

CHAPTER 15 MULTIPLE INTEGRALS

15.1 DOUBLE AND ITERATED INTEGRALS OVER RECTANGLES

1. $\int_1^2 \int_0^4 2xy \, dy \, dx = \int_1^2 [xy^2]_0^4 \, dx = \int_1^2 16x \, dx = [8x^2]_1^2 = 24$
2. $\int_0^2 \int_{-1}^1 (x-y) \, dy \, dx = \int_0^2 [xy - \frac{1}{2}y^2]_{-1}^1 \, dx = \int_0^2 2x \, dx = [x^2]_0^2 = 4$
3. $\int_{-1}^0 \int_{-1}^1 (x+y+1) \, dx \, dy = \int_{-1}^0 [\frac{x^2}{2} + yx + x]_{-1}^1 \, dy = \int_{-1}^0 (2y+2) \, dy = [y^2 + 2y]_{-1}^0 = 1$
4. $\int_0^1 \int_0^1 (1 - \frac{x^2+y^2}{2}) \, dx \, dy = \int_0^1 [x - \frac{x^3}{6} - \frac{xy^2}{2}]_0^1 \, dy = \int_0^1 (\frac{5}{6} - \frac{y^2}{2}) \, dy = [\frac{5}{6}y - \frac{y^3}{6}]_0^1 = \frac{2}{3}$
5. $\int_0^3 \int_0^2 (4-y^2) \, dy \, dx = \int_0^3 [4y - \frac{y^3}{3}]_0^2 \, dx = \int_0^3 \frac{16}{3} \, dx = [\frac{16}{3}x]_0^3 = 16$
6. $\int_0^3 \int_{-2}^0 (x^2y - 2xy) \, dy \, dx = \int_0^3 [\frac{x^2y^2}{2} - xy^2]_{-2}^0 \, dx = \int_0^3 (4x - 2x^2) \, dx = [2x^2 - \frac{2x^3}{3}]_0^3 = 0$
7. $\int_0^1 \int_0^1 \frac{y}{1+xy} \, dx \, dy = \int_0^1 [\ln|1+xy|]_0^1 \, dy = \int_0^1 \ln|1+y| \, dy = [y \ln|1+y| - y + \ln|1+y|]_0^1 = 2 \ln 2 - 1$
8. $\int_1^4 \int_0^4 (\frac{x}{2} + \sqrt{y}) \, dx \, dy = \int_1^4 [\frac{1}{4}x^2 + x\sqrt{y}]_0^4 \, dy = \int_1^4 (4 + 4y^{1/2}) \, dy = [4y + \frac{8}{3}y^{3/2}]_1^4 = \frac{92}{3}$
9. $\int_0^{\ln 2} \int_1^5 e^{2x+y} \, dy \, dx = \int_0^{\ln 2} [e^{2x+y}]_1^5 \, dx = \int_0^{\ln 2} (5e^{2x} - e^{2x+1}) \, dx = [\frac{5}{2}e^{2x} - \frac{1}{2}e^{2x+1}]_0^{\ln 2} = \frac{3}{2}(5-e)$
10. $\int_0^1 \int_1^2 x \, y \, e^x \, dy \, dx = \int_0^1 [\frac{1}{2}x \, y^2 e^x]_1^2 \, dx = \int_0^1 \frac{3}{2}x \, e^x \, dx = [\frac{3}{2}x \, e^x - \frac{3}{2}e^x]_0^1 = \frac{3}{2}$
11. $\int_{-1}^2 \int_0^{\pi/2} y \sin x \, dx \, dy = \int_{-1}^2 [-y \cos x]_0^{\pi/2} \, dy = \int_{-1}^2 y \, dy = [\frac{1}{2}y^2]_{-1}^2 = \frac{3}{2}$
12. $\int_{\pi}^{2\pi} \int_0^{\pi} (\sin x + \cos y) \, dx \, dy = \int_{\pi}^{2\pi} [-\cos x + x \cos y]_0^{\pi} \, dy = \int_{\pi}^{2\pi} (2 + \pi \cos y) \, dy = [2y + \pi \sin y]_{\pi}^{2\pi} = 2\pi$
13. $\int_1^4 \int_1^e \frac{\ln x}{xy} \, dx \, dy = \int_1^4 \left(\frac{(\ln x)^2}{2y} \right)_{x=1}^{x=e} \, dy = \int_1^4 \frac{1}{2y} \, dy = \left[\frac{\ln y}{2} \right]_1^4 = \ln 2$

14. $\int_{-1}^2 \int_1^2 x \ln y \, dy \, dx = \int_{-1}^2 \left(x(y \ln y - y) \right) \Big|_{y=1}^{y=2} dx = \int_{-1}^2 (2 \ln 2 - 1)x \, dx = (2 \ln 2 - 1) \frac{x^2}{2} \Big|_{-1}^2 = 3 \ln 2 - \frac{3}{2}$
15. $\iint_R (6y^2 - 2x) \, dA = \int_0^1 \int_0^2 (6y^2 - 2x) \, dy \, dx = \int_0^1 [2y^3 - 2xy]_0^2 dx = \int_0^1 (16 - 4x) \, dx = [16x - 2x^2]_0^1 = 14$
16. $\iint_R \frac{\sqrt{x}}{y^2} \, dA = \int_0^4 \int_1^2 \frac{\sqrt{x}}{y^2} \, dy \, dx = \int_0^4 \left[-\frac{\sqrt{x}}{y} \right]_1^2 dx = \int_0^4 \frac{1}{2} x^{1/2} \, dx = \left[\frac{1}{3} x^{3/2} \right]_0^4 = \frac{8}{3}$
17. $\iint_R xy \cos y \, dA = \int_{-1}^1 \int_0^\pi xy \cos y \, dy \, dx = \int_{-1}^1 [xy \sin y + x \cos y]_0^\pi dx = \int_{-1}^1 (-2x) \, dx = [-x^2]_{-1}^1 = 0$
18. $\iint_R y \sin(x+y) \, dA = \int_{-\pi}^0 \int_0^\pi y \sin(x+y) \, dy \, dx = \int_{-\pi}^0 [-y \cos(x+y) + \sin(x+y)]_0^\pi dx$
 $= \int_{-\pi}^0 (\sin(x+\pi) - \pi \cos(x+\pi) - \sin x) \, dx = [-\cos(x+\pi) - \pi \sin(x+\pi) + \cos x]_{-\pi}^0 = 4$
19. $\iint_R e^{x-y} \, dA = \int_0^{\ln 2} \int_0^{\ln 2} e^{x-y} \, dy \, dx = \int_0^{\ln 2} [-e^{x-y}]_0^{\ln 2} dx = \int_0^{\ln 2} (-e^{x-\ln 2} + e^x) \, dx = [-e^{x-\ln 2} + e^x]_0^{\ln 2} = \frac{1}{2}$
20. $\iint_R x y e^{x y^2} \, dA = \int_0^2 \int_0^1 x y e^{x y^2} \, dy \, dx = \int_0^2 \left[\frac{1}{2} e^{x y^2} \right]_0^1 dx = \int_0^2 \left(\frac{1}{2} e^x - \frac{1}{2} \right) dx = \left[\frac{1}{2} e^x - \frac{1}{2} x \right]_0^2 = \frac{1}{2} (e^2 - 3)$
21. $\iint_R \frac{xy^3}{x^2+1} \, dA = \int_0^1 \int_0^2 \frac{xy^3}{x^2+1} \, dy \, dx = \int_0^1 \left[\frac{xy^4}{4(x^2+1)} \right]_0^2 dx = \int_0^1 \frac{4x}{x^2+1} \, dx = [2 \ln |x^2+1|]_0^1 = 2 \ln 2$
22. $\iint_R \frac{y}{x^2 y^2 + 1} \, dA = \int_0^1 \int_0^1 \frac{y}{(xy)^2 + 1} \, dx \, dy = \int_0^1 [\tan^{-1}(xy)]_0^1 dy = \int_0^1 \tan^{-1} y \, dy = \left[y \tan^{-1} y - \frac{1}{2} \ln |1+y^2| \right]_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2$
23. $\int_1^2 \int_1^2 \frac{1}{xy} \, dy \, dx = \int_1^2 \frac{1}{x} (\ln 2 - \ln 1) \, dx = (\ln 2) \int_1^2 \frac{1}{x} \, dx = (\ln 2)^2$
24. $\int_0^1 \int_0^\pi y \cos xy \, dx \, dy = \int_0^1 [\sin xy]_0^\pi dy = \int_0^1 \sin \pi y \, dy = \left[-\frac{1}{\pi} \cos \pi y \right]_0^1 = -\frac{1}{\pi} (-1 - 1) = \frac{2}{\pi}$
25. $V = \iint_R f(x, y) \, dA = \int_{-1}^1 \int_{-1}^1 (x^2 + y^2) \, dy \, dx = \int_{-1}^1 \left[x^2 y + \frac{1}{3} y^3 \right]_{-1}^1 dx = \int_{-1}^1 \left(2x^2 + \frac{2}{3} \right) dx = \left[\frac{2}{3} x^3 + \frac{2}{3} x \right]_{-1}^1 = \frac{8}{3}$
26. $V = \iint_R f(x, y) \, dA = \int_0^2 \int_0^2 (16 - x^2 - y^2) \, dy \, dx = \int_0^2 \left[16y - x^2 y - \frac{1}{3} y^3 \right]_0^2 dx = \int_0^2 \left(\frac{88}{3} - 2x^2 \right) dx = \left[\frac{88}{3} x - \frac{2}{3} x^3 \right]_0^2$
 $= \frac{160}{3}$

$$27. \quad V = \iint_R f(x, y) \, dA = \int_0^1 \int_0^1 (2 - x - y) \, dy \, dx = \int_0^1 \left[2y - xy - \frac{1}{2}y^2 \right]_0^1 dx = \int_0^1 \left(\frac{3}{2} - x \right) dx = \left[\frac{3}{2}x - \frac{1}{2}x^2 \right]_0^1 = 1$$

$$28. \quad V = \iint_R f(x, y) \, dA = \int_0^4 \int_0^2 \frac{y}{2} \, dy \, dx = \int_0^4 \left[\frac{y^2}{4} \right]_0^2 dx = \int_0^4 1 \, dx = [x]_0^4 = 4$$

$$29. \quad V = \iint_R f(x, y) \, dA = \int_0^{\pi/2} \int_0^{\pi/4} 2 \sin x \cos y \, dy \, dx = \int_0^{\pi/2} [2 \sin x \sin y]_0^{\pi/4} dx = \int_0^{\pi/2} (\sqrt{2} \sin x) dx \\ = [-\sqrt{2} \cos x]_0^{\pi/2} = \sqrt{2}$$

$$30. \quad V = \iint_R f(x, y) \, dA = \int_0^1 \int_0^2 (4 - y^2) \, dy \, dx = \int_0^1 \left[4y - \frac{1}{3}y^3 \right]_0^2 dx = \int_0^1 \left(\frac{16}{3} \right) dx = \left[\frac{16}{3}x \right]_0^1 = \frac{16}{3}$$

$$31. \quad \int_1^2 \int_0^3 kx^2 y \, dx \, dy = \int_1^2 \left(\frac{k}{3} x^3 y \right)_{x=0}^{x=3} dx = \int_1^2 9ky \, dy = \left[\frac{9}{2}ky^2 \right]_1^2 = \frac{27}{2}k$$

Thus we choose $k = 2/27$.

$$32. \quad \int_0^{\pi/2} \sin(\sqrt{y}) \, dy \text{ is some number, say } a. \text{ Then } \int_{-1}^1 \int_0^{\pi/2} x \sin(\sqrt{y}) \, dy \, dx = a \int_{-1}^1 x \, dx = 0 \text{ since the integral of the odd function } x \text{ over an interval symmetric to 0 is equal to 0.}$$

33. By Fubini's Theorem,

$$\int_0^2 \int_0^1 \frac{x}{1+xy} \, dx \, dy = \int_0^1 \int_0^2 \frac{x}{1+xy} \, dy \, dx \\ = \int_0^1 \left(\ln(1+xy) \right)_{y=0}^{y=2} dx = \int_0^1 \ln(1+2x) \, dx = \frac{(1+2x)}{2} [\ln(1+2x) - 1]_0^1 = \frac{3}{2} \ln 3 - 1$$

34. By Fubini's Theorem,

$$\int_0^1 \int_0^3 x e^{xy} \, dx \, dy = \int_0^3 \int_0^1 x e^{xy} \, dy \, dx \\ = \int_0^3 \left(e^{xy} \right)_{y=0}^{y=1} dx = \int_0^3 (e^x - 1) \, dx = (e^x - x) \Big|_0^3 = e^3 - 4 \approx 16.086$$

$$35. \quad (a) \quad \text{MAPLE gives } \int_0^1 \int_0^2 \frac{y-x}{(x+y)^3} \, dx \, dy = \frac{1}{3} \text{ and } \int_0^2 \int_0^1 \frac{y-x}{(x+y)^3} \, dy \, dx = -\frac{2}{3}. \text{ This does not contradict}$$

Fubini's Theorem since the integrand is not continuous on the region R : $0 \leq x \leq 2$, $0 \leq y \leq 1$.

36. Since f is continuous on R , for fixed u $f(u, v)$ is a continuous function of v and has an antiderivative with respect to v on R , call it $g(u, v)$. Then $\int_c^y f(u, v) dv = g(u, y) - g(u, c)$ and

$$F(x, y) = \int_a^x \int_c^y f(u, v) dv du = \int_a^x (g(u, y) - g(u, c)) du.$$

$$F_x = \frac{\partial}{\partial x} \int_a^x (g(u, y) - g(u, c)) du = g(x, y) - g(x, c).$$

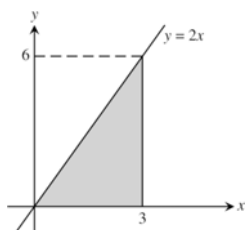
Now taking the derivative with respect to y , we get

$$F_{xy} = \frac{\partial}{\partial y} (g(x, y) - g(x, c)) = f(x, y).$$

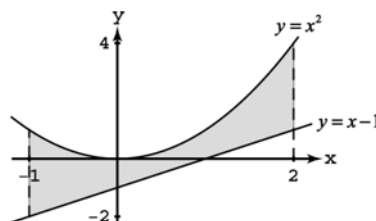
To evaluate F_{yx} we use Fubini's Theorem to rewrite $F(x, y)$ as $\int_c^y \int_a^x f(u, v) du dv$ and make a similar argument. The result is again $f(x, y)$.

15.2 DOUBLE INTEGRALS OVER GENERAL REGIONS

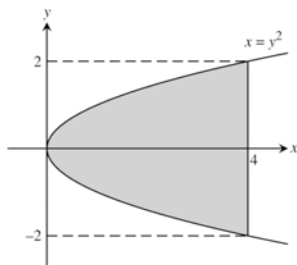
1.



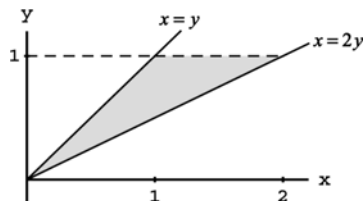
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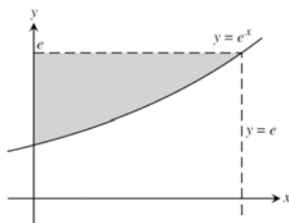
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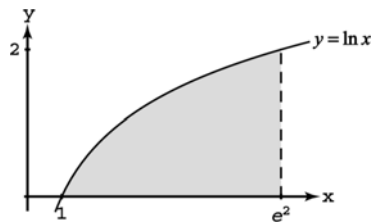
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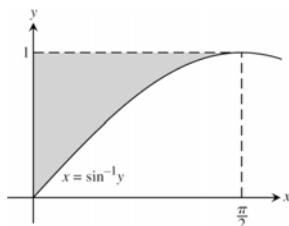
5.



6.



7.



9. (a) $\int_0^2 \int_{x^3}^8 dy dx$

10. (a) $\int_0^3 \int_0^{2x} dy dx$

11. (a) $\int_0^3 \int_{x^2}^{3x} dy dx$

12. (a) $\int_0^2 \int_1^{e^x} dy dx$

13. (a) $\int_0^9 \int_0^{\sqrt{x}} dy dx$

(b) $\int_0^3 \int_{y^2}^9 dx dy$

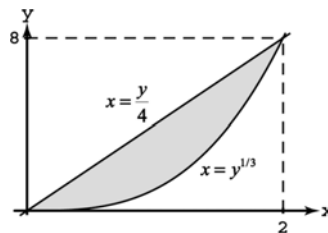
14. (a) $\int_0^{\pi/4} \int_{\tan x}^1 dy dx$

(b) $\int_0^1 \int_0^{\tan^{-1}y} dx dy$

15. (a) $\int_0^{\ln 3} \int_{e^{-x}}^1 dy dx$

(b) $\int_{1/3}^1 \int_{-\ln y}^{\ln 3} dx dy$

8.

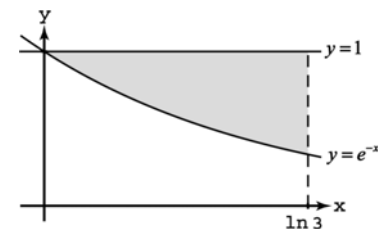
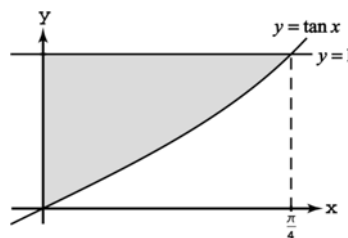


(b) $\int_0^8 \int_0^{y^{1/3}} dx dy$

(b) $\int_0^6 \int_{y/2}^3 dx dy$

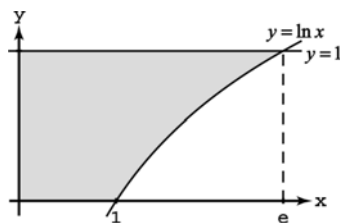
(b) $\int_0^9 \int_{y/3}^{\sqrt{y}} dx dy$

(b) $\int_1^{e^2} \int_{\ln y}^2 dx dy$



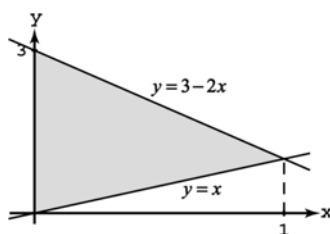
16. (a) $\int_0^1 \int_0^1 dy \, dx + \int_1^e \int_{\ln x}^1 dy \, dx$

(b) $\int_0^1 \int_0^{e^y} dx \, dy$



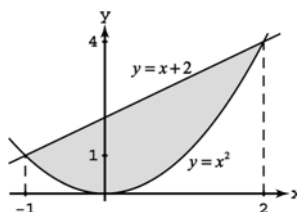
17. (a) $\int_0^1 \int_x^{3-2x} dy \, dx$

(b) $\int_0^1 \int_0^y dx \, dy + \int_1^3 \int_0^{(3-y)/2} dx \, dy$

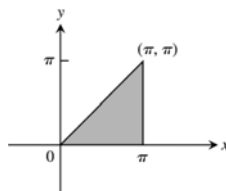


18. (a) $\int_{-1}^2 \int_{x^2}^{x+2} dy \, dx$

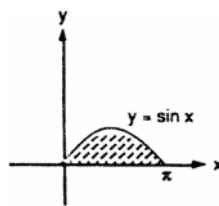
(b) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dy + \int_1^3 \int_{y-2}^{\sqrt{y}} dx \, dy$



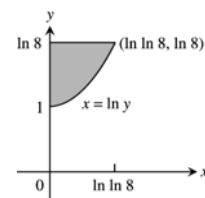
19. $\int_0^\pi \int_0^\pi (x \sin y) \, dy \, dx = \int_0^\pi [-x \cos y]_0^\pi \, dx$
 $= \int_0^\pi (x - x \cos x) \, dx = \left[\frac{x^2}{2} - (\cos x + x \sin x) \right]_0^\pi$
 $= \frac{\pi^2}{2} + 2$



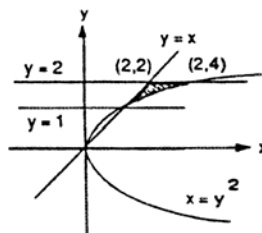
20. $\int_0^\pi \int_0^{\sin x} y \, dy \, dx = \int_0^\pi \left[\frac{y^2}{2} \right]_0^{\sin x} \, dx = \int_0^\pi \frac{1}{2} \sin^2 x \, dx$
 $= \frac{1}{4} \int_0^\pi (1 - \cos 2x) \, dx = \frac{1}{4} \left[x - \frac{1}{2} \sin 2x \right]_0^\pi = \frac{\pi}{4}$



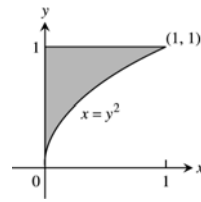
21. $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y} \, dx \, dy = \int_1^{\ln 8} [e^{x+y}]_0^{\ln y} \, dy$
 $= \int_1^{\ln 8} (ye^y - e^y) \, dy = \left[(y-1)e^y - e^y \right]_1^{\ln 8}$
 $= 8(\ln 8 - 1) - 8 + e = 8 \ln 8 - 16 + e$



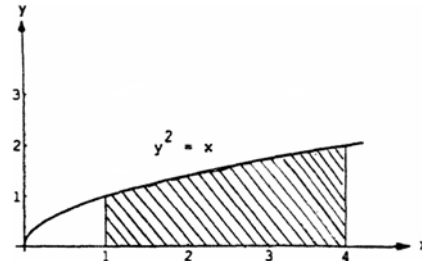
22. $\int_1^2 \int_y^{y^2} dx \, dy = \int_1^2 (y^2 - y) \, dy = \left[\frac{y^3}{3} - \frac{y^2}{2} \right]_1^2$
 $= \left(\frac{8}{3} - 2 \right) - \left(\frac{1}{3} - \frac{1}{2} \right) = \frac{7}{3} - \frac{3}{2} = \frac{5}{6}$



$$\begin{aligned}
 23. \quad \int_0^1 \int_0^{y^2} 3y^3 e^{xy} dx dy &= \int_0^1 \left[3y^2 e^{xy} \right]_0^{y^2} dy \\
 &= \int_0^1 (3y^2 e^{y^3} - 3y^2) dy = \left[e^{y^3} - y^3 \right]_0^1 = e - 2
 \end{aligned}$$



$$\begin{aligned}
 24. \quad \int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} dy dx &= \int_1^4 \left[\frac{3}{2} \sqrt{x} e^{y/\sqrt{x}} \right]_0^{\sqrt{x}} dx \\
 &= \frac{3}{2} (e-1) \int_1^4 \sqrt{x} dx = \left[\frac{3}{2} (e-1) \left(\frac{2}{3} \right) x^{3/2} \right]_1^4 = 7(e-1)
 \end{aligned}$$



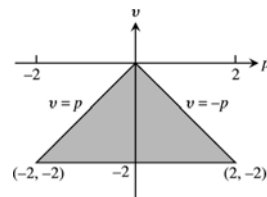
$$25. \quad \int_1^2 \int_x^{2x} \frac{x}{y} dy dx = \int_1^2 [x \ln y]_x^{2x} dx = (\ln 2) \int_1^2 x dx = \frac{3}{2} \ln 2$$

$$\begin{aligned}
 26. \quad \int_0^1 \int_0^{1-x} (x^2 + y^2) dy dx &= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx = \int_0^1 \left[x^2 (1-x) + \frac{(1-x)^3}{3} \right] dx \\
 &= \int_0^1 \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12} \right] dx = \left(\frac{1}{3} - \frac{1}{4} - 0 \right) - \left(0 - 0 - \frac{1}{12} \right) = \frac{1}{6}
 \end{aligned}$$

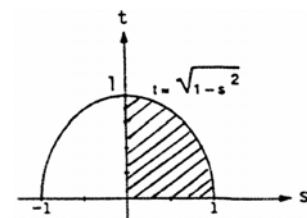
$$\begin{aligned}
 27. \quad \int_0^1 \int_0^{1-u} (v - \sqrt{u}) dv du &= \int_0^1 \left[\frac{v^2}{2} - v\sqrt{u} \right]_0^{1-u} du = \int_0^1 \left[\frac{1-2u+u^2}{2} - \sqrt{u}(1-u) \right] du \\
 &= \int_0^1 \left(\frac{1}{2} - u + \frac{u^2}{2} - u^{1/2} + u^{3/2} \right) du = \left[\frac{u}{2} - \frac{u^2}{2} + \frac{u^3}{6} - \frac{2}{3} u^{3/2} + \frac{2}{5} u^{5/2} \right]_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} - \frac{2}{3} + \frac{2}{5} = -\frac{1}{2} + \frac{2}{5} = -\frac{1}{10}
 \end{aligned}$$

$$\begin{aligned}
 28. \quad \int_1^2 \int_0^{\ln t} e^s \ln t ds dt &= \int_1^2 [e^s \ln t]_0^{\ln t} dt = \int_1^2 (t \ln t - \ln t) dt = \left[\frac{t^2}{2} \ln t - \frac{t^2}{4} - t \ln t + t \right]_1^2 \\
 &= (2 \ln 2 - 1 - 2 \ln 2 + 2) - \left(-\frac{1}{4} + 1 \right) = \frac{1}{4}
 \end{aligned}$$

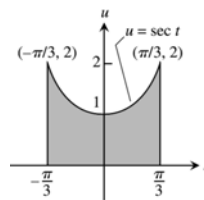
$$\begin{aligned}
 29. \quad \int_{-2}^0 \int_v^{-v} 2 dp dv &= 2 \int_{-2}^0 [p]_v^{-v} dv = 2 \int_{-2}^0 -2v dv \\
 &= -2 \left[v^2 \right]_{-2}^0 = 8
 \end{aligned}$$



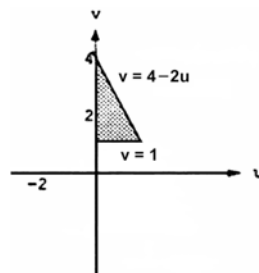
$$\begin{aligned}
 30. \quad \int_0^1 \int_0^{\sqrt{1-s^2}} 8t dt ds &= \int_0^1 \left[4t^2 \right]_0^{\sqrt{1-s^2}} ds \\
 &= \int_0^1 4(1-s^2) ds = 4 \left[s - \frac{s^3}{3} \right]_0^1 = \frac{8}{3}
 \end{aligned}$$



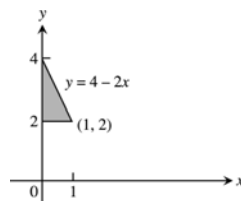
$$31. \int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t \, du \, dt = \int_{-\pi/3}^{\pi/3} [(3 \cos t)u]_0^{\sec t} \, dt \\ = \int_{-\pi/3}^{\pi/3} 3 \, dt = 2\pi$$



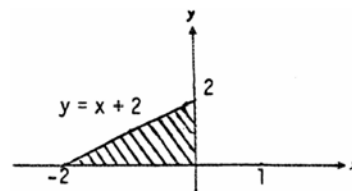
$$32. \int_0^{3/2} \int_1^{4-2u} \frac{4-2u}{v^2} \, dv \, du = \int_0^{3/2} \left[\frac{2u-4}{v} \right]_1^{4-2u} \, du \\ = \int_0^{3/2} (3-2u) \, du = \left[3u - u^2 \right]_0^{3/2} = \frac{9}{2}$$



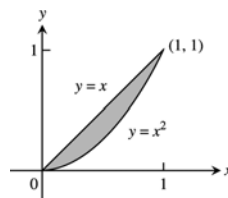
$$33. \int_2^4 \int_0^{(4-y)/2} dx \, dy$$



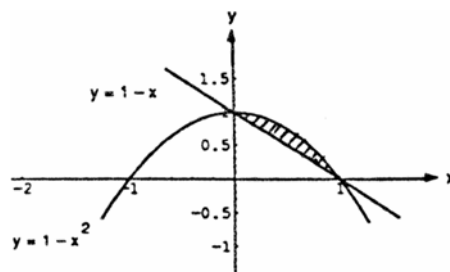
$$34. \int_{-2}^0 \int_0^{x+2} dy \, dx$$



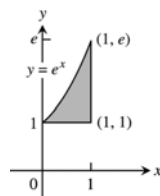
$$35. \int_0^1 \int_{x^2}^x dy \, dx$$



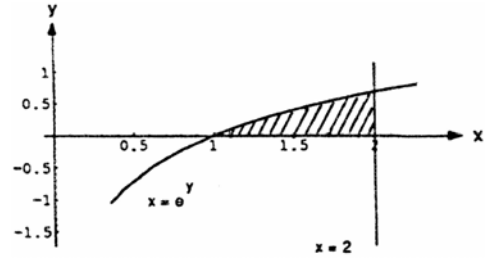
$$36. \int_0^1 \int_{1-y}^{\sqrt{1-y}} dx \, dy$$



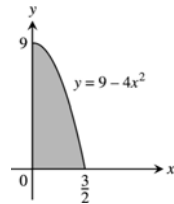
$$37. \int_1^e \int_{\ln y}^1 dx \, dy$$



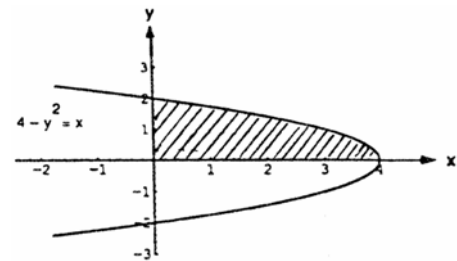
38. $\int_1^2 \int_0^{\ln x} dy \, dx$



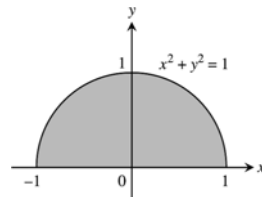
39. $\int_0^9 \int_0^{\frac{1}{2}\sqrt{9-y}} 16x \, dx \, dy$



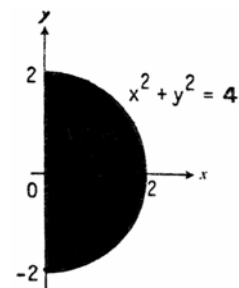
40. $\int_0^4 \int_0^{\sqrt{4-x}} y \, dy \, dx$



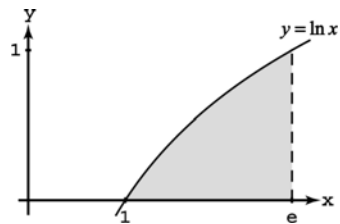
41. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3y \, dy \, dx$



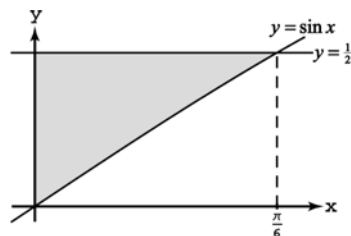
42. $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} 6x \, dx \, dy$



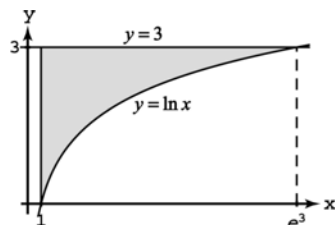
43. $\int_0^1 \int_{e^y}^e xy \, dx \, dy$



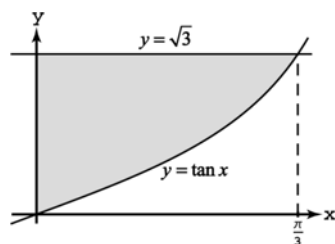
$$44. \int_0^{1/2} \int_0^{\sin^{-1} y} x y^2 dx dy$$



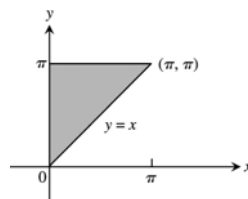
$$45. \int_1^{e^3} \int_{\ln x}^3 (x+y) dy dx$$



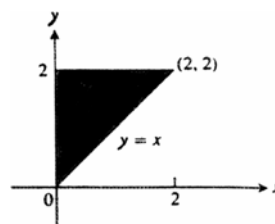
$$46. \int_0^{\pi/3} \int_{\tan x}^{\sqrt{3}} \sqrt{x} y dy dx$$



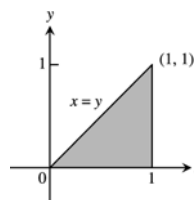
$$47. \int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} dy dx = \int_0^{\pi} \int_0^y \frac{\sin y}{y} dx dy = \int_0^{\pi} \sin y dy = 2$$



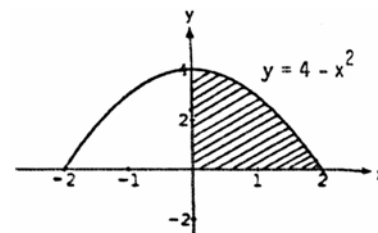
$$\begin{aligned} 48. \int_0^2 \int_x^2 2y^2 \sin xy dy dx &= \int_0^2 \int_0^y 2y^2 \sin xy dx dy \\ &= \int_0^2 [-2y \cos xy]_0^y dy = \int_0^2 (-2y \cos y^2 + 2y) dy \\ &= [-\sin y^2 + y^2]_0^2 = 4 - \sin 4 \end{aligned}$$



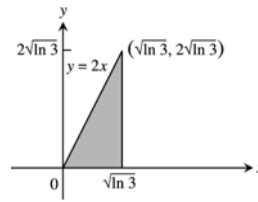
$$\begin{aligned} 49. \int_0^1 \int_y^1 x^2 e^{xy} dx dy &= \int_0^1 \int_0^x x^2 e^{xy} dy dx = \int_0^1 [x e^{xy}]_0^x dx \\ &= \int_0^1 (x e^{x^2} - x) dx = \left[\frac{1}{2} e^{x^2} - \frac{x^2}{2} \right]_0^1 = \frac{e-2}{2} \end{aligned}$$



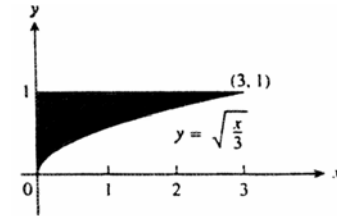
$$\begin{aligned} 50. \int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y} dy dx &= \int_0^2 \int_0^{\sqrt{4-y}} \frac{x e^{2y}}{4-y} dx dy \\ &= \int_0^4 \left[\frac{x^2 e^{2y}}{2(4-y)} \right]_0^{\sqrt{4-y}} dy = \int_0^4 \frac{e^{2y}}{2} dy = \left[\frac{e^{2y}}{4} \right]_0^4 = \frac{e^8 - 1}{4} \end{aligned}$$



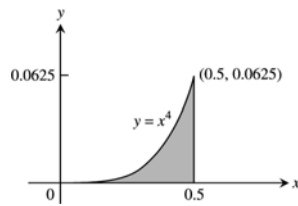
$$\begin{aligned}
 51. \quad \int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2} dx dy &= \int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2} dy dx \\
 &= \int_0^{\sqrt{\ln 3}} 2xe^{x^2} dx = \left[e^{x^2} \right]_0^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2
 \end{aligned}$$



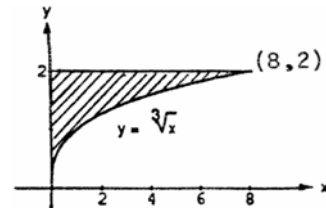
$$\begin{aligned}
 52. \quad \int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} dy dx &= \int_0^1 \int_0^{3y^2} e^{y^3} dx dy \\
 &= \int_0^1 3y^2 e^{y^3} dy = \left[e^{y^3} \right]_0^1 = e - 1
 \end{aligned}$$



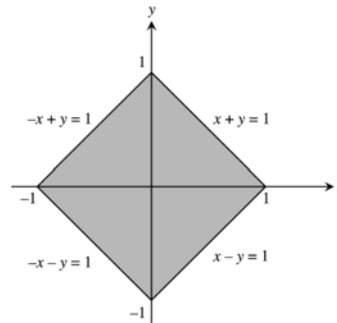
$$\begin{aligned}
 53. \quad \int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5) dx dy &= \int_0^{1/2} \int_0^{x^4} \cos(16\pi x^5) dy dx = \int_0^{1/2} x^4 \cos(16\pi x^5) dx \\
 &= \left[\frac{\sin(16\pi x^5)}{80\pi} \right]_0^{1/2} = \frac{1}{80\pi}
 \end{aligned}$$



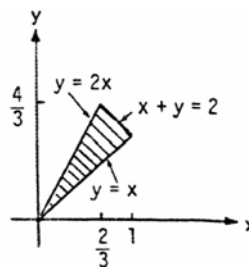
$$\begin{aligned}
 54. \quad \int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1} dy dx &= \int_0^2 \int_0^{y^3} \frac{1}{y^4+1} dx dy \\
 &= \int_0^2 \frac{y^3}{y^4+1} dy = \frac{1}{4} \left[\ln(y^4+1) \right]_0^2 = \frac{\ln 17}{4}
 \end{aligned}$$



$$\begin{aligned}
 55. \quad \iint_R (y - 2x^2) dA &= \int_{-1}^0 \int_{-x-1}^{x+1} (y - 2x^2) dy dx + \int_0^1 \int_{x-1}^{1-x} (y - 2x^2) dy dx \\
 &= \int_{-1}^0 \left[\frac{1}{2} y^2 - 2x^2 y \right]_{-x-1}^{x+1} dx + \int_0^1 \left[\frac{1}{2} y^2 - 2x^2 y \right]_{x-1}^{1-x} dx \\
 &= \int_{-1}^0 \left(\frac{1}{2} (x+1)^2 - 2x^2 (x+1) - \frac{1}{2} (-x-1)^2 + 2x^2 (-x-1) \right) dx \\
 &\quad + \int_0^1 \left(\frac{1}{2} (1-x)^2 - 2x^2 (1-x) - \frac{1}{2} (x-1)^2 + 2x^2 (x-1) \right) dx \\
 &= -4 \int_{-1}^0 (x^3 + x^2) dx + 4 \int_0^1 (x^3 + x^2) dx \\
 &= -4 \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_{-1}^0 + 4 \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_0^1 \\
 &= 4 \left[\frac{(-1)^4}{4} + \frac{(-1)^3}{3} \right] + 4 \left(\frac{1}{4} + \frac{1}{3} \right) = 8 \left(\frac{3}{12} - \frac{4}{12} \right) = -\frac{8}{12} = -\frac{2}{3}
 \end{aligned}$$



$$\begin{aligned}
56. \quad \iint_R xy \, dA &= \int_0^{2/3} \int_x^{2-x} xy \, dy \, dx + \int_{2/3}^1 \int_x^{2-x} xy \, dy \, dx \\
&= \int_0^{2/3} \left[\frac{1}{2} xy^2 \right]_x^{2-x} dx + \int_{2/3}^1 \left[\frac{1}{2} xy^2 \right]_x^{2-x} dx \\
&= \int_0^{2/3} \left(2x^3 - \frac{1}{2} x^3 \right) dx + \int_{2/3}^1 \left[\frac{1}{2} x(2-x)^2 - \frac{1}{2} x^3 \right] dx \\
&= \int_0^{2/3} \frac{3}{2} x^3 \, dx + \int_{2/3}^1 (2x - x^2) \, dx \\
&= \left[\frac{3}{8} x^4 \right]_0^{2/3} + \left[x^2 - \frac{2}{3} x^3 \right]_{2/3}^1 = \left(\frac{3}{8} \right) \left(\frac{16}{81} \right) + \left(1 - \frac{2}{3} \right) - \left[\frac{4}{9} - \left(\frac{2}{3} \right) \left(\frac{8}{27} \right) \right] = \frac{6}{81} + \frac{27}{81} - \left(\frac{36}{81} - \frac{16}{81} \right) = \frac{13}{81}
\end{aligned}$$



$$\begin{aligned}
57. \quad V &= \int_0^1 \int_x^{2-x} (x^2 + y^2) \, dy \, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^{2-x} dx = \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3} \right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12} \right]_0^1 \\
&= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12} \right) - \left(0 - 0 - \frac{16}{12} \right) = \frac{4}{3}
\end{aligned}$$

$$\begin{aligned}
58. \quad V &= \int_{-2}^1 \int_x^{2-x^2} x^2 \, dy \, dx = \int_{-2}^1 \left[x^2 y \right]_x^{2-x^2} dx = \int_{-2}^1 (2x^2 - x^4 - x^3) \, dx = \left[\frac{2}{3} x^3 - \frac{1}{5} x^5 - \frac{1}{4} x^4 \right]_{-2}^1 \\
&= \left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4} \right) - \left(-\frac{16}{3} + \frac{32}{5} - \frac{16}{4} \right) = \left(\frac{40}{60} - \frac{12}{60} - \frac{15}{60} \right) - \left(-\frac{320}{60} + \frac{384}{60} - \frac{240}{60} \right) = \frac{189}{60} = \frac{63}{20}
\end{aligned}$$

$$\begin{aligned}
59. \quad V &= \int_{-4}^1 \int_{3x}^{4-x^2} (x+4) \, dy \, dx = \int_{-4}^1 [xy + 4y]_{3x}^{4-x^2} dx = \int_{-4}^1 \left[x(4-x^2) + 4(4-x^2) - 3x^2 - 12x \right] dx \\
&= \int_{-4}^1 (-x^3 - 7x^2 - 8x + 16) \, dx = \left[-\frac{1}{4} x^4 - \frac{7}{3} x^3 - 4x^2 + 16x \right]_{-4}^1 = \left(-\frac{1}{4} - \frac{7}{3} + 12 \right) - \left(\frac{64}{3} - 64 \right) = \frac{157}{3} - \frac{1}{4} = \frac{625}{12}
\end{aligned}$$

$$\begin{aligned}
60. \quad V &= \int_0^2 \int_0^{\sqrt{4-x^2}} (3-y) \, dy \, dx = \int_0^2 \left[3y - \frac{y^2}{2} \right]_0^{\sqrt{4-x^2}} dx = \int_0^2 \left[3\sqrt{4-x^2} - \left(\frac{4-x^2}{2} \right) \right] dx \\
&= \left[\frac{3}{2} x\sqrt{4-x^2} + 6 \sin^{-1} \left(\frac{x}{2} \right) - 2x + \frac{x^3}{6} \right]_0^2 = 6 \left(\frac{\pi}{2} \right) - 4 + \frac{8}{6} = 3\pi - \frac{16}{6} = \frac{9\pi-8}{3}
\end{aligned}$$

$$61. \quad V = \int_0^2 \int_0^3 (4-y^2) \, dx \, dy = \int_0^2 [4x - y^2 x]_0^3 dy = \int_0^2 (12 - 3y^2) \, dy = [12y - y^3]_0^2 = 24 - 8 = 16$$

$$\begin{aligned}
62. \quad V &= \int_0^2 \int_0^{4-x^2} (4-x^2-y) \, dy \, dx = \int_0^2 \left[(4-x^2)y - \frac{y^2}{2} \right]_0^{4-x^2} dx = \int_0^2 \frac{1}{2} (4-x^2)^2 dx = \int_0^2 \left(8 - 4x^2 + \frac{x^4}{2} \right) dx \\
&= \left[8x - \frac{4}{3} x^3 + \frac{1}{10} x^5 \right]_0^2 = 16 - \frac{32}{3} + \frac{32}{10} = \frac{480-320+96}{30} = \frac{128}{15}
\end{aligned}$$

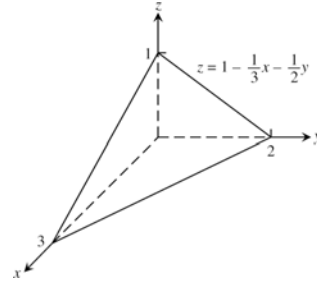
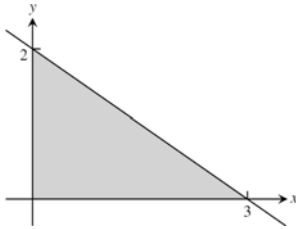
$$63. \quad V = \int_0^2 \int_0^{2-x} (12-3y^2) \, dy \, dx = \int_0^2 [12y - y^3]_0^{2-x} dx = \int_0^2 [24 - 12x - (2-x)^3] dx = \left[24x - 6x^2 + \frac{(2-x)^4}{4} \right]_0^2 = 20$$

$$64. \quad V = \int_{-1}^0 \int_{-x-1}^{x+1} (3-3x) \, dy \, dx + \int_0^1 \int_{x-1}^{1-x} (3-3x) \, dy \, dx = 6 \int_{-1}^0 (1-x^2) dx + 6 \int_0^1 (1-x^2) dx = 4 + 2 = 6$$

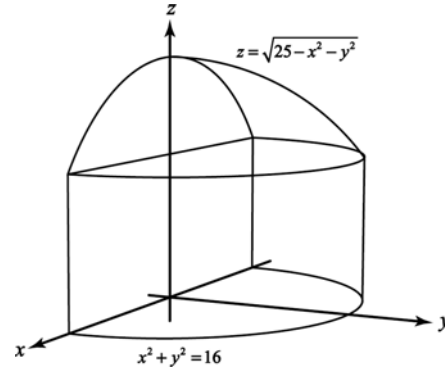
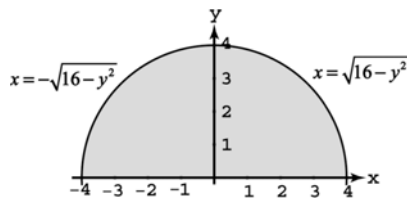
$$65. \quad V = \int_1^2 \int_{-1/x}^{1/x} (x+1) \, dy \, dx = \int_1^2 [xy + y]_{-1/x}^{1/x} \, dx = \int_1^2 \left[1 + \frac{1}{x} - \left(-1 - \frac{1}{x} \right) \right] \, dx = 2 \int_1^2 \left(1 + \frac{1}{x} \right) \, dx = 2[x + \ln x]_1^2 = 2(1 + \ln 2)$$

$$66. \quad V = 4 \int_0^{\pi/3} \int_0^{\sec x} (1+y^2) \, dy \, dx = 4 \int_0^{\pi/3} \left[y + \frac{y^3}{3} \right]_0^{\sec x} \, dx = 4 \int_0^{\pi/3} \left(\sec x + \frac{\sec^3 x}{3} \right) \, dx \\ = \frac{2}{3} [7 \ln |\sec x + \tan x| + \sec x \tan x]_0^{\pi/3} = \frac{2}{3} [7 \ln (2 + \sqrt{3}) + 2\sqrt{3}]$$

67.



68.



$$69. \quad \int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y} \, dy \, dx = \int_1^\infty \left[\frac{\ln y}{x^3} \right]_{e^{-x}}^1 \, dx = \int_1^\infty -\left(\frac{-x}{x^3} \right) \, dx = - \lim_{b \rightarrow \infty} \left[\frac{1}{x} \right]_1^b = - \lim_{b \rightarrow \infty} \left(\frac{1}{b} - 1 \right) = 1$$

$$70. \quad \int_{-1}^1 \int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} (2y+1) \, dy \, dx = \int_{-1}^1 \left[y^2 + y \right]_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} \, dx = \int_{-1}^1 \frac{2}{\sqrt{1-x^2}} \, dx = 4 \lim_{b \rightarrow 1^-} [\sin^{-1} x]_0^b = 4 \lim_{b \rightarrow 1^-} [\sin^{-1} b - 0] \\ = 2\pi$$

$$71. \quad \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{(x^2+1)(y^2+1)} \, dx \, dy = 2 \int_0^\infty \left(\frac{2}{y^2+1} \right) \left(\lim_{b \rightarrow \infty} \tan^{-1} b - \tan^{-1} 0 \right) \, dy = 2\pi \lim_{b \rightarrow \infty} \int_0^b \frac{1}{y^2+1} \, dy \\ = 2\pi \left(\lim_{b \rightarrow \infty} \tan^{-1} b - \tan^{-1} 0 \right) = (2\pi) \left(\frac{\pi}{2} \right) = \pi^2$$

$$72. \quad \int_0^\infty \int_0^\infty x e^{-(x+2y)} \, dx \, dy = \int_0^\infty e^{-2y} \lim_{b \rightarrow \infty} [-x e^{-x} - e^{-x}]_0^b \, dy = \int_0^\infty e^{-2y} \lim_{b \rightarrow \infty} (-b e^{-b} - e^{-b} + 1) \, dy \\ = \int_0^\infty e^{-2y} \, dy = \frac{1}{2} \lim_{b \rightarrow \infty} (-e^{-2b} + 1) = \frac{1}{2}$$

$$73. \iint_R f(x, y) \, dA \approx \frac{1}{4} f\left(-\frac{1}{2}, 0\right) + \frac{1}{8} f(0, 0) + \frac{1}{8} f\left(\frac{1}{4}, 0\right) = \frac{1}{4}\left(-\frac{1}{2}\right) + \frac{1}{8}\left(0 + \frac{1}{4}\right) = -\frac{3}{32}$$

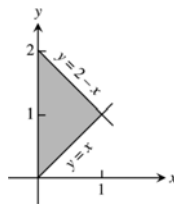
$$74. \iint_R f(x, y) \, dA \approx \frac{1}{4} \left[f\left(\frac{7}{4}, \frac{11}{4}\right) + f\left(\frac{9}{4}, \frac{11}{4}\right) + f\left(\frac{7}{4}, \frac{13}{4}\right) + f\left(\frac{9}{4}, \frac{13}{4}\right) \right] = \frac{1}{16} (29 + 31 + 33 + 35) = \frac{128}{16} = 8$$

75. The ray $\theta = \frac{\pi}{6}$ meets the circle $x^2 + y^2 = 4$ at the point $(\sqrt{3}, 1) \Rightarrow$ the ray is represented by the line $y = \frac{x}{\sqrt{3}}$.

$$\text{Thus, } \iint_R f(x, y) \, dA = \int_0^{\sqrt{3}} \int_{x/\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{4-x^2} \, dy \, dx = \int_0^{\sqrt{3}} \left[(4-x^2) - \frac{x}{\sqrt{3}} \sqrt{4-x^2} \right] dx = \left[4x - \frac{x^3}{3} + \frac{(4-x^2)^{3/2}}{3\sqrt{3}} \right]_0^{\sqrt{3}} \\ = \frac{20\sqrt{3}}{9}$$

$$76. \int_2^\infty \int_0^2 \frac{1}{(x^2-x)(y-1)^{2/3}} \, dy \, dx = \int_2^\infty \left[\frac{3(y-1)^{1/3}}{x^2-x} \right]_0^2 dx = \int_2^\infty \left(\frac{3}{x^2-x} + \frac{3}{x^2-x} \right) dx = 6 \int_2^\infty \frac{dx}{x(x-1)} = 6 \lim_{b \rightarrow \infty} \int_2^b \left(\frac{1}{x-1} - \frac{1}{x} \right) dx \\ = 6 \lim_{b \rightarrow \infty} [\ln(x-1) - \ln x]_2^b = 6 \lim_{b \rightarrow \infty} [\ln(b-1) - \ln b - \ln 1 + \ln 2] = 6 \left[\lim_{b \rightarrow \infty} \ln\left(1 - \frac{1}{b}\right) + \ln 2 \right] = 6 \ln 2$$

$$77. V = \int_0^1 \int_x^{2-x} (x^2 + y^2) \, dy \, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^{2-x} dx \\ = \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3} \right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12} \right]_0^1 \\ = \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12} \right) - \left(0 - 0 - \frac{16}{12} \right) = \frac{4}{3}$$



$$78. \int_0^2 (\tan^{-1} \pi x - \tan^{-1} x) \, dx = \int_0^2 \int_x^{\pi x} \frac{1}{1+y^2} \, dy \, dx = \int_0^2 \int_{y/\pi}^y \frac{1}{1+y^2} \, dx \, dy + \int_2^{2\pi} \int_{y/\pi}^2 \frac{1}{1+y^2} \, dx \, dy \\ = \int_0^2 \frac{(1-\frac{1}{\pi})y}{1+y^2} \, dy + \int_2^{2\pi} \frac{(2-\frac{y}{\pi})}{1+y^2} \, dy = \left(\frac{\pi-1}{2\pi} \right) \left[\ln(1+y^2) \right]_0^2 + \left[2 \tan^{-1} y + \frac{1}{2\pi} \ln(1+y^2) \right]_2^{2\pi} \\ = \left(\frac{\pi-1}{2\pi} \right) \ln 5 + 2 \tan^{-1} 2\pi - \frac{1}{2\pi} \ln(1+4\pi^2) - 2 \tan^{-1} 2 + \frac{1}{2\pi} \ln 5 \\ = 2 \tan^{-1} 2\pi - 2 \tan^{-1} 2 - \frac{1}{2\pi} \ln(1+4\pi^2) + \frac{\ln 5}{2}$$

79. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where the integrand is negative. These criteria are met by the points (x, y) such that $4 - x^2 - 2y^2 \geq 0$ or $x^2 + 2y^2 \leq 4$, which is the ellipse $x^2 + 2y^2 = 4$ together with its interior.

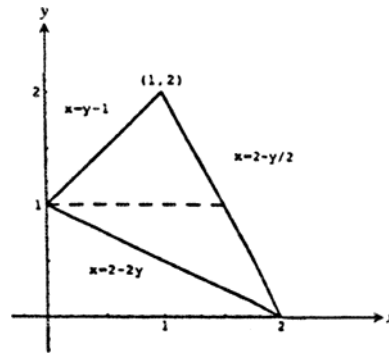
80. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where the integrand is positive. These criteria are met by the points (x, y) such that $x^2 + y^2 - 9 \leq 0$ or $x^2 + y^2 \leq 9$, which is the closed disk of radius 3 centered at the origin.

81. No, it is not possible. By Fubini's theorem, the two orders of integration must give the same result.

82. One way would be to partition R into two triangles with the line $y = 1$. The integral of f over R could then be written as a sum of integrals that could be evaluated by integrating first with respect to x and then with respect to y :

$$\iint_R f(x, y) dA = \int_0^1 \int_{2-2y}^{2-(y/2)} f(x, y) dx dy + \int_1^2 \int_{y-1}^{2-(y/2)} f(x, y) dx dy.$$

Partitioning R with the line $x = 1$ would let us write the integral of f over R as a sum of iterated integrals with order $dy dx$.



83.
$$\int_{-b}^b \int_{-b}^b e^{-x^2-y^2} dx dy = \int_{-b}^b \int_{-b}^b e^{-y^2} e^{-x^2} dx dy = \int_{-b}^b e^{-y^2} \left(\int_{-b}^b e^{-x^2} dx \right) dy = \left(\int_{-b}^b e^{-x^2} dx \right) \left(\int_{-b}^b e^{-y^2} dy \right)$$

$$= \left(\int_{-b}^b e^{-x^2} dx \right)^2 = \left(2 \int_0^b e^{-x^2} dx \right)^2 = 4 \left(\int_0^b e^{-x^2} dx \right)^2$$
; taking limits as $b \rightarrow \infty$ gives the stated result.

84.
$$\int_0^1 \int_0^3 \frac{x^2}{(y-1)^{2/3}} dy dx = \int_0^3 \int_0^1 \frac{x^2}{(y-1)^{2/3}} dx dy = \int_0^3 \frac{1}{(y-1)^{2/3}} \left[\frac{x^3}{3} \right]_0^1 dy = \frac{1}{3} \int_0^3 \frac{dy}{(y-1)^{2/3}}$$

$$= \frac{1}{3} \lim_{b \rightarrow 1^-} \int_0^b \frac{dy}{(y-1)^{2/3}} + \frac{1}{3} \lim_{b \rightarrow 1^+} \int_b^3 \frac{dy}{(y-1)^{2/3}} = \lim_{b \rightarrow 1^-} \left[(y-1)^{1/3} \right]_0^b + \lim_{b \rightarrow 1^+} \left[(y-1)^{1/3} \right]_b^3$$

$$= \left[\lim_{b \rightarrow 1^-} (b-1)^{1/3} - (-1)^{1/3} \right] - \left[\lim_{b \rightarrow 1^+} (b-1)^{1/3} - (2)^{1/3} \right] = (0+1) - (0-\sqrt[3]{2}) = 1 + \sqrt[3]{2}$$

- 85-88. Example CAS commands:

Maple:

```
f := (x,y) -> 1/x/y;
q1 := Int( Int( f(x,y) y=1..x), x=1..3);
evalf(q1);
value(q1);
evalf( value(q1) );
```

- 89-94. Example CAS commands:

Maple:

```
f := (x,y) -> exp(x^2);
c,d := 0,1;
g1 := y -> 2*y;
g2 := y -> 4;
q5 := Int( Int( f(x,y), x=g1(y)..g2(y) ), y=c..d);
value(q5);
plot3d(0, (x=g1(y)..g2(y), y=c..d), color=pink, style=patchnogrid, axes=boxed, orientation=[-90,0]
scaling=constrained, title="#89(Section 15.2)");
r5 := Int( Int( f(x,y), y=0..x/2), x=0..2 ) + Int( Int( f(x,y), x=2..4), y=0..1);
```

```
value(r5);
value(q5-t5);
```

85-94. Example CAS commands:

Mathematica: (functions and bounds will vary)

You can integrate using the built-in integral signs or with the command **Integrate**. In the **Integrate** command, the integration begins with the variable on the right. (In this case, y going from 1 to x).

```
Clear[x, y, f]
f[x_, y_] := 1 / (x y)
Integrate[f[x, y], {x, 1, 3}, {y, 1, x}]
```

To reverse the order of integration, it is best to first plot the region over which the integration extends. This can be done with **ImplicitPlot** and all bounds involving both x and y can be plotted. A graphics package must be loaded. Remember to use the double equal sign for the equations of the bounding curves.

```
Clear[x, y, f]
<<Graphics`ImplicitPlot`
ImplicitPlot[{x == 2y, x == 4, y == 0, y == 1}, {x, 0, 4.1}, {y, 0, 1.1}];
f[x_, y_] := Exp[x^2]
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + Integrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
```

To get a numerical value for the result, use the numerical integrator, **NIntegrate**. Verify that this equals the original.

```
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + NIntegrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
NIntegrate[f[x, y], {y, 0, 1}, {x, 2y, 4}]
```

Another way to show a region is with the **FilledPlot** command. This assumes that functions are given as $y=f(x)$.

```
Clear[x, y, f]
<<Graphics`FilledPlot`
FilledPlot[{x^2, 9}, {x, 0, 3}, AxesLabels -> {x, y}];
f[x_, y_] := x Cos[y^2]
Integrate[f[x, y], {y, 0, 9}, {x, 0, Sqrt[y]}]
```

$$85. \int_1^3 \int_1^x \frac{1}{xy} dy dx \approx 0.603$$

$$86. \int_0^1 \int_0^1 e^{-(x^2+y^2)} dy dx \approx 0.558$$

$$87. \int_0^1 \int_0^1 \tan^{-1} xy dy dx \approx 0.233$$

$$88. \int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3\sqrt{1-x^2-y^2} dy dx \approx 3.142$$

89. Evaluate the integrals:

$$\begin{aligned} & \int_0^1 \int_{2y}^4 e^{x^2} dx dy \\ &= \int_0^2 \int_0^{x/2} e^{x^2} dy dx + \int_2^4 \int_0^1 e^{x^2} dy dx \\ &= -\frac{1}{4} + \frac{1}{4} \left(e^4 - 2\sqrt{\pi} \operatorname{erfi}(2) + 2\sqrt{\pi} \operatorname{erfi}(4) \right) \\ &\approx 1.1494 \times 10^6 \end{aligned}$$

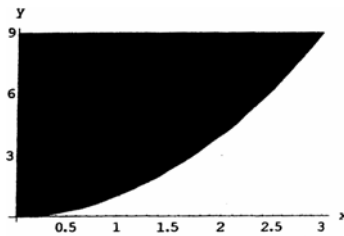
The following graph was generated using Mathematica.



90. Evaluate the integrals:

$$\begin{aligned} & \int_0^3 \int_{x^2}^9 x \cos(y^2) dy dx = \int_0^9 \int_0^{\sqrt{y}} x \cos(y^2) dx dy \\ &= \frac{\sin(81)}{4} \approx -0.157472 \end{aligned}$$

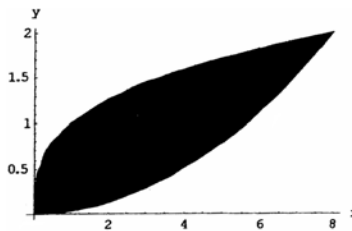
The following graph was generated using Mathematica.



91. Evaluate the integrals:

$$\begin{aligned} & \int_0^2 \int_{y^3}^{4\sqrt{2}y} (x^2 y - xy^2) dx dy = \int_0^8 \int_{x^2/32}^{\sqrt[3]{x}} (x^2 y - xy^2) dy dx \\ &= \frac{67,520}{693} \approx 97.4315 \end{aligned}$$

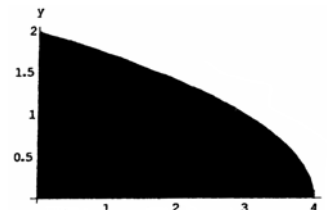
The following graph was generated using Mathematica.



92. Evaluate the integrals:

$$\begin{aligned} & \int_0^2 \int_0^{4-y^2} e^{xy} dx dy = \int_0^4 \int_0^{\sqrt{4-x}} e^{xy} dy dx \\ &\approx 20.5648 \end{aligned}$$

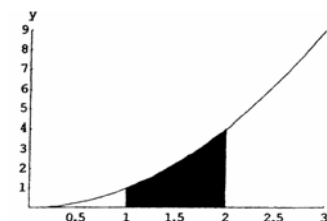
The following graph was generated using Mathematica.



93. Evaluate the integrals:

$$\begin{aligned} & \int_1^2 \int_0^{x^2} \frac{1}{x+y} dy dx \\ &= \int_0^1 \int_1^2 \frac{1}{x+y} dx dy + \int_1^4 \int_{\sqrt{y}}^2 \frac{1}{x+y} dx dy \\ &= -1 + \ln\left(\frac{27}{4}\right) \approx 0.909543 \end{aligned}$$

The following graph was generated using Mathematica.

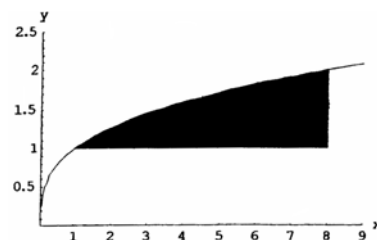


94. Evaluate the integrals:

$$\int_1^2 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}} dx dy = \int_1^8 \int_1^{\sqrt[3]{x}} \frac{1}{\sqrt{x^2+y^2}} dy dx$$

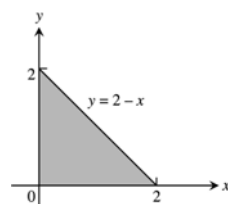
$$\approx 0.866649$$

The following graph was generated using Mathematica.

**15.3 AREA BY DOUBLE INTEGRATION**

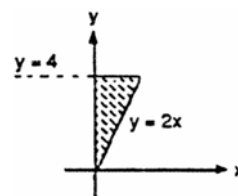
$$1. \int_0^2 \int_0^{2-x} dy dx = \int_0^2 (2-x) dx = \left[2x - \frac{x^2}{2} \right]_0^2 = 2,$$

or $\int_0^2 \int_0^{2-y} dx dy = \int_0^2 (2-y) dy = 2$



$$2. \int_0^2 \int_{2x}^4 dy dx = \int_0^2 (4-2x) dx = \left[4x - x^2 \right]_0^2 = 4,$$

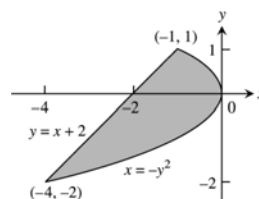
or $\int_0^4 \int_0^{y/2} dx dy = \int_0^4 \frac{y}{2} dy = 4$



$$3. \int_{-2}^1 \int_{y-2}^{-y^2} dx dy = \int_{-2}^1 (-y^2 - y + 2) dy$$

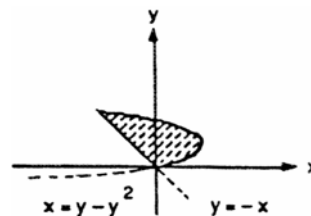
$$= \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y \right]_{-2}^1$$

$$= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) = \frac{9}{2}$$

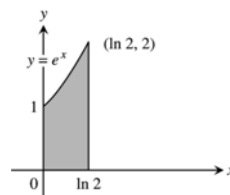


$$4. \int_0^2 \int_{-y}^{y-y^2} dx dy = \int_0^2 (2y - y^2) dy = \left[y^2 - \frac{y^3}{3} \right]_0^2$$

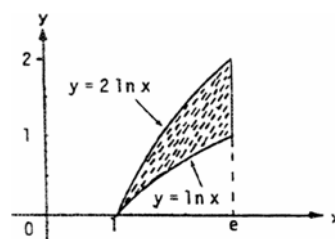
$$= 4 - \frac{8}{3} = \frac{4}{3}$$



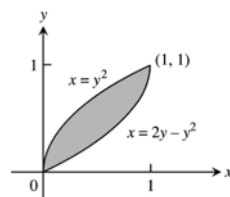
$$5. \int_0^{\ln 2} \int_0^{e^x} dy dx = \int_0^{\ln 2} e^x dx = \left[e^x \right]_0^{\ln 2} = 2 - 1 = 1$$



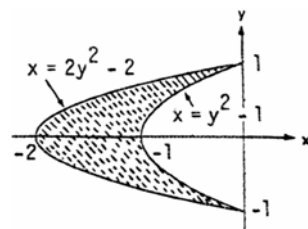
$$\begin{aligned}
 6. \quad \int_1^e \int_{\ln x}^{2 \ln x} dy \, dx &= \int_1^e \ln x \, dx = [x \ln x - x]_1^e \\
 &= (e - e) - (0 - 1) = 1
 \end{aligned}$$



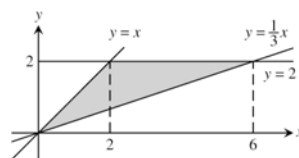
$$\begin{aligned}
 7. \quad \int_0^1 \int_{y^2}^{2y-y^2} dx \, dy &= \int_0^1 (2y - 2y^2) \, dy = \left[y^2 - \frac{2}{3} y^3 \right]_0^1 \\
 &= \frac{1}{3}
 \end{aligned}$$



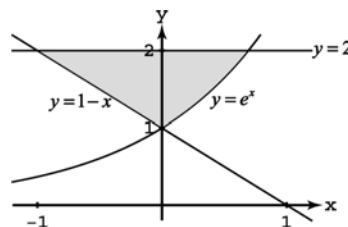
$$\begin{aligned}
 8. \quad \int_{-1}^1 \int_{2y^2-2}^{y^2-1} dx \, dy &= \int_{-1}^1 (y^2 - 1 - 2y^2 + 2) \, dy \\
 &= \int_{-1}^1 (1 - y^2) \, dy = \left[y - \frac{y^3}{3} \right]_{-1}^1 = \frac{4}{3}
 \end{aligned}$$



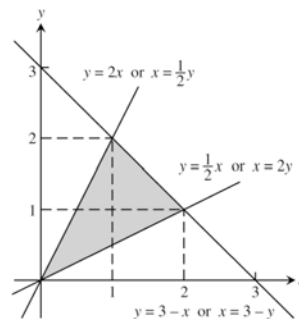
$$\begin{aligned}
 9. \quad \int_0^2 \int_y^{3y} 1 \, dx \, dy &= \int_0^2 [x]_y^{3y} \, dy \\
 &= \int_0^2 (2y) \, dy = \left[y^2 \right]_0^2 = 4
 \end{aligned}$$



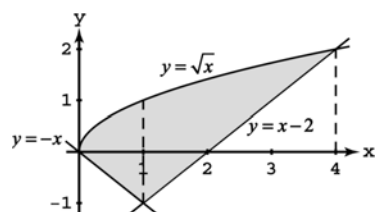
$$\begin{aligned}
 10. \quad \int_1^2 \int_{1-y}^{\ln y} 1 \, dx \, dy &= \int_1^2 [x]_{1-y}^{\ln y} \, dy \\
 &= \int_1^2 (\ln y - 1 + y) \, dy = \left[y \ln y - 2y + \frac{y^2}{2} \right]_1^2 \\
 &= 2 \ln 2 - \frac{1}{2}
 \end{aligned}$$



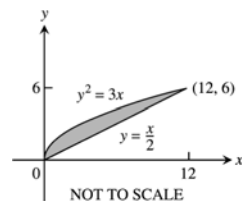
$$\begin{aligned}
 11. \quad \int_0^1 \int_{x/2}^{2x} 1 \, dy \, dx + \int_1^2 \int_{x/2}^{3-x} 1 \, dy \, dx \\
 &= \int_0^1 [y]_{x/2}^{2x} \, dx + \int_1^2 [y]_{x/2}^{3-x} \, dx \\
 &= \int_0^1 \left(\frac{3}{2}x \right) \, dx + \int_1^2 \left(3 - \frac{3}{2}x \right) \, dx \\
 &= \left[\frac{3}{4}x^2 \right]_0^1 + \left[3x - \frac{3}{4}x^2 \right]_1^2 = \frac{3}{2}
 \end{aligned}$$



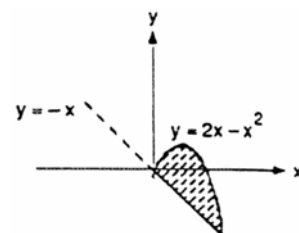
$$\begin{aligned}
 12. \quad & \int_0^1 \int_{-x}^{\sqrt{x}} 1 \, dy \, dx + \int_1^4 \int_{x-2}^{\sqrt{x}} 1 \, dy \, dx \\
 &= \int_0^1 [y]_{-x}^{\sqrt{x}} \, dx + \int_1^4 [y]_{x-2}^{\sqrt{x}} \, dx \\
 &= \int_0^1 (\sqrt{x} + x) \, dx + \int_1^4 (\sqrt{x} - x + 2) \, dx \\
 &= \left[\frac{2}{3} x^{3/2} + \frac{1}{2} x^2 \right]_0^1 + \left[\frac{2}{3} x^{3/2} - \frac{1}{2} x^2 + 2x \right]_1^4 = \frac{13}{3}
 \end{aligned}$$



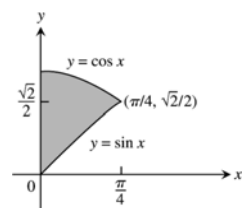
$$\begin{aligned}
 13. \quad & \int_0^6 \int_{y^2/3}^{2y} dx \, dy = \int_0^6 \left(2y - \frac{y^2}{3} \right) dy = \left[y^2 - \frac{y^3}{9} \right]_0^6 \\
 &= 36 - \frac{216}{9} = 12
 \end{aligned}$$



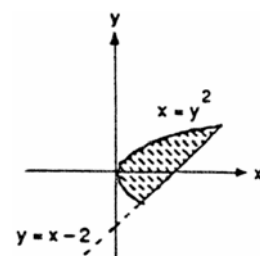
$$\begin{aligned}
 14. \quad & \int_0^3 \int_{-x}^{2x-x^2} dy \, dx = \int_0^3 (3x - x^2) \, dx = \left[\frac{3}{2} x^2 - \frac{1}{3} x^3 \right]_0^3 \\
 &= \frac{27}{2} - 9 = \frac{9}{2}
 \end{aligned}$$



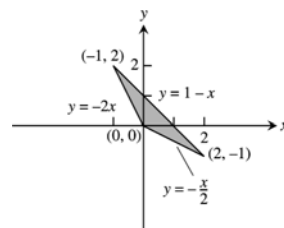
$$\begin{aligned}
 15. \quad & \int_0^{\pi/4} \int_{\sin x}^{\cos x} dy \, dx \\
 &= \int_0^{\pi/4} (\cos x - \sin x) \, dx = [\sin x + \cos x]_0^{\pi/4} \\
 &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) - (0 + 1) = \sqrt{2} - 1
 \end{aligned}$$



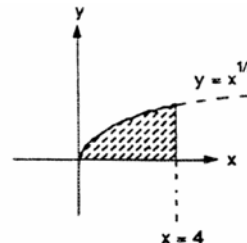
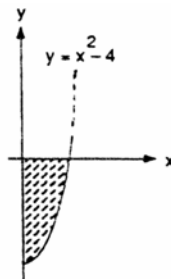
$$\begin{aligned}
 16. \quad & \int_{-1}^2 \int_{y^2}^{y+2} dx \, dy = \int_{-1}^2 (y + 2 - y^2) \, dy \\
 &= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2 = \left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \\
 &= 5 - \frac{1}{2} = \frac{9}{2}
 \end{aligned}$$



$$\begin{aligned}
 17. \quad & \int_{-1}^0 \int_{-2x}^{1-x} dy \, dx + \int_0^2 \int_{-x/2}^{1-x} dy \, dx \\
 &= \int_{-1}^0 (1 + x) \, dx + \int_0^2 \left(1 - \frac{x}{2} \right) dx \\
 &= \left[x + \frac{x^2}{2} \right]_{-1}^0 + \left[x - \frac{x^2}{4} \right]_0^2 = -\left(-1 + \frac{1}{2} \right) + (2 - 1) = \frac{3}{2}
 \end{aligned}$$



$$\begin{aligned}
 18. \quad & \int_0^2 \int_{x^2-4}^0 dy dx + \int_0^4 \int_0^{\sqrt{x}} dy dx \\
 &= \int_0^2 (4 - x^2) dx + \int_0^4 x^{1/2} dx \\
 &= \left[4x - \frac{x^3}{3} \right]_0^2 + \left[\frac{2}{3} x^{3/2} \right]_0^4 = \left(8 - \frac{8}{3} \right) + \frac{16}{3} = \frac{32}{3}
 \end{aligned}$$



$$\begin{aligned}
 19. \quad (a) \quad \text{average} &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \sin(x+y) dy dx = \frac{1}{\pi^2} \int_0^\pi [-\cos(x+y)]_0^\pi dx = \frac{1}{\pi^2} \int_0^\pi [-\cos(x+\pi) + \cos x] dx \\
 &= \frac{1}{\pi^2} [-\sin(x+\pi) + \sin x]_0^\pi = \frac{1}{\pi^2} [(-\sin 2\pi + \sin \pi) - (-\sin \pi + \sin 0)] = 0 \\
 (b) \quad \text{average} &= \frac{1}{\left(\frac{\pi}{2}\right)^2} \int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y) dy dx = \frac{2}{\pi^2} \int_0^{\pi/2} [-\cos(x+y)]_0^{\pi/2} dx = \frac{2}{\pi^2} \int_0^{\pi/2} [-\cos\left(x + \frac{\pi}{2}\right) + \cos x] dx \\
 &= \frac{2}{\pi^2} [-\sin\left(x + \frac{\pi}{2}\right) + \sin x]_0^{\pi/2} = \frac{2}{\pi^2} \left[\left(-\sin \frac{3\pi}{2} + \sin \pi\right) - \left(-\sin \frac{\pi}{2} + \sin 0\right) \right] = \frac{4}{\pi^2}
 \end{aligned}$$

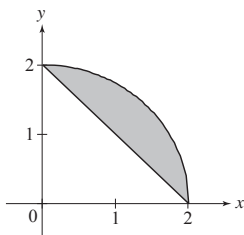
$$20. \quad \text{average value over the square} = \int_0^1 \int_0^1 xy dy dx = \int_0^1 \left[\frac{xy^2}{2} \right]_0^1 dx = \int_0^1 \frac{x}{2} dx = \frac{1}{4} = 0.25;$$

$$\begin{aligned}
 \text{average value over the quarter circle} &= \frac{1}{\left(\frac{\pi}{4}\right)} \int_0^1 \int_0^{\sqrt{1-x^2}} xy dy dx = \frac{4}{\pi} \int_0^1 \left[\frac{xy^2}{2} \right]_0^{\sqrt{1-x^2}} dx = \frac{2}{\pi} \int_0^1 (x - x^3) dx \\
 &= \frac{2}{\pi} \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2\pi} \approx 0.159. \text{ The average value over the square is larger.}
 \end{aligned}$$

$$21. \quad \text{average height} = \frac{1}{4} \int_0^2 \int_0^2 (x^2 + y^2) dy dx = \frac{1}{4} \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_0^2 dx = \frac{1}{4} \int_0^2 \left(2x^2 + \frac{8}{3} \right) dx = \frac{1}{2} \left[\frac{x^3}{3} + \frac{4x}{3} \right]_0^2 = \frac{8}{3}$$

$$\begin{aligned}
 22. \quad \text{average} &= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2 \ln 2} \int_{\ln 2}^{2 \ln 2} \frac{1}{xy} dy dx = \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2 \ln 2} \left[\frac{\ln y}{x} \right]_{\ln 2}^{2 \ln 2} dx = \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2 \ln 2} \frac{1}{x} (\ln 2 + \ln \ln 2 - \ln \ln 2) dx \\
 &= \left(\frac{1}{\ln 2} \right) \int_{\ln 2}^{2 \ln 2} \frac{dx}{x} = \left(\frac{1}{\ln 2} \right) [\ln x]_{\ln 2}^{2 \ln 2} = \left(\frac{1}{\ln 2} \right) (\ln 2 + \ln \ln 2 - \ln \ln 2) = 1
 \end{aligned}$$

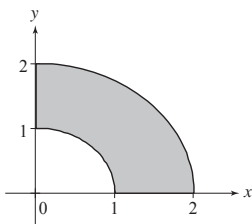
23. The region R is shaded in the following figure.



$$\iint_R dA = \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} 1 dy dx = \int_0^2 \left(\sqrt{4-x^2} - (2-x) \right) dx = \left(\frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \frac{x}{2} + \frac{x^2}{2} - 2x \right) \bigg|_0^2 = \pi - 2, \text{ where}$$

we use integration by parts with $u = \sqrt{4-x^2}$ and $dv = 1/2$ to find $\int \sqrt{4-x^2} dx$. Geometrically, the region R is a quarter of a circle of radius 2 with a triangle of area 2 removed, giving area $\pi - 2$.

24. The area of the region R is 4 times the shaded in the following figure.



The area integral will be easy to compute in polar coordinates, but in rectangular coordinates the calculation is awkward.

$$\iint_R dA = 4 \left[\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} 1 \, dy \, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} 1 \, dy \, dx \right] = 4 \left[\left(\frac{\sqrt{3}}{2} + \frac{\pi}{12} \right) + \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \right] = 4 \frac{3\pi}{4} = 3\pi$$

(As in Exercise 23, use integration by parts to evaluate the integrals $\int \sqrt{4-x^2} \, dx$ and $\int \sqrt{1-x^2} \, dx$.)

Geometrically the area is the difference between the area of a circle of radius 2 and the area of a circle of radius 1, or $4\pi - \pi = 3\pi$.

$$\begin{aligned} 25. \quad \int_{-5}^5 \int_{-2}^0 \frac{10,000e^y}{1+\frac{|x|}{2}} \, dy \, dx &= 10,000(1-e^{-2}) \int_{-5}^5 \frac{dx}{1+\frac{|x|}{2}} = 10,000(1-e^{-2}) \left[\int_{-5}^0 \frac{dx}{1-\frac{x}{2}} + \int_0^5 \frac{dx}{1+\frac{x}{2}} \right] \\ &= 10,000(1-e^{-2}) \left[-2 \ln\left(1-\frac{x}{2}\right) \right]_{-5}^0 + 10,000(1-e^{-2}) \left[2 \ln\left(1+\frac{x}{2}\right) \right]_0^5 \\ &= 10,000(1-e^{-2}) \left[2 \ln\left(1+\frac{5}{2}\right) \right] + 10,000(1-e^{-2}) \left[2 \ln\left(1+\frac{5}{2}\right) \right] = 40,000(1-e^{-2}) \ln\left(\frac{7}{2}\right) \approx 43,329 \end{aligned}$$

$$\begin{aligned} 26. \quad \int_0^1 \int_{y^2}^{2y-y^2} 100(y+1) \, dx \, dy &= \int_0^1 [100(y+1)x]_{y^2}^{2y-y^2} \, dy = \int_0^1 100(y+1)(2y-2y^2) \, dy = 200 \int_0^1 (y-y^3) \, dy \\ &= 200 \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = (200) \left(\frac{1}{4} \right) = 50 \end{aligned}$$

27. Let (x_i, y_i) be the location of the weather station in county i for $i = 1, \dots, 254$. The average temperature in

Texas at time t_0 is approximately $\frac{\sum_{i=1}^{254} T(x_i, y_i) \Delta A_i}{A}$, where $T(x_i, y_i)$ is the temperature at time t_0 at the weather station in county i , ΔA_i is the area of country i , and A is the area of Texas.

$$\begin{aligned} 28. \quad \text{Let } y = f(x) \text{ be a nonnegative, continuous function on } [a, b], \text{ then } A &= \iint_R dA = \int_a^b \int_0^{f(x)} dy \, dx = \int_a^b [y]_0^{f(x)} \, dx \\ &= \int_a^b f(x) \, dx \end{aligned}$$

29. Since f is continuous on R , if $m \leq f(x, y) \leq M$, property 3(b) of double integrals gives us

$$\iint_R m \, dA \leq \iint_R f(x, y) \, dA \leq \iint_R M \, dA \quad \text{and hence} \quad mA(R) \leq \iint_R f(x, y) \, dA \leq MA(R).$$

30. If $f(x, y)$ is positive at some point P in R or on the boundary of R then by the continuity of f there is a disk of positive radius around P (or if P is on the boundary, the intersection of such a disk with R) on which $f(x, y)$ is positive. This sub-region will make a positive contribution to the area $\iint_R f(x, y) dA$, and since $f(x, y)$ is never negative, $\iint_R f(x, y) dA$ will be greater than 0. This contradicts our assumption that $\iint_R f(x, y) dA = 0$, so $f(x, y)$ is positive nowhere on R and is thus equal to 0 at every point of R .

15.4 DOUBLE INTEGRALS IN POLAR FORM

1. $x^2 + y^2 = 9^2 \Rightarrow r = 9 \Rightarrow \frac{\pi}{2} \leq \theta \leq 2\pi, 0 \leq r \leq 9$
2. $x^2 + y^2 = 1^2 \Rightarrow r = 1, x^2 + y^2 = 4^2 \Rightarrow r = 4 \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 4$
3. $y = x \Rightarrow \theta = \frac{\pi}{4}, y = -x \Rightarrow \theta = \frac{3\pi}{4}, y = 1 \Rightarrow r = \csc \theta \Rightarrow \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, 0 \leq r \leq \csc \theta$
4. $x = 1 \Rightarrow r = \sec \theta, y = \sqrt{3}x \Rightarrow \theta = \frac{\pi}{3} \Rightarrow 0 \leq \theta \leq \frac{\pi}{3}, 0 \leq r \leq \sec \theta$
5. $x^2 + y^2 = 1^2 \Rightarrow r = 1, x = 2\sqrt{3} \Rightarrow r = 2\sqrt{3} \sec \theta, y = 2 \Rightarrow r = 2 \csc \theta;$
 $2\sqrt{3} \sec \theta = 2 \csc \theta \Rightarrow \theta = \frac{\pi}{6} \Rightarrow 0 \leq \theta \leq \frac{\pi}{6}, 1 \leq r \leq 2\sqrt{3} \sec \theta; \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 2\sqrt{3} \csc \theta$
6. $x^2 + y^2 = 2^2 \Rightarrow r = 2, x = 1 \Rightarrow r = \sec \theta; 2 = \sec \theta \Rightarrow \theta = \frac{\pi}{3} \text{ or } \theta = -\frac{\pi}{3} \Rightarrow -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}, \sec \theta \leq r \leq 2$
7. $x^2 + y^2 = 2x \Rightarrow r = 2 \cos \theta \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 2 \cos \theta$
8. $x^2 + y^2 = 2y \Rightarrow r = 2 \sin \theta \Rightarrow 0 \leq \theta \leq \pi, 0 \leq r \leq 2 \sin \theta$
9. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy dx = \int_0^{\pi} \int_0^1 r dr d\theta = \frac{1}{2} \int_0^{\pi} d\theta = \frac{\pi}{2}$
10. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx dy = \int_0^{\pi/2} \int_0^1 r^3 dr d\theta = \frac{1}{4} \int_0^{\pi/2} d\theta = \frac{\pi}{8}$
11. $\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2) dx dy = \int_0^{\pi/2} \int_0^2 r^3 dr d\theta = 4 \int_0^{\pi/2} d\theta = 2\pi$
12. $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy dx = \int_0^{2\pi} \int_0^a r dr d\theta = \frac{a^2}{2} \int_0^{2\pi} d\theta = \pi a^2$
13. $\int_0^6 \int_0^y x dx dy = \int_{\pi/4}^{\pi/2} \int_0^{6 \csc \theta} r^2 \cos \theta dr d\theta = 72 \int_{\pi/4}^{\pi/2} \cot \theta \csc^2 \theta d\theta = -36 \left[\cot^2 \theta \right]_{\pi/4}^{\pi/2} = 36$

$$14. \int_0^2 \int_0^x y \, dy \, dx = \int_0^{\pi/4} \int_0^{2 \sec \theta} r^2 \sin \theta \, dr \, d\theta = \frac{8}{3} \int_0^{\pi/4} \tan \theta \sec^2 \theta \, d\theta = \frac{4}{3}$$

$$15. \int_1^{\sqrt{3}} \int_1^x dy \, dx = \int_{\pi/6}^{\pi/4} \int_{\csc \theta}^{\sqrt{3} \sec \theta} r \, dr \, d\theta = \int_{\pi/6}^{\pi/4} \left(\frac{3}{2} \sec^2 \theta - \frac{1}{2} \csc^2 \theta \right) d\theta = \left[\frac{3}{2} \tan \theta + \frac{1}{2} \cot \theta \right]_{\pi/6}^{\pi/4} = 2 - \sqrt{3}$$

$$16. \int_{\sqrt{2}}^2 \int_{\sqrt{4-y^2}}^y dy \, dx = \int_{\pi/4}^{\pi/2} \int_2^{2 \csc \theta} r \, dr \, d\theta = \int_{\pi/4}^{\pi/2} (2 \csc^2 \theta - 2) d\theta = \left[-2 \cot \theta - \frac{1}{2} \theta \right]_{\pi/4}^{\pi/2} = 2 - \frac{\pi}{2}$$

$$17. \int_{-1}^0 \int_{-\sqrt{1-x^2}}^0 \frac{2}{1+\sqrt{x^2+y^2}} dy \, dx = \int_{\pi}^{3\pi/2} \int_0^1 \frac{2r}{1+r} dr \, d\theta = 2 \int_{\pi}^{3\pi/2} \int_0^1 \left(1 - \frac{1}{1+r} \right) dr \, d\theta = 2 \int_{\pi}^{3\pi/2} (1 - \ln 2) d\theta = (1 - \ln 2)\pi$$

$$18. \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} dy \, dx = 4 \int_0^{\pi/2} \int_0^1 \frac{2r}{(1+r^2)^2} dr \, d\theta = 4 \int_0^{\pi/2} \left[-\frac{1}{1+r^2} \right]_0^1 d\theta = 2 \int_0^{\pi/2} d\theta = \pi$$

$$19. \int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}} dx \, dy = \int_0^{\pi/2} \int_0^{\ln 2} r e^r dr \, d\theta = \int_0^{\pi/2} (2 \ln 2 - 1) d\theta = \frac{\pi}{2} (2 \ln 2 - 1)$$

$$20. \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1) dx \, dy = 4 \int_0^{\pi/2} \int_0^1 \ln(r^2 + 1) r \, dr \, d\theta = 2 \int_0^{\pi/2} (\ln 4 - 1) d\theta = \pi(\ln 4 - 1)$$

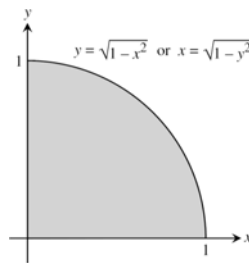
$$21. \int_0^1 \int_x^{\sqrt{2-x^2}} (x+2y) dy \, dx = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} (r \cos \theta + 2r \sin \theta) r \, dr \, d\theta = \int_{\pi/4}^{\pi/2} \left[\frac{r^3}{3} \cos \theta + \frac{2r^3}{3} \sin \theta \right]_0^{\sqrt{2}} d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(\frac{2\sqrt{2}}{3} \cos \theta + \frac{4\sqrt{2}}{3} \sin \theta \right) d\theta = \left[\frac{2\sqrt{2}}{3} \sin \theta - \frac{4\sqrt{2}}{3} \cos \theta \right]_{\pi/4}^{\pi/2} = \frac{2(1+\sqrt{2})}{3}$$

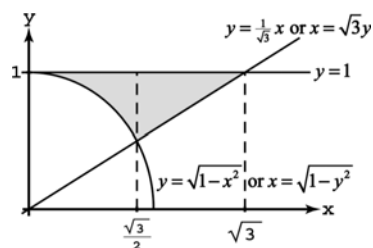
$$22. \int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2} dy \, dx = \int_0^{\pi/4} \int_{\sec \theta}^{2 \cos \theta} \frac{1}{r^4} r \, dr \, d\theta = \int_0^{\pi/4} \left[-\frac{1}{2r^2} \right]_{\sec \theta}^{2 \cos \theta} d\theta = \int_0^{\pi/4} \left(\frac{1}{2} \cos^2 \theta - \frac{1}{8} \sec^2 \theta \right) d\theta$$

$$= \left[\frac{1}{4} \theta + \frac{1}{8} \sin 2\theta - \frac{1}{8} \tan \theta \right]_0^{\pi/4} = \frac{\pi}{16}$$

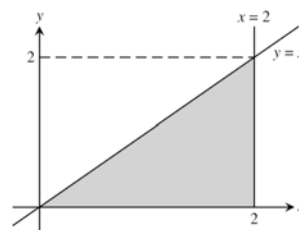
$$23. \int_0^1 \int_0^{\sqrt{1-x^2}} x y \, dy \, dx \text{ or } \int_0^1 \int_0^{\sqrt{1-y^2}} x y \, dy \, dx$$



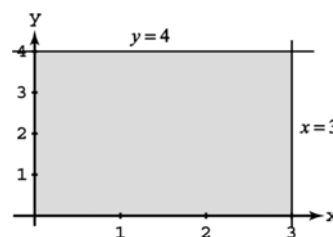
$$24. \int_{1/2}^1 \int_{\sqrt{1-y^2}}^{\sqrt{3}y} x \, dx \, dy \text{ or } \int_0^{\sqrt{3}/2} \int_{\sqrt{1-x^2}}^1 x \, dy \, dx + \int_{\sqrt{3}/2}^1 \int_{x/\sqrt{3}}^1 x \, dy \, dx$$



$$25. \int_0^2 \int_0^x y^2 (x^2 + y^2) \, dy \, dx \text{ or } \int_0^2 \int_y^2 y^2 (x^2 + y^2) \, dx \, dy$$



$$26. \int_0^3 \int_0^4 (x^2 + y^2)^3 \, dy \, dx \text{ or } \int_0^4 \int_0^3 (x^2 + y^2)^3 \, dx \, dy$$



$$27. \int_0^{\pi/2} \int_0^{2\sqrt{2-\sin 2\theta}} r \, dr \, d\theta = 2 \int_0^{\pi/2} (2 - \sin 2\theta) \, d\theta = 2(\pi - 1)$$

$$28. A = 2 \int_0^{\pi/2} \int_1^{1+\cos \theta} r \, dr \, d\theta = \int_0^{\pi/2} (2 \cos \theta + \cos^2 \theta) \, d\theta = \frac{8+\pi}{4}$$

$$29. A = 2 \int_0^{\pi/6} \int_0^{12 \cos 3\theta} r \, dr \, d\theta = 144 \int_0^{\pi/6} \cos^2 3\theta \, d\theta = 12\pi$$

$$30. A = \int_0^{2\pi} \int_0^{4\theta/3} r \, dr \, d\theta = \frac{8}{9} \int_0^{2\pi} \theta^2 \, d\theta = \frac{64\pi^3}{27}$$

$$31. A = \int_0^{\pi/2} \int_0^{1+\sin \theta} r \, dr \, d\theta = \frac{1}{2} \int_0^{\pi/2} \left(\frac{3}{2} + 2 \sin \theta - \frac{\cos 2\theta}{2} \right) \, d\theta = \frac{3\pi}{8} + 1$$

$$32. A = 4 \int_0^{\pi/2} \int_0^{1-\cos \theta} r \, dr \, d\theta = 2 \int_0^{\pi/2} \left(\frac{3}{2} - 2 \cos \theta + \frac{\cos 2\theta}{2} \right) \, d\theta = \frac{3\pi}{2} - 4$$

$$33. \text{average} = \frac{4}{\pi a^2} \int_0^{\pi/2} \int_0^a r \sqrt{a^2 - r^2} \, dr \, d\theta = \frac{4}{3\pi a^2} \int_0^{\pi/2} a^3 \, d\theta = \frac{2a}{3}$$

$$34. \text{average} = \frac{4}{\pi a^2} \int_0^{\pi/2} \int_0^a r^2 \, dr \, d\theta = \frac{4}{3\pi a^2} \int_0^{\pi/2} a^3 \, d\theta = \frac{2a}{3}$$

$$35. \text{ average} = \frac{1}{\pi a^2} \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2} dy dx = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a r^2 dr d\theta = \frac{a}{3\pi} \int_0^{2\pi} d\theta = \frac{2a}{3}$$

$$36. \text{ average} = \frac{1}{\pi} \iint_R [(1-x)^2 + y^2] dy dx = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 [(1-r\cos\theta)^2 + r^2 \sin^2\theta] r dr d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (r^3 - 2r^2 \cos\theta + r) dr d\theta = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{3}{4} - \frac{2\cos\theta}{3} \right) d\theta = \frac{1}{\pi} \left[\frac{3}{4}\theta - \frac{2\sin\theta}{3} \right]_0^{2\pi} = \frac{3}{2}$$

$$37. \int_0^{2\pi} \int_1^{\sqrt{e}} \left(\frac{\ln r^2}{r} \right) r dr d\theta = \int_0^{2\pi} \int_1^{\sqrt{e}} 2 \ln r dr d\theta = 2 \int_0^{2\pi} [r \ln r - r]_1^{e^{1/2}} d\theta = 2 \int_0^{2\pi} \sqrt{e} \left[\left(\frac{1}{2} - 1 \right) + 1 \right] d\theta = 2\pi(2 - \sqrt{e})$$

$$38. \int_0^{2\pi} \int_1^e \left(\frac{\ln r^2}{r} \right) dr d\theta = \int_0^{2\pi} \int_1^e \left(\frac{2 \ln r}{r} \right) dr d\theta = \int_0^{2\pi} [(\ln r)^2]_1^e d\theta = \int_0^{2\pi} d\theta = 2\pi$$

$$39. V = 2 \int_0^{\pi/2} \int_1^{1+\cos\theta} r^2 \cos\theta dr d\theta = \frac{2}{3} \int_0^{\pi/2} (3 \cos^2\theta + 3 \cos^3\theta + \cos^4\theta) d\theta$$

$$= \frac{2}{3} \left[\frac{15\theta}{8} + \sin 2\theta + 3 \sin\theta - \sin^3\theta + \frac{\sin 4\theta}{32} \right]_0^{\pi/2} = \frac{4}{3} + \frac{5\pi}{8}$$

$$40. V = 4 \int_0^{\pi/4} \int_0^{\sqrt{2\cos 2\theta}} r \sqrt{2-r^2} dr d\theta = -\frac{4}{3} \int_0^{\pi/4} [(2-2\cos 2\theta)^{3/2} - 2^{3/2}] d\theta$$

$$= \frac{2\pi\sqrt{2}}{3} - \frac{32}{3} \int_0^{\pi/4} (1-\cos^2\theta) \sin\theta d\theta = \frac{2\pi\sqrt{2}}{3} - \frac{32}{3} \left[\frac{\cos^3\theta}{3} - \cos\theta \right]_0^{\pi/4} = \frac{6\pi\sqrt{2}+40\sqrt{2}-64}{9}$$

$$41. (a) I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^{\pi/2} \int_0^\infty (e^{-r^2}) r dr d\theta = \int_0^{\pi/2} \left[\lim_{b \rightarrow \infty} \int_0^b r e^{-r^2} dr \right] d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \lim_{b \rightarrow \infty} (e^{-b^2} - 1) d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$$

$$(b) \lim_{x \rightarrow \infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} dt = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = \left(\frac{2}{\sqrt{\pi}} \right) \left(\frac{\sqrt{\pi}}{2} \right) = 1, \text{ from part (a)}$$

$$42. \int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2} dx dy = \int_0^{\pi/2} \int_0^\infty \frac{r}{(1+r^2)^2} dr d\theta = \frac{\pi}{2} \lim_{b \rightarrow \infty} \int_0^b \frac{r}{(1+r^2)^2} dr = \frac{\pi}{4} \lim_{b \rightarrow \infty} \left[-\frac{1}{1+r^2} \right]_0^b = \frac{\pi}{4} \lim_{b \rightarrow \infty} \left(1 - \frac{1}{1+b^2} \right)$$

$$= \frac{\pi}{4}$$

$$43. \text{ Over the disk } x^2 + y^2 \leq \frac{3}{4}: \iint_R \frac{1}{1-x^2-y^2} dA = \int_0^{2\pi} \int_0^{\sqrt{3}/2} \frac{r}{1-r^2} dr d\theta = \int_0^{2\pi} \left[-\frac{1}{2} \ln(1-r^2) \right]_0^{\sqrt{3}/2} d\theta$$

$$= \int_0^{2\pi} \left(-\frac{1}{2} \ln \frac{1}{4} \right) d\theta = (\ln 2) \int_0^{2\pi} d\theta = \pi \ln 4$$

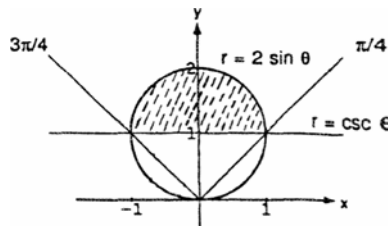
$$\text{Over the disk } x^2 + y^2 \leq 1: \iint_R \frac{1}{1-x^2-y^2} dA = \int_0^{2\pi} \int_0^1 \frac{r}{1-r^2} dr d\theta = \int_0^{2\pi} \left[\lim_{a \rightarrow 1^-} \int_0^a \frac{r}{1-r^2} dr \right] d\theta$$

$$= \int_0^{2\pi} \lim_{a \rightarrow 1^-} \left[-\frac{1}{2} \ln(1-a^2) \right] d\theta = 2\pi \cdot \lim_{a \rightarrow 1^-} \left[-\frac{1}{2} \ln(1-a^2) \right] = 2\pi \cdot \infty, \text{ so the integral does not exist over } x^2 + y^2 \leq 1$$

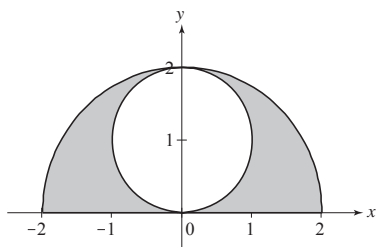
44. The area in polar coordinates is given by $A = \int_{\alpha}^{\beta} \int_0^{f(\theta)} r \, dr \, d\theta = \int_{\alpha}^{\beta} \left[\frac{r^2}{2} \right]_0^{f(\theta)} d\theta = \frac{1}{2} \int_{\alpha}^{\beta} f^2(\theta) \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$, where $r = f(\theta)$

$$\begin{aligned} 45. \text{ average} &= \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \left[(r \cos \theta - h)^2 + r^2 \sin^2 \theta \right] r \, dr \, d\theta = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \left(r^3 - 2r^2 h \cos \theta + rh^2 \right) dr \, d\theta \\ &= \frac{1}{\pi a^2} \int_0^{2\pi} \left(\frac{a^4}{4} - \frac{2a^3 h \cos \theta}{3} + \frac{a^2 h^2}{2} \right) d\theta = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{a^2}{4} - \frac{2ah \cos \theta}{3} + \frac{h^2}{2} \right) d\theta = \frac{1}{\pi} \left[\frac{a^2 \theta}{4} - \frac{2ah \sin \theta}{3} + \frac{h^2 \theta}{2} \right]_0^{2\pi} \\ &= \frac{1}{2} (a^2 + 2h^2) \end{aligned}$$

$$\begin{aligned} 46. \quad A &= \int_{\pi/4}^{3\pi/4} \int_{\csc \theta}^{2 \sin \theta} r \, dr \, d\theta \\ &= \frac{1}{2} \int_{\pi/4}^{3\pi/4} (4 \sin^2 \theta - \csc^2 \theta) \, d\theta \\ &= \frac{1}{2} [2\theta - \sin 2\theta + \cot \theta]_{\pi/4}^{3\pi/4} = \frac{\pi}{2} \end{aligned}$$



47. The region R is shaded in the graph below.



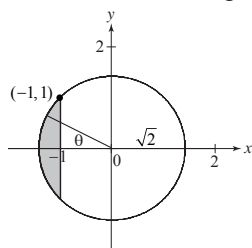
The polar equation of the outer circle is just $r = 2$. The inner circle is $x^2 + (y-1)^2 = 1$ or $x^2 + y^2 = 2y$. This is equivalent to $r^2 = 2r \sin \theta$ or $r = 2 \sin \theta$. The integrand is r in polar coordinates, so

$$\begin{aligned} \iint_R \sqrt{x^2 + y^2} \, dA &= \int_0^{\pi} \int_{2 \sin \theta}^2 r \cdot r \, dr \, d\theta \\ &= \int_0^{\pi} \left[\frac{r^3}{3} \right]_{2 \sin \theta}^2 d\theta = \int_0^{\pi} \frac{8}{3} (1 - \sin^3 \theta) \, d\theta \end{aligned}$$

Write the integrand as $\frac{8}{3} (1 - \sin \theta (1 - \cos^2 \theta))$. The indefinite integral is then $\frac{8}{3} \left(\theta + \cos \theta - \frac{1}{3} \cos^3 \theta \right)$ and the

$$\text{definite integral is } \frac{8}{3} \left(\theta + \cos \theta - \frac{1}{3} \cos^3 \theta \right) \Big|_0^{\pi} = \frac{8}{9} (3\pi - 4)$$

48. The region R is shaded in the graph below.



As θ ranges from $3\pi/4$ to $5\pi/4$ the ray at angle θ enters R at $r = \sec \theta$ and leaves R at $r = \sqrt{2}$. Thus

$$\begin{aligned} \iint_R (x^2 + y^2)^{-2} dA &= \int_{3\pi/4}^{5\pi/4} \int_{\sec \theta}^{\sqrt{2}} r^{-4} \cdot r dr d\theta \\ &= \int_{3\pi/4}^{5\pi/4} \left[-\frac{1}{2} r^{-2} \right]_{\sec \theta}^{\sqrt{2}} d\theta = \int_{3\pi/4}^{5\pi/4} \frac{1}{4} (2 \cos^2 \theta - 1) d\theta = \frac{1}{4} \int_{3\pi/4}^{5\pi/4} \cos 2\theta d\theta = \frac{1}{8} \sin 2\theta \bigg|_{3\pi/4}^{5\pi/4} = \frac{1}{4} \end{aligned}$$

49-52. Example CAS commands:

Maple:

```
f := (x,y) -> y/(x^2+y^2);
a,b := 0,1;
f1 := x -> x;
f2 := x -> 1;
plot3d( f(x,y), y=f1(x)..f2(x), x=a..b, axes=boxed, style=patchnogrid, shading=zhue, orientation=[0,180],
        title="#49(a) (Section 15.4) ); # (a)
q1 := eval( x=a, [x=r*cos(theta),y=r*sin(theta)] ); # (b)
q2 := eval( x=b, [x=r*cos(theta),y=r*sin(theta)] );
q3 := eval( y=f1(x), [x=r*cos(theta),y=r*sin(theta)] );
q4 := eval( y=f2(x), [x=r*cos(theta),y=r*sin(theta)] );
theta1 := solve( q3, theta );
theta2 := solve( q1, theta );
r1 := 0;
r2 := solve( q4, r );

plot3d(0,r=r1..r2, theta=theta1..theta2, axes=boxed, style=patchnogrid, shading=zhue, orientation=[-90,0],
        title="#49(c) (Section 15.4)");

fP := simplify(eval( f(x,y), [x=r*cos(theta),y=r*sin(theta)] )); # (d)
q5 := Int( Int( fP*r, r=r1..r2 ), theta=theta1..theta2 );
value( q5 );
```

Mathematica: (functions and bounds will vary)

For 49 and 50, begin by drawing the region of integration with the **FilledPlot** command.

```
Clear[x, y, r, t]
```

```
<<Graphics`FilledPlot`
```

```
FilledPlot[{x, 1}, {x, 0, 1}, AspectRatio → 1, AxesLabel → {x, y}];
```

The picture demonstrates that r goes from 0 to the line $y=1$ or $r = 1/\sin[t]$, while t goes from $\pi/4$ to $\pi/2$.

```
f:= y / (x^2 + y^2)
```

```
topolar={x → r Cos[t], y → r Sin[t]};
```

```
fp= f/.topolar //Simplify
```

```
Integrate[r fp, {t, π/4, π/2}, {r, 0, 1/Sin[t]}]
```

For 51 and 52, drawing the region of integration with the ImplicitPlot command.

```
Clear[x, y]
```

```
<<Graphics`ImplicitPlot`
```

```
ImplicitPlot[{x==y, x==2-y, y==0, y==1}, {x, 0, 2.1}, {y, 0, 1.1}];
```

The picture shows that as t goes from 0 to $\pi/4$, r goes from 0 to the line $x=2-y$. **Solve** will find the bound for r .

```
bdr=Solve[r Cos[t]==2-r Sin[t], r]//Simplify
```

```
f:=Sqrt[x + y]
```

```
topolar={x → r Cos[t], y → r Sin[t]};
```

```
fp= f/.topolar //Simplify
```

```
Integrate[r fp, {t, 0, π/4}, {r, 0, bdr[[1, 1, 2]]}]
```

15.5 TRIPLE INTEGRALS IN RECTANGULAR COORDINATES

$$1. \int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x, y, z) dy dz dx = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy dz dx = \int_0^1 \int_0^{1-x} (1-x-z) dz dx$$

$$= \int_0^1 \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} \right] dx = \int_0^1 \frac{(1-x)^2}{2} dx = \left[-\frac{(1-x)^3}{6} \right]_0^1 = \frac{1}{6}$$

$$2. \int_0^1 \int_0^2 \int_0^3 dz dy dx = \int_0^1 \int_0^2 3 dy dx = \int_0^1 6 dx = 6, \quad \int_0^2 \int_0^1 \int_0^3 dz dx dy, \quad \int_0^3 \int_0^2 \int_0^1 dx dy dz, \quad \int_0^2 \int_0^3 \int_0^1 dx dz dy,$$

$$\int_0^3 \int_0^1 \int_0^2 dy dx dz, \quad \int_0^1 \int_0^3 \int_0^2 dy dz dx$$

$$3. \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dz dy dx = \int_0^1 \int_0^{2-2x} \left(3-3x-\frac{3}{2}y \right) dy dx$$

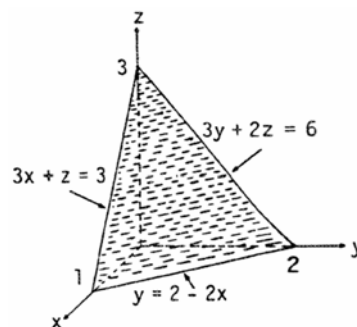
$$= \int_0^1 \left[3(1-x) \cdot 2(1-x) - \frac{3}{4} \cdot 4(1-x)^2 \right] dx$$

$$= 3 \int_0^1 (1-x)^2 dx = \left[-(1-x)^3 \right]_0^1 = 1,$$

$$\int_0^2 \int_0^{1-y/2} \int_0^{3-3x-3y/2} dz dx dy, \quad \int_0^1 \int_0^{3-3x} \int_0^{2-2x-2z/3} dy dz dx,$$

$$\int_0^3 \int_0^{1-z/3} \int_0^{2-2x-2z/3} dy dx dz, \quad \int_0^2 \int_0^{3-3y/2} \int_0^{1-y/2-z/3} dx dz dy,$$

$$\int_0^3 \int_0^{2-2z/3} \int_0^{1-y/2-z/3} dx dy dz$$



$$4. \int_0^2 \int_0^3 \int_0^{\sqrt{4-x^2}} dz dy dx = \int_0^2 \int_0^3 \sqrt{4-x^2} dy dx = \int_0^2 3\sqrt{4-x^2} dx = \frac{3}{2} \left[x\sqrt{4-x^2} + 4 \sin^{-1} \frac{x}{2} \right]_0^2 = 6 \sin^{-1} 1 = 3\pi,$$

$$\int_0^3 \int_0^2 \int_0^{\sqrt{4-x^2}} dz dx dy, \quad \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^3 dy dz dx, \quad \int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^3 dy dx dz, \quad \int_0^2 \int_0^3 \int_0^{\sqrt{4-z^2}} dx dy dz,$$

$$\int_0^3 \int_0^2 \int_0^{\sqrt{4-z^2}} dx dz dy$$

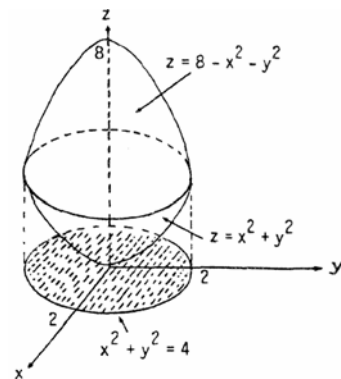
$$5. \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz dy dx = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz dy dx$$

$$= 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \left[8-2(x^2+y^2) \right] dy dx$$

$$= 8 \int_0^2 \int_0^{\sqrt{4-x^2}} (4-x^2-y^2) dy dx = 8 \int_0^{\pi/2} \int_0^2 (4-r^2) r dr d\theta$$

$$= 8 \int_0^{\pi/2} \left[2r^2 - \frac{r^4}{4} \right]_0^2 d\theta = 32 \int_0^{\pi/2} d\theta = 32 \left(\frac{\pi}{2} \right) = 16\pi,$$

$$\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz dx dy,$$



$$\int_{-2}^2 \int_{y^2}^4 \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx dz dy + \int_{-2}^2 \int_4^{8-y^2} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx dz dy, \quad \int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx dy dz + \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx dy dz,$$

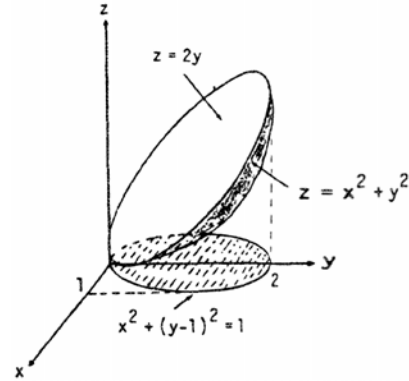
$$\int_{-2}^2 \int_{x^2}^4 \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy dz dx + \int_{-2}^2 \int_4^{8-x^2} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy dz dx, \quad \int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy dx dz + \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy dx dz$$

6. The projection of D onto the xy -plane has the boundary

$$x^2 + y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1, \text{ which is a circle.}$$

Therefore the two integrals are:

$$\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{x^2+y^2}^{2y} dz dx dy \quad \text{and} \quad \int_{-1}^1 \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \int_{x^2+y^2}^{2y} dz dy dx$$



7. $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dz dy dx = \int_0^1 \int_0^1 \left(x^2 + y^2 + \frac{1}{3} \right) dy dx = \int_0^1 \left(x^2 + \frac{2}{3} \right) dx = 1$
8. $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz dx dy = \int_0^{\sqrt{2}} \int_0^{3y} (8-2x^2-4y^2) dx dy = \int_0^{\sqrt{2}} \left[8x - \frac{2}{3} x^3 - 4xy^2 \right]_0^{3y} dy$
 $= \int_0^{\sqrt{2}} (24y - 18y^3 - 12y^3) dy = \left[12y^2 - \frac{15}{2} y^4 \right]_0^{\sqrt{2}} = 24 - 30 = -6$
9. $\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz} dx dy dz = \int_1^e \int_1^{e^2} \left[\frac{\ln x}{yz} \right]_1^{e^2} dy dz = \int_1^e \int_1^{e^2} \frac{3}{yz} dy dz = 3 \int_1^e \left[\frac{\ln y}{z} \right]_1^{e^2} dz = \int_1^e \frac{6}{z} dz = 6$
10. $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz dy dx = \int_0^1 \int_0^{3-3x} (3-3x-y) dy dx = \int_0^1 \left[(3-3x)^2 - \frac{1}{2} (3-3x)^2 \right] dx = \frac{9}{2} \int_0^1 (1-x)^2 dx$
 $= -\frac{3}{2} \left[(1-x)^3 \right]_0^1 = \frac{3}{2}$
11. $\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z dx dy dz = \int_0^{\pi/6} \int_0^1 5y \sin z dy dz = \frac{5}{2} \int_0^{\pi/6} \sin z dz = \frac{5(2-\sqrt{3})}{4}$
12. $\int_{-1}^1 \int_0^1 \int_0^2 (x+y+z) dy dx dz = \int_{-1}^1 \int_0^1 \left[xy + \frac{1}{2} y^2 + zy \right]_0^2 dx dz = \int_{-1}^1 \int_0^1 (2x+2+2z) dx dz$
 $= \int_{-1}^1 \left[x^2 + 2x + 2zx \right]_0^1 dz = \int_{-1}^1 (3+2z) dz = \left[3z + z^2 \right]_{-1}^1 = 6$
13. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz dy dx = \int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9-x^2} dy dx = \int_0^3 (9-x^2) dx = \left[9x - \frac{x^3}{3} \right]_0^3 = 18$
14. $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz dx dy = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x+y) dx dy = \int_0^2 \left[x^2 + xy \right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy = \int_0^2 (4-y^2)^{1/2} (2y) dy$
 $= \left[-\frac{2}{3} (4-y^2)^{3/2} \right]_0^2 = \frac{2}{3} (4)^{3/2} = \frac{16}{3}$

15. $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz \, dy \, dx = \int_0^1 \int_0^{2-x} (2-x-y) \, dy \, dx = \int_0^1 \left[(2-x)^2 - \frac{1}{2}(2-x)^2 \right] dx = \frac{1}{2} \int_0^1 (2-x)^2 \, dx$
 $= \left[-\frac{1}{6}(2-x)^3 \right]_0^1 = -\frac{1}{6} + \frac{8}{6} = \frac{7}{6}$
16. $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x \, dz \, dy \, dx = \int_0^1 \int_0^{1-x^2} x(1-x^2-y) \, dy \, dx = \int_0^1 x \left[(1-x^2)^2 - \frac{1}{2}(1-x^2) \right] dx = \int_0^1 \frac{1}{2} x(1-x^2)^2 \, dx$
 $= \left[-\frac{1}{12}(1-x^2)^3 \right]_0^1 = \frac{1}{12}$
17. $\int_0^\pi \int_0^\pi \int_0^\pi \cos(u+v+w) \, du \, dv \, dw = \int_0^\pi \int_0^\pi [\sin(w+v+\pi) - \sin(w+v)] \, dv \, dw$
 $= \int_0^\pi [(-\cos(w+2\pi) + \cos(w+\pi)) + (\cos(w+\pi) - \cos w)] \, dw$
 $= [-\sin(w+2\pi) + \sin(w+\pi) - \sin w + \sin(w+\pi)]_0^\pi = 0$
18. $\int_0^1 \int_1^{\sqrt{e}} \int_1^e s e^s \ln r \frac{(\ln t)^2}{t} \, dt \, dr \, ds = \int_0^1 \int_1^{\sqrt{e}} (s e^s \ln r) \left[\frac{1}{3}(\ln t)^3 \right]_1^e \, dr \, ds = \int_0^1 \int_1^{\sqrt{e}} \frac{s e^s}{3} \ln r \, dr \, ds$
 $= \int_0^1 \frac{s e^s}{3} [r \ln r - r]_1^{\sqrt{e}} \, ds = \frac{2-\sqrt{e}}{6} \int_0^1 s e^s \, ds = \frac{2-\sqrt{e}}{6} [s e^s - e^s]_0^1 = \frac{2-\sqrt{e}}{6}$
19. $\int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^x \, dx \, dt \, dv = \int_0^{\pi/4} \int_0^{\ln \sec v} \lim_{b \rightarrow -\infty} (e^{2t} - e^b) \, dt \, dv = \int_0^{\pi/4} \int_0^{\ln \sec v} e^{2t} \, dt \, dv$
 $= \int_0^{\pi/4} \left(\frac{1}{2} e^{2 \ln \sec v} - \frac{1}{2} \right) dv = \int_0^{\pi/4} \left(\frac{\sec^2 v}{2} - \frac{1}{2} \right) dv = \left[\frac{\tan v}{2} - \frac{v}{2} \right]_0^{\pi/4} = \frac{1}{2} - \frac{\pi}{8}$
20. $\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1} \, dp \, dq \, dr = \int_0^7 \int_0^2 \frac{q\sqrt{4-q^2}}{r+1} \, dq \, dr = \int_0^7 \frac{1}{3(r+1)} \left[-(4-q^2)^{3/2} \right]_0^2 \, dr = \frac{8}{3} \int_0^7 \frac{1}{r+1} \, dr = \frac{8 \ln 8}{3} = 8 \ln 2$
21. (a) $\int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} dy \, dz \, dx$ (b) $\int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} dy \, dx \, dz$ (c) $\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dy \, dz$
 (d) $\int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dz \, dy$ (e) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} dz \, dx \, dy$
22. (a) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy \, dz \, dx$ (b) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy \, dx \, dz$ (c) $\int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx \, dy \, dz$
 (d) $\int_{-1}^0 \int_0^{y^2} \int_0^1 dx \, dz \, dy$ (e) $\int_{-1}^0 \int_0^1 \int_0^{y^2} dz \, dx \, dy$
23. $V = \int_0^1 \int_{-1}^1 \int_0^{y^2} dz \, dy \, dx = \int_0^1 \int_{-1}^1 y^2 \, dy \, dx = \frac{2}{3} \int_0^1 dx = \frac{2}{3}$
24. $V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dy \, dz \, dx = \int_0^1 \int_0^{1-x} (2-2z) \, dz \, dx = \int_0^1 \left[2z - z^2 \right]_0^{1-x} dx = \int_0^1 (1-x^2) \, dx = \left[x - \frac{x^3}{3} \right]_0^1 = \frac{2}{3}$

25.
$$V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz \, dy \, dx = \int_0^4 \int_0^{\sqrt{4-x}} (2-y) \, dy \, dx = \int_0^4 \left[2\sqrt{4-x} - \left(\frac{4-x}{2} \right) \right] dx = \left[-\frac{4}{3}(4-x)^{3/2} + \frac{1}{4}(4-x)^2 \right]_0^4$$

$$= \frac{4}{3}(4)^{3/2} - \frac{1}{4}(16) = \frac{32}{3} - 4 = \frac{20}{3}$$
26.
$$V = 2 \int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dz \, dy \, dx = -2 \int_0^1 \int_{-\sqrt{1-x^2}}^0 y \, dy \, dx = \int_0^1 (1-x^2) \, dx = \frac{2}{3}$$
27.
$$V = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-\frac{3}{2}y} dz \, dy \, dx = \int_0^1 \int_0^{2-2x} \left(3-3x-\frac{3}{2}y \right) dy \, dx = \int_0^1 \left[6(1-x)^2 - \frac{3}{4} \cdot 4(1-x)^2 \right] dx$$

$$= \int_0^1 3(1-x)^2 \, dx = \left[-(1-x)^3 \right]_0^1 = 1$$
28.
$$V = \int_0^1 \int_0^{1-x} \int_0^{\cos(\pi x/2)} dz \, dy \, dx = \int_0^1 \int_0^{1-x} \cos\left(\frac{\pi x}{2}\right) dy \, dx = \int_0^1 \left(\cos\frac{\pi x}{2} \right) (1-x) \, dx$$

$$= \int_0^1 \cos\left(\frac{\pi x}{2}\right) dx - \int_0^1 x \cos\left(\frac{\pi x}{2}\right) dx = \left[\frac{2}{\pi} \sin\frac{\pi x}{2} \right]_0^1 - \frac{4}{\pi^2} \int_0^{\pi/2} u \cos u \, du = \frac{2}{\pi} - \frac{4}{\pi^2} [\cos u + u \sin u]_0^{\pi/2}$$

$$= \frac{2}{\pi} - \frac{4}{\pi^2} \left(\frac{\pi}{2} - 1 \right) = \frac{4}{\pi^2}$$
29.
$$V = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz \, dy \, dx = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2} \, dy \, dx = 8 \int_0^1 (1-x^2) \, dx = \frac{16}{3}$$
30.
$$V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz \, dy \, dx = \int_0^2 \int_0^{4-x^2} (4-x^2-y) \, dy \, dx = \int_0^2 \left[(4-x^2)^2 - \frac{1}{2}(4-x^2)^2 \right] dx$$

$$= \frac{1}{2} \int_0^2 (4-x^2)^2 \, dx = \int_0^2 \left(8-4x^2+\frac{x^4}{2} \right) dx = \frac{128}{15}$$
31.
$$V = \int_0^4 \int_0^{\sqrt{16-y^2}/2} \int_0^{4-y} dx \, dz \, dy = \int_0^4 \int_0^{\sqrt{16-y^2}/2} (4-y) \, dz \, dy = \int_0^4 \frac{\sqrt{16-y^2}}{2} (4-y) \, dy$$

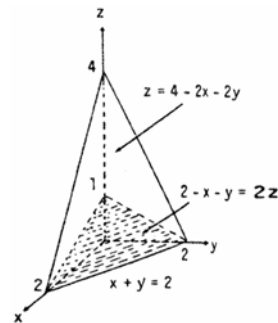
$$= \int_0^4 2\sqrt{16-y^2} \, dy - \frac{1}{2} \int_0^4 y\sqrt{16-y^2} \, dy = \left[y\sqrt{16-y^2} + 16 \sin^{-1} \frac{y}{4} \right]_0^4 + \left[\frac{1}{6}(16-y^2)^{3/2} \right]_0^4$$

$$= 16\left(\frac{\pi}{2}\right) - \frac{1}{6}(16)^{3/2} = 8\pi - \frac{32}{3}$$
32.
$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{3-x} dz \, dy \, dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-x) \, dy \, dx = 2 \int_{-2}^2 (3-x)\sqrt{4-x^2} \, dx$$

$$= 3 \int_{-2}^2 2\sqrt{4-x^2} \, dx - 2 \int_{-2}^2 x\sqrt{4-x^2} \, dx = 3 \left[x\sqrt{4-x^2} + 4 \sin^{-1} \frac{x}{2} \right]_{-2}^2 + \left[\frac{2}{3}(4-x^2)^{3/2} \right]_{-2}^2$$

$$= 12 \sin^{-1} 1 - 12 \sin^{-1}(-1) = 12\left(\frac{\pi}{2}\right) - 12\left(-\frac{\pi}{2}\right) = 12\pi$$

$$\begin{aligned}
 33. \quad & \int_0^2 \int_0^{2-x} \int_{(2-x-y)/2}^{4-2x-2y} dz \, dy \, dx = \int_0^2 \int_0^{2-x} \left(3 - \frac{3x}{2} - \frac{3y}{2} \right) dy \, dx \\
 &= \int_0^2 \left[3 \left(1 - \frac{x}{2} \right) (2-x) - \frac{3}{4} (2-x)^2 \right] dx = \int_0^2 \left[6 - 6x + \frac{3x^2}{2} - \frac{3(2-x)^2}{4} \right] dx \\
 &= \left[6x - 3x^2 + \frac{x^3}{2} + \frac{(2-x)^3}{4} \right]_0^2 = (12 - 12 + 4 + 0) - \frac{2^3}{4} = 2
 \end{aligned}$$



$$\begin{aligned}
 34. \quad V &= \int_0^4 \int_z^8 \int_z^{8-z} dx \, dy \, dz = \int_0^4 \int_z^8 (8-2z) \, dy \, dz = \int_0^4 (8-2z)(8-z) \, dz = \int_0^4 (64 - 24z + 2z^2) \, dz \\
 &= \left[64z - 12z^2 + \frac{2}{3}z^3 \right]_0^4 = \frac{320}{3}
 \end{aligned}$$

$$\begin{aligned}
 35. \quad V &= 2 \int_{-2}^2 \int_0^{\sqrt{4-x^2}/2} \int_0^{x+2} dz \, dy \, dx = 2 \int_{-2}^2 \int_0^{\sqrt{4-x^2}/2} (x+2) \, dy \, dx = \int_{-2}^2 (x+2)\sqrt{4-x^2} \, dx \\
 &= \int_{-2}^2 2\sqrt{4-x^2} \, dx + \int_{-2}^2 x\sqrt{4-x^2} \, dx = \left[x\sqrt{4-x^2} + 4 \sin^{-1} \frac{x}{2} \right]_{-2}^2 + \left[-\frac{1}{3}(4-x^2)^{3/2} \right]_{-2}^2 = 4\left(\frac{\pi}{2}\right) - 4\left(-\frac{\pi}{2}\right) = 4\pi
 \end{aligned}$$

$$\begin{aligned}
 36. \quad V &= 2 \int_0^1 \int_0^{1-y^2} \int_0^{x^2+y^2} dz \, dx \, dy = 2 \int_0^1 \int_0^{1-y^2} (x^2 + y^2) \, dx \, dy = 2 \int_0^1 \left[\frac{x^3}{3} + xy^2 \right]_0^{1-y^2} dy \\
 &= 2 \int_0^1 (1-y^2) \left[\frac{1}{3}(1-y^2)^2 + y^2 \right] dy = 2 \int_0^1 (1-y^2) \left(\frac{1}{3} + \frac{1}{3}y^2 + \frac{1}{3}y^4 \right) dy = \frac{2}{3} \int_0^1 (1-y^6) \, dy \\
 &= \frac{2}{3} \left[y - \frac{y^7}{7} \right]_0^1 = \left(\frac{2}{3} \right) \left(\frac{6}{7} \right) = \frac{4}{7}
 \end{aligned}$$

$$37. \quad \text{average} = \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 (x^2 + 9) \, dz \, dy \, dx = \frac{1}{8} \int_0^2 \int_0^2 (2x^2 + 18) \, dy \, dx = \frac{1}{8} \int_0^2 (4x^2 + 36) \, dx = \frac{31}{3}$$

$$38. \quad \text{average} = \frac{1}{2} \int_0^1 \int_0^1 \int_0^2 (x + y - z) \, dz \, dy \, dx = \frac{1}{2} \int_0^1 \int_0^1 (2x + 2y - 2) \, dy \, dx = \frac{1}{2} \int_0^1 (2x - 1) \, dx = 0$$

$$39. \quad \text{average} = \int_0^1 \int_0^1 \int_0^1 \left(x^2 + y^2 + z^2 \right) \, dz \, dy \, dx = \int_0^1 \int_0^1 \left(x^2 + y^2 + \frac{1}{3} \right) \, dy \, dx = \int_0^1 \left(x^2 + \frac{2}{3} \right) \, dx = 1$$

$$40. \quad \text{average} = \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 xyz \, dz \, dy \, dx = \frac{1}{4} \int_0^2 \int_0^2 xy \, dy \, dx = \frac{1}{2} \int_0^2 x \, dx = 1$$

$$\begin{aligned}
 41. \quad & \int_0^4 \int_0^1 \int_{2y}^2 \frac{4 \cos(x^2)}{2\sqrt{z}} \, dx \, dy \, dz = \int_0^4 \int_0^2 \int_0^{x/2} \frac{4 \cos(x^2)}{2\sqrt{z}} \, dy \, dx \, dz = \int_0^4 \int_0^2 \frac{x \cos(x^2)}{\sqrt{z}} \, dx \, dz = \int_0^4 \left(\frac{\sin 4}{2} \right) z^{-1/2} \, dz \\
 &= \left[(\sin 4) z^{1/2} \right]_0^4 = 2 \sin 4
 \end{aligned}$$

42. $\int_0^1 \int_0^1 \int_{x^2}^1 12xz e^{zy^2} dy dx dz = \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xz e^{zy^2} dx dy dz = \int_0^1 \int_0^1 6yz e^{zy^2} dy dz = \int_0^1 \left[3e^{zy^2} \right]_0^1 dz$
 $= 3 \int_0^1 (e^z - 1) dz = 3 \left[e^z - z \right]_0^1 = 3e - 6$
43. $\int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x} \sin(\pi y^2)}{y^2} dx dy dz = \int_0^1 \int_{\sqrt[3]{z}}^1 \frac{4\pi \sin(\pi y^2)}{y^2} dy dz = \int_0^1 \int_0^{y^3} \frac{4\pi \sin(\pi y^2)}{y^2} dz dy$
 $= \int_0^1 4\pi y \sin(\pi y^2) dy = \left[-2 \cos(\pi y^2) \right]_0^1 = -2(-1) + 2(1) = 4$
44. $\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z} dy dz dx = \int_0^2 \int_0^{4-x^2} \frac{x \sin 2z}{4-z} dz dx = \int_0^4 \int_0^{\sqrt{4-z}} \left(\frac{\sin 2z}{4-z} \right) x dx dz = \int_0^4 \left(\frac{\sin 2z}{4-z} \right) \frac{1}{2} (4-z) dz$
 $= \left[-\frac{1}{4} \cos 2z \right]_0^4 = \left[-\frac{1}{4} + \frac{1}{2} \sin^2 z \right]_0^4 = \frac{\sin^2 4}{2}$
45. $\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dz dy dx = \frac{4}{15} \Rightarrow \int_0^1 \int_0^{4-a-x^2} (4-x^2-y-a) dy dx = \frac{4}{15}$
 $\Rightarrow \int_0^1 \left[(4-a-x^2)^2 - \frac{1}{2} (4-a-x^2)^2 \right] dx = \frac{4}{15} \Rightarrow \frac{1}{2} \int_0^1 (4-a-x^2)^2 dx = \frac{4}{15} \Rightarrow \int_0^1 [(4-a)^2 - 2x^2(4-a) + x^4] dx$
 $= \frac{8}{15} \Rightarrow \left[(4-a)^2 x - \frac{2}{3} x^3 (4-a) + \frac{x^5}{5} \right]_0^1 = \frac{8}{15} \Rightarrow (4-a)^2 - \frac{2}{3} (4-a) + \frac{1}{5} = \frac{8}{15} \Rightarrow 15(4-a)^2 - 10(4-a) - 5 = 0$
 $\Rightarrow 3(4-a)^2 - 2(4-a) - 1 = 0 \Rightarrow [3(4-a)+1][(4-a)-1] = 0 \Rightarrow 4-a = -\frac{1}{3} \text{ or } 4-a = 1 \Rightarrow a = \frac{13}{3} \text{ or } a = 3$
46. The volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4abc\pi}{3}$ so that $\frac{4(1)(2)(c)\pi}{3} = 8\pi \Rightarrow c = 3$.
47. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where it is positive. These criteria are met by the points (x, y, z) such that $4x^2 + 4y^2 + z^2 - 4 \leq 0$ or $4x^2 + 4y^2 + z^2 \leq 4$, which is a solid ellipsoid centered at the origin.
48. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where it is negative. These criteria are met by the points (x, y, z) such that $1 - x^2 - y^2 - z^2 \geq 0$ or $x^2 + y^2 + z^2 \leq 1$, which is a solid sphere of radius 1 centered at the origin.

49-52. Example CAS commands:

Maple:

```
F := (x,y,z) -> x^2*y^2*z;
q1 := Int( Int( Int( F(x,y,z), y=-sqrt(1-x^2)..sqrt(1-x^2) ), x=-1..1 ), z=0..1);
value( q1 );
```

Mathematica: (functions and bounds will vary)

```
Clear[f, x, y, z];
f:= x^2 y^2 z
Integrate[f, {x,-1,1}, {y,-Sqrt[1-x^2], Sqrt[1-x^2]}, {z, 0, 1}]
```

N[%]

topolar={x → r Cos[t], y → r Sin[t]};

fp= f/.topolar //Simplify

Integrate[r fp, {t, 0, 2π}, {r, 0, 1}, {z, 0, 1}]

N[%]

15.6 MOMENTS AND CENTERS OF MASS

1. $M = \int_0^1 \int_x^{2-x^2} 3 \, dy \, dx = 3 \int_0^1 (2 - x^2 - x) \, dx = \frac{7}{2}; \quad M_y = \int_0^1 \int_x^{2-x^2} 3x \, dy \, dx = 3 \int_0^1 [xy]_x^{2-x^2} \, dx$
 $= 3 \int_0^1 (2x - x^3 - x^2) \, dx = \frac{5}{4}; \quad M_x = \int_0^1 \int_x^{2-x^2} 3y \, dy \, dx = \frac{3}{2} \int_0^1 [y^2]_x^{2-x^2} \, dx = \frac{3}{2} \int_0^1 (4 - 5x^2 + x^4) \, dx = \frac{19}{5}$
 $\Rightarrow \bar{x} = \frac{5}{14} \text{ and } \bar{y} = \frac{38}{35}$
2. $M = \delta \int_0^3 \int_0^3 dy \, dx = \delta \int_0^3 3 \, dx = 9\delta; \quad I_x = \delta \int_0^3 \int_0^3 y^2 \, dy \, dx = \delta \int_0^3 \left[\frac{y^3}{3} \right]_0^3 \, dx = 27\delta;$
 $I_y = \delta \int_0^3 \int_0^3 x^2 \, dy \, dx = \delta \int_0^3 [x^2 y]_0^3 \, dx = \delta \int_0^3 3x^2 \, dx = 27\delta$
3. $M = \int_0^2 \int_{y^2/2}^{4-y} dx \, dy = \int_0^2 \left(4 - y - \frac{y^2}{2} \right) dy = \frac{14}{3}; \quad M_y = \int_0^2 \int_{y^2/2}^{4-y} x \, dx \, dy = \frac{1}{2} \int_0^2 [x^2]_{y^2/2}^{4-y} \, dy$
 $= \frac{1}{2} \int_0^2 \left(16 - 8y + y^2 - \frac{y^4}{4} \right) dy = \frac{128}{3}; \quad M_x = \int_0^2 \int_{y^2/2}^{4-y} y \, dx \, dy = \int_0^2 \left(4y - y^2 - \frac{y^3}{2} \right) dy = \frac{10}{3} \Rightarrow \bar{x} = \frac{64}{35} \text{ and } \bar{y} = \frac{5}{7}$
4. $M = \int_0^3 \int_0^{3-x} dy \, dx = \int_0^3 (3 - x) \, dx = \frac{9}{2}; \quad M_y = \int_0^3 \int_0^{3-x} x \, dy \, dx = \int_0^3 [xy]_0^{3-x} \, dx = \int_0^3 (3x - x^2) \, dx = \frac{9}{2}$
 $\Rightarrow \bar{x} = 1 \text{ and } \bar{y} = 1, \text{ by symmetry}$
5. $M = \int_0^a \int_0^{\sqrt{a^2-x^2}} dy \, dx = \frac{\pi a^2}{4}; \quad M_y = \int_0^a \int_0^{\sqrt{a^2-x^2}} x \, dy \, dx = \int_0^a [xy]_0^{\sqrt{a^2-x^2}} \, dx = \int_0^a x\sqrt{a^2-x^2} \, dx = \frac{a^3}{3}$
 $\Rightarrow \bar{x} = \bar{y} = \frac{4a}{3\pi}, \text{ by symmetry}$
6. $M = \int_0^\pi \int_0^{\sin x} dy \, dx = \int_0^\pi \sin x \, dx = 2; \quad M_x = \int_0^\pi \int_0^{\sin x} y \, dy \, dx = \frac{1}{2} \int_0^\pi [y^2]_0^{\sin x} \, dx = \frac{1}{2} \int_0^\pi \sin^2 x \, dx$
 $= \frac{1}{4} \int_0^\pi (1 - \cos 2x) \, dx = \frac{\pi}{4} \Rightarrow \bar{x} = \frac{\pi}{2} \text{ and } \bar{y} = \frac{\pi}{8}$
7. $I_x = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y^2 \, dy \, dx = \int_{-2}^2 \left[\frac{y^3}{3} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \, dx = \frac{2}{3} \int_{-2}^2 (4 - x^2)^{3/2} \, dx = 4\pi; \quad I_y = 4\pi, \text{ by symmetry};$
 $I_o = I_x + I_y = 8\pi$
8. $I_y = \int_\pi^{2\pi} \int_0^{(\sin^2 x)/x^2} x^2 \, dy \, dx = \int_\pi^{2\pi} (\sin^2 x - 0) \, dx = \frac{1}{2} \int_\pi^{2\pi} (1 - \cos 2x) \, dx = \frac{\pi}{2}$

9. $M = \int_{-\infty}^0 \int_0^{e^x} dy \, dx = \int_{-\infty}^0 e^x \, dx = \lim_{b \rightarrow -\infty} \int_b^0 e^x \, dx = 1 - \lim_{b \rightarrow -\infty} e^b = 1$; $M_y = \int_{-\infty}^0 \int_0^{e^x} x \, dy \, dx = \int_{-\infty}^0 x e^x \, dx$
 $= \lim_{b \rightarrow -\infty} \int_b^0 x e^x \, dx = \lim_{b \rightarrow -\infty} \left[x e^x - e^x \right]_b^0 = -1 - \lim_{b \rightarrow -\infty} (b e^b - e^b) = -1$; $M_x = \int_{-\infty}^0 \int_0^{e^x} y \, dy \, dx = \frac{1}{2} \int_{-\infty}^0 e^{2x} \, dx$
 $= \frac{1}{2} \lim_{b \rightarrow -\infty} \int_b^0 e^{2x} \, dx = \frac{1}{4} \Rightarrow \bar{x} = -1 \text{ and } \bar{y} = \frac{1}{4}$
10. $M_y = \int_0^\infty \int_0^{e^{-x^2/2}} x \, dy \, dx = \lim_{b \rightarrow \infty} \int_0^b x e^{-x^2/2} \, dx = - \lim_{b \rightarrow \infty} \left[\frac{1}{e^{x^2/2}} - 1 \right]_0^b = 1$
11. $M = \int_0^2 \int_{-y}^{y-y^2} (x+y) \, dx \, dy = \int_0^2 \left[\frac{x^2}{2} + xy \right]_{-y}^{y-y^2} dy = \int_0^2 \left(\frac{y^4}{2} - 2y^3 + 2y^2 \right) dy = \left[\frac{y^5}{10} - \frac{y^4}{2} + \frac{2y^3}{3} \right]_0^2 = \frac{8}{15}$;
 $I_x = \int_0^2 \int_{-y}^{y-y^2} y^2 (x+y) \, dx \, dy = \int_0^2 \left[\frac{x^2 y^2}{2} + xy^3 \right]_{-y}^{y-y^2} dy = \int_0^2 \left(\frac{y^6}{2} - 2y^5 + 2y^4 \right) dy = \frac{64}{105}$;
12. $M = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \int_{4y^2}^{\sqrt{12-4y^2}} 5x \, dx \, dy = 5 \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \left[\frac{x^2}{2} \right]_{4y^2}^{\sqrt{12-4y^2}} dy = \frac{5}{2} \int_{-\sqrt{3}/2}^{\sqrt{3}/2} (12 - 4y^2 - 16y^4) dy = 23\sqrt{3}$
13. $M = \int_0^1 \int_x^{2-x} (6x+3y+3) \, dy \, dx = \int_0^1 \left[6xy + \frac{3}{2}y^2 + 3y \right]_x^{2-x} dx = \int_0^1 (12 - 12x^2) dx = 8$;
 $M_y = \int_0^1 \int_x^{2-x} x(6x+3y+3) \, dy \, dx = \int_0^1 (12x - 12x^3) dx = 3$; $M_x = \int_0^1 \int_x^{2-x} y(6x+3y+3) \, dy \, dx$
 $= \int_0^1 (14 - 6x - 6x^2 - 2x^3) dx = \frac{17}{2} \Rightarrow \bar{x} = \frac{3}{8} \text{ and } \bar{y} = \frac{17}{16}$
14. $M = \int_0^1 \int_{y^2}^{2y-y^2} (y+1) \, dx \, dy = \int_0^1 (2y - 2y^3) dy = \frac{1}{2}$; $M_x = \int_0^1 \int_{y^2}^{2y-y^2} y(y+1) \, dx \, dy = \int_0^1 (2y^2 - 2y^4) dy = \frac{4}{15}$;
 $M_y = \int_0^1 \int_{y^2}^{2y-y^2} x(y+1) \, dx \, dy = \int_0^1 (2y^2 - 2y^4) dy = \frac{4}{15} \Rightarrow \bar{x} = \frac{8}{15} \text{ and } \bar{y} = \frac{8}{15}$; $I_x = \int_0^1 \int_{y^2}^{2y-y^2} y^2 (y+1) \, dx \, dy$
 $= 2 \int_0^1 (y^3 - y^5) dy = \frac{1}{6}$
15. $M = \int_0^1 \int_0^6 (x+y+1) \, dx \, dy = \int_0^1 (6y+24) dy = 27$; $M_x = \int_0^1 \int_0^6 y(x+y+1) \, dx \, dy = \int_0^1 y(6y+24) dy = 14$;
 $M_y = \int_0^1 \int_0^6 x(x+y+1) \, dx \, dy = \int_0^1 (18y+90) dy = 99 \Rightarrow \bar{x} = \frac{11}{3} \text{ and } \bar{y} = \frac{14}{27}$;
 $I_y = \int_0^1 \int_0^6 x^2 (x+y+1) \, dx \, dy = 216 \int_0^1 \left(\frac{y}{3} + \frac{11}{6} \right) dy = 432$
16. $M = \int_{-1}^1 \int_{x^2}^1 (y+1) \, dy \, dx = - \int_{-1}^1 \left(\frac{x^4}{2} + x^2 - \frac{3}{2} \right) dx = \frac{32}{15}$; $M_x = \int_{-1}^1 \int_{x^2}^1 y(y+1) \, dy \, dx = \int_{-1}^1 \left(\frac{5}{6} - \frac{x^6}{3} - \frac{x^4}{2} \right) dx = \frac{48}{35}$;
 $M_y = \int_{-1}^1 \int_{x^2}^1 x(y+1) \, dy \, dx = \int_{-1}^1 \left(\frac{3x}{2} - \frac{x^5}{2} - x^3 \right) dx = 0 \Rightarrow \bar{x} = 0 \text{ and } \bar{y} = \frac{9}{14}$;
 $I_y = \int_{-1}^1 \int_{x^2}^1 x^2 (y+1) \, dy \, dx = \int_{-1}^1 \left(\frac{3x^2}{2} - \frac{x^6}{2} - x^4 \right) dx = \frac{16}{35}$

17. $M = \int_{-1}^1 \int_0^{x^2} (7y+1) dy dx = \int_{-1}^1 \left(\frac{7x^4}{2} + x^2 \right) dx = \frac{31}{15}$; $M_x = \int_{-1}^1 \int_0^{x^2} y(7y+1) dy dx = \int_{-1}^1 \left(\frac{7x^6}{3} + \frac{x^4}{2} \right) dx = \frac{13}{15}$;
 $M_y = \int_{-1}^1 \int_0^{x^2} x(7y+1) dy dx = \int_{-1}^1 \left(\frac{7x^5}{2} + x^3 \right) dx = 0 \Rightarrow \bar{x} = 0$ and $\bar{y} = \frac{13}{31}$;
 $I_y = \int_{-1}^1 \int_0^{x^2} x^2(7y+1) dy dx = \int_{-1}^1 \left(\frac{7x^6}{2} + x^4 \right) dx = \frac{7}{5}$
18. $M = \int_0^{20} \int_{-1}^1 \left(1 + \frac{x}{20} \right) dy dx = \int_0^{20} \left(2 + \frac{x}{10} \right) dx = 60$; $M_x = \int_0^{20} \int_{-1}^1 y \left(1 + \frac{x}{20} \right) dy dx = \int_0^{20} \left[\left(1 + \frac{x}{20} \right) \left(\frac{y^2}{2} \right) \right]_{-1}^1 dx = 0$;
 $M_y = \int_0^{20} \int_{-1}^1 x \left(1 + \frac{x}{20} \right) dy dx = \int_0^{20} \left(2x + \frac{x^2}{10} \right) dx = \frac{2000}{3} \Rightarrow \bar{x} = \frac{100}{9}$ and $\bar{y} = 0$;
 $I_x = \int_0^{20} \int_{-1}^1 y^2 \left(1 + \frac{x}{20} \right) dy dx = \frac{2}{3} \int_0^{20} \left(1 + \frac{x}{20} \right) dx = 20$
19. $M = \int_0^1 \int_{-y}^y (y+1) dx dy = \int_0^1 (2y^2 + 2y) dy = \frac{5}{3}$; $M_x = \int_0^1 \int_{-y}^y y(y+1) dx dy = 2 \int_0^1 (y^3 + y^2) dy = \frac{7}{6}$;
 $M_y = \int_0^1 \int_{-y}^y x(y+1) dx dy = \int_0^1 0 dy = 0 \Rightarrow \bar{x} = 0$ and $\bar{y} = \frac{7}{10}$; $I_x = \int_0^1 \int_{-y}^y y^2(y+1) dx dy$
 $= \left(\int_0^1 2y^4 + 2y^3 dy \right) = \frac{9}{10}$; $I_y = \int_0^1 \int_{-y}^y x^2(y+1) dx dy = \frac{1}{3} \int_0^1 (2y^4 + 2y^3) dy = \frac{3}{10} \Rightarrow I_o = I_x + I_y = \frac{6}{5}$
20. $M = \int_0^1 \int_{-y}^y (3x^2 + 1) dx dy = \int_0^1 (2y^3 + 2y) dy = \frac{3}{2}$; $M_x = \int_0^1 \int_{-y}^y y(3x^2 + 1) dx dy = \int_0^1 (2y^4 + 2y^2) dy = \frac{16}{15}$;
 $M_y = \int_0^1 \int_{-y}^y x(3x^2 + 1) dx dy = 0 \Rightarrow \bar{x} = 0$ and $\bar{y} = \frac{32}{45}$; $I_x = \int_0^1 \int_{-y}^y y^2(3x^2 + 1) dx dy = \int_0^1 (2y^5 + 2y^3) dy = \frac{5}{6}$;
 $I_y = \int_0^1 \int_{-y}^y x^2(3x^2 + 1) dx dy = 2 \int_0^1 \left(\frac{3}{5} y^5 + \frac{1}{3} y^3 \right) dy = \frac{11}{30} \Rightarrow I_o = I_x + I_y = \frac{6}{5}$
21. $I_x = \int_0^a \int_0^b \int_0^c (y^2 + z^2) dz dy dx = \int_0^a \int_0^b \left(cy^2 + \frac{c^3}{3} \right) dy dx = \int_0^a \left(\frac{cb^3}{3} + \frac{c^3b}{3} \right) dx = \frac{abc(b^2 + c^2)}{3} = \frac{M}{3}(b^2 + c^2)$ where
 $M = abc$; $I_y = \frac{M}{3}(a^2 + c^2)$ and $I_z = \frac{M}{3}(a^2 + b^2)$, by symmetry
22. The plane $z = \frac{4-2y}{3}$ is the top of the wedge $\Rightarrow I_x = \int_{-3}^3 \int_{-2}^4 \int_{-4/3}^{(4-2y)/3} (y^2 + z^2) dz dy dx$
 $= \int_{-3}^3 \int_{-2}^4 \left[\frac{8y^2}{3} - \frac{2y^3}{3} + \frac{8(2-y)^3}{81} + \frac{64}{81} \right] dy dx = \int_{-3}^3 \frac{104}{3} dx = 208$; $I_y = \int_{-3}^3 \int_{-2}^4 \int_{-4/3}^{(4-2y)/3} (x^2 + z^2) dz dy dx$
 $= \int_{-3}^3 \int_{-2}^4 \left[\frac{(4-2y)^3}{81} + \frac{x^2(4-2y)}{3} + \frac{4x^2}{3} + \frac{64}{81} \right] dy dx = \int_{-3}^3 \left(12x^2 + \frac{32}{3} \right) dx = 280$;
 $I_z = \int_{-3}^3 \int_{-2}^4 \int_{-4/3}^{(4-2y)/3} (x^2 + y^2) dz dy dx = \int_{-3}^3 \int_{-2}^4 (x^2 + y^2) \left(\frac{8}{3} - \frac{2y}{3} \right) dy dx = 12 \int_{-3}^3 (x^2 + 2) dx = 360$
23. $M = 4 \int_0^1 \int_0^1 \int_{4y^2}^4 dz dy dx = 4 \int_0^1 \int_0^1 (4 - 4y^2) dy dx = 16 \int_0^1 \frac{2}{3} dx = \frac{32}{3}$; $M_{xy} = 4 \int_0^1 \int_0^1 \int_{4y^2}^4 z dz dy dx$
 $= 2 \int_0^1 \int_0^1 (16 - 16y^4) dy dx = \frac{128}{5} \int_0^1 dx = \frac{128}{5} \Rightarrow \bar{z} = \frac{12}{5}$, and $\bar{x} = \bar{y} = 0$, by symmetry;

$$\begin{aligned}
I_x &= 4 \int_0^1 \int_0^1 \int_{4y^2}^4 (y^2 + z^2) dz dy dx = 4 \int_0^1 \int_0^1 \left[4y^2 + \frac{64}{3} - \left(4y^4 + \frac{64y^6}{3} \right) \right] dy dx = 4 \int_0^1 \frac{1976}{105} dx = \frac{7904}{105}; \\
I_y &= 4 \int_0^1 \int_0^1 \int_{4y^2}^4 (x^2 + z^2) dz dy dx = 4 \int_0^1 \int_0^1 \left[4x^2 + \frac{64}{3} - \left(4x^2 y^4 + \frac{64y^6}{3} \right) \right] dy dx = 4 \int_0^1 \left(\frac{8}{3} x^2 + \frac{128}{7} \right) dx = \frac{4832}{63}; \\
I_z &= 4 \int_0^1 \int_0^1 \int_{4y^2}^4 (x^2 + y^2) dz dy dx = 16 \int_0^1 \int_0^1 (x^2 - x^2 y^2 + y^2 - y^4) dy dx = 16 \int_0^1 \left(\frac{2x^2}{3} + \frac{2}{15} \right) dx = \frac{256}{45}
\end{aligned}$$

24. (a) $M = \int_{-2}^2 \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} \int_0^{2-x} dz dy dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} (2-x) dy dx = \int_{-2}^2 (2-x)\sqrt{4-x^2} dx = 4\pi;$

$M_{yz} = \int_{-2}^2 \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} \int_0^{2-x} x dz dy dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} x(2-x) dy dx = \int_{-2}^2 x(2-x)\sqrt{4-x^2} dx = -2\pi;$

$M_{xz} = \int_{-2}^2 \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} \int_0^{2-x} y dz dy dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} y(2-x) dy dx = \frac{1}{2} \int_{-2}^2 (2-x) \left[\frac{4-x^2}{4} - \frac{4-x^2}{4} \right] dx = 0$

$\Rightarrow \bar{x} = -\frac{1}{2}$ and $\bar{y} = 0$

(b) $M_{xy} = \int_{-2}^2 \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} \int_0^{2-x} z dz dy dx = \frac{1}{2} \int_{-2}^2 \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} (2-x)^2 dy dx = \frac{1}{2} \int_{-2}^2 (2-x)^2 \sqrt{4-x^2} dx$

$= 5\pi \Rightarrow \bar{z} = \frac{5}{4}$

25. (a) $M = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 dz dy dx = 4 \int_0^{\pi/2} \int_0^2 \int_{r^2}^4 r dz dr d\theta = 4 \int_0^{\pi/2} \int_0^2 (4r-r^3) dr d\theta = 4 \int_0^{\pi/2} 4 d\theta = 8\pi;$

$M_{xy} = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 zr dz dr d\theta = \int_0^{2\pi} \int_0^2 \frac{r}{2} (16-r^4) dr d\theta = \frac{32}{3} \int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \bar{z} = \frac{8}{3},$ and $\bar{x} = \bar{y} = 0,$

by symmetry

(b) $M = 8\pi \Rightarrow 4\pi = \int_0^{2\pi} \int_0^{\sqrt{c}} \int_{r^2}^c r dz dr d\theta = \int_0^{2\pi} \int_0^{\sqrt{c}} (cr-r^3) dr d\theta = \int_0^{2\pi} \frac{c^2}{4} d\theta = \frac{c^2\pi}{2} \Rightarrow c^2 = 8 \Rightarrow c = 2\sqrt{2},$

since $c > 0$

26. $M = 8; M_{xy} = \int_{-1}^1 \int_3^5 \int_{-1}^1 z dz dy dx = \int_{-1}^1 \int_3^5 \left[\frac{z^2}{2} \right]_{-1}^1 dy dx = 0; M_{yz} = \int_{-1}^1 \int_3^5 \int_{-1}^1 x dz dy dx = 2 \int_{-1}^1 \int_3^5 x dy dx$

$= 4 \int_{-1}^1 x dx = 0; M_{xz} = \int_{-1}^1 \int_3^5 \int_{-1}^1 y dz dy dx = 2 \int_{-1}^1 \int_3^5 y dy dx = 16 \int_{-1}^1 dx = 32 \Rightarrow \bar{x} = 0, \bar{y} = 4, \bar{z} = 0;$

$I_x = \int_{-1}^1 \int_3^5 \int_{-1}^1 (y^2 + z^2) dz dy dx = \int_{-1}^1 \int_3^5 \left(2y^2 + \frac{2}{3} \right) dy dx = \frac{2}{3} \int_{-1}^1 100 dx = \frac{400}{3};$

$I_y = \int_{-1}^1 \int_3^5 \int_{-1}^1 (x^2 + z^2) dz dy dx = \int_{-1}^1 \int_3^5 \left(2x^2 + \frac{2}{3} \right) dy dx = \frac{4}{3} \int_{-1}^1 (3x^2 + 1) dx = \frac{16}{3};$

$I_z = \int_{-1}^1 \int_3^5 \int_{-1}^1 (x^2 + y^2) dz dy dx = 2 \int_{-1}^1 \int_3^5 (x^2 + y^2) dy dx = 2 \int_{-1}^1 \left(2x^2 + \frac{98}{3} \right) dx = \frac{400}{3}$

27. The plane $y + 2z = 2$ is the top of the wedge $\Rightarrow I_L = \int_{-2}^2 \int_{-2}^4 \int_{-1}^{(2-y)/2} [(y-6)^2 + z^2] dz dy dx$

$= \int_{-2}^2 \int_{-2}^4 \left[\frac{(y-6)^2(4-y)}{2} + \frac{(2-y)^3}{24} + \frac{1}{3} \right] dy dx; \text{ let } t = 2-y \Rightarrow I_L = 4 \int_{-2}^4 \left(\frac{13t^3}{24} + 5t^2 + 16t + \frac{49}{3} \right) dt = 1386;$

$M = \frac{1}{2}(3)(6)(4) = 36$

28. The plane $y + 2z = 2$ is the top of the wedge $\Rightarrow I_L = \int_{-2}^2 \int_{-2}^4 \int_{-1}^{(2-y)/2} [(x-4)^2 + y^2] dz dy dx$
 $= \frac{1}{2} \int_{-2}^2 \int_{-2}^4 (x^2 - 8x + 16 + y^2)(4-y) dy dx = \int_{-2}^2 (9x^2 - 72x + 162) dx = 696; \quad M = \frac{1}{2}(3)(6)(4) = 36$
29. (a) $M = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2x dz dy dx = \int_0^2 \int_0^{2-x} (4x - 2x^2 - 2xy) dy dx = \int_0^2 (x^3 - 4x^2 + 4x) dx = \frac{4}{3}$
 (b) $M_{xy} = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2xz dz dy dx = \int_0^2 \int_0^{2-x} x(2-x-y)^2 dy dx = \int_0^2 \frac{x(2-x)^3}{3} dx = \frac{8}{15}; \quad M_{xz} = \frac{8}{15}$ by symmetry;
 $M_{yz} = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2x^2 dz dy dx = \int_0^2 \int_0^{2-x} 2x^2(2-x-y) dy dx = \int_0^2 (2x - x^2)^2 dx = \frac{16}{15}$
 $\Rightarrow \bar{x} = \frac{4}{5}, \text{ and } \bar{y} = \bar{z} = \frac{2}{5}$
30. (a) $M = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxy dz dy dx = k \int_0^2 \int_0^{\sqrt{x}} xy(4-x^2) dy dx = \frac{k}{2} \int_0^2 (4x^2 - x^4) dx = \frac{32k}{15}$
 (b) $M_{yz} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kx^2 y dz dy dx = k \int_0^2 \int_0^{\sqrt{x}} x^2 y(4-x^2) dy dx = \frac{k}{2} \int_0^2 (4x^3 - x^5) dx = \frac{8k}{3} \Rightarrow \bar{x} = \frac{5}{4};$
 $M_{xz} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxy^2 dz dy dx = k \int_0^2 \int_0^{\sqrt{x}} xy^2(4-x^2) dy dx = \frac{k}{3} \int_0^2 (4x^{5/2} - x^{9/2}) dx = \frac{256\sqrt{2}k}{231}$
 $\Rightarrow \bar{y} = \frac{40\sqrt{2}}{77}; \quad M_{xy} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxyz dz dy dx = \int_0^2 \int_0^{\sqrt{x}} xy(4-x^2)^2 dy dx = \frac{k}{4} \int_0^2 (16x^2 - 8x^4 + x^6) dx$
 $= \frac{256k}{105} \Rightarrow \bar{z} = \frac{8}{7}$
31. (a) $M = \int_0^1 \int_0^1 \int_0^1 (x+y+z+1) dz dy dx = \int_0^1 \int_0^1 (x+y+\frac{3}{2}) dy dx = \int_0^1 (x+2) dx = \frac{5}{2}$
 (b) $M_{xy} = \int_0^1 \int_0^1 \int_0^1 z(x+y+z+1) dz dy dx = \frac{1}{2} \int_0^1 \int_0^1 (x+y+\frac{5}{3}) dy dx = \frac{1}{2} \int_0^1 (x+\frac{13}{6}) dx = \frac{4}{3}$
 $\Rightarrow M_{xy} = M_{yz} = M_{xz} = \frac{4}{3}, \text{ by symmetry } \Rightarrow \bar{x} = \bar{y} = \bar{z} = \frac{8}{15}$
 (c) $I_z = \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2)(x+y+z+1) dz dy dx = \int_0^1 \int_0^1 (x^2 + y^2)(x+y+\frac{3}{2}) dy dx$
 $= \int_0^1 (x^3 + 2x^2 + \frac{1}{3}x + \frac{3}{4}) dx = \frac{11}{6} \Rightarrow I_x = I_y = I_z = \frac{11}{6}, \text{ by symmetry}$
32. The plane $y + 2z = 2$ is the top of the wedge.
 (a) $M = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1) dz dy dx = \int_{-1}^1 \int_{-2}^4 (x+1)(2-\frac{y}{2}) dy dx = 18$
 (b) $M_{yz} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} x(x+1) dz dy dx = \int_{-1}^1 \int_{-2}^4 x(x+1)(2-\frac{y}{2}) dy dx = 6;$
 $M_{xz} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} y(x+1) dz dy dx = \int_{-1}^1 \int_{-2}^4 y(x+1)(2-\frac{y}{2}) dy dx = 0;$
 $M_{xy} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} z(x+1) dz dy dx = \frac{1}{2} \int_{-1}^1 \int_{-2}^4 (x+1)(\frac{y^2}{2} - y) dy dx = 0 \Rightarrow \bar{x} = \frac{1}{3}, \text{ and } \bar{y} = \bar{z} = 0$

$$(c) \quad I_x = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1) (y^2 + z^2) dz dy dx = \int_{-1}^1 \int_{-2}^4 (x+1) \left[2y^2 + \frac{1}{3} - \frac{y^3}{2} + \frac{1}{3} \left(1 - \frac{y}{2} \right)^3 \right] dy dx = 45;$$

$$I_y = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1) (x^2 + z^2) dz dy dx = \int_{-1}^1 \int_{-2}^4 (x+1) \left[2x^2 + \frac{1}{3} - \frac{x^2 y}{2} + \frac{1}{3} \left(1 - \frac{y}{2} \right)^3 \right] dy dx = 15;$$

$$I_z = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1) (x^2 + y^2) dz dy dx = \int_{-1}^1 \int_{-2}^4 (x+1) \left(2 - \frac{y}{2} \right) (x^2 + y^2) dy dx = 42$$

$$33. \quad M = \int_{-1}^1 \int_{z-1}^{1-z} \int_0^{\sqrt{z}} (2y+5) dy dx dz = \int_0^1 \int_{z-1}^{1-z} (z+5\sqrt{z}) dx dz = \int_0^1 2(z+5\sqrt{z})(1-z) dz \\ = 2 \int_0^1 (5z^{1/2} + z - 5z^{3/2} - z^2) dz = 2 \left[\frac{10}{3} z^{3/2} + \frac{1}{2} z^2 - 2z^{5/2} - \frac{1}{3} z^3 \right]_0^1 = 2 \left(\frac{9}{3} - \frac{3}{2} \right) = 3$$

$$34. \quad M = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2(x^2+y^2)}^{16-2(x^2+y^2)} \sqrt{x^2+y^2} dz dy dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2} [16-4(x^2+y^2)] dy dx \\ = 4 \int_0^{2\pi} \int_0^2 r(4-r^2) r dr d\theta = 4 \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 d\theta = 4 \int_0^{2\pi} \frac{64}{15} d\theta = \frac{512\pi}{15}$$

$$35. (a) \quad \bar{x} = \frac{M_{yz}}{M} = 0 \Rightarrow \iiint_R x \delta(x, y, z) dx dy dz = 0 \Rightarrow M_{yz} = 0$$

$$(b) \quad I_L = \iiint_D |\mathbf{v} - h\mathbf{i}|^2 dm = \iiint_D [(x-h)\mathbf{i} + y\mathbf{j}]^2 dm = \iiint_D (x^2 - 2xh + h^2 + y^2) dm \\ = \iiint_D (x^2 + y^2) dm - 2h \iiint_D x dm + h^2 \iiint_D dm = I_x - 0 + h^2 m = I_{c.m.} + h^2 m$$

$$36. \quad I_L = I_{c.m.} + mh^2 = \frac{2}{5} ma^2 + ma^2 = \frac{7}{5} ma^2$$

$$37. (a) \quad (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2} \right) \Rightarrow I_z = I_{c.m.} + abc \left(\sqrt{\frac{a^2}{4} + \frac{b^2}{4}} \right)^2 \Rightarrow I_{c.m.} = I_z - \frac{abc(a^2+b^2)}{4} \\ = \frac{abc(a^2+b^2)}{3} - \frac{abc(a^2+b^2)}{4} = \frac{abc(a^2+b^2)}{12}; \quad R_{c.m.} = \sqrt{\frac{I_{c.m.}}{M}} = \sqrt{\frac{a^2+b^2}{12}}$$

$$(b) \quad I_L = I_{c.m.} + abc \left(\sqrt{\frac{a^2}{4} + \left(\frac{b}{2} - 2b \right)^2} \right)^2 = \frac{abc(a^2+b^2)}{12} + \frac{abc(a^2+9b^2)}{4} = \frac{abc(4a^2+28b^2)}{12} = \frac{abc(a^2+7b^2)}{3}; \\ R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{a^2+7b^2}{3}}$$

$$38. \quad M = \int_{-3}^3 \int_{-2}^4 \int_{-4/3}^{(4-2y)/3} dz dy dx = \int_{-3}^3 \int_{-2}^4 \frac{2}{3} (4-y) dy dx = \int_{-3}^3 \frac{2}{3} \left[4y - \frac{y^2}{2} \right]_{-2}^4 dx = 12 \int_{-3}^2 dx = 72; \quad \bar{x} = \bar{y} = \bar{z} = 0$$

$$\text{from Exercise 22} \Rightarrow I_x = I_{c.m.} + 72 \left(\sqrt{0^2 + 0^2} \right)^2 = I_{c.m.} \Rightarrow I_L = I_{c.m.} + 72 \left(\sqrt{16 + \frac{16}{9}} \right)^2 = 208 + 72 \left(\frac{160}{9} \right) = 1488$$

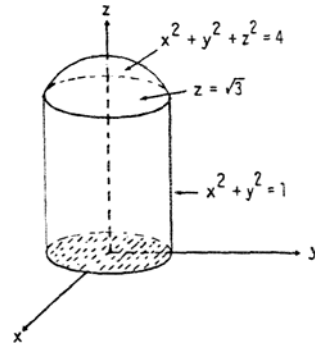
15.7 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

1.
$$\int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \left[r(2-r^2)^{1/2} - r^2 \right] dr \, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^3}{3} \right]_0^1 d\theta$$
$$= \int_0^{2\pi} \left(\frac{2^{3/2}}{3} - \frac{2}{3} \right) d\theta = \frac{4\pi(\sqrt{2}-1)}{3}$$
2.
$$\int_0^{2\pi} \int_0^3 \int_{r^2/3}^{\sqrt{18-r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^3 \left[r(18-r^2)^{1/2} - \frac{r^3}{3} \right] dr \, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(18-r^2)^{3/2} - \frac{r^4}{12} \right]_0^3 d\theta = \frac{9\pi(8\sqrt{2}-7)}{2}$$
3.
$$\int_0^{2\pi} \int_0^{\theta/(2\pi)} \int_0^{3+24r^3} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^{\theta/(2\pi)} (3r+24r^3) dr \, d\theta = \int_0^{2\pi} \left[\frac{3}{2}r^2 + 6r^4 \right]_0^{\theta/(2\pi)} d\theta$$
$$= \frac{3}{2} \int_0^{2\pi} \left(\frac{\theta^2}{4\pi^2} + \frac{4\theta^4}{16\pi^4} \right) d\theta = \frac{3}{2} \left[\frac{\theta^3}{12\pi^2} + \frac{\theta^5}{20\pi^4} \right]_0^{2\pi} = \frac{17\pi}{5}$$
4.
$$\int_0^\pi \int_0^{\theta/\pi} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z \, dz \, r \, dr \, d\theta = \int_0^\pi \int_0^{\theta/\pi} \frac{1}{2} \left[9(4-r^2) - (4-r^2) \right] r \, dr \, d\theta = 4 \int_0^\pi \int_0^{\theta/\pi} (4r-r^3) dr \, d\theta$$
$$= 4 \int_0^\pi \left[2r^2 - \frac{r^4}{4} \right]_0^{\theta/\pi} d\theta = 4 \int_0^\pi \left(\frac{2\theta^2}{\pi^2} - \frac{\theta^4}{4\pi^4} \right) d\theta = \frac{37\pi}{15}$$
5.
$$\int_0^{2\pi} \int_0^1 \int_r^{(2-r^2)^{-1/2}} 3 \, dz \, r \, dr \, d\theta = 3 \int_0^{2\pi} \int_0^1 \left[r(2-r^2)^{-1/2} - r^2 \right] dr \, d\theta = 3 \int_0^{2\pi} \left[-\frac{1}{3}(2-r^2)^{1/2} - \frac{r^3}{3} \right]_0^1 d\theta$$
$$= 3 \int_0^{2\pi} \left(\sqrt{2} - \frac{4}{3} \right) d\theta = \pi(6\sqrt{2}-8)$$
6.
$$\int_0^{2\pi} \int_0^1 \int_{-1/2}^{1/2} (r^2 \sin^2 \theta + z^2) dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \left(r^3 \sin^2 \theta + \frac{r}{12} \right) dr \, d\theta = \int_0^{2\pi} \left(\frac{\sin^2 \theta}{4} + \frac{1}{24} \right) d\theta = \frac{\pi}{3}$$
7.
$$\int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3 \, dr \, dz \, d\theta = \int_0^{2\pi} \int_0^3 \frac{z^4}{324} dz \, d\theta = \int_0^{2\pi} \frac{3}{20} d\theta = \frac{3\pi}{10}$$
8.
$$\int_{-1}^1 \int_0^{2\pi} \int_0^{1+\cos \theta} 4r \, dr \, d\theta \, dz = \int_{-1}^1 \int_0^{2\pi} 2(1+\cos \theta)^2 d\theta \, dz = \int_{-1}^1 6\pi d\theta = 12\pi$$
9.
$$\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2 \theta + z^2) r \, d\theta \, dr \, dz = \int_0^1 \int_0^{\sqrt{z}} \left[\frac{r^2 \theta}{2} + \frac{r^2 \sin 2\theta}{4} + z^2 \theta \right]_0^{2\pi} r \, dr \, dz = \int_0^1 \int_0^{\sqrt{z}} (\pi r^3 + 2\pi r z^2) dr \, dz$$
$$= \int_0^1 \left[\frac{\pi r^4}{4} + \pi r^2 z^2 \right]_0^{\sqrt{z}} dz = \int_0^1 \left(\frac{\pi z^2}{4} + \pi z^3 \right) dz = \left[\frac{\pi z^3}{12} + \frac{\pi z^4}{4} \right]_0^1 = \frac{\pi}{3}$$
10.
$$\int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r \sin \theta + 1) r \, d\theta \, dz \, dr = \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} 2\pi r \, dz \, dr = 2\pi \int_0^2 \left[r(4-r^2)^{1/2} - r^2 + 2r \right] dr$$
$$= 2\pi \left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3} + r^2 \right]_0^2 = 2\pi \left[-\frac{8}{3} + 4 + \frac{1}{3}(4)^{3/2} \right] = 8\pi$$

11. (a) $\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$

(b) $\int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^1 r \, dr \, dz \, d\theta + \int_0^{2\pi} \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta$

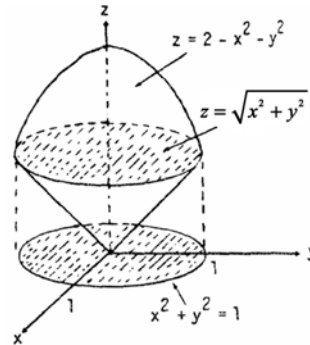
(c) $\int_0^1 \int_0^{\sqrt{4-r^2}} \int_0^{2\pi} r \, d\theta \, dz \, dr$



12. (a) $\int_0^{2\pi} \int_0^1 \int_r^{2-r^2} dz \, r \, dr \, d\theta$

(b) $\int_0^{2\pi} \int_0^1 \int_0^z r \, dr \, dz \, d\theta + \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{2-z}} r \, dr \, dz \, d\theta$

(c) $\int_0^1 \int_r^{2-r^2} \int_0^{2\pi} r \, d\theta \, dz \, dr$



13. $\int_{-\pi/2}^{\pi/2} \int_0^{\cos \theta} \int_0^{3r^2} f(r, \theta, z) \, dz \, r \, dr \, d\theta$

14. $\int_{-\pi/2}^{\pi/2} \int_0^1 \int_0^{r \cos \theta} r^3 \, dz \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^1 r^4 \cos \theta \, dr \, d\theta = \frac{1}{5} \int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta = \frac{2}{5}$

15. $\int_0^{\pi} \int_0^{2 \sin \theta} \int_0^{4-r \sin \theta} f(r, \theta, z) \, dz \, r \, dr \, d\theta$

16. $\int_{-\pi/2}^{\pi/2} \int_0^{3 \cos \theta} \int_0^{5-r \cos \theta} f(r, \theta, z) \, dz \, r \, dr \, d\theta$

17. $\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos \theta} \int_0^4 f(r, \theta, z) \, dz \, r \, dr \, d\theta$

18. $\int_{-\pi/2}^{\pi/2} \int_{\cos \theta}^{2 \cos \theta} \int_0^{3-r \sin \theta} f(r, \theta, z) \, dz \, r \, dr \, d\theta$

19. $\int_0^{\pi/4} \int_0^{\sec \theta} \int_0^{2-r \sin \theta} f(r, \theta, z) \, dz \, r \, dr \, d\theta$

20. $\int_{\pi/4}^{\pi/2} \int_0^{\cos \theta} \int_0^{2-r \sin \theta} f(r, \theta, z) \, dz \, r \, dr \, d\theta$

21. $\int_0^{\pi} \int_0^{\pi} \int_0^{2 \sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{8}{3} \int_0^{\pi} \int_0^{\pi} \sin^4 \phi \, d\phi \, d\theta = \frac{8}{3} \int_0^{\pi} \left(\left[-\frac{\sin^3 \phi \cos \phi}{4} \right]_0^{\pi} + \frac{3}{4} \int_0^{\pi} \sin^2 \phi \, d\phi \right) d\theta$
 $= 2 \int_0^{\pi} \int_0^{\pi} \sin^2 \phi \, d\phi \, d\theta = \int_0^{\pi} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi} d\theta = \int_0^{\pi} \pi \, d\theta = \pi^2$

22. $\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/4} 4 \cos \phi \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \left[2 \sin^2 \phi \right]_0^{\pi/4} d\theta = \int_0^{2\pi} d\theta = 2\pi$

23.
$$\int_0^{2\pi} \int_0^\pi \int_0^{(1-\cos\phi)^{1/2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{24} \int_0^{2\pi} \int_0^\pi (1-\cos\phi)^3 \sin\phi \, d\phi \, d\theta = \frac{1}{96} \int_0^{2\pi} \left[(1-\cos\phi)^4 \right]_0^\pi d\theta$$

$$= \frac{1}{96} \int_0^{2\pi} (2^4 - 0) \, d\theta = \frac{16}{96} \int_0^{2\pi} d\theta = \frac{1}{6} (2\pi) = \frac{\pi}{3}$$
24.
$$\int_0^{3\pi/2} \int_0^\pi \int_0^1 5\rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta = \frac{5}{4} \int_0^{3\pi/2} \int_0^\pi \sin^3\phi \, d\phi \, d\theta = \frac{5}{4} \int_0^{3\pi/2} \left(\left[-\frac{\sin^2\phi \cos\phi}{3} \right]_0^\pi + \frac{2}{3} \int_0^\pi \sin\phi \, d\phi \right) d\theta$$

$$= \frac{5}{6} \int_0^{3\pi/2} [-\cos\phi]_0^\pi d\theta = \frac{5}{3} \int_0^{3\pi/2} d\theta = \frac{5\pi}{2}$$
25.
$$\int_0^{2\pi} \int_0^{\pi/3} \int_{\sec\phi}^2 3\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} (8 - \sec^3\phi) \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \left[-8\cos\phi - \frac{1}{2}\sec^2\phi \right]_0^{\pi/3} d\theta$$

$$= \int_0^{2\pi} \left[(-4-2) - \left(-8-\frac{1}{2}\right) \right] d\theta = \frac{5}{2} \int_0^{2\pi} d\theta = 5\pi$$
26.
$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec\phi} \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta = \frac{1}{4} \int_0^{2\pi} \int_0^{\pi/4} \tan\phi \sec^2\phi \, d\phi \, d\theta = \frac{1}{4} \int_0^{2\pi} \left[\frac{1}{2} \tan^2\phi \right]_0^{\pi/4} d\theta$$

$$= \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4}$$
27.
$$\int_0^2 \int_{-\pi}^0 \int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho = \int_0^2 \int_{-\pi}^0 \rho^3 \left[-\frac{\cos 2\phi}{2} \right]_{\pi/4}^{\pi/2} d\theta \, d\rho = \int_0^2 \int_{-\pi}^0 \frac{\rho^3}{2} d\theta \, d\rho = \int_0^2 \frac{\rho^2 \pi}{2} d\rho$$

$$= \left[\frac{\pi \rho^4}{8} \right]_0^2 = 2\pi$$
28.
$$\int_{\pi/6}^{\pi/3} \int_{\csc\phi}^{2\csc\phi} \int_0^{2\pi} \rho^2 \sin\phi \, d\theta \, d\rho \, d\phi = 2\pi \int_{\pi/6}^{\pi/3} \int_{\csc\phi}^{2\csc\phi} \rho^2 \sin\phi \, d\rho \, d\phi = \frac{2\pi}{3} \int_{\pi/6}^{\pi/3} \left[\rho^3 \sin\phi \right]_{\csc\phi}^{2\csc\phi} d\phi$$

$$= \frac{14\pi}{3} \int_{\pi/6}^{\pi/3} \csc^2\phi \, d\phi = \frac{28\pi}{3\sqrt{3}}$$
29.
$$\int_0^1 \int_0^\pi \int_0^{\pi/4} 12\rho \sin^3\phi \, d\phi \, d\theta \, d\rho = \int_0^1 \int_0^\pi \left(12\rho \left[-\frac{\sin^2\phi \cos\phi}{3} \right]_0^{\pi/4} + 8\rho \int_0^{\pi/4} \sin\phi \, d\phi \right) d\theta \, d\rho$$

$$= \int_0^1 \int_0^\pi \left(-\frac{2\rho}{\sqrt{2}} - 8\rho [\cos\phi]_0^{\pi/4} \right) d\theta \, d\rho = \int_0^1 \int_0^\pi \left(8\rho - \frac{10\rho}{\sqrt{2}} \right) d\theta \, d\rho = \pi \int_0^1 \left(8\rho - \frac{10\rho}{\sqrt{2}} \right) d\rho = \pi \left[4\rho^2 - \frac{5\rho^2}{\sqrt{2}} \right]_0^1$$

$$= \frac{(4\sqrt{2}-5)\pi}{\sqrt{2}}$$
30.
$$\int_{\pi/6}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_{\csc\phi}^2 5\rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi = \int_{\pi/6}^{\pi/2} \int_{-\pi/2}^{\pi/2} (32 - \csc^5\phi) \sin^3\phi \, d\theta \, d\phi$$

$$= \int_{\pi/6}^{\pi/2} \int_{-\pi/2}^{\pi/2} (32 \sin^3\phi - \csc^2\phi) \, d\theta \, d\phi = \pi \int_{\pi/6}^{\pi/2} (32 \sin^3\phi - \csc^2\phi) \, d\phi$$

$$= \pi \left[-\frac{32 \sin^2\phi \cos\phi}{3} \right]_{\pi/6}^{\pi/2} + \frac{64\pi}{3} \int_{\pi/6}^{\pi/2} \sin\phi \, d\phi + \pi [\cot\phi]_{\pi/6}^{\pi/2} = \pi \left(\frac{32\sqrt{3}}{24} \right) - \frac{64\pi}{3} [\cos\phi]_{\pi/6}^{\pi/2} - \pi(\sqrt{3})$$

$$= \frac{\sqrt{3}}{3} \pi + \left(\frac{64\pi}{3} \right) \left(\frac{\sqrt{3}}{2} \right) = \frac{33\pi\sqrt{3}}{3} = 11\pi\sqrt{3}$$

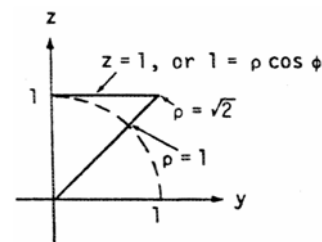
31. (a) $x^2 + y^2 = 1 \Rightarrow \rho^2 \sin^2 \phi = 1$, and $\rho \sin \phi = 1 \Rightarrow \rho = \csc \phi$;

thus $\int_0^{2\pi} \int_0^{\pi/6} \int_0^{\csc \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta + \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{\csc \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

(b) $\int_0^{2\pi} \int_1^2 \int_0^{\sin^{-1}(1/\rho)} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi} \int_0^2 \int_0^{\pi/6} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$

32. (a) $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

(b) $\int_0^{2\pi} \int_0^1 \int_0^{\pi/4} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{\cos^{-1}(1/\rho)}^{\pi/4} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$



33. $V = \int_0^{2\pi} \int_0^{\pi/2} \int_{\cos \phi}^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/2} (8 - \cos^3 \phi) \sin \phi \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \left[-8 \cos \phi + \frac{\cos^4 \phi}{4} \right]_0^{\pi/2} d\theta$
 $= \frac{1}{3} \int_0^{2\pi} \left(8 - \frac{1}{4} \right) d\theta = \left(\frac{31}{12} \right) (2\pi) = \frac{31\pi}{6}$

34. $V = \int_0^{2\pi} \int_0^{\pi/2} \int_1^{1+\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/2} (3 \cos \phi + 3 \cos^2 \phi + \cos^3 \phi) \sin \phi \, d\phi \, d\theta$
 $= \frac{1}{3} \int_0^{2\pi} \left[-\frac{3}{2} \cos^2 \phi - \cos^3 \phi - \frac{1}{4} \cos^4 \phi \right]_0^{\pi/2} d\theta = \frac{1}{3} \int_0^{2\pi} \left(\frac{3}{2} + 1 + \frac{1}{4} \right) d\theta = \frac{11}{12} \int_0^{2\pi} d\theta = \left(\frac{11}{12} \right) (2\pi) = \frac{11\pi}{6}$

35. $V = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} (1 - \cos \phi)^3 \sin \phi \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \left[\frac{(1 - \cos \phi)^4}{4} \right]_0^{\pi} d\theta$
 $= \frac{1}{12} (2)^4 \int_0^{2\pi} d\theta = \frac{4}{3} (2\pi) = \frac{8\pi}{3}$

36. $V = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{1-\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/2} (1 - \cos \phi)^3 \sin \phi \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \left[\frac{(1 - \cos \phi)^4}{4} \right]_0^{\pi/2} d\theta$
 $= \frac{1}{12} \int_0^{2\pi} d\theta = \frac{1}{12} (2\pi) = \frac{\pi}{6}$

37. $V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \cos^3 \phi \sin \phi \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} \left[-\frac{\cos^4 \phi}{4} \right]_{\pi/4}^{\pi/2} d\theta$
 $= \left(\frac{8}{3} \right) \left(\frac{1}{16} \right) \int_0^{2\pi} d\theta = \frac{1}{6} (2\pi) = \frac{\pi}{3}$

38. $V = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \sin \phi \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} [-\cos \phi]_{\pi/3}^{\pi/2} d\theta = \frac{4}{3} \int_0^{2\pi} d\theta = \frac{8\pi}{3}$

39. (a) $8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

(b) $8 \int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$

(c) $8 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$

$$40. \quad (a) \int_0^{\pi/2} \int_0^{3/\sqrt{2}} \int_r^{\sqrt{9-r^2}} dz \, r \, dr \, d\theta \qquad (b) \int_0^{\pi/2} \int_0^{\pi/4} \int_r^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$(c) \int_0^{\pi/2} \int_0^{\pi/4} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 9 \int_0^{\pi/2} \int_0^{\pi/4} \sin \phi \, d\phi \, d\theta = -9 \int_0^{\pi/2} \left(\frac{1}{\sqrt{2}} - 1 \right) d\theta = \frac{9\pi(2-\sqrt{2})}{4}$$

$$41. \quad (a) \, V = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec \phi}^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \qquad (b) \, V = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$$

$$(c) \, V = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$$

$$(d) \, V = \int_0^{2\pi} \int_0^{\sqrt{3}} \left[r(4-r^2)^{1/2} - r \right] dr \, d\theta = \int_0^{2\pi} \left[-\frac{(4-r^2)^{3/2}}{3} - \frac{r^2}{2} \right]_0^{\sqrt{3}} d\theta = \int_0^{2\pi} \left(-\frac{1}{3} - \frac{3}{2} + \frac{4^{3/2}}{3} \right) d\theta$$

$$= \frac{5}{6} \int_0^{2\pi} d\theta = \frac{5\pi}{3}$$

$$42. \quad (a) \, I_z = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} r^2 \, dz \, r \, dr \, d\theta$$

$$(b) \, I_z = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho^2 \sin^2 \phi) (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta,$$

since $r^2 = x^2 + y^2 = \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = \rho^2 \sin^2 \phi$

$$(c) \, I_z = \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{5} \sin^3 \phi \, d\phi \, d\theta = \frac{1}{5} \int_0^{2\pi} \left(\left[-\frac{\sin^2 \phi \cos \phi}{3} \right]_0^{\pi/2} + \frac{2}{3} \int_0^{\pi/2} \sin \phi \, d\phi \right) d\theta = \frac{2}{15} \int_0^{2\pi} [-\cos \phi]_0^{\pi/2} d\theta$$

$$= \frac{2}{15} (2\pi) = \frac{4\pi}{15}$$

$$43. \, V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^4-1}^{4-4r^2} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 (5r - 4r^3 - r^5) \, dr \, d\theta = 4 \int_0^{\pi/2} \left(\frac{5}{2} - 1 - \frac{1}{6} \right) d\theta = 4 \int_0^{\pi/2} d\theta = \frac{8\pi}{3}$$

$$44. \, V = 4 \int_0^{\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^{1-r} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 \left(r - r^2 + r\sqrt{1-r^2} \right) dr \, d\theta = 4 \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^3}{3} - \frac{1}{3} (1-r^2)^{3/2} \right]_0^1 d\theta$$

$$= 4 \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} \right) d\theta = 2 \int_0^{\pi/2} d\theta = 2 \left(\frac{\pi}{2} \right) = \pi$$

$$45. \, V = \int_{3\pi/2}^{2\pi} \int_0^{3\cos \theta} \int_0^{-r\sin \theta} dz \, r \, dr \, d\theta = \int_{3\pi/2}^{2\pi} \int_0^{3\cos \theta} (-r^2 \sin \theta) \, dr \, d\theta = \int_{3\pi/2}^{2\pi} (-9 \cos^3 \theta) (\sin \theta) \, d\theta$$

$$= \left[\frac{9}{4} \cos^4 \theta \right]_{3\pi/2}^{2\pi} = \frac{9}{4} - 0 = \frac{9}{4}$$

$$46. \, V = 2 \int_{\pi/2}^{\pi} \int_0^{-3\cos \theta} \int_0^r dz \, r \, dr \, d\theta = 2 \int_{\pi/2}^{\pi} \int_0^{-3\cos \theta} r^2 \, dr \, d\theta = \frac{2}{3} \int_{\pi/2}^{\pi} (-27 \cos^3 \theta) \, d\theta$$

$$= -18 \left(\left[\frac{\cos^2 \theta \sin \theta}{3} \right]_{\pi/2}^{\pi} + \frac{2}{3} \int_{\pi/2}^{\pi} \cos \theta \, d\theta \right) = -12 [\sin \theta]_{\pi/2}^{\pi} = 12$$

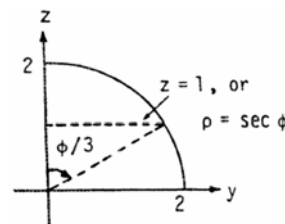
$$\begin{aligned}
 47. \quad V &= \int_0^{\pi/2} \int_0^{\sin \theta} \int_0^{\sqrt{1-r^2}} dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^{\sin \theta} r \sqrt{1-r^2} \, dr \, d\theta = \int_0^{\pi/2} \left[-\frac{1}{3} (1-r^2)^{3/2} \right]_0^{\sin \theta} d\theta \\
 &= -\frac{1}{3} \int_0^{\pi/2} \left[(1-\sin^2 \theta)^{3/2} - 1 \right] d\theta = -\frac{1}{3} \int_0^{\pi/2} (\cos^3 \theta - 1) d\theta = -\frac{1}{3} \left(\left[\frac{\cos^2 \theta \sin \theta}{3} \right]_0^{\pi/2} + \frac{2}{3} \int_0^{\pi/2} \cos \theta \, d\theta \right) + \left[\frac{\theta}{3} \right]_0^{\pi/2} \\
 &= -\frac{2}{9} [\sin \theta]_0^{\pi/2} + \frac{\pi}{6} = \frac{-4+3\pi}{18}
 \end{aligned}$$

$$\begin{aligned}
 48. \quad V &= \int_0^{\pi/2} \int_0^{\cos \theta} \int_0^{3\sqrt{1-r^2}} dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^{\cos \theta} 3r \sqrt{1-r^2} \, dr \, d\theta = \int_0^{\pi/2} \left[-\frac{1}{2} (1-r^2)^{3/2} \right]_0^{\cos \theta} d\theta \\
 &= \int_0^{\pi/2} \left[-\frac{1}{2} (1-\cos^2 \theta)^{3/2} + \frac{1}{2} \right] d\theta = \int_0^{\pi/2} \left(1 - \sin^3 \theta \right) d\theta = \left[\theta + \frac{\sin^2 \theta \cos \theta}{3} \right]_0^{\pi/2} - \frac{2}{3} \int_0^{\pi/2} \sin \theta \, d\theta \\
 &= \frac{\pi}{2} + \frac{2}{3} [\cos \theta]_0^{\pi/2} = \frac{\pi}{2} - \frac{2}{3} = \frac{3\pi-4}{6}
 \end{aligned}$$

$$\begin{aligned}
 49. \quad V &= \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \frac{a^3}{3} \sin \phi \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} [-\cos \phi]_{\pi/3}^{2\pi/3} d\theta = \frac{a^3}{3} \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \right) d\theta \\
 &= \frac{2\pi a^3}{3}
 \end{aligned}$$

$$50. \quad V = \int_0^{\pi/6} \int_0^{\pi/2} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{\pi/6} \int_0^{\pi/2} \sin \phi \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{\pi/6} d\theta = \frac{a^3 \pi}{18}$$

$$\begin{aligned}
 51. \quad V &= \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec \phi}^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} (8 \sin \phi - \tan \phi \sec^2 \phi) d\phi \, d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} \left[-8 \cos \phi - \frac{1}{2} \tan^2 \phi \right]_0^{\pi/3} d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} \left[-4 - \frac{1}{2} (3) + 8 \right] d\theta = \frac{1}{3} \int_0^{2\pi} \frac{5}{2} d\theta = \frac{5}{6} (2\pi) = \frac{5\pi}{3}
 \end{aligned}$$



$$\begin{aligned}
 52. \quad V &= 4 \int_0^{\pi/2} \int_0^{\pi/4} \int_{\sec \phi}^{2 \sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{4}{3} \int_0^{\pi/2} \int_0^{\pi/4} (8 \sec^3 \phi - \sec^3 \phi) \sin \phi \, d\phi \, d\theta \\
 &= \frac{28}{3} \int_0^{\pi/2} \int_0^{\pi/4} \sec^3 \phi \sin \phi \, d\phi \, d\theta = \frac{28}{3} \int_0^{\pi/2} \int_0^{\pi/4} \tan \phi \sec^2 \phi \, d\phi \, d\theta = \frac{28}{3} \int_0^{\pi/2} \left[\frac{1}{2} \tan^2 \phi \right]_0^{\pi/4} d\theta \\
 &= \frac{14}{3} \int_0^{\pi/2} d\theta = \frac{7\pi}{3}
 \end{aligned}$$

$$53. \quad V = 4 \int_0^{\pi/2} \int_0^1 \int_0^{r^2} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 r^3 \, dr \, d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

$$54. \quad V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{r^2+1} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 r \, dr \, d\theta = 2 \int_0^{\pi/2} d\theta = \pi$$

$$55. \quad V = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} \int_0^r dz \, r \, dr \, d\theta = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} r^2 \, dr \, d\theta = 8 \left(\frac{2\sqrt{2}-1}{3} \right) \int_0^{\pi/2} d\theta = \frac{4\pi(2\sqrt{2}-1)}{3}$$

56. $V = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} \int_0^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} r \sqrt{2-r^2} \, dr \, d\theta = 8 \int_0^{\pi/2} \left[-\frac{1}{3} (2-r^2)^{3/2} \right]_1^{\sqrt{2}} d\theta = 8 \int_0^{\pi/2} d\theta = \frac{4\pi}{3}$
57. $V = \int_0^{2\pi} \int_0^2 \int_0^{4-r \sin \theta} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (4r - r^2 \sin \theta) \, dr \, d\theta = 8 \int_0^{2\pi} \left(1 - \frac{\sin \theta}{3} \right) d\theta = 16\pi$
58. $V = \int_0^{2\pi} \int_0^2 \int_0^{4-r \cos \theta - r \sin \theta} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 [4r - r^2 (\cos \theta + \sin \theta)] \, dr \, d\theta = \frac{8}{3} \int_0^{2\pi} (3 - \cos \theta - \sin \theta) \, d\theta = 16\pi$
59. The paraboloids intersect when $4x^2 + 4y^2 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1$ and $z = 4$
 $\Rightarrow V = 4 \int_0^{\pi/2} \int_0^1 \int_{4r^2}^{5-r^2} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 (5r - 5r^3) \, dr \, d\theta = 20 \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta = 5 \int_0^{\pi/2} d\theta = \frac{5\pi}{2}$
60. The paraboloid intersects the xy -plane when $9 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 9 \Rightarrow V = 4 \int_0^{\pi/2} \int_1^3 \int_0^{9-r^2} dz \, r \, dr \, d\theta$
 $= 4 \int_0^{\pi/2} \int_1^3 (9r - r^3) \, dr \, d\theta = 4 \int_0^{\pi/2} \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_1^3 d\theta = 4 \int_0^{\pi/2} \left(\frac{81}{4} - \frac{17}{4} \right) d\theta = 64 \int_0^{\pi/2} d\theta = 32\pi$
61. $V = 8 \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta = 8 \int_0^{2\pi} \int_0^1 r (4-r^2)^{1/2} \, dr \, d\theta = 8 \int_0^{2\pi} \left[-\frac{1}{3} (4-r^2)^{3/2} \right]_0^1 d\theta = -\frac{8}{3} \int_0^{2\pi} (3^{3/2} - 8) \, d\theta$
 $= \frac{4\pi(8-3\sqrt{3})}{3}$
62. The sphere and paraboloid intersect when $x^2 + y^2 + z^2 = 2$ and $z = x^2 + y^2 \Rightarrow z^2 + z - 2 = 0$
 $\Rightarrow (z+2)(z-1) = 0 \Rightarrow z = 1$ or $z = -2 \Rightarrow z = 1$ since $z \geq 0$. Thus, $x^2 + y^2 = 1$ and the volume is given by the triple integral $V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 \left[r (2-r^2)^{1/2} - r^3 \right] \, dr \, d\theta$
 $= 4 \int_0^{\pi/2} \left[-\frac{1}{3} (2-r^2)^{3/2} - \frac{r^4}{4} \right]_0^1 d\theta = 4 \int_0^{\pi/2} \left(\frac{2\sqrt{2}}{3} - \frac{7}{12} \right) d\theta = \frac{\pi(8\sqrt{2}-7)}{6}$
63. $\text{average} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2 \, dz \, dr \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 2r^2 \, dr \, d\theta = \frac{1}{3\pi} \int_0^{2\pi} d\theta = \frac{2}{3}$
64. $\text{average} = \frac{1}{(\frac{4\pi}{3})} \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2 \, dz \, dr \, d\theta = \frac{3}{4\pi} \int_0^{2\pi} \int_0^1 2r^2 \sqrt{1-r^2} \, dr \, d\theta$
 $= \frac{3}{2\pi} \int_0^{2\pi} \left[\frac{1}{8} \sin^{-1} r - \frac{1}{8} r \sqrt{1-r^2} (1-2r^2) \right]_0^1 d\theta = \frac{3}{16\pi} \int_0^{2\pi} \left(\frac{\pi}{2} + 0 \right) d\theta = \frac{3}{32} \int_0^{2\pi} d\theta = \left(\frac{3}{32} \right) (2\pi) = \frac{3\pi}{16}$
65. $\text{average} = \frac{1}{(\frac{4\pi}{3})} \int_0^{2\pi} \int_0^\pi \int_0^1 \rho^3 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{3}{16\pi} \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \, d\theta = \frac{3}{8\pi} \int_0^{2\pi} d\theta = \frac{3}{4}$

$$\begin{aligned}
 66. \text{ average} &= \frac{1}{\left(\frac{2\pi}{3}\right)} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^3 \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta = \frac{3}{8\pi} \int_0^{2\pi} \int_0^{\pi/2} \cos \phi \sin \phi \, d\phi \, d\theta = \frac{3}{8\pi} \int_0^{2\pi} \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/2} d\theta \\
 &= \frac{3}{16\pi} \int_0^{2\pi} d\theta = \left(\frac{3}{16\pi}\right)(2\pi) = \frac{3}{8}
 \end{aligned}$$

$$\begin{aligned}
 67. \quad M &= 4 \int_0^{\pi/2} \int_0^1 \int_0^r dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 r^2 \, dr \, d\theta = \frac{4}{3} \int_0^{\pi/2} d\theta = \frac{2\pi}{3}; \quad M_{xy} = \int_0^{2\pi} \int_0^1 \int_0^r z \, dz \, r \, dr \, d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta = \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\pi}{4}\right)\left(\frac{3}{2\pi}\right) = \frac{3}{8}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}
 \end{aligned}$$

$$\begin{aligned}
 68. \quad M &= \int_0^{\pi/2} \int_0^2 \int_0^r dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 r^2 \, dr \, d\theta = \frac{8}{3} \int_0^{\pi/2} d\theta = \frac{4\pi}{3}; \quad M_{yz} = \int_0^{\pi/2} \int_0^2 \int_0^r x \, dz \, r \, dr \, d\theta \\
 &= \int_0^{\pi/2} \int_0^2 r^3 \cos \theta \, dr \, d\theta = 4 \int_0^{\pi/2} \cos \theta \, d\theta = 4; \quad M_{xz} = \int_0^{\pi/2} \int_0^2 \int_0^r y \, dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 r^3 \sin \theta \, dr \, d\theta \\
 &= 4 \int_0^{\pi/2} \sin \theta \, d\theta = 4; \quad M_{xy} = \int_0^{\pi/2} \int_0^2 \int_0^r z \, dz \, r \, dr \, d\theta = \frac{1}{2} \int_0^{\pi/2} \int_0^2 r^3 \, dr \, d\theta = 2 \int_0^{\pi/2} d\theta = \pi \Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{3}{\pi}, \\
 \bar{y} &= \frac{M_{xz}}{M} = \frac{3}{\pi}, \text{ and } \bar{z} = \frac{M_{xy}}{M} = \frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 69. \quad M &= \frac{8\pi}{3}; \quad M_{xy} = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 z \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \rho^3 \cos \phi \sin \phi \, d\phi \, d\theta \\
 &= 4 \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \cos \phi \sin \phi \, d\phi \, d\theta = 4 \int_0^{2\pi} \left[\frac{\sin^2 \phi}{2} \right]_{\pi/3}^{\pi/2} d\theta = 4 \int_0^{2\pi} \left(\frac{1}{2} - \frac{3}{8} \right) d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi \\
 &\Rightarrow \bar{z} = \frac{M_{xy}}{M} = (\pi) \left(\frac{3}{8\pi} \right) = \frac{3}{8}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}
 \end{aligned}$$

$$\begin{aligned}
 70. \quad M &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \frac{2-\sqrt{2}}{2} d\theta = \frac{\pi a^3 (2-\sqrt{2})}{3}; \\
 M_{xy} &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^3 \sin \phi \cos \phi \, d\rho \, d\phi \, d\theta = \frac{a^4}{4} \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \, d\phi \, d\theta = \frac{a^4}{16} \int_0^{2\pi} d\theta = \frac{\pi a^4}{8} \\
 &\Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\pi a^4}{8} \right) \left[\frac{3}{\pi a^3 (2-\sqrt{2})} \right] = \left(\frac{3a}{8} \right) \left(\frac{2+\sqrt{2}}{2} \right) = \frac{3(2+\sqrt{2})a}{16}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}
 \end{aligned}$$

$$\begin{aligned}
 71. \quad M &= \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{r}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^4 r^{3/2} \, dr \, d\theta = \frac{64}{5} \int_0^{2\pi} d\theta = \frac{128\pi}{5}; \quad M_{xy} = \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{r}} z \, dz \, r \, dr \, d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \int_0^4 r^2 \, dr \, d\theta = \frac{32}{3} \int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \frac{5}{6}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}
 \end{aligned}$$

$$\begin{aligned}
 72. \quad M &= \int_{-\pi/3}^{\pi/3} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz \, r \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \int_0^1 2r\sqrt{1-r^2} \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \left[-\frac{2}{3} (1-r^2)^{3/2} \right]_0^1 d\theta \\
 &= \frac{2}{3} \int_{-\pi/3}^{\pi/3} d\theta = \left(\frac{2}{3} \right) \left(\frac{2\pi}{3} \right) = \frac{4\pi}{9}; \quad M_{yz} = \int_{-\pi/3}^{\pi/3} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2 \cos \theta \, dz \, dr \, d\theta = 2 \int_{-\pi/3}^{\pi/3} \int_0^1 r^2 \sqrt{1-r^2} \cos \theta \, dr \, d\theta \\
 &= 2 \int_{-\pi/3}^{\pi/3} \left[\frac{1}{8} \sin^{-1} r - \frac{1}{8} r \sqrt{1-r^2} (1-2r^2) \right]_0^1 \cos \theta \, d\theta = \frac{\pi}{8} \int_{-\pi/3}^{\pi/3} \cos \theta \, d\theta = \frac{\pi}{8} [\sin \theta]_{-\pi/3}^{\pi/3} = \left(\frac{\pi}{8} \right) \left(2 \cdot \frac{\sqrt{3}}{2} \right) = \frac{\pi\sqrt{3}}{8} \\
 &\Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{9\sqrt{3}}{32}, \text{ and } \bar{y} = \bar{z} = 0, \text{ by symmetry}
 \end{aligned}$$

73. We orient the cone with its vertex at the origin and axis along the z -axis $\Rightarrow \phi = \frac{\pi}{4}$. We use the x -axis which

$$\begin{aligned} & \text{is through the vertex and parallel to the base of the cone} \Rightarrow I_x = \int_0^{2\pi} \int_0^1 \int_r^1 (r^2 \sin^2 \theta + z^2) dz r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(r^3 \sin^2 \theta - r^4 \sin^2 \theta + \frac{r}{3} - \frac{r^4}{3} \right) dr d\theta = \int_0^{2\pi} \left(\frac{\sin^2 \theta}{20} + \frac{1}{10} \right) d\theta = \left[\frac{\theta}{40} - \frac{\sin 2\theta}{80} + \frac{\theta}{10} \right]_0^{2\pi} = \frac{\pi}{20} + \frac{\pi}{5} = \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} 74. \quad I_z &= \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3 dz dr d\theta = \int_0^{2\pi} \int_0^a 2r^3 \sqrt{a^2-r^2} dr d\theta = 2 \int_0^{2\pi} \left[\left(-\frac{r^2}{5} - \frac{2a^2}{15} \right) (a^2-r^2)^{3/2} \right]_0^a d\theta \\ &= 2 \int_0^{2\pi} \frac{2}{15} a^5 d\theta = \frac{8\pi a^5}{15} \end{aligned}$$

$$\begin{aligned} 75. \quad I_z &= \int_0^{2\pi} \int_0^a \int_{\left(\frac{h}{a}\right)r}^h (x^2 + y^2) dz r dr d\theta = \int_0^{2\pi} \int_0^a \int_{\frac{hr}{a}}^h r^3 dz dr d\theta = \int_0^{2\pi} \int_0^a \left(hr^3 - \frac{hr^4}{a} \right) dr d\theta \\ &= \int_0^{2\pi} h \left[\frac{r^4}{4} - \frac{r^5}{5a} \right]_0^a d\theta = \int_0^{2\pi} h \left(\frac{a^4}{4} - \frac{a^5}{5a} \right) d\theta = \frac{ha^4}{20} \int_0^{2\pi} d\theta = \frac{\pi ha^4}{10} \end{aligned}$$

$$\begin{aligned} 76. \quad (a) \quad M &= \int_0^{2\pi} \int_0^1 \int_0^{r^2} z dz r dr d\theta = \int_0^{2\pi} \int_0^1 \frac{1}{2} r^5 dr d\theta = \frac{1}{12} \int_0^{2\pi} d\theta = \frac{\pi}{6}; \\ M_{xy} &= \int_0^{2\pi} \int_0^1 \int_0^{r^2} z^2 dz r dr d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^1 r^7 dr d\theta = \frac{1}{24} \int_0^{2\pi} d\theta = \frac{\pi}{12} \Rightarrow \bar{z} = \frac{1}{2}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by} \\ & \text{symmetry; } I_z = \int_0^{2\pi} \int_0^1 \int_0^{r^2} zr^3 dz dr d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^1 r^7 dr d\theta = \frac{1}{16} \int_0^{2\pi} d\theta = \frac{\pi}{8} \end{aligned}$$

$$\begin{aligned} (b) \quad M &= \int_0^{2\pi} \int_0^1 \int_0^{r^2} r^2 dz dr d\theta = \int_0^{2\pi} \int_0^1 r^4 dr d\theta = \frac{1}{5} \int_0^{2\pi} d\theta = \frac{2\pi}{5}; \\ M_{xy} &= \int_0^{2\pi} \int_0^1 \int_0^{r^2} zr^2 dz dr d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^1 r^6 dr d\theta = \frac{1}{14} \int_0^{2\pi} d\theta = \frac{\pi}{7} \Rightarrow \bar{z} = \frac{5}{14}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by} \\ & \text{symmetry; } I_z = \int_0^{2\pi} \int_0^1 \int_0^{r^2} r^4 dz dr d\theta = \int_0^{2\pi} \int_0^1 r^6 dr d\theta = \frac{1}{7} \int_0^{2\pi} d\theta = \frac{2\pi}{7} \end{aligned}$$

$$\begin{aligned} 77. \quad (a) \quad M &= \int_0^{2\pi} \int_0^1 \int_r^1 z dz r dr d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^1 (r - r^3) dr d\theta = \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4}; \\ M_{xy} &= \int_0^{2\pi} \int_0^1 \int_r^1 z^2 dz r dr d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^1 (r - r^4) dr d\theta = \frac{1}{10} \int_0^{2\pi} d\theta = \frac{\pi}{5} \Rightarrow \bar{z} = \frac{4}{5}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by} \\ & \text{symmetry; } I_z = \int_0^{2\pi} \int_0^1 \int_r^1 zr^3 dz dr d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^1 (r^3 - r^5) dr d\theta = \frac{1}{24} \int_0^{2\pi} d\theta = \frac{\pi}{12} \end{aligned}$$

$$\begin{aligned} (b) \quad M &= \int_0^{2\pi} \int_0^1 \int_r^1 z^2 dz r dr d\theta = \frac{\pi}{5} \text{ from part (a); } M_{xy} = \int_0^{2\pi} \int_0^1 \int_r^1 z^3 dz r dr d\theta = \frac{1}{4} \int_0^{2\pi} \int_0^1 (r - r^5) dr d\theta \\ &= \frac{1}{12} \int_0^{2\pi} d\theta = \frac{\pi}{6} \Rightarrow \bar{z} = \frac{5}{6}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry; } I_z = \int_0^{2\pi} \int_0^1 \int_r^1 z^2 r^3 dz dr d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \int_0^1 (r^3 - r^6) dr d\theta = \frac{1}{28} \int_0^{2\pi} d\theta = \frac{\pi}{14} \end{aligned}$$

$$\begin{aligned} 78. \quad (a) \quad M &= \int_0^{2\pi} \int_0^\pi \int_0^a \rho^4 \sin \phi d\rho d\phi d\theta = \frac{a^5}{5} \int_0^{2\pi} \int_0^\pi \sin \phi d\phi d\theta = \frac{2a^5}{5} \int_0^{2\pi} d\theta = \frac{4\pi a^5}{5}; \\ I_z &= \int_0^{2\pi} \int_0^\pi \int_0^a \rho^6 \sin^3 \phi d\rho d\phi d\theta = \frac{a^7}{7} \int_0^{2\pi} \int_0^\pi (1 - \cos^2 \phi) \sin \phi d\phi d\theta = \frac{a^7}{7} \int_0^{2\pi} \left[-\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^\pi d\theta \\ &= \frac{4a^7}{21} \int_0^{2\pi} d\theta = \frac{8a^7\pi}{21} \end{aligned}$$

$$\begin{aligned}
 \text{(b) } M &= \int_0^{2\pi} \int_0^\pi \int_0^a \rho^3 \sin^2 \phi \, d\rho \, d\phi \, d\theta = \frac{a^4}{4} \int_0^{2\pi} \int_0^\pi \frac{(1-\cos 2\phi)}{2} \, d\phi \, d\theta = \frac{\pi a^4}{8} \int_0^{2\pi} d\theta = \frac{\pi^2 a^4}{4}; \\
 I_z &= \int_0^{2\pi} \int_0^\pi \int_0^a \rho^5 \sin^4 \phi \, d\rho \, d\phi \, d\theta = \frac{a^6}{6} \int_0^{2\pi} \int_0^\pi \sin^4 \phi \, d\phi \, d\theta \\
 &= \frac{a^6}{6} \int_0^{2\pi} \left(\left[\frac{-\sin^3 \phi \cos \phi}{4} \right]_0^\pi + \frac{3}{4} \int_0^\pi \sin^2 \phi \, d\phi \right) d\theta = \frac{a^6}{8} \int_0^{2\pi} \left[\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right]_0^\pi d\theta = \frac{\pi a^6}{16} \int_0^{2\pi} d\theta = \frac{a^6 \pi^2}{8}
 \end{aligned}$$

$$\begin{aligned}
 79. \quad M &= \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^a \frac{h}{a} r \sqrt{a^2-r^2} \, dr \, d\theta = \frac{h}{a} \int_0^{2\pi} \left[-\frac{1}{3} (a^2-r^2)^{3/2} \right]_0^a d\theta = \frac{h}{a} \int_0^{2\pi} \frac{a^3}{3} d\theta \\
 &= \frac{2\pi h a^2}{3}; \quad M_{xy} = \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} z \, dz \, r \, dr \, d\theta = \frac{h^2}{2a^2} \int_0^{2\pi} \int_0^a (a^2 r - r^3) \, dr \, d\theta = \frac{h^2}{2a^2} \int_0^{2\pi} \left(\frac{a^4}{2} - \frac{a^4}{4} \right) d\theta = \frac{a^2 h^2 \pi}{4} \\
 &\Rightarrow \bar{z} = \left(\frac{\pi a^2 h^2}{4} \right) \left(\frac{3}{2\pi h a^2} \right) = \frac{3}{8} h, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}
 \end{aligned}$$

80. Let the base radius of the cone be a and the height h , and place the cone's axis of symmetry along the z -axis with the vertex at the origin. Then $M = \frac{\pi a^2 h}{3}$ and $M_{xy} = \int_0^{2\pi} \int_0^a \int_{\frac{h}{a}r}^h z \, dz \, r \, dr \, d\theta$
- $$\begin{aligned}
 &= \frac{1}{2} \int_0^{2\pi} \int_0^a \left(h^2 r - \frac{h^2}{a^2} r^3 \right) dr \, d\theta = \frac{h^2}{2} \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4a^2} \right]_0^a d\theta = \frac{h^2}{2} \int_0^{2\pi} \left(\frac{a^2}{2} - \frac{a^2}{4} \right) d\theta = \frac{h^2 a^2}{8} \int_0^{2\pi} d\theta = \frac{h^2 a^2 \pi}{4} \\
 &\Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{h^2 a^2 \pi}{4} \right) \left(\frac{3}{\pi a^2 h} \right) = \frac{3}{4} h, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry} \Rightarrow \text{the centroid is one fourth of the way from the base to the vertex}
 \end{aligned}$$

81. The density distribution function is linear so it has the form $\delta(\rho) = k\rho + C$, where ρ is the distance from the center of the planet. Now, $\delta(R) = 0 \Rightarrow kR + C = 0$, and $\delta(\rho) = k\rho - kR$. It remains to determine the constant

$$\begin{aligned}
 k: \quad M &= \int_0^{2\pi} \int_0^\pi \int_0^R (k\rho - kR) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \left[k \frac{\rho^4}{4} - kR \frac{\rho^3}{3} \right]_0^R \sin \phi \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_0^\pi k \left(\frac{R^4}{4} - \frac{R^4}{3} \right) \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \left(-\frac{k}{12} R^4 [-\cos \phi]_0^\pi \right) d\theta = \int_0^{2\pi} \left(-\frac{k}{6} R^4 \right) d\theta = -\frac{k\pi R^4}{3} \Rightarrow k = -\frac{3M}{\pi R^4} \\
 &\Rightarrow \delta(\rho) = -\frac{3M}{\pi R^4} \rho + \frac{3M}{\pi R^4} R. \text{ At the center of the planet } \rho = 0 \Rightarrow \delta(0) = \left(\frac{3M}{\pi R^4} \right) R = \frac{3M}{\pi R^3}.
 \end{aligned}$$

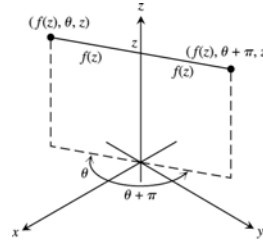
82. The mass of the planet's atmosphere to an altitude h above the surface of the planet is the triple integral

$$\begin{aligned}
 M(h) &= \int_0^{2\pi} \int_0^\pi \int_R^h \mu_0 e^{-c(\rho-R)} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_R^h \int_0^{2\pi} \int_0^\pi \mu_0 e^{-c(\rho-R)} \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho \\
 &= \int_R^h \int_0^{2\pi} \left[\mu_0 e^{-c(\rho-R)} \rho^2 (-\cos \phi) \right]_0^\pi d\theta \, d\rho = 2 \int_R^h \int_0^{2\pi} \mu_0 e^{cR} e^{-c\rho} \rho^2 \, d\theta \, d\rho = 4\pi \mu_0 e^{cR} \int_R^h e^{-c\rho} \rho^2 \, d\rho \\
 &= 4\pi \mu_0 e^{cR} \left[-\frac{\rho^2 e^{-c\rho}}{c} - \frac{2\rho e^{-c\rho}}{c^2} - \frac{2e^{-c\rho}}{c^3} \right]_R^h \text{ (by parts)} \\
 &= 4\pi \mu_0 e^{cR} \left(-\frac{h^2 e^{-ch}}{c} - \frac{2he^{-ch}}{c^2} - \frac{2e^{-ch}}{c^3} + \frac{R^2 e^{-cR}}{c} + \frac{2Re^{-cR}}{c^2} + \frac{2e^{-cR}}{c^3} \right).
 \end{aligned}$$

The mass of the planet's atmosphere is therefore $M = \lim_{h \rightarrow \infty} M(h) = 4\pi \mu_0 \left(\frac{R^2}{c} + \frac{2R}{c^2} + \frac{2}{c^3} \right)$.

83. (a) A plane perpendicular to the x -axis has the form $x = a$ in rectangular coordinates $\Rightarrow r \cos \theta = a$
 $\Rightarrow r = \frac{a}{\cos \theta} \Rightarrow r = a \sec \theta$, in cylindrical coordinates.
- (b) A plane perpendicular to the y -axis has the form $y = b$ in rectangular coordinates $\Rightarrow r \sin \theta = b$
 $\Rightarrow r = \frac{b}{\sin \theta} \Rightarrow r = b \csc \theta$, in cylindrical coordinates.
84. $ax + by = c \Rightarrow a(r \cos \theta) + b(r \sin \theta) = c \Rightarrow r(a \cos \theta + b \sin \theta) = c \Rightarrow r = \frac{c}{a \cos \theta + b \sin \theta}$.

85. The equation $r = f(z)$ implies that the point $(r, \theta, z) = (f(z), \theta, z)$ will lie on the surface for all θ . In particular $(f(z), \theta + \pi, z)$ lies on the surface whenever $(f(z), \theta, z)$ does \Rightarrow the surface is symmetric with respect to the z -axis.



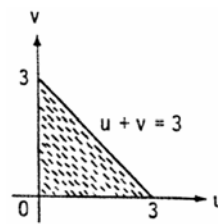
86. The equation $\rho = f(\phi)$ implies that the point $(\rho, \phi, \theta) = (f(\phi), \phi, \theta)$ lies on the surface for all θ . In particular, if $(f(\phi), \phi, \theta)$ lies on the surface, then $(f(\phi), \phi, \theta + \pi)$ lies on the surface, so the surface is symmetric with respect to the z -axis.

15.8 SUBSTITUTIONS IN MULTIPLE INTEGRALS

1. (a) $x - y = u$ and $2x + y = v \Rightarrow 3x = u + v$ and $y = x - u \Rightarrow x = \frac{1}{3}(u + v)$ and $y = \frac{1}{3}(-2u + v)$;

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

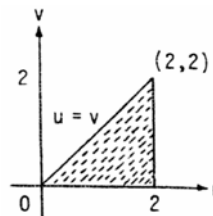
- (b) The line segment $y = x$ from $(0, 0)$ to $(1, 1)$ is $x - y = 0 \Rightarrow u = 0$; the line segment $y = -2x$ from $(0, 0)$ to $(1, -2)$ is $2x + y = 0 \Rightarrow v = 0$; the line segment $x = 1$ from $(1, 1)$ to $(1, -2)$ is $(x - y) + (2x + y) = 3 \Rightarrow u + v = 3$. The transformed region is sketched at the right.



2. (a) $x + 2y = u$ and $x - y = v \Rightarrow 3y = u - v$ and $x = v + y \Rightarrow y = \frac{1}{3}(u - v)$ and $x = \frac{1}{3}(u + 2v)$;

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

- (b) The triangular region in the xy -plane has vertices $(0, 0)$, $(2, 0)$, and $(\frac{2}{3}, \frac{2}{3})$. The line segment $y = x$ from $(0, 0)$ to $(\frac{2}{3}, \frac{2}{3})$ is $x - y = 0 \Rightarrow v = 0$; the line segment $y = 0$ from $(0, 0)$ to $(2, 0) \Rightarrow u = v$; the line segment $x + 2y = 2$ from $(\frac{2}{3}, \frac{2}{3})$ to $(2, 0) \Rightarrow u = 2$. The transformed region is sketched at the right.



3. (a) $3x + 2y = u$ and $x + 4y = v \Rightarrow -5x = -2u + v$ and $y = \frac{1}{2}(u - 3x) \Rightarrow x = \frac{1}{5}(2u - v)$ and $y = \frac{1}{10}(3v - u)$;

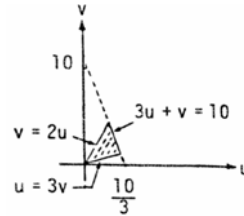
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}$$

- (b) The x -axis $y = 0 \Rightarrow u = 3v$;

the y -axis $x = 0 \Rightarrow v = 2u$;

$$\text{the line } x + y = 1 \Rightarrow \frac{1}{5}(2u - v) + \frac{1}{10}(3v - u) = 1$$

$$\Rightarrow 2(2u - v) + (3v - u) = 10 \Rightarrow 3u + v = 10. \text{ The transformed region is sketched at the right.}$$



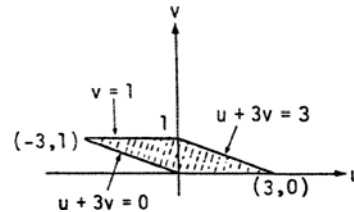
4. (a) $2x - 3y = u$ and $-x + y = v \Rightarrow -x = u + 3v$ and $y = v + x \Rightarrow x = -u - 3v$ and $y = -u - 2v$;

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = 2 - 3 = -1$$

- (b) The line $x = -3 \Rightarrow -u - 3v = -3$ or $u + 3v = 3$;

$$x = 0 \Rightarrow u + 3v = 0; \quad y = x \Rightarrow v = 0; \quad y = x + 1$$

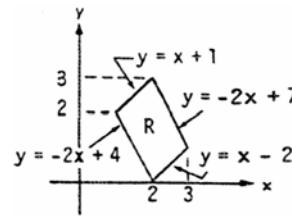
$\Rightarrow v = 1$. The transformed region is the parallelogram sketched at the right.



$$\begin{aligned} 5. \quad \int_0^4 \int_{y/2}^{(y/2)+1} \left(x - \frac{y}{2}\right) dx dy &= \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2} \right]_{y/2}^{(y/2)+1} dy = \frac{1}{2} \int_0^4 \left[\left(\frac{y}{2} + 1\right)^2 - \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2} + 1\right)y + \left(\frac{y}{2}\right)y \right] dy \\ &= \frac{1}{2} \int_0^4 (y + 1 - y) dy = \frac{1}{2} \int_0^4 dy = \frac{1}{2}(4) = 2 \end{aligned}$$

$$\begin{aligned} 6. \quad \iint_R (2x^2 - xy - y^2) dx dy &= \iint_R (x - y)(2x + y) dx dy \\ &= \iint_G uv \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \frac{1}{3} \iint_G uv du dv; \text{ We find the} \end{aligned}$$

boundaries of G from the boundaries of R , shown in the accompanying figure:

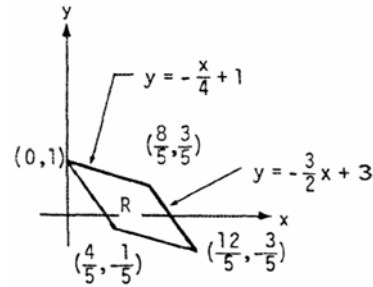


xy -equations for the boundary of R	Corresponding uv -equations for the boundary of G	Simplified uv -equations
$y = -2x + 4$	$\frac{1}{3}(-2u + v) = -\frac{2}{3}(u + v) + 4$	$v = 4$
$y = -2x + 7$	$\frac{1}{3}(-2u + v) = -\frac{2}{3}(u + v) + 7$	$v = 7$
$y = x - 2$	$\frac{1}{3}(-2u + v) = \frac{1}{3}(u + v) - 2$	$u = 2$
$y = x + 1$	$\frac{1}{3}(-2u + v) = \frac{1}{3}(u + v) + 1$	$u = -1$

$$\Rightarrow \frac{1}{3} \iint_G uv du dv = \frac{1}{3} \int_{-1}^2 \int_4^7 uv dv du = \frac{1}{3} \int_{-1}^2 u \left[\frac{v^2}{2} \right]_4^7 du = \frac{11}{2} \int_{-1}^2 u du = \left(\frac{11}{2} \right) \left[\frac{u^2}{2} \right]_{-1}^2 = \left(\frac{11}{4} \right) (4 - 1) = \frac{33}{4}$$

$$\begin{aligned}
 7. \quad & \iint_R (3x^2 + 14xy + 8y^2) dx dy \\
 &= \iint_R (3x + 2y)(x + 4y) dx dy \\
 &= \iint_G uv \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \frac{1}{10} \iint_G uv du dv;
 \end{aligned}$$

We find the boundaries of G from the boundaries of R , shown in the accompanying figure:



xy -equations for the boundary of R	Corresponding uv -equations for the boundary of G	Simplified uv -equations
$y = -\frac{3}{2}x + 1$	$\frac{1}{10}(3v - u) = -\frac{3}{10}(2u - v) + 1$	$u = 2$
$y = -\frac{3}{2}x + 3$	$\frac{1}{10}(3v - u) = -\frac{3}{10}(2u - v) + 3$	$u = 6$
$y = -\frac{1}{4}x$	$\frac{1}{10}(3v - u) = -\frac{1}{20}(2u - v)$	$v = 0$
$y = -\frac{1}{4}x + 1$	$\frac{1}{10}(3v - u) = -\frac{1}{20}(2u - v) + 1$	$v = 4$

$$\Rightarrow \frac{1}{10} \iint_G uv du dv = \frac{1}{10} \int_2^6 \int_0^4 uv dv du = \frac{1}{10} \int_2^6 u \left[\frac{v^2}{2} \right]_0^4 du = \frac{4}{5} \int_2^6 u du = \left(\frac{4}{5} \right) \left[\frac{u^2}{2} \right]_2^6 = \left(\frac{4}{5} \right) (18 - 2) = \frac{64}{5}$$

$$\begin{aligned}
 8. \quad & \iint_R 2(x - y) dx dy = \iint_G (-2v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \iint_G (-2v) du dv; \text{ the region } G \text{ is sketched in Exercise 4} \\
 & \Rightarrow \iint_G (-2v) du dv = \int_0^1 \int_{-3v}^{3-3v} (-2v) du dv = \int_0^1 (-2v)(3 - 3v + 3v) dv = \int_0^1 (-6v) dv = \left[-3v^2 \right]_0^1 = -3
 \end{aligned}$$

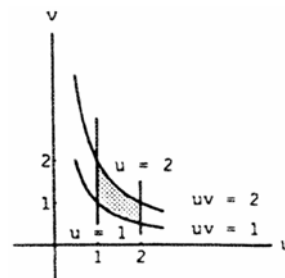
$$9. \quad x = \frac{u}{v} \text{ and } y = uv \Rightarrow \frac{y}{x} = v^2 \text{ and } xy = u^2; \frac{\partial(x, y)}{\partial(u, v)} = J(u, v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v};$$

$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 9 \Rightarrow u = 3$; thus

$$\begin{aligned}
 & \iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dx dy = \int_1^3 \int_1^2 (v + u) \left(\frac{2u}{v} \right) dv du = \int_1^3 \int_1^2 \left(2u + \frac{2u^2}{v} \right) dv du = \int_1^3 \left[2uv + 2u^2 \ln v \right]_1^2 du \\
 &= \int_1^3 \left(2u + 2u^2 \ln 2 \right) du = \left[u^2 + \frac{2}{3} u^2 \ln 2 \right]_1^3 = 8 + \frac{2}{3} (26)(\ln 2) = 8 + \frac{52}{3} (\ln 2)
 \end{aligned}$$

$$10. \quad (a) \quad \frac{\partial(x, y)}{\partial(u, v)} = J(u, v) = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u, \text{ and the region } G$$

is sketched at the right



(b) $x = 1 \Rightarrow u = 1$, and $x = 2 \Rightarrow u = 2$; $y = 1 \Rightarrow uv = 1 \Rightarrow v = \frac{1}{u}$, and $y = 2 \Rightarrow uv = 2 \Rightarrow v = \frac{2}{u}$; thus,

$$\begin{aligned} \int_1^2 \int_1^{2/u} \frac{y}{x} dy dx &= \int_1^2 \int_{1/u}^{2/u} \left(\frac{uv}{u}\right) u dv du = \int_1^2 \int_{1/u}^{2/u} uv dv du = \int_1^2 u \left[\frac{v^2}{2} \right]_{1/u}^{2/u} du = \int_1^2 u \left(\frac{2}{u^2} - \frac{1}{2u^2} \right) du \\ &= \frac{3}{2} \int_1^2 u \left(\frac{1}{u^2} \right) du = \frac{3}{2} [\ln u]_1^2 = \frac{3}{2} \ln 2; \quad \int_1^2 \int_1^{2/u} \frac{y}{x} dy dx = \int_1^2 \left[\frac{1}{x} \cdot \frac{y^2}{2} \right]_1^{2/u} dx = \frac{3}{2} \int_1^2 \frac{dx}{x} = \frac{3}{2} [\ln x]_1^2 = \frac{3}{2} \ln 2 \end{aligned}$$

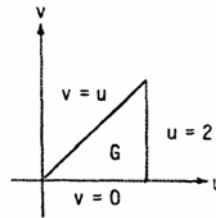
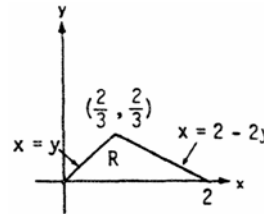
11. $x = ar \cos \theta$ and $y = ar \sin \theta \Rightarrow \frac{\partial(x, y)}{\partial(r, \theta)} = J(r, \theta) = \begin{vmatrix} a \cos \theta & -ar \sin \theta \\ b \sin \theta & br \cos \theta \end{vmatrix} = abr \cos^2 \theta + abr \sin^2 \theta = abr$;

$$\begin{aligned} I_0 &= \iint_R (x^2 + y^2) dA = \int_0^{2\pi} \int_0^1 r^2 (a^2 \cos^2 \theta + b^2 \sin^2 \theta) |J(r, \theta)| dr d\theta \\ &= \int_0^{2\pi} \int_0^1 abr^3 (a^2 \cos^2 \theta + b^2 \sin^2 \theta) dr d\theta = \frac{ab}{4} \left(\int_0^{2\pi} a^2 \cos^2 \theta + b^2 \sin^2 \theta d\theta \right) \\ &= \frac{ab}{4} \left[\frac{a^2 \theta}{2} + \frac{a^2 \sin 2\theta}{4} + \frac{b^2 \theta}{2} - \frac{b^2 \sin 2\theta}{4} \right]_0^{2\pi} = \frac{ab\pi(a^2 + b^2)}{4} \end{aligned}$$

12. $\frac{\partial(x, y)}{\partial(u, v)} = J(u, v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$; $A = \iint_R dy dx = \iint_G ab du dv = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab dv du = 2ab \int_{-1}^1 \sqrt{1-u^2} du$

$$= 2ab \left[\frac{u}{2} \sqrt{1-u^2} + \frac{1}{2} \sin^{-1} u \right]_{-1}^1 = ab \left[\sin^{-1} 1 - \sin^{-1}(-1) \right] = ab \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] = ab\pi$$

13. The region of integration R in the xy -plane is sketched in the figure at the right. The boundaries of the image G are obtained as follows, with G sketched at the right:



xy -equations for the boundary of R	Corresponding uv -equations for the boundary of G	Simplified uv -equations
$x = y$	$\frac{1}{3}(u + 2v) = \frac{1}{3}(u - v)$	$v = 0$
$x = 2 - 2y$	$\frac{1}{3}(u + 2v) = 2 - \frac{2}{3}(u - v)$	$u = 2$
$y = 0$	$0 = \frac{1}{3}(u - v)$	$v = u$

Also, from Exercise 2, $\frac{\partial(x, y)}{\partial(u, v)} = J(u, v) = -\frac{1}{3} \Rightarrow \int_0^{2/3} \int_y^{2-2y} (x+2y)e^{(y-x)} dx dy = \int_0^2 \int_0^u ue^{-v} \left| -\frac{1}{3} \right| dv du$

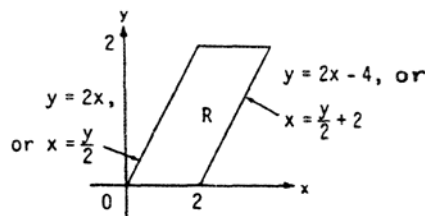
$$= \frac{1}{3} \int_0^2 u \left[-e^{-v} \right]_0^u du = \frac{1}{3} \int_0^2 u (1 - e^{-u}) du = \frac{1}{3} \left[u(u + e^{-u}) - \frac{u^2}{2} + e^{-u} \right]_0^2 = \frac{1}{3} \left[2(2 + e^{-2}) - 2 + e^{-2} - 1 \right]$$

$$= \frac{1}{3} (3e^{-2} + 1) \approx 0.4687$$

14. $x = u + \frac{v}{2}$ and $y = v \Rightarrow 2x - y = (2u + v) - v = 2u$

and $\frac{\partial(x, y)}{\partial(u, v)} = J(u, v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1$; next,

$u = x - \frac{v}{2} = x - \frac{y}{2}$ and $v = y$, so the boundaries of the region of integration R in the xy -plane are transformed to the boundaries of G :



xy -equations for the boundary of R	Corresponding uv -equations for the boundary of G	Simplified uv -equations
$x = \frac{y}{2}$	$u + \frac{v}{2} = \frac{v}{2}$	$u = 0$
$x = \frac{y}{2} + 2$	$u + \frac{v}{2} = \frac{v}{2} + 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

$$\Rightarrow \int_0^2 \int_{y/2}^{(y/2)+2} y^3 (2x - y) e^{(2x-y)^2} dx dy = \int_0^2 \int_0^2 v^3 (2u) e^{4u^2} du dv = \int_0^2 v^3 \left[\frac{1}{4} e^{4u^2} \right]_0^2 dv = \frac{1}{4} \int_0^2 v^3 (e^{16} - 1) dv$$

$$= \frac{1}{4} (e^{16} - 1) \left[\frac{v^4}{4} \right]_0^2 = e^{16} - 1$$

15. $x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$; $\frac{\partial(x, y)}{\partial(u, v)} = J(u, v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$;

$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 4 \Rightarrow u = 2$; thus

$$\int_1^2 \int_{1/y}^y (x^2 + y^2) dx dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2) dx dy = \int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2 v^2 \right) \left(\frac{2u}{v} \right) du dv = \int_1^2 \int_1^2 \left(\frac{2u^3}{v^3} + 2u^3 v \right) du dv$$

$$= \int_1^2 \left[\frac{u^4}{2v^3} + \frac{1}{2} u^4 v \right]_1^2 dv = \int_1^2 \left(\frac{15}{2v^3} + \frac{15v}{2} \right) dv = \left[-\frac{15}{4v^2} + \frac{15v^2}{4} \right]_1^2 = \frac{225}{16}$$

16. $x = u^2 - v^2$ and $y = 2uv$; $\frac{\partial(x, y)}{\partial(u, v)} = J(u, v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 = 4(u^2 + v^2)$;

$y = 2\sqrt{1-x} \Rightarrow y^2 = 4(1-x) \Rightarrow (2uv)^2 = 4(1 - (u^2 - v^2)) \Rightarrow u = \pm 1$; $y = 0 \Rightarrow 2uv = 0 \Rightarrow u = 0$ or $v = 0$;

$x = 0 \Rightarrow u^2 - v^2 = 0 \Rightarrow u = v$ or $u = -v$; This gives us four triangular regions, but only the one in the quadrant where both u, v are positive maps into the region R in the xy -plane.

$$\begin{aligned} \int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2} dx dy &= \int_0^1 \int_0^u \sqrt{(u^2 - v^2)^2 + (2uv)^2} \cdot 4(u^2 + v^2) dv du = 4 \int_0^1 \int_0^u (u^2 + v^2)^2 dv du \\ &= 4 \int_0^1 \left[u^4 v + \frac{2}{3} u^2 v^3 + \frac{1}{5} v^5 \right]_0^u du = \frac{112}{15} \int_0^1 u^5 du = \frac{112}{15} \left[\frac{1}{6} u^6 \right]_0^1 = \frac{56}{45} \end{aligned}$$

$$\begin{aligned} 17. \quad (a) \quad x &= u \cos v \text{ and } y = u \sin v \Rightarrow \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{vmatrix} = u \cos^2 v + u \sin^2 v = u \\ (b) \quad x &= u \sin v \text{ and } y = u \cos v \Rightarrow \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \sin v & u \cos v \\ \cos v & -u \sin v \end{vmatrix} = -u \sin^2 v - u \cos^2 v = -u \end{aligned}$$

$$\begin{aligned} 18. \quad (a) \quad x &= u \cos v, y = u \sin v, z = w \Rightarrow \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \cos v & -u \sin v & 0 \\ \sin v & u \cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u \cos^2 v + u \sin^2 v = u \\ (b) \quad x &= 2u - 1, y = 3v - 4, z = \frac{1}{2}(w - 4) \Rightarrow \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = (2)(3)\left(\frac{1}{2}\right) = 3 \end{aligned}$$

$$\begin{aligned} 19. \quad & \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix} \\ &= (\cos \phi) \begin{vmatrix} \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} + (\rho \sin \phi) \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} \\ &= (\rho^2 \cos \phi) (\sin \phi \cos \phi \cos^2 \theta + \sin \phi \cos \phi \sin^2 \theta) + (\rho^2 \sin \phi) (\sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta) \\ &= \rho^2 \sin \phi \cos^2 \phi + \rho^2 \sin^3 \phi = (\rho^2 \sin \phi) (\cos^2 \phi + \sin^2 \phi) = \rho^2 \sin \phi \end{aligned}$$

$$20. \quad \text{Let } u = g(x) \Rightarrow J(x) = \frac{du}{dx} = g'(x) \Rightarrow \int_a^b f(u) du = \int_{g(a)}^{g(b)} f(g(x)) g'(x) dx \text{ in accordance with Theorem 7 in Section 5.6. Note that } g'(x) \text{ represents the Jacobian of the transformation } u = g(x) \text{ or } x = g^{-1}(u).$$

$$\begin{aligned} 21. \quad & \int_0^3 \int_0^4 \int_{y/2}^{1+(y/2)} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx dy dz = \int_0^3 \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2} + \frac{xz}{3} \right]_{y/2}^{1+(y/2)} dy dz = \int_0^3 \int_0^4 \left[\frac{1}{2}(y+1) - \frac{y}{2} + \frac{z}{3} \right] dy dz \\ &= \int_0^3 \left[\frac{(y+1)^2}{4} - \frac{y^2}{4} + \frac{yz}{3} \right]_0^4 dz = \int_0^3 \left(\frac{9}{4} + \frac{4z}{3} - \frac{1}{4} \right) dz = \int_0^3 \left(2 + \frac{4z}{3} \right) dz = \left[2z + \frac{2z^2}{3} \right]_0^3 = 12 \end{aligned}$$

$$22. \quad J(u, v, w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc; \text{ the transformation takes the ellipsoid region } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \text{ in } xyz\text{-space into}$$

the spherical region $u^2 + v^2 + w^2 \leq 1$ in uvw -space (which has volume $V = \frac{4}{3}\pi$) $\Rightarrow V = \iiint_R dx dy dz$

$$= \iiint_G abc du dv dw = \frac{4\pi abc}{3}$$

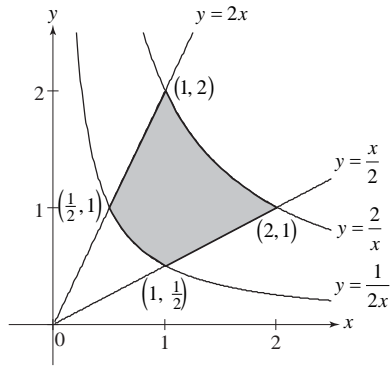
$$\begin{aligned}
23. \quad J(u, v, w) &= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc; \text{ for } R \text{ and } G \text{ as in Exercise 22, } \iiint_R |xyz| \, dx \, dy \, dz \iiint_G a^2 b^2 c^2 uvw \, dw \, dv \, du \\
&= 8a^2 b^2 c^2 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 (\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta)(\rho \cos \phi) (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta \\
&= \frac{4a^2 b^2 c^2}{3} \int_0^{\pi/2} \int_0^{\pi/2} \sin \theta \cos \theta \sin^3 \phi \cos \phi \, d\phi \, d\theta = \frac{a^2 b^2 c^2}{3} \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta = \frac{a^2 b^2 c^2}{6}
\end{aligned}$$

$$\begin{aligned}
24. \quad u = x, v = xy, \text{ and } w = 3z \Rightarrow x = u, y = \frac{v}{u}, \text{ and } z = \frac{1}{3}w \Rightarrow J(u, v, w) &= \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}; \\
\iiint_D (x^2 y + 3xyz) \, dx \, dy \, dz &= \iiint_G \left[u^2 \left(\frac{v}{u} \right) + 3u \left(\frac{v}{u} \right) \left(\frac{w}{3} \right) \right] |J(u, v, w)| \, du \, dv \, dw = \frac{1}{3} \int_0^3 \int_0^2 \int_1^2 \left(v + \frac{vw}{u} \right) \, du \, dv \, dw \\
&= \frac{1}{3} \int_0^3 \int_0^2 (v + vw \ln 2) \, dv \, dw = \frac{1}{3} \int_0^3 (1 + w \ln 2) \left[\frac{v^2}{2} \right]_0^2 \, dw = \frac{2}{3} \int_0^3 (1 + w \ln 2) \, dw = \frac{2}{3} \left[w + \frac{w^2}{2} \ln 2 \right]_0^3 \\
&= \frac{2}{3} \left(3 + \frac{9}{2} \ln 2 \right) = 2 + 3 \ln 2 = 2 + \ln 8
\end{aligned}$$

$$\begin{aligned}
25. \quad \text{The first moment about the } xy\text{-coordinate plane for the semi-ellipsoid, } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ using the} \\
\text{transformation in Exercise 23 is, } M_{xy} &= \iiint_D z \, dz \, dy \, dx = \iiint_G cw |J(u, v, w)| \, du \, dv \, dw \\
&= abc^2 \iiint_G w \, du \, dv \, dw = (abc^2) \cdot (M_{xy} \text{ of the hemisphere } x^2 + y^2 + z^2 = 1, z \geq 0) = \frac{abc^2 \pi}{4}; \\
\text{the mass of the semi-ellipsoid is } \frac{2abc\pi}{3} \Rightarrow \bar{z} &= \left(\frac{abc^2 \pi}{4} \right) \left(\frac{3}{2abc\pi} \right) = \frac{3}{8}c
\end{aligned}$$

$$\begin{aligned}
26. \quad \text{A solid of revolution is symmetric about the axis of revolution, therefore, the height of the solid is solely a} \\
\text{function of } r. \text{ That is, } y = f(x) = f(r). \text{ Using cylindrical coordinates with } x = r \cos \theta, y = y \text{ and } z = r \sin \theta, \\
\text{we have } V = \iiint_G r \, dy \, d\theta \, dr = \int_a^b \int_0^{2\pi} \int_0^{f(r)} r \, dy \, d\theta \, dr = \int_a^b \int_0^{2\pi} [ry]_0^{f(r)} d\theta \, dr = \int_a^b \int_0^{2\pi} r f(r) \, d\theta \, dr \\
= \int_a^b [r\theta f(r)]_0^{2\pi} dr = \int_a^b 2\pi r f(r) dr. \text{ In the last integral, } r \text{ is a dummy or stand-in variable and as such it can be} \\
\text{replaced by any variable name. Choosing } x \text{ instead of } r \text{ we have } V = \int_a^b 2\pi x f(x) dx, \text{ which is the same result} \\
\text{obtained using the shell method.}
\end{aligned}$$

27. The region R is shaded in the graph below.



Solving explicitly for the transformation that gives x and y in terms of u and v yields a complicated expression for $\frac{\partial(x, y)}{\partial(u, v)}$. However, its reciprocal, $\frac{\partial(u, v)}{\partial(x, y)}$ is relatively easy to compute.

Since $u(x, y) = xy$ and $v(x, y) = y/x$, $J(x, y) = \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial v} \\ -\frac{\partial u}{\partial x^2} & \frac{\partial u}{\partial x} \end{vmatrix} = 2 \frac{y}{x} = 2v$. Thus $J(u, v) = 1/2v$. In the uv -plane

the region corresponding to R is $G: \frac{1}{2} \leq u \leq 2, \frac{1}{2} \leq v \leq 2$. Thus v is positive and $|J(u, v)| = 1/2v$.

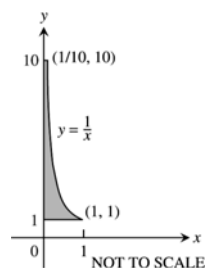
$$\iint_R dA = \int_{1/2}^2 \int_{1/2}^2 \frac{1}{2v} du dv = \int_{1/2}^2 \left(\frac{\ln u}{2} \right)_{1/2}^2 dv = \int_{1/2}^2 \ln 2 dv = \frac{3}{2} \ln 2$$

28. Under the given transformation, $y^2 = uv$, so

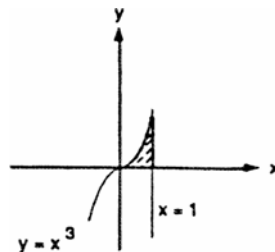
$$\iint_R y^2 dA = \int_{1/2}^2 \int_{1/2}^2 \frac{uv}{2v} du dv = \int_{1/2}^2 \left(\frac{u^2}{4} \right)_{1/2}^2 dv = \int_{1/2}^2 \frac{15}{16} dv = \frac{45}{32}$$

CHAPTER 15 PRACTICE EXERCISES

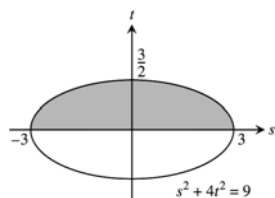
$$\begin{aligned}
 1. \quad \int_1^{10} \int_0^{1/y} ye^{xy} dx dy &= \int_1^{10} \left[e^{xy} \right]_0^{1/y} dy \\
 &= \int_1^{10} (e - 1) dy = 9e - 9
 \end{aligned}$$



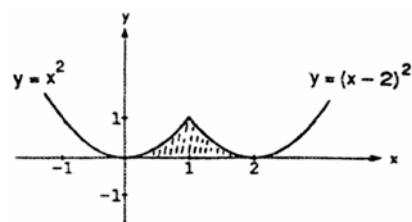
$$\begin{aligned}
 2. \quad \int_0^1 \int_0^{x^3} e^{y/x} dy dx &= \int_0^1 x \left[e^{y/x} \right]_0^{x^3} dx \\
 &= \int_0^1 (xe^{x^2} - x) dx = \left[\frac{1}{2} e^{x^2} - \frac{x^2}{2} \right]_0^1 = \frac{e-2}{2}
 \end{aligned}$$



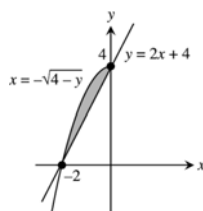
$$\begin{aligned}
 3. \quad \int_0^{3/2} \int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t ds dt &= \int_0^{3/2} [ts]_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} dt \\
 &= \int_0^{3/2} 2t\sqrt{9-4t^2} dt = \left[-\frac{1}{6} (9-4t^2)^{3/2} \right]_0^{3/2} \\
 &= -\frac{1}{6} (0^{3/2} - 9^{3/2}) = \frac{27}{6} = \frac{9}{2}
 \end{aligned}$$



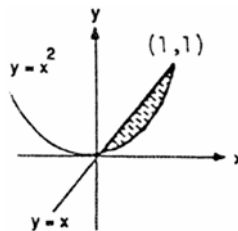
$$\begin{aligned}
 4. \quad \int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy dx dy &= \int_0^1 y \left[\frac{x^2}{2} \right]_{\sqrt{y}}^{2-\sqrt{y}} dy \\
 &= \frac{1}{2} \int_0^1 y(4 - 4\sqrt{y} + y - y) dy \\
 &= \int_0^1 (2y - 2y^{3/2}) dy = \left[y^2 - \frac{4y^{5/2}}{5} \right]_0^1 = \frac{1}{5}
 \end{aligned}$$



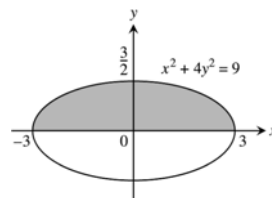
$$\begin{aligned}
 5. \quad \int_{-2}^0 \int_{2x+4}^{4-x^2} dy dx &= \int_{-2}^0 (-x^2 - 2x) dx \\
 &= \left[-\frac{x^3}{3} - x^2 \right]_{-2}^0 = -\left(\frac{8}{3} - 4 \right) = \frac{4}{3} \\
 \int_0^4 \int_{-\sqrt{4-y}}^{(y-4)/2} dx dy &= \int_0^4 \left(\frac{y-4}{2} + \sqrt{4-y} \right) dy \\
 &= \left[\frac{y^2}{2} - 2y - \frac{2}{3} (4-y)^{3/2} \right]_0^4 \\
 &= 4 - 8 + \frac{2}{3} \cdot 4^{3/2} = -4 + \frac{16}{3} = \frac{4}{3}
 \end{aligned}$$



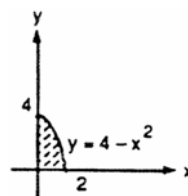
$$\begin{aligned}
6. \quad \int_0^1 \int_y^{\sqrt{y}} \sqrt{x} \, dx \, dy &= \int_0^1 \left[\frac{2}{3} x^{3/2} \right]_y^{\sqrt{y}} dy \\
&= \frac{2}{3} \int_0^1 \left(y^{3/4} - y^{3/2} \right) dy = \frac{2}{3} \left[\frac{4}{7} y^{7/4} - \frac{2}{5} y^{5/2} \right]_0^1 \\
&= \frac{2}{3} \left(\frac{4}{7} - \frac{2}{5} \right) = \frac{4}{35} \\
\int_0^1 \int_{x^2}^x \sqrt{x} \, dy \, dx &= \int_0^1 x^{1/2} (x - x^2) \, dx \\
&= \int_0^1 (x^{3/2} - x^{5/2}) \, dx \\
&= \left[\frac{2}{5} x^{5/2} - \frac{2}{7} x^{7/2} \right]_0^1 = \frac{2}{5} - \frac{2}{7} = \frac{4}{35}
\end{aligned}$$



$$\begin{aligned}
7. \quad \int_{-3}^3 \int_0^{(1/2)\sqrt{9-x^2}} y \, dy \, dx &= \int_{-3}^3 \left[\frac{y^2}{2} \right]_0^{(1/2)\sqrt{9-x^2}} dx \\
&= \int_{-3}^3 \frac{1}{8} (9 - x^2) \, dx = \left[\frac{9x}{8} - \frac{x^3}{24} \right]_{-3}^3 \\
&= \left(\frac{27}{8} - \frac{27}{24} \right) - \left(-\frac{27}{8} + \frac{27}{24} \right) = \frac{27}{6} = \frac{9}{2} \\
\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y \, dx \, dy &= \int_0^{3/2} 2y \sqrt{9-4y^2} \, dy \\
&= \left[-\frac{1}{4} \cdot \frac{2}{3} (9-4y^2)^{3/2} \right]_0^{3/2} = \frac{1}{6} \cdot 9^{3/2} = \frac{27}{6} = \frac{9}{2}
\end{aligned}$$



$$\begin{aligned}
8. \quad \int_0^2 \int_0^{4-x^2} 2x \, dy \, dx &= \int_0^2 [2xy]_0^{4-x^2} dx \\
&= \int_0^2 (2x(4-x^2)) \, dx = \int_0^2 (8x-2x^3) \, dx \\
&= \left[4x^2 - \frac{x^4}{2} \right]_0^2 = 16 - \frac{16}{2} = 8 \\
\int_0^4 \int_0^{\sqrt{4-y}} 2x \, dx \, dy &= \int_0^4 [x^2]_0^{\sqrt{4-y}} dy \\
&= \int_0^4 (4-y) \, dy = \left[4y - \frac{y^2}{2} \right]_0^4 = 16 - \frac{16}{2} = 8
\end{aligned}$$



$$9. \quad \int_0^1 \int_{2y}^2 4 \cos(x^2) \, dx \, dy = \int_0^2 \int_0^{x/2} 4 \cos(x^2) \, dy \, dx = \int_0^2 2x \cos(x^2) \, dx = \left[\sin(x^2) \right]_0^2 = \sin 4$$

$$10. \quad \int_0^2 \int_{y/2}^1 e^{x^2} \, dx \, dy = \int_0^1 \int_0^{2x} e^{x^2} \, dy \, dx = \int_0^1 2xe^{x^2} \, dx = \left[e^{x^2} \right]_0^1 = e - 1$$

$$11. \quad \int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1} \, dy \, dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1} \, dx \, dy = \frac{1}{4} \int_0^2 \frac{4y^3}{y^4+1} \, dy = \frac{\ln 17}{4}$$

$$12. \int_0^1 \int_{\sqrt[3]{y}}^1 \frac{2\pi \sin(\pi x^2)}{x^2} dx dy = \int_0^1 \int_0^{x^3} \frac{2\pi \sin(\pi x^2)}{x^2} dy dx = \int_0^1 2\pi x \sin(\pi x^2) dx = \left[-\cos(\pi x^2) \right]_0^1 = -(-1) - (-1) = 2$$

$$13. A = \int_{-2}^0 \int_{2x+4}^{4-x^2} dy dx = \int_{-2}^0 (-x^2 - 2x) dx = \frac{4}{3} \quad 14. A = \int_1^4 \int_{2-y}^{\sqrt{y}} dx dy = \int_1^4 (\sqrt{y} - 2 + y) dy = \frac{37}{6}$$

$$15. V = \int_0^1 \int_x^{2-x} (x^2 + y^2) dy dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^{2-x} dx = \int_0^1 \left[2x^2 + \frac{(2-x)^3}{3} - \frac{7x^3}{3} \right] dx = \left[\frac{2x^3}{3} - \frac{(2-x)^4}{12} - \frac{7x^4}{12} \right]_0^1 \\ = \left(\frac{2}{3} - \frac{1}{12} - \frac{7}{12} \right) + \frac{2^4}{12} = \frac{4}{3}$$

$$16. V = \int_{-3}^2 \int_x^{6-x^2} x^2 dy dx = \int_{-3}^2 \left[x^2 y \right]_x^{6-x^2} dx = \int_{-3}^2 (6x^2 - x^4 - x^3) dx = \frac{125}{4}$$

$$17. \text{average value} = \int_0^1 \int_0^1 xy dy dx = \int_0^1 \left[\frac{xy^2}{2} \right]_0^1 dx = \int_0^1 \frac{x}{2} dx = \frac{1}{4}$$

$$18. \text{average value} = \left(\frac{\pi}{4} \right) \int_0^1 \int_0^{\sqrt{1-x^2}} xy dy dx = \frac{4}{\pi} \int_0^1 \left[\frac{xy^2}{2} \right]_0^{\sqrt{1-x^2}} dx = \frac{2}{\pi} \int_0^1 (x - x^3) dx = \frac{1}{2\pi}$$

$$19. \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} dy dx = \int_0^{2\pi} \int_0^1 \frac{2r}{(1+r^2)^2} dr d\theta = \int_0^{2\pi} \left[-\frac{1}{1+r^2} \right]_0^1 d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi$$

$$20. \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1) dx dy = \int_0^{2\pi} \int_0^1 r \ln(r^2 + 1) dr d\theta = \int_0^{2\pi} \int_1^2 \frac{1}{2} \ln u du d\theta = \frac{1}{2} \int_0^{2\pi} [u \ln u - u]_1^2 d\theta \\ = \frac{1}{2} \int_0^{2\pi} (2 \ln 2 - 1) d\theta = [\ln(4) - 1]\pi$$

$$21. (x^2 + y^2)^2 - (x^2 - y^2) = 0 \Rightarrow r^4 - r^2 \cos 2\theta = 0 \Rightarrow r^2 = \cos 2\theta \text{ so the integral is } \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2} dr d\theta \\ = \int_{-\pi/4}^{\pi/4} \left[-\frac{1}{2(1+r^2)} \right]_0^{\sqrt{\cos 2\theta}} d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{1+\cos 2\theta} \right) d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{2\cos^2 \theta} \right) d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{\sec^2 \theta}{2} \right) d\theta \\ = \frac{1}{2} \left[\theta - \frac{\tan \theta}{2} \right]_{-\pi/4}^{\pi/4} = \frac{\pi-2}{4}$$

$$22. (a) \iint_R \frac{1}{(1+x^2+y^2)^2} dx dy = \int_0^{\pi/3} \int_0^{\sec \theta} \frac{r}{(1+r^2)^2} dr d\theta = \int_0^{\pi/3} \left[-\frac{1}{2(1+r^2)} \right]_0^{\sec \theta} d\theta = \int_0^{\pi/3} \left[\frac{1}{2} - \frac{1}{2(1+\sec^2 \theta)} \right] d\theta \\ = \frac{1}{2} \int_0^{\pi/3} \frac{\sec^2 \theta}{1+\sec^2 \theta} d\theta; \left[\begin{array}{l} u = \tan \theta \\ du = \sec^2 \theta d\theta \end{array} \right] \rightarrow \frac{1}{2} \int_0^{\sqrt{3}} \frac{du}{2+u^2} = \frac{1}{2} \left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} \right]_0^{\sqrt{3}} = \frac{\sqrt{2}}{4} \tan^{-1} \sqrt{\frac{3}{2}}$$

- (b)
$$\iint_R \frac{1}{(1+x^2+y^2)^2} dx dy = \int_0^{\pi/2} \int_0^\infty \frac{r}{(1+r^2)^2} dr d\theta = \int_0^{\pi/2} \lim_{b \rightarrow \infty} \left[-\frac{1}{2(1+r^2)} \right]_0^b d\theta = \int_0^{\pi/2} \lim_{b \rightarrow \infty} \left[\frac{1}{2} - \frac{1}{2(1+b^2)} \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4}$$
23.
$$\int_0^\pi \int_0^\pi \int_0^\pi \cos(x+y+z) dx dy dz = \int_0^\pi \int_0^\pi [\sin(z+y+\pi) - \sin(z+y)] dy dz$$

$$= \int_0^\pi [-\cos(z+2\pi) + \cos(z+\pi) - \cos z + \cos(z+\pi)] dz = 0$$
24.
$$\int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)} dz dy dx = \int_{\ln 6}^{\ln 7} \int_0^{\ln 2} e^{(x+y)} dy dx = \int_{\ln 6}^{\ln 7} e^x dx = 1$$
25.
$$\int_0^1 \int_0^{x^2} \int_0^{x+y} (2x-y-z) dz dy dx = \int_0^1 \int_0^{x^2} \left(\frac{3x^2}{2} - \frac{3y^2}{2} \right) dy dx = \int_0^1 \left(\frac{3x^4}{2} - \frac{x^6}{2} \right) dx = \frac{8}{35}$$
26.
$$\int_1^e \int_1^x \int_0^{\frac{2y}{z}} dy dz dx = \int_1^e \int_1^x \frac{1}{z} dz dx = \int_1^e \ln x dx = [x \ln x - x]_1^e = 1$$
27.
$$V = 2 \int_0^{\pi/2} \int_{-\cos y}^0 \int_0^{-2x} dz dx dy = 2 \int_0^{\pi/2} \int_{-\cos y}^0 (-2x) dx dy = 2 \int_0^{\pi/2} \cos^2 y dy = 2 \left[\frac{y}{2} + \frac{\sin 2y}{4} \right]_0^{\pi/2} = \frac{\pi}{2}$$
28.
$$V = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2} dz dy dx = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (4-x^2) dy dx = 4 \int_0^2 (4-x^2)^{3/2} dx$$

$$= \left[x(4-x^2)^{3/2} + 6x\sqrt{4-x^2} + 24 \sin^{-1} \frac{x}{2} \right]_0^2 = 24 \sin^{-1} 1 = 12\pi$$
29.
$$\text{average} = \frac{1}{3} \int_0^1 \int_0^3 \int_0^1 30xz\sqrt{x^2+y} dz dy dx = \frac{1}{3} \int_0^1 \int_0^3 15x\sqrt{x^2+y} dy dx = \frac{1}{3} \int_0^1 \int_0^1 15x\sqrt{x^2+y} dx dy$$

$$= \frac{1}{3} \int_0^3 \left[5(x^2+y)^{3/2} \right]_0^1 dy = \frac{1}{3} \int_0^3 [5(1+y)^{3/2} - 5y^{3/2}] dy = \frac{1}{3} [2(1+y)^{5/2} - 2y^{5/2}]_0^3 = \frac{1}{3} [2(4)^{5/2} - 2(3)^{5/2} - 2]$$

$$= \frac{1}{3} [2(31 - 3^{5/2})]$$
30.
$$\text{average} = \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^\pi \int_0^a \rho^3 \sin \phi d\rho d\phi d\theta = \frac{3a}{16\pi} \int_0^{2\pi} \int_0^\pi \sin \phi d\phi d\theta = \frac{3a}{8\pi} \int_0^{2\pi} d\theta = \frac{3a}{4}$$
31. (a)
$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3 dz dx dy$$
- (b)
$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 3\rho^2 \sin \phi d\rho d\phi d\theta$$
- (c)
$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3 dz r dr d\theta = 3 \int_0^{2\pi} \int_0^{\sqrt{2}} \left[r(4-r^2)^{1/2} - r^2 \right] dr d\theta = 3 \int_0^{2\pi} \left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3} \right]_0^{\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} (-2^{3/2} - 2^{3/2} + 4^{3/2}) d\theta = (8 - 4\sqrt{2}) \int_0^{2\pi} d\theta = 2\pi(8 - 4\sqrt{2})$$

32. (a) $\int_{-\pi/2}^{\pi/2} \int_0^1 \int_{-r^2}^r 21(r \cos \theta)(r \sin \theta)^2 dz r dr d\theta = \int_{-\pi/2}^{\pi/2} \int_0^1 \int_{-r^2}^r 21r^3 \cos \theta \sin^2 \theta dz r dr d\theta$
 (b) $\int_{-\pi/2}^{\pi/2} \int_0^1 \int_{-r^2}^r 21r^3 \cos \theta \sin^2 \theta dz r dr d\theta = 84 \int_0^{\pi/2} \int_0^1 r^6 \sin^2 \theta \cos \theta dr d\theta = 12 \int_0^{\pi/2} \sin^2 \theta \cos \theta d\theta = 4$
33. (a) $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^2 \sin \phi d\rho d\phi d\theta$
 (b) $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^2 \sin \phi d\rho d\phi d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} (\sec \phi)(\sec \phi \tan \phi) d\phi d\theta = \frac{1}{3} \int_0^{2\pi} \left[\frac{1}{2} \tan^2 \phi \right]_0^{\pi/4} d\theta$
 $= \frac{1}{6} \int_0^{2\pi} d\theta = \frac{\pi}{3}$
34. (a) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} (6+4y) dz dy dx$ (b) $\int_0^{\pi/2} \int_0^1 \int_0^r (6+4r \sin \theta) dz r dr d\theta$
 (c) $\int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^{\csc \phi} (6+4\rho \sin \phi \sin \theta) (\rho^2 \sin \phi) d\rho d\phi d\theta$
 (d) $\int_0^{\pi/2} \int_0^1 \int_0^r (6+4r \sin \theta) dz r dr d\theta = \int_0^{\pi/2} \int_0^1 (6r^2 + 4r^3 \sin \theta) dr d\theta = \int_0^{\pi/2} \left[2r^3 + r^4 \sin \theta \right]_0^1 d\theta$
 $= \int_0^{\pi/2} (2 + \sin \theta) d\theta = [2\theta - \cos \theta]_0^{\pi/2} = \pi + 1$
35. $\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2 yx dz dy dx + \int_1^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2 yx dz dy dx$
36. (a) Bounded on the top and bottom by the sphere $x^2 + y^2 + z^2 = 4$, on the right by the right circular cylinder $(x-1)^2 + y^2 = 1$, on the left by the plane $y = 0$
 (b) $\int_0^{\pi/2} \int_0^{2\cos \theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz r dr d\theta$
37. (a) $V = \int_0^{2\pi} \int_0^2 \int_2^{\sqrt{8-r^2}} dz r dr d\theta = \int_0^{2\pi} \int_0^2 (r\sqrt{8-r^2} - 2r) dr d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(8-r^2)^{3/2} - r^2 \right]_0^2 d\theta$
 $= \int_0^{2\pi} \left[-\frac{1}{3}(4)^{3/2} - 4 + \frac{1}{3}(8)^{3/2} \right] d\theta = \int_0^{2\pi} \frac{4}{3}(-2-3+2\sqrt{8}) d\theta = \frac{4}{3}(4\sqrt{2}-5) \int_0^{2\pi} d\theta = \frac{8\pi(4\sqrt{2}-5)}{3}$
 (b) $V = \int_0^{2\pi} \int_0^{\pi/4} \int_{2\sec \phi}^{\sqrt{8}} \rho^2 \sin \phi d\rho d\phi d\theta = \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} (2\sqrt{2} \sin \phi - \sec^3 \phi \sin \phi) d\phi d\theta$
 $= \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} (2\sqrt{2} \sin \phi - \tan \phi \sec^2 \phi) d\phi d\theta = \frac{8}{3} \int_0^{2\pi} \left[-2\sqrt{2} \cos \phi - \frac{1}{2} \tan^2 \phi \right]_0^{\pi/4} d\theta$
 $= \frac{8}{3} \int_0^{2\pi} \left(-2 - \frac{1}{2} + 2\sqrt{2} \right) d\theta = \frac{8}{3} \int_0^{2\pi} \left(\frac{-5+4\sqrt{2}}{2} \right) d\theta = \frac{8\pi(4\sqrt{2}-5)}{3}$
38. $I_z = \int_0^{2\pi} \int_0^{\pi/3} \int_0^2 (\rho \sin \phi)^2 (\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^2 \rho^4 \sin^3 \phi d\rho d\phi d\theta$
 $= \frac{32}{5} \int_0^{2\pi} \int_0^{\pi/3} (\sin \phi - \cos^2 \phi \sin \phi) d\phi d\theta = \frac{32}{5} \int_0^{2\pi} \left[-\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^{\pi/3} d\theta = \frac{8\pi}{3}$

39. With the centers of the spheres at the origin, $I_z = \int_0^{2\pi} \int_0^\pi \int_a^b \delta(\rho \sin \phi)^2 (\rho^2 \sin \phi) d\rho d\phi d\theta$
 $= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \int_0^\pi \sin^3 \phi d\phi d\theta = \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \int_0^\pi (\sin \phi - \cos^2 \phi \sin \phi) d\phi d\theta$
 $= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \left[-\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^\pi d\theta = \frac{4\delta(b^5 - a^5)}{15} \int_0^{2\pi} d\theta = \frac{8\pi\delta(b^5 - a^5)}{15}$
40. $I_z = \int_0^{2\pi} \int_0^\pi \int_0^{1-\cos \theta} (\rho \sin \phi)^2 (\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{2\pi} \int_0^\pi \int_0^{1-\cos \theta} \rho^4 \sin^3 \phi d\rho d\phi d\theta$
 $= \frac{1}{5} \int_0^{2\pi} \int_0^\pi (1 - \cos \phi)^5 \sin^3 \phi d\phi d\theta = \int_0^{2\pi} \int_0^\pi (1 - \cos \phi)^6 (1 + \cos \phi) \sin \phi d\phi d\theta; \begin{cases} u = 1 - \cos \phi \\ du = \sin \phi d\phi \end{cases}$
 $\rightarrow \frac{1}{5} \int_0^{2\pi} \int_0^2 u^6 (2 - u) du d\theta = \frac{1}{5} \int_0^{2\pi} \left[\frac{2u^7}{7} - \frac{u^8}{8} \right]_0^2 d\theta = \frac{1}{5} \int_0^{2\pi} \left(\frac{1}{7} - \frac{1}{8} \right) 2^8 d\theta = \frac{1}{5} \int_0^{2\pi} \frac{2^3 \cdot 2^5}{56} d\theta = \frac{32}{35} \int_0^{2\pi} d\theta = \frac{64\pi}{35}$
41. $M = \int_1^2 \int_{2/x}^2 dy dx = \int_1^2 \left(2 - \frac{2}{x} \right) dx = 2 - \ln 4$; $M_y = \int_1^2 \int_{2/x}^2 x dy dx = \int_1^2 x \left(2 - \frac{2}{x} \right) dx = 1$;
 $M_x = \int_1^2 \int_{2/x}^2 y dy dx = \int_1^2 \left(2 - \frac{2}{x^2} \right) dx = 1 \Rightarrow \bar{x} = \bar{y} = \frac{1}{2 - \ln 4}$
42. $M = \int_0^4 \int_{-2y}^{2y-y^2} dx dy = \int_0^4 (4y - y^2) dy = \frac{32}{3}$; $M_x = \int_0^4 \int_{-2y}^{2y-y^2} y dx dy = \int_0^4 (4y^2 - y^3) dy = \left[\frac{4y^3}{3} - \frac{y^4}{4} \right]_0^4 = \frac{64}{3}$;
 $M_y = \int_0^4 \int_{-2y}^{2y-y^2} x dx dy = \int_0^4 \left[\frac{(2y-y^2)^2}{2} - 2y^2 \right] dy = \left[\frac{y^5}{10} - \frac{y^4}{2} \right]_0^4 = -\frac{128}{5} \Rightarrow \bar{x} = \frac{M_y}{M} = -\frac{12}{5} \text{ and } \bar{y} = \frac{M_x}{M} = 2$
43. $I_o = \int_0^2 \int_{2x}^4 (x^2 + y^2)(3) dy dx = 3 \int_0^2 \left(4x^2 + \frac{64}{3} - \frac{14x^3}{3} \right) dx = 104$
44. (a) $I_o = \int_{-2}^2 \int_{-1}^1 (x^2 + y^2) dy dx = \int_{-2}^2 \left(2x^2 + \frac{2}{3} \right) dx = \frac{40}{3}$
 (b) $I_x = \int_{-a}^a \int_{-b}^b y^2 dy dx = \int_{-a}^a \frac{2b^3}{3} dx = \frac{4ab^3}{3}$;
 $I_y = \int_{-b}^b \int_{-a}^a x^2 dx dy = \int_{-b}^b \frac{2a^3}{3} dy = \frac{4a^3b}{3} \Rightarrow I_o = I_x + I_y = \frac{4ab^3}{3} + \frac{4a^3b}{3} = \frac{4ab(b^2 + a^2)}{3}$
45. $M = \delta \int_0^3 \int_0^{2x/3} dy dx = \delta \int_0^3 \frac{2x}{3} dx = 3\delta$; $I_x = \delta \int_0^3 \int_0^{2x/3} y^2 dy dx = \frac{8\delta}{81} \int_0^3 x^3 dx = \left(\frac{8\delta}{81} \right) \left(\frac{3^4}{4} \right) = 2\delta$
46. $M = \int_0^1 \int_{x^2}^x (x+1) dy dx = \int_0^1 (x - x^3) dx = \frac{1}{4}$; $M_x = \int_0^1 \int_{x^2}^x y(x+1) dy dx = \frac{1}{2} \int_0^1 (x^3 - x^5 + x^2 - x^4) dx = \frac{13}{120}$;
 $M_y = \int_0^1 \int_{x^2}^x x(x+1) dy dx = \int_0^1 (x^2 - x^4) dx = \frac{2}{15} \Rightarrow \bar{x} = \frac{8}{15} \text{ and } \bar{y} = \frac{13}{30}$; $I_x = \int_0^1 \int_{x^2}^x y^2(x+1) dy dx$
 $= \frac{1}{3} \int_0^1 (x^4 - x^7 + x^3 - x^6) dx = \frac{17}{280} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{17}{70}}$; $I_y = \int_0^1 \int_{x^2}^x x^2(x+1) dy dx = \int_0^1 (x^3 - x^5) dx = \frac{1}{12}$

$$47. \quad M = \int_{-1}^1 \int_{-1}^1 \left(x^2 + y^2 + \frac{1}{3} \right) dy dx = \int_{-1}^1 \left(2x^2 + \frac{4}{3} \right) dx = 4; \quad M_x = \int_{-1}^1 \int_{-1}^1 y \left(x^2 + y^2 + \frac{1}{3} \right) dy dx = \int_{-1}^1 0 dx = 0;$$

$$M_y = \int_{-1}^1 \int_{-1}^1 x \left(x^2 + y^2 + \frac{1}{3} \right) dy dx = \int_{-1}^1 \left(2x^3 + \frac{4}{3}x \right) dx = 0$$

48. Place the $\triangle ABC$ with its vertices at $A(0, 0)$, $B(b, 0)$ and $C(a, h)$. The line through the points A and C is

$$y = \frac{h}{a}x; \text{ the line through the points } C \text{ and } B \text{ is } y = \frac{h}{a-b}(x-b). \text{ Thus, } M = \int_0^h \int_{ay/h}^{(a-b)y/h+b} \delta dx dy$$

$$= b\delta \int_0^h \left(1 - \frac{y}{h} \right) dy = \frac{\delta b h}{2}; \quad I_x = \int_0^h \int_{ay/h}^{(a-b)y/h+b} y^2 \delta dx dy = b\delta \int_0^h \left(y^2 - \frac{y^3}{h} \right) dy = \frac{\delta b h^3}{12}$$

$$49. \quad M = \int_{-\pi/3}^{\pi/3} \int_0^3 r dr d\theta = \frac{9}{2} \int_{-\pi/3}^{\pi/3} d\theta = 3\pi; \quad M_y = \int_{-\pi/3}^{\pi/3} \int_0^3 r^2 \cos \theta dr d\theta = 9 \int_{-\pi/3}^{\pi/3} \cos \theta d\theta = 9\sqrt{3} \Rightarrow \bar{x} = \frac{3\sqrt{3}}{\pi}, \text{ and } \bar{y} = 0 \text{ by symmetry}$$

$$50. \quad M = \int_0^{\pi/2} \int_1^3 r dr d\theta = 4 \int_0^{\pi/2} d\theta = 2\pi; \quad M_y = \int_0^{\pi/2} \int_1^3 r^2 \cos \theta dr d\theta = \frac{26}{3} \int_0^{\pi/2} \cos \theta d\theta = \frac{26}{3} \Rightarrow \bar{x} = \frac{13}{3\pi}, \text{ and } \bar{y} = \frac{13}{3\pi} \text{ by symmetry}$$

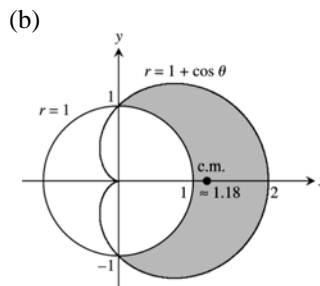
$$51. \quad (a) \quad M = 2 \int_0^{\pi/2} \int_1^{1+\cos \theta} r dr d\theta$$

$$= \int_0^{\pi/2} \left(2 \cos \theta + \frac{1+\cos 2\theta}{2} \right) d\theta = \frac{8+\pi}{4};$$

$$M_y = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos \theta} (r \cos \theta) r dr d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\cos^2 \theta + \cos^3 \theta + \frac{\cos^4 \theta}{3} \right) d\theta$$

$$= \frac{32+15\pi}{24} \Rightarrow \bar{x} = \frac{15\pi+32}{6\pi+48}, \text{ and } \bar{y} = 0 \text{ by symmetry}$$



$$52. \quad (a) \quad M = \int_{-\alpha}^{\alpha} \int_0^a r dr d\theta = \int_{-\alpha}^{\alpha} \frac{a^2}{2} d\theta = a^2 \alpha;$$

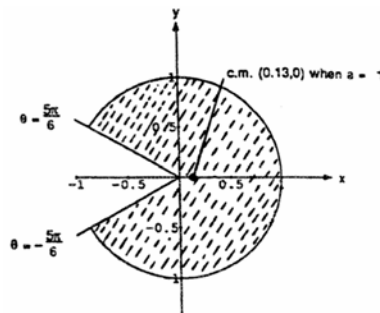
$$M_y = \int_{-\alpha}^{\alpha} \int_0^a (r \cos \theta) r dr d\theta$$

$$= \int_{-\alpha}^{\alpha} \frac{a^3 \cos \theta}{3} d\theta = \frac{2a^3 \sin \alpha}{3}$$

$$\Rightarrow \bar{x} = \frac{2a \sin \alpha}{3\alpha}, \text{ and } \bar{y} = 0 \text{ by symmetry;}$$

$$\lim_{\alpha \rightarrow \pi^-} \bar{x} = \lim_{\alpha \rightarrow \pi^-} \frac{2a \sin \alpha}{3\alpha} = 0$$

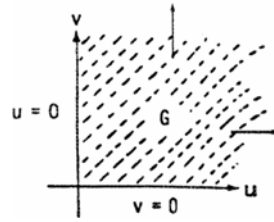
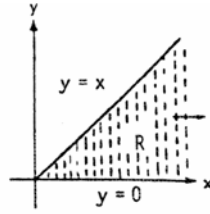
$$(b) \quad \bar{x} = \frac{2a}{5\pi} \text{ and } \bar{y} = 0$$



- 53.
- $x = u + y$
- and
- $y = v \Rightarrow x = u + v$
- and
- $y = v$

$$\Rightarrow J(u, v) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1; \text{ the boundary of the image } G$$

is obtained from the boundary of R as follows:



xy-equations for the boundary of R	Corresponding uv -equations for the boundary of G	Simplified uv -equations
$y = x$	$v = u + v$	$u = 0$
$y = 0$	$v = 0$	$v = 0$

$$\Rightarrow \int_0^\infty \int_0^x e^{-sx} f(x - y, y) dy dx = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u, v) du dv$$

54. If
- $s = \alpha x + \beta y$
- and
- $t = \gamma x + \delta y$
- where
- $(\alpha\delta - \beta\gamma)^2 = ac - b^2$
- , then
- $x = \frac{\delta s - \beta t}{\alpha\delta - \beta\gamma}$
- ,
- $y = \frac{-\gamma s + \alpha t}{\alpha\delta - \beta\gamma}$
- , and

$$\begin{aligned} J(s, t) &= \frac{1}{(\alpha\delta - \beta\gamma)^2} \begin{vmatrix} \delta & -\beta \\ -\gamma & \alpha \end{vmatrix} = \frac{1}{\alpha\delta - \beta\gamma} \Rightarrow \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(s^2+t^2)} \frac{1}{\sqrt{ac-b^2}} ds dt = \frac{1}{\sqrt{ac-b^2}} \int_0^{2\pi} \int_0^\infty re^{-r^2} dr d\theta \\ &= \frac{1}{2\sqrt{ac-b^2}} \int_0^{2\pi} d\theta = \frac{\pi}{\sqrt{ac-b^2}}. \text{ Therefore, } \frac{\pi}{\sqrt{ac-b^2}} = 1 \Rightarrow ac - b^2 = \pi^2. \end{aligned}$$

CHAPTER 15 ADDITIONAL AND ADVANCED EXERCISES

- $V = \int_{-3}^2 \int_x^{6-x^2} x^2 dy dx$
 - $V = \int_{-3}^2 \int_x^{6-x^2} \int_0^{x^2} dz dy dx$
 - $V = \int_{-3}^2 \int_x^{6-x^2} x^2 dy dx = \int_{-3}^2 \int_x^{6-x^2} (6x^2 - x^4 - x^3) dx = \left[2x^3 - \frac{x^5}{5} - \frac{x^4}{4} \right]_{-3}^2 = \frac{125}{4}$
- Place the sphere's center at the origin with the surface of the water at $z = -3$.
Then $9 = 25 - x^2 - y^2 \Rightarrow x^2 + y^2 = 16$ is the projection of the volume of water onto the xy -plane

$$\Rightarrow V = \int_0^{2\pi} \int_0^4 \int_{-\sqrt{25-r^2}}^{-3} dz r dr d\theta = \int_0^{2\pi} \int_0^4 \left(r\sqrt{25-r^2} - 3r \right) dr d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(25-r^2)^{3/2} - \frac{3}{2}r^2 \right]_0^4 d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}(9)^{3/2} - 24 + \frac{1}{3}(25)^{3/2} \right] d\theta = \int_0^{2\pi} \frac{26}{3} d\theta = \frac{52\pi}{3}$$
- Using cylindrical coordinates, $V = \int_0^{2\pi} \int_0^1 \int_0^{2-r(\cos\theta+\sin\theta)} dz r dr d\theta = \int_0^{2\pi} \int_0^1 (2r - r^2 \cos\theta - r^2 \sin\theta) dr d\theta$

$$= \int_0^{2\pi} \left(1 - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta \right) d\theta = \left[\theta - \frac{1}{3}\sin\theta + \frac{1}{3}\cos\theta \right]_0^{2\pi} = 2\pi$$

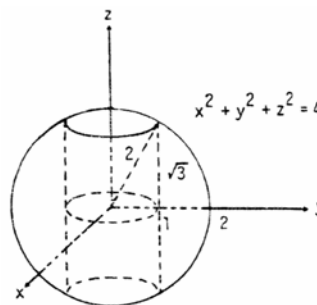
$$\begin{aligned}
 4. \quad V &= 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 \left(r\sqrt{2-r^2} - r^3 \right) dr \, d\theta = 4 \int_0^{\pi/2} \left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^4}{4} \right]_0^1 d\theta \\
 &= 4 \int_0^{\pi/2} \left(-\frac{1}{3} - \frac{1}{4} + \frac{2\sqrt{2}}{3} \right) d\theta = \left(\frac{8\sqrt{2}-7}{3} \right) \int_0^{\pi/2} d\theta = \frac{\pi(8\sqrt{2}-7)}{6}
 \end{aligned}$$

5. The surfaces intersect when $3-x^2-y^2=2x^2+2y^2 \Rightarrow x^2+y^2=1$. Thus the volume is

$$V = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{2x^2+2y^2}^{3-x^2-y^2} dz \, dy \, dx = 4 \int_0^{\pi/2} \int_0^1 \int_{2r^2}^{3-r^2} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 (3r-3r^3) dr \, d\theta = 3 \int_0^{\pi/2} d\theta = \frac{3\pi}{2}$$

$$\begin{aligned}
 6. \quad V &= 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{64}{3} \int_0^{\pi/2} \int_0^{\pi/2} \sin^4\phi \, d\phi \, d\theta \\
 &= \frac{64}{3} \int_0^{\pi/2} \left[-\frac{\sin^3\phi \cos\phi}{4} \right]_0^{\pi/2} + \frac{3}{4} \int_0^{\pi/2} \sin^2\phi \, d\phi \, d\theta = 16 \int_0^{\pi/2} \left[\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right]_0^{\pi/2} d\theta = 4\pi \int_0^{\pi/2} d\theta = 2\pi^2
 \end{aligned}$$

7. (a) The radius of the hole is 1, and the radius of the sphere is 2.



$$(b) \quad V = 2 \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} (3-z^2) dz \, d\theta = 2\sqrt{3} \int_0^{2\pi} d\theta = 4\sqrt{3}\pi$$

$$\begin{aligned}
 8. \quad V &= \int_0^\pi \int_0^{3\sin\theta} \int_0^{\sqrt{9-r^2}} dz \, r \, dr \, d\theta = \int_0^\pi \int_0^{3\sin\theta} r\sqrt{9-r^2} dr \, d\theta = \int_0^\pi \left[-\frac{1}{3}(9-r^2)^{3/2} \right]_0^{3\sin\theta} d\theta \\
 &= \int_0^\pi \left[-\frac{1}{3}(9-9\sin^2\theta)^{3/2} + \frac{1}{3}(9)^{3/2} \right] d\theta = 9 \int_0^\pi \left[1 - (1-\sin^2\theta)^{3/2} \right] d\theta = 9 \int_0^\pi (1-\cos^3\theta) d\theta \\
 &= \int_0^\pi (1-\cos\theta + \sin^2\theta \cos\theta) d\theta = 9 \left[\theta - \sin\theta + \frac{\sin^3\theta}{3} \right]_0^\pi = 9\pi
 \end{aligned}$$

9. The surfaces intersect when $x^2+y^2 = \frac{x^2+y^2+1}{2} \Rightarrow x^2+y^2=1$. Thus the volume in cylindrical coordinates is

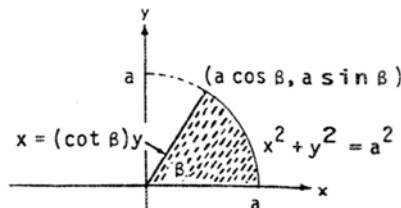
$$V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{(r^2+1)/2} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 \left(\frac{r}{2} - \frac{r^3}{2} \right) dr \, d\theta = 4 \int_0^{\pi/2} \left[\frac{r^2}{4} - \frac{r^4}{8} \right]_0^1 d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

$$\begin{aligned}
 10. \quad V &= \int_0^{\pi/2} \int_1^2 \int_0^{r^2 \sin\theta \cos\theta} dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_1^2 r^3 \sin\theta \cos\theta \, dr \, d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_1^2 \sin\theta \cos\theta \, d\theta \\
 &= \frac{15}{4} \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta = \frac{15}{4} \left[\frac{\sin^2\theta}{2} \right]_0^{\pi/2} = \frac{15}{8}
 \end{aligned}$$

$$11. \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \int_0^\infty \int_a^b e^{-xy} dy dx = \int_a^b \int_0^\infty e^{-xy} dx dy = \int_a^b \left(\lim_{t \rightarrow \infty} \int_0^t e^{-xy} dx \right) dy = \int_a^b \lim_{t \rightarrow \infty} \left[-\frac{e^{-xy}}{y} \right]_0^t dy \\ = \int_a^b \lim_{t \rightarrow \infty} \left(\frac{1}{y} - \frac{e^{-yt}}{y} \right) dy = \int_a^b \frac{1}{y} dy = [\ln y]_a^b = \ln \left(\frac{b}{a} \right)$$

12. (a) The region of integration is sketched at the right

$$\Rightarrow \int_0^{a \sin \beta} \int_{y \cot \beta}^{\sqrt{a^2 - y^2}} \ln(x^2 + y^2) dx dy \\ = \int_0^\beta \int_0^a r \ln(r^2) dr d\theta; \\ \left[\begin{array}{l} u = r^2 \\ du = 2r dr \end{array} \right] \rightarrow \frac{1}{2} \int_0^\beta \int_0^{a^2} \ln u du d\theta \\ = \frac{1}{2} \int_0^\beta [u \ln u - u]_0^{a^2} d\theta = \frac{1}{2} \int_0^\beta \left[2a^2 \ln a - a^2 - \lim_{t \rightarrow 0} t \ln t \right] d\theta = \frac{a^2}{2} \int_0^\beta (2 \ln a - 1) d\theta = a^2 \beta \left(\ln a - \frac{1}{2} \right)$$



$$(b) \int_0^{a \cos \beta} \int_0^{(\tan \beta)x} \ln(x^2 + y^2) dy dx + \int_{a \cos \beta}^a \int_0^{\sqrt{a^2 - x^2}} \ln(x^2 + y^2) dy dx$$

$$13. \int_0^x \int_0^u e^{m(x-t)} f(t) dt du = \int_0^x \int_t^x e^{m(x-t)} f(t) du dt = \int_0^x (x-t) e^{m(x-t)} f(t) dt; \text{ also} \\ \int_0^x \int_0^v \int_0^u e^{m(x-t)} f(t) dt du dv = \int_0^x \int_t^x \int_t^v e^{m(x-t)} f(t) du dv dt = \int_0^x \int_t^x (v-t) e^{m(x-t)} f(t) dv dt \\ = \int_0^x \left[\frac{1}{2} (v-t)^2 e^{m(x-t)} f(t) \right]_t^x dt = \int_0^x \frac{(x-t)^2}{2} e^{m(x-t)} f(t) dt$$

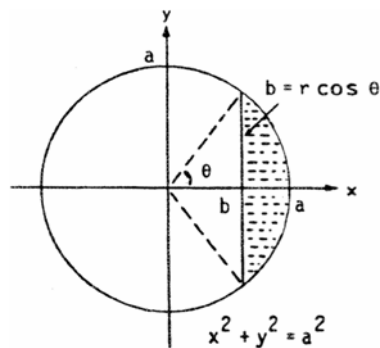
$$14. \int_0^1 f(x) \left(\int_0^x g(x-y) f(y) dy \right) dx = \int_0^1 \int_0^x g(x-y) f(x) f(y) dy dx = \int_0^1 \int_y^1 g(x-y) f(x) f(y) dx dy \\ = \int_0^1 f(y) \left(\int_y^1 g(x-y) f(x) dx \right) dy; \\ \int_0^1 \int_0^1 g(|x-y|) f(x) f(y) dx dy = \int_0^1 \int_0^x g(x-y) f(x) f(y) dy dx + \int_0^1 \int_x^1 g(y-x) f(x) f(y) dy dx \\ = \int_0^1 \int_y^1 g(x-y) f(x) f(y) dx dy + \int_0^1 \int_x^1 g(y-x) f(x) f(y) dy dx \\ = \int_0^1 \int_y^1 g(x-y) f(x) f(y) dx dy + \underbrace{\int_0^1 \int_y^1 g(x-y) f(y) f(x) dx dy}_{\text{simply interchange } x \text{ and } y \text{ variable names}} = 2 \int_0^1 \int_y^1 g(x-y) f(x) f(y) dx dy,$$

and the statement now follows.

$$15. I_o(a) = \int_0^a \int_0^{x/a^2} (x^2 + y^2) dy dx = \int_0^a \left[x^2 y + \frac{y^3}{3} \right]_0^{x/a^2} dx = \int_0^a \left(\frac{x^3}{a^2} + \frac{x^3}{3a^6} \right) dx = \left[\frac{x^4}{4a^2} + \frac{x^4}{12a^6} \right]_0^a = \frac{a^2}{4} + \frac{1}{12} a^{-2}; \\ I'_o(a) = \frac{1}{2} a - \frac{1}{6} a^{-3} = 0 \Rightarrow a^4 = \frac{1}{3} \Rightarrow a = \sqrt[4]{\frac{1}{3}} = \frac{1}{\sqrt[4]{3}}. \text{ Since } I''_o(a) = \frac{1}{2} + \frac{1}{2} a^{-4} > 0, \text{ the value of } a \text{ does provide a} \\ \text{minimum for the polar moment of inertia } I_o(a).$$

$$16. I_o = \int_0^2 \int_{2x}^4 (x^2 + y^2) (3) dy dx = 3 \int_0^2 \left(4x^2 - \frac{14x^3}{3} + \frac{64}{3} \right) dx = 104$$

$$\begin{aligned}
 17. \quad M &= \int_{-\theta}^{\theta} \int_{b \sec \theta}^a r \, dr \, d\theta = \int_{-\theta}^{\theta} \left(\frac{a^2}{2} - \frac{b^2}{2} \sec^2 \theta \right) d\theta \\
 &= a^2 \theta - b^2 \tan \theta = a^2 \cos^{-1} \left(\frac{b}{a} \right) - b^2 \left(\frac{\sqrt{a^2 - b^2}}{b} \right) \\
 &= a^2 \cos^{-1} \left(\frac{b}{a} \right) - b \sqrt{a^2 - b^2}; \\
 I_o &= \int_{-\theta}^{\theta} \int_{b \sec \theta}^a r^3 \, dr \, d\theta = \frac{1}{4} \int_{-\theta}^{\theta} (a^4 + b^4 \sec^4 \theta) d\theta \\
 &= \frac{1}{4} \int_{-\theta}^{\theta} [a^4 + b^4 (1 + \tan^2 \theta) (\sec^2 \theta)] d\theta \\
 &= \frac{1}{4} \left[a^4 \theta - b^4 \tan \theta - \frac{b^4 \tan^3 \theta}{3} \right]_{-\theta}^{\theta} = \frac{a^4 \theta}{2} - \frac{b^4 \tan \theta}{2} - \frac{b^4 \tan^3 \theta}{6} = \frac{1}{2} a^4 \cos^{-1} \left(\frac{b}{a} \right) - \frac{1}{2} b^3 \sqrt{a^2 - b^2} - \frac{1}{6} b^3 (a^2 - b^2)^{3/2}
 \end{aligned}$$



$$\begin{aligned}
 18. \quad M &= \int_{-2}^2 \int_{1-(y^2/4)}^{2-(y^2/2)} x \, dx \, dy = \int_{-2}^2 \left(1 - \frac{y^2}{4} \right) dy = \left[y - \frac{y^3}{12} \right]_{-2}^2 = \frac{8}{3}; \quad M_y = \int_{-2}^2 \int_{1-(y^2/4)}^{2-(y^2/2)} x^2 \, dx \, dy \\
 &= \int_{-2}^2 \left[\frac{x^3}{3} \right]_{1-(y^2/4)}^{2-(y^2/2)} dy = \int_{-2}^2 \frac{3}{32} (4 - y^2) dy = \frac{3}{32} \int_{-2}^2 (16 - 8y^2 + y^4) dy = \frac{3}{16} \left[16y - \frac{8y^3}{3} + \frac{y^5}{5} \right]_0^2 \\
 &= \frac{3}{16} \left(32 - \frac{64}{3} + \frac{32}{5} \right) = \left(\frac{3}{16} \right) \left(\frac{328}{15} \right) = \frac{48}{15} = \frac{3}{2} \int_{-2}^2 (16 - 8y^2 + y^4) dy = \frac{3}{16} \left[16y - \frac{8y^3}{3} + \frac{y^5}{5} \right]_0^2 \text{ and } \bar{y} = 0 \text{ by symmetry}
 \end{aligned}$$

$$19. \quad = \left[\frac{1}{2ab} e^{b^2 x^2} \right]_0^a + \left[\frac{1}{2ba} e^{a^2 y^2} \right]_0^b = \frac{1}{2ab} (e^{b^2 a^2} - 1) + \frac{1}{2ab} (e^{a^2 b^2} - 1) = \frac{1}{ab} (e^{a^2 b^2} - 1)$$

$$\begin{aligned}
 20. \quad \int_{y_0}^{y_1} \int_{x_0}^{x_1} \frac{\partial^2 F(x, y)}{\partial x \partial y} \, dx \, dy &= \int_{y_0}^{y_1} \left[\frac{\partial F(x, y)}{\partial y} \right]_{x_0}^{x_1} dy = \int_{y_0}^{y_1} \left[\frac{\partial F(x_1, y)}{\partial y} - \frac{\partial F(x_0, y)}{\partial y} \right] dy = [F(x_1, y) - F(x_0, y)]_{y_0}^{y_1} \\
 &= F(x_1, y_1) - F(x_0, y_1) - F(x_1, y_0) + F(x_0, y_0)
 \end{aligned}$$

21. (a) (i) Fubini's Theorem
 (ii) Treating $G(y)$ as a constant
 (iii) Algebraic rearrangement
 (iv) The definite integral is a constant number

$$(b) \quad \int_0^{\ln 2} \int_0^{\pi/2} e^x \cos y \, dy \, dx = \left(\int_0^{\ln 2} e^x \, dx \right) \left(\int_0^{\pi/2} \cos y \, dy \right) = (e^{\ln 2} - e^0) (\sin \frac{\pi}{2} - \sin 0) = (1)(1) = 1$$

$$(c) \quad \int_1^2 \int_{-1}^1 \frac{x}{y^2} \, dx \, dy = \left(\int_1^2 \frac{1}{y^2} \, dy \right) \left(\int_{-1}^1 x \, dx \right) = \left[-\frac{1}{y} \right]_{-1}^2 \left[\frac{x^2}{2} \right]_{-1}^1 = \left(-\frac{1}{2} + 1 \right) \left(\frac{1}{2} - \frac{1}{2} \right) = 0$$

22. (a) $\nabla f = x\mathbf{i} + y\mathbf{j} \Rightarrow D_u f = u_1 x + u_2 y$; the area of the region of integration is $\frac{1}{2}$

$$\begin{aligned}
 \Rightarrow \text{average} &= 2 \int_0^1 \int_0^{1-x} (u_1 x + u_2 y) \, dy \, dx = 2 \int_0^1 \left[u_1 x(1-x) + \frac{1}{2} u_2 (1-x)^2 \right] dx \\
 &= 2 \left[u_1 \left(\frac{x^2}{2} - \frac{x^3}{3} \right) - \left(\frac{1}{2} u_2 \right) \frac{(1-x)^3}{3} \right]_0^1 = 2 \left(\frac{1}{6} u_1 + \frac{1}{6} u_2 \right) = \frac{1}{3} (u_1 + u_2)
 \end{aligned}$$

$$(b) \quad \text{average} = \frac{1}{\text{area}} \iint_R (u_1 x + u_2 y) \, dA = \frac{u_1}{\text{area}} \iint_R x \, dA + \frac{u_2}{\text{area}} \iint_R y \, dA = u_1 \left(\frac{M_y}{M} \right) + u_2 \left(\frac{M_x}{M} \right) = u_1 \bar{x} + u_2 \bar{y}$$

$$\begin{aligned}
 23. \quad (a) \quad I^2 &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^{\pi/2} \int_0^\infty (e^{-r^2}) r dr d\theta = \int_0^{\pi/2} \left[\lim_{b \rightarrow \infty} \int_0^b r e^{-r^2} dr \right] d\theta \\
 &= -\frac{1}{2} \int_0^{\pi/2} \lim_{b \rightarrow \infty} (e^{-b^2} - 1) d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2} \\
 (b) \quad \Gamma\left(\frac{1}{2}\right) &= \int_0^\infty t^{-1/2} e^{-t} dt = \int_0^\infty (y^2)^{-1/2} e^{-y^2} (2y) dy = 2 \int_0^\infty e^{-y^2} dy = 2 \left(\frac{\sqrt{\pi}}{2} \right) = \sqrt{\pi}, \text{ where } y = \sqrt{t}
 \end{aligned}$$

$$24. \quad Q = \int_0^{2\pi} \int_0^R kr^2(1 - \sin \theta) dr d\theta = \frac{kR^3}{3} \int_0^{2\pi} (1 - \sin \theta) d\theta = \frac{kR^3}{3} [\theta + \cos \theta]_0^{2\pi} = \frac{2\pi kR^3}{3}$$

$$\begin{aligned}
 25. \quad \text{For a height } h \text{ in the bowl the volume of water is } V &= \int_{-\sqrt{h}}^{\sqrt{h}} \int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} \int_{x^2+y^2}^h dz dy dx \\
 &= \int_{-\sqrt{h}}^{\sqrt{h}} \int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} (h - x^2 - y^2) dy dx = \int_0^{2\pi} \int_0^{\sqrt{h}} (h - r^2) r dr d\theta = \int_0^{2\pi} \left[\frac{hr^2}{2} - \frac{r^4}{4} \right]_0^{\sqrt{h}} d\theta = \int_0^{2\pi} \frac{h^2}{4} d\theta = \frac{h^2\pi}{2}.
 \end{aligned}$$

Since the top of the bowl has area 30π , then we calibrate the bowl by comparing it to a right circular cylinder whose cross sectional area is 30π from $z = 0$ to $z = 30$. If such a cylinder contains $\frac{h^2\pi}{2}$ cubic centimeters of water to a depth w then we have $30\pi w = \frac{h^2\pi}{2} \Rightarrow w = \frac{h^2}{60}$. So for 3 cm of rain, $w = 3$ and $h = \sqrt{180}$; for 9 cm of rain, $w = 9$ and $h = \sqrt{540}$.

26. (a) An equation for the satellite dish in standard position is $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Since the axis is tilted 30° , a unit vector $\mathbf{v} = 0\mathbf{i} + a\mathbf{j} + b\mathbf{k}$ normal to the plane of the water level satisfies

$$\begin{aligned}
 b &= \mathbf{v} \cdot \mathbf{k} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\
 \Rightarrow a &= -\sqrt{1-b^2} = -\frac{1}{2} \\
 \Rightarrow \mathbf{v} &= -\frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k} \\
 \Rightarrow -\frac{1}{2}(y-1) + \frac{\sqrt{3}}{2}\left(z - \frac{1}{2}\right) &= 0
 \end{aligned}$$

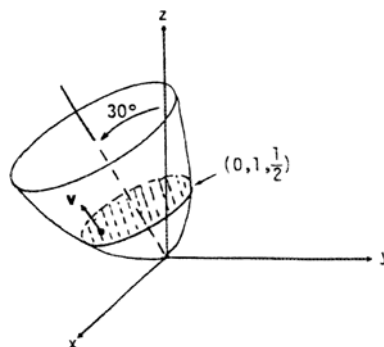
$\Rightarrow z = \frac{1}{\sqrt{3}}y + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)$ is an equation of the plane of the water level. Therefore the volume of water is

$$V = \iiint_R \int_{\frac{1}{2}x^2 + \frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y + \frac{1}{2} - \frac{1}{\sqrt{3}}} dz dy dx, \text{ where } R \text{ is the interior of the ellipse } x^2 + y^2 - \frac{2}{\sqrt{3}}y - 1 + \frac{2}{\sqrt{3}} = 0. \text{ When } x = 0,$$

$$\text{then } y = \alpha \text{ or } y = \beta, \quad \alpha = \frac{\frac{2}{\sqrt{3}} + \sqrt{\frac{4}{3} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2} \text{ and } \beta = \frac{\frac{2}{\sqrt{3}} - \sqrt{\frac{4}{3} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2}$$

$$\Rightarrow V = \int_\alpha^\beta \int_{-\left(\frac{2}{3}y + 1 - \frac{2}{\sqrt{3}}y^2\right)^{1/2}}^{\left(\frac{2}{3}y + 1 - \frac{2}{\sqrt{3}}y^2\right)^{1/2}} \int_{\frac{1}{2}x^2 + \frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y + \frac{1}{2} - \frac{1}{\sqrt{3}}} 1 dz dx dy$$

- (b) $x = 0 \Rightarrow z = \frac{1}{2}y^2$ and $\frac{dz}{dy} = y$; $y = 1 \Rightarrow \frac{dz}{dy} = 1 \Rightarrow$ the tangent line has slope 1 or a 45° slant \Rightarrow at 45° and thereafter, the dish will not hold water.



27. The cylinder is given by $x^2 + y^2 = 1$ from $z = 1$ to $\infty \Rightarrow \iiint_D z(r^2 + z^2)^{-5/2} dV$

$$\begin{aligned}
 &= \int_0^{2\pi} \int_0^1 \int_1^\infty \frac{z}{(r^2 + z^2)^{5/2}} dz r dr d\theta = \lim_{a \rightarrow \infty} \int_0^{2\pi} \int_0^1 \int_1^a \frac{r z}{(r^2 + z^2)^{5/2}} dz dr d\theta = \lim_{a \rightarrow \infty} \int_0^{2\pi} \int_0^1 \left[\left(-\frac{1}{3}\right) \frac{r}{(r^2 + z^2)^{3/2}} \right]_1^a dr d\theta \\
 &= \lim_{a \rightarrow \infty} \int_0^{2\pi} \int_0^1 \left[\left(-\frac{1}{3}\right) \frac{r}{(r^2 + a^2)^{3/2}} + \left(\frac{1}{3}\right) \frac{r}{(r^2 + 1)^{3/2}} \right] dr d\theta = \lim_{a \rightarrow \infty} \int_0^{2\pi} \left[\frac{1}{3} (r^2 + a^2)^{-1/2} - \frac{1}{3} (r^2 + 1)^{-1/2} \right]_0^1 d\theta \\
 &= \lim_{a \rightarrow \infty} \int_0^{2\pi} \left[\frac{1}{3} (1 + a^2)^{-1/2} - \frac{1}{3} (2^{-1/2}) - \frac{1}{3} (a^2)^{-1/2} + \frac{1}{3} \right] d\theta = \lim_{a \rightarrow \infty} 2\pi \left[\frac{1}{3} (1 + a^2)^{-1/2} - \frac{1}{3} \left(\frac{\sqrt{2}}{2}\right) - \frac{1}{3} \left(\frac{1}{a}\right) + \frac{1}{3} \right] \\
 &= 2\pi \left[\frac{1}{3} - \left(\frac{1}{3}\right) \frac{\sqrt{2}}{2} \right].
 \end{aligned}$$

28. Let's see?

The length of the "unit" line segment is: $L = 2 \int_0^1 dx = 2$.

The area of the unit circle is: $A = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} dy dx = \pi$.

The volume of the unit sphere is: $V = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dy dx = \frac{4}{3} \pi$.

Therefore, the hypervolume of the unit 4-sphere should be:

$$V_{\text{hyper}} = 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \int_0^{\sqrt{1-x^2-y^2-z^2}} dw dz dy dx.$$

Mathematica is able to handle this integral, but we'll use the brute force approach.

$$\begin{aligned}
 V_{\text{hyper}} &= 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \int_a^{\sqrt{1-x^2-y^2-z^2}} dw dz dy dx = 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2-z^2} dz dy dx \\
 &= 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2} \sqrt{1-\frac{z^2}{1-x^2-y^2}} dz dy dx; \left[\begin{array}{l} \frac{z}{\sqrt{1-x^2-y^2}} = \cos \theta \\ dz = -\sqrt{1-x^2-y^2} \sin \theta d\theta \end{array} \right] \\
 &= 16 \int_0^1 \int_0^{\sqrt{1-x^2}} (1-x^2-y^2) \int_{\pi/2}^0 \left(-\sqrt{1-\cos^2 \theta} \sin \theta \right) d\theta dy dx = 16 \int_0^1 \int_0^{\sqrt{1-x^2}} (1-x^2-y^2) \int_{\pi/2}^0 \left(-\sin^2 \theta \right) d\theta dy dx \\
 &= 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{4} (1-x^2-y^2) dy dx = 4\pi \int_0^1 \left(\sqrt{1-x^2} - x^2 \sqrt{1-x^2} - \frac{1}{3} (1-x^2)^{3/2} \right) dx \\
 &= 4\pi \int_0^1 \sqrt{1-x^2} \left[(1-x^2) - \frac{1-x^3}{3} \right] dx = \frac{8}{3} \pi \int_0^1 (1-x^2)^{3/2} dx; \left[\begin{array}{l} x = \cos \theta \\ dx = -\sin \theta d\theta \end{array} \right] \\
 &= -\frac{8}{3} \pi \int_{\pi/2}^0 \sin^4 \theta d\theta = -\frac{8}{3} \pi \int_{\pi/2}^0 \left(\frac{1-\cos 2\theta}{2} \right)^2 d\theta = -\frac{2}{3} \pi \int_{\pi/2}^0 (1-2\cos 2\theta + \cos^2 2\theta) d\theta \\
 &= -\frac{2}{3} \pi \int_{\pi/2}^0 \left(\frac{3}{2} - 2\cos 2\theta + \frac{\cos 4\theta}{2} \right) d\theta = \frac{\pi^2}{2}
 \end{aligned}$$