Summary of Attitude Transformation

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I. Rodrigues' Rotation Formula

For $\mathbf{v}_{rot} \in \mathbb{R}^3$, the rotation axis $\mathbf{k}$ is a unit vector with a rotation angle of θ. Thus, the rotated vector can be obtained through the Rodriguez rotation formula:

$$\mathbf{v}_{rot} = \cos\theta\mathbf{v} + (1 - \cos\theta)(\mathbf{k} \cdot \mathbf{v})\mathbf{k} + \sin\theta\mathbf{k} \times \mathbf{v} \tag{1}$$

It can also be expressed in the Matrix Formula, let $\mathbf{v}_{rot} = \mathbf{R}\mathbf{v}$, in which the $\mathbf{R}$ is the rotation matrix.

$$\mathbf{R} = \cos\theta\mathbf{I} + (1 - \cos\theta)\mathbf{k}\mathbf{k}^T + \sin\theta\mathbf{k}^\wedge \tag{2}$$

A. Geometry

Decompose vector $\mathbf{v}$ in to $\mathbf{v}_{\|}$, which is parallel to $\mathbf{k}$ and $\mathbf{v}_{\perp}$, which is orthogonal to $\mathbf{k}$.

$$\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp} \tag{3}$$

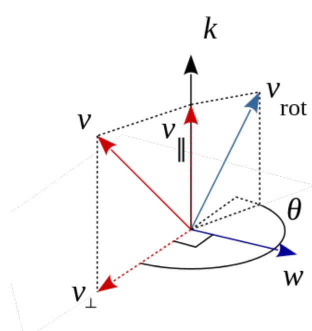

Fig. 1. Vector Decomposition. [1]

$\mathbf{k}$ is the unit vector, thus, the vector $\mathbf{v}_{\|}$ and $\mathbf{v}_{\perp}$ can be expressed as:

$$\mathbf{v}_{\|} = (\mathbf{v} \cdot \mathbf{k})\mathbf{k} \tag{4}$$

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\|} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{k})\mathbf{k} = -\mathbf{k} \times (\mathbf{k} \times \mathbf{v}) \tag{5}$$

The explanation for equation (5) is in the appendix.

After rotation, we know that

$$\mathbf{v}_{\parallel rot} = \mathbf{v}_{\parallel} \tag{6}$$

$$|\mathbf{v}_{\perp rot}| = |\mathbf{v}_{\perp}|$$

$$\mathbf{v}_{\perp rot} = \cos\theta \mathbf{v}_{\perp} + \sin\theta \mathbf{k} \times \mathbf{v}_{\perp} = \cos\theta \mathbf{v}_{\perp} + \sin\theta \mathbf{k} \times \mathbf{v} \tag{7}$$

With the equation (3), (4), (7)

$$\begin{aligned}
\mathbf{v}_{rot} &= \mathbf{v}_{\parallel rot} + \mathbf{v}_{\perp rot} \\
&= \mathbf{v}_{\parallel} + \cos\theta \mathbf{v}_{\perp} + \sin\theta \mathbf{k} \times \mathbf{v} \\
&= \mathbf{v}_{\parallel} + \cos\theta(\mathbf{v} - \mathbf{v}_{\parallel}) + \sin\theta \mathbf{k} \times \mathbf{v} \\
&= \cos\theta \mathbf{v} + (1 - \cos\theta)\mathbf{v}_{\parallel} + \sin\theta \mathbf{k} \times \mathbf{v} \\
&= \cos\theta \mathbf{v} + (1 - \cos\theta)(\mathbf{k} \cdot \mathbf{v})\mathbf{k} + \sin\theta \mathbf{k} \times \mathbf{v}
\end{aligned} \tag{8}$$

With the same method

$$\begin{aligned}
\mathbf{v}_{rot} &= \mathbf{v}_{\parallel rot} + \mathbf{v}_{\perp rot} \\
&= \mathbf{v}_{\parallel} + \cos\theta \mathbf{v}_{\perp} + \sin\theta \mathbf{k} \times \mathbf{v} \\
&= \mathbf{v} - \mathbf{v}_{\perp} + \cos\theta \mathbf{v}_{\perp} + \sin\theta \mathbf{k} \times \mathbf{v} \\
&= \mathbf{v} + (\sin\theta)\mathbf{k} \times \mathbf{v} + (\cos\theta - 1) \times \mathbf{v}_{\perp} \\
&= \mathbf{v} + (\sin\theta)\mathbf{k} \times \mathbf{v} + (1 - \cos\theta)\mathbf{k} \times (\mathbf{k} \times \mathbf{v})
\end{aligned} \tag{9}$$

The Rodrigues'Rotation Formula is

$$\mathbf{v}_{rot} = \cos\theta \mathbf{v} + (1 - \cos\theta)(\mathbf{v} \cdot \mathbf{k})\mathbf{k} + \sin\theta \mathbf{k} \times \mathbf{v} \tag{10}$$

$$\mathbf{v}_{rot} = \mathbf{v} + (\sin\theta)\mathbf{k} \times \mathbf{v} + (1 - \cos\theta)\mathbf{k} \times (\mathbf{k} \times \mathbf{v}) \tag{11}$$

B. Algebraic

In algebraic methods, to avoid confusion with the summation index k, the rotation axis is represented by ω **Lie Group**: $SO(3)$ is the group of all 3x3 orthogonal matrices with determinant 1, which represents the set of all possible rotations in 3D space.

$$SO(3) = \{R \in \mathbb{R}^{3\times3} | R^T R = I, \det(R) = 1\} \tag{12}$$

in which, the limit of $R^T R = I$ ensures that the rotation does not change the length and $\det(R) = 1$ make sure it is a rotation matrix and not a reflection matrix. Since reflection matrices reverse volume, for a reflection matrix $\det(R) = -1$.

Lie Algebra: $\mathfrak{so}(3)$ is the set of all skew-symmetric 3×3 matrices, which corresponds to the infinitesimal generators of the rotations in 3D space.

$$\mathfrak{so}(3) = \{A \in \mathbb{R}^{3\times 3} | A^T = -A\} \tag{13}$$

in which any A can be represented in an axis-angle form

$$A = \omega^\wedge = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

The exponential map of the Lie algebra $\mathfrak{so}(3)$ on the Lie Group $SO(3)$:

$$\mathbf{R}_\omega(\theta) = e^{\theta\omega^\wedge} \tag{14}$$

By using the Maclaurin Series:

$$\mathbf{R}_\omega(\theta) = e^{\theta\omega^\wedge} = \sum_{k=0}^{\infty} \frac{(\theta\omega^\wedge)^k}{k!} = \mathbf{I} + \theta\omega^\wedge + \frac{1}{2}(\theta\omega^\wedge)^2 + \frac{1}{6}(\theta\omega^\wedge)^3 + ... \tag{15}$$

$$\omega^{\wedge 2} = -\mathbf{I} + \omega^\wedge \omega^{\wedge T} \tag{16}$$

$$\omega^{\wedge 4} = \omega^{\wedge 2}\omega^{\wedge 2} = (-\mathbf{I} + \omega^\wedge \omega^{\wedge T})(-\mathbf{I} + \omega^\wedge \omega^{\wedge T}) = -\mathbf{I} + \omega^\wedge \omega^{\wedge T} \tag{17}$$

With the equation (16) and (17)

$$\omega^{\wedge 2k} = \omega^{\wedge 2} = -\mathbf{I} + \omega^\wedge \omega^{\wedge T} \tag{18}$$

$$\omega^{\wedge 3} = (-\mathbf{I} + \omega^\wedge \omega^{\wedge T})\omega^\wedge = -\omega^\wedge \tag{19}$$

With the equation (19)

$$\omega^{\wedge 2k+1} = -\omega^\wedge \tag{20}$$

With the equation (18) and (20), the equation (15) is equal to

$$\mathbf{R}_\omega(\theta) = \mathbf{I} + \omega^\wedge(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...) + \omega^{\wedge 2}(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - ...) = \mathbf{I} + \sin\theta\omega^\wedge + (1 - \cos\theta)\omega^{\wedge 2} \tag{21}$$

Thus, we have the Rodrigues formula, with $\omega^\wedge = \frac{[\phi]_\times}{\theta}$ the rotation matrix is

$$\mathbf{R} = \mathbf{I} + \frac{\sin\theta}{\theta}[\phi]_\times + \frac{1 - \cos\theta}{\theta^2}[\phi]_\times^2 \tag{22}$$

II. Equivalent Rotation Vector Method

ERV describes the rotation with a rotation axis and angle.

$$\phi = \theta\omega^\wedge \tag{23}$$

in which, ϕ is the equivalent rotation vector, $\phi = [\phi_x, \phi_y, \phi_z]^T$. $\omega^\wedge$ is unit rotation axis and θ is the angle of rotation around this axis.

To derive the rotation matrix $\mathbf{R}$ based on **Rodrigues' rotation formula**.

$$\mathbf{R} = \mathbf{I} + \frac{\sin\theta}{\theta}[\phi]_\times + \frac{1-\cos\theta}{\theta^2}[\phi]_\times^2 \tag{24}$$

in which, $[\phi]_\times$ is the skew-symmetric matrix of the rotation vector ϕ.

$$[\phi]_\times = \begin{bmatrix} 0 & -\phi_z & \phi_y \\ \phi_z & 0 & -\phi_x \\ -\phi_y & \phi_x & 0 \end{bmatrix}$$

While the θ is small, by using the first-order approximation, $\frac{\sin\theta}{\theta} = 1$ and $\frac{1-\cos\theta}{\theta^2} = 0$.

$$\mathbf{R} \approx \mathbf{I} + [\phi]_\times \tag{25}$$

The calculation is very efficient for small-angle rotations and is suitable for small attitude updates in inertial navigation.

III. Quaternion Method

With the Rodrigues' Rotation Formula,

$$\mathbf{v}_{rot} = \mathbf{R}\mathbf{v} \tag{26}$$

$$\begin{aligned} \mathbf{R} &= \mathbf{I} + \sin\theta\omega^\wedge + (1-\cos\theta){\omega^\wedge}^2 \\ &= \mathbf{I} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\omega^\wedge + 2\sin^2\frac{\theta}{2}{\omega^\wedge}^2 \end{aligned} \tag{27}$$

in which,

$$\omega = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

$$\omega^\wedge = \begin{bmatrix} 0 & -n & m \\ n & 0 & -l \\ -m & l & 0 \end{bmatrix}$$

With equation (27)

$$\mathbf{R}_b^R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\cos\frac{\theta}{2} \begin{bmatrix} 0 & -n\sin\frac{\theta}{2} & m\sin\frac{\theta}{2} \\ n\sin\frac{\theta}{2} & 0 & -l\sin\frac{\theta}{2} \\ -m\sin\frac{\theta}{2} & l\sin\frac{\theta}{2} & 0 \end{bmatrix} + 2 \begin{bmatrix} -(m^2+n^2)\sin^2\frac{\theta}{2} & lm\sin^2\frac{\theta}{2} & ln\sin^2\frac{\theta}{2} \\ lm\sin^2\frac{\theta}{2} & -(l^2+n^2)\sin^2\frac{\theta}{2} & mn\sin^2\frac{\theta}{2} \\ ln\sin^2\frac{\theta}{2} & mn\sin^2\frac{\theta}{2} & -(m^2+l^2)\sin^2\frac{\theta}{2} \end{bmatrix} \tag{28}$$

Construct a quaternion with q_0, q_1, q_2, q_3. Let q_0, q_1, q_2, q_3 are as follows.

$$\begin{aligned} q_0 &= \cos\frac{\theta}{2} \\ q_1 &= l\sin\frac{\theta}{2} \\ q_2 &= m\sin\frac{\theta}{2} \\ q_3 &= n\sin\frac{\theta}{2} \end{aligned} \tag{29}$$

in which, q_0, q_1, q_2, q_3 satisfies;

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

$$\begin{aligned} Q &= q_0 + q_1\mathbf{i_0} + q_2\mathbf{j_0} + q_3\mathbf{k_0} \\ &= \cos\frac{\theta}{2} + (l\mathbf{i_0} + m\mathbf{j_0} + n\mathbf{k_0})\sin\frac{\theta}{2} \\ &= \cos\frac{\theta}{2} + \omega\sin\frac{\theta}{2} \end{aligned} \tag{30}$$

With equation (28) and (29), the rotation matrix in the Quaternion can be expressed as:

$$\mathbf{R}_b^R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_0q_1 - q_2q_3) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \tag{31}$$

The multiplication of two quaternions $\mathbf{q_1}, \mathbf{q_2}$ is defined as:

$$\begin{aligned} \mathbf{q_1} \otimes \mathbf{q_2} &= \begin{bmatrix} q_{01}q_{02} - \mathbf{q_{v1}} \cdot \mathbf{q_{v2}} \\ q_{01}\mathbf{q_{v2}} + q_{02}\mathbf{q_{v1}} + \mathbf{q_{v1}} \times \mathbf{q_{v2}} \end{bmatrix} \\ &= \begin{bmatrix} q_{0_1} & -q_{1_1} & -q_{2_1} & -q_{3_1} \\ q_{1_1} & q_{0_1} & -q_{3_1} & q_{2_1} \\ q_{2_1} & q_{3_1} & q_{0_1} & -q_{1_1} \\ q_{3_1} & -q_{2_1} & q_{1_1} & q_{0_1} \end{bmatrix} \begin{bmatrix} q_{0_2} \\ q_{1_2} \\ q_{2_2} \\ q_{3_2} \end{bmatrix} \end{aligned} \tag{32}$$

Thus, if we consider the vectors $\mathbf{r}_R$ and $\mathbf{r}_b$ as quaternions with zero scalar part, then their transformation relationship can be represented by quaternion multiplication as follows:

$$\mathbf{r}_R = Q \otimes \mathbf{r}_b \otimes Q^* \tag{33}$$

in which Q^* is the conjugate quaternion of Q.

With the value of q_0, q_1, q_2, q_3, we can also get the angle of roll, pitch, yaw.

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \arctan\frac{2(q_0q_1 + q_2q_3)}{q_0^2 - q_1^2 - q_2^2 + q_3^2} \\ \arcsin[2(q_0q_2 - q_1q_3)] \\ \arctan\frac{2(q_0q_3 + q_1q_2)}{q_0^2 + q_1^2 - q_2^2 - q_3^2} \end{bmatrix} \tag{34}$$

IV. Summary

Assume Euler angles, quaternion, and equivalent rotation vector are as follows.

$$A = \begin{bmatrix} \theta & \gamma & \psi \end{bmatrix}^T$$

$$\mathbf{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T$$

$$\Phi = \phi \mathbf{u}$$

We have the transformation matrix as follows.

$$\mathbf{C}_b^n = \begin{bmatrix} \cos\gamma\cos\psi - \sin\theta\sin\gamma\sin\psi & -\cos\theta\sin\psi & \sin\gamma\cos\psi + \sin\theta\cos\gamma\sin\psi \\ \cos\gamma\sin\psi + \sin\theta\sin\gamma\cos\psi & \cos\theta\cos\psi & \sin\gamma\sin\psi - \sin\theta\cos\gamma\cos\psi \\ -\cos\theta\sin\gamma & \sin\theta & \cos\theta\cos\gamma \end{bmatrix} \tag{35}$$

$$\mathbf{C}_b^n = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_0q_1 - q_2q_3) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \tag{36}$$

$$\mathbf{C}_b^n = \mathbf{I} + \frac{\sin\theta}{\theta}[\phi]_\times + \frac{1-\cos\theta}{\theta^2}[\phi]_\times^2 = \mathbf{I} + [\mathbf{u}]_\times \sin\phi + [\mathbf{u}]_\times^2 (1 - \cos\phi) \tag{37}$$

V. Appendix

A. Vector projection

$$\mathbf{a}_1 = a_1\hat{\mathbf{b}} = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||}\frac{\mathbf{b}}{||\mathbf{b}||} = (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} \tag{38}$$

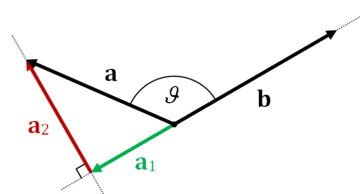

Fig. 2. Projection. [1]

B. Cross Product

$$\mathbf{a}\times\mathbf{b} = ||\mathbf{a}||||\mathbf{b}||\sin\theta\mathbf{n} \tag{39}$$

in which θ is the angle of $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{n}$ shows the direction of the cross product, which is decided by the right-hand's rule. $\mathbf{n}$ is orthogonal to the plane of $\mathbf{a}$ and $\mathbf{b}$.

Besides, the cross product can be expressed in the matrix.

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_3 & b_3 \end{vmatrix} \tag{40}$$

With $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$, the cross product can be expressed as

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [\mathbf{b}]_\times^T \mathbf{a} \tag{41}$$

in which, both of $[a]_\times$ and $\mathbf{a}^\wedge$ represent an anti-symmetric matrix.

$$\mathbf{k} \times \mathbf{v} = \mathbf{k} \times (\mathbf{v}_\| + \mathbf{v}_\perp) = 0 + \mathbf{k} \times \mathbf{v}_\perp = \mathbf{k} \times \mathbf{v}_\perp \tag{42}$$

$\mathbf{k} \times \mathbf{v}$ shows the vector $\mathbf{v}_\perp$ rotates $90°$ counterclockwise about the axis $\mathbf{k}$. Thus, the $\mathbf{v}_\perp$ can be expressed as

$$\mathbf{v}_\perp = -\mathbf{k} \times (\mathbf{k} \times \mathbf{v}) \tag{43}$$

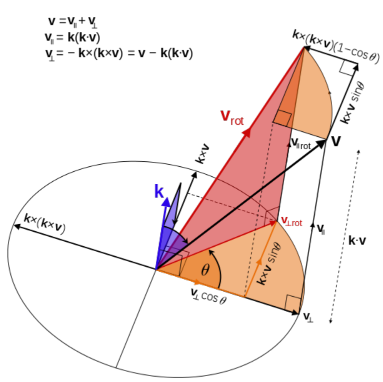

Fig. 3. Cross product. [1]

References

[1] https://www.cnblogs.com/wtyuan/p/12324495.html

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[3] Schaub H, Junkins J L. Analytical mechanics of space systems[M]. Aiaa, 2003.