition 3. *For any measurable function F rapidly decaying on infinity satisfying* (I.2.1) *we have*

$$\left\langle C(F)\ln\left(\frac{1-F}{F}\right)\right\rangle \geq 0.$$

Proof. We apply the proposition 1 with W defined in (I.3.2) and $h = \ln\left(\frac{1-F}{F}\right)$ and obtain that

$$\left\langle C(F)\ln\left(\frac{1-F}{F}\right)\right\rangle = \frac{1}{4}\left\langle C(F)\ln\left(\frac{1-F}{F}\frac{1-F_*}{F_*}\frac{F'}{1-F'}\frac{F'_*}{1-F'_*}\right)\right\rangle.$$

Denote $X = (1-F)(1-F_*)F'F'_*$ and $Y = FF_*(1-F')(1-F'_*)$, then we can rewrite the above expression as

$$\frac{1}{4}\iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(X-Y)\ln(X/Y)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

The function $(X,Y) \to (X-Y)\ln(X/Y)$ is non-negative and so is the collision kernel b, therefore we can conclude that

$$\left\langle C(F)\ln\left(\frac{1-F}{F}\right)\right\rangle \geq 0.$$

□

Proposition 4 (H-theorem)**.** *For every measurable rapidly decaying F with at most polynomially increasing $\left|\ln\left(\frac{1-F}{F}\right)\right|$ satisfying* (I.2.1) *the following properties are equivalent:*

$$\begin{aligned} &1)\quad C(F) = 0,\\ &2)\quad \left\langle C(F)\ln\left(\tfrac{1-F}{F}\right)\right\rangle = 0,\\ &3)\quad F \textit{ is a Fermi-Dirac distribution of the form } \text{(I.2.5)}. \end{aligned} \tag{I.3.3}$$

Proof. If F is of the form (I.2.5), then

$$\ln\left(\frac{1-F}{F}\right) \in \operatorname{span}\left\{1, v_1, v_2, v_3, |v^2|\right\},$$

hence the entropy production rate is zero. On the other hand, by direct substitution one can show that in this case $C(F) = 0$.

If the entropy production rate is zero, then applying the proposition 1 with W defined in (I.3.2) and $h = \ln\left(\frac{1-F}{F}\right)$ yields

$$\left\langle C(F)\ln\left(\frac{1-F}{F}\frac{1-F_*}{F_*}\frac{F'}{1-F'}\frac{F'_*}{1-F'_*}\right)\right\rangle = 0.$$

Let for simplicity $G = \frac{F}{1-F}$, then we can rewrite this expression as

$$\iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(1-F)(1-F_*)(1-F')(1-F'_*) \cdot (G'_*G' - G_*G)\ln\left(\frac{G'_*G'}{G_*G}\right) \mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

Since the function $(X, Y) \to (X - Y)\ln(X/Y)$ is non-negative on $\mathbb{R}^2$ and vanishes if and only if $X = Y$, we deduce that $G'_*G' = G_*G$ almost everywhere in v, or, in other words

$$\ln G'_* + \ln G' = \ln G_* + \ln G \quad \text{a.e.}$$

By Boltzmann-Gronwall theorem (see, for example, [8, 18, 41]) this implies that

$$\ln G(v) = a + w \cdot v + c|v|^2$$

for some constants $a, c \in \mathbb{R}$ and $w \in \mathbb{R}^3$. In its turn, this implies that G is a local Maxwellian distribution and therefore F is a local Fermi-Dirac distribution.

Finally, if $C(F) = 0$,

$$0 = C(F) = \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b(1-F)(1-F_*)(1-F')(1-F'_*)$$

$$\cdot\,(G'_*G' - G_*G)\ \mathrm{d}v_*\ \mathrm{d}\omega$$

with $G = \frac{F}{1-F}$. Since the factor $b(1-F)(1-F_*)(1-F')(1-F'_*)$ is non-negative, we obtain that $G'_*G' - G_*G = 0$, which, as above, implies that G is a local Maxwellian distribution, and therefore F is a Fermi-Dirac distribution. □

4 Moments and parameters of Fermi-Dirac distributions

Recall the definition of Maxwellian distributions:

$$M(v) = M_{\rho^M,u^M,\theta^M}(v) = \frac{\rho^M}{(2\pi\theta^M)^{3/2}} e^{-\frac{\left|v-u^M\right|^2}{2\theta^M}}$$

for parameters $\rho^M \geq 0$, $u^M \in \mathbb{R}^3$, $\theta^M > 0$. Define the vector of moments for such a distribution:

$$\begin{pmatrix}\mu_0\\ \mu_1\\ \mu_2\end{pmatrix} = \begin{pmatrix}\langle 1\rangle_M\\ \langle v\rangle_M\\ \langle |v|^2\rangle_M\end{pmatrix} = \begin{pmatrix}\rho^M\\ \rho^M u^M\\ \rho^M|u^M|^2 + 3\rho^M\theta^M\end{pmatrix}. \tag{I.4.1}$$

It is easy to see that the space of possible moments is described by

$$\mu_0 \geq 0, \quad \mu_1 \in \mathbb{R}^3, \quad \mu_0\mu_2 \geq |\mu_1|^2.$$

Moreover, the map

$$T_M : \mathbb{R}^5 \to \mathbb{R}^5, \quad T_M(\rho^M, u^M, \theta^M) = (\mu_0, \mu_1, \mu_2)$$

is a diffeomorphism; indeed, the expression (I.4.1) clearly shows that T_M is smooth. In addition, the same expression allows us to say that

$$\begin{aligned}\rho^M &= \mu_0,\\ u^M &= \frac{\mu_1}{\mu_0},\\ \theta^M &= \frac{\mu_2 - |\mu_1|^2/\mu_0}{3\mu_0},\end{aligned}$$

hence the inverse application exists and is also smooth.

Another observation requires a slightly different parametrisation of a Maxwellian distribution. If

$$M(v) = e^{-\frac{|v-u^M|^2}{2\theta^M} - f^M},$$

and if the moments are defined as

$$\rho^M = \langle M(v) \rangle, \quad \mathcal{E}^M = \frac{1}{3}\langle |v-u|^2 M(v) \rangle,$$

then the following equations hold:

$$\rho^M \partial_{\rho^M} f^M + \frac{5}{3}\mathcal{E}^M \partial_{\mathcal{E}^M} f^M = 0,$$

$$\rho^M \partial_{\rho^M} \theta^M + \frac{5}{3}\mathcal{E}^M \partial_{\mathcal{E}^M} \theta^M = \frac{2}{3}\theta^M.$$

This result quickly follows from the relations

$$f^M = -\ln\left(\frac{\rho^M}{(2\pi\theta^M)^{3/2}}\right),$$

$$\rho^M = \mu_0, \quad \mathcal{E}^M = \frac{\mu_2 - |\mu_1|^2/\mu_0}{3}.$$

The goal of this section is to establish the similar properties of moments of Fermi-Dirac distributions and find under which conditions the parameters of this distribution can be found via its moments.

We study the Fermi-Dirac distribution

$$F_{f,u,\theta}(v) = \frac{1}{1 + \exp\left(\frac{|v-u|^2}{2\theta} - f\right)}.$$

The admissible parameters form an open convex set $D \subset \mathbb{R}^5$:

$$(f, u, \theta) \in D = \mathbb{R} \times \mathbb{R}^3 \times (0, \infty).$$

The moments of the distribution F are

$$\langle F(v) \rangle \geq 0, \quad \langle vF(v) \rangle \in \mathbb{R}^3, \quad \langle |v|^2 F(v) \rangle \geq 0.$$

Therefore, we can define a map

$$T_F : D \to \mathbb{R}^5, \quad T_F(f, u, \theta) = \Big(\langle F(v) \rangle, \langle vF(v) \rangle, \langle |v|^2 F(v) \rangle\Big).$$

Let U be the range of T_F, i.e. $U = T_F(D)$. We will examine the properties of U and establish that T_F is a diffeomorphism $D \to U$. To simplify the calculations, we will introduce the following notations:

$$\rho = \int_{\mathbb{R}^3} \frac{\mathrm{d}v}{1+\exp\left(\frac{|v-u|^2}{2\theta} - f\right)} = \theta^{3/2} \int_{\mathbb{R}^3} \frac{\mathrm{d}v}{1+\exp\left(\frac{|v|^2}{2} - f\right)}, \tag{I.4.2}$$

$$\mathcal{E} = \frac{1}{3}\int_{\mathbb{R}^3} \frac{|v-u|^2\,\mathrm{d}v}{1+\exp\left(\frac{|v-u|^2}{2\theta} - f\right)} = \frac{\theta^{5/2}}{3} \int_{\mathbb{R}^3} \frac{|v|^2\,\mathrm{d}v}{1+\exp\left(\frac{|v|^2}{2} - f\right)}. \tag{I.4.3}$$

It is easy to see that

$$T_F(f,u,\theta) = (\rho, \rho u, \rho|u|^2 + 3\mathcal{E}).$$

The quantities ρ and $\mathcal{E}$ can be expressed in terms of polylogarithms assuming $\langle F(v)\rangle \neq 0$. We discuss the properties of these special functions in appendix A.

The following theorem establishes the invertibility of the map T_F:

Theorem I.2. *If ρ and $\mathcal{E}$ are given by* (I.4.2) *and* (I.4.3), *respectively, then the following statements hold:*

1. *the ratio $\frac{\rho}{\mathcal{E}^{3/5}}$ depends only on f,*
2. *setting $J = \frac{(8\pi\sqrt{2})^{2/5}}{3}\left(\frac{5}{2}\right)^{3/5}$, then the map*

$$f \to \frac{\rho}{\mathcal{E}^{3/5}}$$

   is $\mathcal{C}^\infty(\mathbb{R};(0,J))$ and strictly monotone; there exists a function

$$\bar{f} \in \mathcal{C}^1((0,J),\mathbb{R}), \quad f = \bar{f}\left(\frac{\rho(f,\theta)}{(\mathcal{E}(f,\theta))^{3/5}}\right),$$

3. *the function θ is given by $\theta = \left(\frac{\rho}{4\sqrt{2}\pi\Gamma(3/2)\mathcal{F}_{3/2}(f)}\right)^{2/3}$*
4. *the map T_F is a diffeomorphism,*
5. *f and θ seen as functions of ρ and $\mathcal{E}$ satisfy*

$$\rho\partial_\rho f + \frac{5}{3}\mathcal{E}\partial_{\mathcal{E}} f = 0, \tag{I.4.4}$$

$$\rho\partial_\rho \theta + \frac{5}{3}\mathcal{E}\partial_{\mathcal{E}}\theta = \frac{2}{3}\theta. \tag{I.4.5}$$

Proof. We can express ρ and $\mathcal{E}$ in terms of polylogarithms:

$$\rho(f,\theta) = 4\pi\sqrt{2}\theta^{3/2}\Gamma(3/2)\mathcal{F}_{3/2}(f), \tag{I.4.6}$$

$$\mathcal{E}(f,\theta) = 8\pi\sqrt{2}\theta^{5/2}\Gamma(5/2)\mathcal{F}_{5/2}(f),$$

with

$$\mathcal{F}_p(w) = \frac{1}{\Gamma(p)}\int_0^\infty \frac{t^{p-1}}{e^{t-w}+1}\mathrm{d}t, \quad \mathcal{G}_p(w) = (\mathcal{F}_p(w))^{1/p},$$

which yields (1). Let

$$a = \frac{\rho}{\mathcal{E}^{3/5}}.$$

The expressions for moments together with properties of polylogarithms yield

$$a = \frac{4\pi\sqrt{2}\Gamma(3/2)}{(8\pi\sqrt{2}\Gamma(5/2))^{3/5}}\frac{\mathcal{F}_{3/2}(f)}{(\mathcal{F}_{5/2}(f))^{3/5}} = \frac{5}{2}\frac{4\pi\sqrt{2}\Gamma(3/2)}{(8\pi\sqrt{2}\Gamma(5/2))^{3/5}}\frac{\mathrm{d}}{\mathrm{d}f}\mathcal{G}_{5/2}(f).$$

As established in the theorem A.I (see appendix A), the function $f \to \frac{\mathrm{d}}{\mathrm{d}f}\mathcal{G}_{5/2}(f)$ is strictly monotone, hence the map $f \mapsto a$ is invertible. In order to obtain possible values of a, we study

$$\lim_{f\to\pm\infty}\frac{\mathcal{F}_{3/2}(f)}{(\mathcal{F}_{5/2}(f))^{3/5}}.$$

The Taylor development of polylogarithms for $|z| < 1$ writes (see, for example, [13],[35],[37])

$$-\operatorname{Li}_s(-z) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}\mathrm{d}t}{e^t/z+1} = \sum_{k\geq 1}\frac{z^k}{k^s},$$

in other words, $-\operatorname{Li}_s(-z) = z + \mathcal{O}(z^2)$ near $z = 0$, which implies that

$$\lim_{f\to-\infty}\frac{\mathcal{F}_{3/2}(f)}{(\mathcal{F}_{5/2}(f))^{3/5}} = 0.$$

On the other hand, we know the limiting behaviour of polylogarithms when $f \to +\infty$ and $p \neq -1, -2, -3, \dots$ (see, for example, [43]):

$$\lim_{f\to+\infty}\frac{\mathcal{F}_p(f)}{f^p} = \frac{1}{\Gamma(p+1)},$$

which yields

$$\lim_{f\to\infty}\frac{\mathcal{F}_{3/2}(f)}{(\mathcal{F}_{5/2}(f))^{3/5}} = \frac{(\Gamma(7/2))^{3/5}}{\Gamma(5/2)},$$

hence

$$a < \frac{4\pi\sqrt{2}\Gamma(3/2)}{(8\pi\sqrt{2}\Gamma(5/2))^{3/5}} \frac{(\Gamma(7/2))^{3/5}}{\Gamma(5/2)} = \frac{(8\pi\sqrt{2})^{2/5}}{3} \left(\frac{5}{2}\right)^{3/5} = J.$$

Thanks to the monotonicity of the function $f \to \frac{\mathrm{d}}{\mathrm{d}f}\mathcal{G}_{5/2}(f)$, we obtain that $a \in (0, J)$. The regularity of the function $\bar{f}$ follows from the regularity of polylogarithms; the expression for θ is a direct consequence of (I.4.6) and the previous point.

Finally, the following equations show that the map T_F is invertible:

$$u = \frac{\langle vF(v)\rangle}{\rho},$$

$$\mathcal{E} = \frac{1}{3}\left(\langle |v|^2 F(v) - \rho|u|^2\rangle\right),$$

$$f = \bar{f}\left(\frac{\rho}{\mathcal{E}^{3/5}}\right),$$

$$\theta = \left(\frac{\rho}{4\sqrt{2}\pi\Gamma(3/2)\mathcal{F}_{3/2}(f)}\right)^{2/3}.$$

The differential equations (I.4.4) (I.4.5) follow from (1) and (2), respectively. □

Remark. The inequality

$$\frac{\rho}{\mathcal{E}^{3/5}} < J = \left(\frac{5}{2}\right)^{3/5} \left(4\pi\frac{2\sqrt{2}}{3}\right)^{2/5}$$

comes from the form of Fermi-Dirac distributions; they are bounded by 1, therefore this inequality can be interpreted as that a Fermi-Dirac distribution cannot accumulate an arbitrary number of particles with small velocities in one point.

The result of the theorem I.2 allows us to say that the image of the map T_F is a convex set

$$U = \left\{(x, v, z) \in (0, \infty) \times \mathbb{R}^3 \times \mathbb{R} : z > \frac{|v|^2}{x} + J^{5/3}x^{5/3}\right\}. \tag{I.4.7}$$

Indeed, U is the subset of $\mathbb{R}^5$ above the graph of the function

$$h : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}, \quad h(x, v) = \frac{|v|^2}{x} + J^{5/3}x^{5/3}.$$

The Hessian of h writes

$$\mathrm{Hess}(h) = \frac{2}{x^2}\begin{pmatrix} |v|^2 + \frac{5}{9}J^{5/3}x^{5/3} & -v_1 & -v_2 & -v_3 \\ -v_1 & x & 0 & 0 \\ -v_2 & 0 & x & 0 \\ -v_3 & 0 & 0 & x \end{pmatrix}.$$

By Sylvester's criterion, this matrix is positive definite for $(x, v) \in (0, \infty) \times \mathbb{R}^3$; hence, the function h is convex and, therefore, the set U is convex.

5 The entropy and the map T_F

From now on we drop the subscript in the notation T_F. We define the vector of conserved quantities $\vec{e}$ as follows:

$$\vec{e}(v) = \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix}.$$

Let us define the operator $\odot$ as the standard scalar product in $\mathbb{R}^5$. Then we can put every Fermi-Dirac distribution in the form

$$F[\vec{\beta}] = \left(1 + e^{\vec{\beta}\odot\vec{e}}\right)^{-1}, \quad \vec{\beta} \in D = \mathbb{R} \times \mathbb{R}^3 \times (0, \infty).$$

It is important to notice that $\vec{\beta}$ does not depend on v.

One can easily see that the parameters (f, u, θ) of the representation (I.2.5) and $\vec{\beta}$ are connected by

$$\begin{array}{ll} \beta_4 = \frac{1}{2\theta}, & \theta = \frac{1}{2\beta_4}, \\ \beta_3 = -\frac{u_3}{\theta}, & u_3 = -\frac{\beta_3}{2\beta_4}, \\ \beta_2 = -\frac{u_2}{\theta}, & u_2 = -\frac{\beta_2}{2\beta_4}, \\ \beta_1 = -\frac{u_1}{\theta}, & u_1 = -\frac{\beta_1}{2\beta_4}, \\ \beta_0 = -f + \frac{|u|^2}{2\theta}, & f = -\beta_0 + (\beta_1^2 + \beta_2^2 + \beta_3^2)\frac{1}{4\beta_4}. \end{array} \tag{I.5.1}$$

Let us consider the vector of conserved densities

$$\vec{\rho} = -\langle \vec{e}F[\vec{\beta}]\rangle. \tag{I.5.2}$$

Clearly, the image of the map $T : \vec{\beta} \to \vec{\rho}$ is the set $U' = -U$ where U is defined in (I.4.7). The relations (I.5.1) and theorem I.2 imply that the map T is a diffeomorphism $D \to U'$.

The following proposition establishes the form of entropy associated with the Fermi-Dirac distribution.

Theorem I.3.

- *There exists a positive and strictly convex function* $\sigma^* : D \to \mathbb{R}$ *such that*
$$\vec{\rho}(\vec{\beta}) = \nabla_{\vec{\beta}} \sigma^*(\vec{\beta})$$
for $\vec{\rho}$ *given by* (I.5.2).

- *There exists a strictly convex function* $\sigma : U' \to \mathbb{R}$ *such that*
$$\vec{\beta}(\vec{\rho}) = \nabla_{\vec{\rho}} \sigma(\vec{\rho});$$
moreover, this function is the entropy density of the Fermi-Dirac distribution associated with $T^{-1}(\vec{\rho})$.

Proof. Put
$$\sigma^*(\vec{\beta}) = -\left\langle \ln\left(1 - F[\vec{\beta}]\right)\right\rangle,$$
then
$$\nabla_{\vec{\beta}} \sigma^*(\vec{\beta}) = -\nabla_{\vec{\beta}} \left\langle \ln\left(1 - F[\vec{\beta}]\right)\right\rangle = \left\langle \frac{\nabla_{\vec{\beta}} F[\vec{\beta}]}{1 - F[\vec{\beta}]} \right\rangle$$
$$= -\left\langle \frac{F[\vec{\beta}](1 - F[\vec{\beta}])\vec{e}}{1 - F[\vec{\beta}]} \right\rangle = \vec{\rho}.$$
Note that σ^* is a strictly positive and convex function of $\vec{\beta}$, because
$$\nabla_{\vec{\beta}} \vec{\rho} = \langle \vec{e} \otimes \vec{e}\, F[\vec{\beta}](1 - F[\vec{\beta}])\rangle$$
is a positive definite matrix. Indeed, take a vector $\vec{w} \in \mathbb{R}^5$ and suppose that
$$0 = \langle \vec{e} \otimes \vec{e}\, F(\vec{\beta})(1 - F[\vec{\beta}])\rangle \vec{w} \odot \vec{w} = \left\langle |\vec{e} \odot \vec{w}|^2 F[\vec{\beta}](1 - F[\vec{\beta}])\right\rangle.$$
Since $F[\vec{\beta}](1 - F[\vec{\beta}])$ is positive, then $\vec{e} \odot \vec{w} = 0$ for almost all v; the functions 1, v_1, v_2, v_3, and $|v|^2$ are linearly independent on $\mathbb{R}^3$, which results in $\vec{w} = 0$. We can therefore conclude that σ^* is strictly convex.

Let the function σ be the Legendre transform of the function σ^*, then σ is also a convex function and
$$\vec{\beta} = \nabla_{\vec{\rho}} \sigma(\vec{\rho}).$$
The function σ is defined on U' and can be expressed via the relation
$$\sigma(\vec{\rho}) + \sigma^*(\vec{\beta}) = \vec{\beta} \odot \vec{\rho}$$

with $\vec{\beta} = (\nabla_{\vec{\beta}}\sigma^*)^{-1}(\vec{\rho}) = T^{-1}(\vec{\rho})$, so

$$\sigma(\vec{\rho}) = \left\langle -\vec{\beta} \odot \vec{e}F[\vec{\beta}]) + \ln\left(1 - F[\vec{\beta}])\right)\right\rangle$$
$$= \left\langle \left(1 - F[\vec{\beta}])\right) \ln\left(1 - F[\vec{\beta}])\right) + F[\vec{\beta}]) \ln F[\vec{\beta}])\right\rangle .$$

In other words, this function coincides with the expression for the entropy density of the Fermi-Dirac distribution associated with $T^{-1}(\vec{\rho})$. □

The following theorem establishes the form of the entropy flux associated with entropy density σ defined in theorem I.3.

Theorem I.4.

- *The flux corresponding to the Fermi-Dirac distribution $F[\vec{\beta}]$ for each $\vec{\rho} \in U'$, $\vec{\beta} = T^{-1}(\vec{\rho})$ is a map $\vec{v} : U' \to \mathbb{R}^{5\times 3}$, written as*

  $$\vec{v}(\vec{\rho}) = -\langle \vec{e} \otimes vF[T^{-1}(\vec{\rho})]) \rangle.$$

  There exists a function $\tau^* : D \to \mathbb{R}^3$ such that

  $$\vec{v}(T(\vec{\beta})) = (\nabla_{\vec{\beta}}\tau^*(\vec{\beta}))^T.$$

- *The function $\tau : U' \to \mathbb{R}^3$ for every $\vec{\rho} \in U'$ and $\vec{\beta} = T^{-1}(\vec{\rho})$ defined by*

  $$\tau(\vec{\rho}) = \left\langle v\left(1 - F[\vec{\beta}])\right) \ln\left(1 - F[\vec{\beta}])\right) + vF[\vec{\beta}]) \ln F[\vec{\beta}])\right\rangle ,$$

  $\tau(\vec{\rho})$ is the entropy flux for the Fermi-Dirac distribution associated with conserved densities $\vec{\rho}$.

Proof. The expression for $\vec{v}(\vec{\rho})$ quickly follows from the form of local conservation laws (I.2.3).

If

$$\tau^* : D \to \mathbb{R}^3, \quad \tau^*(\vec{\beta}) = -\left\langle v \ln\left(1 - F[\vec{\beta}]\right)\right\rangle ,$$

then

$$\vec{v}(\vec{\rho}) = (\nabla_{\vec{\beta}}\tau^*(\vec{\beta}))^T,$$

with $\vec{\beta} = T^{-1}(\vec{\rho})$. Indeed,

$$\nabla_{\vec{\beta}}\tau^*(\vec{\beta}) = -\nabla_{\vec{\beta}}\left\langle v \ln\left(1 - F[\vec{\beta}]\right)\right\rangle$$

$$= -\left\langle v\nabla_{\vec{\beta}} \ln\left(1 - F[\vec{\beta}]\right)\right\rangle = \left\langle v\frac{\nabla_{\vec{\beta}}F[\vec{\beta}]}{1 - F[\vec{\beta}]}\right\rangle$$

$$= -\left\langle \frac{F[\vec{\beta}](1 - F[\vec{\beta}])v \otimes \vec{e}}{1 - F[\vec{\beta}]} \right\rangle = \left\langle F[\vec{\beta}](1 - F[\vec{\beta}])v \otimes \vec{e} \right\rangle = \vec{v}(\vec{\rho})^T.$$

Consider the function $\tau(\vec{\rho})$ given by

$$\tau(\vec{\rho}) + \tau^*(\vec{\beta}) = \vec{\beta} \odot \vec{v}(\vec{\rho})$$

with $\vec{\beta} = T^{-1}(\vec{\rho})$. We can simplify it by writing

$$\tau(\vec{\rho}) = \vec{\beta} \odot \vec{v}(\vec{\rho}) - \tau^*(\vec{\beta}) = \left\langle v \ln \left(1 - F[\vec{\beta}]\right) - \vec{\beta} \odot \vec{e} v F[\vec{\beta}]) \right\rangle.$$

Since $\vec{\beta} \odot \vec{e} = \ln \left(\frac{1}{F} - 1\right)$, we can further simplify as

$$\tau(\vec{\rho}) = \left\langle v \left(1 - F[\vec{\beta}]\right) \ln \left(1 - F[\vec{\beta}]\right) + vF[\vec{\beta}] \ln(F[\vec{\beta}]) \right\rangle.$$

We conclude that τ is the entropy flux of the Fermi-Dirac distribution associated with the conserved densities $\vec{\rho}$. □

6 Applications to the Eulerian limit

Theorem I.5. *Assume that the collision operator satisfies properties* (I.2.2), (I.2.4), *and* (I.2.6). *We consider F_ε — a sequence of non-negative solutions of*

$$(\partial_t + v \cdot \nabla_x)F_\varepsilon = \frac{1}{\varepsilon}C(F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \tag{I.6.1}$$

such that $F_\varepsilon(t, x, v)$ converges to a non-negative function F almost everywhere in $\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ as ε tends to zero. Assume also that the moments

$$\langle F_\varepsilon \rangle, \quad \langle vF_\varepsilon \rangle, \quad \langle v \otimes vF_\varepsilon \rangle, \quad \langle |v|^2 vF_\varepsilon \rangle$$

converge in the sense of distributions to the corresponding moments

$$\langle F \rangle, \quad \langle vF \rangle, \quad \langle v \otimes vF \rangle, \quad \langle |v|^2 vF \rangle,$$

that the entropy density and entropy flux converge in the sense of distributions

$$\lim_{\varepsilon \to 0} \langle F_\varepsilon \ln F_\varepsilon + (1 - F_\varepsilon) \ln(1 - F_\varepsilon) \rangle = \langle F \ln F + (1 - F) \ln(1 - F) \rangle,$$

$$\lim_{\varepsilon \to 0} \langle vF_\varepsilon \ln F_\varepsilon + v(1 - F_\varepsilon) \ln(1 - F_\varepsilon) \rangle = \langle vF \ln F + v(1 - F) \ln(1 - F) \rangle,$$

and finally that the entropy production rate satisfy

$$\liminf_{\varepsilon \to 0} \left\langle C(F_\varepsilon) \ln \left(\frac{1 - F_\varepsilon}{F_\varepsilon}\right) \right\rangle \geq \left\langle C(F) \ln \left(\frac{1 - F}{F}\right) \right\rangle.$$

Then the limit F is a local Fermi-Dirac distribution

$$F(t,x,v) = F[\nabla_{\vec{\rho}}\sigma(\vec{\rho}(t,x))],$$

where the vector of conserved densities $\vec{\rho}(t,x)$ satisfies the system of conservation laws

$$\partial_t \vec{\rho} + \nabla_x \cdot \vec{v}(\vec{\rho}) = 0 \tag{I.6.2}$$

together with the entropy inequality

$$\partial_t \sigma(\vec{\rho}) + \nabla_x \cdot \tau(\vec{\rho}) \leq 0 \tag{I.6.3}$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3$.

Proof. Multiplying the equation (I.6.1) by $\varepsilon \ln\left(\frac{F_\varepsilon}{1-F_\varepsilon}\right)$ and integrating with respect to v gives

$$\varepsilon\partial_t\langle F_\varepsilon \ln F_\varepsilon + (1-F_\varepsilon)\ln(1-F_\varepsilon)\rangle + \varepsilon\nabla_x \cdot \langle vF_\varepsilon \ln F_\varepsilon + v(1-F_\varepsilon)\ln(1-F_\varepsilon)\rangle \\ \leq \left\langle C(F_\varepsilon)\ln\left(\frac{F_\varepsilon}{1-F_\varepsilon}\right)\right\rangle. \tag{I.6.4}$$

The left hand side of the above expression tends to zero in the sense of distributions as ε tends to zero. On the other hand, $\left\langle C(F_\varepsilon)\ln\left(\frac{F_\varepsilon}{1-F_\varepsilon}\right)\right\rangle \leq 0$, therefore it is a Radon measure. Hence,

$$\iint_{\mathbb{R}_+^*\times\mathbb{R}^3} \left\langle C(F_\varepsilon)\ln\left(\frac{F_\varepsilon}{1-F_\varepsilon}\right)\right\rangle \phi(t,x)dxdt \to 0 \quad \forall\phi \in C_c(\mathbb{R}_+^*\times\mathbb{R}^3).$$

In particular, this holds for $\phi \geq \mathbf{1}_{[a,b]\times B(0,R)}$, so

$$\iint_{[a,b]\times B(0,R)} \left\langle C(F_\varepsilon)\ln\left(\frac{F_\varepsilon}{1-F_\varepsilon}\right)\right\rangle dxdt \to 0,$$

by Fatou's lemma we conclude that

$$\iint_{[a,b]\times B(0,R)} \left\langle C(F)\ln\left(\frac{F}{1-F}\right)\right\rangle dxdt \to 0, \quad \forall a,b,\ \forall R.$$

The characterisation of equilibria (I.3.3) allows us to say that for almost every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$ the distribution F is a solution of $C(F) = 0$ and has therefore the form (I.2.5).

In the system of local conservation laws

$$\begin{aligned} \partial_t\langle F_\varepsilon\rangle + \nabla_x \cdot \langle vF_\varepsilon\rangle &= 0, \\ \partial_t\langle vF_\varepsilon\rangle + \nabla_x \cdot \langle v\otimes vF_\varepsilon\rangle &= 0, \\ \partial_t\langle |v|^2F_\varepsilon\rangle + \nabla_x \cdot \langle v|v|^2F_\varepsilon\rangle &= 0 \end{aligned}$$

we pass to the the limit in the sense of distributions. Thanks to the convergence assumptions of this theorem, we obtain the compressible Euler system (I.6.2).

We rewrite (I.6.4) as

$$\partial_t \langle F_\varepsilon \ln F_\varepsilon + (1-F_\varepsilon)\ln(1-F_\varepsilon)\rangle + \nabla_x \cdot \langle v F_\varepsilon \ln F_\varepsilon + v(1-F_\varepsilon)\ln(1-F_\varepsilon)\rangle \\ \leq \frac{1}{\varepsilon}\left\langle C(F_\varepsilon)\ln\left(\frac{F_\varepsilon}{1-F_\varepsilon}\right)\right\rangle.$$

The right hand side is non-positive by the characterization (I.3.3) and the left hand side converges in the sense of distributions to $\partial_t\sigma(\vec{\rho}) + \nabla_x \cdot \tau(\vec{\rho})$, which yields (I.6.3). □

Observe that the system (I.6.2) has the convex entropy σ, therefore, it has Godounov's structure and hence is hyperbolic (see, for example, [19],[20],[21], and [27]). The hyperbolicity of the system (I.6.2) implies several useful properties, notably that it has a unique smooth local solution.

A Properties of polylogarithms

We study the polylogarithms for the argument $p > 0$ defined by the formula

$$\mathrm{Li}_p(-e^w) = -\frac{1}{\Gamma(p)}\int_0^\infty \frac{t^{p-1}}{e^{t-w}+1}\mathrm{d}t.$$

In order to simplify the reasoning, we introduce the notations

$$\mathcal{F}_p(w) = -\mathrm{Li}_p(-e^w), \quad \mathcal{G}_p(w) = (\mathcal{F}_p(w))^{1/p}.$$

We will use the following properties:

- $\frac{\mathrm{d}}{\mathrm{d}w}\mathcal{F}_p(w) = \mathcal{F}_{p-1}(w)$;
- for $p \geq 0$ the function $w \to \mathcal{F}_p(w)$ is positive and monotonically increasing. Moreover, $\mathcal{F}_p(w) \sim \frac{w^p}{\Gamma(p+1)}$ as $w \to +\infty$;
- if $p > 0$, then

$$\frac{\mathrm{d}^2}{\mathrm{d}w^2}\mathcal{G}_p(w) > 0 \iff \frac{p\mathcal{F}_p(w)}{\mathcal{F}_{p-1}(w)} > \frac{(p-1)\mathcal{F}_{p-1}(w)}{\mathcal{F}_{p-2}(w)}.$$

Theorem A.I. *If $p \geq 2$, then the function $w \to \frac{\mathrm{d}^2}{\mathrm{d}w^2}\mathcal{G}_p(w)$ is strictly positive.*

Proof. For each fixed w, we examine the function

$$\Psi(p) = \frac{p\mathcal{F}_p(w)}{\mathcal{F}_{p-1}(w)} = \frac{\int_0^\infty t^{p-1}\mathrm{d}\mu(t)}{\int_0^\infty t^{p-2}\mathrm{d}\mu(t)},$$

with

$$d\mu(t) = \frac{\mathrm{d}t}{e^{t-w}+1}.$$

Clearly, we have $t^s \in L^1(\mathrm{d}\mu(t))$ for all $s > -1$, so that $\Psi(p)$ is defined for $p > 1$. By continuity, we can put $\Psi(1) = 0$.

Indeed, $\int_0^\infty t^{p-1}\mathrm{d}\mu(t)$ converges to $\int_0^\infty \mathrm{d}\mu(t)$ as $p \to 1$ by Lebesgue theorem; on the other hand,

$$\lim_{p\to 1}\int_0^\infty t^{p-2}\mathrm{d}\mu(t) = +\infty$$

by the comparison test, which implies that

$$\lim_{p\to 1,\, p>1} \Psi(p) = 0.$$

We want to prove that $\Psi(p) > \Psi(p-1)$ for $p \geq 2$.

Let us consider the function

$$\Phi : x \to \int_0^\infty t^x \mathrm{d}\mu(t).$$

We claim that this function is strictly log-convex. Indeed, $\Phi > 0$, so $\ln\circ\Phi$ is defined.

Let us take $\frac{1}{m} + \frac{1}{n} = 1$ with $m, n \in [1, \infty]$ and $y > x > -1$. By Hölder's inequality,

$$\int_0^\infty t^{\frac{x}{n}+\frac{y}{m}}\mathrm{d}\mu(t) \leq \left(\int_0^\infty t^x\mathrm{d}\mu(t)\right)^{\frac{1}{n}} \left(\int_0^\infty t^y\mathrm{d}\mu(t)\right)^{\frac{1}{m}},$$

hence

$$\ln(\Phi(x/m + y/n)) = \ln\int_0^\infty t^{x/m+y/n}\mathrm{d}\mu(t)$$

$$< \frac{1}{m}\ln\int_0^\infty t^x\mathrm{d}\mu(t) + \frac{1}{n}\ln\int_0^\infty t^y\mathrm{d}\mu(t) = \frac{1}{m}\ln\Phi(x) + \frac{1}{n}\ln\Phi(y).$$

The inequality is strict because the functions $t \to t^x$ and $t \to t^y$ are linearly independent since $x \neq y$.

Thus, the function $p \to \ln(\Phi(p))$ is strictly convex. Clearly, this function is $\mathcal{C}^\infty(-1, \infty)$. In particular, the function

$$p \to \frac{\ln(\Phi(p)) - \ln(\Phi(p-1))}{p-(p-1)} = \ln(\Psi(p))$$

is strictly increasing wherever it is defined. Indeed, let us suppose that its derivative is zero at some point:

$$0 = \ln(\Phi(p))' - \ln(\Phi(p-1))' = \int_{p-1}^{p} \ln(\Phi(s))'' \mathrm{d}s.$$

As $\ln(\Phi(s))'' \geq 0$, we conclude that $\ln(\Phi(s))'' = 0$ on the interval $[p-1, p]$, which contradicts the strict convexity of $\ln \circ \Phi$ on its domain of definition, hence $\ln(\Psi(p))$ is strictly increasing. It follows immediately that Ψ itself is strictly increasing, which assures that $\Psi(p+1) > \Psi(p)$ on its domain of definition.

Thus, we conclude that $w \to \mathcal{G}_p(w)$ has a strictly positive second derivative for $p \geq 2$. □

Remark. We also conjecture that $w \to \mathcal{G}_p(w)$ is convex for $p \geq 1$, which is supported by numerical evidence and the fact that $\mathcal{G}_1(w) = \ln(1 + e^w)$ is a strictly convex function.

B Technical result

Proposition B.1. *Suppose that we have functions* $W : (\mathbb{R}^3)^4 \to \mathbb{C}$ *and* $h : \mathbb{R}^3 \to \mathbb{C}$ *such that the integral*

$$I = \iiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} W(v, v_*, v - (v - v_*, \omega)\omega, v_* + (v - v_*, \omega)\omega) h(v) \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\omega$$

exists. Then the following identity holds:

$$I = \iiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} W(v_*, v, v_* + (v - v_*, \omega)\omega, v - (v - v_*, \omega)\omega) h(v_*) \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\omega.$$

$$= \iiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} W(v', v'_*, v, v_*) h(v) \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\omega$$

$$= \iiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} W(v'_*, v', v_*, v) h(v_*) \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\omega.$$

Proof. First, let us apply the change of variables $(v, v_*) \to (v_*, v)$. The Jacobian of such change of variables is unity, hence

$$I = \iiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} W(v_*, v, v_* + (v - v_*, \omega)\omega, v - (v - v_*, \omega)\omega) h(v_*) \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\omega.$$

In the next change of variables we will express v and v_* in terms of v' and v'_*:

$$v = v' - (v' - v'_*, \omega)\omega, \quad v_* = v'_* - (v' - v'_*, \omega)\omega.$$

This change of variables also has the Jacobian equal to one, hence

$$I = \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} W(v' - (v' - v'_*, \omega)\omega, v'_* + (v' - v'_*, \omega)\omega, v', v'_*)h(v')\,\mathrm{d}v'\,\mathrm{d}v'_*\,\mathrm{d}\omega$$

$$= \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} W(v - (v - v_*, \omega)\omega, v_* + (v - v_*, \omega)\omega, v, v_*)h(v)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega$$

$$= \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} W(v', v'_*, v, v_*)h(v)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

Applying the first change of variables to the above expression, we obtain that

$$I = \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} W(v'_*, v', v_*, v)h(v_*)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

□

Chapter II

The Navier-Stokes limit for kinetic models with Fermi-Dirac statistics

1 Introduction

In this chapter we establish the connection between kinetic theory for Fermi-Dirac statistics and macroscopic fluid dynamics. We derive formal limits; as in the chapter I, we introduce a scaling for standard kinetic equations (see [28]) of the form (I.1.1)

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} C(F_\varepsilon).$$

Here F_ε is a nonnegative function representing the density of particles with position x and velocity v in the single-particle phase space $\mathbb{R}^3_x \times \mathbb{R}^3_v$ at time t. The interaction of particles through collisions is given by the operator $C(F)$; this operator acts only on the variable v and is nonlinear in the general case. In the present work we rely, in particular, on the specific form of this operator.

We base the connection between kinetic and macroscopic dynamics on the following two properties of the collision operator:

(a) conservation properties and entropy relations implying that the equilibria are Fermi-Dirac (i.e. of the form $1/(1+\exp(c_1 + c_2|v-u|^2))$ distributions;

(b) the derivative of $C(F)$ satisfies the Fredholm alternative with a nullspace related to the conservation properties of C.

We obtain the macroscopic limits when the fluid is dense enough so that particles undergo many collisions per unit of time. In order to describe this situation, we introduce a small parameter ε, called the Knudsen number, that is the ratio of the mean free path of particles between collisions to some characteristic length of the flow.

We use the connection between the parameters of a Fermi-Dirac distribution and its moments, which was established in I.

Properties (b) are used to obtain Navier-Stokes equations, which depend on a more detailed knowledge of the collision operator. The compressible Navier-Stokes equations appear as a correction to Euler equations at the next order of the Chapman-Enskog expansion; see section 2. We generally follow the ideas presented in [4]. We need strong assumptions on the regularity of the solutions of the compressible Navier-Stokes equations in order to make sense of these expansions (see theorem II.5).

2 Compressible Navier-Stokes equations

In the chapter I it was established that the form of the limiting Euler equations is independent of the choice of the collision operator C in the class of operators satisfying conservation and entropy properties. This choice appears at the macroscopic level only in the construction of the Navier-Stokes limit. We obtain the compressible Navier-Stokes equations by the classical Chapman-Enskog expansion. We give a description of this method below.

Given (μ, u, θ), define the corresponding Fermi-Dirac distribution

$$F_{(\mu,u,\theta)}(v) = \left(1 + \exp\left(-\frac{\mu}{\theta} + \frac{|v-u|^2}{2\theta}\right)\right)^{-1}. \tag{II.2.1}$$

The parameter $\mu \in \mathbb{R}$ is the chemical potential, $u \in \mathbb{R}^3$ is the bulk velocity, and $\theta > 0$ is the local temperature.

However, for computational reasons it is useful to write Fermi-Dirac distributions in the form

$$F_{(f,u,\theta)}(v) = \left(1 + \exp\left(-f + \frac{|v-u|^2}{2\theta}\right)\right)^{-1} \tag{II.2.2}$$

with $f = \frac{\mu}{\theta}$. The subscript is omitted wherever it is convenient. The Navier-Stokes equations act on the hydrodynamic quantities ρ and $\mathcal{E}$ defined in (I.4.2) and (I.4.3):

$$\rho = \int\limits_{\mathbb{R}^3} \frac{\mathrm{d}v}{1 + \exp\left(\frac{|v-u|^2}{2\theta} - f\right)} = \theta^{3/2} \int\limits_{\mathbb{R}^3} \frac{\mathrm{d}v}{1 + \exp\left(\frac{|v|^2}{2} - f\right)},$$

$$\mathcal{E} = \frac{1}{3} \int\limits_{\mathbb{R}^3} \frac{|v-u|^2 \,\mathrm{d}v}{1 + \exp\left(\frac{|v-u|^2}{2\theta} - f\right)} = \frac{\theta^{5/2}}{3} \int\limits_{\mathbb{R}^3} \frac{|v|^2 \,\mathrm{d}v}{1 + \exp\left(\frac{|v|^2}{2} - f\right)}.$$

Note that ρ and $\mathcal{E}$ do not depend on u and are $\mathcal{C}^1$ functions of f and θ. We remind the result established in theorem I.2 for moments of Fermi-Dirac distributions of the form (II.2.2):

Theorem. *If ρ and $\mathcal{E}$ are given, respectively, by* (I.4.2) *and* (I.4.3)*, then the following statements hold:*

1. *the ratio $\frac{\rho}{\mathcal{E}^{3/5}}$ depends only on f,*
2. *setting $J = \frac{(8\pi\sqrt{2})^{2/5}}{3}\left(\frac{5}{2}\right)^{3/5}$, then the map*

$$f \to \frac{\rho}{\mathcal{E}^{3/5}}$$

is $\mathcal{C}^\infty(\mathbb{R}; (0, J))$ and strictly monotone,

3. *the map $(f, \theta) \to (\rho, \mathcal{E})$ is a diffeomorphism,*

4. f and θ seen as functions of ρ and $\mathcal{E}$ satisfy

$$\rho\partial_\rho f + \frac{5}{3}\mathcal{E}\partial_{\mathcal{E}} f = 0,$$
$$\rho\partial_\rho \theta + \frac{5}{3}\mathcal{E}\partial_{\mathcal{E}} \theta = \frac{2}{3}\theta.$$

In this work we will consider the class of collision operators given by (see [14, 32])

$$C(F) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} b\left(F'F'_*(1-F)(1-F_*) - FF_*(1-F')(1-F'_*)\right) \mathrm{d}v_*\,\mathrm{d}\omega, \quad \text{(II.2.3)}$$

where, for each $\omega \in \mathbb{S}^2$

$$v' = v - (v - v_*, \omega)\omega, \quad v'_* = v_* + (v - v_*, \omega)\omega,$$
$$F' = F(t, x, v'), \quad F_* = F(t, x, v_*), \quad F'_* = F(t, x, v'_*)$$

and $b = b(|v - v_*|, |(\omega, v - v_*)|) > 0$ is a collision kernel. We will assume that the collision kernel has separated form, i.e.

$$b(z, \omega) = |z|^\beta \hat{b}\left(\omega \cdot n\right), \quad \text{with } n = \frac{z}{|z|}$$

and satisfies the weak cut-off condition (see [26]):

$$b_1 = \int_{\mathbb{S}^2} \hat{b}(\omega \cdot n)\,\mathrm{d}\omega < \infty. \quad \text{(II.2.4)}$$

Such a collision kernel will be said to correspond to a "hard" potential for the molecular interaction if $\beta \in (0, 1]$, and to a "soft" potential if $\beta \in (-d, 0)$. The case $\beta = 0$ corresponds to an assumption made by Maxwell in [33], and is referred to as the case of "Maxwell molecules". The case of hard sphere collisions is the case where $b(z, \omega) = |z \cdot \omega|$.

We denote by L and Q the first two Fréchet derivatives of the operator $G \to (F(1-F))^{-1}C(F + F(1-F)G)$ at $G = 0$ for a Fermi-Dirac distribution F:

$$L(g) = \frac{DC(F)\cdot(F(1-F)g)}{F(1-F)},$$
$$Q(g, g) = \frac{D^2C(F) : (F(1-F)g \otimes F(1-F)g)}{F(1-F)}.$$

By Taylor's formula

$$\frac{C(F + F(1-F)\varepsilon g)}{F(1-F)} = \varepsilon L(g) + \frac{\varepsilon^2}{2}Q(g, g) + \mathcal{O}(\varepsilon^3). \quad \text{(II.2.5)}$$

In general, the operator L is unbounded, but it is defined as a linear operator in the space $L^2(F(1-F)\mathrm{d}v)$.

One can show that for the collision operator (II.2.3) the linearised collision operator writes

$$L[h](v) = \frac{1}{(1-F)F} \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}\{h\}\,\mathrm{d}v_*\,\mathrm{d}\omega \tag{II.2.6}$$

with

$$\mathcal{N} = FF_*(1-F')(1-F'_*) = F'F'_*(1-F)(1-F_*)$$

and

$$\{\phi\} = \phi'_* + \phi' - \phi_* - \phi.$$

The latter notation is connected to the following result:

Proposition 2. *Let a continuous function ϕ satisfy*

$$\phi(v) + \phi(v_*) = \phi(v') + \phi(v'_*)$$

almost everywhere in $v \in \mathbb{R}^3$, $v_* \in \mathbb{R}^3$, and $\omega \in \mathbb{S}^2$. Then there exist constant $a \in \mathbb{R}$, $b \in \mathbb{R}^3$, and $c \in \mathbb{R}$ such that

$$\phi(v) = a + b \cdot v + c|v|^2.$$

The proof can be found in in [8, 18]. Moreover, in [9] the authors prove this result just for a measurable function ϕ.

Note that $FF_*(1-F')(1-F'_*) = F'F'_*(1-F)(1-F_*)$ if and only if F is a Fermi-Dirac distribution. Indeed, Fermi-Dirac distributions satisfy this identity. If this identity is satisfied, then by considering the logarithm of this expression, we obtain that a.e. in v, v_*, and ω

$$\left\{\ln\left(\frac{F}{1-F}\right)\right\} = \ln\left(\frac{F'_*}{1-F'_*}\right) + \ln\left(\frac{F'}{1-F'}\right) - \ln\left(\frac{F_*}{1-F_*}\right) - \ln\left(\frac{F}{1-F}\right) = 0.$$

By Proposition 2, this implies that $\frac{F}{1-F}$ is a local Maxwellian distribution and, therefore, that F is a local Fermi-Dirac distribution.

We will assume that the operator L is symmetric and satisfies the Fredholm alternative. In the following subsection we establish that these properties hold for a wide class of collision kernels.

In order to simplify the expressions we introduce the notation

$$\langle s\rangle_k = \int_{\mathbb{R}^3} s(v)k(v)\,\mathrm{d}v$$

for all functions $s \in L^1(k(v)\,\mathrm{d}v)$ and

$$\langle s\rangle = \int_{\mathbb{R}^3} s(v)\,\mathrm{d}v$$

for $s \in L^1(\,\mathrm{d}v)$.

Analytical properties of the linearised collision operator

Compactness of loss and gain parts of the linearised collision operator

Theorem II.1. *Suppose that the collision kernel has separate form and satisfies the weak cut-off condition* (II.2.4)*. Then the following results hold:*

1. *in the hard sphere case $\beta = 1$ the linearised collision operator L given by* (II.2.6) *satisfies the Fredholm alternative in $L^2(F(v)(1-F(v))\,\mathrm{d}v)$.*
2. *If $\beta > -3$, then the linearised collision operator $\frac{1}{a^F}L$ satisfies the Fredholm alternative in $L^p(a^F(v)F(v)(1-F(v))\,\mathrm{d}v)$, where a^F is given by*

$$a^F(v) = \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b(\omega, v-v_*)\frac{F_*(1-F')(1-F'_*)}{1-F}\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

Proof. The strategy of the proof consists in representing the linearised collision operator as a sum

$$L = a(v)I + K_1 - K_2 - K_3,$$

where $a(v)$ is a positive attenuation coefficient, the operators K_i are positivity preserving. The operator K_1 is called the loss part and $K_2 + K_3$ is called the gain part.

Then we establish that the operators K_i are compact and the inverse of the attenuation coefficient $\frac{1}{a(v)}$ is bounded, which allows us to conclude that L satisfies the Fredholm alternative.

In order to do so, we establish that given a collision kernel, the compactness of gain and loss parts of the linearised collision operator for the Maxwellian case is equivalent to the compactness of the corresponding part of the linearised collision operator in the quantum case.

Lemma 1. *Let $F(v) = \frac{1}{1+e^{|v|^2-f}}$ and $M(v) = e^{-|v|^2}$. Denote $F_* = F(v_*)$, $F' = F(v')$, $F'_* = F(v'_*)$ and similarly for M. The attenuation coefficients for the Fermi-Dirac and the Maxwellian cases are given, respectively, by*

$$a^F(v) = \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b(\omega, v-v_*)\frac{F_1(1-F')(1-F'_*)}{1-F}\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

and

$$a^M(v) = \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b(\omega, v-v_*)M_1\,\mathrm{d}\omega\,\mathrm{d}v_*.$$

The loss and gain parts of the linearised collision operators for Fermi-Dirac and Maxwellian cases are given, respectively, by

$$K_1^F[f](v) = \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b(\omega, v - v_*)\frac{F_1(1-F')(1-F'_*)}{1-F} f(v_*)\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

$$K_2^F[f](v) = \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b(\omega, v - v_*)\frac{F_1(1-F')(1-F'_*)}{1-F} f(v')\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

$$K_3^F[f](v) = \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b(\omega, v - v_*)\frac{F_1(1-F')(1-F'_*)}{1-F} f(v'_*)\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

$$K_1^M[f](v) = \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b(\omega, v - v_*)M(v_*)f(v_*)\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

$$K_2^M[f](v) = \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b(\omega, v - v_*)M(v_*)f(v')\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

$$K_3^M[f](v) = \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b(\omega, v - v_*)M(v_*)f(v'_*)\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

The following statements hold:

1. *there exist positive constants c_1, c_2 such that*

$$c_1 a^M(v) \le a^F(v) \le c_2 a^M(v) \quad a.e. \tag{II.2.7}$$

2. *For each $i = 1, 2, 3$ the operator*

$$K_i^F : L^2(F(v)(1-F(v))\,\mathrm{d}v) \to L^2(F(v)(1-F(v))\,\mathrm{d}v)$$

is compact if and only if the operator

$$K_i^M : L^2(M(v)\,\mathrm{d}v) \to L^2(M(v)\,\mathrm{d}v)$$

is compact.

3. *The operator*

$$(K_2^F + K_3^F) : L^2(F(v)(1-F(v))\,\mathrm{d}v) \to L^2(F(v)(1-F(v))\,\mathrm{d}v)$$

is compact if and only if the operator

$$(K_2^M + K_3^M) : L^2(M(v)\,\mathrm{d}v) \to L^2(M(v)\,\mathrm{d}v)$$

is compact.

4. *For each $i = 1, 2, 3$ the operator*

$$\frac{1}{a^F} K_i^F : L^2(a^F(v)F(v)(1 - F(v))\,\mathrm{d}v) \to L^2(a^F(v)F(v)(1 - F(v))\,\mathrm{d}v)$$

is compact if and only if the operator

$$\frac{1}{a^M} K_i^M : L^2(a^M(v)M(v)\,\mathrm{d}v) \to L^2(a^M(v)M(v)\,\mathrm{d}v)$$

is compact.

5. *The operator*

$$\frac{1}{a^F}(K_2^F + K_3^F) : L^2(a^F(v)F(v)(1 - F(v))\,\mathrm{d}v) \\ \to L^2(a^F(v)F(v)(1 - F(v))\,\mathrm{d}v)$$

is compact if and only if the operator

$$\frac{1}{a^M}(K_2^M + K_3^M) : L^2(a^M(v)M(v)\,\mathrm{d}v) \to L^2(a^M(v)M(v)\,\mathrm{d}v)$$

is compact.

Proof. We recall the following compactness criterion (see [36]):

Theorem II.2. *Let $(X, \mathrm{d}\mu_X(x))$, $(Y, \mathrm{d}\mu_Y(y))$ be spaces with separable measures*[1]*, $p \in (1, \infty)$, and $q \in [1, \infty)$. Suppose also that K_1 and K_2 are linear operators from $L^p(\mathrm{d}\mu_X)$ to $L^q(\mathrm{d}\mu_Y)$. If K_2 is compact, positivity preserving, and*

$$\forall f \in L^p(\mathrm{d}\mu_X) \quad 0 \le |K_1[f](y)| \le K_2[|f|](y), \quad \mathrm{d}\mu_Y - a.e.,$$

then the operator K_1 is also compact.

The inequalities

$$\frac{1}{1 + e^f} \le 1 - F(v) \le 1, \quad \frac{1}{2} e^{-|v|^2 + f} \le F(v) \le e^{-|v|^2 + f} \tag{II.2.8}$$

allow us to immediately deduce the inequality (II.2.7). The same inequalities imply that the Lebesgue norms

$$\|\cdot, L^p(a^F(v)F(v)\,\mathrm{d}v)\|, \quad \|\cdot, L^p(a^F(v)F(v)(1 - F(v))\,\mathrm{d}v)\|,$$

[1]A measure m is called separable, if the σ-algebra of measurable sets with the distance $d(A, B) = m(A \triangle B)$ is a separable metric space. Lebesgue measure and absolutely continuous measures with smooth weights are separable measures.

$$\left\|\cdot, L^p\left(a^F(v)\frac{F(v)}{1-F(v)}\,\mathrm{d}v\right)\right\|, \quad \|\cdot, L^p(a^F(v)M(v)\,\mathrm{d}v)\|$$
$$\|\cdot, L^p(a^M(v)F(v)\,\mathrm{d}v)\|, \quad \|\cdot, L^p(a^M(v)F(v)(1-F(v))\,\mathrm{d}v)\|,$$
$$\left\|\cdot, L^p\left(a^M(v)\frac{F(v)}{1-F(v)}\,\mathrm{d}v\right)\right\|, \quad \|\cdot, L^p(a^M(v)M(v)\,\mathrm{d}v)\|$$

are equivalent for $p \in [1, +\infty]$.

Obviously, the operators K_i^F and K_i^M are positivity preserving, therefore applying theorem II.2 together with inequalities (II.2.7), (II.2.8) yields statements 2 and 3 of the lemma 1. Since the attenuation coefficients are positive, the same reasoning leads to statements 4 and 5 of the same lemma.

As an illustration for this theorem, for the hard sphere case (see [18]) the operators K_i^M are compact in $L^2(M\,\mathrm{d}v)$; therefore, in this case K_i^F are compact operators in $L^2(F(v)(1-F(v))\,\mathrm{d}v)$. Moreover, a^M is separated from zero, and so is a^F. If we recall the form of the operator L given in (II.2.6), then we see that

$$L = -\left(a^F(v)I + K_1^F - K_2^F - K_3^F\right), \tag{II.2.9}$$

hence we conclude that (1) holds.

If for the Maxwellian case this can be done by showing that the operators K_1 and $K_2 + K_3$ are compact, then for the quantum case this also can be done and, hence, the Fredholm alternative holds.

As a further example, in [31], the authors show that operators $\frac{1}{a^M}K_i^M$ are compact in $L^p(a^M(v)M(v)\,\mathrm{d}v)$ for $p \in (1,\infty)$ for collision kernels of separate form satisfying $\beta \in (-3,0)$ and (II.2.4). Lemma 1 allows us to conclude immediately that for this class of collision kernels the operators $\frac{1}{a^F}K_i^F$ are compact in $L^p(a^F(v)F(v)(1-F(v))\,\mathrm{d}v)$ for $p \in (1,\infty)$ and therefore we can deduce (2).

The representation (II.2.9) enables us to prove that the operator L is self-adjoint. In this theorem we established that in the hard sphere case the operators K_1^F and $(K_2^F + K_3^F)$ are symmetric and compact, therefore they are bounded self-adjoint everywhere defined operators in $L^2(F(1-F)\,\mathrm{d}v)$. The operator $a^F(v)I$ is obviously a symmetric densely-defined operator in the same space with natural domain $L^2\left((a^F)^2F(1-F)\,\mathrm{d}v\right)$. Moreover, in [18] the authors find an explicit view of the function $a^M(v)$; among other things, this function belongs to the space $L^2_{loc}(\mathbb{R}^3)$. By lemma 1, so does the function $a^F(v)$; by a standard argument (see, for example, [17]), this implies that the operator $a^F(v)I$ is self-adjoint and therefore the operator L is self-adjoint in $L^2(F(1-F)\,\mathrm{d}v)$ with domain $L^2\left((a^F)^2F(1-F)\,\mathrm{d}v\right)$.

In the same spirit, for soft potentials $\beta \in (-3,0)$ the operators $\frac{1}{a^F(v)}K_1^F$ and $\frac{1}{a^F(v)}(K_2^F + K_3^F)$ are compact and symmetric in the space $L^2(a^F F(1-F)\,\mathrm{d}v)$, therefore they are self-adjoint in this space. It is easy to conclude that the operator $\frac{1}{a^F}L$ is self-adjoint in the same space.

$\square$

$\square$

Obviously, the general case with arbitrary $\theta > 0$ and $u \in \mathbb{R}^3$ can be reduced to the above case by translating and rescaling the variable v.

The nullspace of L and its orthogonal complement

Theorem II.3. *The linearised collision operator L defined in* (II.2.6) *is symmetric, nonpositive, and its nullspace is* $\ker L = span\{1, v_1, v_2, v_3, |v|^2\}$.

Proof. We remind the reader of expression for the operator L given in (II.2.6):

$$L[h](v) = \frac{1}{(1-F)F} \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}\{h\} \mathrm{d}v_* \,\mathrm{d}\omega,$$

with

$$\mathcal{N} = FF_*(1-F')(1-F'_*) = F'F'_*(1-F)(1-F_*)$$

and

$$\{h\} = h'_* + h' - h_* - h.$$

Consider the inner product in $L^2(F(1-F)\,\mathrm{d}v)$

$$(L[g], q) = \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}\{g\}\bar{q}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

We apply the proposition 1 with $W = b\mathcal{N}\{g\}$ and $h = \bar{q}$ to obtain that

$$\begin{aligned}
(L[g], q) &= \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}\{g\}\bar{q}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \\
&= \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}\{g\}\bar{q}_*\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \\
&= -\iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}\{g\}\bar{q}'\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \\
&= -\iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}\{g\}\bar{q}'_*\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega,
\end{aligned}$$

which allows us to conclude that

$$(L[g], q) = -\frac{1}{4} \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}\{g\}\{\bar{q}\}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

This expression immediately gives us that L is symmetric. If we take $q = g$ and take into account that the factors b and $\mathcal{N}$ are strictly positive, then

$$(L[g], g) \leq 0.$$

If in the above expression the equality sign holds, then $\{g\}$ is zero a.e., which implies that g belongs to $\ker L$; on the contrary, if $g \in \ker L$, then $(L[g], g) = 0$.

Finally, the proposition 2 shows that $\{g\} = 0$ whenever it has the form $g(v) = a + w \cdot v + c|v|^2$ for some constants $a, c \in \mathbb{R}$ and $w \in \mathbb{R}^3$, or, in other words, that $\ker L = \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}$. □

Note that the collision invariants of the collision operator coincide with $\ker L$. As was established in the chapter I, we have

$$\forall e \in \ker L \quad \langle C(F)e\rangle = 0$$

for all measurable F satisfying $0 \leq F \leq 1$ almost everywhere in $\mathbb{R}^3$.

Theorem II.4. *If*

$$V = \frac{v-u}{\sqrt{\theta}}, \quad A(V) = \left(\frac{|V|^2}{2} - \frac{5}{2}\frac{\mathcal{E}}{\rho\theta}\right)V, \quad B(V) = V\otimes V - \frac{1}{3}|V|^2 I,$$

where ρ and $\mathcal{E}$ are given by (I.4.2) *and* (I.4.3)*, then A_i and B_{ij} are orthogonal to* $\ker L$ *in* $L^2(F(1-F)\,\mathrm{d}v)$.

Proof. Clearly, it is sufficient to prove orthogonality to 1, V_i, $|V|^2$. By symmetry, B_{ij} is orthogonal to radial and odd functions, hence $B_{ij}\perp\ker L$; in the same spirit, $A_i\perp 1$, $A_i\perp|V|^2$, and $A_i\perp V_j$, $j\neq i$, because $A_i(V)$ is odd in V_i. Thus, the only part requiring proof is $A_i\perp V_i$.

We write

$$\langle A_i(V)V_i\rangle_{F(1-F)} = \left\langle\left(\frac{|V|^2}{2} - \frac{5}{2}\frac{\mathcal{E}}{\rho\theta}\right)V_i^2\right\rangle_{F(1-F)}.$$

In order to prove this theorem we will introduce the following integrals:

$$p_0^0 = \int_{\mathbb{R}^3}\left(1+\exp\left(-f+\frac{|v|^2}{2}\right)\right)^{-1} dv,$$

$$p_2^0 = \int_{\mathbb{R}^3} |v|^2 \left(1 + \exp\left(-f + \frac{|v|^2}{2}\right)\right)^{-1} dv,$$

$$p_0^1 = \int_{\mathbb{R}^3} \frac{\exp\left(-f + \frac{|v|^2}{2}\right) dv}{\left(1 + \exp\left(-f + \frac{|v|^2}{2}\right)\right)^2},$$

$$p_2^1 = \int_{\mathbb{R}^3} \frac{|v|^2 \exp\left(-f + \frac{|v|^2}{2}\right) dv}{\left(1 + \exp\left(-f + \frac{|v|^2}{2}\right)\right)^2},$$

$$p_4^1 = \int_{\mathbb{R}^3} \frac{|v|^4 \exp\left(-f + \frac{|v|^2}{2}\right) dv}{\left(1 + \exp\left(-f + \frac{|v|^2}{2}\right)\right)^2}.$$

It is easy to see that

$$p_4^1 = 5p_2^0, \quad p_2^1 = 3p_0^0,$$
$$\frac{d}{df}p_0^0 = p_0^1, \quad \frac{d}{df}p_2^0 = p_2^1,$$
$$\rho = \theta^{3/2}p_0^0, \quad \mathcal{E} = \frac{1}{3}\theta^{5/2}p_2^0.$$

In addition,

$$\left\langle V_i^2 \right\rangle_{F(1-F)} = \theta^{3/2}\frac{p_2^1}{3},$$
$$\left\langle \frac{|V|^2}{2} V_i^2 \right\rangle_{F(1-F)} = \left\langle \frac{|V|^4}{6} \right\rangle_{F(1-F)} = \theta^{3/2}\frac{p_4^1}{6},$$

which allows us to conclude

$$\langle A_i(V)V_i\rangle_{F(1-F)} = \theta^{3/2}\frac{p_4^1}{6} - \theta^{3/2}\frac{5}{6}\frac{p_2^0}{p_0^0}\frac{p_2^1}{3} = 0.$$

□

Remark 1. The expression for A agrees perfectly with the Maxwellian case. Indeed, since the Maxwellian distribution has the form

$$\mathcal{M}_{\rho^M,u,\theta^M}(v) = \frac{\rho^M}{(2\pi\theta^M)^{3/2}} e^{-\frac{|v-u|^2}{2\theta^M}},$$

the expression for local energy $\mathcal{E}_M$ writes

$$\mathcal{E}^M = \frac{1}{3}\left\langle \mathcal{M}_{\rho^M,u,\theta^M}(v)|v-u|^2 \right\rangle = \rho^M\theta^M.$$

Moreover, if we define

$$V^M = V = \frac{v-u}{\sqrt{\theta^M}}, \quad A^M(V) = \left(\frac{|V|^2}{2} - \frac{5}{2}\right) V,$$

$$B^M(V) = B(V) = V \otimes V - \frac{|V|^2}{3} I,$$

then $A_i(V)$ and $B_{ij}(V)$ are orthogonal to the nullspace of the linearised collision operator L_M (this nullspace is generated by the basis $\{1, v_1, v_2, v_3, |v|^2\}$) in the space $L^2(M\,\mathrm{d}v)$ (see [4]). It is easy to see that if we put $\mathcal{E} = \mathcal{E}^M$, $\rho = \rho^M$, $\theta = \theta^M$ into the expression for A, then we obtain A^M.

With the formalism established in this theorem, we obtain an expression for $A(V) \cdot \frac{\nabla_x \theta}{\sqrt{\theta}}$:

Corollary.

$$A(V) \cdot \frac{\nabla_x \theta}{\sqrt{\theta}} = \frac{|V|^2 V}{2} \cdot \frac{\nabla_x \theta}{\sqrt{\theta}} - \sqrt{\theta} V \cdot \left(-\nabla_x f + \frac{\nabla_x \mathcal{E}}{\rho\theta}\right).$$

Proof. It is sufficient to examine the following term:

$$\begin{aligned}
\sqrt{\theta}\left(-\nabla_x f + \frac{\nabla_x \mathcal{E}}{\rho\theta}\right) &= \sqrt{\theta}\left(-\nabla_x f + \frac{\nabla_x \left(\frac{1}{3}\theta^{5/2} p_2^0\right)}{\rho\theta}\right) \\
&= \sqrt{\theta}\left(-\nabla_x f + \frac{\frac{5}{2}\frac{\mathcal{E}}{\theta}\nabla_x \theta + \frac{1}{3}\theta^{5/2} p_2^1 \nabla_x f}{\rho\theta}\right) \\
&= \sqrt{\theta}\left(-\nabla_x f + \frac{\frac{5}{2}\frac{\mathcal{E}}{\theta}\nabla_x \theta + \theta^{5/2} p_0^0 \nabla_x f}{\theta^{5/2} p_0^0}\right) \\
&= \frac{5}{2}\frac{\mathcal{E}}{\rho\theta}\frac{\nabla_x \theta}{\sqrt{\theta}}.
\end{aligned}$$

□

Lemma 2. *There exist unique functions A_i', B_{ij}' in $L^2(F(1-F)\,\mathrm{d}v)$, such that $A_i' \in (\ker L)^\perp$, $B_{ij}' \in (\ker L)^\perp$ and*

$$L[A_i'] = A_i, \quad L[B_{ij}'] = B_{ij}.$$

Moreover, these functions have the form

$$A'(V) = -\alpha_L(\rho, \mathcal{E}, |V|) A(V), \quad B'(V) = -\beta_L(\rho, \mathcal{E}, |V|) B(V)$$

for some positive functions α_L and β_L.

The existence of such functions relies upon Fredholm alternative; the detailed proof can be found in lemma 3.

The limiting result

A function H_ε is said to be an approximate solution of order p to the kinetic equation (I.1.1), if

$$\partial_t H_\varepsilon + v \cdot \nabla_x H_\varepsilon = \frac{1}{\varepsilon} C(H_\varepsilon) + \mathcal{O}(\varepsilon^p), \tag{II.2.10}$$

where $\mathcal{O}(\varepsilon^p)$ denotes a term bounded by ε^p in some convenient norm. We will construct an approximate solution of order 2 in the form

$$H_\varepsilon = F_\varepsilon + \varepsilon F_\varepsilon (1 - F_\varepsilon)(g_\varepsilon + \varepsilon w_\varepsilon), \tag{II.2.11}$$

where $(\rho_\varepsilon, u_\varepsilon, \mathcal{E}_\varepsilon)$ solve the compressible Navier-Stokes equations with dissipation of order ε:

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\ \rho(\partial_t + u \cdot \nabla_x) u + \nabla_x \mathcal{E} &= \varepsilon \nabla_x \cdot (\mu \sigma), \\ \frac{3}{2}(\partial_t \mathcal{E} + \nabla_x \cdot (\mathcal{E} u)) + \mathcal{E}(\nabla_x \cdot u) &= \varepsilon \frac{1}{2} \mu \sigma : \sigma - \varepsilon \nabla_x \cdot q, \end{aligned} \tag{II.2.12}$$

where $\sigma = \nabla_x u + \nabla_x u^T - \frac{2}{3}(\nabla_x \cdot u) I$ is the strain-rate tensor and

$$\begin{aligned} \mu(\rho, \mathcal{E}) &= \theta \left\langle \beta_L(\rho, \mathcal{E}, |V|) B_{12}^2(V) \right\rangle_{F(1-F)}, \\ \kappa(\rho, \mathcal{E}) &= \theta \left\langle \alpha_L(\rho, \mathcal{E}, |V|) A_1^2(V) \right\rangle_{F(1-F)}, \\ q &= -\kappa(\rho, \mathcal{E}) \nabla_x \theta. \end{aligned} \tag{II.2.13}$$

We also impose additional constraint that

$$\frac{\rho_\varepsilon}{\mathcal{E}_\varepsilon^{3/5}} < J = \frac{(8\pi\sqrt{2})^{2/5}}{3} \left(\frac{5}{2}\right)^{3/5}. \tag{II.2.14}$$

We formulate the Chapman-Enskog derivation according to the following theorem following [4].

Theorem II.5. *Assume that $(\rho_\varepsilon, u_\varepsilon, \mathcal{E}_\varepsilon)$ solve the compressible Navier-Stokes equations* (II.2.12) *with viscosity μ and thermal diffusivity κ given by* (II.2.13) *and satisfy the condition* (II.2.14)*. Then there exist g_ε and w_ε in* $(\ker L)^\perp$ *such that H_ε given by* (II.2.11) *is an approximate solution of the order* 2 *to the equation* (I.1.1)*. Moreover, g_ε is given by the formula*

$$g_\varepsilon = -\alpha_L(\rho_\varepsilon, \mathcal{E}_\varepsilon, |V|) A(V) \cdot \frac{\nabla_x \theta_\varepsilon}{\sqrt{\theta_\varepsilon}} - \frac{1}{2} \beta_L(\rho_\varepsilon, \mathcal{E}_\varepsilon, |V|) B(V) : \sigma(u_\varepsilon). \tag{II.2.15}$$

Proof. We omit the subscript ε. In addition, thanks to the existence of diffeomorphism between (μ, θ) and $(\rho, \mathcal{E})$ established in lemma I.2 (see chapter I), we can conduct our reasoning in the set of variables of our choice.

We put the expression (II.2.10) into the equation (II.2.11), this yields the formula

$$\begin{aligned} \frac{(\partial_t + v\cdot\nabla_x)F}{F(1-F)} &+ \varepsilon \frac{(\partial_t + v\cdot\nabla_x)(F(1-F)g)}{F(1-F)} \\ &= L(g) + \varepsilon\left(L(w) + \frac{1}{2}Q(g,g)\right) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{II.2.16}$$

First, we prove the following proposition:

Proposition 3. *For all $g \in L^2(F(1-F)\,\mathrm{d}v)$, the functions $Q(g,g)$ and $\mathcal{O}$ in the above expansion belong to $(\ker L)^\perp$.*

Proof. Indeed, let us take $e \in \ker L$, and ϵ – small parameter. We can write

$$\frac{C(F + \epsilon F(1-F)g)}{F(1-F)} = \epsilon L(g) + \frac{\epsilon^2}{2}Q(g,g) + \mathcal{O}(\epsilon^3).$$

One has

$$\left\langle \frac{C(F+\epsilon F(1-F)g)e}{F(1-F)} \right\rangle_{F(1-F)} = \langle C(F+\epsilon F(1-F)g)e\rangle = 0$$

by the conservation properties of the operator C. On the other hand,

$$\langle L(g)e\rangle_{F(1-F)} = 0,$$

because $L = L^*$ and $e \in \ker L$. Hence, dividing by ϵ^2, we arrive at

$$\left\langle \frac{\vec{e}}{2}Q(g,g) + \mathcal{O}(\epsilon)e \right\rangle_{F(1-F)} = 0.$$

By passing to the limit $\epsilon \to 0$ we obtain that $Q(g,g) \perp e$, or, in other words, $Q(g,g) \in (\ker L)^\perp$. In the same spirit, the term denoted by $\mathcal{O}(\epsilon)$ also belongs to $(\ker L)^\perp$. □

We take $f = \frac{\mu}{\theta}$, derive (II.2.1), and, with the notation $V = \frac{v-u}{\sqrt{\theta}}$, we obtain formulas

$$
\begin{aligned}
\frac{\nabla_u F}{F(1-F)} &= \frac{V}{\sqrt{\theta}}, \\
\frac{\partial_\rho F}{F(1-F)} &= \partial_\rho f + \frac{|V|^2}{2\theta}\partial_\rho \theta, \\
\frac{\partial_\mathcal{E} F}{F(1-F)} &= \partial_\mathcal{E} f + \frac{|V|^2}{2\theta}\partial_\mathcal{E} \theta.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{(\partial_t + v\cdot\nabla_x)F}{F(1-F)} = &\left(\frac{|V|^2}{2\theta}\partial_\rho\theta + \partial_\rho f\right)(\partial_t + v\cdot\nabla_x)\rho \\
&+ \left(\frac{|V|^2}{2\theta}\partial_\mathcal{E}\theta + \partial_\mathcal{E} f\right)(\partial_t + v\cdot\nabla_x)\mathcal{E} \\
&+ \frac{V}{\sqrt{\theta}}\cdot((\partial_t + v\cdot\nabla_x)u).
\end{aligned} \tag{II.2.17}
$$

By solving (II.2.12) with respect to time derivative and substituting it into (II.2.17), we obtain

$$
\begin{aligned}
\frac{(\partial_t + v\cdot\nabla_x)F}{F(1-F)} = &\left(\frac{|V|^2}{2\theta}\partial_\rho\theta + \partial_\rho f\right)(\sqrt{\theta}V\cdot\nabla_x\rho - \rho\nabla_x\cdot u) \\
&+ \left(\frac{|V|^2}{2\theta}\partial_\mathcal{E}\theta + \partial_\mathcal{E} f\right)\left(\sqrt{\theta}V\cdot\nabla_x\mathcal{E} - \frac{5}{3}\mathcal{E}\nabla_x\cdot u\right) \\
&+ \frac{V}{\sqrt{\theta}}\cdot\left(\left(\sqrt{\theta}V\cdot\nabla_x\right)u - \frac{\nabla_x\mathcal{E}}{\rho}\right) + \varepsilon R
\end{aligned} \tag{II.2.18}
$$

where

$$
R = \left(\frac{|V|^2}{2\theta}\partial_\mathcal{E}\theta - \partial_\mathcal{E} f\right)\left(\frac{1}{3}\mu\sigma:\sigma - \frac{2}{3}\nabla_x\cdot q\right) + \frac{V}{\sqrt{\theta}}\cdot\left(\frac{\nabla_x\cdot(\mu\sigma)}{\rho}\right).
$$

By using the formula

$$
\begin{aligned}
(V\otimes V):\nabla_x u &= \frac{1}{2}(V\otimes V):\left(\nabla_x u + \nabla_x u^T - \frac{2}{3}(\nabla_x\cdot u)I\right) + \frac{1}{3}|V|^2(\nabla_x\cdot u) \\
&= \frac{1}{2}B(V):\sigma(u) + \frac{1}{3}|V|^2(\nabla_x\cdot u)
\end{aligned}
$$

together with the equations (I.4.4), (I.4.5), and corollary 2, we can further simplify the expression (II.2.18) to

$$
\frac{(\partial_t\rho + v\cdot\nabla_x)F}{F(1-F)} = \frac{1}{2}B(V):\sigma(u) + A(V)\cdot\frac{\nabla_x\theta}{\sqrt{\theta}} + \varepsilon R. \tag{II.2.19}
$$

From (II.2.16) and (II.2.19) it immediately follows that the term of order one with respect to ε has to be given by the formula (II.2.15). To complete the proof, we need to show that there exists a function w such that the terms of order one in (II.2.16) are cancelled.

This is equivalent to saying that there exists a solution w of the equation

$$L(w) = R + \frac{(\partial_t + v\cdot\nabla_x)(F(1-F)g)}{F(1-F)} - \frac{1}{2}Q(g,g) \tag{II.2.20}$$

where F, g, and R are given. By the Fredholm alternative, the solution exists if and only if the right side of (II.2.20) belongs to $(\ker L)^\perp$. As we have already seen, the term $Q(g,g)$ belongs to $(\ker L)^\perp$. The term $\frac{\partial_t(F(1-F)g)}{F(1-F)}$ is in $(\ker L)^\perp$, too. Indeed, let us again take $e\in\ker L$, then

$$\begin{aligned}&\left\langle e\frac{\partial_t(F(1-F)g)}{F(1-F)}\right\rangle_{F(1-F)}\\ &= \langle e\partial_t(F(1-F)g)\rangle_1 = \partial_t\langle e(F(1-F)g)\rangle\\ &= \partial_t\langle eg\rangle_{F(1-F)} = 0\end{aligned}$$

by the construction of g.

Let us study the scalar products of the terms R and $\frac{v\cdot\nabla_x(F(1-F)g)}{F(1-F)}$ with the vectors 1, V, $|V|^2$ in $L^2(F(1-F)dv)$. With the notations introduced in the proof of theorem II.4, one has

$$\langle R\rangle_{F(1-F)} = \left(\frac{\sqrt{\theta}\partial_{\mathcal{E}}\theta}{2}p_2^1 - \theta^{3/2}p_0^1\partial_{\mathcal{E}}f\right)\left(\frac{1}{3}\mu\sigma:\sigma - \frac{2}{3}\nabla_x\cdot q\right).$$

Since

$$\begin{aligned}\frac{\sqrt{\theta}\partial_{\mathcal{E}}\theta}{2}p_2^1 - \theta^{3/2}p_0^1\partial_{\mathcal{E}}f &= \frac{3\sqrt{\theta}\partial_{\mathcal{E}}\theta}{2}p_0^0 + \theta^{3/2}\frac{d}{df}p_0^0\partial_{\mathcal{E}}f\\ &= \partial_{\mathcal{E}}(\theta^{3/2}p_0^0) = \partial_{\mathcal{E}}\rho = 0,\end{aligned}$$

we conclude

$$\langle R\rangle_{F(1-F)} = 0.$$

On the other hand,

$$\begin{aligned}&\left\langle \frac{v\cdot\nabla_x(F(1-F)g)}{F(1-F)}\right\rangle_{F(1-F)}\\ &= \nabla_x\cdot\langle v(F(1-F)g)\rangle_1 = \nabla_x\cdot\langle vg\rangle_{F(1-F)} = 0.\end{aligned}$$

Then

$$\langle\sqrt{\theta}VR\rangle_{F(1-F)} = \frac{1}{3}\theta^{3/2}p_2^1\frac{\nabla_x\cdot(\mu\sigma)}{\rho} = \nabla_x\cdot(\mu\sigma),$$

and

$$\left\langle \sqrt{\theta} V \frac{v \cdot \nabla_x (F(1-F)g)}{F(1-F)} \right\rangle_{F(1-F)}$$

$$= \nabla_x \cdot \left\langle \sqrt{\theta} V \otimes vg \right\rangle_{F(1-F)} - \left\langle \sqrt{\theta} g (v \cdot \nabla_x) V \right\rangle_{F(1-F)}$$

$$= \nabla_x \cdot \langle \theta V \otimes Vg \rangle_{F(1-F)} + \nabla_x \cdot \left(\left\langle \sqrt{\theta} Vg \right\rangle_{F(1-F)} \otimes u \right) + \left(\left\langle \sqrt{\theta} gv \right\rangle_{F(1-F)} \cdot \nabla_x \right) \left(\frac{u}{\sqrt{\theta}} \right)$$

$$= \nabla_x \cdot \langle \theta V \otimes Vg \rangle_{F(1-F)}$$

$$= \nabla_x \cdot \langle \theta B(V) g \rangle_{F(1-F)} + \nabla_x \cdot \left\langle \theta \frac{|V|^2}{3} g \right\rangle_{F(1-F)} = \nabla_x \cdot \langle \theta B(V) g \rangle_{F(1-F)} .$$

Finally,

$$\frac{1}{2} \langle \theta |V|^2 R \rangle_{F(1-F)} = \left(\frac{1}{3} \mu \sigma : \sigma - \frac{2}{3} \nabla_x \cdot q \right) \left(\frac{\theta^{3/2} \partial_{\mathcal{E}} \theta}{2} p_4^1 - \theta^{5/2} p_2^1 \partial_{\mathcal{E}} f \right)$$

$$= \left(\frac{1}{3} \mu \sigma : \sigma - \frac{2}{3} \nabla_x \cdot q \right) \left(\frac{5 \theta^{3/2} p_2^0 \partial_{\mathcal{E}} \theta}{2} + \theta^{5/2} \frac{d}{df} p_2^0 \partial_{\mathcal{E}} f \right)$$

$$= \frac{3}{2} \left(\frac{1}{3} \mu \sigma : \sigma - \frac{2}{3} \nabla_x \cdot q \right) \partial_{\mathcal{E}} \mathcal{E} = \frac{1}{2} \mu \sigma : \sigma - \nabla_x \cdot q$$

and

$$\left\langle \frac{\theta |V|^2}{2} \frac{v \cdot \nabla_x (F(1-F)g)}{F(1-F)} \right\rangle_{F(1-F)}$$

$$= \nabla_x \cdot \left\langle \frac{\theta |V|^2}{2} gv \right\rangle_{F(1-F)} - \left\langle gv \cdot \nabla_x \frac{\theta |V|^2}{2} \right\rangle_{F(1-F)}$$

$$= \nabla_x \cdot \left\langle \theta^{3/2} \frac{|V|^2 V}{2} g \right\rangle_{F(1-F)} + \nabla_x \cdot \left\langle \frac{\theta |V|^2}{2} gu \right\rangle_{F(1-F)} + \langle gv \cdot ((v-u) \cdot \nabla_x u \rangle_{F(1-F)}$$

$$= \nabla_x \cdot \left\langle \theta^{3/2} A(V) g \right\rangle_{F(1-F)} + \langle g(v-u) \otimes v \rangle_{F(1-F)} : \nabla_x u$$

$$= \nabla_x \cdot \left\langle \theta^{3/2} A(V) g \right\rangle_{F(1-F)} + \langle \theta g V \otimes V \rangle_{F(1-F)} : \nabla_x u$$
$$+ \langle g(v-u) \otimes u \rangle_{F(1-F)} : \nabla_x u$$

$$= \nabla_x \cdot \left\langle \theta^{3/2} A(V) g \right\rangle_{F(1-F)} + \frac{1}{2} \langle \theta g B(V) \rangle_{F(1-F)} : \sigma(u).$$

Since we know the expression for g

$$g = -\alpha_L(\rho, \mathcal{E}, |V|) A(V) \cdot \frac{\nabla_x \theta}{\sqrt{\theta}} - \frac{1}{2} \beta_L(\rho, \mathcal{E}, |V|) B(V) : \sigma(u),$$

we can write explicitly

$$\theta^{3/2} \langle A(V) g \rangle_{F(1-F)} \cdot \frac{\nabla_x \theta}{\sqrt{\theta}} = -\theta \langle \alpha_L(\rho, \mathcal{E}, |V|) A(V) \otimes A(V) \rangle_{F(1-F)} \cdot \nabla_x \theta$$

$$= -\frac{\theta}{3} \left\langle \alpha_L(\rho, \mathcal{E}, |V|) |A(V)|^2 \right\rangle_{F(1-F)} \nabla_x \theta,$$

$$\frac{1}{2} \theta \langle g B(V) \rangle_{F(1-F)} = -\frac{1}{2} \theta \langle \beta_L(\rho, \mathcal{E}, |V|) B(V) \otimes B(V) \rangle_{F(1-F)} : \sigma(u)$$

$$= -\theta \left\langle \beta_L(\rho, \mathcal{E}, |V|) B_{12}^2(V) \right\rangle_{F(1-F)} \sigma(u).$$

Thus, the existence of solutions of the equation (II.2.20) is equivalent to the following:

$$\nabla_x \cdot (\mu \sigma) - \nabla_x \cdot (\theta \left\langle \beta_L(\rho, \mathcal{E}, |V|) B_{12}^2(V) \right\rangle_{F(1-F)} \sigma(u)) = 0,$$

$$\left(\frac{1}{2} \mu \sigma : \sigma - \nabla_x \cdot q \right) - \nabla_x \cdot \left(\frac{\theta}{3} \left\langle \alpha_L(\rho, \mathcal{E}, |V|) |A(V)|^2 \right\rangle_{F(1-F)} \nabla_x \theta \right)$$
$$-\theta \left\langle \beta_L(\rho, \mathcal{E}, |V|) B_{12}^2(V) \right\rangle_{F(1-F)} \sigma(u) : \sigma(u) = 0.$$

Clearly, the viscosity and thermal conductivity given by (II.2.13) satisfies these relations, and we can conclude that the function w exists. Thus, the theorem is proven.

□

A Inversion of the linearised collision operator on a certain subspace

Lemma 3. *Consider the equations*

$$L[A_i'](v) = A_i(v), \tag{II.A.1}$$

$$\forall k = 0, \dots, 4 \quad \int_{\mathbb{R}^3} A'(v) e_k(v) F(v)(1 - F(v))\,\mathrm{d}v = 0, \tag{II.A.2}$$

and

$$L[B_{ij}'](v) = B_{ij}(v), \tag{II.A.3}$$

$$\forall k = 0, \dots, 4 \quad \int_{\mathbb{R}^3} B'(v) e_k(v) F(v)(1 - F(v))\,\mathrm{d}v = 0, \tag{II.A.4}$$

where A_i and B_{ij} are polynomials defined for $i, j \in \{1, 2, 3\}$ in theorem II.4:

$$A_i(v) = \left(\frac{|v|^2}{2} - \frac{5}{2}\frac{\rho}{\mathcal{E}\theta}\right) v_i,$$

$$B_{ij}(v) = v_i v_j - \frac{|v|^2}{3}\delta_{ij},$$

with

$$\rho = \int_{\mathbb{R}^3} F(v)\,\mathrm{d}v, \quad \mathcal{E} = \frac{1}{3}\int_{\mathbb{R}^3} |v|^2 F(v)\,\mathrm{d}v.$$

Then there exist positive functions $\alpha_L(\rho, \mathcal{E}, |V|)$ and $\beta_L(\rho, \mathcal{E}, |V|)$ such that

$$A'(V) = -\alpha_L(\rho, \mathcal{E}, |V|) A(V),$$
$$B'(V) = -\beta_L(\rho, \mathcal{E}, |V|) B(V).$$

Proof. Note that A_i and B_{ij} are orthogonal to vectors e_k in the sense of $L^2(F(1-F)\,\mathrm{d}v)$ (see theorem II.4), hence the lemma holds whenever the operator L satisfies the Fredholm alternative. Theorem II.1 states that this is the case for a collision kernel corresponding to hard sphere collisions (see subsection 2 for further discussion).

By translating and rescaling the variable v, we can without losing generality consider the case $u = 0$, $\theta = 1$. We decompose the proof into several propositions following [11].

By translating and rescaling the variable v, we can without losing generality consider the case $u = 0$, $\theta = 1$. We decompose the proof into several propositions following [11].

Proposition 4. *Introduce for each isometry $R \in O(\mathbb{R}^3)$ and for each function $h : \mathbb{R}^3 \to \mathbb{R}$ the operator T_R defined by*

$$T_R f(v) = (f \circ R)(v) = f(Rv).$$

Then

$$L \circ T_R = T_R \circ L.$$

Proof. We write

$$(L \circ T_R)[f](v) = L[T_R f](v)$$

$$= \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} \frac{F(v_*)(1 - F(v - (v - v_*, \omega)\omega)(1 - F(v_* + (v - v_*, \omega)\omega)}{1 - F(v)}$$
$$\Big(f(R(v - (v - v_*, \omega)\omega)) + f(R(v_* + (v - v_*, \omega)\omega))$$
$$- f(Rv_*) - f(Rv)\Big) b\left(|v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \omega\right) \mathrm{d}v_* \, \mathrm{d}\omega.$$

We introduce the change of variables

$$\nu = R\omega, \quad w_* = Rv_*.$$

By observing that $F(v)$ depends only on the norm of v, we can say that $F(w) = F(Rw)$ for all $w \in \mathbb{R}^3$ and rewrite $L[T_R f](v)$ as

$$\iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} \frac{F(w_*)(1 - F(Rv - (Rv - w_*, \nu)\nu)(1 - F(w_* + (Rv - w_*, \nu)\nu)}{1 - F(Rv)}$$
$$\times \Big(f(Rv - (Rv - w_*, \nu)\nu) + f(v_* + (Rv - w_*, \nu)\nu) - f(w_*) - f(Rv)\Big)$$
$$\times\, b\left(|Rv - w_*|, \frac{Rv - w_*}{|Rv - w_*|} \cdot \nu\right) \mathrm{d}w_* \, \mathrm{d}\omega$$
$$= (L[f])(Rv) = (T_R \circ L)[f](v).$$

□

Proposition 5. *For all isometries $R \in O(\mathbb{R}^3)$ the function A' defined in* (II.A.1) *and* (II.A.2) *satisfies*

$$(T_R A')(v) = RA'(v). \tag{II.A.5}$$

Moreover, the function B' defined in (II.A.3) *and* (II.A.4) *satisfies the following properties:*

1. *for all* $v \in \mathbb{R}^3$ $B'(v)$ *is a symmetric tensor with zero trace,*
2. *for all isometries* $R \in O(\mathbb{R}^3)$

$$(T_R B')(v) = RB'(v)R^{-1}$$

in the sense of matrix product.

Proof. We note that, according to proposition 4,

$$L(T_R A') = T_R(LA') = T_R A = A \circ R = R \circ A$$

and that

$$L(R \circ A') = L \circ R \circ A' = R \circ (L[A']) = R \circ A.$$

Denote

$$e_0(v) = 1,\ e_1(v) = v_1,\ e_2(v) = v_2,\ e_3(v) = v_3,\ e_4(v) = |v|^2,$$

then

$$\forall j = 0, \ldots, 4 \quad \int_{\mathbb{R}^3} (T_R A')(v) e_j(v) F(v)(1 - F(v))\, \mathrm{d}v = 0$$

if and only if

$$\forall j = 0, \ldots, 4 \quad \int_{\mathbb{R}^3} (A'(v) e_j(v) F(v)(1 - F(v))\, \mathrm{d}v = 0,$$

which is equivalent to

$$\forall j = 0, \ldots, 4 \quad R \int_{\mathbb{R}^3} (A'(v) e_j(v) F(v)(1 - F(v))\, \mathrm{d}v = 0,$$

or

$$\forall j = 0, \ldots, 4 \quad R \int_{\mathbb{R}^3} (R \circ A'(v) e_j(v) F(v)(1 - F(v))\, \mathrm{d}v = 0.$$

Now we use the uniqueness of the solutions p of the system

$$L[p] = R \circ A, \quad \forall j = 0, \ldots, 4 \int_{\mathbb{R}^3} (p(v) e_j(v) F(v)(1 - F(v))\, \mathrm{d}v = 0$$

to deduce (II.A.5).

In order to prove (1), we first note that

$$L(trB') = trL(B') = trB = 0,$$

i.e. that trB' belongs to the kernel of L. On the other hand,

$$\int_{\mathbb{R}^3} trB'(v) e_j(v) F(v)(1 - F(v))\, \mathrm{d}v = tr \int_{\mathbb{R}^3} B'(v) e_j(v) F(v)(1 - F(v))\, \mathrm{d}v = 0$$

for all $j = 0, \dots, 4$ by definition of B', which implies that trB' is orthogonal to the kernel of L; we conclude that $trB' = 0$.

In the same spirit,

$$L(B' - B'^T) = L(B') - L(B'^T) = B - B^T = 0$$

and

$$\int_{\mathbb{R}^3} (B'(v) - B'^T(v)) e_j(v) F(v)(1 - F(v))\,\mathrm{d}v$$

$$= \int_{\mathbb{R}^3} B'(v) e_j(v) F(v)(1 - F(v))\,\mathrm{d}v - \left(\int_{\mathbb{R}^3} B'(v) e_j(v) F(v)(1 - F(v))\,\mathrm{d}v \right)^T = 0$$

for all $j = 0, \dots, 4$, which allows us to conclude that B' is symetric.

Finally, we prove (2). We can observe that

$$L[T_R B'] = T_R(L[B']) = T_R B = RBR^{-1}$$

and

$$L(RB'R^{-1}) = RL(B')R^{-1} = RBR^{-1}.$$

Moreover,

$$\int_{\mathbb{R}^3} T_R B'(v) e_j(v) F(v)(1 - F(v))\,\mathrm{d}v = 0$$

for all $j = 0, \dots, 4$ is equivalent to

$$\int_{\mathbb{R}^3} B'(v) e_j(v) F(v)(1 - F(v))\,\mathrm{d}v = 0 \quad \forall j = 0, \dots, 4$$

$$\iff \int_{\mathbb{R}^3} RB'(v)R^{-1} e_j(v) F(v)(1 - F(v))\,\mathrm{d}v = 0 \quad \forall j = 0, \dots, 4.$$

Therefore, the uniqueness of the solutions B' of the system

$$Lq = RB'R^{-1}, \quad \int_{\mathbb{R}^3} q(v) e_j(v) F(v)(1 - F(v))\,\mathrm{d}v = 0 \quad \forall j = 0, \dots, 4$$

implies that (2) holds. □

Proposition 6. *Let $n \geq 2$ and $s : \mathbb{R}^n \to \mathbb{R}^n$ such that for all isometries $R \in O(\mathbb{R}^n)$ one has*

$$s \circ R = R \circ s.$$

Then there exists $t : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\forall x \in \mathbb{R}^n \quad s(x) = t(|x|)x.$$

The proof of this proposition can be found in [11].

Proposition 7. *Let $n \geq 2$ and $m : \mathbb{R}^n \to M_n(\mathbb{R})$ be a function such that for all isometry R of $O(\mathbb{R}^n)$ one has*

$$\forall x \in \mathbb{R}^n \quad m(Rx) = Rm(x)R^{-1}.$$

We suppose moreover that for all x in $\mathbb{R}^n$ $m(x)$ is a symmetric matrix with zero trace.

Then there exists $n : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\forall x \in \mathbb{R}^n \quad m(x) = n(|x|)\left(x \otimes x - \frac{|x|^2}{n} I\right).$$

The proof of this proposition also can be found in [11].

Lemma 3 is now a straightforward consequence of propositions 5 and 6 on one hand and 5 and 7 on the other hand.

The sign of functions α_L and β_L is a direct consequence of non-positivity of the operator L; the rescaling of the variable v implies that, in fact, these functions have $V = \frac{v-u}{\sqrt{\theta}}$ as one argument and f, θ as another. The result in lemma I.2 allows us to say that indeed

$$\alpha_L = \alpha_L(\rho, \mathcal{E}, |V|), \quad \beta_L = \beta_L(\rho, \mathcal{E}, |V|).$$

$\square$

Chapter III

The incompressible fluid dynamic limit for kinetic models with Fermi-Dirac statistics

1 Introduction

In this chapter we establish the connection between kinetic theory for Fermi-Dirac statistics and macroscopic fluid dynamics. We derive formal limits; in order to do that, we introduce several scalings for standard kinetic equations of the form

$$\partial_t F + v \cdot \nabla_x F = C(F).$$

Here F is a non-negative function representing the density of particles with position x and velocity v in the single-particle phase space $\mathbb{R}^3_x \times \mathbb{R}^3_v$ at time t. The interaction of particles through collisions is given by the operator $C(F)$; this operator acts only on the variable v and is non-linear in the general case.

As in the chapter II we will consider the class of collision operators given by

$$C(F) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} b \left(F' F'_*(1-F)(1-F_*) - F F_*(1-F')(1-F'_*)\right) \, \mathrm{d}v_* \, \mathrm{d}\omega,$$

where for $\omega \in \mathbb{S}^2$

$$v' = v - (v - v_*, \omega)\omega, \quad v'_* = v_* + (v - v_*, \omega)\omega,$$

$$F' = F(t, x, v'), \quad F_* = F(t, x, v_*), \quad F'_* = F(t, x, v'_*)$$

and $b = b(|v - v_*|, |(\omega, v - v_*)|)$ is the collision kernel.

This chapter extends the results obtained in chapters I and II by establishing the limiting form of the fluid dynamic equations in the incompressible case. This regime requires stronger assumptions than II. Form the other point of view, this work extends the results obtained in [4] for the Fermi-Dirac statistics.

In a compressible fluid one can introduce the Mach number *Ma*, which is the ratio of the bulk velocity to the speed of sound, and the Reynolds number *Re*, which is a ratio of inertial forces to the viscous forces and can be written as $Re = \frac{\mathbf{v}L}{\nu}$, where $\mathbf{v}$ is the bulk velocity, L is the characteristic linear dimension, and ν is the dynamic viscosity of the fluid. The Mach and Reynolds numbers are related to the Knudsen number Kn by the von Kármán relation (see [42]):

$$\mathrm{Kn} = \frac{Ma}{Re}.$$

Clearly, in order to obtain a fluid dynamic limit with finite Reynolds number as Kn tends to zero, the Mach number must vanish. This fact was already observed in [40]. The only such hydrodynamic limits are, therefore, incompressible limits. The derivation of the incompressible Navier-Stokes equations is done in section 2 (see theorem III.1). We assume only a formally consistent convergence for the fluid dynamical moments.

2 Incompressible case

We will build a connection between the kinetic equation and the incompressible Navier-Stokes equations. As previously, we describe the range of parameters for which the incompressible Navier-Stokes equations are a good approximation to the solution of the kinetic equation. However, in this case we study macroscopic fluid dynamics with a finite Reynolds number.

One way of realizing distributions with a small Mach number is to think of them as perturbations around a given uniform Fermi-Dirac distribution, i.e. a Fermi-Dirac distribution that is constant in space and time. By some appropriate choice of Galilean frame and dimensional units, this Fermi-Dirac distribution can be taken with zero velocity, unit chemical potential and temperature. We will denote it by F. The initial data $F_\varepsilon(0, x, v)$ is assumed to be close to F with the order of the distance measured in terms of the Knudsen number. Since we want to obtain the incompressible flow in the limit, the kinetic energy of the flow in the acoustic modes must be smaller than that in the vortical modes. The acoustic modes vary on a faster time scale that vortical ones, we may suppress them by assuming that the initial data is consistent with motion on a slow time scale; we will measure this scale separation in terms of the Knudsen number.

We quantify the scaling with a small parameter ε such that the time scale is of order ε^{-1}, the Knudsen number is of order ε^q, and the distance to the absolute Fermi-Dirac distribution F is of order ε^r with $q \geq 1$, $r \geq 1$.

We seek the solutions of the scaled kinetic equation

$$\varepsilon \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon^q} C(F_\varepsilon) \tag{III.2.1}$$

in the form

$$F_\varepsilon = F(1 + \varepsilon^r (1 - F) g_\varepsilon). \tag{III.2.2}$$

We recall several properties of the moments of Fermi-Dirac distributions and of the linearised collision operator. In the chapter I it was established that the form of the limiting Euler equations is independent of the choice of the collision operator C in the class of operators satisfying conservation and entropy properties. This choice appears on the macroscopic level only in the construction of the Navier-Stokes limit. We obtain the compressible Navier-Stokes equations by classical Chapman-Enskog expansion. We give a description of this method below.

As in chapter II we write Fermi-Dirac distributions in the form

$$F_{(f,u,\theta)}(v) = \left(1 + \exp\left(-f + \frac{|v-u|^2}{2\theta}\right)\right)^{-1}.$$

The subscript is omitted wherever it is convenient. The Navier-Stokes equations operate in hydrodynamic variables ρ and $\mathcal{E}$ defined in (I.4.2) and (I.4.3):

$$\rho = \int_{\mathbb{R}^3} \frac{\mathrm{d}v}{1+\exp\left(\frac{|v-u|^2}{2\theta} - f\right)} = \theta^{3/2} \int_{\mathbb{R}^3} \frac{\mathrm{d}v}{1+\exp\left(\frac{|v|^2}{2} - f\right)},$$

$$\mathcal{E} = \frac{1}{3} \int_{\mathbb{R}^3} \frac{|v-u|^2\,\mathrm{d}v}{1+\exp\left(\frac{|v-u|^2}{2\theta} - f\right)} = \frac{\theta^{5/2}}{3} \int_{\mathbb{R}^3} \frac{|v|^2\,\mathrm{d}v}{1+\exp\left(\frac{|v|^2}{2} - f\right)}.$$

Note that ρ and $\mathcal{E}$ do not depend on u and are $\mathcal{C}^1$ functions of f and θ.

As previously, we denote by L and Q the first and the second Fréchet derivatives of the operator $G \to (F(1-F))^{-1}C(F + F(1-F)G)$ at $G = 0$ for a Fermi-Dirac distribution F:

$$L(g) = \frac{DC(F)\cdot(F(1-F)g)}{F(1-F)},$$
$$Q(g,g) = \frac{D^2C(F) : (F(1-F)g \otimes F(1-F)g)}{F(1-F)}.$$

By Taylor's formula

$$\frac{C(F+F(1-F)\epsilon g)}{F(1-F)} = \epsilon L(g) + \frac{\epsilon^2}{2} Q(g,g) + \mathcal{O}(\epsilon^3).$$

Note that this expansion enables us to say that the case $r = q = 1$ is the unique scaling compatible with the usual incompressible Navier-Stokes equations. Indeed, if $u(t,x)$ is a solution of the incompressible Navier-Stokes equation

$$\partial_t u + (u\cdot\nabla_x)u = -\nabla_x p + \nu\Delta u,$$

then so is the function $\lambda^{-1}u(\lambda^{-2}t, \lambda^{-1}x)$ for a positive constant λ. By putting the same rescaling for the function g_ε with $\lambda = \varepsilon$ and denoting the rescaled function by $\tilde{g}_\varepsilon$ we obtain that the following identity must hold:

$$\varepsilon^{r-2}\tilde{g}_\varepsilon + \varepsilon^{r-2} v\cdot\nabla_x \tilde{g}_\varepsilon = \varepsilon^{r-1-q} L(\tilde{g}_\varepsilon) + \frac{\varepsilon^{2r-2-q}}{2} Q(\tilde{g}_\varepsilon, \tilde{g}_\varepsilon) + \mathcal{O}(\varepsilon^{3r-3-q}).$$

This relation implies the equation

$$r - 2 = r - 1 - q = 2r - 2 - q,$$

which has only one solution $r = q = 1$. See also [6, 10] for the discussion of compatible scalings.

As we already said, in general, the operator L is unbounded, but it is defined as an unbounded linear operator on the space $L^2(F(1-F)\mathrm{d}v)$.

Recall that for the collision operator (II.2.3) the linearised collision operator can be written as

$$L[h](v) = \frac{1}{(1-F)F} \iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}\{h\}\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

with

$$\mathcal{N} = FF_*(1-F')(1-F'_*) = F'F'_*(1-F)(1-F_*)$$

and

$$\{\phi\} = \phi'_* + \phi' - \phi_* - \phi.$$

Note that $FF_*(1-F')(1-F'_*) = F'F'_*(1-F)(1-F_*)$ if and only if F is a Fermi-Dirac distribution.

We also recall the following result, established in chapter II:

Theorem.
- *The linearised collision operator L defined in* (II.2.6) *self-adjoint, non-positive, and its nullspace is* $\ker L = span\{1, v_1, v_2, v_3, |v|^2\}$.
- *Denoting*

$$V = \frac{v-u}{\sqrt{\theta}}, \quad A(V) = \left(\frac{|V|^2}{2} - \frac{5}{2}\frac{\mathcal{E}}{\rho\theta}\right)V, \quad B(V) = V\otimes V - \frac{1}{3}|V|^2 I,$$

where ρ and $\mathcal{E}$ are given by (I.4.2) *and* (I.4.3)*, then A_i and $B_{i,j}$ are orthogonal to* $\ker L$ *in* $L^2(F(1-F)\,\mathrm{d}v)$.

With the notation $\langle s\rangle_k = \int_{\mathbb{R}^3} s(v)k(v)\,\mathrm{d}v$ for all functions $s \in L^1(k(v)\,\mathrm{d}v)$ and $\langle s\rangle = \int_{\mathbb{R}^3} s(v)\,\mathrm{d}v$ for $s \in L^1(\,\mathrm{d}v)$, we introduce the following constants:

$$E_0 = \langle 1\rangle_{F(1-F)}, \quad E_2 = \langle |v_1|^2\rangle_{F(1-F)},$$
$$E_4 = \langle |v_1|^4\rangle_{F(1-F)}, \quad E_{22} = \langle |v_1v_2|^2\rangle_{F(1-F)},$$

$$C_A = \left\langle \left(\frac{|v|^2}{2} - K_A\right)^2 v_1^2 \right\rangle_{F(1-F)(1-2F)},$$

$$k_* = \left\langle \alpha_L(\rho_0, \mathcal{E}_0, |v|)\left(\frac{|v|^2}{2} - K_A\right)^2 v_1^2 \right\rangle_{F(1-F)}, \quad k = \frac{k_*}{C_A},$$

$$\mu_* = \langle \beta_L(\rho_0, \mathcal{E}_0, |v|)v_1^2v_2^2\rangle_{F(1-F)}, \quad \mu = \frac{\mu_*}{E_2}.$$

We also recall that $K_A = \frac{5}{2}\frac{\mathcal{E}}{\rho\theta}$ and is defined by the condition $A_1 \perp V_1$ in $L^2(F(1-F)\,\mathrm{d}v)$ (see theorem II.4). In our case $\mu = 1$, $\theta = 1$, we denote the corresponding density and energy by ρ_0, $\mathcal{E}_0$, so $K_A = \frac{5}{2}\frac{\mathcal{E}_0}{\rho_0}$.

Theorem III.1. *Let $F_\varepsilon(t,x,v)$ be a sequence of non-negative solutions of the scaled kinetic equation* (III.2.1) *such that, if written in the form* (III.2.2)*, the sequence g_ε converges in the sense of distributions and almost everywhere to a function g as ε goes to zero. In addition, assume that moments*

$$\langle g_\varepsilon \rangle_{F(1-F)}, \quad \langle v g_\varepsilon \rangle_{F(1-F)}, \quad \langle v \otimes v g_\varepsilon \rangle_{F(1-F)}, \quad \langle v|v|^2 g_\varepsilon \rangle_{F(1-F)},$$

$$\langle L^{-1}(A(v)) \otimes v g_\varepsilon \rangle_{F(1-F)}, \quad \langle L^{-1}(A(v)) Q(g_\varepsilon, g_\varepsilon) \rangle_{F(1-F)},$$

$$\langle L^{-1}(A(v)) \otimes v g_\varepsilon \rangle_{F(1-F)}, \quad \langle L^{-1}(B(v)) Q(g_\varepsilon, g_\varepsilon) \rangle_{F(1-F)}$$

converge in $D'(\mathbb{R}^+_t \times \mathbb{R}^3_x)$ to the corresponding moments

$$\langle g \rangle_{F(1-F)}, \quad \langle v g \rangle_{F(1-F)}, \quad \langle v \otimes v g \rangle_{F(1-F)}, \quad \langle v|v|^2 g \rangle_{F(1-F)},$$

$$\langle L^{-1}(A(v)) \otimes v g \rangle_{F(1-F)}, \quad \langle L^{-1}(A(v)) Q(g, g) \rangle_{F(1-F)},$$

$$\langle L^{-1}(A(v)) \otimes v g \rangle_{F(1-F)}, \quad \langle L^{-1}(B(v)) Q(g, g) \rangle_{F(1-F)}$$

and all formally small in ε terms vanish. Then the limiting g has the form

$$g = \rho + v \cdot u + \theta \left(\frac{|v|^2}{2} - K_g \right), \tag{III.2.3}$$

where the velocity u is divergence-free, $K_g = K_A - 1$ and the density and temperature fluctuations ρ and θ satisfy the Boussinesq relation

$$\nabla_x \cdot u = 0, \quad \nabla_x(\rho + \theta) = 0. \tag{III.2.4}$$

Moreover, the functions ρ, u and θ are weak solutions of the following fluid dynamic equations

- $r = 1$, $q = 1$

$$\partial_t u + (u \cdot \nabla_x) u + \nabla_x p = \mu \Delta u, \tag{III.2.5}$$

$$\partial_t \theta + u \cdot \nabla_x \theta = k \Delta \theta, \tag{III.2.6}$$

- $r > 1$, $q = 1$

$$\partial_t u + \nabla_x p = \Delta u, \tag{III.2.7}$$

$$\partial_t \theta = k \Delta \theta, \tag{III.2.8}$$

- $r = 1,\ q > 1$

$$\partial_t u + (u \cdot \nabla_x)u + \nabla_x p = 0, \tag{III.2.9}$$

$$\partial_t \theta + u \cdot \nabla_x \theta = 0, \tag{III.2.10}$$

- $r > 1,\ q > 1$

$$\partial_t u + \nabla_x p = 0,$$

$$\partial_t \theta = 0.$$

Proof. We split the technical results required for this theorem in several propositions. Their proofs can be found in the appendix A.

Proposition 8. *If $s = s(|v|)$ is a measurable function such that*

$$|v|^4 \in L^1(s(|v|)dv),$$

then

$$\left\langle v_1^4 - 3v_1^2 v_2^2 \right\rangle_s = 0.$$

As a direct corollary of this proposition one can establish that $E_4 = 3E_{22}$.

Proposition 9. *Define*

$$\tilde{E}_{22} = \langle |v_1 v_2|^2 \rangle_{F(1-F)(1-2F)},$$

then

$$\tilde{E}_{22} = E_2.$$

We recall the definition of the following fields which are orthogonal to the nullspace of the operator L:

$$B(v) = v \otimes v - \frac{1}{3}|v|^3 I, \quad A(v) = \left(\frac{|v|^2}{2} - K_A \right) v.$$

The constant K_A is defined from the orthogonality relation $A(v) \perp_{F(1-F)} v_i$ and is equivalent to

$$\left\langle \frac{|v|^2 |v_1|^2}{2} - K_A |v_1|^2 \right\rangle_{F(1-F)} = 0,$$

$$\frac{E_4 + 2E_{22}}{2} - K_A E_2 = 0$$

or

$$K_A = \frac{E_4 + 2E_{22}}{2E_2}.$$

Proposition 10. *One has*

$$K_A E_0 - \frac{3}{2} E_2 > 0. \tag{III.2.11}$$

Proposition 11. *One has*

$$C_A = \left\langle \left(\frac{|v|^2}{2} - K_A \right)^2 v_1^2 \right\rangle_{F(1-F)(1-2F)} = \left(K_A E_0 - \frac{3}{2} E_2 \right) K_A.$$

Proposition 12. *Let $s : \mathbb{R}^3 \to \mathbb{R}_+$ be a measurable function such that $s(v) = s(|v|)$. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be such that $|v|^2 f(|v|) \in L^1(s(v)\,\mathrm{d}v)$. Then*

$$\int_{\mathbb{R}^3} B(v) f(|v|) s(v)\,\mathrm{d}v = 0,$$

where we recall that $B = v \otimes v - \frac{1}{3}|v|^2 I$.

Step 1. If we insert the form (III.2.2) of the distribution F into (III.2.1) and use Taylor expansion of the collision operator, we obtain

$$\varepsilon g_\varepsilon + v \cdot \nabla_x g_\varepsilon = \varepsilon^{-q} L(g_\varepsilon) + \frac{1}{2} \varepsilon^{r-q} Q(g_\varepsilon, g_\varepsilon) + O(\varepsilon^{2r-q}). \tag{III.2.12}$$

Multiplying both sides by ε^q, letting ε go to zero, and using the assumptions on the convergence of moments, we obtain

$$L(g) = 0.$$

Hence g belongs to the nullspace of L and can be written in the form (III.2.3).

Step 2. We derive (III.2.4) from the equations for the conservation of mass and momentum associated with (III.2.12); we arrive at the identity

$$\varepsilon \partial_t \langle g_\varepsilon \rangle_{F(1-F)} + \nabla_x \cdot \langle v g_\varepsilon \rangle_{F(1-F)} = 0,$$

$$\varepsilon \partial_t \langle v g_\varepsilon \rangle_{F(1-F)} + \nabla_x \cdot \langle v \otimes v g_\varepsilon \rangle_{F(1-F)} = 0.$$

Letting ε go to zero in the above formulas, and passing to the limit in the sense of distributions, we conclude that

$$\nabla_x \cdot \langle v g \rangle_{F(1-F)} = 0, \quad \nabla_x \cdot \langle v \otimes v g \rangle_{F(1-F)} = 0.$$

The first equality can be put in the form

$$\nabla_x \cdot \Big(\langle v \otimes v \rangle_{F(1-F)}\, u \Big) = 0$$

or

$$0 = \nabla_x \cdot E_2 I u = E_2 \nabla_x \cdot u$$

so that

$$\nabla_x \cdot u = 0,$$

since $E_2 > 0$. This is the incompressibility condition.

Let us proceed with the second relation, taking g as in (III.2.3) and studying the resulting expression term by term. First,

$$\nabla_x \cdot \Big(\langle v \otimes v \rangle_{F(1-F)} \, \rho \Big) = E_2 \nabla_x \rho,$$

and

$$\langle v \otimes v (v \cdot u) \rangle_{F(1-F)} = 0,$$

since the integrand is odd in v. It remains to study the expression

$$\nabla_x \cdot \left(\left\langle v \otimes v \left(\frac{|v|^2}{2} - K_g \right) \right\rangle_{F(1-F)} \theta \right) = \left\langle |v_1|^2 \left(\frac{|v|^2}{2} - K_g \right) \right\rangle_{F(1-F)} \nabla_x \theta.$$

If we take the definitions of K_A and K_g, we can rewrite this expression as

$$\left\langle |v_1|^2 \left(\frac{|v|^2}{2} - K_A \right) \right\rangle_{F(1-F)} \nabla_x \theta + E_2 \nabla_x \theta = E_2 \nabla_x \theta.$$

Therefore

$$0 = \frac{1}{E_2} \nabla_x \cdot \langle v \otimes v g \rangle_{F(1-F)} = \nabla_x (\rho + \theta)$$

which is precisely the Boussinesq relation (III.2.4). If we assume that ρ and θ belong to, say, the space $L^2(\mathbb{R}^3)$, then this relation immediately implies that $\rho + \theta = 0$. The same conclusion would follow in the case where $\rho,\, \theta \in L^2(\mathbb{T}^3)$ with $\int_{\mathbb{T}^3} \rho \, \mathrm{d}x = \int_{\mathbb{T}^3} \theta \, \mathrm{d}x = 0$.

Step 3. The limiting momentum equation is obtained from

$$\partial_t \langle v g_\varepsilon \rangle_{F(1-F)} + \frac{1}{\varepsilon} \nabla_x \cdot \langle v \otimes v g_\varepsilon \rangle_{F(1-F)} = 0.$$

We separate the flux tensor into its traceless and scalar components:

$$\begin{aligned} \partial_t \langle v g_\varepsilon \rangle_{F(1-F)} + \frac{1}{\varepsilon} \nabla_x \cdot \left\langle \left(v \otimes v - \frac{1}{3} |v|^2 I \right) g_\varepsilon \right\rangle_{F(1-F)} \\ + \frac{1}{\varepsilon} \nabla_x \left\langle \frac{1}{3} |v|^2 g_\varepsilon \right\rangle_{F(1-F)} = 0, \end{aligned}$$

or equivalently

$$\partial_t \langle v g_\varepsilon \rangle_{F(1-F)} + \frac{1}{\varepsilon} \nabla_x \cdot \langle B(v) g_\varepsilon \rangle_{F(1-F)} + \nabla_x p_\varepsilon = 0, \tag{III.2.13}$$

where the pressure is defined as

$$p_\varepsilon = \varepsilon^{-1} \left\langle \frac{1}{3} |v|^2 g_\varepsilon \right\rangle_{F(1-F)}.$$

Likewise, we combine the density and energy equations to get the limiting temperature equation, as follows. Observe that

$$\partial_t \left\langle \left(\frac{1}{2} |v|^2 - K_A \right) g_\varepsilon \right\rangle_{F(1-F)} + \frac{1}{\varepsilon} \nabla_x \cdot \langle A(v) g_\varepsilon \rangle_{F(1-F)} = 0. \tag{III.2.14}$$

By the convergence assumptions and the limiting form (III.2.3) of g, we first see that

$$\lim_{\varepsilon \to 0} \partial_t \langle v g_\varepsilon \rangle_{F(1-F)} = \partial_t \langle v g \rangle_{F(1-F)} = E_2 \partial_t u$$

in the sense of distributions, while

$$\begin{aligned} &\lim_{\varepsilon \to 0} \partial_t \left\langle \left(\frac{1}{2} |v|^2 - K_A \right) g_\varepsilon \right\rangle_{F(1-F)} \\ &= \partial_t \left\langle \left(\frac{1}{2} |v|^2 - K_A \right) \left(\rho + (v \cdot u) + \theta \left(\frac{|v|^2}{2} - K_g \right) \right) \right\rangle_{F(1-F)} \\ &= \partial_t \left\langle \left(\frac{1}{2} |v|^2 - K_A \right) (\rho - K_g \theta) \right\rangle_{F(1-F)} \\ &= \left(K_A E_0 - \frac{3}{2} E_2 \right) \partial_t (K_g \theta - \rho), \end{aligned}$$

also in the sense of distributions.

Since $\rho + \theta = 0$ we can rewrite the expression above as

$$\partial_t \left\langle \left(\frac{1}{2} |v|^2 - K_A \right) g \right\rangle_{F(1-F)} = \left(K_A E_0 - \frac{3}{2} E_2 \right) K_A \partial_t \theta.$$

The pressure term p_ε in the right side of (III.2.13) may fail to have a limit in the sense of distributions as $\varepsilon \to 0$. However, this does not matter since it is eliminated upon integrating the equation against a divergence-free test function.

Step 4. To complete the proof of theorem III.1, we need to estimate the limits of the moments

$$\varepsilon^{-1} \langle B(v) g_\varepsilon \rangle_{F(1-F)}, \quad \varepsilon^{-1} \langle A(v) g_\varepsilon \rangle_{F(1-F)},$$

which appear in (III.2.13) and (III.2.14) respectively.

Bearing in mind that L is symmetric in the space $L^2(F(1-F)\,\mathrm{d}v)$, we start with the identities

$$\langle A(v)g_\varepsilon\rangle_{F(1-F)} = \left\langle L^{-1}(A(v))L(g_\varepsilon)\right\rangle_{F(1-F)},$$

$$\langle B(v)g_\varepsilon\rangle_{F(1-F)} = \left\langle L^{-1}(B(v))L(g_\varepsilon)\right\rangle_{F(1-F)},$$

and eliminate $L(g_\varepsilon)$ from (III.2.12)

$$\varepsilon g_\varepsilon + v\cdot\nabla_x g_\varepsilon = \varepsilon^{-q}L(g_\varepsilon) + \frac{1}{2}\varepsilon^{r-q}Q(g_\varepsilon, g_\varepsilon) + O(\varepsilon^{2r-q}).$$

Our assumptions on the convergence allow us to evaluate the limiting moments as

$$\begin{aligned}
\lim_{\varepsilon\to 0}\varepsilon^{-1}\langle A(v)g_\varepsilon\rangle_{F(1-F)} &= \lim_{\varepsilon\to 0}\varepsilon^{q-1}\nabla_x\cdot\langle L^{-1}(A(v))\otimes v g_\varepsilon\rangle_{F(1-F)}\\
&\quad -\frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon^{r-1}\langle L^{-1}(A(v))Q(g_\varepsilon, g_\varepsilon)\rangle_{F(1-F)},\\
\lim_{\varepsilon\to 0}\varepsilon^{-1}\langle B(v)g_\varepsilon\rangle_{F(1-F)} &= \lim_{\varepsilon\to 0}\varepsilon^{q-1}\nabla_x\cdot\langle L^{-1}(B(v))\otimes v g_\varepsilon\rangle_{F(1-F)}\\
&\quad -\frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon^{r-1}\langle L^{-1}(B(v))Q(g_\varepsilon, g_\varepsilon)\rangle_{F(1-F)}.
\end{aligned}$$

We neglected the terms that are formally $O(\varepsilon^q)$ and $O(\varepsilon^{2r-1})$. For $r > 1$ and $q > 1$ all the limits on the right side vanish by the moment convergence assumptions. For the case $r > q = 1$ we need to compute the moments

$$\nabla_x\cdot\langle L^{-1}(A(v))\otimes vg\rangle_{F(1-F)} \quad\text{and}\quad \nabla_x\cdot\langle L^{-1}(B(v))\otimes vg\rangle_{F(1-F)}.$$

Similarly, for the case $q > r = 1$ we need

$$\langle L^{-1}(A(v))Q(g,g)\rangle_{F(1-F)} \quad\text{and}\quad \langle L^{-1}(B(v))Q(g,g)\rangle_{F(1-F)}.$$

For the case $r = q = 1$ one needs to know all four moments above.

The limiting form of g and the Boussinesq relation allow us to write

$$\begin{aligned}
\nabla_x\cdot\langle L^{-1}(A(v))\otimes vg\rangle_{F(1-F)} &= \langle L^{-1}(A(v))(v\cdot\nabla_x g)\rangle_{F(1-F)}\\
&= \left\langle L^{-1}(A(v))\left(v\cdot\nabla_x\left(\rho+\theta\left(\frac{|v|^2}{2}-K_g\right)\right)\right)\right\rangle_{F(1-F)}\\
&= \left\langle L^{-1}(A(v))\left(\frac{|v|^2}{2}-K_A\right)(v\cdot\nabla_x\theta)\right\rangle_{F(1-F)}
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle L^{-1}(A(v)) \otimes A(v) \right\rangle_{F(1-F)} \cdot \nabla_x \theta \\
&= -\left\langle \alpha_L(\rho_0, \mathcal{E}_0, |v|) A(v) \otimes A(v) \right\rangle_{F(1-F)} \cdot \nabla_x \theta \\
&= -\left\langle \alpha_L(\rho_0, \mathcal{E}_0, |v|) \left(\frac{|v|^2}{2} - K_A \right) v_1^2 \right\rangle_{F(1-F)} \nabla_x \theta = -k_* \nabla_x \theta.
\end{aligned}
$$

Applying the divergence operator results in the term $\Delta\theta$ in the expressions (III.2.6),(III.2.8).

In the same spirit,

$$
\begin{aligned}
&\nabla_x \cdot \langle L^{-1}(B(v)) \otimes vg \rangle_{F(1-F)} \\
&= \nabla_x \cdot \langle L^{-1}(B(v)) \otimes v(v \cdot u) \rangle_{F(1-F)} \\
&= \langle L^{-1}(B(v)) \otimes (v \otimes v) \rangle_{F(1-F)} : \nabla_x u \\
&= \langle L^{-1}(B(v)) \otimes (v \otimes v - \frac{1}{3}|v|^2 I) \rangle_{F(1-F)} : \nabla_x u \\
&\quad + \langle L^{-1}(B(v)) \frac{1}{3} |v|^2 \rangle_{F(1-F)} \nabla_x \cdot u \\
&= \langle L^{-1}(B(v)) \otimes B(v) \rangle_{F(1-F)} : \nabla_x u \\
&= -\langle \beta_L(\rho_0, \mathcal{E}_0, |v|) B(v) \otimes B(v) \rangle_{F(1-F)} : \nabla_x u.
\end{aligned}
$$

The following lemma explains the presence of Δu in equations (III.2.5), (III.2.7).

Lemma 4.

$$
\nabla_x \cdot \Big(\langle \beta_L(\rho_0, \mathcal{E}_0, |v|) B(v) \otimes B(v) \rangle_{F(1-F)} : \nabla_x u \Big) = \mu_* \Delta u
$$

Proof. We consider the terms of $B(v) \otimes B(v)$ that do not vanish after integration with respect to $\beta_L(\rho_0, \mathcal{E}_0, |v|) F(v)(1 - F(v))\, \mathrm{d}v$. It is easy to see that such terms can only have the following forms:

$$
\beta_L(\rho_0, \mathcal{E}_0, |v|) B_{ij}^2 (\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_i \otimes \vec{e}_j + \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_j \otimes \vec{e}_i), \quad i \neq j,
$$

$$
\beta_L(\rho_0, \mathcal{E}_0, |v|) B_{ii} B_{jj} \vec{e}_i \otimes \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_j.
$$

After taking into account that $\nabla_x \cdot u = 0$, we arrive to the divergence of the form

$$
\mu_* \Delta u + \sum_i (DD - ED - 2\mu_*) \frac{\partial^2 u_i}{\partial x_i^2} \vec{e}_i
$$

with

$$
DD = \left\langle B_{11}^2 \right\rangle_{F(1-F)\beta_L(\rho_0, \mathcal{E}_0, |v|)},
$$

$$
ED = \langle B_{11} B_{22} \rangle_{F(1-F)\beta_L(\rho_0, \mathcal{E}_0, |v|)}.
$$

Hence

$$\begin{aligned}
DD - ED - 2\mu_* &= \left\langle B_{11}(B_{11} - B_{22}) - 2B_{12}^2 \right\rangle_{F(1-F)\beta_L(\rho_0,\mathcal{E}_0,|v|)} \\
&= \left\langle B_{11}(v_1^2 - v_2^2) - 2v_1^2 v_2^2 \right\rangle_{F(1-F)\beta_L(\rho_0,\mathcal{E}_0,|v|)} \\
&= \left\langle v_1^2(v_1^2 - v_2^2) - 2v_1^2 v_2^2 \right\rangle_{F(1-F)\beta_L(\rho_0,\mathcal{E}_0,|v|)} \\
&= \left\langle v_1^4 - 3v_1^2 v_2^2 \right\rangle_{F(1-F)\beta_L(\rho_0,\mathcal{E}_0,|v|)} = 0
\end{aligned}$$

by Proposition 8. □

Proposition 13. *If $g \in \ker L$, then $Q(g,g) = -L[(1-2F)g^2]$.*

Proof. From the expansion (II.2.5) it quickly follows that

$$\begin{aligned}
\frac{F(1-F)}{2} Q(g,g) = &\iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} bF'F'_*(1-F')(1-F'_*)(1-F)(1-F_*)g'g'_* \,\mathrm{d}v_*\,\mathrm{d}\omega \\
&- \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} bFF_*F'F'_*(1-F')(1-F'_*)g'g'_* \,\mathrm{d}v_*\,\mathrm{d}\omega \\
&- \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} bF'F'_*F(1-F)(1-F')(1-F_*)g'g \,\mathrm{d}v_*\,\mathrm{d}\omega \\
&+ \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} bFF_*F'(1-F)(1-F')(1-F'_*)g'g \,\mathrm{d}v_*\,\mathrm{d}\omega \\
&- \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} bF'F'_*F_*(1-F)(1-F')(1-F_*)g'g_* \,\mathrm{d}v_*\,\mathrm{d}\omega \\
&+ \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} bFF_*F'(1-F')(1-F'_*)(1-F_*)g'g_* \,\mathrm{d}v_*\,\mathrm{d}\omega \\
&- \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} bF'F'_*F(1-F)(1-F_*)(1-F'_*)g'_*g \,\mathrm{d}v_*\,\mathrm{d}\omega \\
&+ \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} bFF_*F'_*(1-F')(1-F'_*)(1-F)g'_*g \,\mathrm{d}v_*\,\mathrm{d}\omega \\
&- \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} bF'F'_*F_*(1-F)(1-F_*)(1-F'_*)g'_*g_* \,\mathrm{d}v_*\,\mathrm{d}\omega \\
&+ \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} bFF_*F'_*(1-F')(1-F'_*)(1-F_*)g'_*g_* \,\mathrm{d}v_*\,\mathrm{d}\omega
\end{aligned}$$

$$-\iint_{\mathbb{R}^3\times\mathbb{S}^2} bFF_*(1-F')(1-F'_*)(1-F)(1-F_*)gg_*\,\mathrm{d}v_*\,\mathrm{d}\omega$$
$$+\iint_{\mathbb{R}^3\times\mathbb{S}^2} bFF_*F'F'_*(1-F)(1-F_*)gg_*\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

By factoring out $\mathcal{N}=F'F'_*(1-F)(1-F_*)=FF_*(1-F')(1-F'_*)$ we obtain a simpler expression for $\frac{F(1-F)}{2}Q(g,g)$:

$$\begin{aligned}&\iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}(1-F'-F'_*)g'g'_*\,\mathrm{d}v_*\,\mathrm{d}\omega+\iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}(F'-F)g'g\,\mathrm{d}v_*\,\mathrm{d}\omega\\&+\iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}(F'-F_*)g'g_*\,\mathrm{d}v_*\,\mathrm{d}\omega+\iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}(F'_*-F)g'_*g\,\mathrm{d}v_*\,\mathrm{d}\omega\\&+\iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}(F'_*-F_*)g'_*g_*\,\mathrm{d}v_*\,\mathrm{d}\omega-\iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}(1-F-F_*)gg_*\,\mathrm{d}v_*\,\mathrm{d}\omega\end{aligned}\tag{III.2.15}$$

We write the similar expression for

$$\begin{aligned}\frac{F(1-F)}{2}L[(1-2F)g^2]=&\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}((g'_*)^2+(g')^2-g_*^2-g^2)\,\mathrm{d}v_*\,\mathrm{d}\omega\\&-\iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}(F'_*(g'_*)^2+F'(g')^2-F_*g_*^2-Fg^2)\,\mathrm{d}v_*\,\mathrm{d}\omega\end{aligned}\tag{III.2.16}$$

We represent the sum of the expressions (III.2.15) and (III.2.16) as the sum of several integrals, namely

$$\frac{F(1-F)}{2}Q(g,g)+\frac{F(1-F)}{2}L[(1-2F)g^2]=\sum_{i=1}^{5}\mathrm{I}_i,$$

where

$$\mathrm{I}_1=\iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}\left((g'_*)^2+(g')^2+\frac{1}{2}g'g'_*-g_*^2-g^2-\frac{1}{2}gg_*\right)\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

$$\mathrm{I}_2=\iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}F'_*g'_*\left(-g'_*-g'+g+g_*\right)\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

$$\mathrm{I}_3=\iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}F'g'\left(-g'_*-g'+g+g_*\right)\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

$$\mathrm{I}_4=\iint_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}F_*g_*\left(-g'_*-g'+g+g_*\right)\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

$$\mathrm{I}_5 = \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}Fg\,(-g'_* - g' + g + g_*)\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

However, we can further simplify

$$\mathrm{I}_1 = \frac{1}{2}\iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} b\mathcal{N}((g'_* + g')^2 - (g_* + g)^2)\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

Since $g \in \ker L$, we obtain that $g'_* + g' = g_* + g$ a.e. and therefore $\mathrm{I}_1 = 0$. With the same observation, we deduce that all other integrals I_i, $i = 2, \ldots, 5$ are also equal to zero. We can now conclude that

$$\frac{F(1-F)}{2}Q(g,g) + \frac{F(1-F)}{2}L[(1-2F)g^2] = 0$$

or

$$Q(g,g) = -L[(1-2F)g^2]$$

whenever $g \in \ker L$. $\square$

The above Proposition enables us to simplify the following expressions:

$$\begin{aligned}\langle L^{-1}(A(v))Q(g,g)\rangle_{F(1-F)} &= -\langle L^{-1}(A(v))L((1-2F)g^2)\rangle_{F(1-F)}\\ &= -\langle A(v)g^2\rangle_{F(1-F)(1-2F)},\end{aligned}$$

and

$$\begin{aligned}\langle L^{-1}(B(v))Q(g,g)\rangle_{F(1-F)} &= -\langle L^{-1}(B(v))L((1-2F)g^2)\rangle_{F(1-F)}\\ &= -\langle B(v)g^2\rangle_{F(1-F)(1-2F)}.\end{aligned}$$

Now we can rewrite

$$\begin{aligned}-\langle A(v)g^2\rangle_{F(1-F)(1-2F)} &= -2\left\langle A(v)(v\cdot u)\theta\left(\frac{|v|^2}{2} - K_A\right)\right\rangle_{F(1-F)(1-2F)}\\ &= -2\left\langle\left(\frac{|v|^2}{2} - K_A\right)^2 v\otimes v\right\rangle_{F(1-F)(1-2F)}\theta u = -2C_A u\theta.\end{aligned}$$

Applying the divergence operator to both sides of this equality and bearing in mind that $\nabla_x \cdot u = 0$, we obtain the term $u \cdot \nabla_x\theta$ in the expressions (III.2.6), (III.2.10).

Finally, we evaluate

$$-\langle B(v)g^2\rangle_{F(1-F)(1-2F)}$$

$$= -\left\langle B(v)\left((v\cdot u)^2+\theta^2\left(\frac{|v|^2}{2}-K_A\right)\right)\right\rangle_{F(1-F)(1-2F)}.$$

Proposition 12 allows us to conclude that the term with θ in the above expression disappears. We need to estimate

$$-\langle B(v)(v\cdot u)^2\rangle_{F(1-F)(1-2F)}.$$

Applying Lemma 8 and Proposition 9, this expression can be simplified to

$$-2E_2B(u).$$

Applying the divergence operator together with the incompressibility condition $\nabla_x\cdot u=0$ gives the terms $(u\cdot\nabla_x)u$ in (III.2.5) and (III.2.9).

Finally, we divide the equation on u by E_2, the equation on θ by C_A and incorporate these divisors in the pressure term and in the coefficients k and μ to obtain the final equations of this theorem. □

A Integration and symmetry lemmas

Proposition A.1. *If $s=s(|v|)$ is a measurable function such that $|v|^4\in L^1(s(|v|)dv)$, then*

$$\left\langle v_1^4-3v_1^2v_2^2\right\rangle_s=0.$$

Proof. We observe that

$$\left\langle v_1^4-3v_1^2v_2^2\right\rangle_s=\left\langle \frac{1}{2}v_1^4-\frac{1}{2}v_2^4-3v_1^2v_2^2\right\rangle_s.$$

Notice that the function

$$h:\mathbb{R}^3\to\mathbb{R},\quad h(v)=\frac{1}{2}v_1^4-\frac{1}{2}v_2^4-3v_1^2v_2^2$$

is harmonic and vanishes in zero, therefore its integral on the unit sphere is zero by the mean value property for harmonic functions. Since the function s depens only on $|v|$, we can conclude that

$$\int_{\mathbb{S}^2}h(v)s(|v|)\mathrm{d}s=0,$$

which implies that

$$\left\langle v_1^4-3v_1^2v_2^2\right\rangle_s=\langle h(v)\rangle_s=0.$$

□

Proposition A.2. *Define*

$$\tilde{E}_{22} = \langle |v_1 v_2|^2 \rangle_{F(1-F)(1-2F)},$$

then

$$\tilde{E}_{22} = E_2.$$

Proof. Define also

$$\tilde{E}_4 = \langle |v_1|^4 \rangle_{F(1-F)(1-2F)},$$

then by previous proposition

$$\tilde{E}_4 = 3\tilde{E}_{22}.$$

By the rotational invariance of the distribution F, we have

$$3\tilde{E}_4 + 6\tilde{E}_{22} = 15\tilde{E}_{22} = \langle |v|^4 \rangle_{F(1-F)(1-2F)}.$$

Now denote $w = |v|$; we can write $F = \frac{1}{1+\exp(w^2/2-f)}$ and

$$\frac{\mathrm{d}}{\mathrm{d}w}(F(1-F)) = -\frac{1}{w}F(1-F)(1-2F),$$

which allows us to deduce

$$\langle |v|^4 \rangle_{F(1-F)(1-2F)} = \int_{\mathbb{R}^3} |v|^4 F(1-F)(1-2F)\,\mathrm{d}v$$

$$= -4\pi \int_0^\infty w^5 \frac{\mathrm{d}}{\mathrm{d}w}(F(1-F))\,\mathrm{d}w$$

$$= 4\pi \int_0^\infty 5w^4 F(1-F)\,\mathrm{d}w = \int_{\mathbb{R}^3} 5|v|^2 F(1-F)\,\mathrm{d}w = 15E_2.$$

Therefore we can conclude that $\tilde{E}_{22} = E_2$. □

Proposition A.3. *One has*

$$K_A E_0 - \frac{3}{2}E_2 > 0.$$

Proof. We can rewrite K_A as

$$K_A = \frac{E_4 + 2E_{22}}{2E_2} = \frac{\langle v_1^4 + 2v_1^2 v_2^2 \rangle_{F(1-F)}}{2\,\langle v_1^2 \rangle_{F(1-F)}}$$

$$\frac{\langle |v|^4 \rangle_{F(1-F)}}{2\,\langle |v|^2 \rangle_{F(1-F)}}.$$

Moreover,

$$E_2 = \left\langle v_1^2 \right\rangle_{F(1-F)} = \frac{1}{3}\left\langle |v|^2 \right\rangle_{F(1-F)},$$

therefore the inequality (III.2.11) is equivalent to

$$\frac{\langle |v|^4 \rangle_{F(1-F)}}{\langle |v|^2 \rangle_{F(1-F)}} \langle 1 \rangle_{F(1-F)} - \left\langle |v|^2 \right\rangle_{F(1-F)} > 0,$$

and the last inequality holds by the Cauchy-Schwarz inequality, therefore, the proposition is proven. □

Proposition A.4. *One has*

$$C_A = \left\langle \left(\frac{|v|^2}{2} - K_A \right)^2 v_1^2 \right\rangle_{F(1-F)(1-2F)} = \left(K_A E_0 - \frac{3}{2} E_2 \right) K_A.$$

Proof. Denote, as previously, $w = |v|$, so $F = \frac{1}{1+\exp(w^2/2-f)}$ and

$$\frac{\mathrm{d}}{\mathrm{d}w}(F(1-F)) = -\frac{1}{w} F(1-F)(1-2F).$$

In addition, by rotational symmetry, we have

$$C_A = \frac{1}{3}\left\langle \left(\frac{|v|^2}{2} - K_A \right)^2 |v|^2 \right\rangle_{F(1-F)(1-2F)}$$

$$= \frac{1}{3}\left\langle \left(\frac{|v|^6}{4} - K_A|v|^4 + K_A^2|v|^2 \right) \right\rangle_{F(1-F)(1-2F)}$$

$$= \frac{1}{12}\int_{\mathbb{R}^3} |v|^6 F(1-F)(1-2F)\,\mathrm{d}v$$

$$-\frac{1}{3}K_A \int_{\mathbb{R}^3} |v|^4 F(1-F)(1-2F)\,\mathrm{d}v$$

$$+\frac{1}{3}K_A^2 \int_{\mathbb{R}^3} |v|^2 F(1-F)(1-2F)\,\mathrm{d}v$$

$$= \frac{\pi}{3}\int_0^\infty w^8 F(1-F)(1-2F)\,\mathrm{d}w$$

$$-\frac{4\pi}{3}K_A \int_0^\infty w^6 F(1-F)(1-2F)\,\mathrm{d}w$$

$$+\frac{4\pi}{3}K_A^2 \int_0^\infty w^4 F(1-F)(1-2F)\,\mathrm{d}v$$

$$
\begin{aligned}
&= \frac{7\pi}{3}\int_0^\infty w^6 F(1-F)\,\mathrm{d}w \\
&-\frac{20\pi}{3}K_A\int_0^\infty w^4 F(1-F)\,\mathrm{d}w \\
&+4\pi K_A^2\int_0^\infty w^2 F(1-F)\,\mathrm{d}v \\
= \frac{7}{12}\int_{\mathbb{R}^3}|v|^4 F(1-F)\,\mathrm{d}v &- \frac{5}{3}K_A\int_{\mathbb{R}^3}|v|^2 F(1-F)\,\mathrm{d}v \\
&+K_A^2\int_{\mathbb{R}^3} F(1-F)\,\mathrm{d}v \\
&= \frac{7}{12}(E_4+2E_{22}) - 5K_A E_2 + K_A^2 E_0,
\end{aligned}
$$

which can be further simplified with the help of the identity $K_A = \frac{E_4+2E_{22}}{2E_2}$:

$$
= \frac{7}{2}E_2K_A - 5K_AE_2 + K_A^2E_0 = \left(K_AE_0 - \frac{3}{2}E_2\right).
$$

□

Proposition A.5. *Let $s : \mathbb{R}^3 \to \mathbb{R}_+$ be a measurable function such that $s(v) = s(|v|)$. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be such that $|v|^2 f(|v|) \in L^1(s(v)\,\mathrm{d}v)$. Then*

$$
\int_{\mathbb{R}^3} B(v) f(|v|) s(v)\,\mathrm{d}v = 0,
$$

where $B = v\otimes v - \frac{1}{3}|v|^2 I$.

Proof. Thanks to the symmetry of the problem, it is sufficient to prove this result only for elements $B_{11}(v) = \frac{2}{3}v_1^2 - \frac{1}{3}v_2^2 - \frac{1}{3}v_3^2$ and $B_{12} = v_1v_2$.

We write

$$
\int_{\mathbb{R}^3} B_{11}(v) f(|v|) s(v)\,\mathrm{d}v = \int_{\mathbb{R}^3}\left(\frac{2}{3}v_1^2 - \frac{1}{3}v_2^2 - \frac{1}{3}v_3^2\right) f(|v|)s(|v|)\,\mathrm{d}v
$$

$$
= \frac{2}{3}\int_{\mathbb{R}^3} v_1^2 f(|v|)s(|v|)\,\mathrm{d}v - \frac{1}{3}\int_{\mathbb{R}^3} v_2^2 f(|v|)s(|v|)\,\mathrm{d}v - \frac{1}{3}\int_{\mathbb{R}^3} v_3^2 f(|v|)s(|v|)\,\mathrm{d}v = 0
$$

thanks to the rotational symmetry.

Moreover,

$$
\int_{\mathbb{R}^3} B_{12}(v) f(|v|) s(v)\,\mathrm{d}v = \int_{\mathbb{R}^3} v_1v_2 f(|v|)s(|v|)\,\mathrm{d}v = 0
$$

because the function $v \to v_1v_2 f(|v|)s(|v|)$ is odd in v_1. □

Chapter IV

Solutions of the Boltzmann equation over $\mathbb{R}^D$ linearised about global Maxwellians with small mass

1 Introduction

In this chapter we study the kinetic theory of gases in the classical case. We know that for the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = C(F)$$

there two competing mechanisms in the density evolution for this equation: on the one hand, the dissipation increases entropy and therefore the solution relaxes to the thermodynamic equilibrium; on the other hand, the dispersion rarefies the collisions between particles and hence diminishes the effect of dispersion. The authors of [1] show that particular choice of the function M allows us to find a balance between these two mechanisms.

We study long time behaviour of the solutions $g = g(t, x, v)$ of the linearised Boltzmann equation near a global Maxwellian M in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. The global Maxwellian functions $M = M(t, x, v)$ are both local Maxwellians in v (i.e. of the form $\exp(a(t, x) + \vec{b}(t, x) \cdot v + c(t, x)|v|^2))$ and satisfy the free transport equation

$$\partial_t M(t, x, v) + v \cdot \nabla_x M(t, x, v) = 0.$$

An example of a global Maxwellian is a function $(t, x, v) \to \exp(-|x - tv|^2)$. The class of global Maxwellians of finite mass is known to be the source of an interesting dynamics for the Boltzmann equation where the effects of dispersion and of relaxation to the local equilibrium are in a balanced competition. For a further discussion of this subject, see [1].

In [22], Golse considers the question on comparability of two entities: on the one side, the difference $H(f) - H(\mathcal{S}[f])$, where $\mathcal{S}$ is the scattering operator for nonlinear Boltzmann equation and H is the associated H-function for this equation, and $H(f) - H(M_f(0))$, where M_f is a global Maxwellian function admitting the same moments as the function f. It was shown that

$$0 \leq H(f) - H(\mathcal{S}[f]) \leq H(f) - H(M_f(0)).$$

The question is whether one can obtain an inequality of the form

$$c(H(f) - H(M_f(0)))^\alpha \leq H(f) - H(\mathcal{S}[f])$$

for some constants $\alpha > 0$ and $c > 0$, which is analogous to the Cercignani's conjecture on the relation between entropy production rate and relative entropy in the context of the Boltzmann equation over $\mathbb{R}^d$ in the scattering regime (this question is discussed, for example, in [39]).

On the other hand, this implies that thanks to the particular choice of the global Maxwellian function M the solutions of the equation (3.2) are not relaxing to a thermodynamic equilibrium.

Our result can be viewed as a negative answer to the same question but for the linearised Boltzmann equation and suggests that this is also the case for the non-linear Boltzmann equation for soft motentials with angular cut-off.

This chapter is organised as follows: in the section 2 we recall the properties of the global Maxwellians, introduce necessary functional spaces, and prove the continuity of the linearised Boltzmann operator L_t. In the section 3 we prove the existence and unicity of the Cauchy problem and the large-time limit problem for the linearised Boltzmann equation (IV.2.2). We also study the large-time limits of these solutions. In the section 4 we define the scattering operator and show its properties, most notably, the existence of a spectral gap in a weighted Hilbert space.

2 Linearised equation

Global Maxwellians and the linearised Boltzmann equation

Global Maxwellians with finite mass are functions of $(t, x, v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ of the form

$$\frac{m}{(2\pi)^d} \sqrt{\det(Q)} \exp(-q(t, x - x_0 - tv_0, v - v_0)),$$

where $m > 0$, x_0, $v_0 \in \mathbb{R}^d$ and

$$q(t, x, v) = \frac{1}{2} \Big(c|v|^2 + a|x - tv|^2 + 2b(x - tv) \cdot v + 2v \cdot B(x - tv) \Big)$$

such that $a, c > 0$, $b \in \mathbb{R}$, $B = -B^T$ a matrix such that

$$Q = (ac - b^2)I + B^2 > 0.$$

If local Maxwellian writes

$$M[\rho, u, \theta](v) = \frac{\rho}{(2\pi)^{d/2}} \exp\left(-\frac{|v - u|^2}{2\theta} \right),$$

then global Maxwellians can be represented as local Maxwellians with

$$\begin{aligned} \theta(t) &= \frac{1}{at^2 - 2bt + c}, \\ \rho(t, x) &= m \sqrt{\det\left(\frac{Q}{2\pi}\right)} \theta^{d/2}(t) \exp(-\theta(t)(x - x_0 - tv_0)^T Q (x - x_0 - tv_0)), \end{aligned} \tag{IV.2.1}$$

$$u(t,x) = \theta(t)\left((at-b)I - B\right)(x - x_0 - tv_0) + v_0.$$

For a complete characterisation of global Maxwellians see [30].

The linearised Boltzmann equation writes as follows:

$$(\partial_t + v \cdot \nabla_x) g(t,x,v) = L_t[g](t,x,v). \tag{IV.2.2}$$

As previously, for vectors $\omega \in \mathbb{S}^{d-1}$ and $v, v_* \in \mathbb{R}^d$ we denote

$$v' = v - (v - v_*, \omega)\omega, \quad v'_* = v_* + (v - v_*, \omega)\omega.$$

We also denote

$$g' = g(t,x,v'), \quad g_* = g(t,x,v_*), \quad g'_* = g(t,x,v'_*)$$

and similarly for M. The linearised collision operator L_t writes

$$L_t[g](t,x,v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} b(v - v_*, \omega) M(t,x,v_*) \{g\}(t,x,v,v_*) \, \mathrm{d}v_* \, \mathrm{d}\omega$$

with

$$\{g\}(t,x,v,v_*) = g'_* + g' - g_* - g,$$

and b is the collision kernel.

This operator is well-studied for a large class of collision kernels b. Most notably, this operator is local in (t,x), and acts only on the v-dependence of the distribution function only. It is self-adjoint non-positive in the space $L^2(M \, \mathrm{d}v)$. Moreover, the null-space is spanned by functions $1, v_1, \ldots, v_d, |v|^2$ (see, for example, [7], [18], [23], [24], [31] for further discussion of properties of this operator).

We will assume that the collision kernel has separated form, i.e.

$$b(z,\omega) = |z|^\beta \hat{b}\left(\omega \cdot n\right), \quad \text{with } n = \frac{z}{|z|}$$

and satisfies the weak cut-off condition (see [26]):

$$b_1 = \int_{\mathbb{S}^{d-1}} \hat{b}(\omega \cdot n) \, \mathrm{d}\omega < \infty. \tag{IV.2.3}$$

Such a collision kernel will be said to correspond to a "hard" potential for the molecular interaction if $\beta \in (0,1]$, and to a "soft" potential if $\beta \in (-d, 0)$. The case $\beta = 0$ corresponds to an assumption made by Maxwell in [33], and is referred to as the case of "Maxwell molecules". The case of hard sphere collisions is the case where $b(z,\omega) = |z \cdot \omega|$.

Functional spaces

For a given global Maxwellian M we introduce the Hilbert space

$$\mathcal{X}_M(t) = L^2(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v).$$

We will study the solutions of the equation (IV.2.2) in the space $\mathcal{Y}_M$ given by the norm

$$\|g(t,x,v),\mathcal{Y}_M\| = \sup_{t\in\mathbb{R}} \|g(t,\cdot,\cdot),\mathcal{X}_M(t)\|.$$

We will also denote by $B(\mathcal{X}_M(t))$ the space of linear bounded operators on the space $\mathcal{X}_M(t)$. The $(\cdot,\cdot)_{\mathcal{X}_M(t)}$ will stand for scalar product in $\mathcal{X}_M(t)$.

Conservative solutions

In this subsection we will study the space of solutions of the equation (IV.2.2) satisfying both the free transport equation

$$(\partial_t + v\cdot\nabla_x)g(t,x,v) = 0$$

and belonging to the null-space of L_t a.e. in t and x. We will call it the space of conservative solutions.

Lemma 1. *A basis of the space of conservative solutions is*

$$1, \quad (x-tv)_i, \quad |x-tv|^2, \quad v_i, \quad |v|^2$$

$$v\cdot(x-tv), \quad S_j(x-tv)\cdot v,$$

where $i=1,\dots,d$ and the antisymmetric matrices S_j form a basis of the space of antisymmetric matrices in $M_d(\mathbb{R})$.

The proof of this result is given in appendix A.

We will denote this space of functions over $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ by $\mathcal{V}$ and the basis vectors defined in the above lemma by $r_j(t;x,v)$. We will also introduce the space $\mathcal{V}_0$ of functions over $\mathbb{R}^d\times\mathbb{R}^d$ generated by $r_j(t;x,v)$ for a fixed time t. Clearly, the space $\mathcal{V}_0$ does not depend on the choice of time t.

Projection on the space of conservative solutions

We will introduce the operator $A = v\cdot\nabla_x$ defined over the space of functions over $\mathbb{R}^d\times\mathbb{R}^d$:

$$e^{tA}[h](x,v) = h(x+tv,v).$$

We will also extend this definition on functions defined on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ as

$$e^{tA}[g](s,x,v) = g(s, x+tv, v).$$

Note that for each global Maxwellian M we have the identity

$$e^{tA}[M(t,x,v)] = M(0,x,v).$$

In addition, for each time $s \in \mathbb{R}$ we can define an orthogonal projection $P_s = P_s^* = P_s^2$ on $\mathcal{V}_0$ in the space $\mathcal{X}_M(s)$. Note that this operator commutes with the operator e^{sA} in the sense that

$$\forall g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \quad g \in \mathcal{X}_M(0), \quad P_s[e^{sA}[g]] = e^{sA}[P_0[g]].$$

Indeed, the basis vectors of $\mathcal{V}_0$ for $t = 0$ and for $t = s$ are related by

$$r_j(s;x,v) = e^{-sA}[r_j(0;x,v)].$$

Moreover, the orthogonal projection is uniquely determined by the scalar products with basis vectors r_j, and it is easy to see that

$$(r_j(s), e^{-sA}[g])_{\mathcal{X}_M(s)} = (e^{-sA}[r_j(0)], e^{-sA}[g])_{\mathcal{X}_M(s)} = (r_j(0), g)_{\mathcal{X}_M(0)},$$

which allows us to conclude.

Lemma 2. *Let M be a global Maxwellian, then for each function $h(x,v) \in \mathcal{X}_M(t)$ there exists a unique function $p(x,v) \in \mathcal{V}_0$ such that*

$$\forall j \quad (h, r_j(t))_{\mathcal{X}_M(t)} = (p, r_j(t))_{\mathcal{X}_M(t)}.$$

Proof. The vectors $r_j(t)$ are linearly independent, hence the Gram matrix N given by $N_{ij} = (r_i(t), r_j(t))_{\mathcal{X}_M(t)}$ is non-singular. Then again, since for a fixed t the vectors $r_j(t)$ generate the same linear space of functions over x and v, we can assume that p is a linear combination of $r_j(t)$:

$$p(x,v) = \sum_j \alpha_j r_j(t),$$

therefore the problem reduces to finding the solution of the system of linear equations

$$\sum_j N_{ij}\alpha_j = (h, r_i(t))_{\mathcal{X}_M(t)},$$

which obviously has a unique solution. $\square$

Continuity of the operator L_t

Theorem IV.1. *Suppose that the collision kernel b has the separate form with $\beta \in (-d, 0]$. For any global Maxwellian M and any time t the operator L_t is continuous on the space $\mathcal{X}_M(t)$.*

Moreover, if in addition we have $\beta \in (1-d, 0]$, then

$$\int_{\mathbb{R}} \|L_t, B(\mathcal{X}_M(t))\| \, \mathrm{d}t < \infty.$$

We give the proof of this theorem in appendix A.
For a global Maxwellian M and collision kernel b of separate form satisfying the condition (IV.2.3) with $\beta \in (1-d, 0]$ we will denote

$$\mu(M) = \int_{\mathbb{R}} \|L_t, B(\mathcal{X}_M(t))\| \, \mathrm{d}t < \infty.$$

Remark. It is important to notice that the norm of the operator L_t and $\mu(M)$ can be made arbitrarily small by choosing an appropriate global Maxwellian, for example, with sufficiently small mass.
Remark 2. In the case $\beta \in (0, 1]$ the operator L_t is naturally defined as an unbounded operator on $L^2(M \, \mathrm{d}v)$ (see [24]) and therefore one will need to conduct a similar analysis in a Hilbert space with different weight in order to obtain similar results.

3 Solutions of the linearised equation

Definition 1. *A mild solution of the equation* (IV.2.2) *is a function*

$$g = g(t, x, v) \in L^1_{loc}(I \times \mathbb{R}^d \times \mathbb{R}^d)$$

where I is an interval in $\mathbb{R}$, such that $L_t[g] \in L^1_{loc}(I \times \mathbb{R}^d \times \mathbb{R}^d)$ and

$$e^{t_2 A}[g](t_2, x, v) - e^{t_1 A}[g](t_1, x, v) = \int_{t_1}^{t_2} e^{sA}[L_s[g]](s, x, v) \, \mathrm{d}s.$$

Conservation properties

In this subsection we state the conservation properties of solutions of the linearised Boltzmann equation (IV.2.2).

Theorem IV.2. *Assume that $M(t, x, v)$ is a global Maxwellian in v. Let $g = g(t, x, v)$ be a measurable function defined a.e. on $I \times \mathbb{R}^d \times \mathbb{R}^d$ where I is an open interval in $\mathbb{R}$ satisfying the condition*

$$\sup_{t \in I} \|g(t, x, v), \mathcal{X}_M(t)\| < +\infty.$$

1. *For a.e.* $(t,x) \in I \times \mathbb{R}^d$ *the following identity holds:*

$$\int_{\mathbb{R}^d} M(t,x,v) L_t[h](t,x,v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v = 0.$$

2. *Assume moreover that* g *is a mild solution of the linearised Boltzmann equation* (IV.2.2) *in the sense of distributions on* $I \times \mathbb{R}^d \times \mathbb{R}^d$ *satisfying*

$$\sup_{t \in I} \|g(t), \mathcal{X}_M(t)\| < +\infty.$$

Then the function g *satisfies the global conservation laws*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d \times \mathbb{R}^d} M(t,x,v) g(t,x,v) \begin{pmatrix} 1 \\ x - tv \\ |x - tv|^2 \\ v \\ |v|^2 \\ (x - tv) \cdot v \\ x \wedge v \end{pmatrix} \mathrm{d}x \, \mathrm{d}v = 0$$

in the sense of distributions on I.

Theorem IV.3. *Let* M *be a global Maxwellian. Let also* $g = g(t,x,v)$ *be a measurable function defined a.e. on* $I \times \mathbb{R}^d \times \mathbb{R}^d$ *where* I *is an interval in* $\mathbb{R}$ *such that*

$$\sup_{t \in I} \|g(t), \mathcal{X}_M(t)\| < \infty.$$

Then

1. *for a.e.* $(t,x) \in I \times \mathbb{R}$

$$\int_{\mathbb{R}^d} M(t) L[g(t)] g(t) \, \mathrm{d}v \le 0.$$

2. *the inequality above becomes an equality if and only if* $L[g] = 0$ *a.e. on* $I \times \mathbb{R}^d \times \mathbb{R}^d$*, or equivalently,* g *is locally a linear combination of functions* $1, v_1, \ldots, v_d, |v|^2$*, i.e. there exist functions* $a(t,x)$*,* $c(t,x)$ *and a vector field* $b(t,x)$ *in* $\mathbb{R}^d$ *such that*

$$g(t,x,v) = a(t,x) + b(t,x) \cdot v + c(t,x)|v|^2.$$

3. *Assume moreover that g is a mild solution of the linearised Boltzmann equation* (IV.2.2) *on $I \times \mathbb{R}^d \times \mathbb{R}^d$, then the Boltzmann H-function associated with g defined as*

$$H[g](t) = \|g(t), \mathcal{X}_M(t)\|^2$$

satisfies

$$\frac{\mathrm{d}H[g](t)}{\mathrm{d}t} = \int_{\mathbb{R}^d \times \mathbb{R}^d} M(t) L[g(t)] g(t) \, \mathrm{d}x \, \mathrm{d}v \leq 0.$$

We put proofs of these two theorems in the appendix A.

Existence of solutions and their limiting behaviour

In this subsection we will study the existence of mild solutions for the Cauchy problem for the equation (IV.2.2), as well as for the boundary problems. We will also establish the results on the behaviour of these solutions for $|t| \to \infty$.

First, we state the following variant of Gronwall's lemma:

Lemma 3. *Let $\Delta > 0$, let also $m \in L^1(\mathbb{R})$ satisfy $m(t) > 0$ a.e., then the following statements hold:*

1. *if $\phi_+ \in L^\infty([t_0, \infty))$ satisfies the integral inequality*

$$0 \leq \phi_0(t) \leq \Delta + \int_t^{+\infty} \phi_0(s) m(s) \, \mathrm{d}s \text{ for a.e. } t > t_0,$$

then

$$\phi_0(t) \leq \Delta \exp\left(\int_{t_0}^t m(s) \, \mathrm{d}s \right)$$

for a.e. $t \geq t_0$.

2. *if $\phi_- \in L^\infty((-\infty, t_0])$ satisfies the integral inequality*

$$0 \leq \phi_-(t) \leq \Delta + \int_{-\infty}^t \phi_-(s) m(s) \, \mathrm{d}s \text{ for a.e. } t < t_0,$$

then

$$\phi_-(t) \leq \Delta \exp\left(\int_t^{t_0} m(s) \, \mathrm{d}s \right)$$

for a.e. $t < t_0$.

3. *let $T > t_0$; if $\psi_+ \in L^\infty([t_0, T])$ satisfies the integral inequality*

$$0 \leq \psi_+(t) \leq \Delta + \int_{t_0}^t \psi_+(s) m(s) \, \mathrm{d}s \text{ for a.e. } t \in [t_0, T],$$

then

$$\psi_+(t) \le \Delta \exp\Big(\int_{t_0}^{t} m(s)\,\mathrm{d}s\Big)$$

for a.e. $t \in [t_0, T]$.

4. *let* $T < t_0$*; if* $\psi_- \in L^\infty([T, t_0])$ *satisfies the integral inequality*

$$0 \le \psi_-(t) \le \Delta + \int_{t}^{t_0} \psi_-(s) m(s)\,\mathrm{d}s \text{ for a.e. } t \in [T, t_0],$$

then

$$\psi_-(t) \le \Delta \exp\Big(\int_{t}^{t_0} m(s)\,\mathrm{d}s\Big)$$

for a.e. $t \in [T, t_0]$.

We give the proof of this lemma in appendix A.

Theorem IV.4. *Assume that the collision kernel* b *has separated form with* $\beta \in (1-d, 0]$*. Let* M *be a global Maxwellian such that* $\mu(M) < 1$*.*

1. *For each* $g^{in} \in \mathcal{X}_M(0)$ *there exists a unique mild solution* $g \in \mathcal{Y}_M$ *of the linearised Boltzmann equation* (IV.2.2) *such that* $g(0) = g^{in}$*.*
2. *This solution satisfies the estimation:*

$$\|g, \mathcal{Y}_M\| \le \|g^{in}, \mathcal{X}_M(0)\| e^{\mu(M)}.$$

Remark. If the initial value $g^{in}(x, v)$ belongs to the space $\mathcal{V}_0$, then the unique solution $g(t, x, v)$ of the equation (IV.2.2) with initial data $g(0, x, v) = g^{in}(x, v)$ writes, obviously,

$$g(t, x, v) = e^{-tA}[g^{in}(x, v)].$$

Proof. By the definition of mild solutions, we need to find $g \in \mathcal{Y}_M$ such that

$$g(t, x, v) = e^{-tA}[g^{in}] + \int_0^t e^{(s-t)A}[L_s[g(s, x, v)]]\,\mathrm{d}s. \tag{IV.3.1}$$

We will introduce the linear maps

$$G : \mathcal{X}_M(0) \to \mathcal{Y}_M, \quad G[h](t, x, v) = e^{-tA}[h](t, x, v) = h(x - tv, v)$$

and

$$B_0 : \mathcal{Y}_M \to \mathcal{Y}_M, \quad B_0[f](t, x, v) = \int_0^t e^{(s-t)A}[L_s[f(s, x, v)]]\,\mathrm{d}s.$$

In terms of these linear operators we can rewrite the identity (IV.3.1) as

$$g = G[g^{in}] + B_0[g]$$

or

$$(I - B_0)[g] = G[g^{in}].$$

Under the hypotheses of theorem IV.4 the operator B_0 is well-defined on $\mathcal{Y}_M$ and

$$\|B_0\| \leq \mu(M) < 1.$$

Indeed,

$$\begin{aligned}
&\|B_0[f], \mathcal{Y}_M\| \\
&= \left\| \int_0^t e^{(s-t)A}[L_s[f(s,x,v)]]\,\mathrm{d}s, \mathcal{Y}_M \right\| = \sup_{t\in\mathbb{R}} \left\| \int_0^t e^{(s-t)A}[L_s[f(s,x,v)]]\,\mathrm{d}s, \mathcal{X}_M(t) \right\| \\
&= \sup_{t\in\mathbb{R}} \left\| \int_0^t L_s[f(s,x,v)]\,\mathrm{d}s, \mathcal{X}_M(s) \right\| \leq \sup_{t\in\mathbb{R}} \int_0^t \|L_s[f(s,x,v)], \mathcal{X}_M(s)\|\,\mathrm{d}s \\
&\leq \int_{\mathbb{R}} \|L_s[f(s,x,v)], \mathcal{X}_M(s)\|\,\mathrm{d}s \\
&\leq \int_{\mathbb{R}} \|L_s, B(\mathcal{X}_M(s))\| \|f(s), \mathcal{X}_M(s)\|\,\mathrm{d}s \leq \mu(M)\|f, \mathcal{Y}_M\|.
\end{aligned}$$

Since $\|B_0\| < 1$, the operator $I - B_0$ is continuously invertible and

$$g = (I - B_0)^{-1}[G[g^{in}]].$$

This expression allows us to write an estimate

$$\|g, \mathcal{Y}_M\| \leq \frac{\|g^{in}, \mathcal{X}_M(0)\|}{1 - \mu(M)}. \tag{IV.3.2}$$

However, since g is a mild solution of the equation (IV.2.2), it satisfies

$$g(t) = e^{-tA}[g^{in}] + \int_0^t e^{(s-t)A}[L_s[g(s)]]\,\mathrm{d}s,$$

which leads to

$$\begin{aligned}
\|g(t), \mathcal{X}_M(t)\| &\leq \|e^{-tA}[g^{in}], \mathcal{X}_M(t)\| + \int_0^t \left\| e^{(s-t)A}[L_s[g(s)]], \mathcal{X}_M(t) \right\|\,\mathrm{d}s \\
&\leq \|g^{in}, \mathcal{X}_M(0)\| + \int_0^t \|L_s[g(s)], \mathcal{X}_M(s)\|\,\mathrm{d}s \\
&\leq \|g^{in}, \mathcal{X}_M(0)\| + \int_0^t \|L_s, B(\mathcal{X}_M(s))\|\, \|g(s), \mathcal{X}_M(s)\|\,\mathrm{d}s.
\end{aligned}$$

Lemma 3, applied to the function $t \to \|g(t), \mathcal{X}_M(t)\|$, results in

$$\|g(t), \mathcal{X}_M(t)\| \leq \|g^{in}, \mathcal{X}_M(0)\| \exp\left(\left| \int_0^t \|L_s, B(\mathcal{X}_M(s))\|\,\mathrm{d}s \right| \right),$$

and

$$\|g, \mathcal{Y}_M\| \leq \|g^{in}, \mathcal{X}_M(0)\| e^{\mu(M)},$$

which is a better estimation than (IV.3.2). □

Similarly, we can introduce the boundary value problem: instead of condition at time $t = 0$ we impose the condition on $t \to -\infty$ or $t \to +\infty$.

Theorem IV.5. *Assume that the collision kernel b has separated form with $\beta \in (1-d, 0]$. Let M be a global Maxwellian such that $\mu(M) < 1$.*

1. *For each $g^{-\infty} \in \mathcal{X}_M(0)$ there exists a unique mild solution $g_- \in \mathcal{Y}_M$ of the linearised Boltzmann equation* (IV.2.2) *such that*

$$\lim_{t\to-\infty} \left\| e^{tA}[g_-(t)] - g^{-\infty}, \mathcal{X}_M(0) \right\| = 0.$$

2. *For each $g^{+\infty} \in \mathcal{X}_M(0)$ there exists a unique mild solution $g_+ \in \mathcal{Y}_M$ of the linearised Boltzmann equation* (IV.2.2) *such that*

$$\lim_{t\to+\infty} \left\| e^{tA}[g_+(t)] - g^{+\infty}, \mathcal{X}_M(0) \right\| = 0.$$

3. *These solutions satisfy the estimations*

$$\|g_-, \mathcal{Y}_M\| \le \|g^{-\infty}, \mathcal{X}_M(0)\| e^{\mu(M)},$$
$$\|g_+, \mathcal{Y}_M\| \le \|g^{+\infty}, \mathcal{X}_M(0)\| e^{\mu(M)}.$$

We will denote the linear maps $\mathcal{X}_M(0) \to \mathcal{Y}_M$ given by

$$g^{-\infty} \mapsto g_- \quad \textit{and} \quad g^{+\infty} \mapsto g_+$$

as $\mathcal{F}_-$ and $\mathcal{F}_+$, respectively.

Remark. If $h \in \mathcal{V}_0$ then

$$\mathcal{F}_-[h](t,x,v) = e^{-tA}[h(x,v)], \quad \mathcal{F}_+[h](t,x,v) = e^{-tA}[h(x,v)].$$

Indeed, the function $(t,x,v) \to e^{-tA}[h(x,v)]$ belongs to $\mathcal{Y}_M$, is a mild solution of the equation (IV.2.2) and its limiting behaviour for $t \to \pm\infty$ is evident.

Proof. As in the proof of theorem IV.4, we introduce the operators

$$B_- : \mathcal{Y}_M \to \mathcal{Y}_M, \quad B_-[f](t,x,v) = \int_{-\infty}^{t} e^{(s-t)A}[L_s[f(s)]]\,\mathrm{d}s,$$

$$B_+ : \mathcal{Y}_M \to \mathcal{Y}_M, \quad B_+[f](t,x,v) = \int_{t}^{+\infty} e^{(s-t)A}[L_s[f(s)]]\,\mathrm{d}s.$$

The norms of these operators satisfy the estimations

$$\|B_-\| \le \mu(M) < 1, \quad \|B_+\| \le \mu(M) < 1$$

for the same reasons as the operator B_0 introduced in theorem IV.4.

If a mild solution $f \in \mathcal{Y}_M$ of the equation (IV.2.2) satisfies

$$\lim_{t\to-\infty} e^{tA}[f] = g^{-\infty},$$

then we can pass to the limit $t_1 \to -\infty$ in the identity

$$e^{t_2 A}[f](t_2) - e^{t_1 A}[f](t_1) = \int_{t_1}^{t_2} e^{sA}[L_s[f(s)]]\,\mathrm{d}s,$$

implying that

$$e^{t_2 A}[f](t_2) - g^{-\infty} = \int_{-\infty}^{t_2} e^{sA}[L_s[f(s)]]\,\mathrm{d}s.$$

The last integral converges in $\mathcal{X}_M(0)$, because

$$\int_{-\infty}^{t_2} \left\| e^{sA}[L_s[f(s)]], \mathcal{X}_M(0) \right\| \,\mathrm{d}s = \int_{-\infty}^{t_2} \| L_s[f(s)], \mathcal{X}_M(s) \| \,\mathrm{d}s$$

$$\leq \int_{-\infty}^{t_2} \| L_s, B(\mathcal{X}_M(s)) \| \| [f(s)], \mathcal{X}_M(s) \| \,\mathrm{d}s \leq \mu(M) \| f, \mathcal{Y}_M \|.$$

In other words, the function g_- solves the equation

$$g_- = G[g^{-\infty}] + B_-[g_-].$$

The operator $(I - B_-)^{-1}$ is well-defined, hence g_- writes:

$$g_- = (I - B_-)^{-1}[G[g^{-\infty}]].$$

In particular, this leads to

$$e^{t_2 A}[g](t_2) - e^{t_1 A}[g](t_1) = \int_{t_1}^{t_2} e^{sA}[L_s[g(s)]]\,\mathrm{d}s,$$

hence g_- is a mild solution of the equation (IV.2.2). Moreover,

$$\left\| e^{tA}[g(t)] - g^{-\infty}, \mathcal{X}_M(0) \right\| = \left\| \int_{-\infty}^{t} e^{sA}[L_s[g(s)]]\,\mathrm{d}s, \mathcal{X}_M(0) \right\|$$

$$\leq \int_{-\infty}^{t} \left\| e^{sA}[L_s[g(s)]], \mathcal{X}_M(0) \right\| \,\mathrm{d}s = \int_{-\infty}^{t} \| L_s[g(s)], \mathcal{X}_M(s) \| \,\mathrm{d}s$$

$$\leq \| g, \mathcal{Y}_M \| \int_{-\infty}^{t} \| L_s, B(\mathcal{X}_M(s)) \| \,\mathrm{d}s \to 0 \text{ as } t \to -\infty,$$

which concludes the proof of the first part of the theorem.

The function g_- satisfies

$$e^{tA}[g_-(t)] = g^{-\infty} + \int_{-\infty}^{t} e^{sA}[L[g_-(s)]]\,\mathrm{d}s.$$

When we consider the norm in $\mathcal{X}_M(0)$ of both parts of this equality, we get

$$\begin{aligned}\|g_-(t), \mathcal{X}_M(t)\| &\leq \|g^{-\infty}, \mathcal{X}_M(0)\| + \int_{-\infty}^{t} \|L[g_-(s)], \mathcal{X}_M(s)\|\,\mathrm{d}s \\ &\leq \|g^{-\infty}, \mathcal{X}_M(0)\| + \int_{-\infty}^{t} \|L_s, B(\mathcal{X}_M(s))\| \|g_-(s), \mathcal{X}_M(s)\|\,\mathrm{d}s.\end{aligned}$$

By applying lemma 3 we conclude that

$$\begin{aligned}\|g_-(t), \mathcal{X}_M(t)\| &\leq \|g^{-\infty}, \mathcal{X}_M(0)\| \exp\left(\int_{-\infty}^{t} \|L_s, B(\mathcal{X}_M(s))\|\,\mathrm{d}s\right) \\ &\leq \|g^{-\infty}, \mathcal{X}_M(0)\| e^{\mu(M)}.\end{aligned}$$

The proof for the case $t \to +\infty$ is done likewise. $\square$

Large time behaviour

Theorem IV.6. *Assume that the collision kernel has separated form and $\beta \in (1-d, 0]$. Assume that for some global Maxwellian M the function $g = g(t, x, v)$ is the mild solution of the equation* (IV.2.2) *defined a.e. on $(t_0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ (resp. $(-\infty, t_0) \times \mathbb{R}^3 \times \mathbb{R}^3$) for some $t_0 \in \mathbb{R}$. Suppose that*

$$\sup_{t>t_0} \|g, \mathcal{X}_M(t)\| < \infty$$

(resp. $\sup_{t<t_0} \|g, \mathcal{X}_M(t)\| < \infty$), then there exists a unique $g^{+\infty} = g^{+\infty}(x, v)$ (resp. $g^{-\infty} = g^{-\infty}(x, v)$) such that

$$\|g(t) - e^{-tA}[g^{+\infty}], \mathcal{X}_M(t)\| \to 0$$

as $t \to +\infty$ — resp.

$$\|g(t) - e^{-tA}[g^{-\infty}], \mathcal{X}_M(t)\| \to 0$$

as $t \to -\infty$.

Proof. Let us study the case for $t \to +\infty$. By the definition of mild solutions we have

$$e^{(t-t_0)A}[g(t)] = g(t_0) + \int_{t_0}^{t} e^{(s-t_0)A}[L[g(s)]]\,\mathrm{d}s.$$

The last integral converges in $\mathcal{X}_M(t_0)$ as $t \to \infty$, because

$$\left\| \int_{t_0}^{t} e^{(s-t_0)A}[L[g(s)]] \, \mathrm{d}s, \mathcal{X}_M(t_0) \right\| \leq \int_{t_0}^{t} \left\| e^{(s-t_0)A}[L[g(s)]], \mathcal{X}_M(t_0) \right\| \, \mathrm{d}s$$

$$= \int_{t_0}^{t} \|L[g(s)], \mathcal{X}_M(s)\| \, \mathrm{d}s \leq \mu(M) \sup_{s>t_0} \|g(s), \mathcal{X}_M(s)\|.$$

Hence $e^{(t-t_0)A}[g(t)]$ converges in $\mathcal{X}_M(t_0)$ and therefore $e^{(t-t_0)A}[g(t)]$ converges to a limit in $\mathcal{X}_M(t_0)$.

The case $t \to -\infty$ is treated similarly. □

Definition 2. *Let M be a global Maxwellian. Let g^{in} and $g^{+\infty}$ (resp. $g^{-\infty}$) be two elements in $\mathcal{X}_M(0)$. We will say that*

$$g^{+\infty} = \mathcal{T}_+[g^{in}] \text{— } \textit{resp. } g^{-\infty} = \mathcal{T}_+[g^{in}],$$

if there exists a unique mild solution g of the linearised Boltzmann equation (IV.2.2) *on $t \in [0, \infty)$ (resp. on $t \in (-\infty, 0]$) such that*

$$\|g(t) - e^{-tA}[g^{+\infty}], \mathcal{X}_M(t)\| \to 0 \ \textit{as } t \to \infty$$

(resp. $\|g(t) - e^{-tA}[g^{-\infty}], \mathcal{X}_M(t)\| \to 0$ as $t \to -\infty$) and $g|_{t=0} = g^{in}$.

Theorem IV.7. *Assume that the collision kernel b has separated form with $\beta \in (1-d, 0]$. Let M be a global Maxwellian. Let also $g_1(t, x, v)$ defined a.e on $(0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and $g_2(t, x, v)$ defined a.e on $(-\infty, 0) \times \mathbb{R}^d \times \mathbb{R}^d$ be mild solutions of the corresponding linearised Boltzmann equation satisfying*

$$\sup_{t>0} \|g_1(t), \mathcal{X}_M(t)\| < \infty,$$
$$\sup_{t<0} \|g_2(t), \mathcal{X}_M(t)\| < \infty.$$

Let $g^{+\infty}(x, v)$ and $g^{-\infty}(x, v)$ be such that

$$\|g_1(t) - e^{-tA}[g^{+\infty}], \mathcal{X}_M(t)\| \to 0 \quad \textit{as } t \to +\infty,$$
$$\|g_2(t) - e^{-tA}[g^{-\infty}], \mathcal{X}_M(t)\| \to 0 \quad \textit{as } t \to -\infty,$$

then

$$\|g_1(t), \mathcal{X}_M(t)\| \leq \|g^{+\infty}, \mathcal{X}_M(0)\| e^{\mu(M)}$$

a.e. on $(0, \infty)$ and

$$\|g_2(t), \mathcal{X}_M(t)\| \leq \|g^{-\infty}, \mathcal{X}_M(0)\| e^{\mu(M)} \tag{IV.3.3}$$

a.e. on $(-\infty, 0)$.

Proof. By the definition of mild solutions we can write

$$e^{t_1 A}[g_1(t_1)] = e^{tA}[g_1(t)] + \int_t^{t_1} e^{sA}[L[g(s)]]\,\mathrm{d}s$$

for each $t_1 > t > 0$. By letting $t_1 \to +\infty$ we obtain

$$e^{tA}[g_1(t)] = g^{+\infty} - \int_t^{+\infty} e^{sA}[L[g_1(s)]]\,\mathrm{d}s.$$

We take the norm in $\mathcal{X}_M(0)$ of the above expression to obtain the inequality

$$\|g_1, \mathcal{X}_M(t)\| \leq \|g^{+\infty}, \mathcal{X}_M(0)\| + \int_t^{+\infty} \|L_s, B(\mathcal{X}_M(s))\| \|g_1(s), \mathcal{X}_M(s)\|\,\mathrm{d}s.$$

By lemma 3 we deduce that

$$\begin{aligned}\|g_1, \mathcal{X}_M(t)\| &\leq \|g^{+\infty}, \mathcal{X}_M(0)\| \exp\left(\int_t^{+\infty} \|L_s, B(\mathcal{X}_M(s))\|\,\mathrm{d}s\right)\\ &\leq \|g^{+\infty}, \mathcal{X}_M(0)\| e^{\mu(M)}.\end{aligned}$$

The proof of the inequality (IV.3.3) is done likewise. □

In particular, theorem IV.7 implies that the operators $\mathcal{T}_\pm$ are injective.

4 Scattering operator

Definition 3. *Let M be a global Maxwellian; let also $g^{-\infty}$ and $g^{+\infty}$ be two functions in $\mathcal{X}_M(0)$. We say that $g^{+\infty} = \mathcal{S}[g^{-\infty}]$ if there exists a unique mild solution of the linearised Boltzmann equation* (IV.2.2) *in the space $\mathcal{Y}_M$ such that*

$$\|e^{tA}[g(t)] - g^{-\infty}, \mathcal{X}_M(0)\| \to 0 \text{ as } t \to -\infty$$

and

$$\|e^{tA}[g(t)] - g^{+\infty}, \mathcal{X}_M(0)\| \to 0 \text{ as } t \to +\infty.$$

Theorem IV.8. *Assume that collision kernel b has the separated form with $\beta \in (1-d, 0]$ and let M be a global Maxwellian with $\mu(M) < 1$. Then*

1. *for all $g^{-\infty} \in \mathcal{X}_M(0)$ there exists a unique $g^{+\infty} \in \mathcal{X}_M(0)$ such that $\mathcal{S}[g^{-\infty}] = g^{+\infty}$. In particular, if $g^{-\infty} \in \mathcal{V}_0$, then $\mathcal{S}[g^{-\infty}] = g^{-\infty}$.*

2. *for all $g^{+\infty} \in \mathcal{X}_M(0)$ there exists a unique $g^{-\infty} \in \mathcal{X}_M(0)$ such that $\mathcal{S}[g^{-\infty}] = g^{+\infty}$. The function $g^{-\infty}$ so obtained will be noted $\mathcal{S}^{-1}[g^{+\infty}]$.*

3. *the maps $\mathcal{S}$ and $\mathcal{S}^{-1}$ are linear bounded on $\mathcal{X}_M(0)$.*

4. *the operator $\mathcal{S}$ satisfies the global conservation laws in the sense*

$$(r_j(0), \mathcal{S}[g^{-\infty}])_{\mathcal{X}_M(0)} = (r_j(0), g^{-\infty})_{\mathcal{X}_M(0)}$$

where the vectors r_j are defined in lemma 1.

5. *the operator $\mathcal{S}$ decreases the norm of the function in $\mathcal{X}_M(0)$, i.e.*

$$\|\mathcal{S}[g^{-\infty}], \mathcal{X}_M(0)\| \leq \|g^{-\infty}, \mathcal{X}_M(0)\|$$

with equality if and only if $g^{-\infty} \in \mathcal{V}_0$.

Proof. By theorem IV.5 for a given $g^{-\infty}$ the function $g = \mathcal{F}_-[g^{-\infty}]$ is the unique mild solution of the equation (IV.2.2) in the space $\mathcal{Y}_M$ satisfying

$$\|e^{tA}[g(t)] - g^{-\infty}, \mathcal{X}_M(0)\| \to 0 \text{ as } t \to -\infty.$$

Then, by theorem IV.6 there exists a unique limit in $\mathcal{X}_M(0)$ of the function $(t, x, v) \to e^{tA}[g(t, x, v)]$ as $t \to \infty$ which gives us $g^{+\infty}$, which proves the first point of theorem IV.8. The second point of this theorem is treated similarly, which allows us to say that the linear operators $\mathcal{S}$ and $\mathcal{S}^{-1}$ are bounded on $\mathcal{X}_M(0)$.

The conservation properties of the operator $\mathcal{S}$ quickly follow from the conservation properties of the functions $\mathcal{F}_\pm[g^{\pm\infty}]$ shown in theorem IV.2.

Let $g^{+\infty}$ and $g^{-\infty}$ be functions in $\mathcal{X}_M(0)$ such that $\mathcal{S}[g^{-\infty}] = g^{+\infty}$. We consider the corresponding mild solution $g \in \mathcal{Y}_M$ from the definition 3. Given that the function $M(t, x, v)$ satisfies the free transport equation, we can write

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (\partial_t g(t, x, v) + v \cdot \nabla_x g(t, x, v)) M(t, x, v) g(t, x, v) \, \mathrm{d}x \, \mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t} \|g(t), \mathcal{X}_M(t)\|^2$$
$$= (L_t[g(t)], g(t))_{\mathcal{X}_M(t)}.$$

We integrate this equality with respect to time on $[-T, T]$, $T > 0$ to get

$$\|g(T), \mathcal{X}_M(T)\|^2 - \|g(-T), \mathcal{X}_M(-T)\|^2 = \int_{-T}^{T} (L_t[g(t)], g(t))_{\mathcal{X}_M(t)} \, \mathrm{d}t,$$

which gives

$$\|e^{TA}[g(T)], \mathcal{X}_M(0)\|^2 - \|e^{-TA}[g(-T)], \mathcal{X}_M(0)\|^2 = \int_{-T}^{T} (L_t[g(t)], g(t))_{\mathcal{X}_M(t)} \, \mathrm{d}t$$

By the convergence properties

$$\lim_{T \to +\infty} e^{TA}[g(T)] = g^{+\infty}, \quad \lim_{T \to +\infty} e^{-TA}[g(-T)] = g^{-\infty},$$

we can write

$$\|\mathcal{S}[g^{-\infty}], \mathcal{X}_M(0)\|^2 - \|g^{-\infty}, \mathcal{X}_M(0)\|^2 = \int_{\mathbb{R}} (L_t[g(t)], g(t))_{\mathcal{X}_M(t)} \, \mathrm{d}t.$$

Since the operator L_t is non-positive, we immediately conclude that

$$\|\mathcal{S}[g^{-\infty}], \mathcal{X}_M(0)\| \leq \|g^{-\infty}, \mathcal{X}_M(0)\|,$$

and the point (5) of this theorem holds.

The above inequality becomes equality if and only if for a.e. $t \in \mathbb{R}$

$$(L_t[g(t)], g(t))_{\mathcal{X}_M(t)} = 0,$$

which implies that for a.e. t and x the function $v \to g(t,x,v)$ belongs to the null-space of L_t. This result, together with the equation (IV.2.2), implies that g is a conservative solution, i.e. $g \in \mathcal{V}$. This, in its turn, implies that

$$g(t,x,v) = e^{-tA}[g(0,x,v)]$$

and therefore

$$g^{\pm\infty} = \lim_{t \to \pm\infty} e^{tA}[g(t,x,v)] = g(0,x,v).$$

□

The following theorem describes the main result of this chapter:

Theorem IV.9. *Assume that collision kernel b has the separated form with $\beta \in (1-d, 0]$ and let M be a global Maxwellian with $\mu(M) < 1$. Then*

1. *the operator $\mathcal{S}$ satisfies*

$$\|\mathcal{S}[g^{-\infty}], \mathcal{X}_M(0)\| \geq \|P_0[g^{-\infty}], \mathcal{X}_M(0)\|$$

where P_0 is the orthogonal projector on $\mathcal{V}_0$ in $\mathcal{X}_M(0)$. The above expression is an equality if and only if $g^{-\infty} = \mathcal{S}[g^{-\infty}] \in \mathcal{V}_0$.

2. *if, in addition, the collision kernel satisfies the inequality*

$$\inf_{\omega_1, \omega_2 \in \mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \min(\hat{b}(\omega_1 \cdot \omega_3), \hat{b}(\omega_2 \cdot \omega_3)) \, \mathrm{d}\omega_3 > 0,$$

then there exists a constant $C > 0$ such that for each $g^{-\infty} \in \mathcal{X}_M(0)$ orthogonal to $\mathcal{V}_0$ we have

$$\begin{aligned} &\|g^{-\infty}, \mathcal{X}_M(0)\|^2 - \|\mathcal{S}[g^{-\infty}], \mathcal{X}_M(0)\|^2 \\ &\geq C \int_{\mathbb{R}} \left\| g, L^2 \left(M(t,x,v) \rho^{1-\frac{\beta}{2d}}(t,x) \theta^{\frac{d}{2}+\frac{\beta}{4}}(t) (1+|v|^2)^{\beta/2} \, \mathrm{d}x \, \mathrm{d}v \right) \right\|^2 \mathrm{d}t \end{aligned} \tag{IV.4.1}$$

with $\rho(t,x)$, $u(t,x)$ and $\theta(t)$ defined in (IV.2.1).

Proof. The conservation properties of the operator $\mathcal{S}$ imply that

$$(r_j(0), g^{-\infty})_{\mathcal{X}_M(0)} = (r_j(0), \mathcal{S}[g^{-\infty}])_{\mathcal{X}_M(0)},$$

therefore

$$P_0[g^{-\infty}] = P_0[\mathcal{S}[g^{-\infty}]]$$

and

$$\begin{aligned}\|\mathcal{S}[g^{-\infty}], \mathcal{X}_M(0)\|^2 &= \|P_0[\mathcal{S}[g^{-\infty}]], \mathcal{X}_M(0)\|^2 + \|(I-P_0)[\mathcal{S}[g^{-\infty}]], \mathcal{X}_M(0)\|^2 \\ &\geq \|P_0[g^{-\infty}], \mathcal{X}_M(0)\|^2,\end{aligned}$$

so

$$\|g^{-\infty}, \mathcal{X}_M(0)\|^2 - \|\mathcal{S}[g^{-\infty}], \mathcal{X}_M(0)\|^2 \leq \|g^{-\infty}, \mathcal{X}_M(0)\|^2 - \|P_0[g^{-\infty}], \mathcal{X}_M(0)\|^2.$$

On the other hand, we want to obtain a below estimation for the quantity $\|g^{-\infty}, \mathcal{X}_M(0)\|^2 - \|\mathcal{S}[g^{-\infty}], \mathcal{X}_M(0)\|^2$, in other words, on the integral

$$-\int_{\mathbb{R}} (L_t[g(t)], g(t))_{\mathcal{X}_M(t)}\, \mathrm{d}t.$$

We will base our reasoning on the result established in [34]:

Lemma 4. *Suppose that $\bar{M}$ is a local Maxwellian of the form $M[1,0,1](v) = \exp(-|v|^2/2)$. Suppose also that the collision kernel b has separated form*

$$b(|v-v_*|, \omega) = |v-v_*|^\beta \hat{b}\left(\frac{v-v_*}{|v-v_*|} \cdot \omega\right)$$

satisfying $\beta \in (-d, 1]$ and

$$\inf_{\omega_1, \omega_2 \in \mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \min(\hat{b}(\omega_1 \cdot \omega_3), \hat{b}(\omega_2 \cdot \omega_3))\, \mathrm{d}\omega_3 > 0.$$

Then there exists a constant $C > 0$ such that for each function $h \in L^2(\bar{M}(v)\, \mathrm{d}v)$ satisfying

$$\int h(v)\bar{M}(v)\, \mathrm{d}v = \int |v|^2 h(v)\bar{M}(v)\, \mathrm{d}v = 0,$$
$$\forall i = 1, \dots, d \int v_i h(v)\bar{M}(v)\, \mathrm{d}v = 0,$$

one has the estimate

$$\begin{aligned}-\int b(|v-v_*|, \omega)\bar{M}(v)\bar{M}(v_*)\{h\}h(v)\, \mathrm{d}\omega\, \mathrm{d}v\, \mathrm{d}v_* \\ \geq C \int \bar{M}(v)|h(v)|^2(1+|v|^2)^{\beta/2}\, \mathrm{d}v.\end{aligned}$$

In particular, for a local Maxwellian $M[\rho,u,\theta](v)$ this would imply that for the same constant C we can write

$$\begin{aligned} -\int b(|v-v_*|,\omega)M[\rho,u,\theta](v)M[\rho,u,\theta](v_*)\{h\}h(v)\,\mathrm{d}\omega\,\mathrm{d}v\,\mathrm{d}v_* \\ \geq C\rho\theta^{\frac{\beta+d}{2}}\int M[\rho,u,\theta](v)|h(v)|^2\left(1+\frac{|v-u|^2}{2\theta}\right)^{\beta/2}\mathrm{d}v \end{aligned} \tag{IV.4.2}$$

for functions $h\in L^2(M[\rho,u,\theta]\,\mathrm{d}v)$ orthogonal to 1, v_i, $|v|^2$.

Now let us suppose that M is a global Maxwellian, $g\in\mathcal{Y}_M$, such that for a.e. $t\in\mathbb{R}$

$$\int_{\mathbb{R}^d} g(t,x,v)M(t,x,v)\,\mathrm{d}v=\int_{\mathbb{R}^d}|v|^2 g(t,x,v)M(t,x,v)\,\mathrm{d}v=0,$$

$$\forall i=1,\dots,d\int_{\mathbb{R}^d} v_i g(t,x,v)M(t,x,v)\,\mathrm{d}v=0.$$

Since we can represent the global Maxwellian M as a local Maxwellian given by $M[\rho,u,\theta](v)$ with $\rho(t,x)$, $u(t,x)$, $\theta(t)$ defined in (IV.2.1), the inequality (IV.4.2) applies.

Upon integrating with respect to $\mathrm{d}x$ we obtain

$$\begin{aligned} &-(L_t[g(t)],g(t))_{\mathcal{X}_{M(t)}} \\ &\geq C\int_{\mathbb{R}^d\times\mathbb{R}^d}\rho(t,x)\theta^{\frac{\beta+d}{2}}(t)M(t)|g(t,x,v)|^2\left(1+\frac{|v-u|^2}{2\theta(t)}\right)^{\beta/2}\mathrm{d}v\,\mathrm{d}x, \end{aligned}$$

therefore

$$\begin{aligned} &-\int_{\mathbb{R}}(L_t[g(t)],g(t))_{\mathcal{X}_{M(t)}}\,\mathrm{d}t \\ &\geq C\int_{\mathbb{R}}\left\|g,L^2\left(M(t,x,v)\rho(t,x)\theta^{\frac{\beta+d}{2}}(t)\left(1+\frac{|v-u|^2}{2\theta(t)}\right)^{\beta/2}\mathrm{d}x\,\mathrm{d}v\right)\right\|^2\mathrm{d}t. \end{aligned} \tag{IV.4.3}$$

If $\beta=0$, then the last inequality immediately gives (IV.4.1); if $\beta<0$, an extra step is required:

Proposition 6. *If $\beta<0$, then there exists an explicit constant $c>0$ such that $\forall x\in\mathbb{R}^3$ and $\forall t\in\mathbb{R}$*

$$\begin{aligned} \theta^{\beta/2}(t)\exp\left(\frac{-\beta}{2d}\theta(t)(x-x_0-tv_0)^TQ(x-x_0-tv_0)\right)\left(1+\frac{|v-u|^2}{2\theta(t)}\right)^{\beta/2} \\ \geq c\left(1+|v|^2\right)^{\beta/2}. \end{aligned}$$

Proof. Since $\beta < 0$, it is sufficient to prove that

$$\sup_{x\in\mathbb{R}^3,\, t\in\mathbb{R}} \theta(t) \exp\left(-\frac{1}{d}\theta(t)(x-x_0-tv_0)^T Q(x-x_0-tv_0)\right)\left(1+\frac{|v-u|^2}{2\theta(t)}\right) \leq c(1+|v|^2).$$

Recall the expression (IV.2.1) of u:

$$u(t,x) = \theta(t)\left((at-b)I - B\right)(x-x_0-tv_0) + v_0.$$

By making a change of variables $y = \sqrt{\theta(t)}(x-x_0-tv_0)$ we can reduce our problem to proving that

$$\sup_{y\in\mathbb{R}^3,\, t\in\mathbb{R}} \exp\left(-\frac{1}{d}y^T Q y\right)\left(\theta(t)+\frac{|v-u_1|^2}{2}\right) \leq c(1+|v|^2)$$

with $u_1 = \sqrt{\theta(t)}\left((at-b)I - B\right) y + v_0$. First, we can say that

$$\sup_{y\in\mathbb{R}^3,\, t\in\mathbb{R}} \theta(t) = \sup_{t\in\mathbb{R}} \theta(t) = \sup_{t\in\mathbb{R}} \frac{1}{at^2-2bt+c} = \frac{a}{ac-b^2} = l_1 > 0$$

and that

$$\sup_{y\in\mathbb{R}^3,\, t\in\mathbb{R}} \exp\left(-\frac{1}{d}y^T Q y\right) = 1.$$

Second, we can write an estimate

$$\theta(t) + \frac{|v-u_1|^2}{2} \leq \theta(t) + |v|^2 + |u_1|^2,$$

which allows us to say that

$$\sup_{y\in\mathbb{R}^3,\, t\in\mathbb{R}} \exp\left(-\frac{1}{d}y^T Q y\right)(\theta(t)+|v|^2) \leq l_1 + |v|^2. \tag{IV.4.4}$$

On the other hand,

$$|u_1|^2 = \left|\sqrt{\theta(t)}((at-b)I-B)y + v_0\right|^2 \leq 2\theta(t)\|(at-b)I-B\|^2|y|^2 + 2|v_0|^2$$
$$= 2\left\|\frac{(at-b)I-B}{\sqrt{at^2-2bt+c}}\right\|^2 |y|^2 + 2|v_0|^2.$$

Clearly, $0 < \sup_{t\in\mathbb{R}}\left\|\frac{(at-b)I-B}{\sqrt{at^2-2bt+c}}\right\| = l_2 < +\infty$, hence

$$|u_1|^2 \leq 2l_2^2|y|^2 + 2|v_0|^2.$$

Since Q is a positive definite matrix, we can, by denoting $\lambda_{min}(Q) > 0$ the smallest eigenvalue of Q, write $x^T Qx \geq \lambda_{min}(Q)|x|^2$. Together with the inequality

$$\sup_{r\geq 0}\Big(r\exp(-r)\Big) = \frac{1}{e}$$

this allows us to estimate

$$\exp\left(-\frac{1}{d}\theta(t)y^T Qy\right)|u_1|^2 \leq 2(l_2^2|y|^2 + |v_0|^2)\exp\left(-\frac{1}{d}y^T Qy\right)$$

$$\leq 2|v_0|^2 + 2l_2^2|y|^2\exp\left(-\frac{\lambda_{min}(Q)}{d}|y|^2\right) \leq 2|v_0|^2 + 2l_2^2\frac{d}{e\lambda_{min}(Q)}.$$

The last inequality together with (IV.4.4) and the relation for any constants $c_i > 0$

$$c_1 + c_2|v|^2 \leq \max(c_1, c_2)(1 + |v|^2)$$

are sufficient conclude the proof. $\square$

Now we can rewrite the inequality (IV.4.3) with the help of definition of $\rho(t,x)$ and Proposition 6 (we put all multiplicative constants in C) :

$$CM\rho\theta^{\frac{\beta+d}{2}}\left(1 + \frac{|v-u|^2}{2\theta}\right)^{\beta/2}$$

$$= CM\theta^{d/2}\theta^{\frac{d}{2}\left(1-\frac{\beta}{2d}\right)}\theta^{\beta/4}\exp\left(-\left(1-\frac{\beta}{2d}\right)\theta x^T Qx\right)$$

$$\times\theta^{\beta/2}(t)\exp\left(-\frac{\beta}{2d}\theta x^T Qx\right)\left(1 + \frac{|v-u|^2}{2\theta}\right)^{\beta/2}$$

$$\geq CM\theta^{\frac{2d+\beta}{4}}\rho^{1-\frac{\beta}{2d}}(1 + |v|^2)^{\beta/2}.$$

The above inequality allows us to obtain the relation (IV.4.1) and the theorem is proven. $\square$

A Appendices

Proof of lemma 1

First, we remind two easy results:

Lemma 5. *A matrix $H \in M_d(\mathbb{R})$ is antisymmetric $H = -H^T$ if and only if*

$$\forall v \in \mathbb{R}^d \quad (Hv, v) = 0.$$

Lemma 6. *Let $H \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ such that $\nabla H = -(\nabla H)^T$, then there exists a constant antisymmetric matrix $\mathcal{H} \in M_d(\mathbb{R})$ and a constant vector $h \in \mathbb{R}^d$ such that $H(x) = \mathcal{H}x + h$.*

Proof of lemma 1. Since conservative solutions are in the null-space of L, they have the form

$$A(t,x) - B(t,x) \cdot v + C(t,x)|v|^2.$$

We will find the form of functions A, B, and C by substituting them into the transport equation. We will obtain a polynomial with respect to v, which has to be identically zero, which implies that all its coefficients are zero, therefore we can write the following system:

$$\partial_t A(t,x) = 0 \tag{IV.A.1}$$

$$\nabla_x A(t,x) = \partial_t B(t,x) \tag{IV.A.2}$$

$$\forall v \in \mathbb{R}^3 \; ((\partial_t C(t,x) I - \nabla_x B)\, v, v) = 0 \tag{IV.A.3}$$

$$\nabla_x C(t,x) = 0 \tag{IV.A.4}$$

The equation (IV.A.1) implies that

$$A = A(x), \tag{IV.A.5}$$

the equation (IV.A.4) implies that

$$C = C(t).$$

The equation (IV.A.2) together with (IV.A.5) implies that

$$\nabla_x A(x) = \partial_t B(t,x),$$

therefore

$$B(t,x) = B_0(x) + t\nabla_x A(x). \tag{IV.A.6}$$

The equation (IV.A.3) implies that the the matrix $\partial_t C(t) - \nabla_x B(t,x)$ is antisymmetric, therefore we can say that

$$\forall i \quad (\nabla_x B)_{i,i} = \partial_t C(t) \tag{IV.A.7}$$

and, by taking into account (IV.A.6), that

$$C(t) = c_0 + c_1 t + c_2 \frac{t^2}{2}. \tag{IV.A.8}$$

Moreover, the form (IV.A.8) together with (IV.A.7) and (IV.A.6) give us

$$c_1 I - \nabla_x B_0(x) \text{ antisymmetric,} \tag{IV.A.9}$$

$$c_2 I - \nabla_x \nabla_x A(x) \text{ antisymmetric.} \tag{IV.A.10}$$

The equation (IV.A.10) immediately results in $A_{,ij} = -A_{,ji}$ for $i \neq j$, which implies $A_{,ij} = 0$ for that case, hence $\forall i A_{,ii} = c_2$, thus

$$A(x) = \frac{1}{2} c_2 |x|^2 + a_1 \cdot x + a_0$$

for some constant $a_0 \in \mathbb{R}$ and $a_1 \in \mathbb{R}^d$. This also simplifies B:

$$B(t, x) = B_0(x) + t(c_2 x + a_1).$$

The lemma 6 applied to the equation (IV.A.9) gives us a constant antisymmetric matrix $\mathcal{H}$ such that

$$c_1 I - \nabla_x B_0(x) = \mathcal{H},$$

which results in

$$B_0(x) = c_1 x - \mathcal{H} x + b_0$$

for some constant vector $b_0 \in \mathbb{R}^d$.

Therefore, a conservative solution writes

$$\frac{1}{2} c_2 |x|^2 + a_1 \cdot x + a_0 - (c_1 x - \mathcal{H} x + b_0 + t(c_2 x + a_1)) \cdot v + \left(c_0 + c_1 t + c_2 \frac{t^2}{2} \right) |v|^2$$

or

$$\frac{1}{2} c_2 |x - tv|^2 + (a_1 - c_1 v) \cdot (x - tv) + c_0 |v|^2 - b_0 \cdot v + a_0 + \mathcal{H} x \cdot v.$$

for some constant a_0, c_0, c_1, $c_2 \in \mathbb{R}$, b_0, $a_1 \in \mathbb{R}^d$, and $\mathcal{H} = -\mathcal{H}^T \in M_d(\mathbb{R})$.

If the matrices $\mathcal{H}_j$ for a basis of antisymmetric matrices in $M_d(\mathbb{R})$, then the functions

$$1, \quad (x - tv)_i, \quad |x - tv|^2, \quad v_i, \quad |v|^2$$
$$v \cdot (x - tv), \quad \mathcal{H}_j (x - tv) \cdot v$$

are linearly independent. Indeed, suppose that a certain linear combination

$$\frac{1}{2} c_2 |x|^2 + a_1 \cdot x + a_0 - (c_1 x - \mathcal{H} x + b_0 + t(c_2 x + a_1)) \cdot v + \left(c_0 + c_1 t + c_2 \frac{t^2}{2} \right) |v|^2$$

is identically zero for all x and v. By taking $x = v = 0$ we immediately conclude that $a_0 = 0$. Then, by choosing $x = tv$ we obtain an expression

$$\forall v \in \mathbb{R}^d \quad c_0 |v|^2 - b_0 \cdot v,$$

which leads to $c_0 = 0$ and $b_0 = 0$. Then again, after fixing $v = 0$ we get

$$\forall x \in \mathbb{R}^d \quad \frac{1}{2}c_2|x|^2 + a_1 \cdot x = 0,$$

therefore $a_1 = 0$ and $c_2 = 0$. Now we can chose $x = (t+1)v$ to obtain

$$\forall v \in \mathbb{R}^d \quad -c_1 v \cdot v = 0,$$

hence $c_1 = 0$, too.

Finally, we are left with

$$\forall x,\, v \in \mathbb{R}^d \quad \mathcal{H}x \cdot v = 0,$$

which immediately results in $\mathcal{H} = 0$. We conclude that all coefficients in the linear combination are zero, therefore, the functions $r_j(t)$ are indeed linearly independent. $\square$

Proof of theorem IV.1

Proof of theorem IV.1. We will use the estimations obtained in [1]:

Lemma 7. *Let $\beta \in (-d, 0]$ and*

$$C(t,x,v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} M_* \hat{b}\left(\omega \cdot \frac{v - v_*}{|v - v_*|}\right) |v - v_*|^\beta \,\mathrm{d}\omega \,\mathrm{d}v_*,$$

then

$$\sup_{x,v} C(t,x,v) \leq m b_1 \sqrt{\det\left(\frac{Q}{2\pi}\right)} \frac{2^{\beta/2}\Gamma\left(\frac{d+\beta}{2}\right)}{\Gamma(d)} \theta^{\frac{d+\beta}{2}}(t) > 0,$$

where $\theta(t)$ is given by (IV.2.1)*.*

First, we prove that the operator L_t is bounded on the space $L^1(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v)$. Indeed, we write the estimation for any function $g \in L^1(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v)$:

$$\begin{aligned}
&\|L_t[g], L^1(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v)\| \\
&\leq \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} M M_* b(\omega, v - v_*)(|g'| + |g'_*| + |g| + |g_*|)\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}v_* \\
&\leq 4 \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} M M_* |v - v_*|^\beta |g| \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}v_* \\
&\leq 4 \sup_{x,v}\left(\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} b(\omega, v - v_*) M(t,x,v_*)\,\mathrm{d}\omega\,\mathrm{d}v_*\right) \|g, L^1(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v)\| \\
&\leq 4 \sup_{x,v} C(t,x,v) \|g, L^1(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v)\|.
\end{aligned}$$

By lemma 7 the factor $\sup_{x,v} C(t,x,v)$ is finite for $\beta > -d$, therefore the operator L_t is continuous in $L^1(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v)$ and its norm satisfies

$$\|L_t, B(L^1(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v))\| \leq 4 \sup_{x,v} C(t,x,v).$$

Second, we prove that L_t is bounded on the space

$$L^\infty(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v) = L^\infty(\,\mathrm{d}x\,\mathrm{d}v).$$

We can write an estimation for any function $g \in L^\infty(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v)$:

$$\|L_t[g], L^\infty(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v)\| = \|L_t[g], L^\infty(\,\mathrm{d}x\,\mathrm{d}v)\|$$

$$\leq \sup_{x,v} \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} M_* b(\omega, v-v_*)(|g'| + |g'_*| + |g| + |g_*|)\,\mathrm{d}\omega\,\mathrm{d}v_*$$

$$\leq 4 \sup_{x,v} |g(t,x,v)| \sup_{x,v} \left(\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} b(\omega, v-v_*) M(t,x,v_*)\,\mathrm{d}\omega\,\mathrm{d}v_* \right).$$

$$\leq 4 \sup_{x,v} C(t,x,v) \|g, L^\infty(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v)\|.$$

Again, by the same proposition 7 the above inequality shows that the operator L_t is bounded in $L^\infty(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v)$ and its norm satisfies

$$\|L_t, B(L^\infty(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v))\| \leq 4 \sup_{x,v} C(t,x,v).$$

By the Riesz-Thorin's theorem the operator is bounded in the space $L^p(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v)$ for any $p \in [1,\infty]$; we are interested in the particular case $p = 2$. By the same theorem we can obtain an estimation on the norm of L_t in $B(\mathcal{X}_M(t))$, namely:

$$\begin{aligned}
&\|L_t, B(\mathcal{X}_M(t))\| \\
&\quad \leq \|L_t, B(L^1(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v))\|^{1/2} \|L_t, B(L^\infty(M(t,x,v)\,\mathrm{d}x\,\mathrm{d}v))\|^{1/2} \\
&\qquad\qquad \leq 4 \sup_{x,v} C(t,x,v) \leq 4 b_1 m \sqrt{\det\left(\frac{Q}{2\pi}\right)} \frac{2^{\beta/2}\Gamma\left(\frac{d+\beta}{2}\right)}{\Gamma(d)} \theta^{\frac{d+\beta}{2}}(t).
\end{aligned}$$

Moreover, if in addition $\beta \in (1-d, 0]$, then the function $t \to \|L_t, B(\mathcal{X}_M(t))\|$ belongs to $L^1(\mathbb{R})$. This result quickly follows from the asymptotics $\theta(t) = O(|t|^{-2})$ as $|t| \to \infty$ and the above estimations. $\square$

Proof of theorem IV.2

Proof of theorem IV.2. The first part of this theorem is a direct consequence of the continuity and self-adjointness of the operator L_t on the space $\mathcal{X}_M(t)$, as well as the form of the null-space of this opeator.

For the second part we take $p(t,x,v) \in \mathcal{V}$, therefore

$$\forall t \in \mathbb{R} \quad e^{tA}[p(t,x,v)] = p(0,x,v)$$

and hence

$$e^{t_2 A}[pg](t_2,x,v) - e^{t_1 A}[pg](t_1,x,v) = \int_{t_1}^{t_2} e^{sA}\left[pL[g]\right](s,x,v)\,\mathrm{d}s$$

for a.e. $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$ and $t_1,\ t_2 \in \mathbb{R}$. Since $\sup_{t\in I} \|g(t), \mathcal{X}_M(t)\| < \infty$ and $p \in \mathcal{Y}_M$, we obtain that $pg \in L^1(M(t,x,v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v)$ on $[t_1,t_2] \times \mathbb{R}^d \times \mathbb{R}^d$. By lemma IV.1 the operator L_t is linear bounded on $\mathcal{X}_M(t)$ for all time t. We conclude that $p(t,x,v)L_t[g](t,x,v) \in L^1(M(t,x,v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v)$ on $[t_1,t_2] \times \mathbb{R}^d \times \mathbb{R}^d$. Therefore

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{t_2 A}[pg](t_2)M(0)\,\mathrm{d}x\,\mathrm{d}v - \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{t_1 A}[pg](t_1)M(0)\,\mathrm{d}x\,\mathrm{d}v$$

$$= \int_{t_1}^{t_2}\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{sA}\left[pL_t[g]\right](s)M(0)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v,$$

or

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} p(t_2)g(t_2)M(t_2)\,\mathrm{d}x\,\mathrm{d}v - \int_{\mathbb{R}^d\times\mathbb{R}^d} p(t_1)g(t_1)M(t_1)\,\mathrm{d}x\,\mathrm{d}v$$

$$= \int_{t_1}^{t_2}\int_{\mathbb{R}^d\times\mathbb{R}^d} p(s)L_t[g](s)M(s)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v.$$

Finally, for a.e. $(t,x) \in I \times \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} p(s)L_t[g](s)M(s)\,\mathrm{d}v = 0$$

by the first statement in theorem IV.2 because the function $v \to p(t,x,v)$ is a linear combination of functions $1, v_1, \ldots, v_d, |v|^2$. Hence, for all $t_1,\ t_2$

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} p(t_2)g(t_2)M(t_2)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^d\times\mathbb{R}^d} p(t_1)g(t_1)M(t_1)\,\mathrm{d}x\,\mathrm{d}v.$$

□

Proof of theorem IV.3. The first two points of this theorem are direct consequences of the well-known properties of the operator L_t seen as the operator on $L^2(M\,\mathrm{d}v)$.

In order to prove the last point, we take a mild solution of the Boltzmann equation g defined on $I \times \mathbb{R}^d \times \mathbb{R}^d$; by definition for a.e. $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ the function $t \mapsto g(t, x, v)$ is absolutely continuous in t. By the chain rule we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} g^2(t, x+tv, v) = g(t, x+tv, v) L_t[g](t, x+tv, v) \quad \text{for a.e. } (t, x, v) \in I \times \mathbb{R}^d \times \mathbb{R}^d.$$

Since $g(t) \in \mathcal{X}_M(t)$ by the conditions of the theorem and L_t is a bounded operator on this space with an explicit bound on its norm (see estimates in the proof of theorem IV.1), we conclude that $gL_t[g] \in L^1(M \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t, I \times \mathbb{R}^d \times \mathbb{R}^d)$, so, after multiplying by $M(0, x, v)$ and integrating with respect to x and v we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H[g](t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d \times \mathbb{R}^d} M(t, x, v) g(t, x, v) L_t[g](t, x, v) \,\mathrm{d}x \,\mathrm{d}v \leq 0$$

by the non-positivity of the operator L_t on the space $\mathcal{X}_M(t)$. □

Proof of lemma 3

Proof of lemma 3. We can rewrite the inequality

$$0 \leq \phi_+(t) \leq \Delta + \int_t^{+\infty} \phi_+(s) m(s) \,\mathrm{d}s$$

as

$$\frac{m(t)\phi_+(t)}{\Delta + \int_t^{+\infty} \phi_+(s) m(s) \,\mathrm{d}s} \leq m(t),$$

which upon integration with respect to t leads to

$$-\ln\left(\Delta + \int_t^{+\infty} \phi_+(s) m(s) \,\mathrm{d}s\right)\Bigg|_{t=\tau}^{t=+\infty} = \int_\tau^{+\infty} m(t) \,\mathrm{d}t,$$

$$\ln\left(\Delta + \int_\tau^{+\infty} \phi_+(s) m(s) \,\mathrm{d}s\right) = \ln \Delta + \int_\tau^{+\infty} m(t) \,\mathrm{d}t,$$

and

$$\Delta + \int_\tau^{+\infty} \phi_+(s) m(s) \,\mathrm{d}s = \Delta \exp\left(\int_\tau^{+\infty} m(t) \,\mathrm{d}t\right).$$

When we put this inequality into the initial inequality, we obtain

$$0 \leq \phi_+(t) \leq \Delta \exp\left(\int_t^{+\infty} m(s) \,\mathrm{d}s\right).$$

The proof for ϕ_- and $\psi_\pm$ is done likewise. □

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