CHAPTER 8 TECHNIQUES OF INTEGRATION

8.1 USING BASIC INTEGRATION FORMULAS

1. $\displaystyle\int_0^1 \frac{16x}{8x^2+2}\,dx$

$u = 8x^2 + 2 \quad du = 16x\,dx$

$u = 2$ when $x = 0,\ u = 10$ when $x = 1$

$$\int_0^1 \frac{16x}{8x^2+2}\,dx = \int_2^{10} \frac{1}{u}\,du = \ln|u|\Big]_2^{10}$$
$$= \ln 10 - \ln 2 = \ln 5$$

2. $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

Use long division to write the integrand as $1 - \dfrac{1}{x^2+1}$.

$$\int \frac{x^2}{x^2+1}\,dx = \int 1\,dx - \int \frac{1}{x^2+1}\,dx$$
$$= x - \tan^{-1} x + C$$

3. $\displaystyle\int (\sec x - \tan x)^2\,dx$

Expand the integrand: $(\sec x - \tan x)^2 = \sec^2 x - 2\sec x \tan x + \tan^2 x$

$$= \sec^2 x - 2\sec x \tan x + (\sec^2 x - 1)$$
$$= 2\sec^2 x - 2\sec x \tan x - 1$$

$$\int (\sec x - \tan x)^2\,dx = 2\int \sec^2 x\,dx - 2\int \sec x \tan x\,dx - \int 1\,dx$$
$$= 2\tan x - 2\sec x - x + C$$

We have used Formulas 8 and 10 from Table 8.1.

4. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{1}{\cos^2 x \tan x}\,dx$

$u = \tan x \quad du = \sec^2 x\,dx = \dfrac{1}{\cos^2 x}\,dx$

$u = 1$ when $x = \pi/4,\ u = \sqrt{3}$ when $x = \pi/3$

$$\int_{\pi/4}^{\pi/3} \frac{1}{\cos^2 x \tan x}\,dx = \int_1^{\sqrt{3}} \frac{1}{u}\,du = \ln|u|\Big]_1^{\sqrt{3}}$$
$$= \ln\sqrt{3} - \ln 1 = \frac{1}{2}\ln 3$$

5. $\displaystyle\int \frac{1-x}{\sqrt{1-x^2}}\,dx$

Write as the sum of two integrals:

$$\int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

For the first integral use Formula 18 in Table 8.1 with $a = 1$.

For the second:

$u = 1 - x^2 \quad du = -2x\,dx$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{1}{u^{1/2}}\,du$$

$$= -\sqrt{u} = -\sqrt{1-x^2}$$

So $\displaystyle\int \frac{1-x}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + \sqrt{1-x^2} + C$

6. $\displaystyle\int \frac{1}{x-\sqrt{x}}\,dx$

$u = \sqrt{x} - 1 \quad du = \dfrac{1}{2\sqrt{x}}\,dx$

$$\int \frac{1}{x-\sqrt{x}}\,dx = 2\int \frac{1}{u}\,du$$

$$= 2\ln|u| + C = 2\ln\left|\sqrt{x} - 1\right| + C$$

7. $\displaystyle\int \frac{e^{-\cot z}}{\sin^2 z}\,dz$

$u = -\cot z \quad du = -\csc^2 z\,dz = \dfrac{1}{\sin^2 z}\,dz$

$$\int \frac{e^{-\cot z}}{\sin^2 z}\,dz = \int e^{-u}\,du$$

$$= e^{-u} + C = e^{-\cot z} + C$$

8. $\displaystyle\int \frac{2^{\ln z^3}}{16z}\,dz$

$u = \ln z^3 = 3\ln z \quad du = \dfrac{3}{z}\,dz$

Using Formula 5 in Table 8.1,

$$\int \frac{2^{\ln z^3}}{16z}\,dz = \frac{1}{48}\int 2^u\,du$$

$$= \frac{2^u}{48\ln 2} + C = \frac{2^{\ln z^3}}{48\ln 2} + C$$

9. $\int \frac{1}{e^z + e^{-z}}\, dz$

Multiply the integrand by $\frac{e^z}{e^z}$.

$$\int \frac{1}{e^z + e^{-z}}\, dz = \int \frac{e^z}{e^{2z} + 1}\, dx$$

$u = e^z \quad du = e^z\, du$

$$\int \frac{e^z}{e^{2z} + 1}\, dx = \int \frac{1}{u^2 + 1}\, du$$
$$= \tan^{-1} u + C = \tan^{-1} e^z + C$$

10. $\int_1^2 \frac{8}{x^2 - 2x + 2}\, dx$

$u = x - 1 \quad du = dx$

$u = 0$ when $x = 1$, $u = 1$ when $x = 2$

$$\int_1^2 \frac{8}{x^2 - 2x + 2}\, dx = 8\int_0^1 \frac{1}{u^2 + 1}\, du$$
$$= 8\tan^{-1} u\Big]_0^1 = 8\left(\frac{\pi}{4} - 0\right) = 2\pi$$

11. $\int_{-1}^0 \frac{4}{1 + (2x + 1)^2}\, dx$

$u = 2x + 1 \quad du = 2dx$

$u = -1$ when $x = -1$, $u = 1$ when $x = 0$

$$\int_{-1}^0 \frac{4}{1 + (2x + 1)^2}\, dx = 2\int_{-1}^1 \frac{1}{1 + u^2}\, du$$
$$= 2\tan^{-1} u\Big]_{-1}^1 = 2\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \pi$$

12. $\int_{-1}^3 \frac{4x^2 - 7}{2x + 3}\, dx$

Use long division to write the integrand as $2x - 3 + \frac{2}{2x + 3}$.

$$\int_{-1}^3 \frac{4x^2 - 7}{2x + 3}\, dx = \int_{-1}^3 2x\, dx - \int_{-1}^3 3\, dx + \int_{-1}^3 \frac{2}{2x + 3}\, dx$$

$$\int_{-1}^3 2x\, dx - \int_{-1}^3 3\, dx = x^2\Big]_{-1}^3 - 3x\Big]_{-1}^3 = 8 - 12 = -4$$

For the last integral,
$u = 2x + 3 \quad du = 2\,dx$

$u = 1$ when $x = -1$, $u = 9$ when $x = 3$

$$\int_{-1}^{3} \frac{2}{2x+3}\,dx = \int_{1}^{9} \frac{1}{u}\,du$$
$$= \left|\ln u\right|\Big]_1^9 = \ln 9 - \ln 1 = 2\ln 3$$

So $\displaystyle\int_{-1}^{3} \frac{4x^2-7}{2x+3}\,dx = -4 + 2\ln 3$

13. $\displaystyle\int \frac{1}{1-\sec t}\,dt$

Multiply the integrand by $\dfrac{1+\sec t}{1+\sec t}$.

$$\frac{1}{1-\sec t}\cdot\frac{1+\sec t}{1+\sec t} = \frac{-1-\sec t}{\tan^2 t} = -\cot^2 t - \frac{\cos t}{\sin^2 t} = 1-\csc^2 t - \frac{\cos t}{\sin^2 t}$$
$$\int \frac{1}{1-\sec t}\,dt = \int 1\,dt - \int \csc^2 t\,dt - \int \frac{\cos t}{\sin^2 t}\,dt$$
$$= t + \cot t + \csc t + C$$

Here we have used Formula 9 in Table 8.1 for the second integral, and the substitution $u = \sin t,\ du = \cos t\,dt$ for the third integral, which gives it the form $\displaystyle\int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{\sin t}$.

14. $\displaystyle\int \csc t \sin 3t\,dt$

Write $\sin 3t$ as $\sin(2t+t)$ and expand.

$$\csc t \sin 3t = \frac{\cos 2t \sin t + (2\sin t\cos t)\cos t}{\sin t}$$
$$= \cos 2t + 2\cos^2 t = 2\cos 2t + 1$$
$$\int \csc t \sin 3t\,dt = \int 2\cos 2t\,dt + \int 1\,dt$$
$$= \sin 2t + t + C$$

15. $\displaystyle\int_0^{\pi/4} \frac{1+\sin\theta}{\cos^2\theta}\,d\theta$

Split into two integrals.

$$\int_0^{\pi/4} \frac{1+\sin\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4} \frac{1}{\cos^2\theta}\,d\theta + \int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\,d\theta$$
$$= \int_0^{\pi/4} \sec^2\theta\,d\theta + \int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\,d\theta$$
$$= \left[\tan\theta + \sec\theta\right]_0^{\pi/4} = (1+\sqrt{2}) - (0+1) = \sqrt{2}$$

The second integral is evaluated with the substitution $u = \cos\theta \quad du = -\sin\theta\,d\theta$, which gives

$$\int \frac{\sin\theta}{\cos^2\theta}\,d\theta = -\int \frac{1}{u^2}\,du = \frac{1}{u} = \frac{1}{\cos\theta}.$$

16. $\int \frac{1}{\sqrt{2\theta-\theta^2}}\,d\theta$

Write the integrand as $\frac{1}{\sqrt{1-(\theta-1)^2}}$. With $u=\theta-1,\ du=d\theta,$

$$\int \frac{1}{\sqrt{2\theta-\theta^2}}\,d\theta = \int \frac{1}{\sqrt{1-(\theta-1)^2}}\,d\theta$$
$$= \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C = \sin^{-1}(\theta-1)+C$$

We have used Formula 18 in Table 8.1 with $a=1$.

17. $\int \frac{\ln y}{y+4\ln^2 y}\,dy$

Write the integrand as $\frac{\ln y}{y}\cdot\frac{1}{1+4\ln^2 y}$.

$u=1+4\ln^2 y \quad du=\frac{8\ln y}{y}\,dy$

$$\int \frac{\ln y}{y+4\ln^2 y}\,dy = \int \frac{\ln y}{y}\cdot\frac{1}{1+4\ln^2 y}\,dy$$
$$=\frac{1}{8}\int\frac{1}{u}\,du=\frac{1}{8}\ln|u|+C=\frac{1}{8}\ln(1+4\ln^2 y)+C$$

Note that the argument of the logarithm is positive, so we don’t need absolute value bars.

18. $\int \frac{2^{\sqrt{y}}}{2\sqrt{y}}\,dy$

$u=\sqrt{y} \quad du=\frac{1}{2\sqrt{y}}\,dy$

Using Formula 5 in Table 8.1,

$$\int \frac{2^{\sqrt{y}}}{2\sqrt{y}}\,dy = \int 2^u\,du$$
$$=\frac{1}{\ln 2}2^u+C=\frac{1}{\ln 2}2^{\sqrt{y}}+C$$

19. $\int \frac{1}{\sec\theta+\tan\theta}\,d\theta$

Multiply the integrand by $\frac{\cos\theta}{\cos\theta}$.

$$\int \frac{1}{\sec\theta+\tan\theta}\cdot\frac{\cos\theta}{\cos\theta}\,d\theta=\int\frac{\cos\theta}{1+\sin\theta}\,d\theta$$

$u=1+\sin\theta \quad du=\cos\theta\,d\theta$

$$\int \frac{\cos\theta}{1+\sin\theta}\,d\theta = \int \frac{1}{u}\,du = \ln|u| + C$$
$$= \ln(1+\sin\theta) + C$$

We can discard the absolute value because $1+\sin\theta$ is never negative.

20. $\displaystyle\int \frac{1}{t\sqrt{3+t^2}}\,dt$

Use Formula 5 in Table 7.10, with $a = \sqrt{3}$.

$$\int \frac{1}{t\sqrt{3+t^2}}\,dt = -\frac{1}{\sqrt{3}}\operatorname{csch}^{-1}\left|\frac{t}{\sqrt{3}}\right| + C$$

21. $\displaystyle\int \frac{4t^3 - t^2 + 16t}{t^2+4}\,dt$

Use long division to write the integrand as $4t - 1 + \dfrac{4}{t^2+4}$.

$$\int \frac{4t^3 - t^2 + 16t}{t^2+4}\,dt = \int 4t\,dt - \int 1\,dt + 4\int \frac{1}{t^2+4}\,dt$$
$$= 2t^2 - t + 2\tan^{-1}\left(\frac{t}{2}\right) + C$$

To evaluate the third integral we used Formula 19 in Table 8.1 with $a = 2$.

22. $\displaystyle\int \frac{x + 2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

Split into two integrals.

$$\int \frac{x + 2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx = \int \frac{1}{2\sqrt{x-1}}\,dx + \int \frac{1}{x}\,dx$$
$$= \sqrt{x-1} + \ln|x| + C$$

For the first integral we used $u = \sqrt{x-1}$, $du = \dfrac{1}{2\sqrt{x-1}}\,dx$, $\int du = u + C$

23. $\displaystyle\int_0^{\pi/2} \sqrt{1-\cos\theta}\,d\theta$

Multiply the integrand by $\dfrac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}$.

$$\int_0^{\pi/2} \sqrt{1-\cos\theta}\,d\theta = \int_0^{\pi/2} \frac{\sqrt{1-\cos^2\theta}}{\sqrt{1+\cos\theta}}\,d\theta = \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta.$$

(Note that when $0 \le \theta \le \pi/2$, $\sin\theta \ge 0$ so $\sqrt{\sin^2\theta} = \sin\theta$.)

$u = 1 + \cos\theta \quad du = -\sin\theta\,d\theta$

$u = 2$ when $\theta = 0$, $u = 1$ when $\theta = \pi/2$

$$\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta = -\int_2^1 \frac{1}{\sqrt{u}}\,du = \int_1^2 \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big]_1^2 = 2\sqrt{2} - 2$$

24. $\int (\sec t + \cot t)^2 \, dt$

Expand the integrand:

$$\begin{aligned}(\sec t + \cot t)^2 &= \sec^2 t + 2\sec t \cot t + \cot^2 t \\ &= \sec^2 t + 2\sec t \cot t + \csc^2 t - 1\end{aligned}$$

$$\begin{aligned}\int (\sec t + \cot t)^2 \, dt &= \int \sec^2 t \, dt + 2\int \csc t \, dt + \int \csc^2 t \, dt - \int 1 \, dt \\ &= \tan t - 2\ln\left|\csc t + \cot t\right| - \cot t - t + C\end{aligned}$$

We have used Formulas 8, 9 and 15 from Table 8.1.

25. $\displaystyle\int \frac{1}{\sqrt{e^{2y}-1}} \, dy$

Multiply the integrand by $\dfrac{e^y}{e^y}$.

$$\int \frac{1}{\sqrt{e^{2y}-1}} \, dy = \int \frac{e^y}{e^y\sqrt{e^{2y}-1}} \, dy \, ; \quad u = e^y \quad du = e^y \, dy$$

$$\begin{aligned}\int \frac{e^y}{e^y\sqrt{e^{2y}-1}} \, dy &= \int \frac{1}{u\sqrt{u^2-1}} \, du \\ &= \sec^{-1}\left|u\right| + C = \sec^{-1} e^y + C\end{aligned}$$

We have used Formula 20 in Table 8.1.

26. $\displaystyle\int \frac{6}{\sqrt{y}\,(1+y)} \, dy$

$$u = \sqrt{y} \quad du = \frac{1}{2\sqrt{y}} \, dy$$

$$\begin{aligned}\int \frac{6}{\sqrt{y}\,(1+y)} \, dy &= 12\int \frac{1}{1+u^2} \, du \\ &= 12\tan^{-1}\sqrt{y} + C\end{aligned}$$

27. $\displaystyle\int \frac{2}{x\sqrt{1-4\ln^2 x}} \, dx$

$$u = 2\ln x \quad du = \frac{2}{x} dx$$

$$\begin{aligned}\int \frac{2}{x\sqrt{1-4\ln^2 x}} \, dx &= \int \frac{1}{\sqrt{1-u^2}} \, du \\ &= \sin^{-1} u + C = \sin^{-1}(2\ln x) + C\end{aligned}$$

28. $\displaystyle\int \frac{1}{(x-2)\sqrt{x^2-4x+3}}\,dx$

$u = x-2 \quad du = dx$

$$\int \frac{1}{(x-2)\sqrt{x^2-4x+3}}\,dx = \int \frac{1}{u\sqrt{u^2-1}}\,du$$
$$= \sec^{-1}|u| + C = \sec^{-1}|x-2| + C$$

We have used Formula 20 in Table 8.1 with $a = 1$.

29. $\displaystyle\int (\csc x - \sec x)(\sin x + \cos x)\,dx$

Expand the integrand and separate into two integrals.

$(\csc x - \sec x)(\sin x + \cos x) = 1 + \cot x - \tan x - 1 = \cot x - \tan x$

$$\int (\csc x - \sec x)(\sin x + \cos x)\,dx = \int \cot x\,dx - \int \tan x\,dx$$
$$= \ln|\sin x| - \ln|\sec x| + C = \ln|\sin x| + \ln|\cos x| + C$$

We have used Formulas 12 and 13 from Table 8.1.

30. $\displaystyle\int 3\sinh\left(\frac{x}{2} + \ln 5\right)dx$

$u = \dfrac{x}{2} + \ln 5 \quad du = \dfrac{1}{2}dx$

$$\int 3\sinh\left(\frac{x}{2} + \ln 5\right)dx = 6\int \sinh u\,du$$
$$= 6\cosh u + C = 6\cosh\left(\frac{x}{2} + \ln 5\right) + C$$

31. $\displaystyle\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\,dx$

Use long division to write the integrand as $2x + \dfrac{2x}{x^2-1}$.

$$\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\,dx = \int_{\sqrt{2}}^{3}\left(2x + \frac{2x}{x^2-1}\right)dx = \int_{\sqrt{2}}^{3} 2x\,dx + \int_{\sqrt{2}}^{3} \frac{2x}{x^2-1}\,dx$$

For the second integral we use $u = x^2$, $du = 2x\,dx$.

$$\int_{\sqrt{2}}^{3} 2x\,dx + \int_{\sqrt{2}}^{3} \frac{2x}{x^2-1}\,dx = x^2\Big]_{\sqrt{2}}^{3} + \ln\left|x^2-1\right|\Big]_{\sqrt{2}}^{3}$$
$$= (9-2) + (\ln 8 - \ln 1)$$
$$= 7 + \ln 8 \approx 9.079$$

32. $\int_{-1}^{1}\sqrt{1+x^2}\sin x\,dx$ is the integral of an odd function over an interval symmetric to 0, so its value is 0.

33. $\displaystyle\int_{-1}^{0}\sqrt{\frac{1+y}{1-y}}\,dy$

Multiply the integrand by $\dfrac{\sqrt{1+y}}{\sqrt{1+y}}$ and split the indefinite integral into a sum.

$$\begin{aligned}\int\sqrt{\frac{1+y}{1-y}}\,dy &= \int\frac{1+y}{\sqrt{1-y^2}}\,dy = \int\frac{1}{\sqrt{1-y^2}}\,dy + \int\frac{y}{\sqrt{1-y^2}}\,dy\\ &= \sin^{-1}y - \sqrt{1-y^2} + C\end{aligned}$$

The first integral is Formula 18 in Section 8.1, and for the second we use the substitution $u = 1-y^2$, $du = -2y\,dy$. So

$$\begin{aligned}\int_{-1}^{0}\sqrt{\frac{1+y}{1-y}}\,dy &= \left[\sin^{-1}y - \sqrt{1-y^2}\,\right]_{-1}^{0}\\ &= (0-1) - \left(-\frac{\pi}{2} - 0\right) = \frac{\pi}{2} - 1\end{aligned}$$

34. $\int e^{z+e^z}\,dz$

Write the integrand as $e^z e^{e^z}$ and use the substitution $u = e^z$, $du = e^z dz$.

$$\begin{aligned}\int e^{z+e^z}\,dz &= \int e^z e^{e^z}\,dz = \int e^u\,du\\ &= e^u + C = e^{e^z} + C\end{aligned}$$

35. $\displaystyle\int\frac{7}{(x-1)\sqrt{x^2-2x-48}}\,dx$

$u = x-1,\quad du = dx;\quad x^2 - 2x - 48 = u^2 - 7^2$

We use Formula 20 in Table 8.1.

$$\begin{aligned}\int\frac{7}{(x-1)\sqrt{x^2-2x-48}}\,dx &= \int\frac{7}{u\sqrt{u^2-7^2}}\,du\\ &= \frac{1}{7}\left(7\sec^{-1}\left|\frac{u}{7}\right|\right) + C = \sec^{-1}\left|\frac{x-1}{7}\right| + C\end{aligned}$$

36. $\displaystyle\int \frac{1}{(2x+1)\sqrt{4x+4x^2}}\,dx$

$u = 2x+1, \quad du = 2dx; \quad 4x+4x^2 = u^2 - 1^2$

We use Formula 20 in Table 8.1.

$$\int \frac{1}{(2x+1)\sqrt{4x+4x^2}}\,dx = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-1^2}}\,du$$
$$= \frac{1}{2}\sec^{-1}|u| + C = \frac{1}{2}\sec^{-1}|2x+1| + C$$

37. $\displaystyle\int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5}\,d\theta$

Use long division to write the integrand as $\theta^2 - \theta + 1 + \dfrac{5}{2\theta - 5}$.

$$\int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5}\,d\theta = \int \theta^2\,d\theta - \int \theta\,d\theta + \int 1\,d\theta + \int \frac{5}{2\theta - 5}\,d\theta$$
$$= \frac{1}{3}\theta^3 - \frac{1}{2}\theta^2 + \theta + \frac{5}{2}\ln|2\theta - 5| + C$$

In the last integral we have used the substitution $u = 2\theta - 5,\ du = 2\,d\theta$.

38. $\displaystyle\int \frac{1}{\cos\theta - 1}\,d\theta$

Multiply the integrand by $\dfrac{\cos\theta + 1}{\cos\theta + 1}$.

$$\frac{1}{\cos\theta - 1}\frac{\cos\theta + 1}{\cos\theta + 1} = \frac{\cos\theta + 1}{\cos^2\theta - 1} = -\frac{1 + \cos\theta}{\sin^2\theta} = -\csc^2\theta - \csc\theta\cot\theta$$

$$\int\left(-\csc^2\theta - \csc\theta\cot\theta\right)d\theta = -\int \csc^2\theta\,d\theta - \int \csc\theta\cot\theta\,d\theta$$
$$= \cot\theta + \csc\theta + C$$

We have used Formulas 9 and 11 from Table 8.1.

39. $\displaystyle\int \frac{1}{1+e^x}\,dx$

Use one step of long division to write the integrand as $1 - \dfrac{e^x}{1+e^x}$.

$$\int \frac{1}{1+e^x}\,dx = \int 1\,dx - \int \frac{e^x}{1+e^x}\,dx = x - \ln\left(1+e^x\right) + C$$

For the second integral we have used the substitution $u = 1+e^x,\ du = e^x dx.$ Note that $1+e^x$ is always positive.

40. $\int \frac{\sqrt{x}}{1+x^3}\, dx$

$u = x^{3/2}, \quad du = \frac{3}{2}x^{1/2}dx$

$$\int \frac{\sqrt{x}}{1+x^3}\, dx = \frac{2}{3}\int \frac{1}{1+u^2}\, du$$
$$= \frac{2}{3}\tan^{-1} u + C = \frac{2}{3}\tan^{-1} x^{3/2} + C$$

41. The area is $\int_{-\pi/4}^{\pi/4}(2\cos x - \sec x)\, dx = \left[2\sin x - \ln\left|\sec x + \tan x\right|\right]_{-\pi/4}^{\pi/4}$

$$= \left(\sqrt{2} - \ln\left|\sqrt{2}+1\right|\right) - \left(-\sqrt{2} - \ln\left|\sqrt{2}-1\right|\right)$$
$$= 2\sqrt{2} + \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) = 2\sqrt{2} - \ln\left(3+2\sqrt{2}\right) \approx 1.066$$

42. The volume using the washer method is $\pi\int_{-\pi/4}^{\pi/4}\left(4\cos^2 x - \sec^2 x\right) dx.$

Split into two integrals; for the first write $4\cos^2 x$ as $2(1+\cos 2x)$ and for the second use Formula 8 in Table 8.1.

$$\pi\int_{-\pi/4}^{\pi/4}\left(4\cos^2 x - \sec^2 x\right) dx = \pi\int_{-\pi/4}^{\pi/4} 4\cos^2 x\, dx - \pi\int_{-\pi/4}^{\pi/4}\sec^2 x\, dx$$
$$= \pi\int_{-\pi/4}^{\pi/4} 2(1+\cos 2x)\, dx - \pi\int_{-\pi/4}^{\pi/4}\sec^2 x\, dx$$
$$= \pi\left[2x + \sin 2x\right]_{-\pi/4}^{\pi/4} - \pi\tan x\Big]_{-\pi/4}^{\pi/4}$$
$$= \pi\left(\left(\frac{\pi}{2}+1\right) - \left(-\frac{\pi}{2}-1\right)\right) - \pi(1-(-1)) = \pi^2$$

43. For $y = \ln(\cos x)$, $dy/dx = -\tan x$. The arc length is given by

$\int_0^{\pi/3}\sqrt{1+(-\tan x)^2}\, dx = \int_0^{\pi/3}\sec x\, dx$ since $\sec x$ is positive on the interval of integration.

$$\int_0^{\pi/3}\sec x\, dx = \ln\left|\sec x + \tan x\right|\Big]_0^{\pi/3}$$
$$= \ln\left(2+\sqrt{3}\right) - \ln(1+0) = \ln\left(2+\sqrt{3}\right)$$

44. For $y = \ln(\sec x)$, $dy/dx = \tan x$. The arc length is given by

$\int_0^{\pi/4}\sqrt{1+(\tan x)^2}\, dx = \int_0^{\pi/4}\sec x\, dx$ since $\sec x$ is positive on the interval of integration.

$$\int_0^{\pi/4}\sec x\, dx = \ln\left|\sec x + \tan x\right|\Big]_0^{\pi/4}$$
$$= \ln\left(\sqrt{2}+1\right) - \ln(1+0) = \ln\left(\sqrt{2}+1\right)$$

45. Since secant is an even function and the domain is symmetric to 0, $\overline{x} = 0.$

For the y-coordinate:

$$\overline{y} = \frac{\frac{1}{2}\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx}{\int_{-\pi/4}^{\pi/4} \sec x\,dx} = \frac{\frac{1}{2}\tan x\Big]_{-\pi/4}^{\pi/4}}{\ln\left|\sec x + \tan x\right|\Big]_{-\pi/4}^{\pi/4}}$$

$$= \frac{1}{\left(\ln\left(\sqrt{2}+1\right) - \ln\left(\sqrt{2}-1\right)\right)}$$

$$= \frac{1}{\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)} = \frac{1}{\ln\left(3+2\sqrt{2}\right)} \approx 0.567$$

46. Since both cosecant and the domain are symmetric around $\pi/2$, $\overline{x} = \pi/2.$

$$\overline{y} = \frac{\frac{1}{2}\int_{\pi/6}^{5\pi/6} \csc^2 x\,dx}{\int_{\pi/6}^{5\pi/6} \csc x\,dx} = \frac{-\frac{1}{2}\cot x\Big]_{\pi/6}^{5\pi/6}}{-\ln\left|\csc x + \cot x\right|\Big]_{\pi/6}^{5\pi/6}}$$

$$= \frac{-\frac{1}{2}\left(\sqrt{3} - \left(-\sqrt{3}\right)\right)}{-\left(\ln\left(2+\sqrt{3}\right) - \ln\left(2-\sqrt{3}\right)\right)}$$

$$= \frac{\sqrt{3}}{\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)} = \frac{\sqrt{3}}{\ln\left(7+4\sqrt{3}\right)} \approx 0.658$$

47. $\int \left(1+3x^3\right)e^{x^3}\,dx = xe^{x^3} + C$

48. $\int \frac{1}{1+\sin^2 x}\,dx$

Multiply the integrand by $\frac{\sec^2 x}{\sec^2 x}$.

$$\int \frac{1}{1+\sin^2 x}\,dx = \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\,dx = \int \frac{\sec^2 x}{1+2\tan^2 x}\,dx$$

$u = \tan x, \quad du = \sec^2 x\,dx$

$$\int \frac{\sec^2 x}{1+2\tan^2 x}\,dx = \int \frac{1}{1+2u^2}\,du$$

$v = \sqrt{2}u, \quad dv = \sqrt{2}\,du$

$$\int \frac{1}{1+2u^2}\,du = \frac{1}{\sqrt{2}}\int \frac{1}{1+v^2}\,dv$$

$$= \frac{1}{\sqrt{2}}\tan^{-1} v + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right) + C$$

49. $\int x^7 \sqrt{x^4+1}\, dx$

$u = x^4+1,\ du = 4x^3 dx; \qquad x^7 dx = \frac{u-1}{4} du$

$$\int x^7 \sqrt{x^4+1}\, dx = \frac{1}{4}\int (u-1)\sqrt{u}\ du$$
$$= \frac{1}{4}\int u^{3/2}\ du - \frac{1}{4}\int u^{1/2}\ du$$
$$= \frac{1}{10}u^{5/2} - \frac{1}{6}u^{3/2} + C$$
$$= \frac{1}{30}u^{3/2}(3u-5) + C = \frac{1}{30}\left(x^4+1\right)^{3/2}\left(3x^4-2\right) + C$$

50. $\int \left((x^2-1)(x+1)\right)^{-2/3} dx$

The easiest substitution to use is probably $u = \frac{x-1}{x+1},\ du = \frac{2}{(1+x)^2}dx.$

The integral can be written as

$$\int \frac{1}{\left(\frac{x-1}{x+1}\right)^{2/3}(x+1)^2}\, dx = \frac{1}{2}\int u^{-2/3}\, du$$
$$= \frac{3}{2}u^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$$

8.2 INTEGRATION BY PARTS

1. $u = x,\ du = dx;\ dv = \sin\frac{x}{2}dx,\ v = -2\cos\frac{x}{2};$
$\int x\sin\frac{x}{2}dx = -2x\cos\frac{x}{2} - \int\left(-2\cos\frac{x}{2}\right)dx = -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C$

2. $u = \theta,\ du = d\theta;\ dv = \cos\pi\theta\ d\theta,\ v = \frac{1}{\pi}\sin\pi\theta;$
$\int\theta\cos\pi\theta\ d\theta = \frac{\theta}{\pi}\sin\pi\theta - \int\frac{1}{\pi}\sin\pi\theta\ d\theta = \frac{\theta}{\pi}\sin\pi\theta + \frac{1}{\pi^2}\cos\pi\theta + C$

3.
$$\begin{array}{lcl} & & \cos t \\ t^2 & \xrightarrow{(+)} & \sin t \\ 2t & \xrightarrow{(-)} & -\cos t \\ 2 & \xrightarrow{(+)} & -\sin t \\ 0 & & \end{array}$$

$$\int t^2\cos t\, dt = t^2\sin t + 2t\cos t - 2\sin t + C$$

4.

$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

5. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x\,dx$, $v = \frac{x^2}{2}$;

$$\int_1^2 x \ln x\, dx = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x^2}{2}\frac{dx}{x} = 2\ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2\ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$$

6. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x^3\,dx$, $v = \frac{x^4}{4}$;

$$\int_1^e x^3 \ln x\, dx = \left[\frac{x^4}{4}\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4+1}{16}$$

7. $u = x$, $du = dx$; $dv = e^x dx$, $v = e^x$;

$$\int x\, e^x dx = xe^x - \int e^x\, dx = xe^x - e^x + C$$

8. $u = x$, $du = dx$; $dv = e^{3x}dx$, $v = \frac{1}{3}e^{3x}$;

$$\int x\, e^{3x}dx = \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x}dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C$$

9.

$$\begin{array}{lcl} & & e^{-x} \\ x^2 & \xrightarrow{(+)} & -e^{-x} \\ 2x & \xrightarrow{(-)} & e^{-x} \\ 2 & \xrightarrow{(+)} & -e^{-x} \\ 0 & & \end{array}$$

$$\int x^2 e^{-x}\, dx = -x^2 e^{-x} - 2x\, e^{-x} - 2e^{-x} + C$$

10.

$$\begin{array}{lcl} & & e^{2x} \\ x^2 - 2x + 1 & \xrightarrow{(+)} & \frac{1}{2}e^{2x} \\ 2x - 2 & \xrightarrow{(-)} & \frac{1}{4}e^{2x} \\ 2 & \xrightarrow{(+)} & \frac{1}{8}e^{2x} \\ 0 & & \end{array}$$

$$\int \left(x^2 - 2x + 1\right)e^{2x}dx = \frac{1}{2}\left(x^2 - 2x + 1\right)e^{2x} - \frac{1}{4}(2x-2)e^{2x} + \frac{1}{4}e^{2x} + C$$

$$= \left(\frac{1}{2}x^2 - \frac{3}{2}x + \frac{5}{4}\right)e^{2x} + C$$

11. $u = \tan^{-1} y$, $du = \frac{dy}{1+y^2}$; $dv = dy$, $v = y$;

$$\int \tan^{-1} y\, dy = y\tan^{-1} y - \int \frac{y\,dy}{\left(1+y^2\right)} = y\tan^{-1} y - \frac{1}{2}\ln\left(1+y^2\right) + C = y\tan^{-1} y - \ln\sqrt{1+y^2} + C$$

12. $u = \sin^{-1} y,\ du = \frac{dy}{\sqrt{1-y^2}};\ dv = dy,\ v = y;$

$\int \sin^{-1} y\, dy = y \sin^{-1} y - \int \frac{y\, dy}{\sqrt{1-y^2}} = y \sin^{-1} y + \sqrt{1-y^2} + C$

13. $u = x,\ du = dx;\ dv = \sec^2 x\, dx,\ v = \tan x;$

$\int x \sec^2 x\, dx = x \tan x - \int \tan x\, dx = x \tan x + \ln|\cos x| + C$

14. $\int 4x \sec^2 2x\, dx;\ [y = 2x, dy = 2dx] \to \int y \sec^2 y\, dy = y \tan y - \int \tan y\, dy = y \tan y - \ln|\sec y| + C$
$= 2x \tan 2x - \ln|\sec 2x| + C$

15.

$$\begin{array}{lcl} & & e^x \\ x^3 & \xrightarrow{(+)} & e^x \\ 3x^2 & \xrightarrow{(-)} & e^x \\ 6x & \xrightarrow{(+)} & e^x \\ 6 & \xrightarrow{(-)} & e^x \\ 0 & & \end{array}$$

$$\int x^3 e^x\, dx = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C = \left(x^3 - 3x^2 + 6x - 6\right)e^x + C$$

16.

$$\begin{array}{lcl} & & e^{-p} \\ p^4 & \xrightarrow{(+)} & -e^{-p} \\ 4p^3 & \xrightarrow{(-)} & e^{-p} \\ 12p^2 & \xrightarrow{(+)} & -e^{-p} \\ 24p & \xrightarrow{(-)} & e^{-p} \\ 24 & \xrightarrow{(+)} & -e^{-p} \\ 0 & & \end{array}$$

$$\int p^4 e^{-p}\, dp = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24pe^{-p} - 24e^{-p} + C$$
$$= \left(-p^4 - 4p^3 - 12p^2 - 24p - 24\right) e^{-p} + C$$

17.

$$\begin{array}{lcl} & & e^x \\ x^2 - 5x & \xrightarrow{(+)} & e^x \\ 2x - 5 & \xrightarrow{(-)} & e^x \\ 2 & \xrightarrow{(+)} & e^x \\ 0 & & \end{array}$$

$$\int \left(x^2 - 5x\right)e^x\, dx = \left(x^2 - 5x\right)e^x - (2x-5)e^x + 2e^x + C = x^2 e^x - 7xe^x + 7e^x + C$$
$$= \left(x^2 - 7x + 7\right)e^x + C$$

18.

$$\begin{array}{lcl} & & e^r \\ r^2+r+1 & \xrightarrow{(+)} & e^r \\ 2r+1 & \xrightarrow{(-)} & e^r \\ 2 & \xrightarrow{(+)} & e^r \\ 0 & & \end{array}$$

$$\int \left(r^2+r+1\right) e^r\,dr = \left(r^2+r+1\right)e^r - (2r+1)e^r + 2e^r + C$$
$$= \left[\left(r^2+r+1\right) - (2r+1) + 2\right] e^r + C = \left(r^2-r+2\right) e^r + C$$

19.

$$\begin{array}{lcl} & & e^x \\ x^5 & \xrightarrow{(+)} & e^x \\ 5x^4 & \xrightarrow{(-)} & e^x \\ 20x^3 & \xrightarrow{(+)} & e^x \\ 60x^2 & \xrightarrow{(-)} & e^x \\ 120x & \xrightarrow{(+)} & e^x \\ 120 & \xrightarrow{(-)} & e^x \\ 0 & & \end{array}$$

$$\int x^5 e^x\,dx = x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + C$$
$$= \left(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120\right) e^x + C$$

20.

$$\begin{array}{lcl} & & e^{4t} \\ t^2 & \xrightarrow{(+)} & \frac{1}{4}e^{4t} \\ 2t & \xrightarrow{(-)} & \frac{1}{16}e^{4t} \\ 2 & \xrightarrow{(+)} & \frac{1}{64}e^{4t} \\ 0 & & \end{array}$$

$$\int t^2 e^{4t}\,dt = \frac{t^2}{4}e^{4t} - \frac{2t}{16}e^{4t} + \frac{2}{64}e^{4t} + C = \frac{t^2}{4}e^{4t} - \frac{t}{8}e^{4t} + \frac{1}{32}e^{4t} + C$$
$$= \left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32}\right) e^{4t} + C$$

21. $I = \int e^\theta \sin\theta\,d\theta$; $[u = \sin\theta,\ du = \cos\theta\,d\theta;\ dv = e^\theta\,d\theta,\ v = e^\theta] \Rightarrow I \Rightarrow e^\theta \sin\theta - \int e^\theta \cos\theta\,d\theta$;
$[u = \cos\theta, du = -\sin\theta\,d\theta;\ dv = e^\theta\,d\theta,\ v = e^\theta] \Rightarrow I = e^\theta \sin\theta - \left(e^\theta \cos\theta + \int e^\theta \sin\theta\,d\theta\right)$
$= e^\theta \sin\theta - e^\theta\cos\theta - I + C' \Rightarrow 2I = \left(e^\theta \sin\theta - e^\theta \cos\theta\right) + C' \Rightarrow I = \frac{1}{2}\left(e^\theta \sin\theta - e^\theta\cos\theta\right) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

22. $I = \int e^{-y}\cos y\,dy$; $[u = \cos y,\ du = -\sin y\,dy;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$
$\Rightarrow I = -e^{-y}\cos y - \int\left(-e^{-y}\right)(-\sin y)\,dy = -e^{-y}\cos y - \int e^{-y}\sin y\,dy$;
$[u = \sin y,\ du = \cos y\,dy;\ dv = e^{-y}dy,\ v = -e^{-y}] \Rightarrow I = -e^{-y}\cos y - \left(-e^{-y}\sin y - \int\left(-e^{y}\right)\cos y\,dy\right)$

$= -e^{-y}\cos y + e^{-y}\sin y - I + C' \Rightarrow 2I = e^{-y}(\sin y - \cos y) + C' \Rightarrow I = \frac{1}{2}\left(e^{-y}\sin y - e^{-y}\cos y\right) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

23. $I = \int e^{2x}\cos 3x\,dx$; $[u = \cos 3x;\ du = -3\sin 3x\,dx,\ dv = e^{2x}\,dx;\ v = \frac{1}{2}e^{2x}]$
$\Rightarrow I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx$; $[u = \sin 3x,\ du = 3\cos 3x,\ dv = e^{2x}\,dx;\ v = \frac{1}{2}e^{2x}]$
$\Rightarrow I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left(\frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx\right) = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}I + C'$
$\Rightarrow \frac{13}{4}I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C' \Rightarrow I = \frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$, where $C = \frac{4}{13}C'$

24. $\int e^{-2x}\sin 2x\,dx$; $[y = 2x, du = 2dx] \to \frac{1}{2}\int e^{-y}\sin y\,dy = I$; $[u = \sin y, du = \cos y\,dy;\ dv = e^{-y}\,dy, v = -e^{-y}]$
$\Rightarrow I = \frac{1}{2}\left(-e^{-y}\sin y + \int e^{-y}\cos y\,dy\right)$ $[u = \cos y, du = -\sin y;\ dv = e^{-y}\,dy, v = -e^{-y}]$
$\Rightarrow I = -\frac{1}{2}e^{-y}\sin y + \frac{1}{2}\left(-e^{-y}\cos y - \int\left(-e^{-y}\right)(-\sin y)\,dy\right) = -\frac{1}{2}e^{-y}(\sin y + \cos y) - I + C'$
$\Rightarrow 2I = -\frac{1}{2}e^{-y}(\sin y + \cos y) + C' \Rightarrow I = -\frac{1}{4}e^{-y}(\sin y + \cos y) + C = -\frac{e^{-2x}}{4}(\sin 2x + \cos 2x) + C$, where $C = \frac{C'}{2}$

25. $\int e^{\sqrt{3s+9}}\,ds$; $\begin{bmatrix} 3s + 9 = x^2 \\ ds = \frac{2}{3}x\,dx \end{bmatrix} \to \int e^x \cdot \frac{2}{3}x\,dx = \frac{2}{3}\int xe^x\,dx$; $[u = x,\ du = dx;\ dv = e^x\,dx, v = e^x]$;
$\frac{2}{3}\int xe^x\,dx = \frac{2}{3}\left(xe^x - \int e^x\,dx\right) = \frac{2}{3}\left(xe^x - e^x\right) + C = \frac{2}{3}\left(\sqrt{3s+9}\,e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$

26. $u = x, du = dx;\ dv = \sqrt{1-x}\,dx, v = -\frac{2}{3}\sqrt{(1-x)^3}$;
$\int_0^1 x\sqrt{1-x}\,dx = \left[-\frac{2}{3}x\sqrt{(1-x)^3}\right]_0^1 + \frac{2}{3}\int_0^1\sqrt{(1-x)^3}\,dx = 0 + \frac{2}{3}\left[-\frac{2}{5}(1-x)^{5/2}\right]_0^1 = \frac{4}{15}$

27. $u = x, du = dx;\ dv = \tan^2 x\,dx, v = \int \tan^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \frac{dx}{\cos^2 x} - \int dx = \tan x - x$;
$\int_0^{\pi/3} x\tan^2 x\,dx = \left[x(\tan x - x)\right]_0^{\pi/3} - \int_0^{\pi/3}(\tan x - x)\,dx = \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \left[\ln|\cos x| + \frac{x^2}{2}\right]_0^{\pi/3}$
$= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \ln\frac{1}{2} + \frac{\pi^2}{18} = \frac{\pi\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18}$

28. $u = \ln\left(x + x^2\right), du = \frac{(2x+1)\,dx}{x+x^2};\ dv = dx, v = x;\ \int \ln\left(x + x^2\right)dx = x\ln\left(x + x^2\right) - \int \frac{2x+1}{x(x+1)} \cdot x\,dx$
$= x\ln\left(x + x^2\right) - \int \frac{(2x+1)\,dx}{x+1} = x\ln\left(x + x^2\right) - \int\left(2 - \frac{1}{x+1}\right)dx = x\ln\left(x + x^2\right) - 2x + \ln|x+1| + C$

29. $\int \sin(\ln x)\,dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{1}{x}dx \\ dx = e^u\,du \end{bmatrix} \to \int(\sin u)\,e^u\,du$. From Exercise 21, $\int(\sin u)\,e^u\,du = e^u\left(\frac{\sin u - \cos u}{2}\right) + C$
$= \frac{1}{2}\left[-x\cos(\ln x) + x\sin(\ln x)\right] + C$

30. $\int z(\ln z)^2\, dz; \left[\begin{array}{c} u = \ln z \\ du = \frac{1}{z}\, dz \\ dz = e^u du \end{array}\right] \to \int e^u \cdot u^2 \cdot e^u\, du = \int e^{2u} \cdot u^2\, du;$

$$\begin{array}{lll} & & e^{2u} \\ u^2 & \xrightarrow{(+)} & \frac{1}{2}e^{2u} \\ 2u & \xrightarrow{(-)} & \frac{1}{4}e^{2u} \\ 2 & \xrightarrow{(+)} & \frac{1}{8}e^{2u} \\ 0 & & \end{array}$$

$$\int u^2 e^{2u}\, du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}\left(2u^2 - 2u + 1\right) + C$$
$$= \frac{z^2}{4}\left[2(\ln z)^2 - 2\ln z + 1\right] + C$$

31. $\int x \sec x^2\, dx \left[\text{Let } u = x^2,\, du = 2x\, dx \Rightarrow \frac{1}{2}du = x\, dx\right] \to \int x \sec x^2\, dx = \frac{1}{2}\int \sec u\, du = \frac{1}{2}\ln|\sec u + \tan u| + C$
$= \frac{1}{2}\ln|\sec x^2 + \tan x^2| + C$

32. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx \left[\text{Let } u = \sqrt{x},\, du = \frac{1}{2\sqrt{x}}\, dx \Rightarrow 2du = \frac{1}{\sqrt{x}}\, dx\right] \to \int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx = 2\int \cos u\, du = 2\sin u + C = 2\sin\sqrt{x} + C$

33. $\int x(\ln x)^2\, dx; \left[\begin{array}{c} u = \ln x \\ du = \frac{1}{x}\, dx \\ dx = e^u\, du \end{array}\right] \to \int e^u \cdot u^2 \cdot e^u\, du = \int e^{2u} \cdot u^2\, du;$

$$\begin{array}{lll} & & e^{2u} \\ u^2 & \xrightarrow{(+)} & \frac{1}{2}e^{2u} \\ 2u & \xrightarrow{(-)} & \frac{1}{4}e^{2u} \\ 2 & \xrightarrow{(+)} & \frac{1}{8}e^{2u} \\ 0 & & \end{array}$$

$$\int u^2 e^{2u}\, du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}\left(2u^2 - 2u + 1\right) + C$$
$$= \frac{x^2}{4}\left[2\left(\ln x\right)^2 - 2\ln x + 1\right] + C = \frac{x^2}{2}\left(\ln x\right)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C$$

34. $\int \frac{1}{x(\ln x)^2}\, dx \left[\text{Let } u = \ln x,\, du = \frac{1}{x}\, dx\right] \to \int \frac{1}{x(\ln x)^2}\, dx = \int \frac{1}{u^2}\, du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$

35. $u = \ln x,\, du = \frac{1}{x}\, dx;\, dv = \frac{1}{x^2}dx,\, v = -\frac{1}{x};$
$\int \frac{\ln x}{x^2}\, dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\, dx = -\frac{\ln x}{x} - \frac{1}{x} + C$

36. $\int \frac{(\ln x)^3}{x}\, dx \left[\text{Let } u = \ln x,\, du = \frac{1}{x}\, dx\right] \to \int \frac{(\ln x)^3}{x}\, dx = \int u^3\, du = \frac{1}{4}u^4 + C = \frac{1}{4}(\ln x)^4 + C$

37. $\int x^3 e^{x^4}\, dx \left[\text{Let } u = x^4,\ du = 4x^3 dx \Rightarrow \frac{1}{4} du = x^3\, dx\right] \to \int x^3 e^{x^4}\, dx = \frac{1}{4}\int e^u\, du = \frac{1}{4}e^u + C = \frac{1}{4}e^{x^4} + C$

38. $u = x^3,\ du = 3x^2\, dx;\ dv = x^2 e^{x^3}\, dx,\ v = \frac{1}{3}e^{x^3};$

$\int x^5 e^{x^3}\, dx = \int x^3 e^{x^3} x^2\, dx = \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}\int e^{x^3}\, 3x^2 dx = \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3} + C$

39. $u = x^2,\ du = 2x\, dx;\ dv = \sqrt{x^2+1}\, x\, dx,\ v = \frac{1}{3}\left(x^2+1\right)^{3/2};$

$\int x^3\sqrt{x^2+1}\, dx = \frac{1}{3}x^2\left(x^2+1\right)^{3/2} - \frac{1}{3}\int\left(x^2+1\right)^{3/2} 2x\, dx = \frac{1}{3}x^2\left(x^2+1\right)^{3/2} - \frac{2}{15}\left(x^2+1\right)^{5/2} + C$

40. $\int x^2 \sin x^3\, dx \left[\text{Let } u = x^3,\ du = 3x^2\, dx \Rightarrow \frac{1}{3}du = x^2\, dx\right] \to \int x^2 \sin x^3\, dx = \frac{1}{3}\int \sin u\, du = -\frac{1}{3}\cos u + C$

$= -\frac{1}{3}\cos x^3 + C$

41. $u = \sin 3x,\ du = 3\cos 3x\, dx;\ dv = \cos 2x\, dx,\ v = \frac{1}{2}\sin 2x;$

$\int \sin 3x \cos 2x\, dx = \frac{1}{2}\sin 3x \sin 2x - \frac{3}{2}\int \cos 3x \sin 2x\, dx$

$u = \cos 3x,\ du = -3\sin 3x\, dx;\ dv = \sin 2x\, dx,\ v = -\frac{1}{2}\cos 2x;$

$\int \sin 3x \cos 2x\, dx = \frac{1}{2}\sin 3x \sin 2x - \frac{3}{2}\left[-\frac{1}{2}\cos 3x \cos 2x - \frac{3}{2}\int \sin 3x \cos 2x\, dx\right]$

$= \frac{1}{2}\sin 3x \sin 2x + \frac{3}{4}\cos 3x \cos 2x + \frac{9}{4}\int \sin 3x\ \cos 2x\, dx$

$\Rightarrow -\frac{5}{4}\int \sin 3x \cos 2x\, dx = \frac{1}{2}\sin 3x \sin 2x + \frac{3}{4}\cos 3x \cos 2x$

$\Rightarrow \int \sin 3x \cos 2x\, dx = -\frac{2}{5}\sin 3x \sin 2x - \frac{3}{5}\cos 3x \cos 2x + C$

42. $u = \sin 2x,\ du = 2\cos 2x\, dx;\ dv = \cos 4x\, dx,\ v = \frac{1}{4}\sin 4x;$

$\int \sin 2x \cos 4x\, dx = \frac{1}{4}\sin 2x \sin 4x - \frac{1}{2}\int \cos 2x \sin 4x\, dx$

$u = \cos 2x,\ du = -2\sin 2x\, dx;\ dv = \sin 4x\, dx,\ v = -\frac{1}{4}\cos 4x;$

$\int \sin 2x \cos 4x\, dx = \frac{1}{4}\sin 2x \sin 4x - \frac{1}{2}\left[-\frac{1}{4}\cos 2x \cos 4x - \frac{1}{2}\int \sin 2x \cos 4x\, dx\right]$

$= \frac{1}{4}\sin 2x \sin 4x + \frac{1}{8}\cos 2x \cos 4x + \frac{1}{4}\int \sin 2x \cos 4x\, dx$

$\Rightarrow \frac{3}{4}\int \sin 2x \cos 4x\, dx = \frac{1}{4}\sin 2x \sin 4x + \frac{1}{8}\cos 2x \cos 4x$

$\Rightarrow \int \sin 2x \cos 4x\, dx = \frac{1}{3}\sin 2x \sin 4x + \frac{1}{6}\cos 2x \cos 4x + C$

43. $\int \sqrt{x}\ln x\, dx \qquad \left[\text{Let } u = \ln x,\ du = \frac{1}{x}dx,\ dv = \sqrt{x}\, dx,\ v = \frac{2}{3}x^{3/2}\right]$

$\int \sqrt{x}\ln x\, dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int \sqrt{x}\, dx$

$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C = \frac{2}{9}x^{3/2}\left(3\ln x - 2\right) + C$

44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx\left[\text{Let } u=\sqrt{x},\, du=\frac{1}{2\sqrt{x}}\,dx \Rightarrow 2du=\frac{1}{\sqrt{x}}\,dx\right] \to \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$

45. $\int \cos\sqrt{x}\,dx;\ \begin{bmatrix} y=\sqrt{x} \\ dy=\frac{1}{2\sqrt{x}}\,dx \\ dx=2y\,dy \end{bmatrix} \to \int \cos y\ 2y\,dy = \int 2y\cos y\,dy;$

$u=2y,\ du=2dy;\ dv=\cos y\,dy,\ v=\sin y;$

$\int 2y\ \cos y\,dy = 2y\sin y - \int 2\sin y\,dy = 2y\sin y + 2\cos y + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$

46. $\int \sqrt{x}\,e^{\sqrt{x}}\,dx;\ \begin{bmatrix} y=\sqrt{x} \\ dy=\frac{1}{2\sqrt{x}}\,dx \\ dx=2y\,dy \end{bmatrix} \to \int y\,e^y\ 2y\,dy = \int 2y^2\ e^y\,dy;$

e^y

$2y^2 \xrightarrow{(+)} e^y$

$4y \xrightarrow{(-)} e^y$

$4 \xrightarrow{(+)} e^y$

0

$\int 2y^2e^y\,dy = 2y^2e^y - 4ye^y + 4e^y + C = 2xe^{\sqrt{x}} - 4\sqrt{x}\,e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$

47. $\sin 2\theta$

$\theta^2 \xrightarrow{(+)} -\frac{1}{2}\cos 2\theta$

$2\theta \xrightarrow{(-)} -\frac{1}{4}\sin 2\theta$

$2 \xrightarrow{(+)} \frac{1}{8}\cos 2\theta$

0

$\int_0^{\pi/2}\theta^2\sin 2\theta\,d\theta = \left[-\frac{\theta^2}{2}\cos 2\theta + \frac{\theta}{2}\sin 2\theta + \frac{1}{4}\cos 2\theta\right]_0^{\pi/2}$

$= \left[-\frac{\pi^2}{8}\cdot(-1) + \frac{\pi}{4}\cdot 0 + \frac{1}{4}\cdot(-1)\right] - \left[0+0+\frac{1}{4}\cdot 1\right] = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2-4}{8}$

48. $\cos 2x$

$x^3 \xrightarrow{(+)} \frac{1}{2}\sin 2x$

$3x^2 \xrightarrow{(-)} -\frac{1}{4}\cos 2x$

$6x \xrightarrow{(+)} -\frac{1}{8}\sin 2x$

$6 \xrightarrow{(-)} \frac{1}{16}\cos 2x$

0

$\int_0^{\pi/2} x^3\cos 2x\,dx = \left[\frac{x^3}{2}\sin 2x + \frac{3x^2}{4}\cos 2x - \frac{3x}{4}\sin 2x - \frac{3}{8}\cos 2x\right]_0^{\pi/2}$

$= \left[\frac{\pi^3}{16}\cdot 0 + \frac{3\pi^2}{16}\cdot(-1) - \frac{3\pi}{8}\cdot 0 - \frac{3}{8}\cdot(-1)\right] - \left[0+0-0-\frac{3}{8}\cdot 1\right] = -\frac{3\pi^2}{16} + \frac{3}{4} = \frac{3\left(4-\pi^2\right)}{16}$

49. $u=\sec^{-1} t,\ du=\frac{dt}{t\sqrt{t^2-1}};\ dv=t\,dt,\ v=\frac{t^2}{2};$

$$\int_{2/\sqrt{3}}^{2} t\sec^{-1} t\,dt=\left[\frac{t^2}{2}\sec^{-1} t\right]_{2/\sqrt{3}}^{2}-\int_{2/\sqrt{3}}^{2}\left(\frac{t^2}{2}\right)\frac{dt}{t\sqrt{t^2-1}}=\left(2\cdot\frac{\pi}{3}-\frac{2}{3}\cdot\frac{\pi}{6}\right)-\int_{2/\sqrt{3}}^{2}\frac{t\,dt}{2\sqrt{t^2-1}}$$

$$=\frac{5\pi}{9}-\left[\frac{1}{2}\sqrt{t^2-1}\right]_{2/\sqrt{3}}^{2}=\frac{5\pi}{9}-\frac{1}{2}\left(\sqrt{3}-\sqrt{\frac{4}{3}-1}\right)=\frac{5\pi}{9}-\frac{1}{2}\left(\sqrt{3}-\frac{\sqrt{3}}{3}\right)=\frac{5\pi}{9}-\frac{\sqrt{3}}{3}=\frac{5\pi-3\sqrt{3}}{9}$$

50. $u=\sin^{-1}\left(x^2\right),\ du=\frac{2x\,dx}{\sqrt{1-x^4}};\ dv=2x\,dx,\ v=x^2;$

$$\int_0^{1/\sqrt{2}} 2x\sin^{-1}\left(x^2\right)dx=\left[x^2\sin^{-1}\left(x^2\right)\right]_0^{1/\sqrt{2}}-\int_0^{1/\sqrt{2}} x^2\cdot\frac{2x\,dx}{\sqrt{1-x^4}}=\left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right)+\frac{1}{2}\int_0^{1/\sqrt{2}}\left(1-x^4\right)^{-1/2}\left(4x^3\right)dx$$

$$=\frac{\pi}{12}+\left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}}=\frac{\pi}{12}+\sqrt{\frac{3}{4}}-1=\frac{\pi+6\sqrt{3}-12}{12}$$

51. $\int x\tan^{-1} x\,dx$ $\left[\text{Let } u=\tan^{-1} x,\ du=\frac{1}{1+x^2}dx,\ dv=x\,dx,\ v=\frac{x^2}{2}\right]$

$$\begin{aligned}\int x\tan^{-1} x\,dx&=\frac{1}{2}x^2\tan^{-1} x-\frac{1}{2}\int\frac{x^2}{1+x^2}\,dx\\&=\frac{1}{2}x^2\tan^{-1} x-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx\\&=\frac{1}{2}x^2\tan^{-1} x-\frac{1}{2}x+\frac{1}{2}\tan^{-1} x+C\\&=\frac{1}{2}\left(x^2+1\right)\tan^{-1} x-\frac{x}{2}+C\end{aligned}$$

52. $\int x^2\tan^{-1}\left(\frac{x}{2}\right)dx$ $\left[\text{Let } u=\tan^{-1}\frac{x}{2},\ du=\frac{1/2}{1+(x/2)^2}dx,\ dv=x^2dx,\ v=\frac{x^3}{3}\right]$

$$\begin{aligned}\int x^2\tan^{-1}\left(\frac{x}{2}\right)dx&=\frac{x^3}{3}\tan^{-1}\frac{x}{2}-\frac{1}{3}\int\frac{\frac{1}{2}x^3}{1+\frac{x^2}{4}}\,dx\\&=\frac{x^3}{3}\tan^{-1}\frac{x}{3}-\frac{1}{3}\int\left(2x-\frac{2x}{1+\frac{x^2}{4}}\right)dx\\&=\frac{x^3}{3}\tan^{-1}\frac{x}{2}-\frac{1}{3}x^2+\frac{1}{3}\int\left(\frac{2x}{1+\frac{x^2}{4}}\right)dx\end{aligned}$$

In the remaining integral, let $w=1+\frac{x^2}{4},\ dw=\frac{x}{2}.$

$$\frac{1}{3}\int\left(\frac{2x}{1+\frac{x^2}{4}}\right)dx=\frac{1}{3}\int\frac{4}{w}\,dw=\frac{4}{3}\ln|w|=\frac{4}{3}\ln\left(1+\frac{x^2}{4}\right)$$

Thus the original integral is equal to

$$\frac{x^3}{3}\tan^{-1}\frac{x}{2}-\frac{1}{3}x^2+\frac{4}{3}\ln\left(1+\frac{x^2}{4}\right)+C$$

53. (a) $u=x,\ du=dx;\ dv=\sin x\,dx,\ v=-\cos x;$

$$S_1=\int_0^{\pi}x\sin x\,dx=\left[-x\cos x\right]_0^{\pi}+\int_0^{\pi}\cos x\,dx=\pi+\left[\sin x\right]_0^{\pi}=\pi$$

(b) $S_2=-\int_0^{2\pi}x\sin x\,dx=-\left[\left[-x\cos x\right]_{\pi}^{2\pi}+\int_{\pi}^{2\pi}\cos x\,dx\right]=-\left[-3\pi+\left[\sin x\right]_{\pi}^{2\pi}\right]=3\pi$

(c) $S_3=\int_{2\pi}^{3\pi}x\sin x\,dx=\left[-x\cos x\right]_{2\pi}^{3\pi}+\int_{2\pi}^{3\pi}\cos x\,dx=5\pi+\left[\sin x\right]_{2\pi}^{3\pi}=5\pi$

(d) $S_{n+1}=(-1)^{n+1}\int_{n\pi}^{(n+1)\pi}x\sin x\,dx=(-1)^{n+1}\left[\left[-x\cos x\right]_{n\pi}^{(n+1)\pi}+\left[\sin x\right]_{n\pi}^{(n+1)\pi}\right]$

$$=(-1)^{n+1}\left[-(n+1)\pi(-1)^n+n\pi(-1)^{n+1}\right]+0=(2n+1)\pi$$

54. (a) $u=x,\ du=dx;\ dv=\cos x\,dx,\ v=\sin x;$

$$S_1=-\int_{\pi/2}^{3\pi/2}x\cos x\,dx=-\left[\left[x\sin x\right]_{\pi/2}^{3\pi/2}-\int_{\pi/2}^{3\pi/2}\sin x\,dx\right]=-\left(-\tfrac{3\pi}{2}-\tfrac{\pi}{2}\right)-\left[\cos x\right]_{\pi/2}^{3\pi/2}=2\pi$$

(b) $S_2=\int_{3\pi/2}^{5\pi/2}x\cos x\,dx=\left[x\sin x\right]_{3\pi/2}^{5\pi/2}-\int_{3\pi/2}^{5\pi/2}\sin x\,dx=\left[\tfrac{5\pi}{2}-\left(-\tfrac{3\pi}{2}\right)\right]-\left[\cos x\right]_{3\pi/2}^{5\pi/2}=4\pi$

(c) $S_3=-\int_{5\pi/2}^{7\pi/2}x\cos x\,dx=-\left[\left[x\sin x\right]_{5\pi/2}^{7\pi/2}-\int_{5\pi/2}^{7\pi/2}\sin x\,dx\right]=-\left(-\tfrac{7\pi}{2}-\tfrac{5\pi}{2}\right)-\left[\cos x\right]_{5\pi/2}^{7\pi/2}=6\pi$

(d) $S_n=(-1)^n\int_{(2n-1)\pi/2}^{(2n+1)\pi/2}x\cos x\,dx=(-1)^n\left[\left[x\sin x\right]_{(2n-1)\pi/2}^{(2n+1)\pi/2}-\int_{(2n-1)\pi/2}^{(2n+1)\pi/2}\sin x\,dx\right]$

$$=(-1)^n\left[\tfrac{(2n+1)\pi}{2}(-1)^n-\tfrac{(2n-1)\pi}{2}(-1)^{n-1}\right]-\left[\cos x\right]_{(2n-1)\pi/2}^{(2n+1)\pi/2}=\tfrac{1}{2}\left(2n\pi+\pi+2n\pi-\pi\right)=2n\pi$$

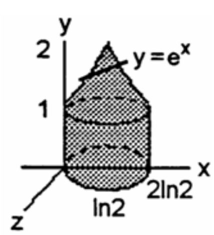
55. $V=\int_0^{\ln 2}2\pi\left(\ln 2-x\right)e^x\,dx$

$=2\pi\ln 2\int_0^{\ln 2}e^x\,dx-2\pi\int_0^{\ln 2}xe^x\,dx$

$=(2\pi\ln 2)\left[e^x\right]_0^{\ln 2}-2\pi\left(\left[xe^x\right]_0^{\ln 2}-\int_0^{\ln 2}e^x\,dx\right)$

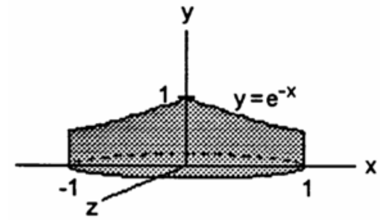
$=2\pi\ln 2-2\pi\left(2\ln 2-\left[e^x\right]_0^{\ln 2}\right)=-2\pi\ln 2+2\pi$

$=2\pi(1-\ln 2)$



56. (a) $V = \int_0^1 2\pi x e^{-x}\,dx = 2\pi\left(\left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx\right)$

$= 2\pi\left(-\frac{1}{e} + \left[-e^{-x}\right]_0^1\right) = 2\pi\left(-\frac{1}{e} - \frac{1}{e} + 1\right) = 2\pi - \frac{4\pi}{e}$

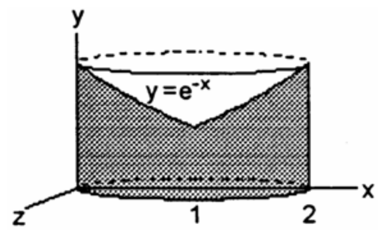


(b) $V = \int_0^1 2\pi(1-x)e^{-x}\,dx;$

$u = 1-x,\ du = -dx;\ dv = e^{-x}\,dx, v = -e^{-x};$

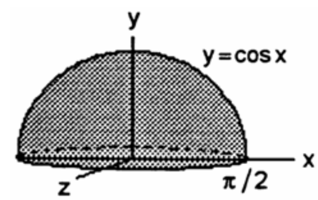
$V = 2\pi\left[\left[(1-x)\left(-e^{-x}\right)\right]_0^1 - \int_0^1 e^{-x}\,dx\right]$

$= 2\pi\left[[0-1(-1)] + \left[e^{-x}\right]_0^1\right] = 2\pi\left(1 + \frac{1}{e} - 1\right) = \frac{2\pi}{e}$



57. (a) $V = \int_0^{\pi/2} 2\pi x \cos x\,dx = 2\pi\left(\left[x\sin x\right]_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx\right)$

$= 2\pi\left(\frac{\pi}{2} + \left[\cos x\right]_0^{\pi/2}\right) = 2\pi\left(\frac{\pi}{2} + 0 - 1\right) = \pi(\pi - 2)$



(b) $V = \int_0^{\pi/2} 2\pi\left(\frac{\pi}{2} - x\right)\cos x\,dx;\quad u = \frac{\pi}{2} - x,\ du = -dx;\ dv = \cos x\,dx,\ v = \sin x;$

$V = 2\pi\left[\left(\frac{\pi}{2} - x\right)\sin x\right]_0^{\pi/2} + 2\pi\int_0^{\pi/2}\sin x\,dx = 0 + 2\pi\left[-\cos x\right]_0^{\pi/2} = 2\pi(0+1) = 2\pi$

58. (a) $V = \int_0^{\pi} 2\pi x(x\sin x)\,dx;$

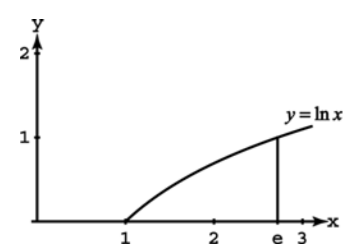
$$\begin{array}{lll} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$\Rightarrow V = 2\pi\int_0^{\pi} x^2\ \sin x\,dx = 2\pi\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\pi} = 2\pi\left(\pi^2 - 4\right)$

(b) $V = \int_0^{\pi} 2\pi(\pi - x)x\sin x\,dx = 2\pi^2\int_0^{\pi} x\sin x\,dx - 2\pi\int_0^{\pi} x^2\sin x\,dx = 2\pi^2\left[-x\cos x + \sin x\right]_0^{\pi} - \left(2\pi^3 - 8\pi\right)$
$= 8\pi$

59. (a) $A = \int_1^e \ln x\,dx = \left[x\ln x\right]_1^e - \int_1^e dx$

$= (e\ln e - 1\ln 1) - \left[x\right]_1^e = e - (e-1) = 1$



(b) $V = \int_1^e \pi(\ln x)^2\,dx = \pi\left(\left[x(\ln x)^2\right]_1^e - \int_1^e 2\ln x\,dx\right)$

$= \pi\left[\left(e(\ln e)^2 - 1(\ln 1)^2\right) - \left(\left[2x\ln x\right]_1^e - \int_1^e 2\,dx\right)\right]$

$= \pi\left[e - \left((2e\ln e - 2(1)\ln 1) - \left[2x\right]_1^e\right)\right] = \pi\left[e - \left(2e - (2e-2)\right)\right] = \pi(e-2)$

(c) $V=\int_1^e 2\pi(x+2)\ln x\,dx = 2\pi\int_1^e (x+2)\ln x\,dx = 2\pi\left(\left[\left(\frac12 x^2+2x\right)\ln x\right]_1^e - \int_1^e\left(\frac12 x+2\right)dx\right)$

$= 2\pi\left(\left(\frac12 e^2+2e\right)\ln e - \left(\frac12+2\right)\ln 1 - \left[\left(\frac14 x^2+2x\right)\right]_1^e\right) = 2\pi\left(\left(\frac12 e^2+2e\right)-\left(\left(\frac14 e^2+2e\right)-\frac94\right)\right)=\frac{\pi}{2}\left(e^2+9\right)$

(d) $M=\int_1^e \ln x\,dx=1$ (from part (a)); $\quad \bar{x}=\frac11\int_1^e x\ln x\,dx = \left[\frac12 x^2\ln x\right]_1^e - \int_1^e \frac12 x\,dx$

$=\left(\frac12 e^2\ln e - \frac12(1)^2\ln 1\right)-\left[\frac14 x^2\right]_1^e = \frac12 e^2 - \left(\frac14 e^2-\frac14(1)^2\right)=\frac14\left(e^2+1\right)$;

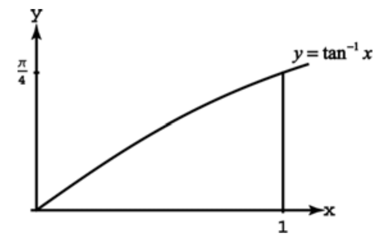
$\bar{y}=\frac11\int_1^e\frac12(\ln x)^2\,dx = \frac12\left(\left[x(\ln x)^2\right]_1^e - \int_1^e 2\ln x\,dx\right)=\frac12\left(\left(e(\ln e)^2-1\cdot(\ln 1)^2\right)-\left(\left[2x\ln x\right]_1^e-\int_1^e 2\,dx\right)\right)$

$=\frac12\left(e-\left((2e\ln e-2(1)\ln 1)-\left[2x\right]_1^e\right)\right)=\frac12(e-2e+2e-2)=\frac12(e-2)\Rightarrow(\bar x,\bar y)=\left(\frac{e^2+1}{4},\frac{e-2}{2}\right)$ is the centroid.

60. (a) $A=\int_0^1\tan^{-1}x\,dx=\left[x\tan^{-1}x\right]_0^1-\int_0^1\frac{x}{1+x^2}\,dx$

$=\left(\tan^{-1}1-0\right)-\frac12\left[\ln\left(1+x^2\right)\right]_0^1$

$=\frac{\pi}{4}-\frac12(\ln 2-\ln 1)=\frac{\pi}{4}-\frac12\ln 2$

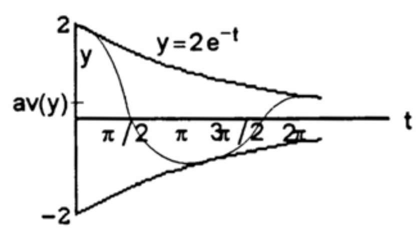


(b) $V=\int_0^1 2\pi x\tan^{-1}x\,dx=2\pi\left(\left[\frac{x^2}{2}\tan^{-1}x\right]_0^1-\frac12\int_0^1\frac{x^2}{1+x^2}\,dx\right)=2\pi\left(\frac12\tan^{-1}1-0-\frac12\int_0^1\left(1-\frac{1}{1+x^2}\right)dx\right)$

$=2\pi\left(\frac{\pi}{8}-\frac12\left[x-\tan^{-1}x\right]_0^1\right)=2\pi\left(\frac{\pi}{8}-\frac12\left(1-\tan^{-1}1-(0-0)\right)\right)=2\pi\left(\frac{\pi}{8}-\frac12\left(1-\frac{\pi}{4}\right)\right)=\frac{\pi(\pi-2)}{2}$

61. $av(y)=\frac{1}{2\pi}\int_0^{2\pi}2e^{-t}\cos t\,dt=\frac{1}{\pi}\left[e^{-t}\left(\frac{\sin t-\cos t}{2}\right)\right]_0^{2\pi}$

(see Exercise 22) $\Rightarrow av(y)=\frac{1}{2\pi}\left(1-e^{-2\pi}\right)$

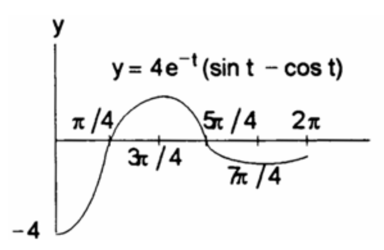


62. $av(y)=\frac{1}{2\pi}\int_0^{2\pi}4e^{-t}\left(\sin t-\cos t\right)dt$

$=\frac{2}{\pi}\int_0^{2\pi}e^{-t}\sin t\,dt-\frac{2}{\pi}\int_0^{2\pi}e^{-t}\cos t\,dt$

$=\frac{2}{\pi}\left[e^{-t}\left(\frac{-\sin t-\cos t}{2}\right)-e^{-t}\left(\frac{\sin t-\cos t}{2}\right)\right]_0^{2\pi}$

$=\frac{2}{\pi}\left[-e^{-t}\sin t\right]_0^{2\pi}=0$



63. $I=\int x^n\cos x\,dx$; $[u=x^n,\ du=nx^{n-1}\,dx;\ dv=\cos x\,dx,\ v=\sin x]$

$\Rightarrow I=x^n\sin x-\int nx^{n-1}\sin x\,dx$

64. $I = \int x^n \sin x\,dx$; $[u = x^n,\ du = nx^{n-1}\,dx;\ dv = \sin x\,dx,\ v = -\cos x]$

$\Rightarrow I = -x^n \cos x + \int nx^{n-1} \cos x\,dx$

65. $I = \int x^n e^{ax}\,dx$; $[u = x^n,\ du = nx^{n-1}\,dx;\ dv = e^{ax}\,dx,\ v = \frac{1}{a}e^{ax}]$

$\Rightarrow I = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx,\ a \neq 0$

66. $I = \int (\ln x)^n\,dx$; $[u = (\ln x)^n,\ du = \frac{n(\ln x)^{n-1}}{x}dx;\ dv = 1\,dx,\ v = x]$

$\Rightarrow I = x(\ln x)^n - \int n(\ln x)^{n-1}\,dx$

67. $u = (\ln x)^n,\ du = \frac{n}{x}(\ln x)^{n-1}dx,\ dv = x^m dx,\ v = \frac{x^{m+1}}{m+1}$

$uv = \frac{1}{m+1}x^{m+1}(\ln x)^{n-1}$ and $\int v\,du = \frac{n}{m+1}\int x^m (\ln x)^{n-1}\,dx$

68. First to show that $\int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \sin^n x\,dx$ note that $\cos x$ over the interval $[0, \pi/2]$ is the reflection of $\sin x$ over the same interval around the line $x = \pi/4$.

Each iteration of the reduction formula in Example 5 for the definite integral produces an expression like

$$\left.\frac{\left(\cos^{n-1} x\right)\left(\sin x\right)}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx$$

The evaluation on the left will be 0 as long as $n \geq 2$, and factors of the form $\frac{n}{n-1}$ accumulate in front of the integral on the right. When the initial n is even, the last iteration will have $n = 2$ and the remaining integral will be $\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{1}{2}\int_0^{\pi/2} 1\,dx = \frac{\pi}{2}\cdot\frac{1\cdot 3\cdot\cdots\cdot(n-1)}{2\cdot 4\cdot\cdots\cdot n}$. When the initial n is odd the last iteration will have $n = 3$ and the remaining integral will be $\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{2}{3}\int_0^{\pi/2} \cos x\,dx = 1\cdot\frac{2\cdot 4\cdot\cdots\cdot(n-1)}{3\cdot\cdots\cdot n}$.

69. $\int_a^b (x-a)\ f(x)\,dx$; $\left[u = x - a,\ du = dx;\ dv = f(x)\,dx,\ v = \int_b^x f(t)\,dt = -\int_x^b f(t)\,dt\right]$

$= \left[(x-a)\int_b^x f(t)\,dt\right]_a^b - \int_a^b \left(\int_b^x f(t)\,dt\right) dx = \left((b-a)\int_b^b f(t)\,dt - (a-a)\int_b^a f(t)\,dt\right) - \int_a^b \left(-\int_x^b f(t)\,dt\right) dx$

$= 0 + \int_a^b \left(\int_x^b f(t)\,dt\right) dx = \int_a^b \left(\int_x^b f(t)\,dt\right) dx$

70. $\int \sqrt{1-x^2}\,dx$; $\left[u = \sqrt{1-x^2},\ du = \frac{-x}{\sqrt{1-x^2}}\,dx;\ dv = dx,\ v = x\right]$

$= x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\,dx = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx = x\sqrt{1-x^2} - \left(\int \frac{1-x^2}{\sqrt{1-x^2}}\,dx - \int \frac{1}{\sqrt{1-x^2}}\,dx\right)$

$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$

$$\Rightarrow \int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \sqrt{1-x^2}\,dx \Rightarrow 2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx$$
$$\Rightarrow \int \sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx + C$$

71. $\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \sin y\,dy = x\sin^{-1} x + \cos y + C = x\sin^{-1} x + \cos\left(\sin^{-1} x\right) + C$

72. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \tan y\,dy = x\tan^{-1} x + \ln|\cos y| + C = x\tan^{-1} x + \ln\left|\cos\left(\tan^{-1} x\right)\right| + C$

73. $\int \sec^{-1} x\,dx = x\sec^{-1} x - \int \sec y\,dy = x\sec^{-1} x - \ln|\sec y + \tan y| + C$
$= x\sec^{-1} x - \ln\left|\sec\left(\sec^{-1} x\right) + \tan\left(\sec^{-1} x\right)\right| + C = x\sec^{-1} x - \ln\left|x + \sqrt{x^2-1}\right| + C$

74. $\int \log_2 x\,dx = x\log_2 x - \int 2^y\,dy = x\log_2 x - \frac{2y}{\ln 2} + C = x\log_2 x - \frac{x}{\ln 2} + C$

75. Yes, $\cos^{-1} x$ is the angle whose cosine is x which implies $\sin\left(\cos^{-1} x\right) = \sqrt{1-x^2}$.

76. Yes, $\tan^{-1} x$ is the angle whose tangent is x which implies $\sec\left(\tan^{-1} x\right) = \sqrt{1+x^2}$.

77. (a) $\int \sinh^{-1} x\,dx = x\sinh^{-1} x - \int \sinh y\,dy = x\sinh^{-1} x - \cosh y + C = x\sinh^{-1} x - \cosh\left(\sinh^{-1} x\right) + C$;

check: $d\left[x\sinh^{-1} x - \cosh\left(\sinh^{-1} x\right) + C\right] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \sinh\left(\sinh^{-1} x\right)\frac{1}{\sqrt{1+x^2}}\right]dx = \sinh^{-1} x\,dx$

(b) $\int \sinh^{-1} x\,dx = x\sinh^{-1} x - \int x\left(\frac{1}{\sqrt{1+x^2}}\right)dx = x\sinh^{-1} x - \frac{1}{2}\int\left(1+x^2\right)^{-1/2} 2x\,dx = x\sinh^{-1} x - \left(1+x^2\right)^{1/2} + C$

check: $d\left[x\sinh^{-1} x - \left(1+x^2\right)^{1/2} + C\right] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}\right]dx = \sinh^{-1} x\,dx$

78. (a) $\int \tanh^{-1} x\,dx = x\tanh^{-1} x - \int \tanh y\,dy = x\tanh^{-1} x - \ln|\cosh y| + C$
$= x\tanh^{-1} x - \ln\left|\cosh\left(\tanh^{-1} x\right)\right| + C$;

check: $d\left[x\tanh^{-1} x - \ln\left|\cosh\left(\tanh^{-1} x\right)\right| + C\right] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{\sinh\left(\tanh^{-1} x\right)}{\cosh\left(\tanh^{-1} x\right)}\frac{1}{1-x^2}\right]dx$
$= \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\right]dx = \tanh^{-1} x\,dx$

(b) $\int \tanh^{-1} x\,dx = x\tanh^{-1} x - \int \frac{x}{1-x^2}\,dx = \tanh^{-1} x - \frac{1}{2}\int \frac{2x}{1-x^2}\,dx = x\tanh^{-1} x + \frac{1}{2}\ln\left|1-x^2\right| + C$

check: $d\left[x\tanh^{-1} x + \frac{1}{2}\ln\left|1-x^2\right| + C\right] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\right]dx = \tanh^{-1} x\,dx$

8.3 TRIGONOMETRIC INTEGRALS

1. $\int \cos 2x\, dx = \frac{1}{2}\int \cos 2x \cdot 2dx = \frac{1}{2}\sin 2x + C$

2. $\int_0^{\pi} 3\sin\frac{x}{3}\, dx = 9\int_0^{\pi}\sin\frac{x}{3}\cdot\frac{1}{3}dx = 9\left[-\cos\frac{x}{3}\right]_0^{\pi} = 9\left(-\cos\frac{\pi}{3}+\cos 0\right) = 9\left(-\frac{1}{2}+1\right) = \frac{9}{2}$

3. $\int \cos^3 x \sin x\, dx = -\int \cos^3 x(-\sin x)\, dx = -\frac{1}{4}\cos^4 x + C$

4. $\int \sin^4 2x \cos 2x\, dx = \frac{1}{2}\int \sin^4 2x \cos 2x \cdot 2dx = \frac{1}{10}\sin^5 2x + C$

5. $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = \int\left(1-\cos^2 x\right)\sin x\, dx = \int \sin x\, dx - \int \cos^2 x \sin x\, dx = -\cos x + \frac{1}{3}\cos^3 x + C$

6. $\int \cos^3 4x\, dx = \int \cos^2 4x \cos 4x\, dx = \frac{1}{4}\int\left(1-\sin^2 4x\right)\cos 4x\cdot 4dx = \frac{1}{4}\int \cos 4x\cdot 4dx - \frac{1}{4}\int \sin^2 4x \cos 4x\cdot 4dx$
$= \frac{1}{4}\sin 4x - \frac{1}{12}\sin^3 4x + C$

7. $\int \sin^5 x\, dx = \int\left(\sin^2 x\right)^2 \sin x\, dx = \int\left(1-\cos^2 x\right)^2 \sin x\, dx = \int\left(1-2\cos^2 x+\cos^4 x\right)\sin x\, dx$
$= \int \sin x\, dx - \int 2\cos^2 x \sin x\, dx + \int \cos^4 x \sin x\, dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$

8. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)dx$ (using Exercise 7) $= \int_0^{\pi}\sin\left(\frac{x}{2}\right)dx - \int_0^{\pi} 2\cos^2\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx + \int_0^{\pi}\cos^4\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx$
$= \left[-2\cos\left(\frac{x}{2}\right) + \frac{4}{3}\cos^3\left(\frac{x}{2}\right) - \frac{2}{5}\cos^5\left(\frac{x}{2}\right)\right]_0^{\pi} = (0) - \left(-2+\frac{4}{3}-\frac{2}{5}\right) = \frac{16}{15}$

9. $\int \cos^3 x\, dx = \int\left(\cos^2 x\right)\cos x\, dx = \int\left(1-\sin^2 x\right)\cos x\, dx = \int \cos x\, dx - \int \sin^2 x \cos x\, dx = \sin x - \frac{1}{3}\sin^3 x + C$

10. $\int_0^{\pi/6} 3\cos^5 3x\, dx = \int_0^{\pi/6}\left(\cos^2 3x\right)^2 \cos 3x\cdot 3dx = \int_0^{\pi/6}\left(1-\sin^2 3x\right)^2 \cos 3x\cdot 3dx$
$= \int_0^{\pi/6}\left(1-2\sin^2 3x+\sin^4 3x\right)\cos 3x\cdot 3dx$
$= \int_0^{\pi/6}\cos 3x\cdot 3dx - 2\int_0^{\pi/6}\sin^2 3x\cos 3x\cdot 3dx + \int_0^{\pi/6}\sin^4 3x \cos 3x\cdot 3dx = \left[\sin 3x - 2\frac{\sin^3 3x}{3} + \frac{\sin^5 3x}{5}\right]_0^{\pi/6}$
$= \left(1-\frac{2}{3}+\frac{1}{5}\right) - (0) = \frac{8}{15}$

11. $\int \sin^3 x \cos^3 x\, dx = \int \sin^3 x \cos^2 x \cos x dx = \int \sin^3 x\left(1-\sin^2 x\right)\cos x\, dx = \int \sin^3 x \cos x\, dx - \int \sin^5 x \cos x\, dx$
$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$

12. $\int \cos^3 2x \sin^5 2x\, dx = \frac{1}{2}\int \cos^3 2x \sin^5 2x \cdot 2dx = \frac{1}{2}\int \cos 2x \cos^2 2x \sin^5 2x \cdot 2dx$

$= \frac{1}{2}\int \left(1-\sin^2 2x\right)\sin^5 2x \cos 2x \cdot 2dx = \frac{1}{2}\int \sin^5 2x \cos 2x \cdot 2dx - \frac{1}{2}\int \sin^7 2x \cos 2x \cdot 2dx$

$\frac{1}{12}\sin^6 2x - \frac{1}{16}\sin^8 2x + C$

13. $\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\, dx = \frac{1}{2}\int dx + \frac{1}{4}\int \cos 2x \cdot 2dx$

$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

14. $\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}\int_0^{\pi/2}(1-\cos 2x)\,dx = \frac{1}{2}\int_0^{\pi/2} dx - \frac{1}{2}\int_0^{\pi/2}\cos 2x\, dx$

$= \frac{1}{2}\int_0^{\pi/2} dx - \frac{1}{4}\int_0^{\pi/2}\cos 2x \cdot 2dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \left(\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)\right) - \left(\frac{1}{2}(0) - \frac{1}{4}\sin 2(0)\right)$

$= \left(\frac{\pi}{4} - 0\right) - (0-0) = \frac{\pi}{4}$

15. $\int_0^{\pi/2} \sin^7 y\, dy = \int_0^{\pi/2} \sin^6 y \sin y\, dy = \int_0^{\pi/2} \left(1-\cos^2 y\right)^3 \sin y\, dy$

$= \int_0^{\pi/2} \sin y\, dy - 3\int_0^{\pi/2} \cos^2 y \sin y\, dy + 3\int_0^{\pi/2}\cos^4 y \sin y\, dy - \int_0^{\pi/2}\cos^6 y \sin y\, dy$

$= \left[-\cos y + 3\frac{\cos^3 y}{3} - 3\frac{\cos^5 y}{5} + \frac{\cos^7 y}{7}\right]_0^{\pi/2} = (0) - \left(-1+1-\frac{3}{5}+\frac{1}{7}\right) = \frac{16}{35}$

16. $\int 7\cos^7 t\, dt$ (using Exercise 15) $= 7\left[\int \cos t\, dt - 3\int \sin^2 t \cos t\, dt + 3\int \sin^4 t \cos t\, dt - \int \sin^6 t \cos t\, dt\right]$

$= 7\left(\sin t - 3\frac{\sin^3 t}{3} + 3\frac{\sin^5 t}{5} - \frac{\sin^7 t}{7}\right) + C = 7\sin t - 7\sin^3 t + \frac{21}{5}\sin^5 t - \sin^7 t + C$

17. $\int_0^{\pi} 8\sin^4 x\, dx = 8\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)^2 dx = 2\int_0^{\pi}\left(1-2\cos 2x+\cos^2 2x\right)dx$

$= 2\int_0^{\pi} dx - 2\int_0^{\pi}\cos 2x \cdot 2dx + 2\int_0^{\pi}\frac{1+\cos 4x}{2}\, dx = \left[2x - 2\sin 2x\right]_0^{\pi} + \int_0^{\pi} dx + \int_0^{\pi}\cos 4x\, dx$

$= 2\pi + \left[x + \frac{1}{4}\sin 4x\right]_0^{\pi} = 2\pi + \pi = 3\pi$

18. $\int 8\cos^4 2\pi x\, dx = 8\int\left(\frac{1+\cos 4\pi x}{2}\right)^2 dx = 2\int\left(1+2\cos 4\pi x + \cos^2 4\pi x\right)dx = 2\int dx + 4\int \cos 4\pi x\, dx + 2\int \frac{1+\cos 8\pi x}{2}\, dx$

$= 3\int dx + 4\int \cos 4\pi x\, dx + \int \cos 8\pi x\, dx = 3x + \frac{1}{\pi}\sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C$

19. $\int 16\sin^2 x\cos^2 x\, dx = 16\int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx = 4\int\left(1-\cos^2 2x\right)dx = 4\int dx - 4\int\left(\frac{1+\cos 4x}{2}\right)dx$

$= 4x - 2\int dx - 2\int \cos 4x\, dx = 4x - 2x - \frac{1}{2}\sin 4x + C = 2x - \frac{1}{2}\sin 4x + C = 2x - \sin 2x \cos 2x + C$

$= 2x - 2\sin x\cos x\,(2\cos^2 x - 1) + C = 2x - 4\sin x\cos^3 x + 2\sin x\cos x + C$

20. $\int_0^{\pi} 8\sin^4 y\cos^2 y\,dy = 8\int_0^{\pi}\left(\frac{1-\cos 2y}{2}\right)^2\left(\frac{1+\cos 2y}{2}\right)dy = \int_0^{\pi} dy - \int_0^{\pi}\cos 2y\,dy - \int_0^{\pi}\cos^2 2y\,dy + \int_0^{\pi}\cos^3 2y\,dy$

$= \left[y - \frac{1}{2}\sin 2y\right]_0^{\pi} - \int_0^{\pi}\left(\frac{1+\cos 4y}{2}\right)dy + \int_0^{\pi}\left(1-\sin^2 2y\right)\cos 2y\,dy$

$= \pi - \frac{1}{2}\int_0^{\pi} dy - \frac{1}{2}\int_0^{\pi}\cos 4y\,dy + \int_0^{\pi}\cos 2y\,dy - \int_0^{\pi}\sin^2 2y\cos 2y\,dy$

$= \pi + \left[-\frac{1}{2}y - \frac{1}{8}\sin 4y + \frac{1}{2}\sin 2y - \frac{1}{2}\cdot\frac{\sin^3 2y}{3}\right]_0^{\pi} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

21. $\int 8\cos^3 2\theta\sin 2\theta\,d\theta = 8\left(-\frac{1}{2}\right)\frac{\cos^4 2\theta}{4} + C = -\cos^4 2\theta + C$

22. $\int_0^{\pi/2}\sin^2 2\theta\cos^3 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta\left(1-\sin^2 2\theta\right)\cos 2\theta\,d\theta$

$= \int_0^{\pi/2}\sin^2 2\theta\cos 2\theta\,d\theta - \int_0^{\pi/2}\sin^4 2\theta\cos 2\theta\,d\theta = \left[\frac{1}{2}\cdot\frac{\sin^3 2\theta}{3} - \frac{1}{2}\cdot\frac{\sin^5 2\theta}{5}\right]_0^{\pi/2} = 0$

23. $\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}\,dx = \int_0^{2\pi}\left|\sin\frac{x}{2}\right|dx = \int_0^{2\pi}\sin\frac{x}{2}\,dx = \left[-2\cos\frac{x}{2}\right]_0^{2\pi} = 2+2 = 4$

24. $\int_0^{\pi}\sqrt{1-\cos 2x}\,dx = \int_0^{\pi}\sqrt{2}\,|\sin x|\,dx = \int_0^{\pi}\sqrt{2}\sin x\,dx = \left[-\sqrt{2}\cos x\right]_0^{\pi} = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

25. $\int_0^{\pi}\sqrt{1-\sin^2 t}\,dt = \int_0^{\pi}|\cos t|\,dt = \int_0^{\pi/2}\cos t\,dt - \int_{\pi/2}^{\pi}\cos t\,dt = \left[\sin t\right]_0^{\pi/2} - \left[\sin t\right]_{\pi/2}^{\pi} = 1-0-0+1 = 2$

26. $\int_0^{\pi}\sqrt{1-\cos^2\theta}\,d\theta = \int_0^{\pi}|\sin\theta|\,d\theta = \int_0^{\pi}\sin\theta\,d\theta = \left[-\cos\theta\right]_0^{\pi} = 1+1 = 2$

27. $\int_{\pi/3}^{\pi/2}\frac{\sin^2 x}{\sqrt{1-\cos x}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sin^2 x}{\sqrt{1-\cos x}}\frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sin^2 x\sqrt{1+\cos x}}{\sqrt{1-\cos^2 x}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sin^2 x\sqrt{1+\cos x}}{\sqrt{\sin^2 x}}\,dx$

$= \int_{\pi/3}^{\pi/2}\sin x\sqrt{1+\cos x}\,dx = \left[-\frac{2}{3}(1+\cos x)^{3/2}\right]_{\pi/3}^{\pi/2} = -\frac{2}{3}\left(1+\cos\left(\frac{\pi}{2}\right)\right)^{3/2} + \frac{2}{3}\left(1+\cos\left(\frac{\pi}{3}\right)\right)^{3/2} = -\frac{2}{3}+\frac{2}{3}\left(\frac{3}{2}\right)^{3/2}$

$= \sqrt{\frac{3}{2}} - \frac{2}{3}$

28. $\int_0^{\pi/6}\sqrt{1+\sin x}\,dx = \int_0^{\pi/6}\frac{\sqrt{1+\sin x}}{1}\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6}\frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6}\frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6}\frac{\cos x}{\sqrt{1-\sin x}}\,dx$

$= \left[-2(1-\sin x)^{1/2}\right]_0^{\pi/6} = -2\sqrt{1-\sin\left(\frac{\pi}{6}\right)} + 2\sqrt{1-\sin 0} = -2\sqrt{\frac{1}{2}} + 2\sqrt{1} = 2-\sqrt{2}$

29. $\int_{5\pi/6}^{\pi}\frac{\cos^4 x}{\sqrt{1-\sin x}}\,dx = \int_{5\pi/6}^{\pi}\frac{\cos^4 x}{\sqrt{1-\sin x}}\frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}\,dx = \int_{5\pi/6}^{\pi}\frac{\cos^4 x\sqrt{1+\sin x}}{\sqrt{1-\sin^2 x}}\,dx = \int_{5\pi/6}^{\pi}\frac{\cos^4 x\sqrt{1+\sin x}}{\sqrt{\cos^2 x}}\,dx$

$= \int_{5\pi/6}^{\pi}\frac{\cos^4 x\sqrt{1+\sin x}}{-\cos x}\,dx = -\int_{5\pi/6}^{\pi}\cos^3 x\sqrt{1+\sin x}\,dx = -\int_{5\pi/6}^{\pi}\cos x\left(1-\sin^2 x\right)\sqrt{1+\sin x}\,dx$

$= -\int_{5\pi/6}^{\pi}\cos x\sqrt{1+\sin x}\,dx + \int_{5\pi/6}^{\pi}\cos x\sin^2 x\sqrt{1+\sin x}\,dx;$

$$\left[\text{Let } u = 1+\sin x \Rightarrow u-1 = \sin x \Rightarrow du = \cos x\, dx,\ x = \tfrac{5\pi}{6} \Rightarrow u = 1+\sin\left(\tfrac{5\pi}{6}\right) = \tfrac{3}{2},\ x = \pi \Rightarrow u = 1+\sin \pi = 1\right]$$

$$= \left[-\tfrac{2}{3}(1+\sin x)^{3/2}\right]_{5\pi/6}^{\pi} + \int_{3/2}^{1}(u-1)^2\sqrt{u}\, du = \left[-\tfrac{2}{3}(1+\sin x)^{3/2}\right]_{5\pi/6}^{\pi} + \int_{3/2}^{1}\left(u^{5/2} - 2u^{3/2} + \sqrt{u}\right) du$$

$$= \left(-\tfrac{2}{3}(1+\sin \pi)^{3/2} + \tfrac{2}{3}\left(1+\sin\left(\tfrac{5\pi}{6}\right)\right)^{3/2}\right) + \left[\tfrac{2}{7}u^{7/2} - \tfrac{4}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right]_{3/2}^{1}$$

$$= \left(-\tfrac{2}{3} + \tfrac{2}{3}\left(\tfrac{3}{2}\right)^{3/2}\right) + \left(\tfrac{2}{7} - \tfrac{4}{5} + \tfrac{2}{3}\right) - \left(\tfrac{2}{7}\left(\tfrac{3}{2}\right)^{7/2} - \tfrac{4}{5}\left(\tfrac{3}{2}\right)^{5/2} + \tfrac{2}{3}\left(\tfrac{3}{2}\right)^{3/2}\right) = \tfrac{4}{5}\left(\tfrac{3}{2}\right)^{5/2} - \tfrac{2}{7}\left(\tfrac{3}{2}\right)^{7/2} - \tfrac{18}{35}$$

30. $$\int_{\pi/2}^{7\pi/12}\sqrt{1-\sin 2x}\, dx = \int_{\pi/2}^{7\pi/12}\sqrt{\tfrac{1-\sin 2x}{1}}\,\tfrac{\sqrt{1+\sin 2x}}{\sqrt{1+\sin 2x}}\, dx = \int_{\pi/2}^{7\pi/12}\tfrac{\sqrt{1-\sin^2 2x}}{\sqrt{1+\sin 2x}}\, dx = \int_{\pi/2}^{7\pi/12}\tfrac{\sqrt{\cos^2 2x}}{\sqrt{1+\sin 2x}}\, dx$$

$$= \int_{\pi/2}^{7\pi/12}\tfrac{-\cos 2x}{\sqrt{1+\sin 2x}}\, dx = \left[-\sqrt{1+\sin 2x}\right]_{\pi/2}^{7\pi/12} = -\sqrt{1+\sin 2\left(\tfrac{7\pi}{12}\right)} + \sqrt{1+\sin 2\left(\tfrac{\pi}{2}\right)} = -\sqrt{\tfrac{1}{2}} + 1 = 1 - \tfrac{1}{\sqrt{2}} = \tfrac{\sqrt{2}-1}{\sqrt{2}}$$

31. $$\int_0^{\pi/2}\theta\sqrt{1-\cos 2\theta}\, d\theta = \int_0^{\pi/2}\theta\sqrt{2}\,|\sin\theta|\, d\theta = \sqrt{2}\int_0^{\pi/2}\theta\sin\theta\, d\theta = \sqrt{2}\left[-\theta\cos\theta + \sin\theta\right]_0^{\pi/2} = \sqrt{2}(1) = \sqrt{2}$$

32. $$\int_{-\pi}^{\pi}\left(1-\cos^2 t\right)^{3/2} dt = \int_{-\pi}^{\pi}\left(\sin^2 t\right)^{3/2} dt = \int_{-\pi}^{\pi}\left|\sin^3 t\right| dt = -\int_{-\pi}^{0}\sin^3 t\, dt + \int_0^{\pi}\sin^3 t\, dt$$

$$= -\int_{-\pi}^{0}\left(1-\cos^2 t\right)\sin t\, dt + \int_0^{\pi}\left(1-\cos^2 t\right)\sin t\, dt$$

$$= -\int_{-\pi}^{0}\sin t\, dt + \int_{-\pi}^{0}\cos^2 t\sin t\, dt + \int_0^{\pi}\sin t\, dt - \int_0^{\pi}\cos^2 t\sin t\, dt$$

$$= \left[\cos t - \tfrac{\cos^3 t}{3}\right]_{-\pi}^{0} + \left[-\cos t + \tfrac{\cos^3 t}{3}\right]_0^{\pi} = \left(1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\right) + \left(1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\right) = \tfrac{8}{3}$$

33. $$\int \sec^2 x\tan x\, dx = \int \tan x\sec^2 x\, dx = \tfrac{1}{2}\tan^2 x + C$$

34. $$\int \sec x\tan^2 x\, dx = \int \sec x\tan x\ \tan x\, dx;\ u = \tan x,\ du = \sec^2 x\, dx,\ dv = \sec x\tan x\, dx,\ v = \sec x;$$

$$= \sec x\tan x - \int \sec^3 x\, dx = \sec x\tan x - \int \sec^2 x\sec x\, dx = \sec x\tan x - \int\left(\tan^2 x + 1\right)\sec x\, dx$$

$$= \sec x\tan x - \left(\int \tan^2 x\ \sec x\, dx + \int \sec x\, dx\right) = \sec x\tan x - \ln|\sec x + \tan x| - \int \tan^2 x\sec x\, dx$$

$$\Rightarrow \int \sec x\tan^2 x\, dx = \sec x\tan x - \ln|\sec x + \tan x| - \int \tan^2 x\sec x\, dx$$

$$\Rightarrow 2\int \tan^2 x\sec x\, dx = \sec x\tan x - \ln|\sec x + \tan x|$$

$$\Rightarrow \int \tan^2 x\sec x\, dx = \tfrac{1}{2}\sec x\tan x - \tfrac{1}{2}\ln|\sec x + \tan x| + C$$

35. $$\int \sec^3 x\tan x\, dx = \int \sec^2 x\sec x\tan x\, dx = \tfrac{1}{3}\sec^3 x + C$$

36. $$\int \sec^3 x\tan^3 x\, dx = \int \sec^2 x\tan^2 x\sec x\tan x\, dx = \int \sec^2 x\left(\sec^2 x - 1\right)\sec x\tan x\, dx$$

$$= \int \sec^4 x\sec x\tan x\, dx - \int \sec^2 x\sec x\tan x\, dx = \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C$$

37. $$\int \sec^2 x\tan^2 x\, dx = \int \tan^2 x\sec^2 x\, dx = \tfrac{1}{3}\tan^3 x + C$$

38. $\int \sec^4 x \tan^2 x\, dx = \int \sec^2 x \tan^2 x \sec^2 x\, dx = \int \left(\tan^2 x + 1\right)\tan^2 x \sec^2 x\, dx$
$= \int \tan^4 x \sec^2 x\, dx + \int \tan^2 x \sec^2 x\, dx = \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$

39. $\int_{-\pi/3}^{0} 2\sec^3 x\, dx;\;\; u = \sec x,\, du = \sec x \tan x\, dx,\, dv = \sec^2 x\, dx,\, v = \tan x;$
$\int_{-\pi/3}^{0} 2\sec^3 x\, dx = \left[2\sec x \tan x\right]_{-\pi/3}^{0} - 2\int_{-\pi/3}^{0} \sec x \tan^2 x\, dx = 2\cdot 1\cdot 0 - 2\cdot 2\cdot\left(-\sqrt{3}\right) - 2\int_{-\pi/3}^{0} \sec x\left(\sec^2 x - 1\right) dx$
$= 4\sqrt{3} - 2\int_{-\pi/3}^{0} \sec^3 x\, dx + 2\int_{-\pi/3}^{0} \sec x\, dx;$
$2\int_{-\pi/3}^{0} 2\sec^3 x\, dx = 4\sqrt{3} + \left[2\ln\left|\sec x + \tan x\right|\right]_{-\pi/3}^{0} \Rightarrow 2\int_{-\pi/3}^{0} 2\sec^3 x\, dx = 4\sqrt{3} + 2\ln|1+0| - 2\ln\left|2-\sqrt{3}\right|$
$= 4\sqrt{3} - 2\ln\left(2-\sqrt{3}\right) \Rightarrow \int_{-\pi/3}^{0} 2\sec^3 x\, dx = 2\sqrt{3} - \ln\left(2-\sqrt{3}\right)$

40. $\int e^x \sec^3\left(e^x\right) dx;\, u = \sec\left(e^x\right),\, du = \sec\left(e^x\right)\tan\left(e^x\right)e^x dx,\, dv = \sec^2\left(e^x\right)e^x dx,\, v = \tan\left(e^x\right);$
$\int e^x \sec^3\left(e^x\right) dx = \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec\left(e^x\right)\tan^2\left(e^x\right)e^x\, dx$
$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec\left(e^x\right)\left(\sec^2\left(e^x\right) - 1\right)e^x dx$
$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \int \sec\left(e^x\right)e^x dx$
$2\int e^x \sec^3\left(e^x\right) dx = \sec\left(e^x\right)\tan\left(e^x\right) + \ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right| + C$
$\int e^x \sec^3\left(e^x\right) dx = \frac{1}{2}\left[\sec\left(e^x\right)\tan\left(e^x\right) + \ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right|\right] + C$

41. $\int \sec^4\theta\, d\theta = \int\left(1 + \tan^2\theta\right)\sec^2\theta\, d\theta = \int \sec^2\theta\, d\theta + \int \tan^2\theta \sec^2\theta\, d\theta = \tan\theta + \frac{1}{3}\tan^3\theta + C$
$= \tan\theta + \frac{1}{3}\tan\theta\left(\sec^2\theta - 1\right) + C = \frac{1}{3}\tan\theta\sec^2\theta + \frac{2}{3}\tan\theta + C$

42. $\int 3\sec^4(3x)\, dx = \int\left(1 + \tan^2(3x)\right)\sec^2(3x)\, 3dx = \int \sec^2(3x)\, 3dx + \int \tan^2(3x)\sec^2(3x)\, 3dx$
$= \tan(3x) + \frac{1}{3}\tan^3(3x) + C$

43. $\int_{\pi/4}^{\pi/2} \csc^4\theta\, d\theta = \int_{\pi/4}^{\pi/2}\left(1 + \cot^2\theta\right)\csc^2\theta\, d\theta = \int_{\pi/4}^{\pi/2}\csc^2\theta\, d\theta + \int_{\pi/4}^{\pi/2}\cot^2\theta \csc^2\theta\, d\theta = \left[-\cot\theta - \frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2}$
$= (0) - \left(-1 - \frac{1}{3}\right) = \frac{4}{3}$

44. $\int \sec^6 x\, dx = \int \sec^4 x \sec^2 x\, dx = \int\left(\tan^2 x + 1\right)^2 \sec^2 x\, dx = \int\left(\tan^4 x + 2\tan^2 x + 1\right)\sec^2 x\, dx$
$= \int \tan^4 x \sec^2 x\, dx + 2\int \tan^2 x \sec^2 x\, dx + \int \sec^2 x\, dx = \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

45. $\int 4\tan^3 x\, dx = 4\int\left(\sec^2 x - 1\right)\tan x\, dx = 4\int \sec^2 x \tan x\, dx - 4\int \tan x\, dx = 4\frac{\tan^2 x}{2} - 4\ln|\sec x| + C$
$= 2\tan^2 x - 4\ln|\sec x| + C = 2\tan^2 x - 2\ln\left|\sec^2 x\right| + C = 2\tan^2 x - 2\ln\left(1 + \tan^2 x\right) + C$

46. $\int_{-\pi/4}^{\pi/4} 6\tan^4 x\,dx = 6\int_{-\pi/4}^{\pi/4}\left(\sec^2 x-1\right)\tan^2 x\,dx = 6\int_{-\pi/4}^{\pi/4}\sec^2 x\tan^2 x\,dx - 6\int_{-\pi/4}^{\pi/4}\tan^2 x\,dx$

$= 6\int_{-\pi/4}^{\pi/4}\sec^2 x\tan^2 x\,dx - 6\int_{-\pi/4}^{\pi/4}\left(\sec^2 x-1\right)dx = \left[6\frac{\tan^3 x}{3}\right]_{-\pi/4}^{\pi/4} - 6\int_{-\pi/4}^{\pi/4}\sec^2 x\,dx + 6\int_{-\pi/4}^{\pi/4}dx$

$= 2\left(1-(-1)\right) - \left[6\tan x\right]_{-\pi/4}^{\pi/4} + \left[6x\right]_{-\pi/4}^{\pi/4} = 4 - 6\left(1-(-1)\right) + \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi - 8$

47. $\int \tan^5 x\,dx = \int \tan^4 x\tan x\,dx = \int\left(\sec^2 x-1\right)^2 \tan x\,dx = \int\left(\sec^4 x - 2\sec^2 x + 1\right)\tan x\,dx$

$= \int \sec^4 x\tan x\,dx - 2\int \sec^2 x\tan x\,dx + \int \tan x\,dx$

$= \int \sec^3 x\sec x\tan x\,dx - 2\int \sec x\sec x\tan x\,dx + \int \tan x\,dx = \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C$

$= \frac{1}{4}\left(\tan^2 x+1\right)^2 - \left(\tan^2 x+1\right) + \ln|\sec x| + C = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C$

48. $\int \cot^6 2x\,dx = \int \cot^4 2x\cot^2 2x\,dx = \int \cot^4 2x\left(\csc^2 2x-1\right)dx = \int \cot^4 2x\csc^2 2x\,dx - \int \cot^4 2x\,dx$

$= \int \cot^4 2x\csc^2 2x\,dx - \int \cot^2 2x\cot^2 2x\,dx = \int \cot^4 2x\csc^2 2x\,dx - \int \cot^2 2x\left(\csc^2 2x-1\right)dx$

$= \int \cot^4 2x\csc^2 2x\,dx - \int \cot^2 2x\csc^2 2x\,dx + \int \cot^2 2x\,dx$

$= \int \cot^4 2x\csc^2 2x\,dx - \int \cot^2 2x\csc^2 2x\,dx + \int\left(\csc^2 2x-1\right)dx$

$= \int \cot^4 2x\csc^2 2x\,dx - \int \cot^2 2x\csc^2 2x\,dx + \int \csc^2 2x\,dx - \int dx = -\frac{1}{10}\cot^5 2x + \frac{1}{6}\cot^3 2x - \frac{1}{2}\cot 2x - x + C$

49. $\int_{\pi/6}^{\pi/3}\cot^3 x\,dx = \int_{\pi/6}^{\pi/3}\left(\csc^2 x-1\right)\cot x\,dx = \int_{\pi/6}^{\pi/3}\csc^2 x\cot x\,dx - \int_{\pi/6}^{\pi/3}\cot x\,dx = \left[-\frac{\cot^2 x}{2} + \ln|\csc x|\right]_{\pi/6}^{\pi/3}$

$= -\frac{1}{2}\left(\frac{1}{3}-3\right) + \left(\ln\frac{2}{\sqrt{3}} - \ln 2\right) = \frac{4}{3} - \ln\sqrt{3}$

50. $\int 8\cot^4 t\,dt = 8\int\left(\csc^2 t-1\right)\cot^2 t\,dt = 8\int \csc^2 t\cot^2 t\,dt - 8\int \cot^2 t\,dt = -\frac{8}{3}\cot^3 t - 8\int\left(\csc^2 t-1\right)dt$

$= -\frac{8}{3}\cot^3 t + 8\cot t + 8t + C$

51. $\int \sin 3x\cos 2x\,dx = \frac{1}{2}\int(\sin x + \sin 5x)\,dx = -\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$

52. $\int \sin 2x\cos 3x\,dx = \frac{1}{2}\int\left(\sin(-x) + \sin 5x\right)dx = \frac{1}{2}\int\left(-\sin x + \sin 5x\right)dx = \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$

53. $\int_{-\pi}^{\pi}\sin 3x\sin 3x\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\left(\cos 0 - \cos 6x\right)dx = \frac{1}{2}\int_{-\pi}^{\pi}dx - \frac{1}{2}\int_{-\pi}^{\pi}\cos 6x\,dx = \frac{1}{2}\left[x - \frac{1}{12}\sin 6x\right]_{-\pi}^{\pi}$

$= \frac{\pi}{2} + \frac{\pi}{2} - 0 = \pi$

54. $\int_0^{\pi/2}\sin x\cos x\,dx = \frac{1}{2}\int_0^{\pi/2}\left(\sin 0 + \sin 2x\right)dx = \frac{1}{2}\int_0^{\pi/2}\sin 2x\,dx = -\frac{1}{4}\left[\cos 2x\right]_0^{\pi/2} = -\frac{1}{4}(-1-1) = \frac{1}{2}$

55. $\int \cos 3x\cos 4x\,dx = \frac{1}{2}\int\left(\cos(-x) + \cos 7x\right)dx = \frac{1}{2}\int\left(\cos x + \cos 7x\right)dx = \frac{1}{2}\sin x + \frac{1}{14}\sin 7x + C$

56. $\int_{-\pi/2}^{\pi/2}\cos 7x\cos x\,dx=\frac{1}{2}\int_{-\pi/2}^{\pi/2}(\cos 6x+\cos 8x)\,dx=\frac{1}{2}\left[\frac{1}{6}\sin 6x+\frac{1}{8}\sin 8x\right]_{-\pi/2}^{\pi/2}=0$

57. $\int\sin^2\theta\cos 3\theta\,d\theta=\int\frac{1-\cos 2\theta}{2}\cos 3\theta\,d\theta=\frac{1}{2}\int\cos 3\theta\,d\theta-\frac{1}{2}\int\cos 2\theta\cos 3\theta\,d\theta$
$=\frac{1}{2}\int\cos 3\theta\,d\theta-\frac{1}{2}\int\frac{1}{2}\left(\cos(2-3)\theta+\cos(2+3)\theta\right)d\theta=\frac{1}{2}\int\cos 3\theta\,d\theta-\frac{1}{4}\int\left(\cos(-\theta)+\cos 5\theta\right)d\theta$
$=\frac{1}{2}\int\cos 3\theta\,d\theta-\frac{1}{4}\int\cos\theta\,d\theta-\frac{1}{4}\int\cos 5\theta\,d\theta=\frac{1}{6}\sin 3\theta-\frac{1}{4}\sin\theta-\frac{1}{20}\sin 5\theta+C$

58. $\int\cos^2 2\theta\sin\theta\,d\theta=\int\left(2\cos^2\theta-1\right)^2\sin\theta\,d\theta=\int\left(4\cos^4\theta-4\cos^2\theta+1\right)\sin\theta\,d\theta$
$=\int 4\cos^4\theta\sin\theta\,d\theta-\int 4\cos^2\theta\sin\theta\,d\theta+\int\sin\theta\,d\theta=-\frac{4}{5}\cos^5\theta+\frac{4}{3}\cos^3\theta-\cos\theta+C$

59. $\int\cos^3\theta\sin 2\theta\,d\theta=\int\cos^3\theta\left(2\sin\theta\cos\theta\right)d\theta=2\int\cos^4\theta\sin\theta\,d\theta=-\frac{2}{5}\cos^5\theta+C$

60. $\int\sin^3\theta\cos 2\theta\,d\theta=\int\sin^2\theta\cos 2\theta\sin\theta\,d\theta=\int\left(1-\cos^2\theta\right)\left(2\cos^2\theta-1\right)\sin\theta\,d\theta$
$=\int\left(-2\cos^4\theta+3\cos^2\theta-1\right)\sin\theta\,d\theta=-2\int\cos^4\theta\sin\theta\,d\theta+3\int\cos^2\theta\sin\theta\,d\theta-\int\sin\theta\,d\theta$
$=\frac{2}{5}\cos^5\theta-\cos^3\theta+\cos\theta+C$

61. $\int\sin\theta\cos\theta\cos 3\theta\,d\theta=\frac{1}{2}\int 2\sin\theta\cos\theta\cos 3\theta\,d\theta=\frac{1}{2}\int\sin 2\theta\cos 3\theta\,d\theta$
$=\frac{1}{2}\int\frac{1}{2}\left(\sin(2-3)\theta+\sin(2+3)\theta\right)d\theta=\frac{1}{4}\int\left(\sin(-\theta)+\sin 5\theta\right)d\theta=\frac{1}{4}\int\left(-\sin\theta+\sin 5\theta\right)d\theta$
$=\frac{1}{4}\cos\theta-\frac{1}{20}\cos 5\theta+C$

62. $\int\sin\theta\sin 2\theta\sin 3\theta\,d\theta=\int\frac{1}{2}\left(\cos(1-2)\theta-\cos(1+2)\theta\right)\sin 3\theta\,d\theta=\frac{1}{2}\int\left(\cos(-\theta)-\cos 3\theta\right)\sin 3\theta\,d\theta$
$=\frac{1}{2}\int\sin 3\theta\cos\theta\,d\theta-\frac{1}{2}\int\sin 3\theta\cos 3\theta\,d\theta=\frac{1}{2}\int\frac{1}{2}\left(\sin(3-1)\theta+\sin(3+1)\theta\right)d\theta-\frac{1}{4}\int 2\sin 3\theta\cos 3\theta\,d\theta$
$=\frac{1}{4}\int\left(\sin 2\theta+\sin 4\theta\right)d\theta-\frac{1}{4}\int\sin 6\theta\,d\theta=-\frac{1}{8}\cos 2\theta-\frac{1}{16}\cos 4\theta+\frac{1}{24}\cos 6\theta+C$

63. $\int\frac{\sec^3 x}{\tan x}\,dx=\int\frac{\sec^2 x\sec x}{\tan x}\,dx=\int\frac{\left(\tan^2 x+1\right)\sec x}{\tan x}\,dx=\int\frac{\tan^2 x\sec x}{\tan x}\,dx+\int\frac{\sec x}{\tan x}\,dx=\int\tan x\sec x\,dx+\int\csc x\,dx$
$=\sec x-\ln|\csc x+\cot x|+C$

64. $\int\frac{\sin^3 x}{\cos^4 x}\,dx=\int\frac{\sin^2 x\sin x}{\cos^4 x}\,dx=\int\frac{\left(1-\cos^2 x\right)\sin x}{\cos^4 x}\,dx=\int\frac{\sin x}{\cos^4 x}\,dx-\int\frac{\cos^2 x\sin x}{\cos^4 x}\,dx=\int\sec^3 x\tan x\,dx-\int\sec x\tan x\,dx$
$=\int\sec^2 x\sec x\tan x\,dx-\int\sec x\tan x\,dx=\frac{1}{3}\sec^3 x-\sec x+C$

65. $\int\frac{\tan^2 x}{\csc x}\,dx=\int\frac{\sin^2 x}{\cos^2 x}\sin x\,dx=\int\frac{\left(1-\cos^2 x\right)}{\cos^2 x}\sin x\,dx=\int\frac{1}{\cos^2 x}\sin x\,dx-\int\frac{\cos^2 x}{\cos^2 x}\sin x\,dx$
$=\int\sec x\tan x\,dx-\int\sin x\,dx=\sec x+\cos x+C$

66. $\int\frac{\cot x}{\cos^2 x}\,dx=\int\frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x}\,dx=\int\frac{2}{2\sin x\cos x}\,dx=\int\frac{2}{\sin 2x}\,dx=\int\csc 2x\,2dx=-\ln|\csc 2x+\cot 2x|+C$

67. $\int x\sin^2 x\,dx = \int x\frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx\ \left[u = x,\ du = dx,\ dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x\right]$

$= \frac{1}{4}x^2 - \frac{1}{2}\left[\frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx\right] = \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$

68. $\int x\cos^3 x\,dx = \int x\cos^2 x\cos x\,dx = \int x\left(1-\sin^2 x\right)\cos x\,dx = \int x\cos x\ dx - \int x\sin^2 x\cos x\,dx;$

$\int x\ \cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x;\ [u = x,\ du = dx,\ dv = \cos x\,dx,\ v = \sin x]$

$\int x\sin^2 x\cos x\,dx = \frac{1}{3}x\sin^3 x - \int \frac{1}{3}\sin^3 x\,dx;\ \left[u = x,\ du = dx,\ dv = \sin^2 x\cos x\,dx,\ v = \frac{1}{3}\sin^3 x\right]$

$= \frac{1}{3}x\sin^3 x - \frac{1}{3}\int\left(1-\cos^2 x\right)\sin x\,dx = \frac{1}{3}x\sin^3 x - \frac{1}{3}\int \sin x\,dx + \frac{1}{3}\int \cos^2 x\sin x\,dx$

$= \frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x;$

$\Rightarrow \int x\cos x\,dx - \int x\sin^2 x\cos x\,dx = (x\sin x + \cos x) - \left(\frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x\right) + C$

$= x\sin x - \frac{1}{3}x\sin^3 x + \frac{2}{3}\cos x + \frac{1}{9}\cos^3 x + C$

69. $y = \ln(\sec x);\ y' = \frac{\sec x\tan x}{\sec x} = \tan x;\ (y')^2 = \tan^2 x;\ \int_0^{\pi/4}\sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4}|\sec x|\,dx$

$= \left[\ln|\sec x + \tan x|\right]_0^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln(0+1) = \ln\left(\sqrt{2}+1\right)$

70. $M = \int_{-\pi/4}^{\pi/4}\sec x\,dx = \left[\ln\left|\sec x + \tan x\right|\right]_{-\pi/4}^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln\left|\sqrt{2}-1\right| = \ln\frac{\sqrt{2}+1}{\sqrt{2}-1};$

$\bar{y} = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}\int_{-\pi/4}^{\pi/4}\frac{\sec^2 x}{2}\,dx = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}\left[\tan x\right]_{-\pi/4}^{\pi/4} = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}(1-(-1)) = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}} \Rightarrow (\bar{x},\bar{y}) = \left(0, \left(\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^{-1}\right)$

71. $V = \pi\int_0^{\pi}\sin^2 x\,dx = \pi\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx = \frac{\pi}{2}\int_0^{\pi}dx - \frac{\pi}{2}\int_0^{\pi}\cos 2x\,dx = \frac{\pi}{2}[x]_0^{\pi} - \frac{\pi}{4}[\sin 2x]_0^{\pi}$

$= \frac{\pi}{2}(\pi - 0) - \frac{\pi}{4}(0-0) = \frac{\pi^2}{2}$

72. $A = \int_0^{\pi}\sqrt{1+\cos 4x}\,dx = \int_0^{\pi}\sqrt{2}\left|\cos 2x\right|dx = \sqrt{2}\int_0^{\pi/4}\cos 2x\,dx - \sqrt{2}\int_{\pi/4}^{3\pi/4}\cos 2x\,dx + \sqrt{2}\int_{3\pi/4}^{\pi}\cos 2x\,dx$

$= \frac{\sqrt{2}}{2}[\sin 2x]_0^{\pi/4} - \frac{\sqrt{2}}{2}[\sin 2x]_{\pi/4}^{3\pi/4} + \frac{\sqrt{2}}{2}[\sin 2x]_{3\pi/4}^{\pi} = \frac{\sqrt{2}}{2}(1-0) - \frac{\sqrt{2}}{2}(-1-1) + \frac{\sqrt{2}}{2}(0+1) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

73. $M = \int_0^{2\pi}(x + \cos x)\,dx = \left[\frac{1}{2}x^2 + \sin x\right]_0^{2\pi} = \left(\frac{1}{2}(2\pi)^2 + \sin(2\pi)\right) - \left(\frac{1}{2}(0)^2 + \sin(0)\right) = 2\pi^2;$

$\bar{x} = \frac{1}{2\pi^2}\int_0^{2\pi}x(x+\cos x)\,dx = \frac{1}{2\pi^2}\int_0^{2\pi}\left(x^2 + x\cos x\right)dx = \frac{1}{2\pi^2}\int_0^{2\pi}x^2dx + \frac{1}{2\pi^2}\int_0^{2\pi}x\cos x\,dx$

$[u = x,\ du = dx,\ dv = \cos x\,dx,\ v = \sin x]$

$= \frac{1}{6\pi^2}\left[x^3\right]_0^{2\pi} + \frac{1}{2\pi^2}\left([x\sin x]_0^{2\pi} - \int_0^{2\pi}\sin x\,dx\right) = \frac{1}{6\pi^2}\left(8\pi^3 - 0\right) + \frac{1}{2\pi^2}\left(2\pi\sin 2\pi - 0 - \int_0^{2\pi}\sin x\,dx\right)$

$= \frac{4\pi}{3} + \frac{1}{2\pi^2}[\cos x]_0^{2\pi} = \frac{4\pi}{3} + \frac{1}{2\pi^2}(\cos 2\pi - \cos 0) = \frac{4\pi}{3} + 0 = \frac{4\pi}{3};\ \ \bar{y} = \frac{1}{2\pi^2}\int_0^{2\pi}\frac{1}{2}(x+\cos x)^2\,dx$

$= \frac{1}{4\pi^2}\int_0^{2\pi}\left(x^2 + 2x\cos x + \cos^2 x\right)dx = \frac{1}{4\pi^2}\int_0^{2\pi}x^2dx + \frac{1}{2\pi^2}\int_0^{2\pi}x\cos x\,dx + \frac{1}{4\pi^2}\int_0^{2\pi}\cos^2 x\,dx$

$= \frac{1}{12\pi^2}\left[x^3\right]_0^{2\pi} + \frac{1}{2\pi^2}\left[x \sin x + \cos x\right]_0^{2\pi} + \frac{1}{4\pi^2}\int_0^{2\pi} \frac{\cos 2x+1}{2}\, dx = \frac{2\pi}{3} + 0 + \frac{1}{8\pi^2}\int_0^{2\pi} \cos 2x\, dx + \frac{1}{8\pi^2}\int_0^{2\pi} dx$

$= \frac{2\pi}{3} + \frac{1}{16\pi^2}\left[\sin 2x\right]_0^{2\pi} + \frac{1}{8\pi^2}\left[x\right]_0^{2\pi} = \frac{2\pi}{3} + 0 + \frac{1}{4\pi} = \frac{8\pi^2+3}{12\pi} \Rightarrow$ The centroid is $\left(\frac{4\pi}{3}, \frac{8\pi^2+3}{12\pi}\right)$.

74. $V = \int_0^{\pi/3} \pi\left(\sin x + \sec x\right)^2 dx = \pi\int_0^{\pi/3}\left(\sin^2 x + 2\sin x \sec x + \sec^2 x\right) dx$

$= \pi\int_0^{\pi/3} \sin^2 x\, dx + \pi\int_0^{\pi/3} 2\tan x\, dx + \pi\int_0^{\pi/3} \sec^2 x\, dx = \pi\int_0^{\pi/3} \frac{1-\cos 2x}{2}\, dx + 2\pi\left[\ln|\sec x|\right]_0^{\pi/3} + \pi\left[\tan x\right]_0^{\pi/3}$

$= \frac{\pi}{2}\int_0^{\pi/3} dx - \frac{\pi}{2}\int_0^{\pi/3} \cos 2x\, dx + 2\pi\left(\ln\left|\sec\frac{\pi}{3}\right| - \ln\left|\sec 0\right|\right) + \pi\left(\tan\frac{\pi}{3} - \tan 0\right)$

$= \frac{\pi}{2}\left[x\right]_0^{\pi/3} - \frac{\pi}{4}\left[\sin 2x\right]_0^{\pi/3} + 2\pi \ln 2 + \pi\sqrt{3} = \frac{\pi}{2}\left(\frac{\pi}{3} - 0\right) - \frac{\pi}{4}\left(\sin 2\left(\frac{\pi}{3}\right) - \sin 2(0)\right) + 2\pi \ln 2 + \pi\sqrt{3}$

$= \frac{\pi^2}{6} - \frac{\pi\sqrt{3}}{8} + 2\pi \ln 2 + \pi\sqrt{3} = \frac{\pi\left(4\pi + 21\sqrt{3} - 48\ln 2\right)}{24}$

8.4 TRIGONOMETRIC SUBSTITUTIONS

1. $x = 3\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \frac{3\,d\theta}{\cos^2\theta},\ 9 + x^2 = 9\left(1 + \tan^2\theta\right) = 9\sec^2\theta \Rightarrow \frac{1}{\sqrt{9+x^2}} = \frac{1}{3|\sec\theta|} = \frac{|\cos\theta|}{3} = \frac{\cos\theta}{3}$;

because $\cos\theta > 0$ when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$;

$\int \frac{dx}{\sqrt{9+x^2}} = 3\int \frac{\cos\theta\, d\theta}{3\cos^2\theta} = \int \frac{d\theta}{\cos\theta} = \ln|\sec\theta + \tan\theta| + C' = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C' = \ln\left|\sqrt{9+x^2} + x\right| + C$

2. $\int \frac{3\,dx}{\sqrt{1+9x^2}}; [3x = u, 3\,dx = du] \to \int \frac{du}{\sqrt{1+u^2}}; u = \tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}, du = \frac{dt}{\cos^2 t}, \sqrt{1+u^2} = |\sec t| = \sec t$;

$\int \frac{du}{\sqrt{1+u^2}} = \int \frac{dt}{\cos^2 t\,(\sec t)} = \int \sec t\, dt = \ln|\sec t + \tan t| + C = \ln\left|\sqrt{u^2+1} + u\right| + C = \ln\left|\sqrt{1+9x^2} + 3x\right| + C$

3. $\int_{-2}^{2} \frac{dx}{\left(4+x^2\right)} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2} = \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}(-1) = \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) - \left(\frac{1}{2}\right)\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$

4. $\int_0^2 \frac{dx}{8+2x^2} = \frac{1}{2}\int_0^2 \frac{dx}{4+x^2} = \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 = \frac{1}{2}\left(\frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{16}$

5. $\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{3/2} = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}$

6. $\int_0^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}; \left[t = 2x, dt = 2\,dx\right] \to \int_0^{1/\sqrt{2}} \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1}t\right]_0^{1/\sqrt{2}} = \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

7. $t = 5\sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dt = 5\cos\theta\, d\theta, \sqrt{25-t^2} = 5\cos\theta$;

$\int \sqrt{25-t^2}\, dt = \int\left(5\cos\theta\right)\left(5\cos\theta\right) d\theta = 25\int \cos^2\theta\, d\theta = 25\int \frac{1+\cos 2\theta}{2}\, d\theta = 25\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C$

$= \frac{25}{2}\left(\theta + \sin\theta\cos\theta\right) + C = \frac{25}{2}\left[\sin^{-1}\left(\frac{t}{5}\right) + \left(\frac{t}{5}\right)\left(\frac{\sqrt{25-t^2}}{5}\right)\right] + C = \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25-t^2}}{2} + C$

8. $t=\frac{1}{3}\sin\theta, -\frac{\pi}{2}<\theta<\frac{\pi}{2}, dt=\frac{1}{3}\cos\theta\,d\theta, \sqrt{1-9t^2}=\cos\theta;$

$\int\sqrt{1-9t^2}\,dt=\frac{1}{3}\int(\cos\theta)(\cos\theta)\,d\theta=\frac{1}{3}\int\cos^2\theta\,d\theta=\frac{1}{6}(\theta+\sin\theta\cos\theta)+C=\frac{1}{6}\left[\sin^{-1}(3t)+3t\sqrt{1-9t^2}\right]+C$

9. $x=\frac{7}{2}\sec\theta, 0<\theta<\frac{\pi}{2}, dx=\frac{7}{2}\sec\theta\tan\theta\,d\theta, \sqrt{4x^2-49}=\sqrt{49\sec^2\theta-49}=7\tan\theta;$

$\int\frac{dx}{\sqrt{4x^2-49}}=\int\frac{\left(\frac{7}{2}\sec\theta\tan\theta\right)d\theta}{7\tan\theta}=\frac{1}{2}\int\sec\theta\,d\theta=\frac{1}{2}\ln|\sec\theta+\tan\theta|+C=\frac{1}{2}\ln\left|\frac{2x}{7}+\frac{\sqrt{4x^2-49}}{7}\right|+C$

10. $x=\frac{3}{5}\sec\theta, 0<\theta<\frac{\pi}{2}, dx=\frac{3}{5}\sec\theta\tan\theta\,d\theta, \sqrt{25x^2-9}=\sqrt{9\sec^2\theta-9}=3\tan\theta;$

$\int\frac{5\,dx}{\sqrt{25x^2-9}}=\int\frac{5\left(\frac{3}{5}\sec\theta\tan\theta\right)d\theta}{3\tan\theta}=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C=\ln\left|\frac{5x}{3}+\frac{\sqrt{25x^2-9}}{3}\right|+C$

11. $y=7\sec\theta, 0<\theta<\frac{\pi}{2}, dy=7\sec\theta\tan\theta\,d\theta, \sqrt{y^2-49}=7\tan\theta;$

$\int\frac{\sqrt{y^2-49}}{y}\,dy=\int\frac{(7\tan\theta)(7\sec\theta\tan\theta)\,d\theta}{7\sec\theta}=7\int\tan^2\theta\,d\theta=7\int\left(\sec^2\theta-1\right)d\theta=7(\tan\theta-\theta)+C$

$=7\left[\frac{\sqrt{y^2-49}}{7}-\sec^{-1}\left(\frac{y}{7}\right)\right]+C$

12. $y=5\sec\theta, 0<\theta<\frac{\pi}{2}, dy=5\sec\theta\tan\theta\,d\theta, \sqrt{y^2-25}=5\tan\theta;$

$\int\frac{\sqrt{y^2-25}}{y^3}\,dy=\int\frac{(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta}{125\sec^3\theta}=\frac{1}{5}\int\tan^2\theta\cos^2\theta\,d\theta=\frac{1}{5}\int\sin^2\theta\,d\theta=\frac{1}{10}\int(1-\cos 2\theta)\,d\theta$

$=\frac{1}{10}(\theta-\sin\theta\cos\theta)+C=\frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right)-\left(\frac{\sqrt{y^2-25}}{y}\right)\left(\frac{5}{y}\right)\right]+C=\left[\frac{\sec^{-1}\left(\frac{y}{5}\right)}{10}-\frac{\sqrt{y^2-25}}{2y^2}\right]+C$

13. $x=\sec\theta, 0<\theta<\frac{\pi}{2}, dx=\sec\theta\tan\theta\,d\theta, \sqrt{x^2-1}=\tan\theta;$

$\int\frac{dx}{x^2\sqrt{x^2-1}}=\int\frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\tan\theta}=\int\frac{d\theta}{\sec\theta}=\sin\theta+C=\frac{\sqrt{x^2-1}}{x}+C$

14. $x=\sec\theta, 0<\theta<\frac{\pi}{2}, dx=\sec\theta\tan\theta\,d\theta, \sqrt{x^2-1}=\tan\theta;$

$\int\frac{2\,dx}{x^3\sqrt{x^2-1}}=\int\frac{2\tan\theta\sec\theta\,d\theta}{\sec^3\theta\tan\theta}=2\int\cos^2\theta\,d\theta=2\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta=\theta+\sin\theta\cos\theta+C$

$=\theta+\tan\theta\cos^2\theta+C=\sec^{-1}x+\sqrt{x^2-1}\left(\frac{1}{x}\right)^2+C=\sec^{-1}x+\frac{\sqrt{x^2-1}}{x^2}+C$

15. $u=9-x^2\Rightarrow du=-2x\,dx\Rightarrow-\frac{1}{2}du=x\,dx;$

$\int\frac{x\,dx}{\sqrt{9-x^2}}=-\frac{1}{2}\int\frac{1}{\sqrt{u}}\,du=-\sqrt{u}+C=-\sqrt{9-x^2}+C$

16. $x = 2\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = 2\sec^2\theta\, d\theta, 4 + x^2 = 4\sec^2\theta;$

$$\int \frac{x^2 dx}{4+x^2} = \int \frac{(4\tan^2\theta)(2\sec^2\theta)d\theta}{4\sec^2\theta} = \int 2\tan^2\theta\, d\theta = 2\left(\int \sec^2\theta - 1\right)d\theta = 2\int \sec^2\theta\, d\theta - 2\int d\theta$$

$$= 2\tan\theta - 2\theta + C = x - 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

17. $x = 2\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \frac{2\,d\theta}{\cos^2\theta}, \sqrt{x^2+4} = \frac{2}{\cos\theta};$

$$\int \frac{x^3 dx}{\sqrt{x^2+4}} = \int \frac{(8\tan^3\theta)(\cos\theta)\,d\theta}{\cos^2\theta} = 8\int \frac{\sin^3\theta\, d\theta}{\cos^4\theta} = 8\int \frac{(\cos^2\theta - 1)(-\sin\theta)\,d\theta}{\cos^4\theta}; \quad [t = \cos\theta, dt = -\sin\theta\, d\theta]$$

$$\to 8\int \frac{t^2-1}{t^4}\,dt = 8\int \left(\frac{1}{t^2} - \frac{1}{t^4}\right)dt = 8\left(-\frac{1}{t} + \frac{1}{3t^3}\right) + C = 8\left(-\sec\theta + \frac{\sec^3\theta}{3}\right) + C = 8\left(-\frac{\sqrt{x^2+4}}{2} + \frac{(x^2+4)^{3/2}}{8\cdot 3}\right) + C$$

$$= \frac{1}{3}\left(x^2+4\right)^{3/2} - 4\sqrt{x^2+4} + C = \frac{1}{3}\left(x^2-8\right)\sqrt{x^2+4} + C$$

18. $x = \tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \sec^2\theta\, d\theta, \sqrt{x^2+1} = \sec\theta;$

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sec\theta} = \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \frac{-\sqrt{x^2+1}}{x} + C$$

19. $w = 2\sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dw = 2\cos\theta\, d\theta, \sqrt{4-w^2} = 2\cos\theta;$

$$\int \frac{8\,dw}{w^2\sqrt{4-w^2}} = \int \frac{8\cdot 2\cos\theta\, d\theta}{4\sin^2\theta\cdot 2\cos\theta} = 2\int \frac{d\theta}{\sin^2\theta} = -2\cot\theta + C = \frac{-2\sqrt{4-w^2}}{w} + C$$

20. $w = 3\sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dw = 3\cos\theta\, d\theta, \sqrt{9-w^2} = 3\cos\theta;$

$$\int \frac{\sqrt{9-w^2}}{w^2}\,dw = \int \frac{3\cos\theta\cdot 3\cos\theta\, d\theta}{9\sin^2\theta} = \int \cot^2\theta\, d\theta = \int \left(\frac{1-\sin^2\theta}{\sin^2\theta}\right)d\theta = \int \left(\csc^2\theta - 1\right)d\theta$$

$$= -\cot\theta - \theta + C = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$$

21. $\int \sqrt{\frac{x+1}{1-x}}\,dx$ Multiply the integrand by $\sqrt{\frac{1+x}{1+x}}$.

$$\int \sqrt{\frac{x+1}{1-x}}\,dx = \int \frac{x+1}{\sqrt{1-x^2}}\,dx \text{ where } -1 < x < 1$$

$x = \sin\theta, dx = \cos\theta\, d\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ so that $\cos\theta > 0$ and $\sqrt{1-x^2} = \cos\theta$.

$$\int \frac{x+1}{\sqrt{1-x^2}}\,dx = \int \frac{\sin\theta + 1}{\cos\theta}\cos\theta\, d\theta$$

$$= \int (\sin\theta + 1)\,d\theta = \theta - \cos\theta + C$$

$$= \sin^{-1} x - \sqrt{1-x^2} + C$$

22. $u = x^2 - 4 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$;

$\int x\sqrt{x^2-4}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}\left(x^2-4\right)^{3/2} + C$

23. $x = \sin\theta,\ 0 \le \theta \le \frac{\pi}{3},\ dx = \cos\theta\,d\theta,\ \left(1-x^2\right)^{3/2} = \cos^3\theta$;

$\int_0^{\sqrt{3}/2} \frac{4x^2dx}{\left(1-x^2\right)^{3/2}} = \int_0^{\pi/3} \frac{4\sin^2\theta\cos\theta\,d\theta}{\cos^3\theta} = 4\int_0^{\pi/3}\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)d\theta = 4\int_0^{\pi/3}\left(\sec^2\theta - 1\right)d\theta = 4\left[\tan\theta - \theta\right]_0^{\pi/3} = 4\sqrt{3} - \frac{4\pi}{3}$

24. $x = 2\sin\theta,\ 0 \le \theta \le \frac{\pi}{6},\ dx = 2\cos\theta\,d\theta,\ \left(4-x^2\right)^{3/2} = 8\cos^3\theta$;

$\int_0^1 \frac{dx}{\left(4-x^2\right)^{3/2}} = \int_0^{\pi/6} \frac{2\cos\theta\,d\theta}{8\cos^3\theta} = \frac{1}{4}\int_0^{\pi/6}\frac{d\theta}{\cos^2\theta} = \frac{1}{4}\left[\tan\theta\right]_0^{\pi/6} = \frac{\sqrt{3}}{12} = \frac{1}{4\sqrt{3}}$

25. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ \left(x^2-1\right)^{3/2} = \tan^3\theta$;

$\int \frac{dx}{\left(x^2-1\right)^{3/2}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\tan^3\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = -\frac{x}{\sqrt{x^2-1}} + C$

26. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ \left(x^2-1\right)^{5/2} = \tan^5\theta$;

$\int \frac{x^2dx}{\left(x^2-1\right)^{5/2}} = \int \frac{\sec^2\theta\cdot\sec\theta\tan\theta\,d\theta}{\tan^5\theta} = \int \frac{\cos\theta}{\sin^4\theta}\,d\theta = -\frac{1}{3\sin^3\theta} + C = -\frac{x^3}{3\left(x^2-1\right)^{3/2}} + C$

27. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ \left(1-x^2\right)^{3/2} = \cos^3\theta$;

$\int \frac{\left(1-x^2\right)^{3/2}dx}{x^6} = \int \frac{\cos^3\theta\cdot\cos\theta\,d\theta}{\sin^6\theta} = \int \cot^4\theta\csc^2\theta\,d\theta = -\frac{\cot^5\theta}{5} + C = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$

28. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ \left(1-x^2\right)^{1/2} = \cos\theta$;

$\int \frac{\left(1-x^2\right)^{1/2}dx}{x^4} = \int \frac{\cos\theta\cdot\cos\theta\,d\theta}{\sin^4\theta} = \int \cot^2\theta\csc^2\theta\,d\theta = -\frac{\cot^3\theta}{3} + C = -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C$

29. $x = \frac{1}{2}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{1}{2}\sec^2\theta\,d\theta,\ \left(4x^2+1\right)^2 = \sec^4\theta$;

$\int \frac{8dx}{\left(4x^2+1\right)^2} = \int \frac{8\left(\frac{1}{2}\sec^2\theta\right)d\theta}{\sec^4\theta} = 4\int \cos^2\theta\,d\theta = 2\left(\theta + \sin\theta\cos\theta\right) + C = 2\tan^{-1}2x + \frac{4x}{4x^2+1} + C$

30. $t = \frac{1}{3}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\sec^2\theta\,d\theta,\ 9t^2+1 = \sec^2\theta$;

$\int \frac{6\,dt}{\left(9t^2+1\right)^2} = \int \frac{6\left(\frac{1}{3}\sec^2\theta\right)d\theta}{\sec^4\theta} = 2\int \cos^2\theta\,d\theta = \theta + \sin\theta\cos\theta + C = \tan^{-1}3t + \frac{3t}{9t^2+1} + C$

31. $u = x^2 - 1 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx;$

$\int \frac{x^3}{x^2-1}dx = \int \left(x + \frac{x}{x^2-1}\right)dx = \int x\,dx + \int \frac{x}{x^2-1}dx = \frac{1}{2}x^2 + \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}x^2 + \frac{1}{2}\ln|u| + C = \frac{1}{2}x^2 + \frac{1}{2}\ln\left|x^2-1\right| + C$

32. $u = 25 + 4x^2 \Rightarrow du = 8x\,dx \Rightarrow \frac{1}{8}du = x\,dx;$

$\int \frac{x}{25+4x^2}dx = \frac{1}{8}\int \frac{1}{u}du = \frac{1}{8}\ln|u| + C = \frac{1}{8}\ln\left(25+4x^2\right) + C$

33. $v = \sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dv = \cos\theta\,d\theta, \left(1-v^2\right)^{5/2} = \cos^5\theta;$

$\int \frac{v^2dv}{\left(1-v^2\right)^{5/2}} = \int \frac{\sin^2\theta\cos\theta\,d\theta}{\cos^5\theta} = \int \tan^2\theta\sec^2\theta\,d\theta = \frac{\tan^3\theta}{3} + C = \frac{1}{3}\left(\frac{v}{\sqrt{1-v^2}}\right)^3 + C$

34. $r = \sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}; dr = \cos\theta\,d\theta, \left(1-r^2\right)^{5/2} = \cos^5\theta;$

$\int \frac{\left(1-r^2\right)^{5/2}dr}{r^8} = \int \frac{\cos^5\theta\cdot\cos\theta\,d\theta}{\sin^8\theta} = \int \cot^6\theta\csc^2\theta\,d\theta = -\frac{\cot^7\theta}{7} + C = -\frac{1}{7}\left[\frac{\sqrt{1-r^2}}{r}\right]^7 + C$

35. Let $e^t = 3\tan\theta, t = \ln(3\tan\theta), \tan^{-1}\left(\frac{1}{3}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right), dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta, \sqrt{e^{2t}+9} = \sqrt{9\tan^2\theta+9} = 3\sec\theta;$

$\int_0^{\ln 4}\frac{e^t\,dt}{\sqrt{e^{2t}+9}} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)}\frac{3\tan\theta\cdot\sec^2\theta\,d\theta}{\tan\theta\cdot3\sec\theta} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)}\sec\theta\,d\theta = \left[\ln\left|\sec\theta+\tan\theta\right|\right]_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)}$

$= \ln\left(\frac{5}{3}+\frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3}+\frac{1}{3}\right) = \ln 9 - \ln\left(1+\sqrt{10}\right)$

36. Let $e^t = \tan\theta, t = \ln(\tan\theta), \tan^{-1}\left(\frac{3}{4}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right), dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta, 1+e^{2t} = 1+\tan^2\theta = \sec^2\theta;$

$\int_{\ln(3/4)}^{\ln(4/3)}\frac{e^t dt}{\left(1+e^{2t}\right)^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)}\frac{(\tan\theta)\left(\frac{\sec^2\theta}{\tan\theta}\right)d\theta}{\sec^3\theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)}\cos\theta\,d\theta = \left[\sin\theta\right]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$

37. $\int_{1/12}^{1/4}\frac{2\,dt}{\sqrt{t}+4t\sqrt{t}};\ \left[u = 2\sqrt{t}, du = \frac{1}{\sqrt{t}}\,dt\right] \to \int_{1/\sqrt{3}}^{1}\frac{2\,du}{1+u^2}; u = \tan\theta, \frac{\pi}{6} \le \theta \le \frac{\pi}{4}, du = \sec^2\theta\,d\theta, 1+u^2 = \sec^2\theta;$

$\int_{1/\sqrt{3}}^{1}\frac{2\,du}{1+u^2} = \int_{\pi/6}^{\pi/4}\frac{2\sec^2\theta\,d\theta}{\sec^2\theta} = \left[2\theta\right]_{\pi/6}^{\pi/4} = 2\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{\pi}{6}$

38. $y = e^{\tan\theta}, 0 \le \theta \le \frac{\pi}{4}, dy = e^{\tan\theta}\sec^2\theta\,d\theta, \sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2\theta} = \sec\theta;$

$\int_1^e\frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4}\frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}\,d\theta = \int_0^{\pi/4}\sec\theta\,d\theta = \left[\ln\left|\sec\theta+\tan\theta\right|\right]_0^{\pi/4} = \ln\left(1+\sqrt{2}\right)$

39. $x = \sec\theta, 0 < \theta < \frac{\pi}{2}, dx = \sec\theta\tan\theta\,d\theta, \sqrt{x^2-1} = \sqrt{\sec^2\theta-1} = \tan\theta;$

$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec\theta\tan\theta} = \theta + C = \sec^{-1}x + C$

40. $x=\tan\theta,\ dx=\sec^2\theta\,d\theta,\ 1+x^2=\sec^2\theta;$

$\int\frac{dx}{x^2+1}=\int\frac{\sec^2\theta\,d\theta}{\sec^2\theta}=\theta+C=\tan^{-1}x+C$

41. $x=\sec\theta,\ dx=\sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-1}=\sqrt{\sec^2\theta-1}=\tan\theta;$

$\int\frac{x\,dx}{\sqrt{x^2-1}}=\int\frac{\sec\theta\cdot\sec\theta\tan\theta\,d\theta}{\tan\theta}=\int\sec^2\theta\,d\theta=\tan\theta+C=\sqrt{x^2-1}+C$

42. $x=\sin\theta,\ dx=\cos\theta\,d\theta,\ -\frac{\pi}{2}<\theta<\frac{\pi}{2},\ \sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\cos\theta;$

$\int\frac{dx}{\sqrt{1-x^2}}=\int\frac{\cos\theta\,d\theta}{\cos\theta}=\theta+C=\sin^{-1}x+C$

43. Let $x^2=\tan\theta,\ 0\le\theta<\frac{\pi}{2},\ 2x\,dx=\sec^2\theta\,d\theta\Rightarrow x\,dx=\frac{1}{2}\sec^2\theta\,d\theta;\ \sqrt{1+x^4}=\sqrt{1+\tan^2\theta}=\sec\theta$

$\int\frac{x}{\sqrt{1+x^4}}dx=\frac{1}{2}\int\frac{\sec^2\theta}{\sec\theta}d\theta=\frac{1}{2}\int\sec\theta\,d\theta=\frac{1}{2}\ln|\sec\theta+\tan\theta|+C=\frac{1}{2}\ln\left|\sqrt{1+x^4}+x^2\right|+C$

44. Let $\ln x=\sin\theta,\ -\frac{\pi}{2}\le\theta<0$ or $0<\theta\le\frac{\pi}{2},\ \frac{1}{x}dx=\cos\theta\,d\theta,\ \sqrt{1-(\ln x)^2}=\cos\theta$

$\int\frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx=\int\frac{\cos^2\theta}{\sin\theta}d\theta=\int\frac{1-\sin^2\theta}{\sin\theta}d\theta=\int\csc\theta\,d\theta-\int\sin\theta\,d\theta=-\ln|\csc\theta+\cot\theta|+\cos\theta+C$

$=-\ln\left|\frac{1}{\ln x}+\frac{\sqrt{1-(\ln x)^2}}{\ln x}\right|+\sqrt{1-(\ln x)^2}+C=-\ln\left|\frac{1+\sqrt{1-(\ln x)^2}}{\ln x}\right|+\sqrt{1-(\ln x)^2}+C$

45. Let $u=\sqrt{x}\Rightarrow x=u^2\Rightarrow dx=2u\,du\Rightarrow\int\sqrt{\frac{4-x}{x}}dx=\int\sqrt{\frac{4-u^2}{u^2}}\,2u\,du=2\int\sqrt{4-u^2}\,du;$

$u=2\sin\theta,\ du=2\cos\theta\,d\theta,\ 0<\theta\le\frac{\pi}{2},\ \sqrt{4-u^2}=2\cos\theta$

$2\int\sqrt{4-u^2}\,du=2\int(2\cos\theta)(2\cos\theta)\,d\theta=8\int\cos^2\theta\,d\theta=8\int\frac{1+\cos2\theta}{2}\,d\theta=4\int d\theta+4\int\cos2\theta\,d\theta$

$=4\theta+2\sin2\theta+C=4\theta+4\sin\theta\cos\theta+C=4\sin^{-1}\left(\frac{u}{2}\right)+4\left(\frac{u}{2}\right)\left(\frac{\sqrt{4-u^2}}{2}\right)+C=4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)+\sqrt{x}\sqrt{4-x}+C$

$=4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)+\sqrt{4x-x^2}+C$

46. Let $u=x^{3/2}\Rightarrow x=u^{2/3}\Rightarrow dx=\frac{2}{3}u^{-1/3}du$

$\int\sqrt{\frac{x}{1-x^3}}\,dx=\int\sqrt{\frac{u^{2/3}}{1-\left(u^{2/3}\right)^3}}\left(\frac{2}{3}u^{-1/3}\right)du=\int\frac{u^{1/3}}{\sqrt{1-u^2}}\left(\frac{2}{3u^{1/3}}\right)du=\frac{2}{3}\int\frac{1}{\sqrt{1-u^2}}\,du=\frac{2}{3}\sin^{-1}u+C=\frac{2}{3}\sin^{-1}\left(x^{3/2}\right)+C$

47. Let $u=\sqrt{x}\Rightarrow x=u^2\Rightarrow dx=2u\,du\Rightarrow\int\sqrt{x}\sqrt{1-x}dx=\int u\sqrt{1-u^2}\,2u\,du=2\int u^2\sqrt{1-u^2}\,du;$

$u=\sin\theta,\ du=\cos\theta\,d\theta,\ -\frac{\pi}{2}<\theta\le\frac{\pi}{2},\ \sqrt{1-u^2}=\cos\theta$

$2\int u^2\sqrt{1-u^2}\,du=2\int\sin^2\theta\cos\theta\cos\theta\,d\theta=2\int\sin^2\theta\cos^2\theta\,d\theta=\frac{1}{2}\int\sin^2 2\theta\,d\theta=\frac{1}{2}\int\frac{1-\cos4\theta}{2}\,d\theta$

$=\frac{1}{4}\int d\theta-\frac{1}{4}\int\cos4\theta\,d\theta=\frac{1}{4}\theta-\frac{1}{16}\sin4\theta+C=\frac{1}{4}\theta-\frac{1}{8}\sin2\theta\cos2\theta+C=\frac{1}{4}\theta-\frac{1}{4}\sin\theta\cos\theta\left(2\cos^2\theta-1\right)+C$

$= \frac{1}{4}\theta - \frac{1}{2}\sin\theta\cos^3\theta + \frac{1}{4}\sin\theta\cos\theta + C = \frac{1}{4}\sin^{-1}u - \frac{1}{2}u\left(1-u^2\right)^{3/2} - \frac{1}{4}u\sqrt{1-u^2} + C$

$= \frac{1}{4}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x}\left(1-x\right)^{3/2} - \frac{1}{4}\sqrt{x}\sqrt{1-x} + C$

48. Let $w = \sqrt{x-1} \Rightarrow w^2 = x-1 \Rightarrow 2w\,dw = dx \Rightarrow \int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx = \int \frac{\sqrt{w^2-1}}{w}2w\,dw = 2\int\sqrt{w^2-1}\,dw$

$w = \sec\theta,\ dx = \sec\theta\tan\theta\,d\theta,\ 0<\theta<\frac{\pi}{2},\ \sqrt{w^2-1} = \tan\theta$

$2\int\sqrt{w^2-1}\,dw = 2\int\tan\theta\sec\theta\tan\theta\,d\theta;\quad u=\tan\theta, du = \sec^2\theta\,d\theta, dv = \sec\theta\tan\theta\,d\theta, v = \sec\theta$

$2\int\tan\theta\sec\theta\tan\theta\,d\theta = 2\sec\theta\tan\theta - 2\int\sec^3\theta\,d\theta = 2\sec\theta\tan\theta - 2\int\sec^2\theta\sec\theta\,d\theta$

$= 2\sec\theta\tan\theta - 2\int\left(\tan^2\theta+1\right)\sec\theta\,d\theta = 2\sec\theta\tan\theta - 2\left(\int\tan^2\theta\sec\theta\,d\theta + \int\sec\theta\,d\theta\right)$

$= 2\sec\theta\tan\theta - 2\ln|\sec\theta+\tan\theta| - 2\int\tan^2\theta\sec\theta\,d\theta$

$\Rightarrow 2\int\tan^2\theta\sec\theta\,d\theta = \sec\theta\tan\theta - \ln|\sec\theta+\tan\theta| + C = w\sqrt{w^2-1} - \ln\left|w+\sqrt{w^2-1}\right| + C$

$= \sqrt{x-1}\sqrt{x-2} - \ln|\sqrt{x-1}+\sqrt{x-2}| + C$

49. $x\frac{dy}{dx} = \sqrt{x^2-4};\ dy = \sqrt{x^2-4}\,\frac{dx}{x};\ y = \int\frac{\sqrt{x^2-4}}{x}\,dx;$

$x = 2\sec\theta,\ 0<\theta<\frac{\pi}{2}, dx = 2\sec\theta\tan\theta\,d\theta, \sqrt{x^2-4} = 2\tan\theta$

$\rightarrow y = \int\frac{(2\tan\theta)(2\sec\theta\tan\theta)\,d\theta}{2\sec\theta} = 2\int\tan^2\theta\,d\theta = 2\int\left(\sec^2\theta-1\right)d\theta = 2\left(\tan\theta-\theta\right)+C$

$= 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right]+C;\quad x=2 \text{ and } y=0 \Rightarrow 0 = 0+C \Rightarrow C=0 \Rightarrow y = 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\frac{x}{2}\right]$

50. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1,\ dy = \frac{dx}{\sqrt{x^2-9}};\ y = \int\frac{dx}{\sqrt{x^2-9}};\quad x = 3\sec\theta,\ 0<\theta<\frac{\pi}{2}, dx = 3\sec\theta\tan\theta\,d\theta, \sqrt{x^2-9} = 3\tan\theta$

$\rightarrow y = \int\frac{3\sec\theta\tan\theta\,d\theta}{3\tan\theta} = \int\sec\theta\,d\theta = \ln|\sec\theta+\tan\theta|+C = \ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|+C;\quad x=5 \text{ and } y=\ln 3$

$\Rightarrow \ln 3 = \ln 3 + C \Rightarrow C=0 \Rightarrow y = \ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|$

51. $\left(x^2+4\right)\frac{dy}{dx} = 3,\ dy = \frac{3\,dx}{x^2+4};\ y = 3\int\frac{dx}{x^2+4} = \frac{3}{2}\tan^{-1}\frac{x}{2}+C;\quad x=2 \text{ and } y=0 \Rightarrow 0 = \frac{3}{2}\tan^{-1}1+C \Rightarrow C = -\frac{3\pi}{8}$

$\Rightarrow y = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{3\pi}{8}$

52. $\left(x^2+1\right)^2\frac{dy}{dx} = \sqrt{x^2+1},\ dy = \frac{dx}{\left(x^2+1\right)^{3/2}};\ x=\tan\theta,\ dx = \sec^2\theta\,d\theta, \left(x^2+1\right)^{3/2} = \sec^3\theta;$

$y = \int\frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \int\cos\theta\,d\theta = \sin\theta+C = \tan\theta\cos\theta+C = \frac{\tan\theta}{\sec\theta}+C = \frac{x}{\sqrt{x^2+1}}+C;\quad x=0 \text{ and } y=1$

$\Rightarrow 1 = 0+C \Rightarrow C=1 \Rightarrow y = \frac{x}{\sqrt{x^2+1}}+1$

53. $A=\int_0^3 \frac{\sqrt{9-x^2}}{3}\,dx$; $x=3\sin\theta,\ 0\le\theta\le\frac{\pi}{2}$, $dx=3\cos\theta\,d\theta$, $\sqrt{9-x^2}=\sqrt{9-9\sin^2\theta}=3\cos\theta$;

$$A=\int_0^{\pi/2}\frac{3\cos\theta\cdot 3\cos\theta\,d\theta}{3}=3\int_0^{\pi/2}\cos^2\theta\,d\theta=\tfrac{3}{2}\left[\theta+\sin\theta\cos\theta\right]_0^{\pi/2}=\tfrac{3\pi}{4}$$

54. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow y=\pm b\sqrt{1-\frac{x^2}{a^2}}$; $A=4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx=4b\int_0^a\sqrt{1-\frac{x^2}{a^2}}\,dx$

$\left[x=a\sin\theta,-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}, dx=a\cos\theta\,d\theta, \sqrt{1-\frac{x^2}{a^2}}=\cos\theta, x=0=a\sin\theta\Rightarrow\theta=0, x=a=a\sin\theta\Rightarrow\theta=\frac{\pi}{2}\right]$

$$4b\int_0^a\sqrt{1-\frac{x^2}{a^2}}\,dx=4b\int_0^{\pi/2}\cos\theta\,(a\cos\theta)\,d\theta=4ab\int_0^{\pi/2}\cos^2\theta\,d\theta=4ab\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$=2ab\int_0^{\pi/2}d\theta+2ab\int_0^{\pi/2}\cos 2\theta\,d\theta=2ab\left[\theta\right]_0^{\pi/2}+ab\left[\sin 2\theta\right]_0^{\pi/2}=2ab\left(\frac{\pi}{2}-0\right)+ab\left(\sin\pi-\sin 0\right)=\pi ab$$

55. (a) $A=\int_0^{1/2}\sin^{-1}x\,dx\left[u=\sin^{-1}x, du=\frac{1}{\sqrt{1-x^2}}dx, dv=dx, v=x\right]$

$$=\left[x\sin^{-1}x\right]_0^{1/2}-\int_0^{1/2}\frac{x}{\sqrt{1-x^2}}dx=\left(\tfrac{1}{2}\sin^{-1}\tfrac{1}{2}-0\right)+\left[\sqrt{1-x^2}\right]_0^{1/2}=\frac{\pi+6\sqrt{3}-12}{12}$$

(b) $M=\int_0^{1/2}\sin^{-1}x\,dx=\frac{\pi+6\sqrt{3}-12}{12}$;

$\overline{x}=\frac{1}{\frac{\pi+6\sqrt{3}-12}{12}}\int_0^{1/2}x\sin^{-1}x\,dx=\frac{12}{\pi+6\sqrt{3}-12}\int_0^{1/2}x\sin^{-1}x\,dx\ \left[u=\sin^{-1}x, du=\frac{1}{\sqrt{1-x^2}}dx, dv=x\,dx, v=\frac{1}{2}x^2\right]$

$$=\frac{12}{\pi+6\sqrt{3}-12}\left(\left[\tfrac{1}{2}x^2\sin^{-1}x\right]_0^{1/2}-\tfrac{1}{2}\int_0^{1/2}\frac{x^2}{\sqrt{1-x^2}}dx\right)$$

$\left[x=\sin\theta,-\frac{\pi}{2}<\theta<\frac{\pi}{2}, dx=\cos\theta\,d\theta, \sqrt{1-x^2}=\cos\theta, x=0=\sin\theta\Rightarrow\theta=0, x=\frac{1}{2}=\sin\theta\Rightarrow\theta=\frac{\pi}{6}\right]$

$$=\frac{12}{\pi+6\sqrt{3}-12}\left(\left(\tfrac{1}{2}\left(\tfrac{1}{2}\right)^2\sin^{-1}\left(\tfrac{1}{2}\right)-0\right)-\tfrac{1}{2}\int_0^{\pi/6}\frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta\right)=\frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48}-\frac{1}{2}\int_0^{\pi/6}\sin^2\theta\,d\theta\right)$$

$$=\frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48}-\frac{1}{2}\int_0^{\pi/6}\frac{1-\cos 2\theta}{2}\,d\theta\right)=\frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48}-\frac{1}{4}\int_0^{\pi/6}d\theta+\frac{1}{4}\int_0^{\pi/6}\cos 2\theta\,d\theta\right)$$

$$=\frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48}+\left[-\frac{\theta}{4}+\frac{1}{8}\sin 2\theta\right]_0^{\pi/6}\right)=\frac{3\sqrt{3}-\pi}{4\left(\pi+6\sqrt{3}-12\right)};$$

$\overline{y}=\frac{1}{\frac{\pi+6\sqrt{3}-12}{12}}\int_0^{1/2}\frac{1}{2}\left(\sin^{-1}x\right)^2dx\ \left[u=\left(\sin^{-1}x\right)^2, du=\frac{2\sin^{-1}x}{\sqrt{1-x^2}}dx, dv=dx, v=x\right]$

$$=\frac{6}{\pi+6\sqrt{3}-12}\left(\left[x\left(\sin^{-1}x\right)^2\right]_0^{1/2}-\int_0^{1/2}\frac{2x\sin^{-1}x}{\sqrt{1-x^2}}dx\right)\ \left[u=\sin^{-1}x, du=\frac{1}{\sqrt{1-x^2}}dx, dv=\frac{2x}{\sqrt{1-x^2}}dx, v=-2\sqrt{1-x^2}\right]$$

$$=\frac{6}{\pi+6\sqrt{3}-12}\left(\left(\tfrac{1}{2}\left(\sin^{-1}\left(\tfrac{1}{2}\right)\right)^2-0\right)+\left[2\sqrt{1-x^2}\,\sin^{-1}x\right]_0^{1/2}-\int_0^{1/2}\frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}}dx\right)$$

$$=\frac{6}{\pi+6\sqrt{3}-12}\left(\frac{\pi^2}{72}+\left(2\sqrt{1-\left(\tfrac{1}{2}\right)^2}\,\sin^{-1}\left(\tfrac{1}{2}\right)-0\right)-\left[2x\right]_0^{1/2}\right)=\frac{6}{\pi+6\sqrt{3}-12}\left(\frac{\pi^2}{72}+\frac{\pi\sqrt{3}}{6}-1\right)=\frac{\pi^2+12\pi\sqrt{3}-72}{12\left(\pi+6\sqrt{3}-12\right)}$$

56. $V = \int_0^1 \pi\left(\sqrt{x\tan^{-1}x}\right)^2 dx = \pi\int_0^1 x\tan^{-1}x\,dx \quad \left[u = \tan^{-1}x,\ du = \frac{1}{1+x^2}dx,\ dv = x\,dx,\ v = \frac{1}{2}x^2\right]$

$= \pi\left(\left[\frac{1}{2}x^2\tan^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}dx\right) = \pi\left(\left(\frac{1}{2}\tan^{-1}1 - 0\right) - \frac{1}{2}\int_0^1\left(1-\frac{1}{1+x^2}\right)dx\right) = \pi\left(\frac{\pi}{8} - \frac{1}{2}\int_0^1\left(1-\frac{1}{1+x^2}\right)dx\right)$

$= \pi\left(\frac{\pi}{8} - \frac{1}{2}\int_0^1 dx + \frac{1}{2}\int_0^1\frac{1}{1+x^2}dx\right) = \pi\left(\frac{\pi}{8} + \left[-\frac{1}{2}x + \frac{1}{2}\tan^{-1}x\right]_0^1\right) = \pi\left(\frac{\pi}{8} + \left(-\frac{1}{2} + \frac{1}{2}\tan^{-1}1 + 0 - 0\right)\right) = \frac{\pi(\pi-2)}{4}$

57. (a) Integration by parts: $u = x^2,\ du = 2x\,dx,\ dv = x\sqrt{1-x^2}\,dx,\ v = -\frac{1}{3}\left(1-x^2\right)^{3/2}$

$\int x^3\sqrt{1-x^2}\,dx = -\frac{1}{3}x^2\left(1-x^2\right)^{3/2} + \frac{1}{3}\int\left(1-x^2\right)^{3/2}2x\,dx = -\frac{1}{3}x^2\left(1-x^2\right)^{3/2} - \frac{2}{15}\left(1-x^2\right)^{5/2} + C$

(b) Substitution: $u = 1-x^2 \Rightarrow x^2 = 1-u \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx$

$\int x^3\sqrt{1-x^2}\,dx = \int x^2\sqrt{1-x^2}\,x\,dx = -\frac{1}{2}\int(1-u)\sqrt{u}\,du = -\frac{1}{2}\int\left(\sqrt{u} - u^{3/2}\right)du = -\frac{1}{3}u^{3/2} + \frac{1}{5}u^{5/2} + C$

$= -\frac{1}{3}\left(1-x^2\right)^{3/2} + \frac{1}{5}\left(1-x^2\right)^{5/2} + C$

(c) Trig substitution: $x = \sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ \sqrt{1-x^2} = \cos\theta$

$\int x^3\sqrt{1-x^2}\,dx = \int\sin^3\theta\cos\theta\cos\theta\,d\theta = \int\sin^2\theta\cos^2\theta\sin\theta\,d\theta = \int\left(1-\cos^2\theta\right)\cos^2\theta\sin\theta\,d\theta$

$= \int\cos^2\theta\sin\theta\,d\theta - \int\cos^4\theta\sin\theta\,d\theta = -\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C = -\frac{1}{3}\left(1-x^2\right)^{3/2} + \frac{1}{5}\left(1-x^2\right)^{5/2} + C$

58. (a) The slope of the line tangent to $y = f(x)$ is given by $f'(x)$. Consider the triangle whose hypotenuse is the 10 m rope, the length of the base is x and height $h = \sqrt{100-x^2}$. The slope of the tangent line is also $-\frac{\sqrt{100-x^2}}{x}$, thus $f'(x) = -\frac{\sqrt{100-x^2}}{x}$.

(b) $f(x) = \int -\frac{\sqrt{100-x^2}}{x}dx \quad \left[x = 10\sin\theta,\ 0 < \theta \le \frac{\pi}{2},\ dx = 10\cos\theta\,d\theta,\ \sqrt{100-x^2} = 10\cos\theta\right]$

$= -\int\frac{10\cos\theta}{10\sin\theta}10\cos\theta\,d\theta = -10\int\frac{\cos^2\theta}{\sin\theta}d\theta = -10\int\frac{\left(1-\sin^2\theta\right)}{\sin\theta}d\theta = -10\int\csc\theta\,d\theta + 10\int\sin\theta\,d\theta$

$= 10\ln\left|\csc\theta + \cot\theta\right| - 10\cos\theta + C = 10\ln\left|\frac{10}{x} + \frac{\sqrt{100-x^2}}{x}\right| - \sqrt{100-x^2} + C;$

$f(10) = 0 \Rightarrow 0 = 10\ln\left|\frac{10}{10} + \frac{\sqrt{100-10^2}}{10}\right| - \sqrt{100-10^2} + C = C \Rightarrow f(x) = 10\ln\left|\frac{10}{x} + \frac{\sqrt{100-x^2}}{x}\right| - \sqrt{100-x^2}$

8.5 INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

1. $\frac{5x-13}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow 5x - 13 = A(x-2) + B(x-3) = (A+B)x - (2A+3B)$

$\Rightarrow \left.\begin{array}{r} A+B=5 \\ 2A+3B=13 \end{array}\right\} \Rightarrow -B = (10-13) \Rightarrow B = 3 \Rightarrow A = 2;$ thus, $\frac{5x-13}{(x-3)(x-2)} = \frac{2}{x-3} + \frac{3}{x-2}$

2. $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 5x-7 = A(x-1) + B(x-2) = (A+B)x - (A+2B)$

$\Rightarrow \left.\begin{array}{r} A+B=5 \\ A+2B=7 \end{array}\right\} \Rightarrow B=2 \Rightarrow A=3;$ thus, $\frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$

3. $\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x+4 = A(x+1) + B = Ax + (A+B) \Rightarrow \left.\begin{array}{r} A=1 \\ A+B=4 \end{array}\right\} \Rightarrow A=1$ and $B=3;$ thus,

$\frac{x+4}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}$

4. $\frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 2x+2 = A(x-1) + B = Ax + (-A+B) \Rightarrow \left.\begin{array}{r} A=2 \\ -A+B=2 \end{array}\right\}$

$\Rightarrow A=2$ and $B=4;$ thus, $\frac{2x+2}{x^2-2x+1} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$

5. $\frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} \Rightarrow z+1 = Az(z-1) + B(z-1) + Cz^2 \Rightarrow z+1 = (A+C)z^2 + (-A+B)z - B \Rightarrow \left.\begin{array}{r} A+C=0 \\ -A+B=1 \\ -B=1 \end{array}\right\}$

$\Rightarrow B=-1 \Rightarrow A=-2 \Rightarrow C=2;$ thus, $\frac{z+1}{z^2(z-1)} = \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1}$

6. $\frac{z}{z^3-z^2-6z} = \frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2} \Rightarrow 1 = A(z+2) + B(z-3) = (A+B)z + (2A-3B) \Rightarrow \left.\begin{array}{r} A+B=0 \\ 2A-3B=1 \end{array}\right\}$

$\Rightarrow -5B=1 \Rightarrow B=-\frac{1}{5} \Rightarrow A=\frac{1}{5};$ thus, $\frac{z}{z^3-z^2-6z} = \frac{\frac{1}{5}}{z-3} + \frac{-\frac{1}{5}}{z+2}$

7. $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2} \Rightarrow 5t+2 = A(t-2) + B(t-3)$

$= (A+B)t + (-2A-3B) \Rightarrow \left.\begin{array}{r} A+B=5 \\ -2A-3B=2 \end{array}\right\} \Rightarrow -B = (10+2) = 12 \Rightarrow B=-12 \Rightarrow A=17;$ thus,

$\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$

8. $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^2\left(t^2+9\right)}$ (after long division); $\frac{-9t^2+9}{t^2\left(t^2+9\right)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}$

$\Rightarrow -9t^2+9 = At\left(t^2+9\right) + B\left(t^2+9\right) + (Ct+D)t^2 = (A+C)t^3 + (B+D)t^2 + 9At + 9B$

$\Rightarrow \left.\begin{array}{r} A+C=0 \\ B+D=-9 \\ 9A=0 \\ 9B=9 \end{array}\right\} \Rightarrow A=0 \Rightarrow C=0; B=1 \Rightarrow D=-10;$ thus, $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{1}{t^2} + \frac{-10}{t^2+9}$

9. $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x) + B(1-x);\ x=1 \Rightarrow A=\frac{1}{2};\ x=-1 \Rightarrow B=\frac{1}{2};$

$\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x} = \frac{1}{2}\left[\ln|1+x| - \ln|1-x|\right] + C$

10. $\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + Bx;\ x = 0 \Rightarrow A = \frac{1}{2};\ x = -2 \Rightarrow B = -\frac{1}{2};$
$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}\left[\ln|x| - \ln|x+2|\right] + C$

11. $\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1) + B(x+6);\ x = 1 \Rightarrow B = \frac{5}{7};\ x = -6 \Rightarrow A = \frac{-2}{-7} = \frac{2}{7};$
$\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C = \frac{1}{7}\ln\left|(x+6)^2(x-1)^5\right| + C$

12. $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3) + B(x-4);\ x = 3 \Rightarrow B = \frac{7}{-1} = -7;\ x = 4 \Rightarrow A = \frac{9}{1} = 9;$
$\int \frac{2x+1}{x^2-7x+12}\,dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3} = 9\ln|x-4| - 7\ln|x-3| + C = \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$

13. $\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1) + B(y-3);\ y = -1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4};\ y = 3 \Rightarrow A = \frac{3}{4};$
$\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8 = \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right)$
$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 = \frac{\ln 15}{2}$

14. $\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By;\ \ y = 0 \Rightarrow A = 4;\ y = -1 \Rightarrow B = \frac{3}{-1} = -3;$
$\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy = 4\int_{1/2}^1 \frac{dy}{y} - 3\int_{1/2}^1 \frac{dy}{y+1} = \left[4\ln|y| - 3\ln|y+1|\right]_{1/2}^1 = (4\ln 1 - 3\ln 2) - \left(4\ln\frac{1}{2} - 3\ln\frac{3}{2}\right)$
$= \ln\frac{1}{8} - \ln\frac{1}{16} + \ln\frac{27}{8} = \ln\left(\frac{27}{8}\cdot\frac{1}{8}\cdot 16\right) = \ln\frac{27}{4}$

15. $\frac{1}{t^3+t^2-2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Rightarrow 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2);\ t = 0 \Rightarrow A = -\frac{1}{2};\ t = -2 \Rightarrow B = \frac{1}{6};$
$t = 1 \Rightarrow C = \frac{1}{3};\ \int \frac{dt}{t^3+t^2-2t} = -\frac{1}{2}\int \frac{dt}{t} + \frac{1}{6}\int \frac{dt}{t+2} + \frac{1}{3}\int \frac{dt}{t-1} = -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| + C$

16. $\frac{x+3}{2x^3-8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \Rightarrow \frac{1}{2}(x+3) = A(x+2)(x-2) + Bx(x-2) + Cx(x+2);\ x = 0 \Rightarrow A = \frac{3}{-8};\ x = -2 \Rightarrow B = \frac{1}{16};$
$x = 2 \Rightarrow C = \frac{5}{16};\ \int \frac{x+3}{2x^3-8x}\,dx = -\frac{3}{8}\int \frac{dx}{x} + \frac{1}{16}\int \frac{dx}{x+2} + \frac{5}{16}\int \frac{dx}{x-2} = -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + C$
$= \frac{1}{16}\ln\left|\frac{(x-2)^5(x+2)}{x^6}\right| + C$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1) + B = Ax + (A+B)$
$\Rightarrow A = 3,\ A+B = 2 \Rightarrow A = 3,\ B = -1;\ \int_0^1 \frac{x^3\,dx}{x^2+2x+1} = \int_0^1 (x-2)\,dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2}$
$= \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1 = \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$

18. $\frac{x^3}{x^2-2x+1}=(x+2)+\frac{3x-2}{(x-1)^2}$ (after long division); $\frac{3x-2}{(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}\Rightarrow 3x-2=A(x-1)+B=Ax+(-A+B)$

$\Rightarrow A=3, -A+B=-2\Rightarrow A=3, B=1;\quad \int_{-1}^{0}\frac{x^3dx}{x^2-2x+1}=\int_{-1}^{0}(x+2)dx+3\int_{-1}^{0}\frac{dx}{x-1}+\int_{-1}^{0}\frac{dx}{(x-1)^2}$

$=\left[\frac{x^2}{2}+2x+3\ln|x-1|-\frac{1}{x-1}\right]_{-1}^{0}=\left(0+0+3\ln 1-\frac{1}{(-1)}\right)-\left(\frac{1}{2}-2+3\ln 2-\frac{1}{(-2)}\right)=2-3\ln 2$

19. $\frac{1}{(x^2-1)^2}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x+1)^2}+\frac{D}{(x-1)^2}\Rightarrow 1=A(x+1)(x-1)^2+B(x-1)(x+1)^2+C(x-1)^2+D(x+1)^2;$

$x=-1\Rightarrow C=\frac{1}{4};\ \ x=1\Rightarrow D=\frac{1}{4};$ coefficient of $x^3=A+B\Rightarrow A+B=0;$ constant $=A-B+C+D$

$\Rightarrow A-B+C+D=1\Rightarrow A-B=\frac{1}{2};$ thus, $A=\frac{1}{4}\Rightarrow B=-\frac{1}{4};$

$\int\frac{dx}{\left(x^2-1\right)^2}=\frac{1}{4}\int\frac{dx}{x+1}-\frac{1}{4}\int\frac{dx}{x-1}+\frac{1}{4}\int\frac{dx}{(x+1)^2}+\frac{1}{4}\int\frac{dx}{(x-1)^2}=\frac{1}{4}\ln\left|\frac{x+1}{x-1}\right|-\frac{x}{2\left(x^2-1\right)}+C$

20. $\frac{x^2}{(x-1)\left(x^2+2x+1\right)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}\Rightarrow x^2=A(x+1)^2+B(x-1)(x+1)+C(x-1);\ \ x=-1\Rightarrow C=-\frac{1}{2};$

$x=1\Rightarrow A=\frac{1}{4};$ coefficient of $x^2=A+B\Rightarrow A+B=1\Rightarrow B=\frac{3}{4};\int\frac{x^2dx}{(x-1)\left(x^2+2x+1\right)}$

$=\frac{1}{4}\int\frac{dx}{x-1}+\frac{3}{4}\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{dx}{(x+1)^2}=\frac{1}{4}\ln|x-1|+\frac{3}{4}\ln|x+1|+\frac{1}{2(x+1)}+C=\frac{\ln\left|(x-1)(x+1)^3\right|}{4}+\frac{1}{2(x+1)}+C$

21. $\frac{1}{(x+1)\left(x^2+1\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}\Rightarrow 1=A\left(x^2+1\right)+(Bx+C)(x+1); x=-1\Rightarrow A=\frac{1}{2};$ coefficient of $x^2=A+B$

$\Rightarrow A+B=0\Rightarrow B=-\frac{1}{2};$ constant $=A+C\Rightarrow A+C=1\Rightarrow C=\frac{1}{2};\ \int_0^1\frac{dx}{(x+1)\left(x^2+1\right)}=\frac{1}{2}\int_0^1\frac{dx}{x+1}+\frac{1}{2}\int_0^1\frac{(-x+1)}{x^2+1}dx$

$=\left[\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln\left(x^2+1\right)+\frac{1}{2}\tan^{-1}x\right]_0^1=\left(\frac{1}{2}\ln 2-\frac{1}{4}\ln 2+\frac{1}{2}\tan^{-1}1\right)-\left(\frac{1}{2}\ln 1-\frac{1}{4}\ln 1+\frac{1}{2}\tan^{-1}0\right)$

$=\frac{1}{4}\ln 2+\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{(\pi+2\ln 2)}{8}$

22. $\frac{3t^2+t+4}{t^3+t}=\frac{A}{t}+\frac{Bt+C}{t^2+1}\Rightarrow 3t^2+t+4=A\left(t^2+1\right)+(Bt+C)t; t=0\Rightarrow A=4;$ coefficient of $t^2=A+B\Rightarrow A+B=3$

$\Rightarrow B=-1;$ coefficient of $t=C\Rightarrow C=1;\int_1^{\sqrt{3}}\frac{3t^2+t+4}{t^3+1}dt=4\int_1^{\sqrt{3}}\frac{dt}{t}+\int_1^{\sqrt{3}}\frac{(-t+1)}{t^2+1}dt$

$=\left[4\ln|t|-\frac{1}{2}\ln\left(t^2+1\right)+\tan^{-1}t\right]_1^{\sqrt{3}}=\left(4\ln\sqrt{3}-\frac{1}{2}\ln 4+\tan^{-1}\sqrt{3}\right)-\left(4\ln 1-\frac{1}{2}\ln 2+\tan^{-1}1\right)$

$=2\ln 3-\ln 2+\frac{\pi}{3}+\frac{1}{2}\ln 2-\frac{\pi}{4}=2\ln 3-\frac{1}{2}\ln 2+\frac{\pi}{12}=\ln\left(\frac{9}{\sqrt{2}}\right)+\frac{\pi}{12}$

23. $\frac{y^2+2y+1}{\left(y^2+1\right)^2}=\frac{Ay+B}{y^2+1}+\frac{Cy+D}{\left(y^2+1\right)^2}\Rightarrow y^2+2y+1=(Ay+B)\left(y^2+1\right)+Cy+D=Ay^3+By^2+(A+C)y+(B+D)$

$\Rightarrow A=0, B=1; A+C=2\Rightarrow C=2; B+D=1\Rightarrow D=0;\ \int\frac{y^2+2y+1}{\left(y^2+1\right)^2}dy=\int\frac{1}{y^2+1}dy+2\int\frac{y}{\left(y^2+1\right)^2}dy$

$=\tan^{-1}y-\frac{1}{y^2+1}+C$

24. $\frac{8x^2+8x+2}{\left(4x^2+1\right)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{\left(4x^2+1\right)^2} \Rightarrow 8x^2+8x+2 = (Ax+B)\left(4x^2+1\right)+Cx+D = 4Ax^3+4Bx^2+(A+C)x+(B+D);$

$A=0,\ B=2;\ A+C=8 \Rightarrow C=8;\ B+D=2 \Rightarrow D=0;\quad \int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}\,dx = 2\int \frac{dx}{4x^2+1} + 8\int \frac{x\,dx}{\left(4x^2+1\right)^2}$

$= \tan^{-1} 2x - \frac{1}{4x^2+1} + C$

25. $\frac{2s+2}{\left(s^2+1\right)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3}$

$\Rightarrow 2s+2 = (As+B)(s-1)^3 + C\left(s^2+1\right)(s-1)^2 + D\left(s^2+1\right)(s-1) + E\left(s^2+1\right)$

$= As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B + C\left(s^4-2s^3+2s^2-2s+1\right) + D\left(s^3-s^2+s-1\right) + E\left(s^2+1\right)$

$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$

$$\Rightarrow \left.\begin{aligned} A+C&=0\\ -3A+B-2C+D&=0\\ 3A-3B+2C-D+E&=0\\ -A+3B-2C+D&=2\\ -B+C-D+E&=2\end{aligned}\right\} \text{ summing all equations } \Rightarrow 2E=4 \Rightarrow E=2;$$

summing eqs (2) and (3) $\Rightarrow -2B+2=0 \Rightarrow B=1$; summing eqs (3) and (4) $\Rightarrow 2A+2=2 \Rightarrow A=0$;
$C=0$ from eq (1); then $-1+0-D+2=2$ from eq (5) $\Rightarrow D=-1$;

$\int \frac{2s+2}{\left(s^2+1\right)(s-1)^3}\,ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1} s + C$

26. $\frac{s^4+81}{s\left(s^2+9\right)^2} = \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{\left(s^2+9\right)^2} \Rightarrow s^4+81 = A\left(s^2+9\right)^2 + (Bs+C)s\left(s^2+9\right) + (Ds+E)s$

$= A\left(s^4+18s^2+81\right) + \left(Bs^4+Cs^3+9Bs^2+9Cs\right) + Ds^2 + Es$

$= (A+B)s^4 + Cs^3 + (18A+9B+D)s^2 + (9C+E)s + 81A \Rightarrow 81A=81$ or $A=1;\ A+B=1 \Rightarrow B=0;\ \ C=0;$

$9C+E=0 \Rightarrow E=0; 18A+9B+D=0 \Rightarrow D=-18;\quad \int \frac{s^4+81}{s\left(s^2+9\right)^2}\,ds = \int \frac{ds}{s} - 18\int \frac{s\,ds}{(s^2+9)^2} = \ln|s| + \frac{9}{s^2+9} + C$

27. $\frac{x^2-x+2}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \Rightarrow x^2-x+2 = A\left(x^2+x+1\right) + (Bx+C)(x-1) = (A+B)x^2 + (A-B+C)x + (A-C)$

$\Rightarrow A+B=1,\ A-B+C=-1,\ A-C=2 \Rightarrow$ adding eq(2) and eq(3) $\Rightarrow 2A-B=1$, add this equation to eq (1)

$\Rightarrow 3A=2 \Rightarrow A=\frac{2}{3} \Rightarrow B=1-A=\frac{1}{3} \Rightarrow C=-1-A+B=-\frac{4}{3};\quad \int \frac{x^2-x+2}{x^3-1}\,dx = \int\left(\frac{2/3}{x-1} + \frac{(1/3)x-4/3}{x^2+x+1}\right)dx$

$= \frac{2}{3}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{x-4}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx\ \left[u = x+\frac{1}{2} \Rightarrow u-\frac{1}{2} = x \Rightarrow du = dx\right]$

$= \frac{2}{3}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{u-\frac{9}{2}}{u^2+\frac{3}{4}}\,du = \frac{2}{3}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}\,du - \frac{3}{2}\int \frac{1}{u^2+\frac{3}{4}}\,du$

$= \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln\left|\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right| - \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + C = \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln\left|x^2+x+1\right| - \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

28. $\frac{1}{x^4+x}=\frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2-x+1}\Rightarrow 1=A(x+1)\left(x^2-x+1\right)+Bx\left(x^2-x+1\right)+(Cx+D)x(x+1)$
$=(A+B+C)x^3+(-B+C+D)x^2+(B+D)x+A\Rightarrow A=1,\ B+D=0\Rightarrow D=-B,\ -B+C+D=0$
$\Rightarrow -2B+C=0\Rightarrow C=2B,\ A+B+C=0\Rightarrow 1+B+2B=0\Rightarrow B=-\frac{1}{3}\Rightarrow C=-\frac{2}{3}\Rightarrow D=\frac{1}{3};$
$\int\frac{1}{x^4+x}\,dx=\int\left(\frac{1}{x}-\frac{1/3}{x+1}+\frac{(-2/3)x+1/3}{x^2-x+1}\right)dx=\int\frac{1}{x}\,dx-\frac{1}{3}\int\frac{1}{x+1}\,dx-\frac{1}{3}\int\frac{2x-1}{x^2-x+1}\,dx$
$=\ln|x|-\frac{1}{3}\ln|x+1|-\frac{1}{3}\ln|x^2-x+1|+C$

29. $\frac{x^2}{x^4-1}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{Cx+D}{x^2+1}\Rightarrow x^2=A(x-1)\left(x^2+1\right)+B(x+1)\left(x^2+1\right)+(Cx+D)(x-1)(x+1)$
$=(A+B+C)x^3+(-A+B+D)x^2+(A+B-C)x-A+B-D\Rightarrow A+B+C=0,\ -A+B+D=1,$
$A+B-C=0,\ -A+B-D=0\Rightarrow$ adding eq (1) to eq (3) gives $2A+2B=0,$ adding eq (2) to eq (4) gives $-2A+2B=1,$ adding these two equations gives $4B=1\Rightarrow B=\frac{1}{4},$ using $2A+2B=0\Rightarrow A=-\frac{1}{4},$ using $-A+B-D=0\Rightarrow D=\frac{1}{2},$ and using $A+B-C=0\Rightarrow C=0;$ $\int\frac{x^2}{x^4-1}\,dx=\int\left(\frac{-1/4}{x+1}+\frac{1/4}{x-1}+\frac{1/2}{x^2+1}\right)dx$
$=-\frac{1}{4}\int\frac{1}{x+1}\,dx+\frac{1}{4}\int\frac{1}{x-1}\,dx+\frac{1}{2}\int\frac{1}{x^2+1}\,dx=-\frac{1}{4}\ln|x+1|+\frac{1}{4}\ln|x-1|+\frac{1}{2}\tan^{-1}x+C=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|+\frac{1}{2}\tan^{-1}x+C$

30. $\frac{x^2+x}{x^4-3x^2-4}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+1}\Rightarrow x^2+x=A(x+2)\left(x^2+1\right)+B(x-2)\left(x^2+1\right)+(Cx+D)(x-2)(x+2)$
$=(A+B+C)x^3+(2A-2B+D)x^2+(A+B-4C)x+2A-2B-4D\Rightarrow A+B+C=0,\ 2A-2B+D=1,$
$A+B-4C=1,\ 2A-2B-4D=0\Rightarrow$ subtracting eq (1) from eq (3) gives $-5C=1\Rightarrow C=-\frac{1}{5},$ subtracting eq (2) from eq (4) gives $-5D=-1\Rightarrow D=\frac{1}{5},$ substituting for C in eq (1) gives $A+B=\frac{1}{5},$ and substituting for D in eq (4) gives $2A-2B=\frac{4}{5}\Rightarrow A-B=\frac{2}{5},$ adding this equation to the previous equation gives $2A=\frac{3}{5}\Rightarrow A=\frac{3}{10}\Rightarrow B=-\frac{1}{10};$ $\int\frac{x^2+x}{x^4-3x^2-4}\,dx=\int\left(\frac{3/10}{x-2}-\frac{1/10}{x+2}+\frac{(-1/5)x+1/5}{x^2+1}\right)dx$
$=\frac{3}{10}\int\frac{1}{x-2}\,dx-\frac{1}{10}\int\frac{1}{x+2}\,dx-\frac{1}{5}\int\frac{x}{x^2+1}\,dx+\frac{1}{5}\int\frac{1}{x^2+1}\,dx=\frac{3}{10}\ln|x-2|-\frac{1}{10}\ln|x+2|-\frac{1}{10}\ln\left|x^2+1\right|+\frac{1}{5}\tan^{-1}x+C$

31. $\frac{2\theta^3+5\theta^2+8\theta+4}{\left(\theta^2+2\theta+2\right)^2}=\frac{A\theta+B}{\theta^2+2\theta+2}+\frac{C\theta+D}{\left(\theta^2+2\theta+2\right)^2}\Rightarrow 2\theta^3+5\theta^2+8\theta+4=(A\theta+B)\left(\theta^2+2\theta+2\right)+C\theta+D$
$=A\theta^3+(2A+B)\theta^2+(2A+2B+C)\theta+(2B+D)\Rightarrow A=2;\ 2A+B=5\Rightarrow B=1;\ 2A+2B+C=8\Rightarrow C=2;$
$2B+D=4\Rightarrow D=2;\ \int\frac{2\theta^3+5\theta^2+8\theta+4}{\left(\theta^2+2\theta+2\right)^2}\,d\theta=\int\frac{2\theta+1}{\theta^2+2\theta+2}\,d\theta+\int\frac{2\theta+2}{\left(\theta^2+2\theta+2\right)^2}\,d\theta$
$=\int\frac{(2\theta+2)d\theta}{\theta^2+2\theta+2}-\int\frac{d\theta}{\theta^2+2\theta+2}+\int\frac{(2\theta+2)d\theta}{\left(\theta^2+2\theta+2\right)^2}=\ln\left(\theta^2+2\theta+2\right)-\int\frac{d\theta}{(\theta+1)^2+1}-\frac{1}{\theta^2+2\theta+2}$
$=\ln\left(\theta^2+2\theta+2\right)-\tan^{-1}(\theta+1)-\frac{1}{\theta^2+2\theta+2}+C$

32. $\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}=\frac{A\theta+B}{\theta^2+1}+\frac{C\theta+D}{\left(\theta^2+1\right)^2}+\frac{E\theta+F}{\left(\theta^2+1\right)^3}$
$\Rightarrow\theta^4-4\theta^3+2\theta^2-3\theta+1=(A\theta+B)\left(\theta^2+1\right)^2+(C\theta+D)\left(\theta^2+1\right)+E\theta+F$
$=(A\theta+B)\left(\theta^4+2\theta^2+1\right)+\left(C\theta^3+D\theta^2+C\theta+D\right)+E\theta+F$

$$= \left(A\theta^5 + B\theta^4 + 2A\theta^3 + 2B\theta^2 + A\theta + B\right) + \left(C\theta^3 + D\theta^2 + C\theta + D\right) + E\theta + F$$

$= A\theta^5 + B\theta^4 + (2A+C)\theta^3 + (2B+D)\theta^2 + (A+C+E)\theta + (B+D+F) \Rightarrow A = 0;\ B = 1;$
$2A + C = -4 \Rightarrow C = -4;\ 2B + D = 2 \Rightarrow D = 0;\ A + C + E = -3 \Rightarrow E = 1;\ B + D + F = 1 \Rightarrow F = 0;$

$$\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2+1\right)^3}\,d\theta = \int \frac{d\theta}{\theta^2+1} - 4\int \frac{\theta\,d\theta}{\left(\theta^2+1\right)^2} + \int \frac{\theta\,d\theta}{\left(\theta^2+1\right)^3} = \tan^{-1}\theta + 2\left(\theta^2+1\right)^{-1} - \frac{1}{4}\left(\theta^2+1\right)^{-2} + C$$

33. $\frac{2x^3-2x^2+1}{x^2-x} = 2x + \frac{1}{x^2-x} = 2x + \frac{1}{x(x-1)};\ \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + Bx;\ x = 0 \Rightarrow A = -1;\ x = 1 \Rightarrow B = 1;$
$\int \frac{2x^3-2x^2+1}{x^2-x} = \int 2x\,dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C = x^2 + \ln\left|\frac{x-1}{x}\right| + C$

34. $\frac{x^4}{x^2-1} = \left(x^2+1\right) + \frac{1}{x^2-1} = \left(x^2+1\right) + \frac{1}{(x+1)(x-1)};\ \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x+1);$
$x = -1 \Rightarrow A = -\frac{1}{2};\ x = 1 \Rightarrow B = \frac{1}{2};\ \int \frac{x^4}{x^2-1}dx = \int\left(x^2+1\right)dx - \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$
$= \frac{1}{3}x^3 + x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C = \frac{x^3}{3} + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

35. $\frac{9x^3-3x+1}{x^3-x^2} = 9 + \frac{9x^2-3x+1}{x^2(x-1)}$ (after long division); $\frac{9x^2-3x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$
$\Rightarrow 9x^2 - 3x + 1 = Ax(x-1) + B(x-1) + Cx^2;\ x = 1 \Rightarrow C = 7;\ x = 0 \Rightarrow B = -1;\ A + C = 9 \Rightarrow A = 2;$
$\int \frac{9x^3-3x+1}{x^3-x^2}dx = \int 9\,dx + 2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 7\int \frac{dx}{x-1} = 9x + 2\ln|x| + \frac{1}{x} + 7\ln|x-1| + C$

36. $\frac{16x^3}{4x^2-4x+1} = (4x+4) + \frac{12x-4}{4x^2-4x+1};\ \frac{12x-4}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} \Rightarrow 12x - 4 = A(2x-1) + B$
$\Rightarrow A = 6;\ -A + B = -4 \Rightarrow B = 2;\ \int \frac{16x^3}{4x^2-4x+1}dx = 4\int(x+1)\,dx + 6\int \frac{dx}{2x-1} + 2\int \frac{dx}{(2x-1)^2}$
$= 2(x+1)^2 + 3\ln|2x-1| - \frac{1}{2x-1} + C_1 = 2x^2 + 4x + 3\ln|2x-1| - (2x-1)^{-1} + C,$ where $C = 2 + C_1$

37. $\frac{y^4+y^2-1}{y^3+y} = y - \frac{1}{y\left(y^2+1\right)};\ \frac{1}{y\left(y^2+1\right)} = \frac{A}{y} + \frac{By+C}{y^2+1} \Rightarrow 1 = A\left(y^2+1\right) + (By+C)y = (A+B)y^2 + Cy + A$
$\Rightarrow A = 1;\ A + B = 0 \Rightarrow B = -1;\ C = 0;\ \int \frac{y^4+y^2-1}{y^3+y}dy = \int y\,dy - \int \frac{dy}{y} + \int \frac{y\,dy}{y^2+1} = \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln\left(1+y^2\right) + C$

38. $\frac{2y^4}{y^3-y^2+y-1} = 2y + 2 + \frac{2}{y^3-y^2+y-1};\ \frac{2}{y^3-y^2+y-1} = \frac{2}{\left(y^2+1\right)(y-1)} = \frac{A}{y-1} + \frac{By+C}{y^2+1}$
$\Rightarrow 2 = A\left(y^2+1\right) + (By+C)(y-1) = \left(Ay^2 + A\right) + \left(By^2 + Cy - By - C\right) = (A+B)y^2 + (-B+C)y + (A-C)$
$\Rightarrow A + B = 0,\ -B + C = 0$ or $C = B,\ A - C = A - B = 2 \Rightarrow A = 1,\ B = -1,\ C = -1;$
$\int \frac{2y^4}{y^3-y^2+y-1}dy = 2\int(y+1)\,dy + \int \frac{dy}{y-1} - \int \frac{y}{y^2+1}dy - \int \frac{dy}{y^2+1} = (y+1)^2 + \ln|y-1| - \frac{1}{2}\ln\left(y^2+1\right) - \tan^{-1}y + C_1$
$= y^2 + 2y + \ln|y-1| - \frac{1}{2}\ln\left(y^2+1\right) - \tan^{-1}y + C,$ where $C = C_1 + 1$

39. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2};[e^t = y, e^t dt = dy] \rightarrow \int \frac{dy}{y^2+3y+2} = \int \frac{dy}{y+1} - \int \frac{dy}{y+2} = \ln\left|\frac{y+1}{y+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

40. $\int \frac{e^{4t}+2e^{2t}-e^{t}}{e^{2t}+1}\,dt = \int \frac{e^{3t}+2e^{t}-1}{e^{2t}+1} e^{t}\,dt; \left[y=e^{t},\ dy=e^{t}\,dt \right] \to \int \frac{y^{3}+2y-1}{y^{2}+1}\,dy = \int \left(y+\frac{y-1}{y^{2}+1} \right) dy = \frac{y^{2}}{2} + \int \frac{y}{y^{2}+1}\,dy - \int \frac{dy}{y^{2}+1}$
$= \frac{y^{2}}{2} + \frac{1}{2}\ln\left(y^{2}+1\right) - \tan^{-1} y + C = \frac{1}{2}e^{2t} + \frac{1}{2}\ln\left(e^{2t}+1\right) - \tan^{-1}\left(e^{t}\right) + C$

41. $\int \frac{\cos y\,dy}{\sin^{2} y+\sin y-6}; [\sin y = t,\ \cos y\,dy = dt] \to \int \frac{dt}{t^{2}+t-6} = \frac{1}{5}\int \left(\frac{1}{t-2} - \frac{1}{t+3} \right) dt = \frac{1}{5}\ln\left|\frac{t-2}{t+3}\right| + C = \frac{1}{5}\ln\left|\frac{\sin y-2}{\sin y+3}\right| + C$

42. $\int \frac{\sin\theta\,d\theta}{\cos^{2}\theta+\cos\theta-2}; [\cos\theta = y, -\sin\theta\,d\theta = dy] \to -\int \frac{dy}{y^{2}+y-2} = \frac{1}{3}\int \frac{dy}{y+2} - \frac{1}{3}\int \frac{dy}{y-1} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C = \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C$
$= -\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

43. $\int \frac{(x-2)^{2}\tan^{-1}(2x)-12x^{3}-3x}{\left(4x^{2}+1\right)(x-2)^{2}}\,dx = \int \frac{\tan^{-1}(2x)}{4x^{2}+1}\,dx - 3\int \frac{x}{(x-2)^{2}}\,dx = \frac{1}{2}\int \tan^{-1}(2x)\frac{2\,dx}{4x^{2}+1} - 3\int \frac{dx}{x-2} - 6\int \frac{dx}{(x-2)^{2}}$
$= \frac{1}{4}\left(\tan^{-1} 2x\right)^{2} - 3\ln|x-2| + \frac{6}{x-2} + C$

44. $\int \frac{(x+1)^{2}\tan^{-1}(3x)+9x^{3}+x}{\left(9x^{2}+1\right)(x+1)^{2}}\,dx = \int \frac{\tan^{-1}(3x)}{9x^{2}+1}\,dx + \int \frac{x}{(x+1)^{2}}\,dx = \frac{1}{3}\int \tan^{-1}(3x)\frac{3\,dx}{9x^{2}+1} + \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^{2}}$
$= \frac{1}{6}\left(\tan^{-1} 3x\right)^{2} + \ln|x+1| + \frac{1}{x+1} + C$

45. $\int \frac{1}{x^{3/2}-\sqrt{x}}\,dx = \int \frac{1}{\sqrt{x}(x-1)}\,dx; \left[\text{Let } u=\sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2du = \frac{1}{\sqrt{x}}\,dx \right] \to \int \frac{2}{u^{2}-1}\,du;$
$\frac{2}{u^{2}-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1)+B(u+1) = (A+B)u - A + B \Rightarrow A+B=0, -A+B=2 \Rightarrow B=1 \Rightarrow A=-1;$
$\int \frac{2}{u^{2}-1}\,du = \int \left(\frac{-1}{u+1} + \frac{1}{u-1} \right) du = -\int \frac{1}{u+1}\,du + \int \frac{1}{u-1}\,du = -\ln|u+1| + \ln|u-1| + C = \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$

46. $\int \frac{1}{\left(x^{1/3}-1\right)\sqrt{x}}\,dx; [\text{Let } x=u^{6} \Rightarrow dx = 6u^{5}du] \to \int \frac{1}{\left(u^{2}-1\right)u^{3}}6u^{5}du = \int \frac{6u^{2}}{u^{2}-1}\,du = \int \left(6+\frac{6}{u^{2}-1} \right) du = 6\int du + \int \frac{6}{u^{2}-1}\,du;$
$\frac{6}{u^{2}-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 6 = A(u-1)+B(u+1) = (A+B)u - A + B \Rightarrow A+B=0, -A+B=6 \Rightarrow B=3 \Rightarrow A=-3;$
$6\int du + \int \frac{6}{u^{2}-1}\,du = 6u + \int \left(\frac{-3}{u+1} + \frac{3}{u-1} \right) du = 6u - 3\int \frac{1}{u+1}\,du + 3\int \frac{1}{u-1}\,du = 6u - 3\ln|u+1| + 3\ln|u-1| + C$
$= 6x^{1/6} + 3\ln\left|\frac{x^{1/6}-1}{x^{1/6}+1}\right| + C$

47. $\int \frac{\sqrt{x+1}}{x}\,dx; \left[\text{Let } x+1 = u^{2} \Rightarrow dx = 2u\,du \right] \to \int \frac{u}{u^{2}-1}2u\,du = \int \frac{2u^{2}}{u^{2}-1}\,du = \int \left(2+\frac{2}{u^{2}-1} \right) du = 2\int du + \int \frac{2}{u^{2}-1}\,du;$
$\frac{2}{u^{2}-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1)+B(u+1) = (A+B)u - A + B \Rightarrow A+B=0, -A+B=2 \Rightarrow B=1 \Rightarrow A=-1;$
$2\int du + \int \frac{2}{u^{2}-1}\,du = 2u + \int \left(\frac{-1}{u+1} + \frac{1}{u-1} \right) du = 2u - \int \frac{1}{u+1}\,du + \int \frac{1}{u-1}\,du = 2u - \ln|u+1| + \ln|u-1| + C$
$= 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

48. $\int \frac{1}{x\sqrt{x+9}}\,dx$; $\left[\text{Let } x+9=u^2 \Rightarrow dx=2u\,du\right] \to \int \frac{1}{(u^2-9)u}\,2u\,du = \int \frac{2}{u^2-9}\,du$; $\frac{2}{u^2-9}=\frac{A}{u-3}+\frac{B}{u+3}$
$\Rightarrow 2 = A(u+3)+B(u-3) = (A+B)u+3A-3B \Rightarrow A+B=0,\ 3A-3B=2 \Rightarrow A=\frac{1}{3} \Rightarrow B=-\frac{1}{3}$;
$\int \frac{2}{u^2-9}\,du = \int\left(\frac{1/3}{u-3}-\frac{1/3}{u+3}\right)du = \frac{1}{3}\int\frac{1}{u-3}\,du - \frac{1}{3}\int\frac{1}{u+3}\,du = \frac{1}{3}\ln|u-3| - \frac{1}{3}\ln|u+3| + C = \frac{1}{3}\ln\left|\frac{\sqrt{x+9}-3}{\sqrt{x+9}+3}\right| + C$

49. $\int \frac{1}{x(x^4+1)}\,dx = \int \frac{x^3}{x^4(x^4+1)}\,dx$; $\left[\text{Let } u=x^4 \Rightarrow du=4x^3dx\right] \to \frac{1}{4}\int \frac{1}{u(u+1)}\,du$; $\frac{1}{u(u+1)}=\frac{A}{u}+\frac{B}{u+1}$
$\Rightarrow 1 = A(u+1)+Bu = (A+B)u + A \Rightarrow A=1 \Rightarrow B=-1$;
$\frac{1}{4}\int\frac{1}{u(u+1)}\,du = \frac{1}{4}\int\left(\frac{1}{u}-\frac{1}{u+1}\right)du = \frac{1}{4}\int\frac{1}{u}\,du - \frac{1}{4}\int\frac{1}{u+1}\,du = \frac{1}{4}\ln|u| - \frac{1}{4}\ln|u+1| + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right)+C$

50. $\int \frac{1}{x^6(x^5+4)}\,dx = \int \frac{x^4}{x^{10}(x^5+4)}\,dx$; $\left[\text{Let } u=x^5 \Rightarrow du=5x^4dx\right] \to \frac{1}{5}\int\frac{1}{u^2(u+4)}\,du$; $\frac{1}{u^2(u+4)} = \frac{A}{u}+\frac{B}{u^2}+\frac{C}{u+4}$
$\Rightarrow 1 = Au(u+4)+B(u+4)+Cu^2 = (A+C)u^2+(4A+B)u+4B \Rightarrow A+C=0,\ 4A+B=0,\ 4B=1 \Rightarrow B=\frac{1}{4}$
$\Rightarrow A=-\frac{1}{16} \Rightarrow C=\frac{1}{16}$; $\frac{1}{5}\int\frac{1}{u^2(u+4)}\,du = \frac{1}{5}\int\left(-\frac{1/16}{u}+\frac{1/4}{u^2}+\frac{1/16}{u+4}\right)du = -\frac{1}{80}\int\frac{1}{u}\,du + \frac{1}{20}\int\frac{1}{u^2}\,du + \frac{1}{80}\int\frac{1}{u+4}\,du$
$= -\frac{1}{80}\ln|u| - \frac{1}{20u} + \frac{1}{80}\ln|u+4| + C = -\frac{1}{80}\ln|x^5| - \frac{1}{20x^5} + \frac{1}{80}\ln|x^5+4| + C = \frac{1}{80}\ln\left|\frac{x^5+4}{x^5}\right| - \frac{1}{20x^5} + C$

51. $\left(t^2-3t+2\right)\frac{dx}{dt}=1$; $x = \int\frac{dt}{t^2-3t+2} = \int\frac{dt}{t-2} - \int\frac{dt}{t-1} = \ln\left|\frac{t-2}{t-1}\right| + C$; $\frac{t-2}{t-1} = Ce^x$; $t=3$ and $x=0 \Rightarrow \frac{1}{2}=C$
$\Rightarrow \frac{t-2}{t-1} = \frac{1}{2}e^x \Rightarrow x = \ln\left|2\left(\frac{t-2}{t-1}\right)\right| = \ln|t-2| - \ln|t-1| + \ln 2$

52. $\left(3t^4+4t^2+1\right)\frac{dx}{dt} = 2\sqrt{3}$; $x = 2\sqrt{3}\int\frac{dt}{3t^4+4t^2+1} = \sqrt{3}\int\frac{dt}{t^2+\frac{1}{3}} - \sqrt{3}\int\frac{dt}{t^2+1} = 3\tan^{-1}\left(\sqrt{3}t\right) - \sqrt{3}\tan^{-1}t + C$; $t=1$ and
$x = \frac{-\pi\sqrt{3}}{4} \Rightarrow -\frac{\sqrt{3}\pi}{4} = \pi - \frac{\sqrt{3}}{4}\pi + C \Rightarrow C = -\pi \Rightarrow x = 3\tan^{-1}\left(\sqrt{3}t\right) - \sqrt{3}\tan^{-1}t - \pi$

53. $\left(t^2+2t\right)\frac{dx}{dt} = 2x+2$; $\frac{1}{2}\int\frac{dx}{x+1} = \int\frac{dt}{t^2+2t} \Rightarrow \frac{1}{2}\ln|x+1| = \frac{1}{2}\int\frac{dt}{t} - \frac{1}{2}\int\frac{dt}{t+2} \Rightarrow \ln|x+1| = \ln\left|\frac{t}{t+2}\right| + C$; $t=1$ and
$x=1 \Rightarrow \ln 2 = \ln\frac{1}{3} + C \Rightarrow C = \ln 2 + \ln 3 = \ln 6 \Rightarrow \ln|x+1| = \ln 6\left|\frac{t}{t+2}\right| \Rightarrow x+1 = \frac{6t}{t+2} \Rightarrow x = \frac{6t}{t+2} - 1,\ t>0$

54. $(t+1)\frac{dx}{dt} = x^2+1 \Rightarrow \int\frac{dx}{x^2+1} = \int\frac{dt}{t+1} \Rightarrow \tan^{-1}x = \ln|t+1| + C$; $t=0$ and $x=0 \Rightarrow \tan^{-1}0 = \ln|1| + C$
$\Rightarrow C = \tan^{-1}0 = 0 \Rightarrow \tan^{-1}x = \ln|t+1| \Rightarrow x = \tan\left(\ln(t+1)\right),\ t>-1$

55. $V = \pi\int_{0.5}^{2.5} y^2\,dx = \pi\int_{0.5}^{2.5}\frac{9}{3x-x^2}\,dx = 3\pi\left(\int_{0.5}^{2.5}\left(-\frac{1}{x-3}+\frac{1}{x}\right)\right)dx = \left[3\pi\ln\left|\frac{x}{x-3}\right|\right]_{0.5}^{2.5} = 3\pi\ln 25$

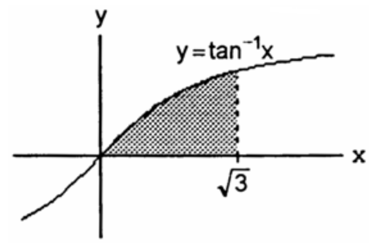
56. $V = 2\pi\int_0^1 xy\,dx = 2\pi\int_0^1\frac{2x}{(x+1)(2-x)}\,dx = 4\pi\int_0^1\left[-\frac{1}{3}\left(\frac{1}{x+1}\right)+\frac{2}{3}\left(\frac{1}{2-x}\right)\right]dx = \left[-\frac{4\pi}{3}\left(\ln|x+1| + 2\ln|2-x|\right)\right]_0^1 = \frac{4\pi}{3}(\ln 2)$

57. $A=\int_0^{\sqrt{3}}\tan^{-1}x\,dx=\left[x\tan^{-1}x\right]_0^{\sqrt{3}}-\int_0^{\sqrt{3}}\frac{x}{1+x^2}dx$

$=\frac{\pi\sqrt{3}}{3}-\left[\frac{1}{2}\ln\left(x^2+1\right)\right]_0^{\sqrt{3}}=\frac{\pi\sqrt{3}}{3}-\ln 2;$

$\overline{x}=\frac{1}{A}\int_0^{\sqrt{3}}x\tan^{-1}x\,dx=\frac{1}{A}\left(\left[\frac{1}{2}x^2\tan^{-1}x\right]_0^{\sqrt{3}}-\frac{1}{2}\int_0^{\sqrt{3}}\frac{x^2}{1+x^2}dx\right)$

$=\frac{1}{A}\left[\frac{\pi}{2}-\left[\frac{1}{2}\left(x-\tan^{-1}x\right)\right]_0^{\sqrt{3}}\right]=\frac{1}{A}\left(\frac{\pi}{2}-\frac{\sqrt{3}}{2}+\frac{\pi}{6}\right)=\frac{1}{A}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)\cong 1.10$



58. $A=\int_3^5\frac{4x^2+13x-9}{x^3+2x^2-3x}dx=3\int_3^5\frac{dx}{x}-\int_3^5\frac{dx}{x+3}+2\int_3^5\frac{dx}{x-1}=\left[3\ln|x|-\ln|x+3|+2\ln|x-1|\right]_3^5=\ln\frac{125}{9};$

$\overline{x}=\frac{1}{A}\int_3^5\frac{x\left(4x^2+13x-9\right)}{x^3+2x^2-3x}dx=\frac{1}{A}\left(\left[4x\right]_3^5+3\int_3^5\frac{dx}{x+3}+2\int_3^5\frac{dx}{x-1}\right)=\frac{1}{A}(8+11\ln 2-3\ln 6)\cong 3.90$

59. (a) $\frac{dx}{dt}=kx(N-x)\Rightarrow\int\frac{dx}{x(N-x)}=\int k\,dt\Rightarrow\frac{1}{N}\int\frac{dx}{x}+\frac{1}{N}\int\frac{dx}{N-x}=\int k\,dt\Rightarrow\frac{1}{N}\ln\left|\frac{x}{N-x}\right|=kt+C;$

$k=\frac{1}{250}$, $N=1000$, $t=0$ and $x=2\Rightarrow\frac{1}{1000}\ln\left|\frac{2}{998}\right|=C\Rightarrow\frac{1}{1000}\ln\left|\frac{x}{1000-x}\right|=\frac{t}{250}+\frac{1}{1000}\ln\left(\frac{1}{499}\right)$

$\Rightarrow\ln\left|\frac{499x}{1000-x}\right|=4t\Rightarrow\frac{499x}{1000-x}=e^{4t}\Rightarrow 499x=e^{4t}(1000-x)\Rightarrow\left(499+e^{4t}\right)x=1000e^{4t}\Rightarrow x=\frac{1000e^{4t}}{499+e^{4t}}$

(b) $x=\frac{1}{2}N=500\Rightarrow 500=\frac{1000e^{4t}}{499+e^{4t}}\Rightarrow 500\cdot 499+500e^{4t}=1000e^{4t}\Rightarrow e^{4t}=499\Rightarrow t=\frac{1}{4}\ln 499\approx 1.55$ days

60. $\frac{dx}{dt}=k(a-x)(b-x)\Rightarrow\frac{dx}{(a-x)(b-x)}=k\,dt$

(a) $a=b$: $\int\frac{dx}{(a-x)^2}=\int k\,dt\Rightarrow\frac{1}{a-x}=kt+C$; $t=0$ and $x=0\Rightarrow\frac{1}{a}=C\Rightarrow\frac{1}{a-x}=kt+\frac{1}{a}$

$\Rightarrow\frac{1}{a-x}=\frac{akt+1}{a}\Rightarrow a-x=\frac{a}{akt+1}\Rightarrow x=a-\frac{a}{akt+1}=\frac{a^2kt}{akt+1}$

(b) $a\neq b$: $\int\frac{dx}{(a-x)(b-x)}=\int k\,dt\Rightarrow\frac{1}{b-a}\int\frac{dx}{a-x}-\frac{1}{b-a}\int\frac{dx}{b-x}=\int k\,dt\Rightarrow\frac{1}{b-a}\ln\left|\frac{b-x}{a-x}\right|=kt+C$; $t=0$ and

$x=0\Rightarrow\frac{1}{b-a}\ln\frac{b}{a}=C\Rightarrow\ln\left|\frac{b-x}{a-x}\right|=(b-a)kt+\ln\left(\frac{b}{a}\right)\Rightarrow\frac{b-x}{a-x}=\frac{b}{a}e^{(b-a)kt}\Rightarrow x=\frac{ab\left[1-e^{(b-a)kt}\right]}{a-be^{(b-a)kt}}$

8.6 INTEGRAL TABLES AND COMPUTER ALGEBRA SYSTEMS

1. $\int\frac{dx}{x\sqrt{x-3}}=\frac{2}{\sqrt{3}}\tan^{-1}\sqrt{\frac{x-3}{3}}+C$

(We used FORMULA 29(b) with $a=1$, $b=3$)

2. $\int\frac{dx}{x\sqrt{x+4}}=\frac{1}{\sqrt{4}}\ln\left|\frac{\sqrt{x+4}-\sqrt{4}}{\sqrt{x+4}+\sqrt{4}}\right|+C=\frac{1}{2}\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right|+C$

(We used FORMULA 20(a) with $a=1$, $b=4$)

3. $\int \frac{x\,dx}{\sqrt{x-2}} = \int \frac{(x-2)dx}{\sqrt{x-2}} + 2\int \frac{dx}{\sqrt{x-2}} = \int \left(\sqrt{x-2}\right)^1 dx + 2\int \left(\sqrt{x-2}\right)^{-1} dx$

$= \left(\frac{2}{1}\right)\frac{\left(\sqrt{x-2}\right)^3}{3} + 2\left(\frac{2}{1}\right)\frac{\left(\sqrt{x-2}\right)^1}{1} = \sqrt{x-2}\left[\frac{2(x-2)}{3} + 4\right] + C$

(We used FORMULA 11 with $a = 1, b = -2, n = 1$ and $a = 1, b = -2, n = -1$)

4. $\int \frac{x\,dx}{(2x+3)^{3/2}} = \frac{1}{2}\int \frac{(2x+3)dx}{(2x+3)^{3/2}} - \frac{3}{2}\int \frac{dx}{(2x+3)^{3/2}} = \frac{1}{2}\int \frac{dx}{\sqrt{2x+3}} - \frac{3}{2}\int \frac{dx}{\left(\sqrt{2x+3}\right)^3}$

$= \frac{1}{2}\int \left(\sqrt{2x+3}\right)^{-1} dx - \frac{3}{2}\int \left(\sqrt{2x+3}\right)^{-3} dx = \left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\frac{\left(\sqrt{2x+3}\right)^1}{1} - \left(\frac{3}{2}\right)\left(\frac{2}{2}\right)\frac{\left(\sqrt{2x+3}\right)^{-1}}{(-1)} + C$

$= \frac{1}{2\sqrt{2x+3}}(2x+3+3) + C = \frac{(x+3)}{\sqrt{2x+3}} + C$

(We used FORMULA 11 with $a = 2, \ b = 3, \ n = -1$ and $a = 2, b = 3, n = -3$)

5. $\int x\sqrt{2x-3}\,dx = \frac{1}{2}\int (2x-3)\sqrt{2x-3}\,dx + \frac{3}{2}\int \sqrt{2x-3}\,dx = \frac{1}{2}\int \left(\sqrt{2x-3}\right)^3 dx + \frac{3}{2}\int \left(\sqrt{2x-3}\right)^1 dx$

$= \left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\frac{\left(\sqrt{2x-3}\right)^5}{5} + \left(\frac{3}{2}\right)\left(\frac{2}{2}\right)\frac{\left(\sqrt{2x-3}\right)^3}{3} + C = \frac{(2x-3)^{3/2}}{2}\left[\frac{2x-3}{5} + 1\right] + C = \frac{(2x-3)^{3/2}(x+1)}{5} + C$

(We used FORMULA 11 with $a = 2, b = -3, n = 3$ and $a = 2, b = -3, n = 1$)

6. $\int x(7x+5)^{3/2}\,dx = \frac{1}{7}\int (7x+5)(7x+5)^{3/2}\,dx - \frac{5}{7}\int (7x+5)^{3/2} dx = \frac{1}{7}\int \left(\sqrt{7x+5}\right)^5 dx - \frac{5}{7}\int \left(\sqrt{7x+5}\right)^3 dx$

$= \left(\frac{1}{7}\right)\left(\frac{2}{7}\right)\frac{\left(\sqrt{7x+5}\right)^7}{7} - \left(\frac{5}{7}\right)\left(\frac{2}{7}\right)\frac{\left(\sqrt{7x+5}\right)^5}{5} + C = \frac{(7x+5)^{5/2}}{49}\left[\frac{2(7x+5)}{7} - 2\right] + C = \frac{(7x+5)^{5/2}}{49}\left(\frac{14x-4}{7}\right) + C$

(We used FORMULA 11 with $a = 7, b = 5, n = 5$ and $a = 7, b = 5, n = 3$)

7. $\int \frac{\sqrt{9-4x}}{x^2}\,dx = -\frac{\sqrt{9-4x}}{x} + \frac{(-4)}{2}\int \frac{dx}{x\sqrt{9-4x}} + C$

(We used FORMULA 14 with $a = -4, b = 9$)

$= -\frac{\sqrt{9-4x}}{x} - 2\left(\frac{1}{\sqrt{9}}\right)\ln\left|\frac{\sqrt{9-4x}-\sqrt{9}}{\sqrt{9-4x}+\sqrt{9}}\right| + C$

(We used FORMULA 29(a) with $a = -4, b = 9$)

$= \frac{-\sqrt{9-4x}}{x} - \frac{2}{3}\ln\left|\frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3}\right| + C$

8. $\int \frac{dx}{x^2\sqrt{4x-9}} = -\frac{\sqrt{4x-9}}{(-9)x} + \frac{4}{18}\int \frac{dx}{x\sqrt{4x-9}} + C$

(We used FORMULA 15 with $a = 4, b = -9$)

$= \frac{\sqrt{4x-9}}{9x} + \left(\frac{2}{9}\right)\left(\frac{2}{\sqrt{9}}\right)\tan^{-1}\sqrt{\frac{4x-9}{9}} + C$

(We used FORMULA 29(b) with $a = 4, b = 9$)

$= \frac{\sqrt{4x-9}}{9x} + \frac{4}{27}\tan^{-1}\sqrt{\frac{4x-9}{9}} + C$

9. $\int x\sqrt{4x-x^2}\,dx = \int x\sqrt{2\cdot 2x-x^2}\,dx = \frac{(x+2)(2x-3\cdot 2)\sqrt{2\cdot 2\cdot x-x^2}}{6} + \frac{2^3}{2}\sin^{-1}\left(\frac{x-2}{2}\right) + C$
$= \frac{(x+2)(2x-6)\sqrt{4x-x^2}}{6} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C = \frac{(x+2)(x-3)\sqrt{4x-x^2}}{3} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$
(We used FORMULA 51 with $a = 2$)

10. $\int \frac{\sqrt{x-x^2}}{x}\,dx = \int \frac{\sqrt{2\cdot\frac{1}{2}x-x^2}}{x}\,dx = \sqrt{2\cdot\frac{1}{2}x-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right) + C = \sqrt{x-x^2} + \frac{1}{2}\sin^{-1}(2x-1) + C$
(We used FORMULA 52 with $a = \frac{1}{2}$)

11. $\int \frac{dx}{x\sqrt{7+x^2}} = \int \frac{dx}{x\sqrt{\left(\sqrt{7}\right)^2+x^2}} = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{\left(\sqrt{7}\right)^2+x^2}}{x}\right| + C = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{7+x^2}}{x}\right| + C$
(We used FORMULA 26 with $a = \sqrt{7}$)

12. $\int \frac{dx}{x\sqrt{7-x^2}} = \int \frac{dx}{x\sqrt{\left(\sqrt{7}\right)^2-x^2}} = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{\left(\sqrt{7}\right)^2-x^2}}{x}\right| + C = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{7-x^2}}{x}\right| + C$
(We used FORMULA 34 with $a = \sqrt{7}$)

13. $\int \frac{\sqrt{4-x^2}}{x}\,dx = \int \frac{\sqrt{2^2-x^2}}{x}\,dx = \sqrt{2^2-x^2} - 2\ln\left|\frac{2+\sqrt{2^2-x^2}}{x}\right| + C = \sqrt{4-x^2} - 2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + C$
(We used FORMULA 31 with $a = 2$)

14. $\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{\sqrt{x^2-2^2}}{x}\,dx = \sqrt{x^2-2^2} - 2\sec^{-1}\left|\frac{x}{2}\right| + C = \sqrt{x^2-4} - 2\sec^{-1}\left|\frac{x}{2}\right| + C$
(We used FORMULA 42 with $a = 2$)

15. $\int e^{2t}\cos 3t\,dt = \frac{e^{2t}}{2^2+3^2}(2\cos 3t + 3\sin 3t) + C = \frac{e^{2t}}{13}(2\cos 3t + 3\sin 3t) + C$
(We used FORMULA 108 with $a = 2, b = 3$)

16. $\int e^{-3t}\sin 4t\,dt = \frac{e^{-3t}}{(-3)^2+4^2}(-3\sin 4t - 4\cos 4t) + C = \frac{e^{-3t}}{25}(-3\sin 4t - 4\cos 4t) + C$
(We used FORMULA 107 with $a = -3, b = 4$)

17. $\int x\cos^{-1}x\,dx = \int x^1\cos^{-1}x\,dx = \frac{x^{1+1}}{1+1}\cos^{-1}x + \frac{1}{1+1}\int \frac{x^{1+1}dx}{\sqrt{1-x^2}} = \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\int \frac{x^2dx}{\sqrt{1-x^2}}$
(We used FORMULA 100 with $a = 1, n = 1$)
$= \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\left(\frac{1}{2}\sin^{-1}x\right) - \frac{1}{2}\left(\frac{1}{2}x\sqrt{1-x^2}\right) + C = \frac{x^2}{2}\cos^{-1}x + \frac{1}{4}\sin^{-1}x - \frac{1}{4}x\sqrt{1-x^2} + C$
(We used FORMULA 33 with $a = 1$)

18. $\int x\tan^{-1}x\,dx = \int x^1\tan^{-1}(1x)\,dx = \frac{x^{1+1}}{1+1}\tan^{-1}(1x) - \frac{1}{1+1}\int \frac{x^{1+1}dx}{1+(1)^2x^2} = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2dx}{1+x^2}$
(We used FORMULA 101 with $a = 1, n = 1$)

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$ (after long division)

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{1}{1+x^2}dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C = \frac{1}{2}\left(\left(x^2+1\right)\tan^{-1}x - x\right) + C$

19. $\int x^2\tan^{-1}x\,dx = \frac{x^{2+1}}{2+1}\tan^{-1}x - \frac{1}{2+1}\int\frac{x^{2+1}}{1+x^2}dx = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\frac{x^3}{1+x^2}dx$

(We used FORMULA 101 with $a=1, n=2$)

$\int\frac{x^3}{1+x^2}dx = \int x\,dx - \int\frac{x\,dx}{1+x^2} = \frac{x^2}{2} - \frac{1}{2}\ln\left(1+x^2\right) + C \Rightarrow \int x^2\tan^{-1}x\,dx = \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\ln\left(1+x^2\right) + C$

20. $\int\frac{\tan^{-1}x}{x^2}dx = \int x^{-2}\tan^{-1}x\,dx = \frac{x^{(-2+1)}}{(-2+1)}\tan^{-1}x - \frac{1}{(-2+1)}\int\frac{x^{(-2+1)}}{1+x^2}dx = \frac{x^{-1}}{(-1)}\tan^{-1}x + \int\frac{x^{-1}}{1+x^2}dx$

(We used FORMULA 101 with $a=1, n=-2$)

$\int\frac{x^{-1}dx}{1+x^2} = \int\frac{dx}{x\left(1+x^2\right)} = \int\frac{dx}{x} - \int\frac{x\,dx}{1+x^2} = \ln|x| - \frac{1}{2}\ln\left(1+x^2\right) + C \Rightarrow \int\frac{\tan^{-1}x}{x^2}dx = -\frac{1}{x}\tan^{-1}x + \ln|x| - \frac{1}{2}\ln\left(1+x^2\right) + C$

21. $\int\sin 3x\cos 2x\,dx = -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$

(We used FORMULA 62(a) with $a=3, b=2$)

22. $\int\sin 2x\cos 3x\,dx = -\frac{\cos 5x}{10} + \frac{\cos x}{2} + C$

(We used FORMULA 62(a) with $a=2, b=3$)

23. $\int 8\sin 4t\sin\frac{t}{2}dt = \frac{8}{7}\sin\left(\frac{7t}{2}\right) - \frac{8}{9}\sin\left(\frac{9t}{2}\right) + C = 8\left[\frac{\sin\left(\frac{7t}{2}\right)}{7} - \frac{\sin\left(\frac{9t}{2}\right)}{9}\right] + C$

(We used FORMULA 62(b) with $a=4, b=\frac{1}{2}$)

24. $\int\sin\frac{t}{3}\sin\frac{t}{6}dt = 3\sin\left(\frac{t}{6}\right) - \sin\left(\frac{t}{2}\right) + C$

(We used FORMULA 62(b) with $a=\frac{1}{3}, b=\frac{1}{6}$)

25. $\int\cos\frac{\theta}{3}\cos\frac{\theta}{4}d\theta = 6\sin\left(\frac{\theta}{12}\right) + \frac{6}{7}\sin\left(\frac{7\theta}{12}\right) + C$

(We used FORMULA 62(c) with $a=\frac{1}{3}, b=\frac{1}{4}$)

26. $\int\cos\frac{\theta}{2}\cos 7\theta\,d\theta = \frac{1}{13}\sin\left(\frac{13\theta}{2}\right) + \frac{1}{15}\sin\left(\frac{15\theta}{2}\right) + C = \frac{\sin\left(\frac{13\theta}{2}\right)}{13} + \frac{\sin\left(\frac{15\theta}{2}\right)}{15} + C$

(We used FORMULA 62(c) with $a=\frac{1}{2}, b=7$)

27. $\int\frac{x^3+x+1}{\left(x^2+1\right)^2}dx = \int\frac{x\,dx}{x^2+1} + \int\frac{dx}{\left(x^2+1\right)^2} = \frac{1}{2}\int\frac{2x\,dx}{x^2+1} + \int\frac{dx}{\left(x^2+1\right)^2} = \frac{1}{2}\ln\left(x^2+1\right) + \frac{x}{2\left(1+x^2\right)} + \frac{1}{2}\tan^{-1}x + C$

(For the second integral we used FORMULA 17 with $a=1$)

28. $\int \frac{x^2+6x}{\left(x^2+3\right)^2}\,dx = \int \frac{dx}{x^2+3} + \int \frac{6x\,dx}{\left(x^2+3\right)^2} - \int \frac{3dx}{\left(x^2+3\right)^2} = \int \frac{dx}{x^2+\left(\sqrt{3}\right)^2} + 3\int \frac{2x\,dx}{\left(x^2+3\right)^2} - 3\int \frac{dx}{\left[x^2+\left(\sqrt{3}\right)^2\right]^2}$

$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{x^2+3} - 3\left(\frac{x}{2\left(\sqrt{3}\right)^2\left(\left(\sqrt{3}\right)^2+x^2\right)} + \frac{1}{2\left(\sqrt{3}\right)^3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right) + C$

(For the first integral we used FORMULA 16 with $a=\sqrt{3}$; for the third integral we used FORMULA 17 with $a=\sqrt{3}$)

$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{x^2+3} - \frac{x}{2\left(x^2+3\right)} + C$

29. $\int \sin^{-1}\sqrt{x}\,dx;\begin{bmatrix} u=\sqrt{x} \\ x=u^2 \\ dx=2u\,du \end{bmatrix} \to 2\int u^1\sin^{-1}u\,du = 2\left(\frac{u^{1+1}}{1+1}\sin^{-1}u - \frac{1}{1+1}\int \frac{u^{1+1}}{\sqrt{1-u^2}}\,du\right) = u^2\sin^{-1}u - \int \frac{u^2du}{\sqrt{1-u^2}}$

(We used FORMULA 99 with $a=1, n=1$)

$= u^2\sin^{-1}u - \left(\frac{1}{2}\sin^{-1}u - \frac{1}{2}u\sqrt{1-u^2}\right) + C = \left(u^2-\frac{1}{2}\right)\sin^{-1}u + \frac{1}{2}u\sqrt{1-u^2} + C$

(We used FORMULA 33 with $a=1$)

$= \left(x-\frac{1}{2}\right)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x-x^2} + C$

30. $\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}\,dx;\begin{bmatrix} u=\sqrt{x} \\ x=u^2 \\ dx=2u\,du \end{bmatrix} \to \int \frac{\cos^{-1}u}{u}\cdot 2u\,du = 2\int \cos^{-1}u\,du = 2\left(u\cos^{-1}u - \frac{1}{1}\sqrt{1-u^2}\right) + C$

(We used FORMULA 97 with $a=1$)

$= 2\left(\sqrt{x}\cos^{-1}\sqrt{x} - \sqrt{1-x}\right) + C$

31. $\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx;\begin{bmatrix} u=\sqrt{x} \\ x=u^2 \\ dx=2u\,du \end{bmatrix} \to \int \frac{u\cdot 2u}{\sqrt{1-u^2}}\,du = 2\int \frac{u^2}{\sqrt{1-u^2}}\,du = 2\left(\frac{1}{2}\sin^{-1}u - \frac{1}{2}u\sqrt{1-u^2}\right) + C = \sin^{-1}u - u\sqrt{1-u^2} + C$

(We used FORMULA 33 with $a=1$)

$= \sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x} + C = \sin^{-1}\sqrt{x} - \sqrt{x-x^2} + C$

32. $\int \frac{\sqrt{2-x}}{\sqrt{x}}\,dx;\begin{bmatrix} u=\sqrt{x} \\ x=u^2 \\ dx=2u\,du \end{bmatrix} \to \int \frac{\sqrt{2-u^2}}{u}\cdot 2u\,du = 2\int \sqrt{\left(\sqrt{2}\right)^2-u^2}\,du = 2\left[\frac{u}{2}\sqrt{\left(\sqrt{2}\right)^2-u^2} + \frac{\left(\sqrt{2}\right)^2}{2}\sin^{-1}\left(\frac{u}{\sqrt{2}}\right)\right] + C$

(We used FORMULA 29 with $a=\sqrt{2}$)

$= u\sqrt{2-u^2} + 2\sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + C = \sqrt{2x-x^2} + 2\sin^{-1}\sqrt{\frac{x}{2}} + C$

33. $\int (\cot t)\sqrt{1-\sin^2 t}\, dt = \int \frac{\sqrt{1-\sin^2 t}\,(\cos t)\,dt}{\sin t};\ \begin{bmatrix} u = \sin t \\ du = \cos t\, dt \end{bmatrix} \to \int \frac{\sqrt{1-u^2}\,du}{u} = \sqrt{1-u^2} - \ln\left|\frac{1+\sqrt{1-u^2}}{u}\right| + C$

(We used FORMULA 31 with $a = 1$)

$= \sqrt{1-\sin^2 t} - \ln\left|\frac{1+\sqrt{1-\sin^2 t}}{\sin t}\right| + C$

34. $\int \frac{dt}{(\tan t)\sqrt{4-\sin^2 t}} = \int \frac{\cos t\, dt}{(\sin t)\sqrt{4-\sin^2 t}};\ [u = \sin t,\ du = \cos t\, dt] \to \int \frac{du}{u\sqrt{4-u^2}} = -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-u^2}}{u}\right| + C$

(We used FORMULA 34 with $a = 2$)

$= -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-\sin^2 t}}{\sin t}\right| + C$

35. $\int \frac{dy}{y\sqrt{3+(\ln y)^2}};\ \begin{bmatrix} u = \ln y \\ y = e^u \\ dy = e^u du \end{bmatrix} \to \int \frac{e^u du}{e^u\sqrt{3+u^2}} = \int \frac{du}{\sqrt{3+u^2}} = \ln\left|u + \sqrt{3+u^2}\right| + C = \ln\left|\ln y + \sqrt{3+(\ln y)^2}\right| + C$

(We used FORMULA 20 with $a = \sqrt{3}$)

36. $\int \tan^{-1}\sqrt{y}\, dy;\ \begin{bmatrix} t = \sqrt{y} \\ y = t^2 \\ dy = 2t\, dt \end{bmatrix} \to 2\int t \tan^{-1} t\, dt = 2\left[\frac{t^2}{2}\tan^{-1} t - \frac{1}{2}\int \frac{t^2}{1+t^2}\, dt\right] = t^2 \tan^{-1} t - \int \frac{t^2}{1+t^2}\, dt$

(We used FORMULA 101 with $n = 1,\ a = 1$)

$= t^2 \tan^{-1} t - \int \frac{t^2+1}{t^2+1}\, dt + \int \frac{dt}{1+t^2} = t^2 \tan^{-1} t - t + \tan^{-1} t + C = y\tan^{-1}\sqrt{y} + \tan^{-1}\sqrt{y} - \sqrt{y} + C$

37. $\int \frac{1}{\sqrt{x^2+2x+5}}\, dx = \int \frac{1}{\sqrt{(x+1)^2+4}}\, dx;\ [t = x+1,\ dt = dx] \to \int \frac{1}{\sqrt{t^2+4}}\, dt = \ln\left|t + \sqrt{t^2+4}\right| + C$

(We used FORMULA 20 with $a = 2$)

$= \ln\left|(x+1) + \sqrt{(x+1)^2+4}\right| + C = \ln\left|(x+1) + \sqrt{x^2+2x+5}\right| + C$

38. $\int \frac{x^2}{\sqrt{x^2-4x+5}}\, dx = \int \frac{x^2}{\sqrt{(x-2)^2+1}}\, dx;\ \begin{bmatrix} t = x-2 \\ dt = dx \end{bmatrix} \to \int \frac{(t+2)^2}{\sqrt{t^2+1}}\, dt = \int \frac{t^2+4t+2}{\sqrt{t^2+1}}\, dt = \int \frac{t^2}{\sqrt{t^2+1}}\, dt + \int \frac{4t}{\sqrt{t^2+1}}\, dt + \int \frac{4}{\sqrt{t^2+1}}\, dt$

(We used FORMULA 25 with $a = 1$) (We used FORMULA 20 with $a = 1$)

$= \left[-\frac{1}{2}\ln\left|t + \sqrt{t^2+1}\right| + \frac{t\sqrt{t^2+1}}{2}\right] + 4\sqrt{t^2+1} + \left[4\ln\left|t + \sqrt{t^2+1}\right|\right] + C$

$= -\frac{1}{2}\ln\left|(x-2) + \sqrt{(x-2)^2+1}\right| + \frac{(x-2)\sqrt{(x-2)^2+1}}{2} + 4\sqrt{(x-2)^2+1} + 4\ln\left|(x-2) + \sqrt{(x-2)^2+1}\right| + C$

$= \frac{7}{2}\ln\left|(x-2) + \sqrt{x^2-4x+5}\right| + \frac{(x+6)\sqrt{x^2-4x+5}}{2} + C$

39. $\int \sqrt{5-4x-x^2}\,dx = \int \sqrt{9-(x+2)^2}\,dx;\ [t = x+2, dt = dx] \to \int \sqrt{9-t^2}\,dt = \frac{t}{2}\sqrt{9-t^2} + \frac{3^2}{2}\sin^{-1}\left(\frac{t}{3}\right) + C$

(We used FORMULA 29 with $a = 3$)

$= \frac{x+2}{2}\sqrt{9-(x+2)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + C = \frac{x+2}{2}\sqrt{5-4x-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + C$

40. $\int x^2\sqrt{2x-x^2}\,dx = \int x^2\sqrt{1-(x-1)^2}\,dx;\ [t = x-1,\ dt = dx] \to \int (t+1)^2\sqrt{1-t^2}\,dt = \int \left(t^2+2t+1\right)\sqrt{1-t^2}\,dt$

$= \int t^2\sqrt{1-t^2}\,dt + \int 2t\sqrt{1-t^2}\,dt + \int \sqrt{1-t^2}\,dt$

(We used FORMULA 30 with $a = 1$) (We used FORMULA 29 with $a = 1$)

$= \left[\frac{1^4}{8}\sin^{-1}\left(\frac{t}{1}\right) - \frac{1}{8}t\sqrt{1-t^2}\left(1^2-2t^2\right)\right] - \frac{2}{3}\left(1-t^2\right)^{3/2} + \left[\frac{t}{2}\sqrt{1-t^2} + \frac{1^2}{2}\sin^{-1}\left(\frac{t}{1}\right)\right] + C$

$= \frac{1}{8}\sin^{-1}(x-1) - \frac{1}{8}(x-1)\sqrt{1-(x-1)^2}\left(1^2-2(x-1)^2\right) - \frac{2}{3}\left(1-(x-1)^2\right)^{3/2} + \frac{x-1}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1) + C$

$= \frac{5}{8}\sin^{-1}(x-1) - \frac{2}{3}\left(2x-x^2\right)^{3/2} + \frac{x-1}{8}\sqrt{2x-x^2}\left(2x^2-4x+5\right) + C$

41. $\int \sin^5 2x\,dx = -\frac{\sin^4 2x\cos 2x}{5\cdot 2} + \frac{5-1}{5}\int \sin^3 2x\,dx = -\frac{\sin^4 2x\cos 2x}{10} + \frac{4}{5}\left[-\frac{\sin^2 2x\cos 2x}{3\cdot 2} + \frac{3-1}{3}\int \sin 2x\,dx\right]$

(We used FORMULA 60 with $a = 2, n = 5$ and $a = 2, n = 3$)

$= -\frac{\sin^4 2x\cos 2x}{10} - \frac{2}{15}\sin^2 2x\cos 2x + \frac{8}{15}\left(-\frac{1}{2}\right)\cos 2x + C = -\frac{\sin^4 2x\cos 2x}{10} - \frac{2\sin^2 2x\cos 2x}{15} - \frac{4\cos 2x}{15} + C$

42. $\int 8\cos^4 2\pi t\,dt = 8\left(\frac{\cos^3 2\pi t\sin 2\pi t}{4\cdot 2\pi} + \frac{4-1}{4}\int \cos^2 2\pi t\,dt\right)$

(We used FORMULA 61 with $a = 2\pi, n = 4$)

$= \frac{\cos^3 2\pi t\sin 2\pi t}{\pi} + 6\left[\frac{t}{2} + \frac{\sin(2\cdot 2\pi\cdot t)}{4\cdot 2\pi}\right] + C$

(We used FORMULA 59 with $a = 2\pi$)

$= \frac{\cos^3 2\pi t\sin 2\pi t}{\pi} + 3t + \frac{3\sin 4\pi t}{4\pi} + C = \frac{\cos^3 2\pi t\sin 2\pi t}{\pi} + \frac{3\cos 2\pi t\sin 2\pi t}{2\pi} + 3t + C$

43. $\int \sin^2 2\theta\cos^3 2\theta\,d\theta = \frac{\sin^3 2\theta\cos^2 2\theta}{2(2+3)} + \frac{3-1}{3+2}\int \sin^2 2\theta\cos 2\theta\,d\theta$

(We used FORMULA 69 with $a = 2, m = 3, n = 2$)

$= \frac{\sin^3 2\theta\cos^2 2\theta}{10} + \frac{2}{5}\int \sin^2 2\theta\cos 2\theta\,d\theta = \frac{\sin^3 2\theta\cos^2 2\theta}{10} + \frac{2}{5}\left[\frac{1}{2}\int \sin^2 2\theta(\cos 2\theta)2d\theta\right] = \frac{\sin^3 2\theta\cos^2 2\theta}{10} + \frac{\sin^3 2\theta}{15} + C$

44. $\int 2\sin^2 t\sec^4 t\,dt = \int 2\sin^2 t\cos^{-4} t\,dt = 2\left(-\frac{\sin t\cos^{-3} t}{2-4} + \frac{2-1}{2-4}\int \cos^{-4} t\,dt\right)$

(We used FORMULA 68 with $a = 1, n = 2, m = -4$)

$= \sin t\cos^{-3} t - \int \cos^{-4} t\,dt = \sin t\cos^{-3} t - \int \sec^4 t\,dt = \sin t\cos^{-3} t - \left(\frac{\sec^2 t\tan t}{4-1} + \frac{4-2}{4-1}\int \sec^2 t\,dt\right)$

(We used FORMULA 92 with $a = 1, n = 4$)

$= \sin t\cos^{-3} t - \left(\frac{\sec^2 t\tan t}{3}\right) - \frac{2}{3}\tan t + C = \frac{2}{3}\sec^2 t\tan t - \frac{2}{3}\tan t + C = \frac{2}{3}\tan t\left(\sec^2 t - 1\right) + C = \frac{2}{3}\tan^3 t + C$

An easy way to find the integral using substitution:

$\int 2\sin^2 t\cos^{-4} t\,dt = \int 2\tan^2 t\sec^2 t\,dt = 2\int (\tan t)^2\sec^2 t\,dt = \frac{2}{3}\tan^3 t + C$

45. $\int 4\tan^3 2x\,dx = 4\left(\frac{\tan^2 2x}{2\cdot 2} - \int \tan 2x\,dx\right) = \tan^2 2x - 4\int \tan 2x\,dx$

(We used FORMULA 86 with $n = 3, a = 2$)

$= \tan^2 2x - \frac{4}{2}\ln|\sec 2x| + C = \tan^2 2x - 2\ln|\sec 2x| + C$

46. $\int 8\cot^4 t\,dt = 8\left(-\frac{\cot^3 t}{3} - \int \cot^2 t\,dt\right)$

(We used FORMULA 87 with $a = 1, n = 4$)

$= 8\left(-\frac{1}{3}\cot^3 t + \cot t + t\right) + C$

(We used FORMULA 85 with $a = 1$)

47. $\int 2\sec^3 \pi x\,dx = 2\left[\frac{\sec \pi x \tan \pi x}{\pi(3-1)} + \frac{3-2}{3-1}\int \sec \pi x\,dx\right]$

(We used FORMULA 92 with $n = 3, a = \pi$)

$= \frac{1}{\pi}\sec \pi x \tan \pi x + \frac{1}{\pi}\ln|\sec \pi x + \tan \pi x| + C$

(We used FORMULA 88 with $a = \pi$)

48. $\int 3\sec^4 3x\,dx = 3\left[\frac{\sec^2 3x \tan 3x}{3(4-1)} + \frac{4-2}{4-1}\int \sec^2 3x\,dx\right]$

(We used FORMULA 92 with $n = 4, a = 3$)

$= \frac{\sec^2 3x \tan 3x}{3} + \frac{2}{3}\tan 3x + C$

(We used FORMULA 90 with $a = 3$)

49. $\int \csc^5 x\,dx = -\frac{\csc^3 x \cot x}{5-1} + \frac{5-2}{5-1}\int \csc^3 x\,dx = -\frac{\csc^3 x \cot x}{4} + \frac{3}{4}\left(-\frac{\csc x \cot x}{3-1} + \frac{3-2}{3-1}\int \csc x\,dx\right)$

(We used FORMULA 93 with $n = 5, a = 1$ and $n = 3, a = 1$)

$= -\frac{1}{4}\csc^3 x \cot x - \frac{3}{8}\csc x \cot x - \frac{3}{8}\ln|\csc x + \cot x| + C$

(We used FORMULA 89 with $a = 1$)

50. $\int 16x^3(\ln x)^2\,dx = 16\left[\frac{x^4(\ln x)^2}{4} - \frac{2}{4}\int x^3 \ln x\,dx\right] = 16\left[\frac{x^4(\ln x)^2}{4} - \frac{1}{2}\left[\frac{x^4(\ln x)}{4} - \frac{1}{4}\int x^3\,dx\right]\right]$

(We used FORMULA 110 with $a = 1, n = 3, m = 2$ and $a = 1, n = 3, m = 1$)

$= 16\left(\frac{x^4(\ln x)^2}{4} - \frac{x^4(\ln x)}{8} + \frac{x^4}{32}\right) + C = 4x^4(\ln x)^2 - 2x^4\ln x + \frac{x^4}{2} + C$

51. $\int e^t \sec^3\left(e^t - 1\right)dt;\ \left[x = e^t - 1,\ dx = e^t\,dt\right] \to \int \sec^3 x\,dx = \frac{\sec x \tan x}{3-1} + \frac{3-2}{3-1}\int \sec x\,dx$

(We used FORMULA 92 with $a = 1, n = 3$)

$= \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + C = \frac{1}{2}\left[\sec\left(e^t - 1\right)\tan\left(e^t - 1\right) + \ln\left|\sec\left(e^t - 1\right) + \tan\left(e^t - 1\right)\right|\right] + C$

52. $\int \frac{\csc^3\sqrt{\theta}}{\sqrt{\theta}}\,d\theta;\begin{bmatrix} t=\sqrt{\theta} \\ \theta=t^2 \\ d\theta=2t\,dt \end{bmatrix} \to 2\int \csc^3 t\,dt = 2\left[-\frac{\csc t\cot t}{3-1}+\frac{3-2}{3-1}\int \csc t\,dt\right] = 2\left[-\frac{\csc t\cot t}{2}-\frac{1}{2}\ln|\csc t+\cot t|\right]+C$

(We used FORMULA 93 with $a=1, n=3$)

$= -\csc\sqrt{\theta}\cot\sqrt{\theta}-\ln\left|\csc\sqrt{\theta}+\cot\sqrt{\theta}\right|+C$

53. $\int_0^1 2\sqrt{x^2+1}\,dx;\left[x=\tan t, dx=\sec^2 t\,dt\right] \to 2\int_0^{\pi/4}\sec t\cdot\sec^2 t\,dt = 2\int_0^{\pi/4}\sec^3 t\,dt$

$= 2\left[\left[\frac{\sec t\cdot\tan t}{3-1}\right]_0^{\pi/4}+\frac{3-2}{3-1}\int_0^{\pi/4}\sec t\,dt\right]$

(We used FORMULA 92 with, $n=3, a=1$)

$=\left[\sec t\cdot\tan t+\ln\left|\sec t+\tan t\right|\right]_0^{\pi/4}=\sqrt{2}+\ln\left(\sqrt{2}+1\right)$

54. $\int_0^{\sqrt{3}/2}\frac{dy}{\left(1-y^2\right)^{5/2}};[y=\sin x, dy=\cos x\,dx]\to\int_0^{\pi/3}\frac{\cos x\,dx}{\cos^5 x}=\int_0^{\pi/3}\sec^4 x\,dx=\left[\frac{\sec^2 x\tan x}{4-1}\right]_0^{\pi/3}+\frac{4-2}{4-1}\int_0^{\pi/3}\sec^2 x\,dx$

(We used FORMULA 92 with $a=1, n=4$)

$=\left[\frac{\sec^2 x\tan x}{3}+\frac{2}{3}\tan x\right]_0^{\pi/3}=\left(\frac{4}{3}\right)\sqrt{3}+\left(\frac{2}{3}\right)\sqrt{3}=2\sqrt{3}$

55. $\int_1^2\frac{\left(r^2-1\right)^{3/2}}{r}\,dr;[r=\sec\theta, dr=\sec\theta\tan\theta\,d\theta]\to\int_0^{\pi/3}\frac{\tan^3\theta}{\sec\theta}(\sec\theta\tan\theta)\,d\theta=\int_0^{\pi/3}\tan^4\theta\,d\theta$

$=\left[\frac{\tan^3\theta}{4-1}\right]_0^{\pi/3}-\int_0^{\pi/3}\tan^2\theta\,d\theta=\left[\frac{\tan^3\theta}{3}-\tan\theta+\theta\right]_0^{\pi/3}=\frac{3\sqrt{3}}{3}-\sqrt{3}+\frac{\pi}{3}=\frac{\pi}{3}$

(We used FORMULA 86 with $a=1, n=4$ and FORMULA 84 with $a=1$)

56. $\int_0^{1/\sqrt{3}}\frac{dt}{\left(t^2+1\right)^{7/2}};[t=\tan\theta, dt=\sec^2\theta\,d\theta]\to\int_0^{\pi/6}\frac{\sec^2\theta\,d\theta}{\sec^7\theta}=\int_0^{\pi/6}\cos^5\theta\,d\theta$

$=\left[\frac{\cos^4\theta\sin\theta}{5}\right]_0^{\pi/6}+\left(\frac{5-1}{5}\right)\int_0^{\pi/6}\cos^3\theta\,d\theta=\left[\frac{\cos^4\theta\sin\theta}{5}\right]_0^{\pi/6}+\frac{4}{5}\left[\left[\frac{\cos^2\theta\sin\theta}{3}\right]_0^{\pi/6}+\left(\frac{3-1}{3}\right)\int_0^{\pi/6}\cos\theta\,d\theta\right]$

$=\left[\frac{\cos^4\theta\sin\theta}{5}+\frac{4}{15}\cos^2\theta\sin\theta+\frac{8}{15}\sin\theta\right]_0^{\pi/6}$

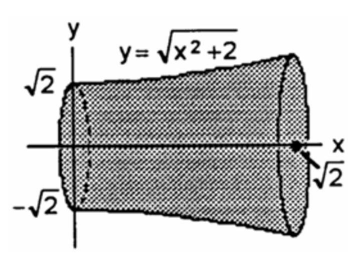
(We used FORMULA 61 with $a=1, n=5$ and $a=1, n=3$)

$=\frac{\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right)}{5}+\left(\frac{4}{15}\right)\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{8}{15}\right)\left(\frac{1}{2}\right)=\frac{9}{160}+\frac{1}{10}+\frac{4}{15}=\frac{3\cdot 9+48+32\cdot 4}{480}=\frac{203}{480}$

57. $S=\int_0^{\sqrt{2}}2\pi y\sqrt{1+\left(y'\right)^2}\,dx$

$=2\pi\int_0^{\sqrt{2}}\sqrt{x^2+2}\sqrt{1+\frac{x^2}{x^2+2}}\,dx$

$=2\sqrt{2}\pi\int_0^{\sqrt{2}}\sqrt{x^2+1}\,dx$



$= 2\sqrt{2}\pi\left[\frac{x\sqrt{x^2+1}}{2} + \frac{1}{2}\ln\left|x + \sqrt{x^2+1}\right|\right]_0^{\sqrt{2}}$

(We used FORMULA 21 with $a = 1$)

$= \sqrt{2}\pi\left[\sqrt{6} + \ln\left(\sqrt{2} + \sqrt{3}\right)\right] = 2\pi\sqrt{3} + \pi\sqrt{2}\ln\left(\sqrt{2} + \sqrt{3}\right)$

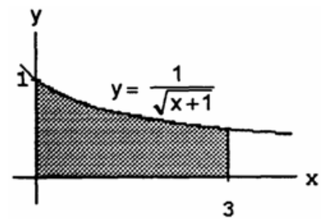
58. $L = \int_0^{\sqrt{3}/2} \sqrt{1+(2x)^2}\,dx = 2\int_0^{\sqrt{3}/2} \sqrt{\frac{1}{4} + x^2}\,dx = 2\left[\frac{x}{2}\sqrt{\frac{1}{4} + x^2} + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\ln\left(x + \sqrt{\frac{1}{4} + x^2}\right)\right]_0^{\sqrt{3}/2}$

(We used FORMULA 2 with $a = \frac{1}{2}$)

$= \left[\frac{x}{2}\sqrt{1+4x^2} + \frac{1}{4}\ln\left(x + \frac{1}{2}\sqrt{1+4x^2}\right)\right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}}{4}\sqrt{1+4\left(\frac{3}{4}\right)} + \frac{1}{4}\ln\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{3}{4}\right)}\right) - \frac{1}{4}\ln\frac{1}{2}$

$= \frac{\sqrt{3}}{4}(2) + \frac{1}{4}\ln\left(\frac{\sqrt{3}}{2} + 1\right) + \frac{1}{4}\ln 2 = \frac{\sqrt{3}}{2} + \frac{1}{4}\ln\left(\sqrt{3} + 2\right)$

59. $A = \int_0^3 \frac{dx}{\sqrt{x+1}} = \left[2\sqrt{x+1}\right]_0^3 = 2;$



$\overline{x} = \frac{1}{A}\int_0^3 \frac{x\,dx}{\sqrt{x+1}} = \frac{1}{A}\int_0^3 \sqrt{x+1}\,dx - \frac{1}{A}\int_0^3 \frac{dx}{\sqrt{x+1}}$

$= \frac{1}{2}\cdot\frac{2}{3}\left[(x+1)^{3/2}\right]_0^3 - 1 = \frac{4}{3};$

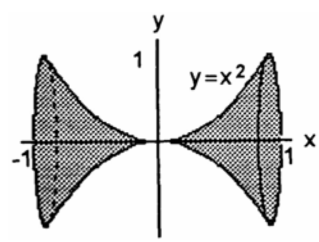
(We used FORMULA 11 with $a = 1, b = 1, n = 1$ and $a = 1, b = 1, n = -1$)

$\overline{y} = \frac{1}{2A}\int_0^3 \frac{dx}{x+1} = \frac{1}{4}\left[\ln(x+1)\right]_0^3 = \frac{1}{4}\ln 4 = \frac{1}{2}\ln 2 = \ln\sqrt{2}$

60. $M_y = \int_0^3 x\left(\frac{36}{2x+3}\right)dx = 18\int_0^3 \frac{2x+3}{2x+3}\,dx - 54\int_0^3 \frac{dx}{2x+3} = \left[18x - 27\ln|2x+3|\right]_0^3 = 18\cdot 3 - 27\ln 9 - (-27\ln 3)$

$= 54 - 27\cdot 2\ln 3 + 27\ln 3 = 54 - 27\ln 3$

61. $S = 2\pi\int_{-1}^1 x^2\sqrt{1+4x^2}\,dx;$ $[u = 2x,\ du = 2\,dx]$



$\to \frac{\pi}{4}\int_{-2}^2 u^2\sqrt{1+u^2}\,du$

$= \frac{\pi}{4}\left[\frac{u}{8}\left(1+2u^2\right)\sqrt{1+u^2} - \frac{1}{8}\ln\left(u + \sqrt{1+u^2}\right)\right]_{-2}^2$

(We used FORMULA 22 with $a = 1$)

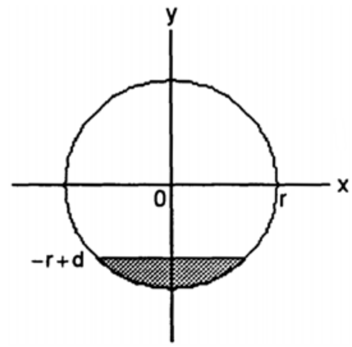
$= \frac{\pi}{4}\left[\frac{2}{8}(1+2\cdot 4)\sqrt{1+4} - \frac{1}{8}\ln\left(2 + \sqrt{1+4}\right) + \frac{2}{8}(1+2\cdot 4)\sqrt{1+4} + \frac{1}{8}\ln\left(-2 + \sqrt{1+4}\right)\right]$

$= \frac{\pi}{4}\left[\frac{9}{2}\sqrt{5} - \frac{1}{8}\ln\left(\frac{2+\sqrt{5}}{-2+\sqrt{5}}\right)\right] \approx 7.62$

62. (a) The volume of the filled part equals the length of the tank times the area of the shaded region shown in the accompanying figure. Consider a layer of gasoline of thickness dy located at height y where $-r < y < -r+d$. The width of this layer is $2\sqrt{r^2-y^2}$. Therefore,

$A = 2\int_{-r}^{-r+d} \sqrt{r^2-y^2}\,dy$ and

$V = L \cdot A = 2L\int_{-r}^{-r+d} \sqrt{r^2-y^2}\,dy$



(b) $2L\int_{-r}^{-r+d} \sqrt{r^2-y^2}\,dy = 2L\left[\frac{y\sqrt{r^2-y^2}}{2} + \frac{r^2}{2}\sin^{-1}\frac{y}{r}\right]_{-r}^{-r+d}$

(We used FORMULA 29 with $a = r$)

$$= 2L\left[\frac{(d-r)}{2}\sqrt{2rd-d^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{d-r}{r}\right) + \frac{r^2}{2}\left(\frac{\pi}{2}\right)\right] = 2L\left[\left(\frac{d-r}{2}\right)\sqrt{2rd-d^2} + \left(\frac{r^2}{2}\right)\left(\sin^{-1}\left(\frac{d-r}{r}\right) + \frac{\pi}{2}\right)\right]$$

63. The integrand $f(x) = \sqrt{x-x^2}$ is nonnegative, so the integral is maximized by integrating over the function's entire domain, which runs from $x = 0$ to $x = 1$

$$\Rightarrow \int_0^1 \sqrt{x-x^2}\,dx = \int_0^1 \sqrt{2\cdot\frac{1}{2}x - x^2}\,dx = \left[\frac{\left(x-\frac{1}{2}\right)}{2}\sqrt{2\cdot\frac{1}{2}x-x^2} + \frac{\left(\frac{1}{2}\right)^2}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

(We used FORMULA 48 with $a = \frac{1}{2}$)

$$= \left[\frac{\left(x-\frac{1}{2}\right)}{2}\sqrt{x-x^2} + \frac{1}{8}\sin^{-1}(2x-1)\right]_0^1 = \frac{1}{8}\cdot\frac{\pi}{2} - \frac{1}{8}\left(-\frac{\pi}{2}\right) = \frac{\pi}{8}$$

64. The integrand is maximized by integrating $g(x) = x\sqrt{2x-x^2}$ over the largest domain on which g is nonnegative, namely $[0, 2]$

$$\Rightarrow \int_0^2 x\sqrt{2x-x^2}\,dx = \left[\frac{(x+1)(2x-3)\sqrt{2x-x^2}}{6} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^2$$

(We used FORMULA 51 with $a = 1$)

$$= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

<u>CAS</u> <u>EXPLORATIONS</u>

65. Example CAS commands:

<u>Maple</u>:

```
q1:= Int( x*ln(x), x );          # (a)
q1 = value( q1 );
q2:= Int( x^2*ln(x), x );        # (b)
q2 = value( q2 );
q3:= Int( x^3*ln(x), x );        # (c)
```

```
q3 = value( q3 );
q4 := Int( x^4*ln(x), x );                      # (d)
q4 = value( q4 );
q5 := Int( x^n*ln(x), x );                      # (e)
q6 = value( q5 );
q7 := simplify(q6) assuming n::integer;
q5 = collect( factor(q7), ln(x) );
```

66. Example CAS commands:

Maple:

```
q1 := Int( ln(x)/x, x );                        # (a)
q1 = value( q1 );
q2 := Int( ln(x)/x^2, x );                      # (b)
q2 = value( q2 );
q3 := Int( ln(x)/x^3, x );                      # (c)
q3 = value( q3 );
q4 := Int( ln(x)/x^4, x );                      # (d)
q4 = value( q4 );
q5 := Int( ln(x)/x^n, x );                                # (e)
q6 := value( q5 );
q7 := simplify(q6) assuming n::integer;
q5 = collect( factor(q7), ln(x) );
```

67. Example CAS commands:

Maple:

```
q := Int( sin(x)^n/sin(x)^n+cos(x)^n), x=0..Pi/2 );     # (a)
q = value( q );
q1 := eval( q, n=1 ):                                   # (b)
q1 = value( q1 );
for N in [1,2,3,5,7] do
  q1 := eval( q, n=N );
  print( q1 = evalf(q1) );
end do:
qq1 := PDEtools[dchange]( x=Pi/2-u, q, [u] );           # (c)
qq2 := subs( u=x, qq1 );
qq3 := q + q = q + qq2;
qq4 := combine( qq3 );
```

```
qq5 := value( qq4 );
simplify( qq5/2 );
```

65-67. Example CAS commands:

Mathematica: (functions may vary)

In Mathematica, the natural log is denoted by Log rather than Ln, Log base 10 is Log[x, 10]

Mathematica does not include an arbitrary constant when computing an indefinite integral,

```
Clear[x, f, n]
f[x_]:=Log[x] / x^n
Integrate[f[x], x]
```

For exercise 67, Mathematica cannot evaluate the integral with arbitrary n. It does evaluate the integral (value is $\pi/4$ in each case) for small values of n, but for large values of n, it identifies this integral as Indeterminate

65. (e) $\int x^n \ln x\, dx = \frac{x^{n+1}\ln x}{n+1} - \frac{1}{n+1}\int x^n\, dx,\ n \neq -1$

(We used FORMULA 110 with $a = 1, m = 1$)

$= \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C$

66. (e) $\int x^{-n} \ln x\, dx = \frac{x^{-n+1}\ln x}{-n+1} - \frac{1}{(-n)+1}\int x^{-n}\, dx,\ n \neq 1$

(We used FORMULA 110 with $a = 1,\ m = 1,\ n = -n$)

$= \frac{x^{1-n}\ln x}{1-n} - \frac{1}{1-n}\left(\frac{x^{1-n}}{1-n}\right) + C = \frac{x^{1-n}}{1-n}\left(\ln x - \frac{1}{1-n}\right) + C$

67. (a) Neither MAPLE nor MATHEMATICA can find this integral for arbitrary n.

(b) MAPLE and MATHEMATICA get stuck at about $n = 5$.

(c) Let $x = \frac{\pi}{2} - u \Rightarrow dx = -du$; $x = 0 \Rightarrow u = \frac{\pi}{2}$, $x = \frac{\pi}{2} \Rightarrow u = 0$;

$$I = \int_0^{\pi/2} \frac{\sin^n x\, dx}{\sin^n x + \cos^n x} = \int_{\pi/2}^{0} \frac{-\sin^n\left(\frac{\pi}{2} - u\right) du}{\sin^n\left(\frac{\pi}{2} - u\right) + \cos^n\left(\frac{\pi}{2} - u\right)} = \int_0^{\pi/2} \frac{\cos^n u\, du}{\cos^n u + \sin^n u} = \int_0^{\pi/2} \frac{\cos^n x\, dx}{\cos^n x + \sin^n x}$$

$$\Rightarrow I + I = \int_0^{\pi/2} \left(\frac{\sin^n x + \cos^n x}{\sin^n + \cos^n x}\right) dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

8.7 NUMERICAL INTEGRATION

1. $\int_1^2 x\,dx$

I. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{2-1}{4}=\frac{1}{4}\Rightarrow\frac{\Delta x}{2}=\frac{1}{8}$;
$\sum mf(x_i)=12\Rightarrow T=\frac{1}{8}(12)=\frac{3}{2}$;
$f(x)=x\Rightarrow f'(x)=1\Rightarrow f''=0$
$\Rightarrow M=0\Rightarrow |E_T|=0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	2	5/2
x_2	3/2	3/2	2	3
x_3	7/4	7/4	2	7/2
x_4	2	2	1	2

(b) $\int_1^2 x\,dx=\left[\frac{x^2}{2}\right]_1^2=2-\frac{1}{2}=\frac{3}{2}$
$\Rightarrow |E_T|=\int_1^2 x\,dx-T=0$

(c) $\frac{|E_T|}{\text{True Value}}\times 100=0\%$

II. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{2-1}{4}=\frac{1}{4}\Rightarrow\frac{\Delta x}{3}=\frac{1}{12}$;
$\sum mf(x_i)=18\Rightarrow S=\frac{1}{12}(18)=\frac{3}{2}$;
$f^{(4)}(x)=0\Rightarrow M=0\Rightarrow |E_s|=0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	4	5
x_2	3/2	3/2	2	3
x_3	7/4	7/4	4	7
x_4	2	2	1	2

(b) $\int_1^2 x\,dx=\frac{3}{2}\Rightarrow |E_s|=\int_1^2 x\,dx-S=\frac{3}{2}-\frac{3}{2}=0$

(c) $\frac{|E_s|}{\text{True Value}}\times 100=0\%$

2. $\int_1^3 (2x-1)\,dx$

I. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{3-1}{4}=\frac{2}{4}=\frac{1}{2}$
$\Rightarrow\frac{\Delta x}{2}=\frac{1}{4}$;
$\sum mf(x_i)=24\Rightarrow T=\frac{1}{4}(24)=6$;
$f(x)=2x-1\Rightarrow f'(x)=2\Rightarrow f''=0$
$\Rightarrow M=0\Rightarrow |E_T|=0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	2	4
x_2	2	3	2	6
x_3	5/2	4	2	8
x_4	3	5	1	5

(b) $\int_1^3 (2x-1)\,dx=\left[x^2-x\right]_1^3=(9-3)-(1-1)=6\Rightarrow |E_T|=\int_1^3(2x-1)\,dx-T=6-6=0$

(c) $\frac{|E_s|}{\text{True Value}}\times 100=0\%$

II. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{3-1}{4}=\frac{2}{4}=\frac{1}{2}\Rightarrow\frac{\Delta x}{3}=\frac{1}{6}$;
$\sum mf(x_i)=36\Rightarrow S=\frac{1}{6}(36)=6$;
$f^{(4)}(x)=0\Rightarrow M=0\Rightarrow |E_s|=0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	4	8
x_2	2	3	2	6
x_3	5/2	4	4	16
x_4	3	5	1	5

(b) $\int_1^3 (2x-1)\,dx=6\Rightarrow |E_s|=\int_1^3(2x-1)\,dx-S$
$=6-6=0$

(c) $\frac{|E_s|}{\text{True Value}}\times 100=0\%$

3. $\int_{-1}^{1}\left(x^2+1\right)dx$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	–1	2	1	2
x_1	–1/2	5/4	2	5/2
x_2	0	1	2	2
x_3	1/2	5/4	2	5/2
x_4	1	2	1	2

I. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{1-(-1)}{4}=\frac{2}{4}=\frac{1}{2}\Rightarrow\frac{\Delta x}{2}=\frac{1}{4}$;

$\sum mf(x_i)=11\Rightarrow T=\frac{1}{4}(11)=2.75$;

$f(x)=x^2+1\Rightarrow f'(x)=2x\Rightarrow f''(x)=2$

$\Rightarrow M=2\Rightarrow\left|E_T\right|\le\frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(2)=\frac{1}{12}$ or 0.08333

(b) $\int_{-1}^{1}\left(x^2+1\right)dx=\left[\frac{x^3}{3}+x\right]_{-1}^{1}=\left(\frac{1}{3}+1\right)-\left(-\frac{1}{3}-1\right)=\frac{8}{3}\Rightarrow E_T=\int_{-1}^{1}\left(x^2+1\right)dx-T=\frac{8}{3}-\frac{11}{4}=-\frac{1}{12}$

$\Rightarrow\left|E_T\right|=\left|-\frac{1}{12}\right|\approx 0.08333$

(c) $\frac{\left|E_T\right|}{\text{True Value}}\times 100=\left(\frac{\frac{1}{12}}{\frac{8}{3}}\right)\times 100\approx 3\%$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	–1	2	1	2
x_1	–1/2	5/4	4	5
x_2	0	1	2	2
x_3	1/2	5/4	4	5
x_4	1	2	1	2

II. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{1-(-1)}{4}=\frac{2}{4}=\frac{1}{2}\Rightarrow\frac{\Delta x}{3}=\frac{1}{6}$;

$\sum mf(x_i)=16\Rightarrow S=\frac{1}{6}(16)\ =\frac{8}{3}=2.66667$;

$f^3(x)=0\Rightarrow f(4)(x)=0\Rightarrow M=0$

$\Rightarrow\left|E_s\right|=0$

(b) $\int_{-1}^{1}\left(x^2+1\right)dx=\left[\frac{x^3}{3}+x\right]_{-1}^{1}=\frac{8}{3}$

$\Rightarrow\left|E_s\right|=\int_{-1}^{1}\left(x^2+1\right)dx-S=\frac{8}{3}-\frac{8}{3}=0$

(c) $\frac{\left|E_s\right|}{\text{True Value}}\times 100=0\%$

4. $\int_{-2}^{0}\left(x^2-1\right)dx$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	–2	3	1	3
x_1	–3/2	5/4	2	5/2
x_2	–1	0	2	0
x_3	–1/2	–3/4	2	–3/2
x_4	0	–1	1	–1

I. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{0-(-2)}{4}=\frac{2}{4}=\frac{1}{2}$

$\Rightarrow\frac{\Delta x}{2}=\frac{1}{4}$; $\sum mf(x_i)=3\Rightarrow T=\frac{1}{4}(3)=\frac{3}{4}$;

$f(x)=x^2-1\Rightarrow f'(x)=2x\Rightarrow f''(x)=2$

$\Rightarrow M=2\Rightarrow\left|E_T\right|\le\frac{0-(-2)}{12}\left(\frac{1}{2}\right)^2(2)=\frac{1}{12}$

≈ 0.08333

(b) $\int_{-2}^{0}\left(x^2-1\right)dx=\left[\frac{x^3}{3}-x\right]_{-2}^{0}=0-\left(-\frac{8}{3}+2\right)=\frac{2}{3}\Rightarrow E_T=\int_{-2}^{0}\left(x^2-1\right)dx-T=\frac{2}{3}-\frac{3}{4}=-\frac{1}{12}\Rightarrow\left|E_T\right|=\frac{1}{12}$

(c) $\frac{\left|E_T\right|}{\text{True Value}}\times 100=\left(\frac{\frac{1}{12}}{\frac{2}{3}}\right)\times 100\approx 13\%$

II. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{0-(-2)}{4}=\frac{2}{4}=\frac{1}{2}\Rightarrow\frac{\Delta x}{3}=\frac{1}{6}$;

$\sum mf(x_i)=4\Rightarrow S=\frac{1}{6}(4)=\frac{2}{3}$; $f^{(3)}(x)=0$

$\Rightarrow f^{(4)}(x)=0\Rightarrow M=0\Rightarrow |E_s|=0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	-2	3	1	3
x_1	$-3/2$	5/4	4	5
x_2	-1	0	2	0
x_3	$-1/2$	$-3/4$	4	-3
x_4	0	-1	1	-1

(b) $\int_{-2}^{0}\left(x^2-1\right)dx=\frac{2}{3}\Rightarrow |E_s|=\int_{-2}^{0}\left(x^2-1\right)dx-S$

$=\frac{2}{3}-\frac{2}{3}=0$

(c) $\frac{|E_s|}{\text{True Value}}\times 100=0\%$

5. $\int_0^2\left(t^3+t\right)dt$

I. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{2-0}{4}=\frac{2}{4}=\frac{1}{2}\Rightarrow\frac{\Delta x}{2}=\frac{1}{4}$;

$\sum mf(t_i)=25\Rightarrow T=\frac{1}{4}(25)=\frac{25}{4}$;

$f(t)=t^3+t\Rightarrow f'(t)=3t^2+1\Rightarrow f''(t)=6t$

$\Rightarrow M=12=f''(2)\Rightarrow |E_T|\le\frac{2-0}{12}\left(\frac{1}{2}\right)^2(12)=\frac{1}{2}$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/2	5/8	2	5/4
t_2	1	2	2	4
t_3	3/2	39/8	2	39/4
t_4	2	10	1	10

(b) $\int_0^2\left(t^3+t\right)dt=\left[\frac{t^4}{4}+\frac{t^2}{2}\right]_0^2=\left(\frac{2^4}{4}+\frac{2^2}{2}\right)-0=6\Rightarrow |E_T|=\int_0^2\left(t^3+t\right)dt-T=6-\frac{25}{4}=-\frac{1}{4}\Rightarrow |E_T|=\frac{1}{4}$

(c) $\frac{|E_T|}{\text{True Value}}\times 100=\frac{\left|-\frac{1}{4}\right|}{6}\times 100\approx 4\%$

II. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{2-0}{4}=\frac{2}{4}=\frac{1}{2}\Rightarrow\frac{\Delta x}{3}=\frac{1}{6}$;

$\sum mf(t_i)=36\Rightarrow S=\frac{1}{6}(36)=6$;

$f^{(3)}(t)=6\Rightarrow f^{(4)}(t)=0\Rightarrow M=0\Rightarrow |E_s|=0$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/2	5/8	4	5/2
t_2	1	2	2	4
t_3	3/2	39/8	4	39/2
t_4	2	10	1	10

(b) $\int_0^2\left(t^3+t\right)dt=6\Rightarrow |E_s|=\int_0^2\left(t^3+t\right)dt-S$

$=6-6=0$

(c) $\frac{|E_s|}{\text{True Value}}\times 100=0\%$

6. $\int_{-1}^{1}\left(t^3+1\right)dt$

I. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{1-(-1)}{4}=\frac{2}{4}=\frac{1}{2}\Rightarrow\frac{\Delta x}{2}=\frac{1}{4}$;

$\sum mf(t_i)=8\Rightarrow T=\frac{1}{4}(8)=2$;

$f(t)=t^3+1\Rightarrow f'(t)=3t^2\Rightarrow f''(t)=6t$

$\Rightarrow M=6=f''(1)\Rightarrow |E_T|\le\frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(6)=\frac{1}{4}$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	-1	0	1	0
t_1	$-1/2$	7/8	2	7/4
t_2	0	1	2	2
t_3	1/2	9/8	2	9/4
t_4	1	2	1	2

(b) $\int_{-1}^{1}\left(t^3+1\right)dt=\left[\frac{t^4}{4}+t\right]_{-1}^{1}=\left(\frac{1^4}{4}+1\right)-\left(\frac{(-1)^4}{4}+(-1)\right)=2\Rightarrow |E_T|=\int_{-1}^{1}\left(t^3+1\right)dt-T=2-2=0$

(c) $\frac{|E_T|}{\text{True Value}}\times 100=0\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;

$\sum mf(t_i) = 12 \Rightarrow S = \frac{1}{6}(12) = 2$;

$f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_s| = 0$

(b) $\int_{-1}^{1}(t^3+1)\,dt = 2 \Rightarrow |E_s| = \int_{-1}^{1}(t^3+1)\,dt - S$
$= 2 - 2 = 0$

(c) $\frac{|E_s|}{\text{True Valule}} \times 100 = 0\%$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	−1	0	1	0
t_1	−1/2	7/8	4	7/2
t_2	0	1	2	2
t_3	1/2	9/8	4	9/2
t_4	1	2	1	2

7. $\int_1^2 \frac{1}{s^2}\,ds$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;

$\sum mf(s_i) = \frac{179{,}573}{44{,}100} \Rightarrow T = \frac{1}{8}\left(\frac{179{,}573}{44{,}100}\right) = \frac{179{,}573}{352{,}800}$

≈ 0.50899; $f(s) = \frac{1}{s^2} \Rightarrow f'(s) = -\frac{2}{s^3}$

$\Rightarrow f''(s) = \frac{6}{s^4} \Rightarrow M = 6 = f''(1)$

$\Rightarrow |E_T| \le \frac{2-1}{12}\left(\frac{1}{4}\right)^2(6) = \frac{1}{32} = 0.03125$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	1	1	1	1
s_1	5/4	16/25	2	32/25
s_2	3/2	4/9	2	8/9
s_3	7/4	16/49	2	32/49
s_4	2	1/4	1	1/4

(b) $\int_1^2 \frac{1}{s^2}\,ds = \int_1^2 s^{-2}\,ds = \left[-\frac{1}{s}\right]_1^2 = -\frac{1}{2} - \left(-\frac{1}{1}\right) = \frac{1}{2} \Rightarrow E_T = \int_1^2 \frac{1}{s^2}\,ds - T = \frac{1}{2} - 0.50899 = -0.00899$
$\Rightarrow |E_T| = 0.00899$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.00899}{0.5} \times 100 \approx 2\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;

$\sum mf(s_i) = \frac{264{,}821}{44{,}100} \Rightarrow S = \frac{1}{12}\left(\frac{264{,}821}{44{,}100}\right)$

$= \frac{264{,}821}{529{,}200} \approx 0.50042$;

$f^{(3)}(s) = -\frac{24}{s^5} \Rightarrow f^{(4)}(s) = \frac{120}{s^6} \Rightarrow M = 120$

$\Rightarrow |E_s| \le \left|\frac{2-1}{180}\right|\left(\frac{1}{4}\right)^4(120) = \frac{1}{384} \approx 0.00260$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	1	1	1	1
s_1	5/4	16/25	4	64/25
s_2	3/2	4/9	2	8/9
s_3	7/4	16/49	4	64/49
s_4	2	1/4	1	1/4

(b) $\int_1^2 \frac{1}{s^2}\,ds = \frac{1}{2} \Rightarrow E_s = \int_1^2 \frac{1}{s^2}\,ds - S = \frac{1}{2} - 0.50042 = -0.00042 \Rightarrow |E_s| = 0.00042$

(c) $\frac{|E_s|}{\text{True Value}} \times 100 = \frac{0.0004}{0.5} \times 100 \approx 0.08\%$

8. $\int_2^4 \frac{1}{(s-1)^2}\,ds$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;

$\sum mf(s_i) = \frac{1269}{450} \Rightarrow T = \frac{1}{4}\left(\frac{1269}{450}\right) = \frac{1269}{1800} = 0.70500$;

$f(s) = (s-1)^{-2} \Rightarrow f'(s) = \frac{-2}{(s-1)^3} \Rightarrow f''(s) = \frac{6}{(s-1)^4}$

$\Rightarrow |E_T| \le \frac{4-2}{12}\left(\frac{1}{2}\right)^2(6) = \frac{1}{4} = 0.25 \Rightarrow M = 6$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	2	1	1	1
s_1	5/2	4/9	2	8/9
s_2	3	1/4	2	1/2
s_3	7/2	4/25	2	8/25
s_4	4	1/9	1	1/9

(b) $\int_2^4 \frac{1}{(s-1)^2}\,ds = \left[\frac{-1}{(s-1)}\right]_2^4 = \left(\frac{-1}{4-1}\right) - \left(\frac{-1}{2-1}\right) = \frac{2}{3} \Rightarrow E_T = \int_2^4 \frac{1}{(s-1)^2}\,ds - T = \frac{2}{3} - 0.705 \approx -0.03833$
$\Rightarrow \left|E_T\right| \approx 0.03833$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03833}{\left(\frac{2}{3}\right)} \times 100 \approx 6\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
$\sum mf(s_i) = \frac{1813}{450} \Rightarrow S = \frac{1}{6}\left(\frac{1813}{450}\right)$
$= \frac{1813}{2700} \approx 0.67148$; $f^{(3)}(s) = \frac{-24}{(s-1)^5}$
$\Rightarrow f^{(4)}(s) = \frac{120}{(s-1)^6} \Rightarrow M = 120$
$\Rightarrow \left|E_s\right| \le \frac{4-2}{180}\left(\frac{1}{2}\right)^4 (120) = \frac{1}{12} \approx 0.08333$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	2	1	1	1
s_1	5/2	4/9	4	16/9
s_2	3	1/4	2	1/2
s_3	7/2	4/25	4	16/25
s_4	4	1/9	1	1/9

(b) $\int_2^4 \frac{1}{(s-1)^2}\,ds = \frac{2}{3} \Rightarrow E_s = \int_2^4 \frac{1}{(s-1)^2}\,ds - S \approx \frac{2}{3} - 0.67148 = -0.00481 \Rightarrow \left|E_s\right| \approx 0.00481$

(c) $\frac{|E_s|}{\text{True Value}} \times 100 = \frac{0.00481}{\left(\frac{2}{3}\right)} \times 100 \approx 1\%$

9. $\int_0^{\pi} \sin t\,dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{\Delta x}{2} = \frac{\pi}{8}$;
$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.8284$;
$\Rightarrow T = \frac{\pi}{8}\left(2 + 2\sqrt{2}\right) \approx 1.89612$; $f(t) = \sin t$
$\Rightarrow f'(t) = \cos t \Rightarrow f''(t) = -\sin t \Rightarrow M = 1$
$\Rightarrow \left|E_T\right| \le \frac{\pi-0}{12}\left(\frac{\pi}{4}\right)^2 (1) = \frac{\pi^3}{192} \approx 0.16149$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	$\pi/4$	$\sqrt{2}/2$	2	$\sqrt{2}$
t_2	$\pi/2$	1	2	2
t_3	$3\pi/4$	$\sqrt{2}/2$	2	$\sqrt{2}$
t_4	π	0	1	0

(b) $\int_0^{\pi} \sin t\,dt = \left[-\cos t\right]_0^{\pi} = (-\cos \pi) - (-\cos 0) = 2 \Rightarrow \left|E_T\right| = \int_0^{\pi} \sin t\,dt - T \approx 2 - 1.89612 = 0.10388$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.10388}{2} \times 100 \approx 5\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{12}$;
$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.6569 \Rightarrow S = \frac{\pi}{12}\left(2 + 4\sqrt{2}\right)$
≈ 2.00456; $f^{(3)}(t) = -\cos t \Rightarrow f^{(4)}(t) = \sin t$
$\Rightarrow M = 1 \Rightarrow \left|E_s\right| \le \frac{\pi-0}{180}\left(\frac{\pi}{4}\right)^4 (1) \approx 0.00664$

(b) $\int_0^{\pi} \sin t\,dt = 2 \Rightarrow E_s = \int_0^{\pi} \sin t\,dt - S \approx 2 - 2.00456$
$= -0.00456 \Rightarrow \left|E_s\right| \approx 0.00456$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	$\pi/4$	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_2	$\pi/2$	1	2	2
t_3	$3\pi/4$	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_4	π	0	1	0

(c) $\frac{|E_s|}{\text{True Value}} \times 100 = \frac{0.00456}{2} \times 100 \approx 0\%$

10. $\int_0^1 \sin \pi t dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;

$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.828$

$\Rightarrow T = \frac{1}{8}\left(2 + 2\sqrt{2}\right) \approx 0.60355$; $f(t) = \sin \pi t$

$\Rightarrow f'(t) = \pi \cos \pi t \Rightarrow f''(t) = -\pi^2 \sin \pi t$

$\Rightarrow M = \pi^2 \Rightarrow |E_T| \le \frac{1-0}{12}\left(\frac{1}{4}\right)^2\left(\pi^2\right) \approx 0.05140$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/4	$\sqrt{2}/2$	2	$\sqrt{2}$
t_2	1/2	1	2	2
t_3	3/4	$\sqrt{2}/2$	2	$\sqrt{2}$
t_4	1	0	1	0

(b) $\int_0^1 \sin \pi t \, dt = \left[-\frac{1}{\pi}\cos \pi t\right]_0^1 = \left(-\frac{1}{\pi}\cos \pi\right) - \left(-\frac{1}{\pi}\cos 0\right) = \frac{2}{\pi} \approx 0.63662 \Rightarrow |E_T| = \int_0^1 \sin \pi t \, dt - T$
$\approx \frac{2}{\pi} - 0.60355 = 0.03307$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03307}{\left(\frac{2}{\pi}\right)} \times 100 \approx 5\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;

$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.65685$

$\Rightarrow S = \frac{1}{12}\left(2 + 4\sqrt{2}\right) \approx 0.63807$;

$f^{(3)}(t) = -\pi^3 \cos \pi t \Rightarrow f^{(4)}(t) = \pi^4 \sin \pi t$

$\Rightarrow M = \pi^4 \Rightarrow |E_s| \le \frac{1-0}{180}\left(\frac{1}{4}\right)^4\left(\pi^4\right) \approx 0.00211$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/4	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_2	1/2	1	2	2
t_3	3/4	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_4	1	0	1	0

(b) $\int_0^1 \sin \pi t \, dt = \frac{2}{\pi} \approx 0.63662 \Rightarrow E_s = \int_0^1 \sin \pi t \, dt - S \approx \frac{2}{\pi} - 0.63807 = -0.00145 \Rightarrow |E_s| \approx 0.00145$

(c) $\frac{|E_s|}{\text{True Value}} \times 100 = \frac{0.00145}{\left(\frac{2}{\pi}\right)} \times 100 \approx 0\%$

11. (a) $M = 0$ (see Exercise 1): Then $n = 1 \Rightarrow \Delta x = 1 \Rightarrow |E_T| = \frac{1}{12}(1)^2(0) = 0 < 10^{-4}$

(b) $M = 0$ (see Exercise 1): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = \frac{1}{2} \Rightarrow |E_s| = \frac{1}{180}\left(\frac{1}{2}\right)^4(0) = 0 < 10^{-4}$

12. (a) $M = 0$ (see Exercise 2): Then $n = 1 \Rightarrow \Delta x = 2 \Rightarrow |E_T| = \frac{2}{12}(2)^2(0) = 0 < 10^{-4}$

(b) $M = 0$ (see Exercise 2): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180}(1)^4(0) < 10^{-4}$

13. (a) $M = 2$ (see Exercise 3): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(2) = \frac{4}{3n^2} < 10^{-4} \Rightarrow n^2 > \frac{4}{3}\left(10^4\right) \Rightarrow n > \sqrt{\frac{4}{3}\left(10^4\right)}$
$\Rightarrow n > 115.4$, so let $n = 116$

(b) $M = 0$ (see Exercise 3): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

14. (a) $M = 2$ (see Exercise 4): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(2) = \frac{4}{3n^2} < 10^{-4} \Rightarrow n^2 > \frac{4}{3}\left(10^4\right) \Rightarrow n > \sqrt{\frac{4}{3}\left(10^4\right)}$
$\Rightarrow n > 115.4$, so let $n = 116$

(b) $M = 0$ (see Exercise 4): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

15. (a) $M = 12$ (see Exercise 5): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2 (12) = \frac{8}{n^2} < 10^{-4} \Rightarrow n^2 > 8\left(10^4\right) \Rightarrow n > \sqrt{8\left(10^4\right)}$ $\Rightarrow n > 282.8$, so let $n = 283$

(b) $M = 0$ (see Exercise 5): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

16. (a) $M = 6$ (see Exercise 6): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2 (6) = \frac{4}{n^2} < 10^{-4} \Rightarrow n^2 > 4\left(10^4\right) \Rightarrow n > \sqrt{4\left(10^4\right)}$ $= 200$, so let $n = 201$

(b) $M = 0$ (Exercise 6): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

17. (a) $M = 6$ (see Exercise 7): Then $\Delta x = \frac{1}{n} \Rightarrow |E_T| \le \frac{1}{12}\left(\frac{1}{n}\right)^2 (6) = \frac{1}{2n^2} < 10^{-4} \Rightarrow n^2 > \frac{1}{2}\left(10^4\right) \Rightarrow n > \sqrt{\frac{1}{2}\left(10^4\right)}$ $\Rightarrow n > 70.7$, so let $n = 71$

(b) $M = 120$ (see Exercise 7): Then $\Delta x = \frac{1}{n} \Rightarrow |E_S| = \frac{1}{180}\left(\frac{1}{n}\right)^4 (120) = \frac{2}{3n^4} < 10^{-4} \Rightarrow n^4 > \frac{2}{3}\left(10^4\right)$ $\Rightarrow n > \sqrt[4]{\frac{2}{3}\left(10^4\right)} \Rightarrow n = 9.04$, so let $n = 10$ (n must be even)

18. (a) $M = 6$ (see Exercise 8): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2 (6) = \frac{4}{n^2} < 10^{-4} \Rightarrow n^2 > 4\left(10^4\right) \Rightarrow n > \sqrt{4\left(10^4\right)}$ $\Rightarrow n > 200$, so let $n = 201$

(b) $M = 120$ (see Exercise 8): Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \le \frac{2}{180}\left(\frac{2}{n}\right)^4 (120) = \frac{64}{3n^4} < 10^{-4} \Rightarrow n^4 > \frac{64}{3}\left(10^4\right)$ $\Rightarrow n > \sqrt[4]{\frac{64}{3}\left(10^4\right)} \Rightarrow n > 21.5$, so let $n = 22$ (n must be even)

19. (a) $f(x) = \sqrt{x+1} \Rightarrow f'(x) = \frac{1}{2}(x+1)^{-1/2} \Rightarrow f''(x) = -\frac{1}{4}(x+1)^{-3/2} = -\frac{1}{4\left(\sqrt{x+1}\right)^3} \Rightarrow M = \frac{1}{4\left(\sqrt{1}\right)^3} = \frac{1}{4}$.

Then $\Delta x = \frac{3}{n} \Rightarrow |E_T| \le \frac{3}{12}\left(\frac{3}{n}\right)^2\left(\frac{1}{4}\right) = \frac{9}{16n^2} < 10^{-4} \Rightarrow n^2 > \frac{9}{16}\left(10^4\right) \Rightarrow n > \sqrt{\frac{9}{16}\left(10^4\right)} \Rightarrow n > 75$, so let $n = 76$

(b) $f^{(3)}(x) = \frac{3}{8}(x+1)^{-5/2} \Rightarrow f^{(4)}(x) = -\frac{15}{16}(x+1)^{-7/2} = -\frac{15}{16\left(\sqrt{x+1}\right)^7} \Rightarrow M = \frac{15}{16\left(\sqrt{1}\right)^7} = \frac{15}{16}$. Then

$\Delta x = \frac{3}{n} \Rightarrow |E_S| \le \frac{3}{180}\left(\frac{3}{n}\right)^4\left(\frac{15}{16}\right) = \frac{3^5(15)}{16(180)n^4} < 10^{-4} \Rightarrow n^4 > \frac{3^5(15)\left(10^4\right)}{16(180)} \Rightarrow n > \sqrt[4]{\frac{3^5(15)\left(10^4\right)}{16(180)}} \Rightarrow n > 10.6$, so let $n = 12$ (n must be even)

20. (a) $f(x) = \frac{1}{\sqrt{x+1}} \Rightarrow f'(x) = -\frac{1}{2}(x+1)^{-3/2} \Rightarrow f''(x) = \frac{3}{4}(x+1)^{-5/2} = \frac{3}{4\left(\sqrt{x+1}\right)^5} \Rightarrow M = \frac{3}{4\left(\sqrt{1}\right)^5} = \frac{3}{4}$. Then

$\Delta x = \frac{3}{n} \Rightarrow |E_T| \le \frac{3}{12}\left(\frac{3}{n}\right)^2\left(\frac{3}{4}\right) = \frac{3^4}{48n^2} < 10^{-4} \Rightarrow n^2 > \frac{3^4\left(10^4\right)}{48} \Rightarrow n > \sqrt{\frac{3^4\left(10^4\right)}{48}} \Rightarrow n > 129.9$, so let $n = 130$

(b) $f^{(3)}(x) = -\frac{15}{8}(x+1)^{-7/2} \Rightarrow f^{(4)}(x) = \frac{105}{16}(x+1)^{-9/2} = \frac{105}{16\left(\sqrt{x+1}\right)^9} \Rightarrow M = \frac{105}{16\left(\sqrt{1}\right)^9} = \frac{105}{16}$. Then

$\Delta x = \frac{3}{n} \Rightarrow |E_S| \le \frac{3}{180}\left(\frac{3}{n}\right)^4\left(\frac{105}{16}\right) = \frac{3^5(105)}{16(180)n^4} < 10^{-4} \Rightarrow n^4 > \frac{3^5(105)\left(10^4\right)}{16(180)} \Rightarrow n > \sqrt[4]{\frac{3^5(105)\left(10^4\right)}{16(180)}} \Rightarrow n > 17.25$, so let $n = 18$ (n must be even)

21. (a) $f(x) = \sin(x+1) \Rightarrow f'(x) = \cos(x+1) \Rightarrow f''(x) = -\sin(x+1) \Rightarrow M = 1$. Then

$\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2 (1) = \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8(10^4)}{12} \Rightarrow n > \sqrt{\frac{8(10^4)}{12}} \Rightarrow n > 81.6$, so let $n = 82$

(b) $f^{(3)}(x) = -\cos(x+1) \Rightarrow f^{(4)}(x) = \sin(x+1) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_s| \le \frac{2}{180}\left(\frac{2}{n}\right)^4 (1)$

$= \frac{32}{180n^4} < 10^{-4} \Rightarrow n^4 > \frac{32(10^4)}{180} \Rightarrow n > \sqrt[4]{\frac{32(10^4)}{180}} \Rightarrow n > 6.49$, so let $n = 8$ (n must be even)

22. (a) $f(x) = \cos(x+\pi) \Rightarrow f'(x) = -\sin(x+\pi) \Rightarrow f''(x) = -\cos(x+\pi) \Rightarrow M = 1$. Then

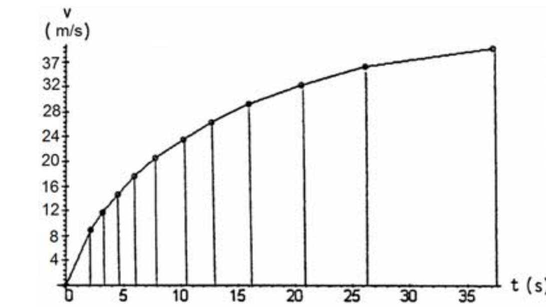
$\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2 (1) = \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8(10^4)}{12} \Rightarrow n > \sqrt{\frac{8(10^4)}{12}} \Rightarrow n > 81.6$, so let $n = 82$

(b) $f^{(3)}(x) = \sin(x+\pi) \Rightarrow f^{(4)}(x) = \cos(x+\pi) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_s| \le \frac{2}{180}\left(\frac{2}{n}\right)^4 (1)$

$= \frac{32}{180^4} < 10^{-4} \Rightarrow n^4 > \frac{32(10^4)}{180} \Rightarrow \sqrt[4]{\frac{32(10^4)}{180}} \Rightarrow n > 6.49$, so let $n = 8$ (n must be even)

23. $1\big(1.2 + 2(1.6) + 2(1.8) + \cdots\ 2(2.5) + 2.61.2 + 2(1.6) + 2(1.8) + \cdots\ 2(2.5) + 2.61.2 + 2(1.6) + 2(1.8) + \cdots$
$2(2.5) + 2.61.2 + 2(1.6) + 2(1.8) + \cdots\ 2(2.5) + 2.6\big)(5) = 106.5\ \text{m}^3$

24. Use the conversion 36 km/h = 10 m/s since time is measured in seconds. The distance traveled as the car accelerates from, say, 40 kmph = 11.11 m/s to 50 km/h = 13. m/s in $(4.5 - 3.2) = 1.3$ s is the area of the trapezoid (see figure) associated with that time interval: ½(11.11 + 13.88)(1.3) = 16.25m The total distance traveled by the car is the sum of all these eleven trapezoids (using $\frac{\Delta t}{2}$ and the table below):



v(km/h)	0	30	40	50	60	70	80	90	100	110	120	130
v(m/s)	0	8.33	11.11	13.89	16.67	19.44	22.22	25	27.78	30.56	33.33	36.11
t(s)	0	2.2	3.2	4.5	5.9	7.8	10.2	12.7	16	20.6	26.2	37.1
$\Delta t/2$	0	1.1	0.5	0.65	0.7	0.95	1.2	1.25	1.65	2.3	2.8	5.45

$(8.33)(1.1) + (19.44)(0.5) + (25)(0.65) + (30.56)(0.7) + (36.11)(0.95) + (41.67)(1.2) + (47.22)(1.25) +$
$(52.78)(1.65) + (58.56)(2.3) + (63.89)(2.8) + (69.44)(5.45) = 978.5\ \text{m} \approx 0.9785\ \text{km}$

25. Using Simpson's Rule, $\Delta x = 3 \Rightarrow \frac{\Delta x}{3} = 0.1$;
$\sum my_i = 11.65 \Rightarrow$ Cross Section Area $\approx 0.1(11.65)$ $= 1.165\ \text{m}^2$ Let x be the length of the tank. Then the Volume $V =$ (Cross Sectional Area) $x = 1.165x$. Now 2000 kg gasoline at 673 kg/m
$\Rightarrow V = \frac{2000}{673} = 2.97\ \text{m}^3$
$\Rightarrow 2.97 = 1.165x \Rightarrow x \approx 2.55$ m

	x_i	y_i	m	my_i
x_0	0	0.5	1	0.5
x_1	1	0.55	4	2.2
x_2	2	0.6	2	1.2
x_3	3	0.65	4	2.6
x_4	4	0.7	2	1.4
x_5	5	0.75	4	3.0
x_6	6	0.75	1	0.75

26. $\frac{24}{2}\left[0.019+2(0.020)+2(0.021)+\ldots+2(0.031)+0.035\right]=4.2\ L$

27. (a) $\left|E_s\right| \le \frac{b-a}{180}\left(\Delta x^4\right)M$; $n=4 \Rightarrow \Delta x=\frac{\frac{\pi}{2}-0}{4}=\frac{\pi}{8}$; $\left|f^{(4)}\right| \le 1 \Rightarrow M=1 \Rightarrow \left|E_s\right| \le \frac{\left(\frac{\pi}{2}-0\right)}{180}\left(\frac{\pi}{8}\right)^4(1) \approx 0.00021$

(b) $\Delta x=\frac{\pi}{8} \Rightarrow \frac{\Delta x}{3}=\frac{\pi}{24}$;

$\sum mf(x_i)=10.47208705$

$\Rightarrow S=\frac{\pi}{24}(10.47208705) \approx 1.37079$

	x_i	$f(x_i)$	m	$mf(x_{1i})$
x_0	0	1	1	1
x_1	$\pi/8$	0.974495358	4	3.897981432
x_2	$\pi/4$	0.900316316	2	1.800632632
x_3	$3\pi/8$	0.784213303	4	3.136853212
x_4	$\pi/2$	0.636619772	1	0.636619772

(c) $\approx\left(\frac{0.00021}{1.37079}\right)\times 100 \approx 0.015\%$

28. (a) $\Delta x=\frac{b-a}{n}=\frac{1-0}{10}=0.1 \Rightarrow \operatorname{erf}(1)=\frac{2}{\sqrt{3}}\left(\frac{0.1}{3}\right)\left(y_0+4y_1+2y_2+4y_3+\ldots+4y_9+y_{10}\right)$

$=\frac{2}{30\sqrt{\pi}}\left(e^0+4e^{-0.01}+2e^{-0.04}+4e^{-0.09}+\ldots+4e^{-0.81}+e^{-1}\right) \approx 0.843$

(b) $\left|E_s\right| \le \frac{1-0}{180}(0.1)^4(12) \approx 6.7\times 10^{-6}$

29. $T=\frac{\Delta x}{2}\left(y_0+2y_1+2y_2+2y_3+\ldots+2y_{n-1}+y_n\right)$ where $\Delta x=\frac{b-a}{n}$ and f is continuous on $[a,b]$. So $T=\frac{b-a}{n}\frac{\left(y_0+y_1+y_1+y_2+y_2+\ldots+y_{n-1}+y_{n-1}+y_n\right)}{2}=\frac{b-a}{n}\left(\frac{f(x_0)+f(x_1)}{2}+\frac{f(x_1)+f(x_2)}{2}+\ldots+\frac{f(x_{n-1})+f(x_n)}{2}\right)$. Since f is continuous on each interval $\left[x_{k-1},x_k\right]$, and $\frac{f(x_{k-1})+f(x_k)}{2}$ is always between $f(x_{k-1})$ and $f(x_k)$, there is a point c_k in $\left[x_{k-1},x_k\right]$ with $f(c_k)=\frac{f(x_{k-1})+f(x_k)}{2}$; this is a consequence of the Intermediate Value Theorem. Thus our sum is $\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f(c_k)$ which has the form $\sum_{k=1}^{n}\Delta x_k\, f(c_k)$ with $\Delta x_k=\frac{b-a}{n}$ for all k. This a Riemann Sum for f on $[a,b]$.

30. $S=\frac{\Delta x}{3}\left(y_0+4y_1+2y_2+4y_3+\ldots+2y_{n-2}+4y_{n-1}+y_n\right)$ where n is even, $\Delta x=\frac{b-a}{n}$ and f is continuous on $[a,b]$. So $S=\frac{b-a}{n}\left(\frac{y_0+4y_1+y_2}{3}+\frac{y_2+4y_3+y_4}{3}+\frac{y_4+4y_5+y_6}{3}+\ldots+\frac{y_{n-2}+4y_{n-1}+y_n}{3}\right)$

$=\frac{b-a}{\frac{n}{2}}\left(\frac{f(x_0)+4f(x_1)+f(x_2)}{6}+\frac{f(x_2)+4f(x_3)+f(x_4)}{6}+\frac{f(x_4)+4f(x_5)+f(x_6)}{6}+\ldots+\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right)$

$\frac{f(x_{2k})+4f(x_{2k+1})+f(x_{2k+2})}{6}$ is the average of the six values of the continuous function on the interval $\left[x_{2k},x_{2k+2}\right]$, so it is between the minimum and maximum of f on this interval. By the Extreme Value Theorem for continuous functions, f takes on its maximum and minimum in this interval, so there are x_a and x_b with $x_{2k} \le x_a, x_b \le x_{2k+2}$ and $f(x_a) \le \frac{f(x_{2k})+4f(x_{2k+1})+f(x_{2k+2})}{6} \le f(x_b)$.

By the Intermediate Value Theorem, there is c_k in $\left[x_{2k},x_{2k+2}\right]$ with $f(c_k)=\frac{f(x_{2k})+4f(x_{2k+1})+f(x_{2k+2})}{6}$.

So our sum has the form $\sum_{k=1}^{n/2}\Delta x_k\, f(c_k)$ with $\Delta x_k=\frac{b-a}{(n/2)}$, a Riemann sum for f on $[a,b]$.

31. (a) $a=1, e=\frac{1}{2} \Rightarrow \text{Length} = 4\int_0^{\pi/2}\sqrt{1-\frac{1}{4}\cos^2 t}\,dt$
$= 2\int_0^{\pi/2}\sqrt{4-\cos^2 t}\,dt = \int_0^{\pi/2} f(t)\,dt$; use the Trapezoid Rule with $n=10$
$\Rightarrow \Delta t = \frac{b-a}{n} = \frac{\left(\frac{\pi}{2}\right)-0}{10} = \frac{\pi}{20}$.
$\int_0^{\pi/2}\sqrt{4-\cos^2 t}\,dt \approx \sum_{n=0}^{10} mf(x_n) = 37.3686183$
$\Rightarrow T = \frac{\Delta t}{2}(37.3686183) = \frac{\pi}{40}(37.3686183)$
$= 2.934924419$
$\Rightarrow \text{Length} = 2(2.934924419) \approx 5.870$

(b) $|f''(t)| < 1 \Rightarrow M = 1$
$\Rightarrow |E_T| \le \frac{b-a}{12}\left(\Delta t^2 M\right) \le \frac{\left(\frac{\pi}{2}\right)-0}{12}\left(\frac{\pi}{20}\right)^2 1 \le 0.0032$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	1.732050808	1	1.732050808
x_1	$\pi/20$	1.739100843	2	3.478201686
x_2	$\pi/10$	1.759400893	2	3.518801786
x_3	$3\pi/20$	1.790560631	2	3.581121262
x_4	$\pi/5$	1.82906848	2	3.658136959
x_5	$\pi/4$	1.870828693	2	3.741657387
x_6	$3\pi/10$	1.911676881	2	3.823353762
x_7	$7\pi/20$	1.947791731	2	3.895583461
x_8	$2\pi/5$	1.975982919	2	3.951965839
x_9	$9\pi/20$	1.993872679	2	3.987745357
x_{10}	$\pi/2$	2	1	2

32. $\Delta x = \frac{\pi-0}{8} = \frac{\pi}{8} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{24}$;
$\sum mf(x_i) = 29.184807792$
$\Rightarrow S = \frac{\pi}{24}(29.18480779) \approx 3.82028$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	1.414213562	1	1.414213562
x_1	$\pi/8$	1.361452677	4	5.445810706
x_2	$\pi/4$	1.224744871	2	2.449489743
x_3	$3\pi/8$	1.070722471	4	4.282889883
x_4	$\pi/2$	1	2	2
x_5	$5\pi/8$	1.070722471	4	4.282889883
x_6	$3\pi/4$	1.224744871	2	2.449489743
x_7	$7\pi/8$	1.361452677	4	5.445810706
x_8	π	1.414213562	1	1.414213562

33. The length of the curve $y=\sin\left(\frac{3\pi}{20}x\right)$ from 0 to 20 is: $L=\int_0^{20}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$; $\frac{dy}{dx}=\frac{3\pi}{20}\cos\left(\frac{3\pi}{20}x\right)$
$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{9\pi^2}{400}\cos^2\left(\frac{3\pi}{20}x\right) \Rightarrow L=\int_0^{20}\sqrt{1+\frac{9\pi^2}{400}\cos^2\left(\frac{3\pi}{20}x\right)}\,dx$. Using numerical integration we find $L \approx 21.07$ cm

34. First, we'll find the length of the cosine curve: $L=\int_{-7.5}^{7.5}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$; $\frac{dy}{dx} = -\frac{7.5\pi}{15}\sin\left(\frac{\pi x}{15}\right)$
$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{15}\right) \Rightarrow L=\int_{-7.5}^{7.5}\sqrt{1+\frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{15}\right)}\,dx$. Using a numerical integrator we find $L \approx 21.9554$ m. Surface area is: $A = \text{length}\cdot\text{width} \approx (21.9554)(90) = 1{,}975.986$ m.
$\text{Cost} = 26.11A = (26.11)(1{,}975.986) = \$51{,}593$. Answers may vary slightly, depending on the numerical integration used.

35. $y=\sin x \Rightarrow \frac{dy}{dx}=\cos x \Rightarrow \left(\frac{dy}{dx}\right)^2=\cos^2 x \Rightarrow S=\int_0^{\pi} 2\pi(\sin x)\sqrt{1+\cos^2 x}\,dx$; a numerical integration gives $S\approx 14.4$

36. $y=\frac{x^2}{4} \Rightarrow \frac{dy}{dx}=\frac{x}{2} \Rightarrow \left(\frac{dy}{dx}\right)^2=\frac{x^2}{4} \Rightarrow S=\int_0^2 2\pi\left(\frac{x^2}{4}\right)\sqrt{1+\frac{x^2}{4}}\,dx$; a numerical integration gives $S\approx 5.28$

37. A calculator or computer numerical integrator yields $\sin^{-1} 0.6 \approx 0.643501109$.

38. A calculator or computer numerical integrator yields $\pi \approx 3.1415929$.

39. The amount of medication absorbed over a 12-h period is given by $\int_0^{12}\left(6-\ln\left(2t^2-3t+3\right)\right)dt$. A numerical integrator yields a value of 28.684 for this integral, so the amount of medication absorbed over a 12-h period is approximately 28.7 milligrams.

40. The average concentration of antihistamine over a 6-h period is given by $\frac{1}{6}\int_0^1\left(12.5-4\ln\left(t^2-3t+4\right)\right)dt$. A numerical integrator yields a value of 6.078 for this integral, so the average concentration is approximately 6.1 grams per liter.

8.8 IMPROPER INTEGRALS

1. $\int_0^{\infty}\frac{dx}{x^2+1}=\lim\limits_{b\to\infty}\int_0^b\frac{dx}{x^2+1}=\lim\limits_{b\to\infty}\left[\tan^{-1}x\right]_0^b=\lim\limits_{b\to\infty}\left(\tan^{-1}b-\tan^{-1}0\right)=\frac{\pi}{2}-0=\frac{\pi}{2}$

2. $\int_1^{\infty}\frac{dx}{x^{1.001}}=\lim\limits_{b\to\infty}\int_1^b\frac{dx}{x^{1.001}}=\lim\limits_{b\to\infty}\left[-1000x^{-0.001}\right]_1^b=\lim\limits_{b\to\infty}\left(\frac{-1000}{b^{0.001}}+1000\right)=1000$

3. $\int_0^1\frac{dx}{\sqrt{x}}=\lim\limits_{b\to 0^+}\int_b^1 x^{-1/2}dx=\lim\limits_{b\to 0^+}\left[2x^{1/2}\right]_b^1=\lim\limits_{b\to 0^+}\left(2-2\sqrt{b}\right)=2-0=2$

4. $\int_0^4\frac{dx}{\sqrt{4-x}}=\lim\limits_{b\to 4^-}\int_0^b(4-x)^{-1/2}dx=\lim\limits_{b\to 4^-}\left[-2\sqrt{4-b}-\left(-2\sqrt{4}\right)\right]=0+4=4$

5. $\int_{-1}^1\frac{dx}{x^{2/3}}=\int_{-1}^0\frac{dx}{x^{2/3}}+\int_0^1\frac{dx}{x^{2/3}}=\lim\limits_{b\to 0^-}\left[3x^{1/3}\right]_{-1}^b+\lim\limits_{c\to 0^+}\left[3x^{1/3}\right]_c^1=\lim\limits_{b\to 0^-}\left[3b^{1/3}-3(-1)^{1/3}\right]+\lim\limits_{c\to 0^+}\left[3(1)^{1/3}-3c^{1/3}\right]$
$=(0+3)+(3-0)=6$

6. $\int_{-8}^1\frac{dx}{x^{1/3}}=\int_{-8}^0\frac{dx}{x^{1/3}}+\int_0^1\frac{dx}{x^{1/3}}=\lim\limits_{b\to 0^-}\left[\frac{3}{2}x^{2/3}\right]_{-8}^b+\lim\limits_{c\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_c^1$
$=\lim\limits_{b\to 0^-}\left[\frac{3}{2}b^{2/3}-\frac{3}{2}(-8)^{2/3}\right]+\lim\limits_{c\to 0^+}\left[\frac{3}{2}(1)^{2/3}-\frac{3}{2}c^{2/3}\right]=\left[0-\frac{3}{2}(4)\right]+\left(\frac{3}{2}-0\right)=-\frac{9}{2}$

7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-}\left[\sin^{-1} x\right]_0^b = \lim_{b\to 1^-}\left(\sin^{-1} b - \sin^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

8. $\int_0^1 \frac{dr}{r^{0.999}} = \lim_{b\to 0^+}\left[1000r^{0.001}\right]_b^1 = \lim_{b\to 0^+}\left(1000 - 1000b^{0.001}\right) = 1000 - 0 = 1000$

9. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1} = \int_{-\infty}^{-2} \frac{dx}{x-1} - \int_{-\infty}^{-2} \frac{dx}{x+1} = \lim_{b\to -\infty}\left[\ln|x-1|\right]_b^{-2} - \lim_{b\to -\infty}\left[\ln|x+1|\right]_b^{-2} = \lim_{b\to -\infty}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_b^{-2}$
$= \lim_{b\to -\infty}\left(\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right) = \ln 3 - \ln\left(\lim_{b\to -\infty}\frac{b-1}{b+1}\right) = \ln 3 - \ln 1 = \ln 3$

10. $\int_{-\infty}^{2} \frac{2\,dx}{x^2+4} = \lim_{b\to -\infty}\left[\tan^{-1}\frac{x}{2}\right]_b^2 = \lim_{b\to -\infty}\left(\tan^{-1} 1 - \tan^{-1}\frac{b}{2}\right) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}$

11. $\int_2^{\infty} \frac{2\,dv}{v^2-v} = \lim_{b\to \infty}\left[2\ln\left|\frac{v-1}{v}\right|\right]_2^b = \lim_{b\to \infty}\left(2\ln\left|\frac{b-1}{b}\right| - 2\ln\left|\frac{2-1}{2}\right|\right) = 2\ln(1) - 2\ln\left(\frac{1}{2}\right) = 0 + 2\ln 2 = \ln 4$

12. $\int_2^{\infty} \frac{2\,dt}{t^2-1} = \lim_{b\to \infty}\left[\ln\left|\frac{t-1}{t+1}\right|\right]_2^b = \lim_{b\to \infty}\left(\ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right) = \ln(1) - \ln\left(\frac{1}{3}\right) = 0 + \ln 3 = \ln 3$

13. $\int_{-\infty}^{\infty} \frac{2x\,dx}{\left(x^2+1\right)^2} = \int_{-\infty}^{0} \frac{2x\,dx}{\left(x^2+1\right)^2} + \int_0^{\infty} \frac{2x\,dx}{\left(x^2+1\right)^2};\ \begin{bmatrix} u = x^2+1 \\ du = 2x\,dx \end{bmatrix} \to \int_{\infty}^{1} \frac{du}{u^2} + \int_1^{\infty} \frac{du}{u^2} = \lim_{b\to \infty}\left[-\frac{1}{u}\right]_b^1 + \lim_{c\to \infty}\left[-\frac{1}{u}\right]_1^c$
$= \lim_{b\to \infty}\left(-1 + \frac{1}{b}\right) + \lim_{c\to \infty}\left[-\frac{1}{c} - (-1)\right] = (-1+0) + (0+1) = 0$

14. $\int_{-\infty}^{\infty} \frac{x\,dx}{\left(x^2+4\right)^{3/2}} = \int_{-\infty}^{0} \frac{x\,dx}{\left(x^2+4\right)^{3/2}} + \int_0^{\infty} \frac{x\,dx}{\left(x^2+4\right)^{3/2}};\ \begin{bmatrix} u = x^2+4 \\ du = 2x\,dx \end{bmatrix} \to \int_{\infty}^{4} \frac{du}{2u^{3/2}} + \int_4^{\infty} \frac{du}{2u^{3/2}} = \lim_{b\to \infty}\left[-\frac{1}{\sqrt{u}}\right]_b^4 + \lim_{c\to \infty}\left[-\frac{1}{\sqrt{u}}\right]_4^c$
$= \lim_{b\to \infty}\left(-\frac{1}{2} + \frac{1}{\sqrt{b}}\right) + \lim_{c\to \infty}\left(-\frac{1}{\sqrt{c}} + \frac{1}{2}\right) = \left(-\frac{1}{2} + 0\right) + \left(0 + \frac{1}{2}\right) = 0$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta;\ \begin{bmatrix} u = \theta^2+2\theta \\ du = 2(\theta+1)d\theta \end{bmatrix} \to \int_0^3 \frac{du}{2\sqrt{u}} = \lim_{b\to 0^+}\int_b^3 \frac{du}{2\sqrt{u}} = \lim_{b\to 0^+}\left[\sqrt{u}\right]_b^3 = \lim_{b\to 0^+}\left(\sqrt{3} - \sqrt{b}\right) = \sqrt{3} - 0 = \sqrt{3}$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds = \frac{1}{2}\int_0^2 \frac{2s\,ds}{\sqrt{4-s^2}} + \int_0^2 \frac{ds}{\sqrt{4-s^2}};\ \begin{bmatrix} u = 4-s^2 \\ du = -2s\,ds \end{bmatrix} \to -\frac{1}{2}\int_4^0 \frac{du}{\sqrt{u}} + \lim_{c\to 2^-}\int_0^c \frac{ds}{\sqrt{4-s^2}}$
$= \lim_{b\to 0^+}\int_b^4 \frac{du}{2\sqrt{u}} + \lim_{c\to 2^-}\int_0^c \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to 0^+}\left[\sqrt{u}\right]_b^4 + \lim_{c\to 2^-}\left[\sin^{-1}\frac{s}{2}\right]_0^c = \lim_{b\to 0^+}\left(2 - \sqrt{b}\right) + \lim_{c\to 2^-}\left(\sin^{-1}\frac{c}{2} - \sin^{-1} 0\right)$
$= (2-0) + \left(\frac{\pi}{2} - 0\right) = \frac{4+\pi}{2}$

17. $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}};\ \begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int_0^{\infty} \frac{2\,du}{u^2+1} = \lim_{b\to \infty}\int_0^b \frac{2\,\mathrm{du}}{u^2+1} = \lim_{b\to \infty}\left[2\tan^{-1} u\right]_0^b = \lim_{b\to \infty}\left(2\tan^{-1} b - 2\tan^{-1} 0\right)$
$= 2\left(\frac{\pi}{2}\right) - 2(0) = \pi$

18. $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{b\to 1^+}\int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty}\int_2^c \frac{dx}{x\sqrt{x^2-1}} = \lim_{b\to 1^+}\left[\sec^{-1}|x|\right]_b^2 + \lim_{c\to\infty}\left[\sec^{-1}|x|\right]_2^c$
$= \lim_{b\to 1^+}\left(\sec^{-1}2 - \sec^{-1}b\right) + \lim_{c\to\infty}\left(\sec^{-1}c - \sec^{-1}2\right) = \left(\frac{\pi}{3} - 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{2}$

19. $\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1}v)} = \lim_{b\to\infty}\left[\ln\left|1+\tan^{-1}v\right|\right]_0^b = \lim_{b\to\infty}\left[\ln\left|1+\tan^{-1}b\right| - \ln\left|1+\tan^{-1}0\right|\right] = \ln\left(1+\frac{\pi}{2}\right) - \ln(1+0)$
$= \ln\left(1+\frac{\pi}{2}\right)$

20. $\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx = \lim_{b\to\infty}\left[8\left(\tan^{-1}x\right)^2\right]_0^b = \lim_{b\to\infty}\left[8\left(\tan^{-1}b\right)^2 - 8\left(\tan^{-1}0\right)^2\right] = 8\left(\frac{\pi}{2}\right)^2 - 8(0) = 2\pi^2$

21. $\int_{-\infty}^0 \theta e^\theta\,d\theta = \lim_{b\to-\infty}\left[\theta e^\theta - e^\theta\right]_b^0 = \lim_{b\to-\infty}\left[\left(0\cdot e^0 - e^0\right) - \left(be^b - e^b\right)\right] = -1 - \lim_{b\to-\infty}\left(\frac{b-1}{e^{-b}}\right) = -1 - \lim_{b\to-\infty}\left(\frac{1}{-e^{-b}}\right)$
(l'Hôpital's rule for $\frac{\infty}{\infty}$ form)
$= -1 - 0 = -1$

22. $\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta = \lim_{b\to\infty}\int_0^b 2e^{-\theta}\sin\theta\,d\theta = \lim_{b\to\infty} 2\left[\frac{e^{-\theta}}{1+1}(-\sin\theta - \cos\theta)\right]_0^b$ (Formula 107 with $a=-1, b=1$)
$= \lim_{b\to\infty}\left[\frac{-2(\sin b+\cos b)}{2e^b} + \frac{2(\sin 0+\cos 0)}{2e^0}\right] = 0 + \frac{2(0+1)}{2} = 1$

23. $\int_{-\infty}^0 e^{-|x|}\,dx = \int_{-\infty}^0 e^x dx = \lim_{b\to-\infty}\left[e^x\right]_b^0 = \lim_{b\to-\infty}\left(1-e^b\right) = (1-0) = 1$

24. $\int_{-\infty}^\infty 2xe^{-x^2}\,dx = \int_{-\infty}^0 2xe^{-x^2}\,dx + \int_0^\infty 2xe^{-x^2}\,dx = \lim_{b\to-\infty}\left[-e^{-x^2}\right]_b^0 + \lim_{c\to\infty}\left[-e^{-x^2}\right]_0^c$
$= \lim_{b\to-\infty}\left[-1-\left(-e^{-b^2}\right)\right] + \lim_{c\to\infty}\left[-e^{-c^2} - (-1)\right] = (-1-0) + (0+1) = 0$

25. $\int_0^1 x\ln x\,dx = \lim_{b\to 0^+}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_b^1 = \lim_{b\to 0^+}\left[\left(\frac{1}{2}\ln 1 - \frac{1}{4}\right) - \left(\frac{b^2}{2}\ln b - \frac{b^2}{4}\right)\right] = -\frac{1}{4} - \lim_{b\to 0^+}\frac{\ln b}{\left(\frac{2}{b^2}\right)} + 0$
$= -\frac{1}{4} - \lim_{b\to 0^+}\frac{\left(\frac{1}{b}\right)}{\left(-\frac{4}{b^3}\right)} = -\frac{1}{4} + \lim_{b\to 0^+}\left(\frac{b^2}{4}\right) = -\frac{1}{4} + 0 = -\frac{1}{4}$

26. $\int_0^1(-\ln x)\,dx = \lim_{b\to 0^+}\left[x - x\ln x\right]_b^1 = \lim_{b\to 0^+}\left[(1 - 1\ln 1) - (b - b\ln b)\right] = 1 - 0 + \lim_{b\to 0^+}\frac{\ln b}{\left(\frac{1}{b}\right)} = 1 + \lim_{b\to 0^+}\frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$
$= 1 - \lim_{b\to 0^+} b = 1 - 0 = 1$

27. $\int_0^2 \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to 2^-}\left[\sin^{-1}\frac{s}{2}\right]_0^b = \lim_{b\to 2^-}\left[\sin^{-1}\frac{b}{2} - \sin^{-1}0\right] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

28. $\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}} = \lim_{b\to 1^-}\left[2\sin^{-1}\left(r^2\right)\right]_0^b = \lim_{b\to 1^-}\left[2\sin^{-1}\left(b^2\right)-2\sin^{-1}0\right] = 2\cdot\frac{\pi}{2}-0=\pi$

29. $\int_1^2 \frac{ds}{s\sqrt{s^2-1}} = \lim_{b\to 1^+}\left[\sec^{-1}s\right]_b^2 = \lim_{b\to 1^+}\left[\sec^{-1}2-\sec^{-1}b\right] = \frac{\pi}{3}-0=\frac{\pi}{3}$

30. $\int_2^4 \frac{dt}{t\sqrt{t^2-4}} = \lim_{b\to 2^+}\left[\frac{1}{2}\sec^{-1}\frac{t}{2}\right]_b^4 = \lim_{b\to 2^+}\left[\left(\frac{1}{2}\sec^{-1}\frac{4}{2}\right)-\frac{1}{2}\sec^{-1}\left(\frac{b}{2}\right)\right] = \frac{1}{2}\left(\frac{\pi}{3}\right)-\frac{1}{2}\cdot 0=\frac{\pi}{6}$

31. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}} = \lim_{b\to 0^-}\int_{-1}^b \frac{dx}{\sqrt{-x}} + \lim_{c\to 0^+}\int_c^4 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^-}\left[-2\sqrt{-x}\right]_{-1}^b + \lim_{c\to 0^+}\left[2\sqrt{x}\right]_c^4$
$= \lim_{b\to 0^-}\left[\left(-2\sqrt{-b}\right)-\left(-2\sqrt{-(-1)}\right)\right] + \lim_{c\to 0^+}\left[2\sqrt{4}-2\sqrt{c}\right] = 0+2+2\cdot 2-0=6$

32. $\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{b\to 1^-}\left[-2\sqrt{1-x}\right]_0^b + \lim_{c\to 1^+}\left[2\sqrt{x-1}\right]_c^2$
$= \lim_{b\to 1^-}\left[\left(-2\sqrt{1-b}\right)-\left(-2\sqrt{1-0}\right)\right] + \lim_{c\to 1^+}\left[2\sqrt{2-1}-\left(2\sqrt{c-1}\right)\right] = 0+2+2-0=4$

33. $\int_{-1}^\infty \frac{d\theta}{\theta^2+5\theta+6} = \lim_{b\to\infty}\left[\ln\left|\frac{\theta+2}{\theta+3}\right|\right]_{-1}^b = \lim_{b\to\infty}\left[\ln\left|\frac{b+2}{b+3}\right|-\ln\left|\frac{-1+2}{-1+3}\right|\right] = 0-\ln\left(\frac{1}{2}\right)=\ln 2$

34. $\int_0^\infty \frac{dx}{(x+1)\left(x^2+1\right)} = \lim_{b\to\infty}\left[\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln\left(x^2+1\right)+\frac{1}{2}\tan^{-1}x\right]_0^b = \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(\frac{x+1}{\sqrt{x^2+1}}\right)+\frac{1}{2}\tan^{-1}x\right]_0^b$
$= \lim_{b\to\infty}\left[\left(\frac{1}{2}\ln\left(\frac{b+1}{\sqrt{b^2+1}}\right)+\frac{1}{2}\tan^{-1}b\right)-\left(\frac{1}{2}\ln\frac{1}{\sqrt{1}}+\frac{1}{2}\tan^{-1}0\right)\right] = \frac{1}{2}\ln 1+\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{2}\ln 1-\frac{1}{2}\cdot 0=\frac{\pi}{4}$

35. $\int_0^{\pi/2}\tan\theta\,d\theta = \lim_{b\to\left(\frac{\pi}{2}\right)^-}\left[-\ln|\cos\theta|\right]_0^b = \lim_{b\to\left(\frac{\pi}{2}\right)^-}\left[-\ln|\cos b|+\ln 1\right] = \lim_{b\to\left(\frac{\pi}{2}\right)^-}\left[-\ln|\cos b|\right]=+\infty,$ the integral diverges

36. $\int_0^{\pi/2}\cot\theta\,d\theta = \lim_{b\to 0^+}\left[\ln|\sin\theta|\right]_b^{\pi/2} = \lim_{b\to 0^+}\left[\ln 1-\ln|\sin b|\right] = -\lim_{b\to 0^+}\left[\ln|\sin b|\right]=+\infty,$ the integral diverges

37. $\int_0^1 \frac{\ln x}{x^2}\,dx$

$\int_{1/3}^1 \frac{\ln x}{x^2}\,dx$ is bounded, so convergence is determined by $\int_0^{1/3}\frac{\ln x}{x^2}\,dx$.

On $(0, 1/3]$, $\ln x < -1$ and $\frac{\ln x}{x^2} < -\frac{1}{x^2}$. Since $\int_0^{1/3} -\frac{1}{x^2}\,dx$ diverges to $-\infty$, so does $\int_0^{1/3}\frac{\ln x}{x^2}\,dx$ and hence $\int_0^1 \frac{\ln x}{x^2}\,dx$ diverges.

38. Since $\int \frac{1}{x\ln x}\,dx = \ln(\ln x)$, $\int_1^2 \frac{1}{x\ln x}\,dx = \lim_{a\to 1^+}\left(\ln(\ln 2)-\ln(\ln a)\right) = \infty$; the integral diverges. (In this case we don't need a comparison test.)

39. $\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$; $\left[\frac{1}{x}=y\right] \to \int_\infty^{1/\ln 2} \frac{y^2e^{-y}\,dy}{-y^3} = \int_{1/\ln 2}^{\infty} e^{-y}\,dy = \lim_{b\to\infty}\left[-e^{-y}\right]_{1/\ln 2}^{b} = \lim_{b\to\infty}\left[-e^{-b}-\left(-e^{-1/\ln 2}\right)\right]$ $= 0+e^{-1/\ln 2} = e^{-1/\ln 2}$, so the integral converges.

40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$; $\left[y=\sqrt{x}\right] \to 2\int_0^1 e^{-y}\,dy = 2-\frac{2}{e}$, so the integral converges.

41. $\int_0^\pi \frac{dt}{\sqrt{t}+\sin t}$. Since for $0 \le t \le \pi, 0 \le \frac{1}{\sqrt{t}+\sin t} \le \frac{1}{\sqrt{t}}$ and $\int_0^\pi \frac{dt}{\sqrt{t}}$ converges, then the original integral converges as well by the Direct Comparison Test.

42. $\int_0^1 \frac{dt}{t-\sin t}$; Let $f(t)=\frac{1}{t-\sin t}$ and $g(t)=\frac{1}{t^3}$, then $\lim_{t\to 0}\frac{f(t)}{g(t)} = \lim_{t\to 0}\frac{t^3}{t-\sin t} = \lim_{t\to 0}\frac{3t^2}{1-\cos t} = \lim_{t\to 0}\frac{6t}{\sin t} = \lim_{t\to 0}\frac{6}{\cos t} = 6.$ Now, $\int_0^1 \frac{dt}{t^3} = \lim_{b\to 0^+}\left[-\frac{1}{2t^2}\right]_b^1 = \lim_{b\to 0^+}\left[-\frac{1}{2}-\left(-\frac{1}{2b^2}\right)\right] = +\infty$, which diverges $\Rightarrow \int_0^1 \frac{dt}{t-\sin t}$ diverges by the Limit Comparison Test.

43. $\int_0^2 \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$ and $\int_0^1 \frac{dx}{1-x^2} = \lim_{b\to 1^-}\left[\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right]_0^b = \lim_{b\to 1^-}\left[\frac{1}{2}\ln\left|\frac{1+b}{1-b}\right|-0\right] = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1-x^2}$ diverges as well.

44. $\int_0^2 \frac{dx}{1-x} = \int_0^1 \frac{dx}{1-x} + \int_1^2 \frac{dx}{1-x}$ and $\int_0^1 \frac{dx}{1-x} = \lim_{b\to 1^-}\left[-\ln(1-x)\right]_0^b = \lim_{b\to 1^-}\left[-\ln(1-b)-0\right] = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1-x}$ diverges as well.

45. $\int_{-1}^1 \ln|x|\,dx = \int_{-1}^0 \ln(-x)\,dx + \int_0^1 \ln x\,dx$; $\int_0^1 \ln x\,dx = \lim_{b\to 0^+}\left[x\ln x - x\right]_b^1 = \lim_{b\to 0^+}\left[(1\cdot 0-1)-(b\ln b-b)\right]$ $= -1-0 = -1$; $\int_{-1}^0 \ln(-x)\,dx = -1 \Rightarrow \int_{-1}^1 \ln|x|\,dx = -2$ converges.

46. $\int_{-1}^1 \left(-x\ln|x|\right)dx = \int_{-1}^0 \left[-x\ln(-x)\right]dx + \int_0^1 \left(-x\ln x\right)dx = \lim_{b\to 0^+}\left[\frac{x^2}{2}\ln x-\frac{x^2}{4}\right]_b^1 - \lim_{c\to 0^+}\left[\frac{x^2}{2}\ln x-\frac{x^2}{4}\right]_c^1$ $= \lim_{b\to 0^+}\left[\left(\frac{1}{2}\ln 1-\frac{1}{4}\right)-\left(\frac{b^2}{2}\ln b-\frac{b^2}{4}\right)\right] - \lim_{c\to 0^+}\left[\left(\frac{1}{2}\ln 1-\frac{1}{4}\right)-\left(\frac{c^2}{2}\ln c-\frac{c^2}{4}\right)\right] = -\frac{1}{4}-0+\frac{1}{4}+0 = 0 \Rightarrow$ the integral converges (see Exercise 25 for the limit calculations).

47. $\int_1^\infty \frac{dx}{1+x^3}; 0 \le \frac{1}{x^3+1} \le \frac{1}{x^3}$ for $1 \le x < \infty$ and $\int_1^\infty \frac{dx}{x^3}$ converges $\Rightarrow \int_1^\infty \frac{dx}{1+x^3}$ converges by the Direct Comparison Test.

48. $\int_4^\infty \frac{dx}{\sqrt{x}-1}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x}-1}\right)}{\left(\frac{1}{\sqrt{x}}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}-1} = \lim\limits_{x\to\infty} \frac{1}{1-\frac{1}{\sqrt{x}}} = \frac{1}{1-0} = 1$ and $\int_4^\infty \frac{dx}{\sqrt{x}} = \lim\limits_{b\to\infty}\left[2\sqrt{x}\right]_4^b = \infty$, which diverges

$\Rightarrow \int_4^\infty \frac{dx}{\sqrt{x}-1}$ diverges by the Limit Comparison Test.

49. $\int_2^\infty \frac{dv}{\sqrt{v-1}}$; $\lim\limits_{v\to\infty} \frac{\left(\frac{1}{\sqrt{v-1}}\right)}{\left(\frac{1}{\sqrt{v}}\right)} = \lim\limits_{v\to\infty} \frac{\sqrt{v}}{\sqrt{v-1}} = \lim\limits_{v\to\infty} \frac{1}{\sqrt{1-\frac{1}{v}}} = \frac{1}{\sqrt{1-0}} = 1$ and $\int_2^\infty \frac{dv}{\sqrt{v}} = \lim\limits_{b\to\infty}\left[2\sqrt{v}\right]_2^b = \infty$, which diverges

$\Rightarrow \int_2^\infty \frac{dv}{\sqrt{v-1}}$ diverges by the Limit Comparison Test.

50. $\int_0^\infty \frac{d\theta}{1+e^\theta}$; $0 \le \frac{1}{1+e^\theta} \le \frac{1}{e^\theta}$ for $0 \le \theta < \infty$ and $\int_0^\infty \frac{d\theta}{e^\theta} = \lim\limits_{b\to\infty}\left[-e^{-\theta}\right]_0^b = \lim\limits_{b\to\infty}\left(-e^{-b}+1\right) = 1 \Rightarrow \int_0^\infty \frac{d\theta}{e^\theta}$ converges

$\Rightarrow \int_0^\infty \frac{d\theta}{1+e^\theta}$ by the Direct Comparison Test.

51. $\int_0^\infty \frac{dx}{\sqrt{x^6+1}} = \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^\infty \frac{dx}{\sqrt{x^6+1}} < \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^\infty \frac{dx}{x^3}$ and $\int_1^\infty \frac{dx}{x^3} = \lim\limits_{b\to\infty}\left[-\frac{1}{2x^2}\right]_1^b = \lim\limits_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2}$

$\Rightarrow \int_0^\infty \frac{dx}{\sqrt{x^6+1}}$ converges by the Direct Comparison Test.

52. $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$; $\int_2^\infty \frac{1}{x}\,dx = \lim\limits_{b\to\infty}\left[\ln b\right]_2^b = \infty$, which diverges

$\Rightarrow \int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges by the Limit Comparison Test.

53. $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$; $\lim\limits_{x\to\infty} \frac{\left(\frac{\sqrt{x}}{x^2}\right)}{\left(\frac{\sqrt{x+1}}{x^2}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+1}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}} = 1$; $\int_1^\infty \frac{\sqrt{x}}{x^2}\,dx = \int_1^\infty \frac{dx}{x^{3/2}} = \lim\limits_{b\to\infty}\left[-2x^{-1/2}\right]_1^b = \lim\limits_{b\to\infty}\left(\frac{-2}{\sqrt{b}}+2\right)$

$= 2 \Rightarrow \int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ converges by the Limit Comparison Test.

54. $\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{x}{\sqrt{x^4-1}}\right)}{\left(\frac{x}{\sqrt{x^4}}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{x^4}}{\sqrt{x^4-1}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^4}}} = 1$; $\int_2^\infty \frac{x\,dx}{\sqrt{x^4}} = \int_2^\infty \frac{dx}{x} = \lim\limits_{x\to\infty}\left[\ln x\right]_2^b = \infty$, which diverges

$\Rightarrow \int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$ diverges by the Limit Comparison Test.

55. $\int_\pi^\infty \frac{2+\cos x}{x}\,dx$; $0 < \frac{1}{x} \le \frac{2+\cos x}{x}$ for $x \ge \pi$ and $\int_\pi^\infty \frac{dx}{x} = \lim\limits_{b\to\infty}\left[\ln x\right]_\pi^b = \infty$, which diverges $\Rightarrow \int_\pi^\infty \frac{2+\cos x}{x}\,dx$ diverges by the Direct Comparison Test.

56. $\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\, dx$; $0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$ for $x \ge \pi$ and $\int_{\pi}^{\infty} \frac{2}{x^2}\, dx = \lim_{b\to\infty}\left[-\frac{2}{x}\right]_{\pi}^{b} = \lim_{b\to\infty}\left(-\frac{2}{b}+\frac{2}{\pi}\right) = \frac{2}{\pi} \Rightarrow \int_{\pi}^{\infty} \frac{2\,dx}{x^2}$ converges $\Rightarrow \int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\, dx$ converges by the Direct Comparison Test.

57. $\int_4^{\infty} \frac{2\,dt}{t^{3/2}-1}$; $\lim_{t\to\infty} \frac{t^{3/2}}{t^{3/2}-1} = 1$ and $\int_4^{\infty} \frac{2\,dt}{t^{3/2}} = \lim_{b\to\infty}\left[-4t^{-1/2}\right]_4^b = \lim_{b\to\infty}\left(\frac{-4}{\sqrt{b}}+2\right) = 2 \Rightarrow \int_4^{\infty} \frac{2\,dt}{t^{3/2}}$ converges $\Rightarrow \int_4^{\infty} \frac{2\,dt}{t^{3/2}-1}$ converges by the Limit Comparison Test.

58. $\int_2^{\infty} \frac{dx}{\ln x}$; $0 < \frac{1}{x} < \frac{1}{\ln x}$ for $x > 2$ and $\int_2^{\infty} \frac{dx}{x}$ diverges $\Rightarrow \int_2^{\infty} \frac{dx}{\ln x}$ diverges by the Direct Comparison Test.

59. $\int_1^{\infty} \frac{e^x}{x}\, dx$; $0 < \frac{1}{x} < \frac{e^x}{x}$ for $x > 1$ and $\int_1^{\infty} \frac{dx}{x}$ diverges $\Rightarrow \int_1^{\infty} \frac{e^x dx}{x}$ diverges by the Direct Comparison Test.

60. $\int_{e^e}^{\infty} \ln(\ln x)\, dx$; $[x = e^y] \to \int_e^{\infty} (\ln y)e^y\, dy$; $0 < \ln y < (\ln y)e^y$ for $y \ge e$ and $\int_e^{\infty} \ln y\, dy = \lim_{b\to\infty}\left[y\ln y - y\right]_e^b = \infty$, which diverges $\Rightarrow \int_e^{\infty} (\ln y)e^y dy$ diverges $\Rightarrow \int_{e^e}^{\infty} \ln(\ln x)\, dx$ diverges by the Direct Comparison Test.

61. $\int_1^{\infty} \frac{dx}{\sqrt{e^x - x}}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{e^x-x}}\right)}{\left(\frac{1}{\sqrt{e^x}}\right)} = \lim_{x\to\infty} \frac{\sqrt{e^x}}{\sqrt{e^x-x}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{x}{e^x}}} = \frac{1}{\sqrt{1-0}} = 1$; $\int_1^{\infty} \frac{dx}{\sqrt{e^x}} = \int_1^{\infty} e^{-x/2}\, dx = \lim_{b\to\infty}\left[-2e^{-x/2}\right]_1^b$ $= \lim_{b\to\infty}\left(-2e^{-b/2}+2e^{-1/2}\right) = \frac{2}{\sqrt{e}} \Rightarrow \int_1^{\infty} e^{-x/2}\, dx$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{\sqrt{e^x-x}}$ converges by the Limit Comparison Test.

62. $\int_1^{\infty} \frac{dx}{e^x - 2^x}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{e^x-2^x}\right)}{\left(\frac{1}{e^x}\right)} = \lim_{x\to\infty} \frac{e^x}{e^x-2^x} = \lim_{x\to\infty} \frac{1}{1-\left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = 1$ and $\int_1^{\infty} \frac{dx}{e^x} = \lim_{b\to\infty}\left[-e^{-x}\right]_1^b = \lim_{b\to\infty}\left(-e^{-b}+e^{-1}\right) = \frac{1}{e}$ $\Rightarrow \int_1^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{e^x-2^x}$ converges by the Limit Comparison Test.

63. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}} = 2\int_0^{\infty} \frac{dx}{\sqrt{x^4+1}}$; $\int_0^{\infty} \frac{dx}{\sqrt{x^4+1}} = \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^{\infty} \frac{dx}{\sqrt{x^4+1}} < \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^{\infty} \frac{dx}{x^2}$ and $\int_1^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b$ $= \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$ converges by the Direct Comparison Test.

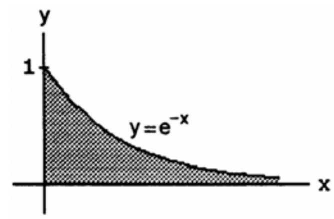
64. $\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}} = 2\int_0^{\infty} \frac{dx}{e^x+e^{-x}}$; $0 < \frac{1}{e^x+e^{-x}} < \frac{1}{e^x}$ for $x > 0$; $\int_0^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow 2\int_0^{\infty} \frac{dx}{e^x+e^{-x}}$ converges by the Direct Comparison Test.

65. (a) $\int_1^2 \frac{dx}{x(\ln x)^p}$; $\left[t = \ln x\right] \to \int_0^{\ln 2} \frac{dt}{t^p} = \lim_{b\to 0^+}\left[\frac{1}{-p+1}t^{1-p}\right]_b^{\ln 2} = \lim_{b\to 0^+}\left[\frac{b^{1-p}}{p-1} + \frac{1}{1-p}(\ln 2)^{1-p}\right] \Rightarrow$ the integral converges for $p < 1$ and diverges for $p \ge 1$

(b) $\int_2^\infty \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \to \int_{\ln 2}^\infty \frac{dt}{t^p}$ and this integral is essentially the same as in Exercise 65(a): it converges for $p > 1$ and diverges for $p \le 1$

66. $\int_0^\infty \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty}\left[\ln\left(x^2+1\right)\right]_0^b = \lim_{b\to\infty}\left[\ln\left(b^2+1\right)-0\right] = \lim_{b\to\infty}\ln\left(b^2+1\right) = \infty \Rightarrow$ the integral $\int_{-\infty}^\infty \frac{2x}{x^2+1}\,dx$ diverges. But $\lim_{b\to\infty}\int_{-\infty}^b \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty}\left[\ln\left(x^2+1\right)\right]_{-b}^b = \lim_{b\to\infty}\left[\ln\left(b^2+1\right)-\ln\left(b^2+1\right)\right] = \lim_{b\to\infty}\ln\left(\frac{b^2+1}{b^2+1}\right)$ $= \lim_{b\to\infty}(\ln 1) = 0$

67. $A = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b$
$= \lim_{b\to\infty}\left[\left(-e^{-b}\right)-\left(-e^{-0}\right)\right] = 0+1 = 1$



68. $\bar{x} = \frac{1}{A}\int_0^\infty xe^{-x}dx = \lim_{b\to\infty}\left[-xe^{-x}-e^{-x}\right]_0^b = \lim_{b\to\infty}\left[\left(-be^{-b}-e^{-b}\right)-\left(-0\cdot e^{-0}-e^{-0}\right)\right] = 0+1 = 1;$
$\bar{y} = \frac{1}{2A}\int_0^\infty \left(e^{-x}\right)^2 dx = \frac{1}{2}\int_0^\infty e^{-2x}dx = \lim_{b\to\infty}\frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_0^b = \lim_{b\to\infty}\left[\frac{1}{2}\left(-\frac{1}{2}e^{-2b}\right)-\frac{1}{2}\left(-\frac{1}{2}e^{-2\cdot 0}\right)\right] = 0+\frac{1}{4} = \frac{1}{4}$

69. $V = \int_0^\infty 2\pi xe^{-x}dx = 2\pi\int_0^\infty xe^{-x}dx = 2\pi\lim_{b\to\infty}\left[-xe^{-x}-e^{-x}\right]_0^b = 2\pi\lim_{b\to\infty}\left[\left(-be^{-b}-e^{-b}\right)-1\right] = 2\pi$

70. $V = \int_0^\infty \pi\left(e^{-x}\right)^2 dx = \pi\int_0^\infty e^{-2x}dx = \pi\lim_{b\to\infty}\left[-\frac{1}{2}e^{-2x}\right]_0^b = \pi\lim_{b\to\infty}\left(-\frac{1}{2}e^{-2b}+\frac{1}{2}\right) = \frac{\pi}{2}$

71. $A = \int_0^{\pi/2}(\sec x - \tan x)\,dx = \lim_{b\to\left(\frac{\pi}{2}\right)^-}\left[\ln|\sec x+\tan x| - \ln|\sec x|\right]_0^b = \lim_{b\to\left(\frac{\pi}{2}\right)^-}\left[\ln\left|1+\frac{\tan b}{\sec b}\right| - \ln|1+0|\right]$
$= \lim_{b\to\left(\frac{\pi}{2}\right)^-}\ln|1+\sin b| = \ln 2$

72. (a) $V\int_0^{\pi/2}\pi\sec^2 x\,dx - \int_0^{\pi/2}\pi\tan^2 x\,dx = \pi\int_0^{\pi/2}\left(\sec^2 x-\tan^2 x\right)dx = \int_0^{\pi/2}\pi\left[\sec^2 x-\left(\sec^2 x-1\right)\right]dx$
$= \pi\int_0^{\pi/2}dx = \frac{\pi^2}{2}$

(b) $S_{\text{outer}} = \int_0^{\pi/2}2\pi\sec x\sqrt{1+\sec^2 x\tan^2 x}\,dx \ge \int_0^{\pi/2}2\pi\sec x(\sec x\tan x)\,dx = \pi\lim_{b\to\left(\frac{\pi}{2}\right)^-}\left[\tan^2 x\right]_0^b =$
$\pi\lim_{b\to\left(\frac{\pi}{2}\right)^-}\left[\left(\tan^2 b\right)-0\right] = \pi\lim_{b\to\left(\frac{\pi}{2}\right)^-}\left[\tan^2 b\right] = \infty \Rightarrow S_{\text{outer}}$ diverges; $S_{\text{inner}} = \int_0^{\pi/2}2\pi\tan x\sqrt{1+\sec^4 x}\,dx \ge$
$\int_0^{\pi/2}2\pi\tan x\sec^2 x\,dx = \pi\lim_{b\to\left(\frac{\pi}{2}\right)^-}\left[\tan^2 x\right]_0^b = \pi\lim_{b\to\left(\frac{\pi}{2}\right)^-}\left[\left(\tan^2 b\right)-0\right] = \pi\lim_{b\to\left(\frac{\pi}{2}\right)^-}\left[\tan^2 b\right] = \infty \Rightarrow S_{\text{inner}}$ diverges

73. (a) $\int_0^1 \frac{1}{\sqrt{t}(1+t)}\,dt$

With $u=\sqrt{t}$ and $du=\frac{1}{2\sqrt{t}}dt$ the limits of integration are unchanged.

$$\int_0^1 \frac{1}{\sqrt{t}(1+t)}\,dt=\int_0^1 \frac{2}{1+u^2}\,du$$
$$=2\lim_{a\to 0^+}\left(\tan^{-1}1-\tan^{-1}a\right)$$
$$=2\cdot\frac{\pi}{4}=\frac{\pi}{2}$$

(b) $\int_0^\infty \frac{1}{\sqrt{t}(1+t)}\,dt$

With $u=\sqrt{t}$ and $du=\frac{1}{2\sqrt{t}}dt$ the limits of integration are unchanged. We split the integral into two integrals, the first of which was evaluated in (a).

$$\int_0^\infty \frac{1}{\sqrt{t}(1+t)}\,dt=\int_0^1 \frac{2}{1+u^2}\,du+\int_1^\infty \frac{2}{1+u^2}\,du$$
$$=\frac{\pi}{2}+2\lim_{b\to\infty}\left(\tan^{-1}b-\tan^{-1}1\right)$$
$$=\frac{\pi}{2}+2\left(\frac{\pi}{2}-\frac{\pi}{2}\right)=\pi$$

74. Let c be any number in $(3,\infty)$.

$\int_3^\infty \frac{1}{x\sqrt{x^2-9}}\,dx=\int_3^c \frac{1}{x\sqrt{x^2-9}}\,dx+\int_c^\infty \frac{1}{x\sqrt{x^2-9}}\,dx$ provided both integrals on the right converge.

Formula 20 in Table 8.1 gives $\int \frac{1}{x\sqrt{x^2-9}}\,dx=\frac{1}{3}\sec^{-1}\left|\frac{x}{3}\right|$. (The definition of the inverse secant is given in Section 3.9.) Both integrals do converge:

$$\int_3^c \frac{1}{x\sqrt{x^2-9}}\,dx=\lim_{a\to 3^+}\left(\frac{1}{3}\sec^{-1}\left|\frac{c}{3}\right|-\frac{1}{3}\sec^{-1}\left|\frac{a}{3}\right|\right)=\frac{1}{3}\sec^{-1}\frac{c}{3}$$

$$\int_c^\infty \frac{1}{x\sqrt{x^2-9}}\,dx=\lim_{b\to\infty}\left(\frac{1}{3}\sec^{-1}\left|\frac{b}{3}\right|-\frac{1}{3}\sec^{-1}\left|\frac{c}{3}\right|\right)=\frac{\pi}{6}-\frac{1}{3}\sec^{-1}\frac{c}{3}$$

Thus $\int_3^\infty \frac{1}{x\sqrt{x^2-9}}\,dx=\frac{\pi}{6}$.

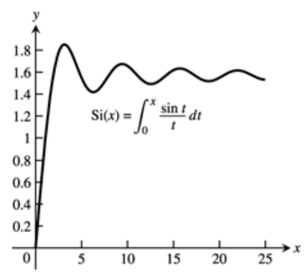
75. (a) $\int_3^\infty e^{-3x}dx = \lim\limits_{b\to\infty}\left[-\frac{1}{3}e^{-3x}\right]_3^b = \lim\limits_{b\to\infty}\left[\left(-\frac{1}{3}e^{-3b}\right)-\left(-\frac{1}{3}e^{-3\cdot 3}\right)\right] = 0+\frac{1}{3}\cdot e^{-9} = \frac{1}{3}e^{-9} \approx 0.0000411$ $< 0.000042.$ Since $e^{-x^2} \le e^{-3x}$ for $x > 3,$ then $\int_3^\infty e^{-x^2}dx < 0.000042$ and therefore $\int_0^\infty e^{-x^2}dx$ can be replaced by $\int_0^3 e^{-x^2}dx$ without introducing an error greater than $0.000042.$

(b) $\int_0^3 e^{-x^2}dx \cong 0.88621$

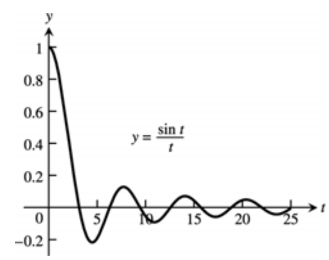
76. (a) $V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi \lim\limits_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \pi \lim\limits_{b\to\infty}\left[\left(-\frac{1}{b}\right)-\left(-\frac{1}{1}\right)\right] = \pi(0+1) = \pi$

(b) When you take the limit to $\infty,$ you are no longer modeling the real world which is finite. The comparison step in the modeling process discussed in Section 4.2 relating the mathematical world to the real world fails to hold.

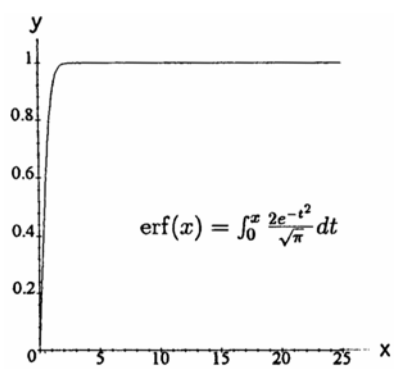
77. (a)



(b) > int((sin(t))/t, t = 0.. infinity); $\left(\text{answer is } \frac{\pi}{2}\right)$



78. (a)



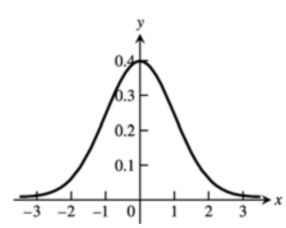
(b) > f:=2*exp(−t^2)/sqrt(Pi);
> int(f, t=0..infinity); (answer is 1)

79. (a) $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

f is increasing on $(-\infty, 0],$

f is decreasing on $[0, \infty),$

f has a local maximum at $(0,\ f(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right)$



(b) Maple commands:

>f: = exp(−x^2/2)(sqrt(2*pi);

>int(f, x = −1..1); ≈ 0.683

>int(f, x = −2..2); ≈ 0.954

>int(f, x = −3..3); ≈ 0.997

(c) Part (b) suggests that as n increases, the integral approaches 1. We can take $\int_{-n}^{n} f(x)\,dx$ as close to 1 as we want by choosing $n > 1$ large enough. Also, we can make $\int_{n}^{\infty} f(x)\,dx$ and $\int_{-\infty}^{-n} f(x)\,dx$ as small as we want by choosing n large enough. This is because $0 < f(x) < e^{-x/2}$ for $x > 1$. (Likewise, $0 < f(x) < e^{x/2}$ for $x < -1$.) Thus, $\int_{n}^{\infty} f(x)\,dx < \int_{n}^{\infty} e^{-x/2}\,dx$.

$\int_{n}^{\infty} e^{-x/2}\,dx = \lim_{c\to\infty} \int_{n}^{c} e^{-x/2}\,dx = \lim_{c\to\infty} \left[-2e^{-x/2}\right]_{n}^{c} = \lim_{c\to\infty} \left[-2e^{-c/2} + 2e^{-n/2}\right] = 2e^{-n/2}$

As $n \to \infty, 2e^{-n/2} \to 0$, for large enough n, $\int_{n}^{\infty} f(x)\,dx$ is as small as we want.

Likewise for large enough n, $\int_{-\infty}^{-n} f(x)\,dx$ is as small as we want.

80. (a) The statement is true since $\int_{-\infty}^{b} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx$, $\int_{b}^{\infty} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx - \int_{a}^{b} f(x)\,dx$ and $\int_{a}^{b} f(x)\,dx$ exists since $f(x)$ is integrable on every interval $[a, b]$.

(b) $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx - \int_{a}^{b} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$

$= \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$

81. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(x);
domain := 0..exp(1);
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=domain, y=-50..10, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9],
      thickness=[3,4,1,2,0], legend=["p= -2","p = -1","p = 0","p = 1","p = 2"], title="#81 (Section 8.8)" );
q1 := Int( f(x,p), x=domain );
q2 := value( q1 );
q3 := simplify( q2 ) assuming p>-1;
q4 := simplify( q2 ) assuming p<-1;
q5 := value( eval( q1, p=-1 ) );
i1 := q1 = piecewise( p<-1, q4, p=-1, q5, p>-1, q3 );
```

82. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(x);
domain := exp(1)..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=exp(1)..10, y=0..100, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9],
      thickness=[3,4,1,2,0], legend=["p= -2","p = -1","p = 0","p = 1","p = 2"], title="#82 (Section 8.8)" );
```

```
q6 := Int( f(x,p), x=domain );
q7 := value( q6 );
q8 := simplify( q7 ) assuming p>-1;
q9 := simplify( q7 ) assuming p<-1;
q10 := value( eval( q6, p=-1 ) );
i2 := q6 = piecewise( p<-1, q9, p=-1, q10, p>-1, q8 );
```

83. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(x);
domain := 0..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=0..10, y=-50..50, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9],
    thickness=[3,4,1,2,0], legend=["p= -2","p = -1","p = 0","p = 1","p = 2"], title="#83 (Section 8.8)" );
q11 := Int( f(x,p), x=domain ):
q11 = lhs(i1+i2);
`` = rhs(i1+i2);
`` = piecewise( p<-1, q4+q9, p=-1, q5+q10, p>-1, q3+q8 );
`` = piecewise( p<-1, -infinity, p=-1, undefined, p>-1, infinity );
```

84. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(abs(x));
domain := -infinity..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=4..4, y=-20..10, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9],
    legend=["p= -2","p = -1","p = 0","p = 1","p = 2"], title="#84 (Section 8.8)" );
q12 := Int( f(x,p), x=domain );
q12p := Int( f(x,p), x=0..infinity );
q12n := Int( f(x,p), x=-infinity..0 );
q12 = q12p + q12n;
`` = simplify( q12p+q12n );
```

81-84. Example CAS commands:

Mathematica: (functions and domains may vary)

```
Clear[x, f, p]
f[x_]:= x^p Log[Abs[x]]
int = Integrate[f[x], {x, e, 100)]
int /. p → 2.5
```

In order to plot the function, a value for p must be selected.

p = 3;

Plot[f[x], {x, 2.72, 10}]

85. Maple gives $\int_0^{2/\pi} \sin\left(\frac{1}{x}\right) dx = \frac{1}{\pi}\left(2 - \pi \cdot \mathrm{Ci}\left(\frac{\pi}{2}\right)\right) \approx 0.16462,$ where Ci is the cosine integral function defined by $\mathrm{Ci}(t) = -\int_t^{\infty} \frac{\cos x}{x}\, dx.$

86. Maple gives $\int_0^{2/\pi} x \sin\left(\frac{1}{x}\right) dx = \frac{2}{\pi^2} + \frac{1}{2}\mathrm{Si}\left(\frac{\pi}{2}\right) - \frac{\pi}{4} \approx 0.10276,$ where Si is the sine integral function defined by $\mathrm{Si}(t) = \int_0^{t} \frac{\sin x}{x}\, dx.$

8.9 PROBABILITY

1. $\int_4^8 \frac{1}{18} x\, dx = \frac{4}{3} \neq 1;$ not a probability density.

2. $\int_0^2 \frac{1}{2}(2 - x)\, dx = 1;$ a probability density.

3. $$\int_0^{\ln(1+\ln 2)/\ln 2} 2^x dx = \left.\frac{2^x}{\ln 2}\right]_0^{\ln(1+\ln 2)/\ln 2}$$
$$= \left(\frac{1+\ln 2}{\ln 2} - \frac{1}{\ln 2}\right) = 1$$
This is a probability density.

4. $x - 1$ is not nonnegative on $[0, 1+\sqrt{3}],$ so not a probability density.

5. $\int_1^{\infty} \frac{1}{x^2}\, dx = 1;$ a probability density.

6. $$\int_0^{\infty} \frac{8}{\pi\left(4 + x^2\right)}\, dx = \lim_{b\to\infty}\left(\left.\frac{4}{\pi}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^b\right)$$
$$= \frac{4}{\pi} \cdot \frac{\pi}{2} = 2$$
This is not a probability density.

7. $\int_0^{\pi/4} 2\cos 2x\, dx = 1;$ a probability density.

8. $\int_0^e \frac{1}{x}\,dx$ diverges; not a probability density.

9. (a) The probability that a tire lasts between 25,000 and 32,000 kilometers
(b) The probability that a tire lasts more than 30,000 kilometers
(c) The probability that a tire lasts less than 20,000 kilometers
(d) The probability that a tire lasts less than 15,000 kilometers

10. (a) $\int_{\pi}^{3\pi/2} \frac{1}{2\pi}\,dx + \int_{\pi/2}^{\pi} \frac{1}{2\pi}\,dx = 0.5$

(b) $\int_2^{2\pi} \frac{1}{2\pi}\,dx = 1 - \frac{1}{\pi} \approx 0.682$

11. $\int_1^3 xe^{-x}dx = -(x+1)e^{-x}\Big]_1^3 = -4e^{-3} + 2e^{-1} \approx 0.537$

12. $\int_2^{15} \frac{\ln x}{x^2}\,dx = -\frac{\ln x + 1}{x}\Big]_2^{15} = -\frac{\ln 15}{15} - \frac{1}{15} + \frac{\ln 2}{2} + \frac{1}{2} \approx 0.599$

13. $\int_0^1 \frac{3}{2}x(2-x)\,dx = -\frac{1}{2}x^2(x-3)\Big]_{1/2}^{1} = \frac{11}{16} \approx 0.688$

14. Using software to evaluate the Sine Integral we find $\int_{\frac{200}{1059}}^{\infty} \frac{\sin^2 \pi x}{\pi x^2}\,dx \approx 1.00004780741$ so the given function is very nearly a probability density over the given interval. Again using software we find that

$\int_{\frac{200}{1059}}^{\pi/6} \frac{\sin^2 \pi x}{\pi x^2}\,dx \approx 0.6732.$

15. $\int_4^9 \frac{2}{x^3}\,dx = \frac{65}{1296} \approx 0.0502$

16. $\int_{\pi/6}^{\pi/4} \sin x\,dx = -\cos x\Big]_{\pi/6}^{\pi/4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \approx 0.159$

17. $\int_3^c \frac{1}{6}x\,dx = \frac{1}{12}c^2 - \frac{3}{4}$. Solving $\frac{1}{12}c^2 - \frac{3}{4} = 1$, we find $c = \sqrt{21}$.

18. $\int_c^{c+1} \frac{1}{x}\,dx = \ln(c+1) - \ln c = \ln\left(\frac{c+1}{c}\right)$. Solving $\ln\left(\frac{c+1}{c}\right) = 1$, we find $\frac{c+1}{c} = e$ and thus $c = \frac{1}{e-1}$.

19. $\int_0^c 4e^{-2x}\,dx = -2e^{-2c} + 2$. Solving $-2e^{-2c} + 2 = 1$, we find $c = \frac{1}{2}\ln 2$.

20. $\int_0^5 cx\sqrt{25-x^2}\,dx = -\frac{1}{3}c\left(25-x^2\right)^{3/2}\Big]_0^5 = \frac{125}{3}c$, so $c = \frac{3}{125}$.

21. We will assume that the given function is to be a probability density over the whole real line.

$\int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = c\pi$ so we take $c = \frac{1}{\pi}$. Then $\int_1^2 \frac{1}{\pi\left(1+x^2\right)}\,dx = \frac{\tan^{-1} x}{\pi}\Big]_1^2 = \frac{\tan^{-1} 2}{\pi} - \frac{1}{4} \approx 0.10242$.

22. $\int_0^1 c\sqrt{x}(1-x)\,dx = c\left(\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right)\Big]_0^1 = \frac{4}{15}c$, so $c = \frac{15}{4}$. Then $\int_{1/4}^{1/2} \frac{15}{4}\sqrt{x}(1-x)\,dx = \frac{7}{16}\sqrt{2} - \frac{17}{64} \approx 0.353$

23. $\int_0^\infty e^{-cx}dx = \frac{1}{c}\lim_{b\to\infty}\left(-e^{-bcx}+1\right) = \frac{1}{c}$. Thus multiplying e^{-cx} by c produces a probability density on $[0,\infty)$.

24. $\operatorname{Var}(X) = \int_{-\infty}^{\infty} (X-\mu)^2 f(X)\,dX$

$= \int_{-\infty}^{\infty} X^2 f(X)\,dX + \int_{-\infty}^{\infty} (-2X\mu) f(X)\,dX + \int_{-\infty}^{\infty} \mu^2 f(X)\,dX$

$\int_{-\infty}^{\infty} (-2X\mu) f(X)\,dX = -2\mu\int_{-\infty}^{\infty} X\, f(X)\,dX = -2\mu^2$

$\int_{-\infty}^{\infty} \mu^2 f(X)\,dX = \mu^2\int_{-\infty}^{\infty} f(X)\,dX = \mu^2(1) = \mu^2$

Thus $\operatorname{Var}(X) = \int_{-\infty}^{\infty} X^2 f(X)\,dX - \mu^2$.

25. mean $= \int_0^4 x\left(\frac{1}{8}x\right)dx = \frac{8}{3}$

To find the median we need to solve $\int_0^c \frac{1}{8}x\,dx = \frac{1}{16}c^2 = \frac{1}{2}$ for c. Thus the median is $\sqrt{8}$.

26. mean $= \int_0^3 x\left(\frac{1}{9}x^2\right)dx = \frac{9}{4}$

To find the median we need to solve $\int_0^c \frac{1}{9}x^2\,dx = \frac{1}{27}c^3 = \frac{1}{2}$ for c. Thus the median is $\frac{3}{2}2^{2/3} \approx 2.381$.

27. mean $= \int_1^\infty x\left(\frac{2}{x^3}\right)dx = \lim_{b\to\infty}\left(-\frac{2}{x}\Big]_1^b\right) = 2$

To find the median we need to solve $\int_1^c \frac{2}{x^3}\,dx = -\frac{1}{c^2}+1 = \frac{1}{2}$ for c. Thus the median is $\sqrt{2}$.

28. mean $= \int_1^e x\left(\frac{1}{x}\right)dx = e-1 \approx 1.718$

To find the median we need to solve $\int_1^c \frac{1}{x}\,dx = \ln c = \frac{1}{2}$ for c. Thus the median is $\sqrt{e} \approx 1.649$.

29. The exponential density with mean 1 is e^{-X}. The probability that the food is digested in less than 30 minutes is $\int_0^{1/2} e^{-X}\,dX = -e^{-1/2}+1 \approx 0.3935.$

30. The exponential density with mean 4 is $(1/4)e^{-X/4}$. The probability that a flower is pollinated within 5 minutes is $\int_0^5 (1/4)e^{-X/4}\,dX = -e^{-5/4}+1 \approx 0.7135$. Out of 1000 flowers we would expect 713 or 714 to be pollinated within 5 minutes.

31. The exponential density with mean 1200 is $(1/1200)e^{-X/1200}$.

(a) The probability that a bulb will last less than 1000 hours is $\int_0^{1000} (1/1200)e^{-X/1200}\,dX = -e^{-5/6}+1 \approx 0.5654.$

(b) By Example 9 the median lifetime is $1200 \ln 2 \approx 831.8$ so the expected time until half the bulbs in a batch fail is 832 h.

32. To find the density, solve $\int_0^3 (1/c)e^{-X/c}\,dX = -e^{-3/c}+1 = \frac{1}{3}$. Then $c = \frac{3}{\ln(3/2)} \approx 7.3989$, so the mean lifetime of the components is 7.4 years. The probability of failure within 1 year is
$$\int_0^1 \frac{\ln(3/2)}{3} e^{-\frac{X\ln(3/2)}{3}}\,dX = -\left(\frac{2}{3}\right)^{1/3}+1 \approx 0.1264.$$

33. To find the density, solve $\int_0^2 (1/c)e^{-X/c}\,dX = -e^{-2/c}+1 = \frac{2}{5}; c = \frac{2}{\ln(5/3)}$. The probability that a hydra dies within 6 months, or half a year, is $\int_0^{1/2} \frac{\ln(5/3)}{2} e^{-\frac{X\ln(5/3)}{2}}\,dX = -\left(\frac{3}{5}\right)^{1/4}+1 \approx 0.1199$, so we would expect $(0.12)(500) = 60$ hydra to die within the first six months.

34. To find the density, solve $\int_0^{50} (1/c)e^{-X/c}\,dX = -e^{-50/c}+1 = \frac{3}{10}; c = \frac{50}{\ln(10/7)}$. The probability that a high-risk driver is involved in an accident in the first 80 days is $\int_0^{80} \frac{\ln(10/7)}{50} e^{-\frac{X\ln(10/7)}{50}}\,dX =$ $-\left(\frac{7}{10}\right)^{8/5}+1 \approx 0.4349$, so we would expect 43 or 44 out of 100 high-risk drivers to be involved in an accident in the first 80 days.

35. Using seconds as the time unit, the density is $(1/30)e^{-X/30}$.

(a) $\int_0^{15} (1/30)e^{-X/30}\,dX = -e^{-1/2}+1 \approx 0.393$

(b) $\int_{60}^{\infty} (1/30)e^{-X/30}\,dX = e^{-2} \approx 0.135$

(c) In a continuous distribution the probability of a particular number is 0.

(d) The probability than a single customer waits less than 3 minutes is $-e^{-6}+1 \approx 0.997521$. The probability that at least one customer out of 200 waits longer than 3 minutes is $1-(0.997521)^{200} \approx 0.391 < 0.5$, so the most likely outcome is that all 200 are served within 3 minutes.

36. For parts (a) and (b) the density is $(1/16)e^{-X/16}$. For parts (c) and (d) the density is $(1/32)e^{-X/32}$.

(a) $\int_{10}^{30}(1/16)e^{-X/16}\,dX = -e^{-15/8}+e^{-5/8} \approx 0.382$

(b) $\int_{25}^{\infty}(1/16)e^{-X/16}\,dX = e^{-25/16} \approx 0.210$

(c) $\int_{35}^{50}(1/32)e^{-X/32}\,dX = -e^{-25/16}+e^{-35/32} \approx 0.125$

(d) $\int_{0}^{20}(1/32)e^{-X/32}\,dX = -e^{-5/8}+1 \approx 0.465$

37. The expected payout per printer is $200\int_0^1(1/2)e^{-X/2}\,dX + 100\int_1^2(1/2)e^{-X/2}\,dX \approx \$102.56$. Thus the expected refund total for 100 machines is \$10,256.

38. To find the density, solve $\displaystyle\int_0^2 \frac{1}{c}e^{-X/c}\,dX = -e^{-2/c}+1 = \frac{1}{2}$, which gives $c = \dfrac{2}{\ln(2)}$. The probability of failure in the first year is $\displaystyle\int_0^1 \frac{\ln 2}{2}e^{-\frac{X\ln 2}{2}}\,dX = -\frac{\sqrt{2}}{2}+1 \approx 0.293$. We expect $(150)(0.293) = 43.934$ or about 44 copiers to fail during the first year.

For Exercises 39–52, the density function is $f(X) = \dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ with μ and σ as given in the solution.

39. $\mu = 4.19,\ \sigma = 0.72$

$\int_{4.27}^{4.99} f(X)\,dX \approx 0.323$; about 323 children

$\int_{3.83}^{4.32} f(X)\,dX \approx 0.262$; about 262 children

40. $\mu = 510,\ \sigma = 120$

$\int_{444}^{\infty} f(X)\,dX = \frac{1}{2}+\int_{444}^{\mu} f(X)\,dX \approx 0.74593$

41. $\mu = 55,\ \sigma = 4$

$\int_0^{60} f(X)\,dX \approx 0.89435$

42. $\mu = 35{,}000,\ \sigma = 6000$

(a) $\int_{29{,}000}^{\infty} f(X)\,dX = \frac{1}{2}+\int_{29{,}000}^{\mu} f(X)\,dX \approx 0.84134$; $(4000)(0.84134) \approx 3365$ tires

(b) We want to find L such that $\int_L^{\infty} f(X)\,dX = 0.9$. A CAS gives $L \approx 27{,}311$, so 90% of tires will have a lifetime of at least 27,311 km.

43. $\mu = 166,\ \sigma = 6$

(a) $\int_{172}^{\infty} f(X)\,dX = \frac{1}{2}-\int_{\mu}^{172} f(X)\,dX \approx 0.159$, or 16%.

(b) $\int_{155}^{163} f(X)\,dX \approx 0.23832$

44. $\mu = 81,\ \sigma = 7$

$\int_{75}^{85} f(X)\,dX \approx 0.520$; one would expect 52 of the babies to live to between 75 and 85.

45. $\mu = 266,\ \sigma = 16;\ (36)(7) = 252,\ (40)(7) = 280$

$\int_{252}^{280} f(X)\,dX \approx 0.6184$; we would expect 618 of the women to have pregnancies lasting between 36 and 40 weeks.

46. $\mu = 1400,\ \sigma = 100$

(a) $\int_{1325}^{1450} f(X)\,dX \approx 0.46484$

(b) $\int_{1480}^{\infty} f(X)\,dX = \frac{1}{2} - \int_{\mu}^{1480} f(X)\,dX \approx 0.21186$

$(500)(0.21186) \approx 106$; we would expect about 106 males to have a brain weight exceeding 1480 g.

47. $\mu = 80,\ \sigma = 12$

$\int_{0}^{70} f(X)\,dX \approx 0.20233$

$(300)(0.20233) \approx 61$; about 61 adults.

48. $\mu = 4.4,\ \sigma = 0.2$

$\int_{4.3}^{4.45} f(X)\,dX \approx 0.29017$

49. $\mu = 35,\ \sigma = 9$

$\int_{40}^{\infty} f(X)\,dX = \frac{1}{2} - \int_{\mu}^{40} f(X)\,dX \approx 0.28926$

About 289 shafts would need more than 45 grams of added weight.

50. $\mu = 300,\ \sigma = 50$

$\int_{400}^{\infty} f(X)\,dX = \frac{1}{2} - \int_{\mu}^{400} f(X)\,dX \approx 0.02275$

$\int_{0}^{250} f(X)\,dX \approx 0.15866$

$0.02275 + 0.15866 = 0.18141$ or about 18%

51. $\mu = (0.8)(2500) = 2000,\ \sigma = (0.4)(50) = 20$

(a) $\int_{1960}^{\infty} f(X)\,dX = \frac{1}{2} + \int_{1960}^{\mu} f(X)\,dX \approx 0.977$

(b) $\int_{0}^{1980} f(X)\,dX \approx 0.159$

(c) $\int_{1940}^{2020} f(X)\,dX \approx 0.840$

52. $\mu = (0.5)(400) = 200,\ \sigma = \frac{20}{2} = 10$

To improve the approximation to the binomial distribution we will modify the interval of integration. We give the true binomial values for comparison.

(a) $\int_{189.5}^{209.5} f(X)\,dX \approx 0.68208$; correct to 5 places

(b) $\int_{0}^{169.5} f(X)\,dX \approx 0.00114$; true value ≈ 0.00112

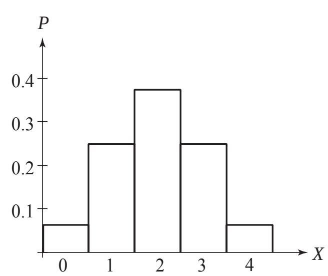
(c) $\int_{220.5}^{\infty} f(X)\,dX = \frac{1}{2} - \int_{\mu}^{220.5} f(X)\,dX \approx 0.02018$; true value ≈ 0.02012

(d) The value is very close to 0, in fact about 10^{-24}.

53. (a) and (b)

Outcome	X
HHHH	0
THHH	1
HTHH	1
HHTH	1
HHHT	1
TTHH	2
THTH	2
THHT	2
HTTH	2
HTHT	2
HHTT	2
TTTH	3
TTHT	3
THTT	3
HTTT	3
TTTT	4

(c)

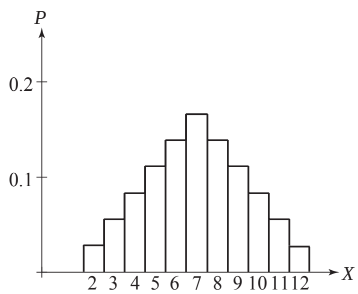


The probability of at least 2 heads is $\frac{1}{16}(1+4+6) = \frac{11}{16}$.

54. (a) The first die is listed first in each pair, the second die second.

1 + 1 = 2	2 + 1 = 3	3 + 1 = 4	4 + 1 = 5	5 + 1 = 6	6 + 1 = 7
1 + 2 = 3	2 + 2 = 4	3 + 2 = 5	4 + 2 = 6	5 + 2 = 7	6 + 2 = 8
1 + 3 = 4	2 + 3 = 5	3 + 3 = 6	4 + 3 = 7	5 + 3 = 8	6 + 3 = 9
1 + 4 = 5	2 + 4 = 6	3 + 4 = 7	4 + 4 = 8	5 + 4 = 9	6 + 4 = 10
1 + 5 = 6	2 + 5 = 7	3 + 5 = 8	4 + 5 = 9	5 + 5 = 10	6 + 5 = 11
1 + 6 = 7	2 + 6 = 8	3 + 6 = 9	4 + 6 = 10	5 + 6 = 11	6 + 6 = 12

(b)



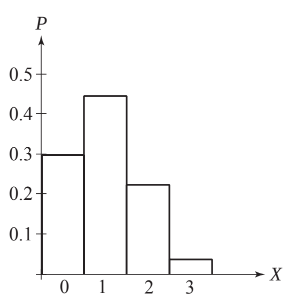
(c) $P(8) = \dfrac{5}{36}$

(d) $P(X \le 5) = \dfrac{1+2+3+4}{36} = \dfrac{5}{18}$

$P(X > 9) = \dfrac{3+2+1}{36} = \dfrac{1}{6}$

55. (a) {LLL, LLD, LDL, DLL, LLU, LUL, ULL, LDD, DLD, DDL,LUU, ULU, UUL, DDU, DUD, UDD, DUU, UDU, UUD,LUD, LDU, ULD, UDL, DLU, DUL, DDD, UUU}

(b) We will assume that the three answers are equally likely, though other assumptions might be reasonable. The plots for the $X =$ number of Ls, the number of Us and the number of Ds are identical.



(c) $P(\text{at least two L}) = \dfrac{1+3+3}{27} = \dfrac{7}{27} \approx 0.26$

(d) $P(\text{no more than one D}) = 1 - P(\text{at least two D})$

$$= 1 - \frac{7}{27} = \frac{20}{27} \approx 0.74$$

56. The probability that both systems fail is 0.0148. Since the two systems have the same performance distribution, the failure probability for a single system is $\sqrt{0.0148} = 0.121655$. The success probability for a single system is $1-\sqrt{0.0148}$ so the probability that both succeed is $\left(1-\sqrt{0.0148}\right)^2 = 0.771489$. The probability that one fails and one succeeds is $1-0.0148-0.771489 = 0.213711$. Since the events "main fails, backup succeeds" and "main succeeds, backup fails" have the same probability, the probability that only the main fails is $0.213711/2 = 0.106856$. Thus the probability that the main fails, either along with the backup or by itself, is $0.0148+0.106856 = 0.121656$.

CHAPTER 8 PRACTICE EXERCISES

1. $\int x\ln(x)\,dx = \ln(x)\int x\,dx - \int\left(\int x\,dx\right)\left(\frac{d}{dx}(\ln(x))\right)dx = \ln(x)\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C$

2. $\int x^2\cos x\,dx = x^2\int\cos x\,dx - \int\left(\int\cos x\,dx\right)\left(\frac{d}{dx}(x^2)\right)dx = x^2(\sin x) - \int(\sin x)(2x)\,dx$

$$= x^2\sin x - 2\int x\sin x\,dx = x^2\sin x - 2\left[x\int\sin x\,dx - -\int\left(\int\sin x\,dx\right)\left(\frac{d}{dx}(x)\right)dx\right]$$
$$= x^2\sin x - 2\left[-x\cos x - \int -\cos x\,dx\right] = x^2\sin x + 2x\cos x - 2\sin x + C$$

3. Put $\sin^{-1}2x = t,\ 2x = \sin t,\ 2\,dx = \cos t\,dt,\ dx = \frac{\cos t}{2}dt;$

$$\int\sin^{-1}2x\,dx = \int t\frac{\cos t}{2}dt = \frac{1}{2}\left[t\int\cos t dt - \int\left(\int\cos t dt\right)\left(\frac{d}{dt}(t)\right)dt\right]$$
$$= \frac{1}{2}[t\sin t + \cos t] + C = \frac{1}{2}\left[2x\sin^{-1}2x + \cos\left(\sin^{-1}2x\right)\right] + C$$

4. Put $\tan^{-1}\left(\frac{x}{3}\right) = \mathrm{t} \Rightarrow \frac{x}{3} = \tan t \Rightarrow \frac{dx}{3} = \sec^2 tdt \Rightarrow dx = 3\sec^2 tdt;$

$$\int\tan^{-1}\left(\frac{x}{3}\right)dx = 3\int t\sec^2 t\,dt = 3\left[t\int\sec^2 t\,dt - \int\left(\int\sec^2 t\,dt\right)\left(\frac{d}{dt}(t)\right)dt\right] = 3\left[t\tan t - \int\tan t\,dt\right]$$
$$= 3\left[t\tan t + \log\left|\cos t\right|\right] + C = 3\left[\frac{x}{3}\tan^{-1}\left(\frac{x}{3}\right) + \log\left|\cos\left(\tan^{-1}\frac{x}{3}\right)\right|\right] + C$$

5. $\int(1+x^2)e^x\,dx = (1+x^2)\int e^x\,dx - \int\left(\int e^x\,dx\right)\left(\frac{d}{dx}(1+x^2)\right)dx = (1+x^2)e^x - 2\int xe^x\,dx$

$$= (1+x^2)e^x - 2xe^x + 2e^x + C$$

6. $\int x^2\cos(2-x)\,dx = x^2\int\cos(2-x)dx - \int\left(\int\cos(2-x)dx\right)\left(\frac{d}{dx}(x^2)\right)dx = x^2\left(\frac{\sin(2-x)}{-1}\right) - \int\left(\frac{\sin(2-x)}{-1}\right)(2x)dx$

$$= -x^2\sin(2-x) + 2\left[x\int\sin(2-x)dx - \int\left(\int\sin(2-x)dx\right)\left(\frac{d}{dx}(x)\right)dx\right]$$
$$= -x^2\sin(2-x) + 2x\cos(2-x) - 2\int\cos(2-x)dx = -x^2\sin(2-x) + 2x\cos(2-x) + 2\sin(2-x) + C$$

7. $\int e^x\sin 3x\,dx = \frac{e^x}{1^2+3^2}\left((1)\sin 3x - 3\cos 3x\right) + C = \frac{e^x}{10}\left(\sin 3x - 3\cos 3x\right) + C$

8. $\int x\sin 2x\cos 2x\,dx = \int x\frac{\sin 2(2x)}{2}\,dx = \frac{1}{2}\int x\sin 4x\,dx = \frac{1}{2}\left[x\left(\frac{-\cos 4x}{4}\right) - \int\left(\frac{-\cos 4x}{4}\right)\left(\frac{d}{dx}(x)\right)dx\right]$

$$= \frac{-x\cos 4x}{8} + \frac{\sin 4x}{32} + C$$

9. Consider $\frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)} \Rightarrow x = A(x-3) + B(x-2)$

when $x = 2$, $2 = A(2-3) + B(2-2) \Rightarrow A = -2$

when $x = 3$, $3 = A(3-3) + B(3-2) \Rightarrow B = 3$

$\int \frac{x}{x^2-5x+6} dx = \int \frac{-2}{x-2} + \frac{3}{x-3} dx = -2\ln(x-2) + 3\ln(x-3) + C$

10. Consider $\frac{x}{x^2+4x-5} = \frac{x}{(x+5)(x-1)} = \frac{A}{(x+5)} + \frac{B}{(x-1)} \Rightarrow x = A(x-1) + B(x+5)$

when $x = 1$, $1 = B(1+5) \Rightarrow B = 1/6$

when $x = -5$, $-5 = A(-5-1) + B(-5+5) \Rightarrow A = 5/6$

$\int \frac{x}{x^2+4x-5} dx = \int \frac{5/6}{x+5} + \frac{1/6}{x-1} dx = 5/6\ln(x+5) + 1/6\ln(x-1) + C$

11. Consider $\frac{1}{x(x-2)^2} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} \Rightarrow 1 = A(x-2)^2 + B(x-2) + Cx$

when $x = 0$, $1 = 4A \Rightarrow A = 1/4$

when $x = 2$, $1 = C(2) \Rightarrow C = 1/2$

coefficient of x^2, $0 = A + B \Rightarrow B = -1/4$

$\int \frac{1}{x(x-2)^2} dx = \int \frac{1/4}{x} dx - \int \frac{1/4}{x-2} dx + \int \frac{1/2}{(x-2)^2} dx = \frac{1}{4}\ln x - \frac{1}{4}\ln|x-2| - \frac{1}{2(x-1)} + C$

12. Consider $\frac{x+3}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \Rightarrow x + 3 = Ax(x-2) + B(x-2) + Cx^2$

when $x = 0$, $3 = B(-2) \Rightarrow B = -3/2$

when $x = 2$, $5 = C(2)^2 \Rightarrow C = 5/4$

coefficient of x^2, $0 = A + C \Rightarrow A = -5/4$

$\int \frac{x+3}{x^2(x-2)} dx = \int \frac{-5/4}{x} dx + \int \frac{-3/2}{x^2} dx + \int \frac{5/4}{x-2} dx = \frac{-5}{4}\ln|x| + \frac{3}{2x} + \frac{5}{4}\ln|x-2| + C$

13. Put $\sin x = t$, $\cos x dx = dt \Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \int \frac{dt}{(1-t)(2-t)}$

consider $\frac{1}{(1-t)(2-t)} = \frac{A}{1-t} + \frac{B}{2-t} \Rightarrow 1 = A(2-t) + B(1-t)$

when $t = 1$, $1 = A(1) \Rightarrow A = 1$

when $t = 2$, $1 = B(-1)^2 \Rightarrow B = -1$

$\int \frac{dt}{(1-t)(2-t)} = \int \frac{1}{(1-t)} dt - \int \frac{1}{(2-t)} dt = -\ln|1-t| + \ln|2-t| + C$

$\int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \int \frac{1}{1-\sin x} dx - \int \frac{1}{2-\sin x} dx = -\ln|1-\sin x| + \ln|2-\sin x| + C$

14. Put $\cos x = t$, $-\sin x dx = dt \Rightarrow \int \frac{\sin x}{\cos^2 x + 3\cos x + 2} dx = \int \frac{-dt}{t^2+3t+2} = -\int \frac{dt}{(t+2)(t+1)}$

consider $\frac{1}{(t+2)(t+1)} = \frac{A}{t+2} + \frac{B}{t+1} \Rightarrow 1 = A(t+1) + B(t+2)$

when $t = -2$, $1 = B(-1) \Rightarrow B = -1$

when $t = -1$, $1 = A(1) \Rightarrow A = 1$

$\int \frac{1}{(t+2)(t+1)} dt = \int \frac{1}{t+1} dt - \int \frac{1}{t+2} dt = \ln|t+1| - \ln|t+2| + C$

$\int \frac{\sin x}{\cos^2 x + 3\cos x + 2} dx = \ln|\cos x + 1| - \ln|\cos x + 2| + C$

15. Put $\cos x = t, -\sin x dx = dt \Rightarrow \int \frac{\sin x}{\cos^2 x + 3\cos x + 2} dx = \int \frac{-dt}{t^2+3t+2} = -\int \frac{dt}{(t+2)(t+1)}$

$\frac{5x^2+7x+9}{x^3-x} = \frac{5x^2+7x+9}{x(x^2-1)} = \frac{A}{x} + \frac{B}{x^2-1} \Rightarrow 5x^2+7x+9 = A(x^2-1)+(Bx+C)(x)$

when $x=0$, $9 = A(-1) \Rightarrow A = -1/9$

coefficient of x^2, $5 = A+B \Rightarrow B = 5-A \Rightarrow B = 5-\left(-\frac{1}{9}\right) = \frac{46}{9}$

coefficient of x, $7 = C$

$\int \frac{5x^2+7x+9}{x^3-x} dx = \int \frac{-1/9}{x} dx + \int \frac{\frac{46}{9}x+7}{x^2-1} dx = \frac{-1}{9}\ln|x| + \frac{46}{9}\frac{1}{2}\ln\left|x^2-1\right| + 7\ln\left|\frac{x-1}{x+1}\right| + C$

16. $\int \frac{9x}{x^3+9x} dx = 9\int \frac{x}{x(x^2+9)} dx = 9\int \frac{1}{x^2+9} dx = 9\frac{1}{3}\tan^{-1}\frac{x}{3} + C = 3\tan^{-1}\frac{x}{3} + C$

17. Consider $\frac{x-5}{x(x^2-4)} = \frac{A}{x} + \frac{Bx+C}{x^2-4} \Rightarrow x-5 = Ax(x^2-4)+(B+C)(x)$

when $x=0$, $-5 = A(-4) \Rightarrow A = 5/4$

coefficient of x^2, $0 = A+B \Rightarrow B = -A \Rightarrow B = -5/4$

coefficient of x, $C = 1$

$\int \frac{x-5}{3x^3-12x} dx = \frac{1}{3}\int \frac{x-5}{x(x^2-4)} dx = \frac{1}{3}\left[\int \frac{5/4}{x} dx + \int \frac{\frac{-5}{4}x}{x^2-4} dx + \int \frac{1}{x^2-4} dx\right] = \frac{1}{3}\left[\frac{5}{4}\ln|x| - \frac{5}{8}\ln\left|x^2-4\right| + \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|\right] + C$

18. Consider $\frac{5x+9}{(x-2)(x+3)(x+4)} = \frac{A}{x-2} + \frac{B}{x+3} + \frac{C}{x+4}$

$\Rightarrow 5x+9 = A(x+3)(x+4)+B(x-2)(x+4)+C(x-2)(x+3)$

when $x=2$, $5(2)+9 = A(2+3)(2+4) \Rightarrow 19 = A(30) \Rightarrow A = \frac{19}{30}$

when $x=-3$, $5(-3)+9 = A(-3-2)(-3+4) \Rightarrow 6 = B(-5) \Rightarrow B = \frac{-6}{5}$

when $x=-4$, $5(-4)+9 = A(-4-2)(-4+3) \Rightarrow -11 = C(6) \Rightarrow C = \frac{-6}{11}$

coefficient of x^2, $0 = A+B \Rightarrow B = -A \Rightarrow B = -5/4$

coefficient of x, $C = 1$

$\int \frac{5x+9}{(x-2)(x+3)(x+4)} dx = \int \frac{19/30}{x-2} dx + \int \frac{-6/5}{x+3} dx + \int \frac{-6/11}{x+4} dx = \frac{19}{30}\ln\left|x-2\right| - \frac{6}{5}\ln\left|x+3\right| - \frac{6}{11}\ln\left|x+4\right| + C$

19. $\int \frac{d\theta}{\theta^4+5\theta^2+4} = \int \frac{d\theta}{(\theta^2+4)(\theta^2+1)}$

consider $\frac{1}{(\theta^2+4)(\theta^2+1)} = \frac{1}{(p+4)(p+1)}$ by replacing θ^2 by p

$\frac{1}{(p+4)(p+1)} = \frac{A}{p+4} + \frac{B}{p+1} \Rightarrow 1 = A(p+1)+B(p+4)$

when $p=-1$, $1 = B(-1+4) \Rightarrow B = \frac{1}{3}$

when $x=-4$, $1 = A(-4+1) \Rightarrow A = \frac{-1}{3}$

$\int \frac{d\theta}{\theta^4+5\theta^2+4} = \int \frac{d\theta}{(\theta^2+4)(\theta^2+1)} = \int \frac{-1/3}{\theta^2+4} d\theta + \int \frac{1/3}{\theta^2+1} d\theta = \frac{-1}{3}\frac{1}{2}\tan^{-1}\left(\frac{\theta}{2}\right) + \frac{1}{2}\tan^{-1}\theta + C$

20. Put $t^2 = p \Rightarrow 2tdt = dp \Rightarrow tdt = \frac{dp}{2}$

$\int \frac{t\,dt}{t^4+t^2-2} = \int \frac{dp/2}{p^2+p-2} = \frac{1}{2}\int \frac{dp}{(p+2)(p-1)}$

consider $\frac{1}{(p+2)(p-1)} = \frac{A}{p+2} + \frac{B}{p-1} \Rightarrow 1 = A(p-1) + B(p+2)$

when $p = 1$, $\quad 1 = B(1+2) \Rightarrow B = \frac{1}{3}$

when $p = -2$, $\quad 1 = A(-2-1) \Rightarrow A = -\frac{1}{3}$

$\therefore \frac{1}{2}\int \frac{dp}{(p+2)(p-1)} = \frac{1}{2}\left[\int \frac{-1/3}{p+2}\,dp + \int \frac{1/3}{p-1}\,dp\right] = \frac{1}{2}\left[-\frac{1}{3}\ln|p+2| + \frac{1}{3}\ln|p-1|\right] + C$

$= \frac{1}{2}\left[-\frac{1}{3}\ln\left|t^2+2\right| + \frac{1}{3}\ln\left|t^2-1\right|\right] + C$

21. $\int \frac{x^3+x^2}{x^2+x-2}\,dx = \int\left(x + \frac{2x}{x^2+x-2}\right)dx = \int x\,dx + \frac{2}{3}\int \frac{dx}{x-1} + \frac{4}{3}\int \frac{dx}{x+2} = \frac{x^2}{2} + \frac{4}{3}\ln|x+2| + \frac{2}{3}\ln|x-1| + C$

22. $\int \frac{x^3+1}{x^3-x}\,dx = \int\left(1 + \frac{x+1}{x^3-x}\right)dx = \int\left[1 + \frac{1}{x(x-1)}\right]dx = \int dx + \int \frac{dx}{x-1} - \int \frac{dx}{x} = x + \ln|x-1| - \ln|x| + C$

23. $\int \frac{x^3+4x^2}{x^2+4x+3}\,dx = \int\left(x - \frac{3x}{x^2+4x+3}\right)dx = \int x\,dx + \frac{3}{2}\int \frac{dx}{x+1} - \frac{9}{2}\int \frac{dx}{x+3} = \frac{x^2}{2} - \frac{9}{2}\ln|x+3| + \frac{3}{2}\ln|x+1| + C$

24. $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx = \int\left[(2x-3) + \frac{x}{x^2+2x-8}\right]dx = \int (2x-3)\,dx + \frac{1}{3}\int \frac{dx}{x-2} + \frac{2}{3}\int \frac{dx}{x+4}$

$= x^2 - 3x + \frac{2}{3}\ln|x+4| + \frac{1}{3}\ln|x-2| + C$

25. $\int \frac{dx}{x\left(3\sqrt{x+1}\right)};\ \begin{bmatrix} u = \sqrt{x+1} \\ du = \frac{dx}{2\sqrt{x+1}} \\ dx = 2u\,du \end{bmatrix} \to \frac{2}{3}\int \frac{u\,du}{\left(u^2-1\right)u} = \frac{1}{3}\int \frac{du}{u-1} - \frac{1}{3}\int \frac{du}{u+1} = \frac{1}{3}\ln|u-1| - \frac{1}{3}\ln|u+1| + C = \frac{1}{3}\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

26. $\int \frac{dx}{x\left(1+\sqrt[3]{x}\right)};\ \begin{bmatrix} u = \sqrt[3]{x} \\ du = \frac{dx}{3x^{2/3}} \\ dx = 3u^2\,du \end{bmatrix} \to \int \frac{3u^2\,du}{u^3(1+u)} = 3\int \frac{du}{u(1+u)} = 3\ln\left|\frac{u}{u+1}\right| + C = 3\ln\left|\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}\right| + C$

27. $\int \frac{ds}{e^s-1};\ \begin{bmatrix} u = e^s - 1 \\ du = e^s ds \\ ds = \frac{du}{u+1} \end{bmatrix} \to \int \frac{du}{u(u+1)} = -\int \frac{du}{u+1} + \int \frac{du}{u} = \ln\left|\frac{u}{u+1}\right| + C = \ln\left|\frac{e^s-1}{e^s}\right| + C = \ln\left|1-e^{-s}\right| + C$

28. $\int \frac{ds}{\sqrt{e^s+1}};\ \begin{bmatrix} u = \sqrt{e^s+1} \\ du = \frac{e^s ds}{2\sqrt{e^s+1}} \\ ds = \frac{2u\,du}{u^2-1} \end{bmatrix} \to \int \frac{2u\,du}{u\left(u^2-1\right)} = 2\int \frac{du}{(u+1)(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u+1} = \ln\left|\frac{u-1}{u+1}\right| + C = \ln\left|\frac{\sqrt{e^s+1}-1}{\sqrt{e^s+1}+1}\right| + C$

29. (a) $\int \frac{y\,dy}{\sqrt{16-y^2}} = -\frac{1}{2}\int \frac{-2y\,dy}{\sqrt{16-y^2}} = -\sqrt{16-y^2} + C$

(b) $\int \frac{y\,dy}{\sqrt{16-y^2}};\ [y = 4\sin x] \to 4\int \frac{\sin x \cos x\,dx}{\cos x} = -4\cos x + C = -\frac{4\sqrt{16-y^2}}{4} + C = -\sqrt{16-y^2} + C$

30. (a) $\int \frac{x\,dx}{\sqrt{4+x^2}} = \frac{1}{2}\int \frac{2x\,dx}{\sqrt{4+x^2}} = \sqrt{4+x^2} + C$

(b) $\int \frac{x\,dx}{\sqrt{4+x^2}}$; $[x = 2\tan y] \to \int \frac{2\tan y \cdot 2\sec^2 y\,dy}{2\sec y} = 2\int \sec y \tan y\,dy = 2\sec y + C = \sqrt{4+x^2} + C$

31. (a) $\int \frac{x\,dx}{4-x^2} = -\frac{1}{2}\int \frac{(-2x)dx}{4-x^2} = -\frac{1}{2}\ln\left|4-x^2\right| + C$

(b) $\int \frac{x\,dx}{4-x^2}$; $[x = 2\sin\theta] \to \int \frac{2\sin\theta \cdot 2\cos\theta\,d\theta}{4\cos^2\theta} = \int \tan\theta\,d\theta = -\ln|\cos\theta| + C = -\ln\left(\frac{\sqrt{4-x^2}}{2}\right) + C$

$= -\frac{1}{2}\ln\left|4-x^2\right| + C$

32. (a) $\int \frac{t\,dt}{\sqrt{4t^2-1}} = \frac{1}{8}\int \frac{8t\,dt}{\sqrt{4t^2-1}} = \frac{1}{4}\sqrt{4t^2-1} + C$

(b) $\int \frac{t\,dt}{\sqrt{4t^2-1}}$; $\left[t = \frac{1}{2}\sec\theta\right] \to \int \frac{\frac{1}{2}\sec\theta\tan\theta \cdot \frac{1}{2}\sec\theta\,d\theta}{\tan\theta} = \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{\tan\theta}{4} + C = \frac{\sqrt{4t^2-1}}{4} + C$

33. $\int \frac{x\,dx}{9-x^2}$; $\begin{bmatrix} u = 9 - x^2 \\ du = -2x\,dx \end{bmatrix} \to -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C = \ln\frac{1}{\sqrt{u}} + C = \ln\frac{1}{\sqrt{9-x^2}} + C$

34. $\int \frac{dx}{x\left(9-x^2\right)} = \frac{1}{9}\int \frac{dx}{x} + \frac{1}{18}\int \frac{dx}{3-x} - \frac{1}{18}\int \frac{dx}{3+x} = \frac{1}{9}\ln|x| - \frac{1}{18}\ln|3-x| - \frac{1}{18}\ln|3+x| + C = \frac{1}{9}\ln|x| - \frac{1}{18}\ln\left|9-x^2\right| + C$

35. $\int \frac{dx}{9-x^2} = \frac{1}{6}\int \frac{dx}{3-x} + \frac{1}{6}\int \frac{dx}{3+x} = -\frac{1}{6}\ln|3-x| + \frac{1}{6}\ln|3+x| + C = \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + C$

36. $\int \frac{dx}{\sqrt{9-x^2}}$; $\begin{bmatrix} x = \sin\theta \\ dx = 3\cos\theta\,d\theta \end{bmatrix} \to \int \frac{3\cos\theta}{3\cos\theta}\,d\theta = \int d\theta = \theta + C = \sin^{-1}\frac{x}{3} + C$

37. $\int \sin^5 x\cos^6 x\,dx = \frac{\sin^{5-1}x\cos^{6+1}x}{5+6} + \frac{5-1}{5+6}\int \sin^{5-2}x\cos^6 x\,dx = \frac{\sin^4 x\cos^7 x}{11} + \frac{4}{11}\int \sin^3 x\cos^6 x\,dx$

$= \frac{\sin^4 x\cos^7 x}{11} + \frac{4}{11}\left[\frac{\sin^{3-1}x\cos^{6+1}x}{3+6} + \frac{3-1}{3+6}\int \sin^{3-2}x\cos^6 x\,dx\right]$

$= \frac{\sin^4 x\cos^7 x}{11} + \frac{4}{99}\sin^2 x\cos^7 x + \frac{8}{99}\left[\int \sin x\cos^6 x\,dx\right]$

$= \frac{\sin^4 x\cos^7 x}{11} + \frac{4}{99}\sin^2 x\cos^7 x + \frac{8}{99}\left(\frac{-\cos^7 x}{7}\right) + C$

38. $\int \cos^3 x\sin^3 x\,dx = \frac{\sin^{3-1}x\cos^{3+1}x}{3+3} + \frac{3-1}{3+3}\int \sin^{3-2}x\cos^4 x\,dx = \frac{\sin^2 x\cos^4 x}{6} + \frac{2}{6}\int \sin x\cos^4 x\,dx$

$= \frac{\sin^2 x\cos^4 x}{6} - \frac{1}{3}\frac{\cos^5 x}{5} + C = \frac{\sin^2 x\cos^4 x}{6} - \frac{\cos^5 x}{15} + C$

39. Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$\int \tan^6 x\sec^2 x\,dx = \int t^6 dt = \frac{t^7}{7} + C = \frac{\tan^7 x}{7} + C$

40. $\int \tan^3 x \sec^3 x\,dx = \int \left(\sec^2 x - 1\right)\sec^2 x \cdot \sec x \cdot \tan x\,dx = \int \sec^4 x \cdot \sec x \cdot \tan x\,dx - \int \sec^2 x \cdot \sec x \cdot \tan x\,dx$
$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

41. $\int \sin 7\theta \cos 5\theta\,d\theta = \frac{1}{2}\int \sin(7\theta + 5\theta) + \sin(7\theta - 5\theta)\,d\theta = \frac{1}{2}\int \sin(12\theta) + \frac{1}{2}\int \sin(2\theta)\,d\theta = \frac{1}{2}\left[-\frac{\cos 12\theta}{12} - \frac{\cos 2\theta}{2}\right] + C$

42. Put $\cot\theta = t \Rightarrow -\operatorname{cosec}^2\theta d\theta = dt$
$\int \csc^2\theta \cot^3\theta\,d\theta = -\int t^3 dt = -\frac{t^4}{4} + C = -\frac{\cot^4\theta}{4} + C$

43. $\int \sqrt{1 - \cos\left(\frac{t}{2}\right)}\,dt = \int \sqrt{2}\sin\frac{t}{4}\,dt = \sqrt{2}\left(\frac{-\cos\left(\frac{t}{4}\right)}{1/4}\right) + C = -4\sqrt{2}\cos\left(\frac{t}{4}\right) + C = -4\sqrt{1 + \cos\left(\frac{t}{2}\right)} + C$

44. Put $e^\theta = t \Rightarrow e^\theta d\theta = dt$
$\int e^\theta \sqrt{1 - \cos^2 e^\theta}\,d\theta = \int \sqrt{1 - \cos^2 t}\,dt = \int \sin t\,dt = -\cos t + C = -\cos e^\theta + C$

45. $|E_s| \le \frac{3-1}{180}(\Delta x)^4 M$ where $\Delta x = \frac{3-1}{n} = \frac{2}{n}$; $f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f''(x) = -6x^{-4}$
$\Rightarrow f^{(4)}(x) = 24x^{-5}$ which is decreasing on $[1, 3] \Rightarrow$ maximum of $f^{(4)}(x)$ on $[1, 3]$ is
$f^{(4)}(1) = 24 \Rightarrow M = 24$. Then $|E_s| \le 0.0001 \Rightarrow \left(\frac{3-1}{180}\right)\left(\frac{2}{n}\right)^4 (24) \le 0.0001 \Rightarrow \left(\frac{768}{180}\right)\left(\frac{1}{n^4}\right) \le 0.0001$
$\Rightarrow \frac{1}{n^4} \le (0.0001)\left(\frac{180}{768}\right) \Rightarrow n^4 \ge 10{,}000\left(\frac{768}{180}\right) \Rightarrow n \ge 14.37 \Rightarrow n \ge 16$ (n must be even)

46. $|E_T| \le \frac{1-0}{12}(\Delta x)^2 M$ where $\Delta x = \frac{1-0}{n} = \frac{1}{n}$; $0 \le f''(x) \le 8 \Rightarrow M = 8$. Then $|E_T| \le 10^{-3} \Rightarrow \frac{1}{12}\left(\frac{1}{n}\right)^2 (8) \le 10^{-3}$
$\Rightarrow \frac{2}{3n^2} \le 10^{-3} \Rightarrow \frac{3n^2}{2} \ge 1000 \Rightarrow n^2 \ge \frac{2000}{3} \Rightarrow n \ge 25.82 \Rightarrow n \ge 26$

47. $\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6} \Rightarrow \frac{\Delta x}{2} = \frac{\pi}{12}$;
$\sum_{i=0}^{6} mf(x_i) = 12 \Rightarrow T = \left(\frac{\pi}{12}\right)(12) = \pi$;

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	$\pi/6$	1/2	2	1
x_2	$\pi/3$	3/2	2	3
x_3	$\pi/2$	2	2	4
x_4	$2\pi/3$	3/2	2	3
x_5	$5\pi/6$	1/2	2	1
x_6	π	0	1	0

$\sum_{i=0}^{6} mf(x_i) = 18$ and $\frac{\Delta x}{3} = \frac{\pi}{18}$

$\Rightarrow S = \left(\frac{\pi}{18}\right)(18) = \pi.$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	$\pi/6$	1/2	4	2
x_2	$\pi/3$	3/2	2	3
x_3	$\pi/2$	2	4	8
x_4	$2\pi/3$	3/2	2	3
x_5	$5\pi/6$	1/2	4	2
x_6	π	0	1	0

48. $\left|f^{(4)}(x)\right| \le 3 \Rightarrow M = 3;$ $\Delta x = \frac{2-1}{n} = \frac{1}{n}$. Hence $|E_s| \le 10^{-5} \Rightarrow \left(\frac{2-1}{180}\right)\left(\frac{1}{n}\right)^4 (3) \le 10^{-5} \Rightarrow \frac{1}{60n^4} \le 10^{-5} \Rightarrow n^4 \ge \frac{10^5}{60}$ $\Rightarrow n \ge 6.38 \Rightarrow n \ge 8$ (n must be even)

49. $y_{av} = \frac{1}{365-0}\int_0^{365}\left[20\sin\left(\frac{2\pi}{365}(x-101)\right)-4\right]dx = \frac{1}{365}\left[-20\left(\frac{365}{2\pi}\cos\left(\frac{2\pi}{365}(x-101)\right)-4x\right)\right]_0^{365}$

$= \frac{1}{365}\left[\left(-20\left(\frac{365}{2\pi}\right)\cos\left[\frac{2\pi}{365}(365-101)\right]-4(365)\right)-\left(-20\left(\frac{365}{2\pi}\right)\cos\left[\frac{2\pi}{365}(0-101)\right]-4(0)\right)\right]$

$= -\frac{20}{2\pi}\cos\left(\frac{2\pi}{365}(264)\right)-4+\frac{20}{2\pi}\cos\left(\frac{2\pi}{365}(-101)\right) = -\frac{20}{2\pi}\left(\cos\left(\frac{2\pi}{365}(264)\right)-\cos\left(\frac{2\pi}{365}(-101)\right)\right)-4$

$\approx -\frac{20}{2\pi}(0.16705-0.16705)-4 = -4°\text{C}$

50. $av(C_v) = \frac{1}{675-20}\int_{20}^{675}\left[8.27+10^{-5}\left(26T-1.87T^2\right)\right]dT = \frac{1}{655}\left[8.27T+\frac{13}{10^5}T^2-\frac{0.62333}{10^5}T^3\right]_{20}^{675}$

$\approx \frac{1}{655}\left[(5582.25+59.23125-1917.03194)-(165.4+0.052-0.04987)\right] \approx 5.434;$

$8.27+10^{-5}\left(26T-1.87T^2\right) = 5.434 \Rightarrow 1.87T^2-26T-283,600 = 0 \Rightarrow T \approx \frac{26+\sqrt{676+4(1.87)(283,600)}}{2(1.87)} \approx 396.45°C$

51. (a) Each interval is 5 min $= \frac{1}{12}$ hour. $\frac{1}{24}[2.5+2(2.4)+2(2.3)+\ldots+2(2.4)+2.3] = \frac{29}{12} \approx 2.42$ liters

(b) (60 km/h) $\left(\frac{12}{29}\text{ hours/liter}\right) \approx 24.83$ km/liter

52. Using the Simpson's rule, $\Delta x = 5 \Rightarrow \frac{\Delta x}{3} = \frac{5}{3}$;

$\sum mf(x_i) = 401 \Rightarrow$ Area $\approx (401)\left(\frac{5}{3}\right) = 668.3 \text{ m}^2$;

The cost is Area $\cdot \left(\$21/\text{m}^2\right) \approx \left(668.3^2\right)\left(\$21/\text{m}^2\right)$

$= \$14{,}035.00 \Rightarrow$ the job cannot be done for $\$11{,}000$.

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	5	12	4	48
x_2	10	18	2	36
x_3	15	17	4	68
x_4	20	16.5	2	33
x_5	25	18	4	72
x_6	30	21	2	42
x_7	35	22	4	88
x_8	40	14	1	14

53. $\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{b\to 3^-} \int_0^b \frac{dx}{\sqrt{9-x^2}} = \lim_{b\to 3^-} \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^b = \lim_{b\to 3^-} \left[\sin^{-1}\left(\frac{b}{3}\right) - \sin^{-1}\left(\frac{0}{3}\right)\right] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

54. $\int_0^1 \ln x\, dx = \lim_{b\to 0^+} \left[x\ln x - x\right]_b^1 = \lim_{b\to 0^+} \left[(1\cdot \ln 1 - 1) - (b\ln b - b)\right] = -1 - \lim_{b\to 0^+} \frac{\ln b}{\left(\frac{1}{b}\right)} = -1 - \lim_{b\to 0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)} = -1 + 0 = -1$

55. $\int_{-1}^1 \frac{dy}{y^{2/3}} = \int_{-1}^0 \frac{dy}{y^{2/3}} + \int_0^1 \frac{dy}{y^{2/3}} = 2\int_0^1 \frac{dy}{y^{2/3}} = 2\cdot 3 \lim_{b\to 0^+} \left[y^{1/3}\right]_b^1 = 6 \lim_{b\to 0^+} \left[1 - b^{1/3}\right] = 6$

56. $\int_{-2}^\infty \frac{d\theta}{(\theta+1)^{3/5}} = \int_{-2}^{-1} \frac{d\theta}{(\theta+1)^{3/5}} + \int_{-1}^2 \frac{d\theta}{(\theta+1)^{3/5}} + \int_2^\infty \frac{d\theta}{(\theta+1)^{3/5}}$ converges if each integral converges, but $\lim_{\theta\to\infty} \frac{\theta^{3/5}}{(\theta+1)^{3/5}} = 1$ and $\int_2^\infty \frac{d\theta}{\theta^{3/5}}$ diverges $\Rightarrow \int_{-2}^\infty \frac{d\theta}{(\theta+1)^{3/5}}$ diverges

57. $\int_3^\infty \frac{2\,du}{u^2-2u} = \int_3^\infty \frac{du}{u-2} - \int_3^\infty \frac{du}{u} = \lim_{b\to\infty} \left[\ln\left|\frac{u-2}{u}\right|\right]_3^b = \lim_{b\to\infty} \left[\ln\left|\frac{b-2}{b}\right| - \ln\left|\frac{3-2}{3}\right|\right] = 0 - \ln\left(\frac{1}{3}\right) = \ln 3$

58. $\int_1^\infty \frac{3v-1}{4v^3-v^2}\, dv = \int_1^\infty \left(\frac{1}{v} + \frac{1}{v^2} - \frac{4}{4v-1}\right) dv = \lim_{b\to\infty} \left[\ln v - \frac{1}{v} - \ln(4v-1)\right]_1^b = \lim_{b\to\infty} \left[\ln\left(\frac{b}{4b-1}\right) - \frac{1}{b} - (\ln 1 - 1 - \ln 3)\right]$
$= \ln\frac{1}{4} + 1 + \ln 3 = 1 + \ln\frac{3}{4}$

59. $\int_0^\infty x^2 e^{-x}\, dx = \lim_{b\to\infty} \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^b = \lim_{b\to\infty} \left[\left(-b^2e^{-b} - 2be^{-b} - 2e^{-b}\right) - (-2)\right] = 0 + 2 = 2$

60. $\int_{-\infty}^0 xe^{3x}\, dx = \lim_{b\to -\infty} \left[\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right]_b^0 = \lim_{b\to -\infty} \left[-\frac{1}{9} - \left(\frac{b}{3}e^{3b} - \frac{1}{9}e^{3b}\right)\right] = -\frac{1}{9} - 0 = -\frac{1}{9}$

61. $\int_{-\infty}^{\infty} \frac{dx}{4x^2+9} = 2\int_0^{\infty} \frac{dx}{4x^2+9} = \frac{1}{2}\int_0^{\infty} \frac{dx}{x^2+\frac{9}{4}} = \frac{1}{2}\lim_{b\to\infty}\left[\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)\right]_0^b = \frac{1}{2}\lim_{b\to\infty}\left[\frac{2}{3}\tan^{-1}\left(\frac{2b}{3}\right) - \frac{2}{3}\tan^{-1}(0)\right]$
$= \frac{1}{2}\left[\left(\frac{2}{3}\cdot\frac{\pi}{2}\right) - 0\right] = \frac{\pi}{6}$

62. $\int_{-\infty}^{\infty} \frac{4dx}{x^2+16} = 2\int_0^{\infty} \frac{4dx}{x^2+16} = 2\lim_{b\to\infty}\left[\tan^{-1}\left(\frac{x}{4}\right)\right]_0^b = 2\lim_{b\to\infty}\left[\tan^{-1}\left(\frac{b}{4}\right) - \tan^{-1}(0)\right] = 2\left[\left(\frac{\pi}{2}\right) - 0\right] = \pi$

63. $\lim_{\theta\to\infty} \frac{\theta}{\sqrt{\theta^2+1}} = 1$ and $\int_6^{\infty} \frac{d\theta}{\theta}$ diverges $\Rightarrow \int_6^{\infty} \frac{d\theta}{\sqrt{\theta^2+1}}$ diverges

64. $I = \int_0^{\infty} e^{-u}\cos u\, du = \lim_{b\to\infty}\left[-e^{-u}\cos u\right]_0^b - \int_0^{\infty} e^{-u}\sin u\, du = 1 + \lim_{b\to\infty}\left[e^{-u}\sin u\right]_0^b - \int_0^{\infty}\left(e^{-u}\right)\cos u\, du$
$\Rightarrow I = 1 + 0 - I \Rightarrow 2I = 1 \Rightarrow I = \frac{1}{2}$ converges

65. $\int_1^{\infty} \frac{\ln z}{z}\, dz = \int_1^{e} \frac{\ln z}{z}\, dz + \int_e^{\infty} \frac{\ln z}{z}\, dz = \left[\frac{(\ln z)^2}{2}\right]_1^e + \lim_{b\to\infty}\left[\frac{(\ln z)^2}{2}\right]_e^b = \left(\frac{1^2}{2} - 0\right) + \lim_{b\to\infty}\left[\frac{(\ln b)^2}{2} - \frac{1}{2}\right] = \infty \Rightarrow$ diverges

66. $0 < \frac{e^{-t}}{\sqrt{t}} \le e^{-t}$ for $t \ge 1$ and $\int_1^{\infty} e^{-t}\, dt$ converges $\Rightarrow \int_1^{\infty} \frac{e^{-t}}{\sqrt{t}}\, dt$ converges

67. $\int_{-\infty}^{\infty} \frac{2\, dx}{e^x+e^{-x}} = 2\int_0^{\infty} \frac{2\, dx}{e^x+e^{-x}} < \int_0^{\infty} \frac{4\, dx}{e^x}$ converges $\Rightarrow \int_{-\infty}^{\infty} \frac{2dx}{e^x+e^{-x}}$ converges

68. $\int_{-\infty}^{\infty} \frac{dx}{x^2\left(1+e^x\right)} = \int_{-\infty}^{-1} \frac{dx}{x^2\left(1+e^x\right)} + \int_{-1}^{0} \frac{dx}{x^2\left(1+e^x\right)} + \int_0^1 \frac{dx}{x^2\left(1+e^x\right)} + \int_1^{\infty} \frac{dx}{x^2\left(1+e^x\right)}$;
$\lim_{x\to 0} \frac{\left(\frac{1}{x^2}\right)}{\left[\frac{1}{x^2\left(1+e^x\right)}\right]} = \lim_{x\to 0} \frac{x^2\left(1+e^x\right)}{x^2} = \lim_{x\to 0}\left(1+e^x\right) = 2$ and $\int_0^1 \frac{dx}{x^2}$ diverges $\Rightarrow \int_0^1 \frac{dx}{x^2\left(1+e^x\right)}$ diverges $\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2\left(1+e^x\right)}$
diverges

69. $\int \frac{x\, dx}{1+\sqrt{x}}$; $\begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int \frac{u^2 \cdot 2u\, du}{1+u} = \int\left(2u^2 - 2u + 2 - \frac{2}{1+u}\right) du = \frac{2}{3}u^3 - u^2 + 2u - 2\ln|1+u| + C$
$= \frac{2x^{3/2}}{3} - x + 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$

70. $\int \frac{x^3+2}{4-x^2}\, dx = -\int\left(x + \frac{4x+2}{x^2-4}\right) dx = -\int x\, dx - \frac{3}{2}\int \frac{dx}{x+2} - \frac{5}{2}\int \frac{dx}{x-2} = -\frac{x^2}{2} - \frac{3}{2}\ln|x+2| - \frac{5}{2}\ln|x-2| + C$

71. $\int \sqrt{2x - x^2}\, dx$; $x - 1 = \sin\theta \quad dx = \cos\theta\, d\theta$

For the integrand to be nonnegative x must be between 0 and 2 so $x-1$ is between -1 and 1 and we can take θ between $-\pi/2$ and $\pi/2$, where cosine is nonnegative. Thus

$$\sqrt{2x - x^2} = \sqrt{1-(x-1)^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta.$$

$$\int \sqrt{2x-x^2}\,dx = \int \cos\theta\cos\theta\,d\theta = \int \cos^2\theta\,d\theta = \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$$
$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$
$$= \frac{1}{2}\sin^{-1}(x-1) + \frac{1}{2}(x-1)\sqrt{2x-x^2} + C$$

72. $\int \frac{dx}{\sqrt{-2x-x^2}} = \int \frac{dx}{\sqrt{1-(x+1)^2}} = \sin^{-1}(x+1) + C$

73. $\int \frac{2-\cos x+\sin x}{\sin^2 x}\,dx = \int 2\csc^2 x\,dx - \int \frac{\cos x\,dx}{\sin^2 x} + \int \csc x\,dx = -2\cot x + \frac{1}{\sin x} - \ln|\csc x + \cot x| + C$
$= -2\cot x + \csc x - \ln|\csc x + \cot x| + C$

74. $\int \sin^2\theta\cos^5\theta\,d\theta$

$$\int \sin^2\theta\cos^5\theta\,d\theta = \int \sin^2\theta\left(1-\sin^2\theta\right)^2\cos\theta\,d\theta; \quad u = \sin\theta \quad du = \cos\theta\,d\theta$$

$$\int \sin^2\theta\left(1-\sin^2\theta\right)^2\cos\theta\,d\theta = \int\left(\sin^2\theta - 2\sin^4\theta + \sin^6\theta\right)\cos\theta\,d\theta$$
$$= \int\left(u^2 - 2u^4 + u^6\right)du = \frac{u^3}{3} - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$
$$= \frac{\sin^3\theta}{3} - \frac{2}{5}\sin^5\theta + \frac{1}{7}\sin^7\theta + C$$

75. $\int \frac{9\,dv}{81-v^4} = \frac{1}{2}\int \frac{dv}{v^2+9} + \frac{1}{12}\int \frac{dv}{3-v} + \frac{1}{12}\int \frac{dv}{3+v} = \frac{1}{12}\ln\left|\frac{3+v}{3-v}\right| + \frac{1}{6}\tan^{-1}\frac{v}{3} + C$

76. $\int_2^\infty \frac{dx}{(x-1)^2} = \lim_{b\to\infty}\left[\frac{1}{1-x}\right]_2^b = \lim_{b\to\infty}\left[\frac{1}{1-b} - (-1)\right] = 0 + 1 = 1$

77.
$$\begin{array}{lll} & \cos(2\theta+1) & \\ \theta \xrightarrow{(+)} & \frac{1}{2}\sin(2\theta+1) & \\ 1 \xrightarrow{(-)} & -\frac{1}{4}\cos(2\theta+1) & \\ 0 & & \Rightarrow \int \theta\cos(2\theta+1)\,d\theta = \frac{\theta}{2}\sin(2\theta+1) + \frac{1}{4}\cos(2\theta+1) + C \end{array}$$

78. $\int \frac{x^3\,dx}{x^2-2x+1} = \int\left(x+2+\frac{3x-2}{x^2-2x+1}\right)dx = \int (x+2)\,dx + 3\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} = \frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C$

79. $\int \frac{\sin 2\theta\,d\theta}{(1+\cos 2\theta)^2} = -\frac{1}{2}\int \frac{(-2\sin 2\theta)\,d\theta}{(1+\cos 2\theta)^2} = \frac{1}{2(1+\cos 2\theta)} + C = \frac{1}{4}\sec^2\theta + C$

80. $\int_{\pi/4}^{\pi/2} \sqrt{1+\cos 4x}\,dx = -\sqrt{2}\int_{\pi/4}^{\pi/2} \cos 2x\,dx = \left[-\frac{\sqrt{2}}{2}\sin 2x\right]_{\pi/4}^{\pi/2} = \frac{\sqrt{2}}{2}$

81. $\int \frac{x\,dx}{\sqrt{2-x}};\ \begin{bmatrix} y=2-x \\ dy=-dx \end{bmatrix} \to -\int \frac{(2-y)\,dy}{\sqrt{y}} = \frac{2}{3}y^{3/2} - 4y^{1/2} + C = \frac{2}{3}(2-x)^{3/2} - 4(2-x)^{1/2} + C$

$= 2\left[\frac{\left(\sqrt{2-x}\right)^3}{3} - 2\sqrt{2-x}\right] + C$

82. $\int \frac{\sqrt{1-v^2}}{v^2}\,dv;\quad [v = \sin\theta] \to \int \frac{\cos\theta \cdot \cos\theta\,d\theta}{\sin^2\theta} = \int \frac{\left(1-\sin^2\theta\right)d\theta}{\sin^2\theta} = \int \csc^2\theta\,d\theta - \int d\theta = \cot\theta - \theta + C$

$= -\sin^{-1} v - \frac{\sqrt{1-v^2}}{v} + C$

83. $\int \frac{dy}{y^2-2y+2} = \int \frac{dy}{(y-1)^2+1} = \tan^{-1}(y-1) + C$

84. $\int \frac{x\,dx}{\sqrt{8-2x^2-x^4}} = \frac{1}{2}\int \frac{(2x)\,dx}{\sqrt{9-\left(x^2+1\right)^2}} = \frac{1}{2}\sin^{-1}\left(\frac{x^2+1}{3}\right) + C$

85. $\int \frac{z+1}{z^2\left(z^2+4\right)}\,dz = \frac{1}{4.}\int\left(\frac{1}{z} + \frac{1}{z^2} - \frac{z+1}{z^2+4}\right)dz = \frac{1}{4}\ln|z| - \frac{1}{4z} - \frac{1}{8}\ln\left(z^2+4\right) - \frac{1}{8}\tan^{-1}\frac{z}{2} + C$

86. $\int x^2(x-1)^{1/3}\,dx;\quad u = x-1 \quad du = dx \quad x^2 = (u+1)^2$

$$\begin{aligned}\int x^2(x-1)^{1/3}\,dx &= \int\left(u^2+2u+1\right)\cdot u^{1/3}du \\ &= \int\left(u^{7/3} + 2u^{4/3} + u^{1/3}\right)du \\ &= \frac{3}{10}u^{10/3} + \frac{6}{7}u^{7/3} + \frac{3}{4}u^{4/3} + C \\ &= \frac{3}{10}(x-1)^{10/3} + \frac{6}{7}(x-1)^{7/3} + \frac{3}{4}(x-1)^{4/3} + C\end{aligned}$$

87. $\int \frac{t\,dt}{\sqrt{9-4t^2}} = -\frac{1}{8}\int \frac{(-8t)dt}{\sqrt{9-4t^2}} = -\frac{1}{4}\sqrt{9-4t^2} + C$

88. $u = \tan^{-1} x,\ du = \frac{dx}{1+x^2};\quad dv = \frac{dx}{x^2}, v = -\frac{1}{x};\ \int \frac{\tan^{-1} x\,dx}{x^2} = -\frac{1}{x}\tan^{-1} x + \int \frac{dx}{x\left(1+x^2\right)} = -\frac{1}{x}\tan^{-1} x + \int \frac{dx}{x} - \int \frac{x\,dx}{1+x^2}$

$= -\frac{1}{x}\tan^{-1} x + \ln|x| - \frac{1}{2}\ln\left(1+x^2\right) + C = -\frac{\tan^{-1} x}{x} + \ln|x| - \ln\sqrt{1+x^2} + C$

89. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2};\ [e^t = x] \to \int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \ln|x+1| - \ln|x+2| + C = \ln\left|\frac{x+1}{x+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

90. $\int \tan^3 t\,dt = \int (\tan t)\left(\sec^2 t - 1\right)dt = \frac{\tan^2 t}{2} - \int \tan t\,dt = \frac{\tan^2 t}{2} - \ln|\sec t| + C$

91. $\int_1^{\infty} \frac{\ln y\, dy}{y^3}$; $\begin{bmatrix} x = \ln y \\ dx = \frac{dy}{y} \\ dy = e^x dx \end{bmatrix} \to \int_0^{\infty} \frac{x \cdot e^x}{e^{3x}}\, dx = \int_0^{\infty} xe^{-2x}\, dx = \lim\limits_{b\to\infty} \left[-\frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} \right]_0^b$

$= \lim\limits_{b\to\infty} \left[\left(\frac{-b}{2e^{2b}} - \frac{1}{4e^{2b}} \right) - \left(0 - \frac{1}{4} \right) \right] = \frac{1}{4}$

92. $\int y^{3/2} (\ln y)^2\, dy$; $u = (\ln y)^2,\ du = \frac{2\ln y}{y} dy,\ dv = y^{3/2} dy,\ v = \frac{2}{5} y^{5/2}$

$\int y^{3/2} (\ln y)^2\, dy = \frac{2}{5} y^{5/2} (\ln y)^2 - \frac{4}{5} \int y^{3/2} \ln y\, dy$

Now we compute $\int y^{3/2} \ln y\, dy$: $u = \ln y,\ du = \frac{1}{y} dy,\ dv = y^{3/2} dy,\ v = \frac{2}{5} y^{5/2}$

$$\begin{aligned} \int y^{3/2} \ln y\, dy &= \frac{2}{5} y^{5/2} \ln y - \frac{2}{5} \int y^{3/2}\, dy \\ &= \frac{2}{5} y^{5/2} \ln y - \frac{4}{5} y^{5/2} \end{aligned}$$

$$\begin{aligned} \int y^{3/2} (\ln y)^2\, dy &= \frac{2}{5} y^{5/2} (\ln y)^2 - \frac{4}{5} \int y^{3/2} \ln y\, dy \\ &= \frac{2}{5} y^{5/2} (\ln y)^2 - \frac{4}{5} \left(\frac{2}{5} y^{5/2} \ln y - \frac{4}{5} y^{5/2} \right) \\ &= y^{5/2} \left(\frac{2}{5} (\ln y)^2 - \frac{8}{25} \ln y + \frac{16}{125} \right) + C \end{aligned}$$

93. $\int e^{\ln \sqrt{x}} dx = \int \sqrt{x}\, dx = \frac{2}{3} x^{3/2} + C$

94. $\int e^{\theta} \sqrt{3 + 4e^{\theta}}\, d\theta$; $\begin{vmatrix} u = 4e^{\theta} \\ du = 4e^{\theta}\, d\theta \end{vmatrix} \to \frac{1}{4} \int \sqrt{3+u}\, du = \frac{1}{4} \cdot \frac{2}{3} (3+u)^{3/2} + C = \frac{1}{6} \left(3 + 4e^{\theta} \right)^{3/2} + C$

95. $\int \frac{\sin 5t\, dt}{1 + (\cos 5t)^2}$; $\begin{bmatrix} u = \cos 5t \\ du = -5 \sin 5t\, dt \end{bmatrix} \to -\frac{1}{5} \int \frac{du}{1+u^2} = -\frac{1}{5} \tan^{-1} u + C = -\frac{1}{5} \tan^{-1} (\cos 5t) + C$

96. $\int \frac{dv}{\sqrt{e^{2v} - 1}}$; $\begin{bmatrix} x = e^v \\ dx = e^v\, dv \end{bmatrix} \to \int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x + C = \sec^{-1} \left(e^v \right) + C$

97. $\int \frac{dr}{1 + \sqrt{r}}$; $\begin{bmatrix} u = \sqrt{r} \\ du = \frac{dr}{2\sqrt{r}} \end{bmatrix} \to \int \frac{2u\, du}{1+u} = \int \left(2 - \frac{2}{1+u} \right) du = 2u - 2\ln|1+u| + C = 2\sqrt{r} - 2\ln\left(1 + \sqrt{r} \right) + C$

98. $\int \frac{4x^3 - 20x}{x^4 - 10x^2 + 9}\, dx = \int \frac{\left(4x^3 - 20x \right) dx}{x^4 - 10x^2 + 9} = \ln\left| x^4 - 10x^2 + 9 \right| + C$

99. $\int \frac{x^3}{1+x^2}\,dx = \int\left(x - \frac{x}{1+x^2}\right)dx = \int x\,dx - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln\left(1+x^2\right) + C$

100. $\int \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int \frac{3x^2}{1+x^3}\,dx = \frac{1}{3}\ln\left|1+x^3\right| + C$

101. $\int \frac{1+x^2}{1+x^3}\,dx$; $\frac{1+x^2}{1+x^3} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2} \Rightarrow 1+x^2 = A\left(1-x+x^2\right) + (Bx+C)(1+x)$

$= (A+B)x^2 + (-A+B+C)x + (A+C) \Rightarrow A+B = 1, -A+B+C = 0, A+C = 1 \Rightarrow A = \frac{2}{3}, B = \frac{1}{3}, C = \frac{1}{3}$;

$\int \frac{1+x^2}{1+x^3}\,dx = \int\left(\frac{2/3}{1+x} + \frac{(1/3)x+1/3}{1-x+x^2}\right)dx = \frac{2}{3}\int \frac{1}{1+x}\,dx + \frac{1}{3}\int \frac{x+1}{1-x+x^2}\,dx = \frac{2}{3}\int \frac{1}{1+x}\,dx + \frac{1}{3}\int \frac{x+1}{\frac{3}{4}+\left(x-\frac{1}{2}\right)^2}\,dx$;

$\begin{bmatrix} u = x - \frac{1}{2} \\ du = dx \end{bmatrix} \to \frac{1}{3}\int \frac{u+\frac{3}{2}}{\frac{3}{4}+u^2}\,du = \frac{1}{3}\int \frac{u}{\frac{3}{4}+u^2}\,du + \frac{1}{2}\int \frac{1}{\frac{3}{4}+u^2}\,du = \frac{1}{6}\ln\left|\frac{3}{4}+u^2\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}/2}\right)$

$= \frac{1}{6}\ln\left|\frac{3}{4}+\left(x-\frac{1}{2}\right)^2\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right) = \frac{1}{6}\ln\left|1-x+x^2\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$

$\Rightarrow \frac{2}{3}\int \frac{1}{1+x}\,dx + \frac{1}{3}\int \frac{x+1}{1-x+x^2}\,dx = \frac{2}{3}\ln|1+x| + \frac{1}{6}\ln\left|1-x+x^2\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

102. $\int \frac{1+x^2}{(1+x)^3}\,dx$; $\begin{bmatrix} u = 1+x \\ du = dx \end{bmatrix} \to \int \frac{1+(u-1)^2}{u^3}\,du = \int \frac{u^2-2u+2}{u^3}\,du = \int \frac{1}{u}\,du - \int \frac{2}{u^2}\,du + \int \frac{2}{u^3}\,du = \ln|u| + \frac{2}{u} - \frac{1}{u^2} + C$

$= \ln|1+x| + \frac{2}{1+x} - \frac{1}{(1+x)^2} + C$

103. $\int \sqrt{x}\sqrt{1+\sqrt{x}}\,dx$; $\begin{bmatrix} w = \sqrt{x} \Rightarrow w^2 = x \\ 2w\,dw = dx \end{bmatrix} \to \int 2w^2\sqrt{1+w}\,dw$

		$\sqrt{1+w}$
$2w^2$	$\xrightarrow{(+)}$	$\frac{2}{3}(1+w)^{3/2}$
$4w$	$\xrightarrow{(-)}$	$\frac{4}{15}(1+w)^{5/2}$
4	$\xrightarrow{(+)}$	$\frac{8}{105}(1+w)^{7/2}$
0		

$\Rightarrow \int 2w^2\sqrt{1+w}\,dw = \frac{4}{3}w^2(1+w)^{3/2} - \frac{16}{15}w(1+w)^{5/2} + \frac{32}{105}(1+w)^{7/2} + C$

$= \frac{4}{3}x\left(1+\sqrt{x}\right)^{3/2} - \frac{16}{15}\sqrt{x}\left(1+\sqrt{x}\right)^{5/2} + \frac{32}{105}\left(1+\sqrt{x}\right)^{7/2} + C$

104. $\int \sqrt{1+\sqrt{1+x}}\,dx$; $\begin{bmatrix} w = \sqrt{1+x} \Rightarrow w^2 = 1+x \\ 2w\,dw = dx \end{bmatrix} \to \int 2w\sqrt{1+w}\,dw$;

$\left[u = 2w, du = 2dw, dv = \sqrt{1+w}\,dw, v = \frac{2}{3}(1+w)^{3/2}\right]$

$\int 2w\sqrt{1+w}\,dw = \frac{4}{3}w(1+w)^{3/2} - \int \frac{4}{3}(1+w)^{3/2}\,dw = \frac{4}{3}w(1+w)^{3/2} - \frac{8}{15}(1+w)^{5/2} + C$

$= \frac{4}{3}\sqrt{1+x}\left(1+\sqrt{1+x}\right)^{3/2} - \frac{8}{15}\left(1+\sqrt{1+x}\right)^{5/2} + C$

105. $\int \frac{1}{\sqrt{x}\sqrt{1+x}}\,dx$; $\begin{bmatrix} u=\sqrt{x} \Rightarrow u^2 = x \\ 2u\,du = dx \end{bmatrix} \to \int \frac{2}{\sqrt{1+u^2}}\,du$; $\left[u=\tan\theta, -\frac{\pi}{2}<\theta<\frac{\pi}{2}, du = \sec^2\theta\,d\theta, \sqrt{1+u^2} = \sec\theta\right]$

$\int \frac{2}{\sqrt{1+u^2}}\,du = \int \frac{2\sec^2\theta}{\sec\theta}\,d\theta = \int 2\sec\theta\,d\theta = 2\ln|\sec\theta+\tan\theta| + C = 2\ln\left|\sqrt{1+u^2}+u\right| + C = 2\ln\left|\sqrt{1+x}+\sqrt{x}\right| + C$

106. $\int_0^{1/2} \sqrt{1+\sqrt{1-x^2}}\,dx$;

$\left[x=\sin\theta, -\frac{\pi}{2}<\theta<\frac{\pi}{2}, dx = \cos\theta\,d\theta, \sqrt{1-x^2} = \cos\theta, x=0=\sin\theta \Rightarrow \theta = 0, x=\frac{1}{2}=\sin\theta \Rightarrow \theta = \frac{\pi}{6}\right]$

$\to \int_0^{\pi/6} \sqrt{1+\cos\theta}\cos\theta\,d\theta = \int_0^{\pi/6} \frac{\sqrt{1-\cos^2\theta}}{\sqrt{1-\cos\theta}}\cos\theta\,d\theta = \int_0^{\pi/6} \frac{\sin\theta\cos\theta}{\sqrt{1-\cos\theta}}\,d\theta = \lim_{c\to 0^+} \int_c^{\pi/6} \frac{\sin\theta\cos\theta}{\sqrt{1-\cos\theta}}\,d\theta$;

$\left[u=\cos\theta, du = -\sin\theta\,d\theta, dv = \frac{\sin\theta}{\sqrt{1-\cos\theta}}\,d\theta, v = 2(1-\cos\theta)^{1/2}\right]$

$= \lim_{c\to 0^+}\left[\left[2\cos\theta(1-\cos\theta)^{1/2}\right]_c^{\pi/6} + \int_c^{\pi/6} 2(1-\cos\theta)^{1/2}\sin\theta\,d\theta\right]$

$= \lim_{c\to 0^+}\left[\left(2\cos\left(\frac{\pi}{6}\right)\left(1-\cos\left(\frac{\pi}{6}\right)\right)^{1/2} - 2\cos c(1-\cos c)^{1/2}\right) + \left[\frac{4}{3}(1-\cos\theta)^{3/2}\right]_c^{\pi/6}\right]$

$= \lim_{c\to 0^+}\left[\sqrt{3}\left(1-\frac{\sqrt{3}}{2}\right)^{1/2} - 2\cos c(1-\cos c)^{1/2} + \left(\frac{4}{3}\left(1-\cos\left(\frac{\pi}{6}\right)\right)^{3/2} - \frac{4}{3}(1-\cos c)^{3/2}\right)\right]$

$= \lim_{c\to 0^+}\left[\sqrt{3}\left(1-\frac{\sqrt{3}}{2}\right)^{1/2} - 2\cos c(1-\cos c)^{1/2} + \frac{4}{3}\left(1-\frac{\sqrt{3}}{2}\right)^{3/2} - \frac{4}{3}(1-\cos c)^{3/2}\right]$

$= \sqrt{3}\left(1-\frac{\sqrt{3}}{2}\right)^{1/2} + \frac{4}{3}\left(1-\frac{\sqrt{3}}{2}\right)^{3/2} = \left(1-\frac{\sqrt{3}}{2}\right)^{1/2}\left(\frac{4+\sqrt{3}}{3}\right) = \frac{(4+\sqrt{3})\sqrt{2-\sqrt{3}}}{3\sqrt{2}}$

107. $\int \frac{\ln x}{x+x\ln x}\,dx = \int \frac{\ln x}{x(1+\ln x)}\,dx$; $\begin{bmatrix} u = 1+\ln x \\ du = \frac{1}{x}\,dx \end{bmatrix} \to \int \frac{u-1}{u}\,du = \int du - \int \frac{1}{u}\,du = u - \ln|u| + C$

$= (1+\ln x) - \ln|1+\ln x| + C = \ln x - \ln|1+\ln x| + C$

108. $\int \frac{1}{x\ln x\cdot\ln(\ln x)}\,dx$; $\begin{bmatrix} u = \ln(\ln x) \\ du = \frac{1}{x\ln x}\,dx \end{bmatrix} \to \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln(\ln x)| + C$

109. $\int \frac{x^{\ln x}\ln x}{x}\,dx$; $\left[u = x^{\ln x} \Rightarrow \ln u = \ln x^{\ln x} = (\ln x)^2 \Rightarrow \frac{1}{u}\,du = \frac{2\ln x}{x}\,dx \Rightarrow du = \frac{2u\ln x}{x}\,dx = \frac{2x^{\ln x}\ln x}{x}\,dx\right]$

$\to \frac{1}{2}\int du = \frac{1}{2}u + C = \frac{1}{2}x^{\ln x} + C$

110. $\int (\ln x)^{\ln x}\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]dx$; $\left[u = (\ln x)^{\ln x} \Rightarrow \ln u = \ln(\ln x)^{\ln x} = (\ln x)\ln(\ln x) \Rightarrow \frac{1}{u}\,du = \left(\frac{(\ln x)}{x\ln x} + \frac{\ln(\ln x)}{x}\right)dx\right.$

$\left.\Rightarrow du = u\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]dx = (\ln x)^{\ln x}\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]dx\right] \to \int du = u + C = (\ln x)^{\ln x} + C$

111. $\int \frac{1}{x\sqrt{1-x^4}}\,dx = \int \frac{x}{x^2\sqrt{1-x^4}}\,dx;\ \left[x^2 = \sin\theta, 0 \le \theta < \frac{\pi}{2}, 2x\,dx = \cos\theta\,d\theta, \sqrt{1-x^4} = \cos\theta\right]$

$\to \frac{1}{2}\int \frac{\cos\theta}{\sin\theta\cos\theta}\,d\theta = \frac{1}{2}\int \csc\theta\,d\theta = -\frac{1}{2}\ln\left|\csc\theta + \cot\theta\right| + C = -\frac{1}{2}\ln\left|\frac{1}{x^2} + \frac{\sqrt{1-x^4}}{x^2}\right| + C = -\frac{1}{2}\ln\left|\frac{1+\sqrt{1-x^4}}{x^2}\right| + C$

112. $\int \frac{\sqrt{1-x}}{x}\,dx;\ \left[u = \sqrt{1-x} \Rightarrow u^2 = 1-x \Rightarrow 2u\,du = -dx\right] \to \int \frac{-2u^2}{1-u^2}\,du = \int \frac{2u^2}{u^2-1}\,du = \int\left(2 + \frac{2}{u^2-1}\right)du;$

$\frac{2}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1} \Rightarrow 2 = A(u+1) + B(u-1) = (A+B)u + A - B \Rightarrow A + B = 0, A - B = 2 \Rightarrow A = 1 \Rightarrow B = -1;$

$\int\left(2 + \frac{2}{u^2-1}\right)du = \int 2\,du + \int\left(\frac{1}{u-1} - \frac{1}{u+1}\right)du = 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1-x} + \frac{1}{2}\ln\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right| + C$

113. (a) $\int_0^a f(a-x)\,dx;\ \left[u = a - x \Rightarrow du = -dx, x = 0 \Rightarrow u = a, x = a \Rightarrow u = 0\right] \to -\int_a^0 f(u)\,du = \int_0^a f(u)\,du,$ which is the same integral as $\int_0^a f(x)\,dx$.

(b) $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)}\,dx = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x}{\sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x + \cos\left(\frac{\pi}{2}\right)\cos x + \sin\left(\frac{\pi}{2}\right)\sin x}\,dx$

$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx \Rightarrow 2\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx$

$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow 2\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{2} \Rightarrow \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{4}$

114. $\int \frac{\sin x}{\sin x + \cos x}\,dx = \int \frac{\sin x + \cos x - \cos x + \sin x - \sin x}{\sin x + \cos x}\,dx = \int \frac{\sin x + \cos x}{\sin x + \cos x}\,dx + \int \frac{-\cos x + \sin x}{\sin x + \cos x}\,dx + \int \frac{-\sin x}{\sin x + \cos x}\,dx$

$= \int dx - \int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx - \int \frac{\sin x}{\sin x + \cos x}\,dx = x - \ln\left|\sin x + \cos x\right| - \int \frac{\sin x}{\sin x + \cos x}\,dx$

$\Rightarrow 2\int \frac{\sin x}{\sin x + \cos x}\,dx = x - \ln\left|\sin x + \cos x\right| \Rightarrow \int \frac{\sin x}{\sin x + \cos x}\,dx = \frac{x}{2} - \frac{1}{2}\ln\left|\sin x + \cos x\right| + C$

115. $\int \frac{\sin^2 x}{1+\sin^2 x}\,dx = \int \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}}\,dx = \int \frac{\tan^2 x}{\sec^2 x + \tan^2 x}\,dx = \int \frac{\tan^2 x + \sec^2 x - \sec^2 x}{\sec^2 x + \tan^2 x}\,dx = \int \frac{\tan^2 x + \sec^2 x}{\sec^2 x + \tan^2 x}\,dx - \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\,dx$

$= \int dx - \int \frac{\sec^2 x}{1 + 2\tan^2 x}\,dx = x - \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right) + C$

116. $\int \frac{1-\cos x}{1+\cos x}\,dx = \int \frac{(1-\cos x)^2}{1-\cos^2 x}\,dx = \int \frac{1 - 2\cos x + \cos^2 x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x}\,dx - \int \frac{2\cos x}{\sin^2 x}\,dx + \int \frac{\cos^2 x}{\sin^2 x}\,dx$

$= \int \csc^2 x\,dx - 2\int \csc x \cot x\,dx + \int \cot^2 x\,dx = -\cot x + 2\csc x + \int\left(\csc^2 x - 1\right)dx = = -2\cot x + 2\csc x - x + C$

CHAPTER 8 ADDITIONAL AND ADVANCED EXERCISES

1. $u = \left(\sin^{-1} x\right)^2, du = \frac{2\sin^{-1} x\,dx}{\sqrt{1-x^2}};\ dv = dx, v = x;$

$\int\left(\sin^{-1} x\right)^2 dx = x\left(\sin^{-1} x\right)^2 - \int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}};\ \ u = \sin^{-1} x, du = \frac{dx}{\sqrt{1-x^2}}; dv = -\frac{2x\,dx}{\sqrt{1-x^2}}, v = 2\sqrt{1-x^2};$

$-\int \frac{2x\sin^{-1}x\,dx}{\sqrt{1-x^2}} = 2\left(\sin^{-1}x\right)\sqrt{1-x^2} - \int 2\,dx = 2\left(\sin^{-1}x\right)\sqrt{1-x^2} - 2x + C$; therefore

$\int \left(\sin^{-1}x\right)^2 dx = x\left(\sin^{-1}x\right)^2 + 2\left(\sin^{-1}x\right)\sqrt{1-x^2} - 2x + C$

2. $\frac{1}{x} = \frac{1}{x}$,

$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$,

$\frac{1}{x(x+1)(x+2)} = \frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)}$,

$\frac{1}{x(x+1)(x+2)(x+3)} = \frac{1}{6x} - \frac{1}{2(x+1)} + \frac{1}{2(x+2)} - \frac{1}{6(x+3)}$,

$\frac{1}{x(x+1)(x+2)(x+3)(x+4)} = \frac{1}{24x} - \frac{1}{6(x+1)} + \frac{1}{4(x+2)} - \frac{1}{6(x+3)} + \frac{1}{24(x+4)}$

$\Rightarrow$ the following pattern: $\frac{1}{x(x+1)(x+2)\cdots(x+m)} = \sum_{k=0}^{m} \frac{(-1)^k}{(k!)(m-k)!(x+k)}$;

therefore $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)} = \sum_{k=0}^{m}\left[\frac{(-1)^k}{(k!)(m-k)!}\ln|x+k|\right] + C$

3. $u = \sin^{-1}x,\ du = \frac{dx}{\sqrt{1-x^2}};\ dv = x\,dx,\ v = \frac{x^2}{2};\ \int x\sin^{-1}x\,dx = \frac{x^2}{2}\sin^{-1}x - \int \frac{x^2dx}{2\sqrt{1-x^2}}$;

$\begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{bmatrix} \to \int x\sin^{-1}x\,dx = \frac{x^2}{2}\sin^{-1}x - \int \frac{\sin^2\theta\cos\theta d\theta}{2\cos\theta} = \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \sin^2\theta\,d\theta$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C = \frac{x^2}{2}\sin^{-1}x + \frac{\sin\theta\cos\theta - \theta}{4} + C = \frac{x^2}{2}\sin^{-1}x + \frac{x\sqrt{1-x^2} - \sin^{-1}x}{4} + C$

4. $\int \sin^{-1}\sqrt{y}\,dy;\ \begin{bmatrix} z = \sqrt{y} \\ dz = \frac{dy}{2\sqrt{y}} \end{bmatrix} \to \int 2z\sin^{-1}z\,dz$; from Exercise 3, $\int z\sin^{-1}z\,dz = \frac{z^2\sin^{-1}z}{2} + \frac{z\sqrt{1-z^2} - \sin^{-1}z}{4} + C$

$\Rightarrow \int \sin^{-1}\sqrt{y}\,dy = y\sin^{-1}\sqrt{y} + \frac{\sqrt{y}\sqrt{1-y} - \sin^{-1}\sqrt{y}}{2} + C = y\sin^{-1}\sqrt{y} + \frac{\sqrt{y-y^2}}{2} - \frac{\sin^{-1}\sqrt{y}}{2} + C$

5. $\int \frac{dt}{t-\sqrt{1-t^2}};\ \begin{bmatrix} t = \sin\theta \\ dt = \cos\theta\,d\theta \end{bmatrix} \to \int \frac{\cos\theta\,d\theta}{\sin\theta - \cos\theta} = \int \frac{d\theta}{\tan\theta - 1};\ \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\,d\theta \\ d\theta = \frac{du}{u^2+1} \end{bmatrix} \to \int \frac{du}{(u-1)\left(u^2+1\right)}$

$= \frac{1}{2}\int \frac{du}{u-1} - \frac{1}{2}\int \frac{du}{u^2+1} - \frac{1}{2}\int \frac{u\,du}{u^2+1} = \frac{1}{2}\ln\left|\frac{u-1}{\sqrt{u^2+1}}\right| - \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\ln\left|\frac{\tan\theta - 1}{\sec\theta}\right| - \frac{1}{2}\theta + C$

$= \frac{1}{2}\ln\left(t - \sqrt{1-t^2}\right) - \frac{1}{2}\sin^{-1}t + C$

6. $\int \frac{1}{x^4+4}\,dx = \int \frac{1}{\left(x^2+2\right)^2 - 4x^2}\,dx = \int \frac{1}{\left(x^2+2x+2\right)\left(x^2-2x+2\right)}\,dx = \frac{1}{16}\int\left[\frac{2x+2}{x^2+2x+2} + \frac{2}{(x+1)^2+1} - \frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1}\right]dx$

$= \frac{1}{16}\ln\left|\frac{x^2+2x+2}{x^2-2x+2}\right| + \frac{1}{8}\left[\tan^{-1}(x+1) + \tan^{-1}(x-1)\right] + C$

7. $\lim\limits_{x\to\infty}\int_{-x}^{x}\sin t\,dt = \lim\limits_{x\to\infty}\left[-\cos t\right]_{-x}^{x} = \lim\limits_{x\to\infty}\left[-\cos x+\cos(-x)\right] = \lim\limits_{x\to\infty}\left(-\cos x+\cos x\right) = \lim\limits_{x\to\infty} 0 = 0$

8. $\lim\limits_{x\to 0^+}\int_x^1 \frac{\cos t}{t^2}\,dt$; $\lim\limits_{t\to 0^+}\frac{\left(\frac{1}{t^2}\right)}{\left(\frac{\cos t}{t^2}\right)} = \lim\limits_{t\to 0^+}\frac{1}{\cos t} = 1 \Rightarrow \lim\limits_{x\to 0^+}\int_x^1 \frac{\cos t}{t^2}\,dt$ diverges since $\int_0^1 \frac{dt}{t^2}$ diverges; thus

$\lim\limits_{x\to 0^+} x\int_x^1 \frac{\cos t}{t^2}\,dt$ is an indeterminate $0\cdot\infty$ form and we apply l′Hôpital′s rule:

$\lim\limits_{x\to 0^+} x\int_x^1 \frac{\cos t}{t^2}\,dt = \lim\limits_{x\to 0^+}\frac{-\int_1^x \frac{\cos t}{t^2}\,dt}{\frac{1}{x}} = \lim\limits_{x\to 0^+}\frac{-\left(\frac{\cos x}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim\limits_{x\to 0^+}\cos x = 1$

9. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\ln\sqrt[n]{1+\frac{k}{n}} = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\ln\left(1+k\left(\frac{1}{n}\right)\right)\left(\frac{1}{n}\right) = \int_0^1 \ln(1+x)\,dx$; $\begin{bmatrix} u=1+x,\ du=dx \\ x=0\Rightarrow u=1,\ x=1\Rightarrow u=2 \end{bmatrix}$

$\to \int_1^2 \ln u\,du = \left[u\ln u - u\right]_1^2 = \left(2\ln 2-2\right)-\left(\ln 1-1\right) = 2\ln 2-1 = \ln 4-1$

10. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n-1}\frac{1}{\sqrt{n^2-k^2}} = \lim\limits_{n\to\infty}\sum\limits_{k=0}^{n-1}\left(\frac{n}{\sqrt{n^2-k^2}}\right)\left(\frac{1}{n}\right) = \lim\limits_{n\to\infty}\sum\limits_{k=0}^{n-1}\left(\frac{1}{\sqrt{1-\left[k\left(\frac{1}{n}\right)\right]^2}}\right)\left(\frac{1}{n}\right) = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1}x\right]_0^1 = \frac{\pi}{2}$

11. $\frac{dy}{dx} = \sqrt{\cos 2x} \Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\cos 2x = 2\cos^2 x$; $L = \int_0^{\pi/4}\sqrt{1+\left(\sqrt{\cos 2t}\right)^2}\,dt = \sqrt{2}\int_0^{\pi/4}\sqrt{\cos^2 t}\,dt$

$= \sqrt{2}\left[\sin t\right]_0^{\pi/4} = 1$

12. $\frac{dy}{dx} = \frac{-2x}{1-x^2} \Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = \frac{\left(1-x^2\right)^2+4x^2}{\left(1-x^2\right)^2} = \frac{1+2x^2+x^4}{\left(1-x^2\right)^2} = \left(\frac{1+x^2}{1-x^2}\right)^2$; $L = \int_0^{1/2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{1/2}\left(\frac{1+x^2}{1-x^2}\right)dx$

$= \int_0^{1/2}\left(-1+\frac{2}{1-x^2}\right)dx = \int_0^{1/2}\left(-1+\frac{1}{1+x}+\frac{1}{1-x}\right)dx = \left[-x+\ln\left|\frac{1+x}{1-x}\right|\right]_0^{1/2} = \left(-\frac{1}{2}+\ln 3\right)-(0+\ln 1) = \ln 3-\frac{1}{2}$

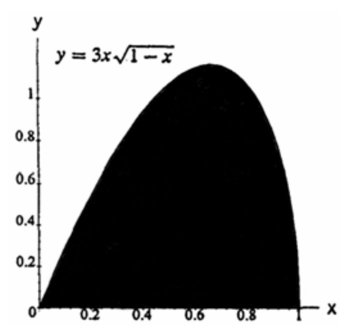
13. $V = \int_a^b 2\pi\left(\text{shell radius}\right)\left(\text{shell height}\right)dx = \int_0^1 2\pi xy\,dx$

$= 6\pi\int_0^1 x^2\sqrt{1-x}\,dx$; $\left[u=1-x,\ du=-dx,\ x^2=(1-u)^2\right]$

$\to -6\pi\int_1^0 (1-u)^2\sqrt{u}\,du = 6\pi\int_0^1\left(u^{1/2}-2u^{3/2}+u^{5/2}\right)du$

$= 6\pi\left[\frac{2}{3}u^{3/2}-\frac{4}{5}u^{5/2}+\frac{2}{7}u^{7/2}\right]_0^1$

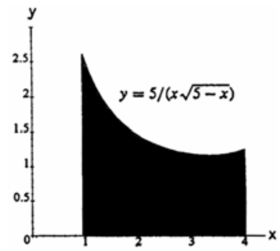
$= 6\pi\left(\frac{2}{3}-\frac{4}{5}+\frac{2}{7}\right) = 6\pi\left(\frac{70-84+30}{105}\right) = 6\pi\left(\frac{16}{105}\right) = \frac{32\pi}{35}$



14. $V=\int_a^b \pi y^2\,dx=\pi\int_1^4 \frac{25\,dx}{x^2(5-x)}=\pi\int_1^4\left(\frac{1}{x}+\frac{5}{x^2}+\frac{1}{5-x}\right)dx$

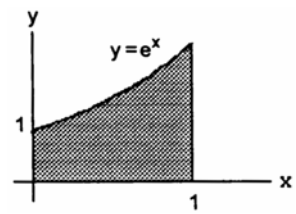
$=\pi\left[\ln\left|\frac{x}{5-x}\right|-\frac{5}{x}\right]_1^4=\pi\left(\ln 4-\frac{5}{4}\right)-\pi\left(\ln\frac{1}{4}-5\right)$

$=\frac{15\pi}{4}+2\pi\ln 4$



15. $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx$

$=\int_0^1 2\pi x e^x\,dx=2\pi\left[xe^x-e^x\right]_0^1=2\pi$



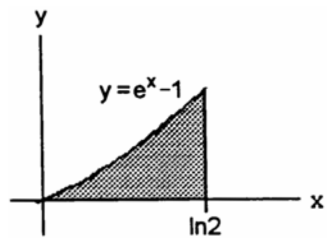
16. $V=\int_0^{\ln 2} 2\pi(\ln 2-x)\left(e^x-1\right)dx$

$=2\pi\int_0^{\ln 2}\left[(\ln 2)e^x-\ln 2-xe^x+x\right]dx$

$=2\pi\left[(\ln 2)e^x-(\ln 2)x-xe^x+e^x+\frac{x^2}{2}\right]_0^{\ln 2}$

$=2\pi\left[2\ln 2-(\ln 2)^2-2\ln 2+2+\frac{(\ln 2)^2}{2}\right]-2\pi(\ln 2+1)$

$=2\pi\left[-\frac{(\ln 2)^2}{2}-\ln 2+1\right]$



17. (a) $V=\int_1^e \pi\left[1-(\ln x)^2\right]dx$

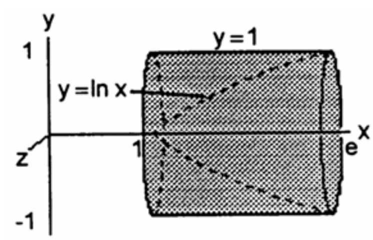
$=\pi\left[x-x(\ln x)^2\right]_1^e+2\pi\int_1^e \ln x\,dx$

(FORMULA 110)

$=\pi\left[x-x(\ln x)^2+2(x\ln x-x)\right]_1^e$

$=\pi\left[-x-x(\ln x)^2+2x\ln x\right]_1^e$

$=\pi[-e-e+2e-(-1)]=\pi$



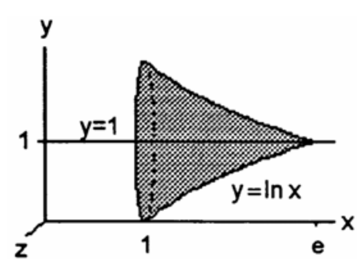
(b) $V=\int_1^e \pi(1-\ln x)^2\,dx=\pi\int_1^e\left[1-2\ln x+(\ln x)^2\right]dx$

$=\pi\left[x-2(x\ln x-x)+x(\ln x)^2\right]_1^e-2\pi\int_1^e \ln x\,dx$

$=\pi\left[x-2(x\ln x-x)+x(\ln x)^2-2(x\ln x-x)\right]_1^e$

$=\pi\left[5x-4x\ln x+x(\ln x)^2\right]_1^e$

$=\pi[(5e-4e+e)-(5)]=\pi(2e-5)$



18. (a) $V=\pi\int_0^1\left[\left(e^y\right)^2-1\right]dy=\pi\int_0^1\left(e^{2y}-1\right)dy=\pi\left[\frac{e^{2y}}{2}-y\right]_0^1=\pi\left[\frac{e^2}{2}-1-\left(\frac{1}{2}\right)\right]=\frac{\pi\left(e^2-3\right)}{2}$

(b) $V=\pi\int_0^1\left(e^y-1\right)^2dy=\pi\int_0^1\left(e^{2y}-2e^y+1\right)dy=\pi\left[\frac{e^{2y}}{2}-2e^y+y\right]_0^1=\pi\left[\left(\frac{e^2}{2}-2e+1\right)-\left(\frac{1}{2}-2\right)\right]$

$=\pi\left(\frac{e^2}{2}-2e+\frac{5}{2}\right)=\frac{\pi\left(e^2-4e+5\right)}{2}$

19. (a) $\lim\limits_{x\to0^+}x\ln x=0\Rightarrow\lim\limits_{x\to0^+}f(x)=0=f(0)\Rightarrow f$ is continuous

(b) $V=\int_0^2\pi x^2(\ln x)^2\,dx;\ \left[u=(\ln x)^2,\,du=(2\ln x)\frac{dx}{x};\,dv=x^2dx,v=\frac{x^3}{3}\right]$

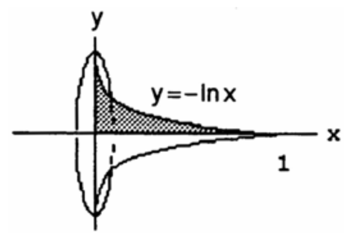
$\to\pi\left(\lim\limits_{b\to0^+}\left[\frac{x^3}{3}(\ln x)^2\right]_b^2-\int_0^2\left(\frac{x^3}{3}\right)(2\ln x)\frac{dx}{x}\right)=\pi\left[\left(\frac{8}{3}\right)(\ln 2)^2-\left(\frac{2}{3}\right)\lim\limits_{b\to0^+}\left[\frac{x^3}{3}\ln x-\frac{x^3}{9}\right]_b^2\right]$

$=\pi\left[\frac{8(\ln 2)^2}{3}-\frac{16(\ln 2)}{9}+\frac{16}{27}\right]$

20. $V=\int_0^1\pi(-\ln x)^2\,dx=\pi\int_0^1(\ln x)^2\,dx$

$=\pi\left(\lim\limits_{b\to0^+}\left[x(\ln x)^2\right]_b^1-2\int_0^1\ln x\,dx\right)$

$=-2\pi\lim\limits_{b\to0^+}\left[x\ln x-x\right]_b^1=2\pi$



21. $M=\int_1^e\ln x\,dx=\left[x\ln x-x\right]_1^e=(e-e)-(0-1)=1;$

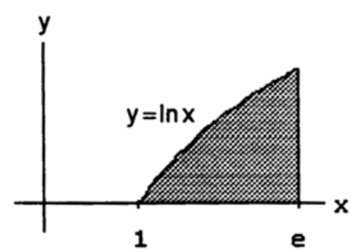
$M_x=\int_1^e(\ln x)\left(\frac{\ln x}{2}\right)dx=\frac{1}{2}\int_1^e(\ln x)^2\,dx$

$=\frac{1}{2}\left(\left[x(\ln x)^2\right]_1^e-2\int_1^e\ln x\,dx\right)=\frac{1}{2}(e-2);$

$M_y=\int_1^e x\ln x\,dx=\left[\frac{x^2\ln x}{2}\right]_1^e-\frac{1}{2}\int_1^e x\,dx$

$=\frac{1}{2}\left[x^2\ln x-\frac{x^2}{2}\right]_1^e=\frac{1}{2}\left[\left(e^2-\frac{e^2}{2}\right)+\frac{1}{2}\right]=\frac{1}{4}\left(e^2+1\right);$

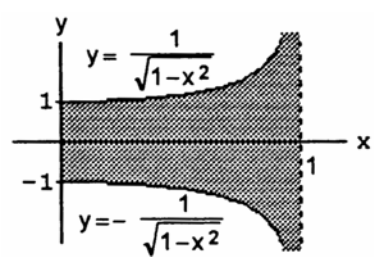
therefore, $\bar{x}=\frac{M_y}{M}=\frac{e^2+1}{4}$ and $\bar{y}=\frac{M_x}{M}=\frac{e-2}{2}$



22. $M=\int_0^1\frac{2dx}{\sqrt{1-x^2}}=\left[2\sin^{-1}x\right]_0^1=\pi;$

$M_y=\int_0^1\frac{2x\,dx}{\sqrt{1-x^2}}=2\left[-\sqrt{1-x^2}\right]_0^1=2;$ therefore,

$\bar{x}=\frac{M_y}{M}=\frac{2}{\pi}$ and $\bar{y}=0$ by symmetry



23. $L=\int_1^e \sqrt{1+\frac{1}{x^2}}\,dx=\int_1^e \frac{\sqrt{x^2+1}}{x}\,dx;\ \begin{bmatrix} x=\tan\theta \\ dx=\sec^2\theta\,d\theta\end{bmatrix}\to L=\int_{\pi/4}^{\tan^{-1}e}\frac{\sec\theta\cdot\sec^2\theta\,d\theta}{\tan\theta}=\int_{\pi/4}^{\tan^{-1}e}\frac{(\sec\theta)\left(\tan^2\theta+1\right)}{\tan\theta}\,d\theta$

$=\int_{\pi/4}^{\tan^{-1}e}(\tan\theta\sec\theta+\csc\theta)\,d\theta=\left[\sec\theta-\ln|\csc\theta+\cot\theta|\right]_{\pi/4}^{\tan^{-1}e}$

$=\left(\sqrt{1+e^2}-\ln\left|\frac{\sqrt{1+e^2}}{e}+\frac{1}{e}\right|\right)-\left[\sqrt{2}-\ln\left(1+\sqrt{2}\right)\right]=\sqrt{1+e^2}-\ln\left(\frac{\sqrt{1+e^2}}{e}+\frac{1}{e}\right)-\sqrt{2}+\ln\left(1+\sqrt{2}\right)$

24. $y=\ln x\Rightarrow 1+\left(\frac{dx}{dy}\right)^2=1+x^2\Rightarrow S=2\pi\int_c^d x\sqrt{1+x^2}\,dy\Rightarrow S=2\pi\int_0^1 e^y\sqrt{1+e^{2y}}\,dy;$

$\begin{bmatrix} u=e^y \\ du=e^y dy\end{bmatrix}\to S=2\pi\int_1^e\sqrt{1+u^2}\,du;\ \begin{bmatrix} u=\tan\theta \\ du=\sec^2\theta\,d\theta\end{bmatrix}\to 2\pi\int_{\pi/4}^{\tan^{-1}e}\sec\theta\cdot\sec^2\theta\,d\theta$

$=2\pi\left(\frac{1}{2}\right)\left[\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right]_{\pi/4}^{\tan^{-1}e}=\pi\left[\left(\sqrt{1+e^2}\right)e+\ln\left|\sqrt{1+e^2}+e\right|\right]-\pi\left[\sqrt{2}\cdot 1+\ln\left(\sqrt{2}+1\right)\right]$

$=\pi\left[e\sqrt{1+e^2}+\ln\left(\frac{\sqrt{1+e^2}+e}{\sqrt{2}+1}\right)-\sqrt{2}\right]$

25. $S=2\pi\int_{-1}^1 f(x)\sqrt{1+[f'(x)]^2}\,dx;\ f(x)=\left(1-x^{2/3}\right)^{3/2}\Rightarrow[f'(x)]^2+1=\frac{1}{x^{2/3}}\Rightarrow S=2\pi\int_{-1}^1\left(1-x^{2/3}\right)^{3/2}\cdot\frac{dx}{\sqrt{x^{2/3}}}$

$=4\pi\int_0^1\left(1-x^{2/3}\right)^{3/2}\left(\frac{1}{x^{1/3}}\right)dx;\ \begin{bmatrix} u=x^{2/3} \\ du=\frac{2}{3}\frac{dx}{x^{1/3}}\end{bmatrix}\to 4\cdot\frac{3}{2}\pi\int_0^1(1-u)^{3/2}\,du=-6\pi\int_0^1(1-u)^{3/2}(-1)du$

$=-6\pi\cdot\frac{2}{5}\left[(1-u)^{5/2}\right]_0^1=\frac{12\pi}{5}$

26. $y=\int_1^x\sqrt{\sqrt{t}-1}\,dt\Rightarrow\frac{dy}{dx}=\sqrt{\sqrt{x}-1}\Rightarrow L=\int_1^{16}\sqrt{1+\left(\sqrt{\sqrt{x}-1}\right)^2}\,dx=\int_1^{16}\sqrt{1+\sqrt{x}-1}\,dx=\int_1^{16}\sqrt[4]{x}\,dx=\left[\frac{4}{5}x^{5/4}\right]_1^{16}$

$=\frac{4}{5}(16)^{5/4}-\frac{4}{5}(1)^{5/4}=\frac{124}{5}$

27. $\int_1^\infty\left(\frac{ax}{x^2+1}-\frac{1}{2x}\right)dx=\lim_{b\to\infty}\int_1^b\left(\frac{ax}{x^2+1}-\frac{1}{2x}\right)dx=\lim_{b\to\infty}\left[\frac{a}{2}\ln\left(x^2+1\right)-\frac{1}{2}\ln x\right]_1^b=\lim_{b\to\infty}\left[\frac{1}{2}\ln\frac{\left(x^2+1\right)^a}{x}\right]_1^b$

$=\lim_{b\to\infty}\frac{1}{2}\left[\ln\frac{\left(b^2+1\right)^a}{b}-\ln 2^a\right];\ \lim_{b\to\infty}\frac{\left(b^2+1\right)^a}{b}>\lim_{b\to\infty}\frac{b^{2a}}{b}=\lim_{b\to\infty}b^{2\left(a-\frac{1}{2}\right)}=\infty$ if $a>\frac{1}{2}\Rightarrow$ the improper integral diverges if $a>\frac{1}{2}$; for $a=\frac{1}{2}$: $\lim_{b\to\infty}\frac{\sqrt{b^2+1}}{b}=\lim_{b\to\infty}\sqrt{1+\frac{1}{b^2}}=1\Rightarrow\lim_{b\to\infty}\frac{1}{2}\left[\ln\frac{\left(b^2+1\right)^{1/2}}{b}-\ln 2^{1/2}\right]$

$=\frac{1}{2}\left(\ln 1-\frac{1}{2}\ln 2\right)=-\frac{\ln 2}{4}$; if $a<\frac{1}{2}$: $0\le\lim_{b\to\infty}\frac{\left(b^2+1\right)^a}{b}<\lim_{b\to\infty}\frac{(b+1)^{2a}}{b+1}=\lim_{b\to\infty}(b+1)^{2a-1}=0$

$\Rightarrow\lim_{b\to\infty}\ln\frac{\left(b^2+1\right)^a}{b}=-\infty\Rightarrow$ the improper integral diverges if $a<\frac{1}{2}$; in summary, the improper integral $\int_1^\infty\left(\frac{ax}{x^2+1}-\frac{1}{2x}\right)dx$ converges only when $a=\frac{1}{2}$ and has the value $-\frac{\ln 2}{4}$

28. $G(x) = \lim_{b\to\infty} \int_0^b e^{-xt}\, dt = \lim_{b\to\infty} \left[-\frac{1}{x}e^{-xt}\right]_0^b = \lim_{b\to\infty} \left(\frac{1-e^{-xb}}{x}\right) = \frac{1-0}{x} = \frac{1}{x}$ if $x > 0 \Rightarrow xG(x) = x\left(\frac{1}{x}\right) = 1$ if $x > 0$

29. $A = \int_1^\infty \frac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \le 1$. Thus, $p \le 1$ for infinite area. The volume of the solid of revolution about the x-axis is $V = \int_1^\infty \pi\left(\frac{1}{x^p}\right)^2 dx = \pi\int_1^\infty \frac{dx}{x^{2p}}$ which converges if $2p > 1$ and diverges if $2p \le 1$. Thus we want $p > \frac{1}{2}$ for finite volume. In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $\frac{1}{2} < p \le 1$.

30. The area is given by the integral $A = \int_0^1 \frac{dx}{x^p}$;

$p = 1: A = \lim_{b\to 0^+} [\ln x]_b^1 = -\lim_{b\to 0^+} \ln b = \infty$, diverges;

$p > 1: A = \lim_{b\to 0^+} \left[x^{1-p}\right]_b^1 = 1 - \lim_{b\to 0^+} b^{1-p} = -\infty$, diverges;

$p < 1: A = \lim_{b\to 0^+} \left[x^{1-p}\right]_b^1 = 1 - \lim_{b\to 0^+} b^{1-p} = 1 - 0$, converges; thus, $p \ge 1$ for infinite area.

The volume of the solid of revolution about the x-axis is $V_x = \pi\int_0^1 \frac{dx}{x^{2p}}$ which converges if $2p < 1$ or $p < \frac{1}{2}$, and diverges if $p \ge \frac{1}{2}$. Thus, V_x is infinite whenever the area is infinite $(p \ge 1)$. The volume of the solid of revolution about the y-axis is $V_y = \pi\int_1^\infty [R(y)]^2\, dy = \pi\int_1^\infty \frac{dy}{y^{2/p}}$ which converges if $\frac{2}{p} > 1 \Leftrightarrow p < 2$ (see Exercise 29). In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $1 \le p < 2$, as described above.

31. See the generalization proved in 32.

32. $0 \le \int_0^a \left(f'(x) + x - \frac{a}{2}\right) dx$

$$= \int_0^a \left(f'(x)\right)^2 dx + \int_0^a (2x-a) f'(x)\, dx + \int_0^a \left(x - \frac{a}{2}\right)^2 dx$$

The last integral is $\left.\frac{1}{3}\left(x - \frac{a}{2}\right)^3\right]_0^a = \frac{a^3}{12}$.

Using integration by parts with $u = 2x - a$, $du = 2dx$, $dv = f'(x)$, $v = f(x)$, and the fact that $f(a) = f(0) = b$, the second integral is $(2x-a)f(x)]_0^a - 2\int_0^a f(x)\,dx = 2ab - 2\int_0^a f(x)\,dx$. Thus

$$\int_0^a \left(f'(x)\right)^2 dx \ge 2\int_0^a f(x)\,dx - \left(2ab + \frac{a^3}{12}\right).$$

33.
$$\begin{array}{lcl} e^{2x} & (+) & \cos 3x \\ 2e^{2x} & (-) & \frac{1}{3}\sin 3x \\ 4e^{2x} & (+) & -\frac{1}{9}\cos 3x \end{array}$$

$I = \frac{e^{2x}}{3}\sin 3x + \frac{2e^{2x}}{9}\cos 3x - \frac{4}{9}I \Rightarrow \frac{13}{9}I = \frac{e^{2x}}{9}\left(3\sin 3x + 2\cos 3x\right) \Rightarrow I = \frac{e^{2x}}{13}\left(3\sin 3x + 2\cos 3x\right) + C$

34.
$$\begin{array}{lcl} e^{3x} & (+) & \sin 4x \\ 3e^{3x} & (-) & -\frac{1}{4}\cos 4x \\ 9e^{3x} & (+) & -\frac{1}{16}\sin 4x \end{array}$$

$I = -\frac{e^{3x}}{4}\cos 4x + \frac{3e^{3x}}{16}\sin 4x - \frac{9}{16}I \Rightarrow \frac{25}{16}I = \frac{e^{3x}}{16}\left(3\sin 4x - 4\cos 4x\right) \Rightarrow I = \frac{e^{3x}}{25}\left(3\sin 4x - 4\cos 4x\right) + C$

35.
$$\begin{array}{lcl} \sin 3x & (+) & \sin x \\ 3\cos 3x & (-) & -\cos x \\ -9\sin 3x & (+) & -\sin x \end{array}$$

$I = -\sin 3x\cos x + 3\cos 3x\sin x + 9I \Rightarrow -8I = -\sin 3x\cos x + 3\cos 3x\sin x \Rightarrow I = \frac{\sin 3x\cos x - 3\cos 3x\sin x}{8} + C$

36.
$$\begin{array}{lcl} \cos 5x & (+) & \sin 4x \\ -5\sin 5x & (-) & -\frac{1}{4}\cos 4x \\ -25\cos 5x & (+) & -\frac{1}{16}\sin 4x \end{array}$$

$I = -\frac{1}{4}\cos 5x\cos 4x - \frac{5}{16}\sin 5x\sin 4x + \frac{25}{16}I \Rightarrow -\frac{9}{16}I = -\frac{1}{4}\cos 5x\cos 4x - \frac{5}{16}\sin 5x\sin 4x$

$\Rightarrow I = \frac{1}{9}\left(4\cos 5x\cos 4x + 5\sin 5x\sin 4x\right) + C$

37.
$$\begin{array}{lcl} e^{ax} & (+) & \sin bx \\ ae^{ax} & (-) & -\frac{1}{b}\cos bx \\ a^2e^{ax} & (+) & -\frac{1}{b^2}\sin bx \end{array}$$

$I = -\frac{e^{ax}}{b}\cos bx + \frac{ae^{ax}}{b^2}\sin bx - \frac{a^2}{b^2}I \Rightarrow \left(\frac{a^2+b^2}{b^2}\right)I = \frac{e^{ax}}{b^2}\left(a\sin bx - b\cos bx\right) \Rightarrow I = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right) + C$

38.
$$\begin{array}{lcl} e^{ax} & (+) & \cos bx \\ ae^{ax} & (-) & \frac{1}{b}\sin bx \\ a^2e^{ax} & (+) & -\frac{1}{b^2}\cos bx \end{array}$$

$I = \frac{e^{ax}}{b}\sin bx + \frac{ae^{ax}}{b^2}\cos bx - \frac{a^2}{b^2} I \Rightarrow \left(\frac{a^2+b^2}{b^2}\right) I = \frac{e^{ax}}{b^2}\left(a\cos bx + b\sin bx\right) \Rightarrow I = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right) + C$

39. $\ln(ax) \quad (+) \quad 1$

$\frac{1}{x} \xrightarrow{\quad (-) \quad} x$

$I = x\ln(ax) - \int\left(\frac{1}{x}\right) x\, dx = x\ln(ax) - x + C$

40. $\ln(ax) \quad (+) \quad x^2$

$\frac{1}{x} \xrightarrow{\quad (-) \quad} \frac{1}{3}x^3$

$I = \frac{1}{3}x^3\ln(ax) - \int\left(\frac{1}{x}\right)\left(\frac{x^3}{3}\right) dx = \frac{1}{3}x^3\ln(ax) - \frac{1}{9}x^3 + C$

41. $\int \frac{dx}{1-\sin x} = \int \frac{\left(\frac{2dz}{1+z^2}\right)}{1-\left(\frac{2z}{1+z^2}\right)} = \int \frac{2\,dz}{(1-z)^2} = \frac{2}{1-z} + C = \frac{2}{1-\tan\left(\frac{x}{2}\right)} + C$

42. $\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\left(\frac{2dz}{1+z^2}\right)}{1+\left(\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{1+z^2+2z+1-z^2} = \int \frac{dz}{1+z} = \ln|1+z| + C = \ln\left|\tan\left(\frac{x}{2}\right)+1\right| + C$

43. $\int_0^{\pi/2} \frac{dx}{1+\sin x} = \int_0^1 \frac{\left(\frac{2dz}{1+z^2}\right)}{1+\left(\frac{2z}{1+z^2}\right)} = \int_0^1 \frac{2\,dz}{(1+z)^2} = -\left[\frac{2}{1+z}\right]_0^1 = -(1-2) = 1$

44. $\int_{\pi/3}^{\pi/2} \frac{dx}{1-\cos x} = \int_{1/\sqrt{3}}^1 \frac{\left(\frac{2dz}{1+z^2}\right)}{1-\left(\frac{1-z^2}{1+z^2}\right)} = \int_{1/\sqrt{3}}^1 \frac{dz}{z^2} = \left[-\frac{1}{z}\right]_{1/\sqrt{3}}^1 = \sqrt{3}-1$

45. $\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta} = \int_0^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{2+\left(\frac{1-z^2}{1+z^2}\right)} = \int_0^1 \frac{2\,dz}{2+2z^2+1-z^2} = \int_0^1 \frac{2\,dz}{z^2+3} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{z}{\sqrt{3}}\right]_0^1 = \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}} = \frac{\sqrt{3}\pi}{9}$

46. $\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\, d\theta}{\sin\theta\cos\theta+\sin\theta} = \int_1^{\sqrt{3}} \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\,dz}{1+z^2}\right)}{\left[\frac{2z\left(1-z^2\right)}{\left(1+z^2\right)^2}+\left(\frac{2z}{1+z^2}\right)\right]} = \int_1^{\sqrt{3}} \frac{2\left(1-z^2\right)dz}{2z-2z^3+2z+2z^3} = \int_1^{\sqrt{3}} \frac{1-z^2}{2z}\, dz = \left[\frac{1}{2}\ln z - \frac{z^2}{4}\right]_1^{\sqrt{3}}$

$= \left(\frac{1}{2}\ln\sqrt{3} - \frac{3}{4}\right) - \left(0 - \frac{1}{4}\right) = \frac{\ln 3}{4} - \frac{1}{2} = \frac{1}{4}(\ln 3 - 2) = \frac{1}{2}\left(\ln\sqrt{3} - 1\right)$

47. $\int \frac{dt}{\sin t-\cos t} = \int \frac{\left(\frac{2dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2}-\frac{1-z^2}{1+z^2}\right)} = \int \frac{2dz}{2z-1+z^2} = \int \frac{2dz}{(z+1)^2-2} = \frac{1}{\sqrt{2}}\ln\left|\frac{z+1-\sqrt{2}}{z+1+\sqrt{2}}\right| + C = \frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{t}{2}\right)+1-\sqrt{2}}{\tan\left(\frac{t}{2}\right)+1+\sqrt{2}}\right| + C$

48. $\int \frac{\cos t\, dt}{1-\cos t} = \int \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2dz}{1+z^2}\right)}{1-\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\left(1-z^2\right)dz}{\left(1+z^2\right)^2-\left(1+z^2\right)\left(1-z^2\right)} = \int \frac{2\left(1-z^2\right)dz}{\left(1+z^2\right)\left(1+z^2-1+z^2\right)} = \int \frac{\left(1-z^2\right)dz}{\left(1+z^2\right)z^2} = \int \frac{dz}{z^2\left(1+z^2\right)} - \int \frac{dz}{1+z^2}$

$= \int \frac{dz}{z^2} - 2\int \frac{dz}{z^2+1} = -\frac{1}{z} - 2\tan^{-1} z + C = -\cot\left(\frac{t}{2}\right) - t + C$

49. $\int \sec\theta\, d\theta = \int \frac{d\theta}{\cos\theta} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{1-z^2} = \int \frac{2dz}{(1+z)(1-z)} = \int \frac{dz}{1+z} + \int \frac{dz}{1-z} = \ln|1+z| - \ln|1-z| + C = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$

50. $\int \csc\theta\, d\theta = \int \frac{d\theta}{\sin\theta} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2}\right)} = \int \frac{dz}{z} = \ln|z| + C = \ln\left|\tan\frac{\theta}{2}\right| + C$

51. (a) $\Gamma(1) = \int_0^\infty e^{-t}dt = \lim\limits_{b\to\infty}\int_0^b e^{-t}dt = \lim\limits_{b\to\infty}\left[-e^{-t}\right]_0^b = \lim\limits_{b\to\infty}\left[-\frac{1}{e^b} - (-1)\right] = 0 + 1 = 1$

(b) $u = t^x,\ du = xt^{x-1}dt;\ dv = e^{-t}\ dt,\ v = -e^{-t};\ x =$ fixed positive real

$\Rightarrow \Gamma(x+1) = \int_0^\infty t^x e^{-t}\ dt = \lim\limits_{b\to\infty}\left[-t^x e^{-t}\right]_0^b + x\int_0^\infty t^{x-1}e^{-t}\ dt = \lim\limits_{b\to\infty}\left(-\frac{b^x}{e^b} + 0^x e^0\right) + x\Gamma(x) = x\Gamma(x)$

(c) $\Gamma(n+1) = n\Gamma(n) = n!$:

$n = 0 : \Gamma(0+1) = \Gamma(1) = 0!$;

$n = k$: Assume $\Gamma(k+1) = k!$ for some $k > 0$;

$n = k+1 : \Gamma(k+1+1) = (k+1)\Gamma(k+1)$ from part (b)

$= (k+1)k!$ induction hypothesis

$= (k+1)!$ definition of factorial

Thus, $\Gamma(n+1) = n\Gamma(n) = n!$ for every positive integer n.

52. (a) $\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}$ and $n\Gamma(n) = n! \Rightarrow n! \approx n\left(\frac{n}{e}\right)^n \sqrt{\frac{2\pi}{n}} = \left(\frac{n}{e}\right)^n \sqrt{2n\pi}$

(b)

n	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}$	calculator
10	3598695.619	3628800
20	2.4227868×10^{18}	2.432902×10^{18}
30	2.6451710×10^{32}	2.652528×10^{32}
40	8.1421726×10^{47}	8.1591528×10^{47}
50	3.0363446×10^{64}	3.0414093×10^{64}
60	8.3094383×10^{81}	8.3209871×10^{81}

(c)

n	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}$	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}e^{1/12n}$	calculator
10	3598695.619	3628810.051	3628800