CHAPTER 16 INTEGRALS AND VECTOR FIELDS

16.1 LINE INTEGRALS

1. $\mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j} \Rightarrow x = t$ and $y = 1-t \Rightarrow y = 1-x \Rightarrow (c)$

2. $\mathbf{r} = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1,\ y = 1,$ and $z = t \Rightarrow (e)$

3. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} \Rightarrow x = 2\cos t$ and $y = 2\sin t \Rightarrow x^2 + y^2 = 4 \Rightarrow (g)$

4. $\mathbf{r} = t\mathbf{i} \Rightarrow x = t,\ y = 0,$ and $z = 0 \Rightarrow (a)$

5. $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \Rightarrow x = t,\ y = t,$ and $z = t \Rightarrow (d)$

6. $\mathbf{r} = t\mathbf{j} + (2-2t)\mathbf{k} \Rightarrow y = t$ and $z = 2-2t \Rightarrow z = 2-2y \Rightarrow (b)$

7. $\mathbf{r} = \left(t^2-1\right)\mathbf{j} + 2t\mathbf{k} \Rightarrow y = t^2-1$ and $z = 2t \Rightarrow y = \frac{z^2}{4} - 1 \Rightarrow (f)$

8. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{k} \Rightarrow x = 2\cos t$ and $z = 2\sin t \Rightarrow x^2 + z^2 = 4 \Rightarrow (h)$

9. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2}\mathbf{j};\ x = t$ and $y = 1-t \Rightarrow x + y = t + (1-t) = 1$
$\Rightarrow \int_C f(x, y, z)\, ds = \int_0^1 f(t, 1-t, 0)\left|\frac{d\mathbf{r}}{dt}\right| dt = \int_0^1 (1)\left(\sqrt{2}\right) dt = \left[\sqrt{2}t\right]_0^1 = \sqrt{2}$

10. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j} + \mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2};\ x = t,\ y = 1-t,$ and $z = 1 \Rightarrow x - y + z - 2$
$= t - (1-t) + 1 - 2 = 2t - 2 \Rightarrow \int_C f(x, y, z)\, ds = \int_0^1 (2t-2)\sqrt{2}\, dt = \sqrt{2}\left[t^2 - 2t\right]_0^1 = -\sqrt{2}$

11. $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2-2t)\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{4+1+4} = 3;\ xy + y + z = (2t)t + t + (2-2t)$
$\Rightarrow \int_C f(x, y, z)\, ds = \int_0^1 \left(2t^2 - t + 2\right) 3\, dt = 3\left[\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t\right]_0^1 = 3\left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{13}{2}$

12. $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k},\ -2\pi \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} + 3\mathbf{k}$
$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = 5;\ \sqrt{x^2 + y^2} = \sqrt{16\cos^2 t + 16\sin^2 t} = 4 \Rightarrow \int_C f(x, y, z)\, ds = \int_{-2\pi}^{2\pi} (4)(5)\, dt$
$= \left[20t\right]_{-2\pi}^{2\pi} = 80\pi$

13. $\mathbf{r}(t) = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + t(-\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = (1-t)\mathbf{i} + (2-3t)\mathbf{j} + (3-2t)\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$
$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+9+4} = \sqrt{14};\ x + y + z = (1-t) + (2-3t) + (3-2t) = 6 - 6t \Rightarrow \int_C f(x, y, z)\, ds$
$= \int_0^1 (6-6t)\sqrt{14}\, dt = 6\sqrt{14}\left[t - \frac{t^2}{2}\right]_0^1 = \left(6\sqrt{14}\right)\left(\frac{1}{2}\right) = 3\sqrt{14}$

14. $\mathbf{r}(t)=t\mathbf{i}+t\mathbf{j}+t\mathbf{k},\ 1\le t\le\infty \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{i}+\mathbf{j}+\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{3};\ \frac{\sqrt{3}}{x^2+y^2+z^2}=\frac{\sqrt{3}}{t^2+t^2+t^2}=\frac{\sqrt{3}}{3t^2}$

$\Rightarrow \int_C f(x,y,z)\,ds=\int_1^\infty\left(\frac{\sqrt{3}}{3t^2}\right)\sqrt{3}\,dt=\left[-\frac{1}{t}\right]_1^\infty=\lim_{b\to\infty}\left(-\frac{1}{b}+1\right)=1$

15. $C_1:\mathbf{r}(t)=t\mathbf{i}+t^2\mathbf{j},\ 0\le t\le 1 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{i}+2t\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{1+4t^2};\ x+\sqrt{y}-z^2=t+\sqrt{t^2}-0=t+|t|=2t$ since

$t\ge 0 \Rightarrow \int_{C_1} f(x,y,z)\,ds=\int_0^1 2t\sqrt{1+4t^2}\,dt=\left[\frac{1}{6}\left(1+4t^2\right)^{3/2}\right]_0^1=\frac{1}{6}(5)^{3/2}-\frac{1}{6}=\frac{1}{6}\left(5\sqrt{5}-1\right);$

$C_2:\mathbf{r}(t)=\mathbf{i}+\mathbf{j}+t\mathbf{k},\ 0\le t\le 1 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=1;\ x+\sqrt{y}-z^2=1+\sqrt{1}-t^2=2-t^2$

$\Rightarrow \int_{C_2} f(x,y,z)\,ds=\int_0^1\left(2-t^2\right)(1)dt=\left[2t-\frac{1}{3}t^3\right]_0^1=2-\frac{1}{3}=\frac{5}{3};$

therefore $\int_C f(x,y,z)\,ds=\int_{C_1} f(x,y,z)\,ds+\int_{C_2} f(x,y,z)\,ds=\frac{5}{6}\sqrt{5}+\frac{3}{2}$

16. $C_1:\mathbf{r}(t)=t\mathbf{k},\ 0\le t\le 1 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=1;\ x+\sqrt{y}-z^2=0+\sqrt{0}-t^2=-t^2$

$\Rightarrow \int_{C_1} f(x,y,z)\,ds=\int_0^1\left(-t^2\right)(1)\,dt=\left[-\frac{t^3}{3}\right]_0^1=-\frac{1}{3};$

$C_2:\mathbf{r}(t)=t\mathbf{j}+\mathbf{k},\ 0\le t\le 1 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=1;\ x+\sqrt{y}-z^2=0+\sqrt{t}-1=\sqrt{t}-1$

$\Rightarrow \int_{C_2} f(x,y,z)\,ds=\int_0^1\left(\sqrt{t}-1\right)(1)\,dt=\left[\frac{2}{3}t^{3/2}-t\right]_0^1=\frac{2}{3}-1=-\frac{1}{3};$

$C_3:\mathbf{r}(t)=t\mathbf{i}+\mathbf{j}+\mathbf{k},\ 0\le t\le 1 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{i} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=1;\ x+\sqrt{y}-z^2=t+\sqrt{1}-1=t$

$\Rightarrow \int_{C_3} f(x,y,z)\,ds=\int_0^1(t)(1)\,dt=\left[\frac{t^2}{2}\right]_0^1=\frac{1}{2}$

$\Rightarrow \int_C f(x,y,z)\,ds=\int_{C_1} f\,ds+\int_{C_2} f\,ds+\int_{C_3} f\,ds=-\frac{1}{3}+\left(-\frac{1}{3}\right)+\frac{1}{2}=-\frac{1}{6}$

17. $\mathbf{r}(t)=t\mathbf{i}+t\mathbf{j}+t\mathbf{k},\ 0<a\le t\le b \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{i}+\mathbf{j}+\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{3};\ \frac{x+y+z}{x^2+y^2+z^2}=\frac{t+t+t}{t^2+t^2+t^2}=\frac{1}{t}$

$\Rightarrow \int_C f(x,y,z)\,ds=\int_a^b\left(\frac{1}{t}\right)\sqrt{3}\,dt=\left[\sqrt{3}\ln|t|\right]_a^b=\sqrt{3}\ln\left(\frac{b}{a}\right)$, since $0<a\le b$

18. $\mathbf{r}(t)=(a\cos t)\mathbf{j}+(a\sin t)\mathbf{k},\ 0\le t\le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt}=(-a\sin t)\mathbf{j}+(a\cos t)\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{a^2\sin^2 t+a^2\cos^2 t}=|a|;$

$-\sqrt{x^2+z^2}=-\sqrt{0+a^2\sin^2 t}=\begin{cases}-|a|\sin t,\ 0\le t\le\pi\\ |a|\sin t,\ \pi\le t\le 2\pi\end{cases} \Rightarrow \int_C f(x,y,z)\,ds=\int_0^\pi -|a|^2\sin t\,dt+\int_\pi^{2\pi}|a|^2\sin t\,dt$

$=\left[a^2\cos t\right]_0^\pi-\left[a^2\cos t\right]_\pi^{2\pi}=\left[a^2(-1)-a^2\right]-\left[a^2-a^2(-1)\right]=-4a^2$

19. (a) $\mathbf{r}(t)=t\mathbf{i}+\frac{1}{2}t\mathbf{j},\ 0\le t\le 4 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{i}+\frac{1}{2}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\frac{\sqrt{5}}{2} \Rightarrow \int_C x\,ds=\int_0^4 t\frac{\sqrt{5}}{2}dt=\frac{\sqrt{5}}{2}\int_0^4 t\,dt=\left[\frac{\sqrt{5}}{4}t^2\right]_0^4=4\sqrt{5}$

(b) $\mathbf{r}(t)=t\mathbf{i}+t^2\mathbf{j},\ 0\le t\le 2 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{i}+2t\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{1+4t^2} \Rightarrow \int_C x\,ds=\int_0^2 t\sqrt{1+4t^2}\,dt$

$=\left[\frac{1}{12}\left(1+4t^2\right)^{3/2}\right]_0^2=\frac{17\sqrt{17}-1}{12}$

20. (a) $\mathbf{r}(t) = t\mathbf{i} + 4t\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 4\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{17} \Rightarrow \int_C \sqrt{x+2y}\ ds = \int_0^1 \sqrt{t+2(4t)}\sqrt{17}\ dt$

$= \sqrt{17}\int_0^1 \sqrt{9t}\ dt = 3\sqrt{17}\int_0^1 \sqrt{t}\ dt = \left[2\sqrt{17}\ t^{2/3}\right]_0^1 = 2\sqrt{17}$

(b) $C_1 : \mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1;\ C_2 : \mathbf{r}(t) = \mathbf{i} + t\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$

$\int_C \sqrt{x+2y}\ ds = \int_{C_1} \sqrt{x+2y}\ ds + \int_{C_2} \sqrt{x+2y}\ ds = \int_0^1 \sqrt{t+2(0)}\ dt + \int_0^2 \sqrt{1+2(t)}\ dt$

$= \int_0^1 \sqrt{t}\ dt + \int_0^2 \sqrt{1+2t}\ dt = \left[\frac{2}{3}t^{2/3}\right]_0^1 + \left[\frac{1}{3}(1+2t)^{2/3}\right]_0^2 = \frac{2}{3} + \left(\frac{5\sqrt{5}}{3} - \frac{1}{3}\right) = \frac{5\sqrt{5}+1}{3}$

21. $\mathbf{r}(t) = 4t\mathbf{i} - 3t\mathbf{j},\ -1 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = 4\mathbf{i} - 3\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 5 \Rightarrow \int_C ye^{x^2}\ ds = \int_{-1}^2 (-3t)\ e^{(4t)^2} \cdot 5dt$

$= -15\int_{-1}^2 t\ e^{16t^2}\,dt = \left[-\frac{15}{32}e^{16t^2}\right]_{-1}^2 = -\frac{15}{32}e^{64} + \frac{15}{32}e^{16} = \frac{15}{32}\left(e^{16} - e^{64}\right)$

22. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} = (\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1$

$\Rightarrow \int_C (x-y+3)\ ds = \int_0^{2\pi} (\cos t - \sin t + 3) \cdot 1\ dt = [\sin t + \cos t + 3t]_0^{2\pi} = 6\pi$

23. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j},\ 1 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + 3t^2\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(2t)^2 + \left(3t^2\right)^2} = t\sqrt{4+9t^2}$

$\Rightarrow \int_C \frac{x^2}{y^{4/3}}\ ds = \int_1^2 \frac{\left(t^2\right)^2}{\left(t^3\right)^{4/3}} \cdot t\sqrt{4+9t^2}\ dt = \int_1^2 t\sqrt{4+9t^2}\ dt = \left[\frac{1}{27}\left(4+9t^2\right)^{3/2}\right]_1^2 = \frac{80\sqrt{10}-13\sqrt{13}}{27}$

24. $\mathbf{r}(t) = t^3\mathbf{i} + t^4\mathbf{j},\ \frac{1}{2} \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 3t^2\mathbf{i} + 4t^3\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\left(3t^2\right)^2 + \left(4t^3\right)^2} = t^2\sqrt{9+16t^2}$

$\Rightarrow \int_C \frac{\sqrt{y}}{x}\ ds = \int_{1/2}^1 \frac{\sqrt{t^4}}{t^3} \cdot t^2\sqrt{9+16t^2}\ dt = \int_{1/2}^1 t\sqrt{9+16t^2}\ dt = \left[\frac{1}{48}\left(9+16t^2\right)^{3/2}\right]_{1/2}^1 = \frac{125-13\sqrt{13}}{48}$

25. $C_1 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+4t^2};\ C_2 : \mathbf{r}(t) = (1-t)\mathbf{i} + (1-t)\mathbf{j},\ 0 \le t \le 1$

$\Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} - \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2} \Rightarrow \int_C \left(x + \sqrt{y}\right)ds = \int_{C_1} \left(x + \sqrt{y}\right)ds + \int_{C_2} \left(x + \sqrt{y}\right)ds$

$= \int_0^1 \left(t + \sqrt{t^2}\right)\sqrt{1+4t^2}\ dt + \int_0^1 \left((1-t) + \sqrt{1-t}\right)\sqrt{2}dt = \int_0^1 2t\sqrt{1+4t^2}dt + \int_0^1 \left(1-t+\sqrt{1-t}\right)\sqrt{2}dt$

$= \left[\frac{1}{6}\left(1+4t^2\right)^{3/2}\right]_0^1 + \sqrt{2}\left[t - \frac{1}{2}t^2 - \frac{2}{3}(1-t)^{3/2}\right]_0^1 = \frac{5\sqrt{5}-1}{6} + \frac{7\sqrt{2}}{6} = \frac{5\sqrt{5}+7\sqrt{2}-1}{6}$

26. $C_1 : \mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1;\ C_2 : \mathbf{r}(t) = \mathbf{i} + t\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1;$

$C_3 : \mathbf{r}(t) = (1-t)\mathbf{i} + \mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1;\ C_4 : \mathbf{r}(t) = (1-t)\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1;$

$\Rightarrow \int_C \frac{1}{x^2+y^2+1}\ ds = \int_{C_1} \frac{1}{x^2+y^2+1}\ ds + \int_{C_2} \frac{1}{x^2+y^2+1}\ ds + \int_{C_3} \frac{1}{x^2+y^2+1}\ ds + \int_{C_4} \frac{1}{x^2+y^2+1}\ ds$

$$= \int_0^1 \frac{dt}{t^2+1} + \int_0^1 \frac{dt}{t^2+2} + \int_0^1 \frac{dt}{(1-t)^2+2} + \int_0^1 \frac{dt}{(1-t)^2+1}$$

$$= \left[\tan^{-1} t\right]_0^1 + \tfrac{1}{\sqrt{2}}\left[\tan^{-1}\left(\tfrac{t}{\sqrt{2}}\right)\right]_0^1 + \tfrac{1}{\sqrt{2}}\left[\tan^{-1}\left(\tfrac{t-1}{\sqrt{2}}\right)\right]_0^1 + \left[-\tan^{-1}(1-t)\right]_0^1 = \tfrac{\pi}{2} + \tfrac{2}{\sqrt{2}}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}\right)$$

27. $\mathbf{r}(x) = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + \frac{x^2}{2}\mathbf{j},\ 0 \le x \le 2 \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + x\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dx}\right| = \sqrt{1+x^2};\ f(x, y) = f\left(x, \frac{x^2}{2}\right) = \frac{x^3}{\left(\frac{x^2}{2}\right)} = 2x$

$$\Rightarrow \int_C f\ ds = \int_0^2 (2x)\sqrt{1+x^2}\ dx = \left[\tfrac{2}{3}\left(1+x^2\right)^{3/2}\right]_0^2 = \tfrac{2}{3}\left(5^{3/2} - 1\right) = \tfrac{10\sqrt{5}-2}{3}$$

28. $\mathbf{r}(t) = (1-t)\mathbf{i} + \frac{1}{2}(1-t)^2\mathbf{j},\ 0 \le t \le 1 \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+(1-t)^2};\ f(x, y) = f\left((1-t), \frac{1}{2}(1-t)^2\right) = \frac{(1-t)+\frac{1}{4}(1-t)^4}{\sqrt{1+(1-t)^2}}$

$$\Rightarrow \int_C f\ ds = \int_0^1 \frac{(1-t)+\frac{1}{4}(1-t)^4}{\sqrt{1+(1-t)^2}}\sqrt{1+(1-t)^2}\,dt = \int_0^1 \left((1-t) + \tfrac{1}{4}(1-t)^4\right) dt = \left[-\tfrac{1}{2}(1-t)^2 - \tfrac{1}{20}(1-t)^5\right]_0^1$$

$$= 0 - \left(-\tfrac{1}{2} - \tfrac{1}{20}\right) = \tfrac{11}{20}$$

29. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j},\ 0 \le t \le \frac{\pi}{2} \Rightarrow \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 2;\ f(x, y) = f(2\cos t, 2\sin t)$

$$= 2\cos t + 2\sin t \Rightarrow \int_C f\ ds = \int_0^{\pi/2} (2\cos t + 2\sin t)(2)\ dt = \left[4\sin t - 4\cos t\right]_0^{\pi/2} = 4 - (-4) = 8$$

30. $\mathbf{r}(t) = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j},\ 0 \le t \le \frac{\pi}{4} \Rightarrow \frac{d\mathbf{r}}{dt} = (2\cos t)\mathbf{i} + (-2\sin t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 2;\ f(x, y) = f(2\sin t, 2\cos t)$

$$= 4\sin^2 t - 2\cos t \Rightarrow \int_C f\ ds = \int_0^{\pi/4} \left(4\sin^2 t - 2\cos t\right)(2)\ dt = \left[4t - 2\sin 2t - 4\sin t\right]_0^{\pi/4} = \pi - 2\left(1+\sqrt{2}\right)$$

31. $y = x^2,\ 0 \le x \le 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+4t^2} \Rightarrow A = \int_C f(x, y)\ ds$

$$= \int_C \left(x + \sqrt{y}\right) ds = \int_0^2 \left(t + \sqrt{t^2}\right)\sqrt{1+4t^2}\ dt = \int_0^2 2t\sqrt{1+4t^2}\ dt = \left[\tfrac{1}{6}\left(1+4t^2\right)^{3/2}\right]_0^2 = \tfrac{17\sqrt{17}-1}{6}$$

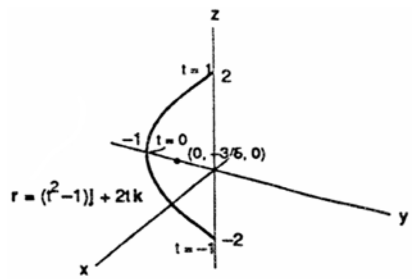
32. $2x + 3y = 6,\ 0 \le x \le 6 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + \left(2 - \frac{2}{3}t\right)\mathbf{j},\ 0 \le t \le 6 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \frac{2}{3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \frac{\sqrt{13}}{3} \Rightarrow A = \int_C f(x, y)\ ds$

$$= \int_C (4 + 3x + 2y)\ ds = \int_0^6 \left(4 + 3t + 2\left(2 - \tfrac{2}{3}t\right)\right)\tfrac{\sqrt{13}}{3}\,dt = \tfrac{\sqrt{13}}{3}\int_0^6 \left(8 + \tfrac{5}{3}t\right)dt = \tfrac{\sqrt{13}}{3}\left[8t + \tfrac{5}{6}t^2\right]_0^6 = 26\sqrt{13}$$

33. $\mathbf{r}(t) = \left(t^2 - 1\right)\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 2\sqrt{t^2+1};\ M = \int_C \delta(x, y, z)\ ds = \int_0^1 \delta(t)\left(2\sqrt{t^2+1}\right) dt$

$$= \int_0^1 \left(\tfrac{3}{2}t\right)\left(2\sqrt{t^2+1}\right) dt = \left[\left(t^2+1\right)^{3/2}\right]_0^1 = 2^{3/2} - 1 = 2\sqrt{2} - 1$$

34. $\mathbf{r}(t)=\left(t^2-1\right)\mathbf{j}+2t\mathbf{k},\ -1\le t\le 1 \Rightarrow \frac{d\mathbf{r}}{dt}=2t\mathbf{j}+2\mathbf{k}$



$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=2\sqrt{t^2+1};\ M=\int_C \delta(x, y, z)\, ds$

$=\int_{-1}^{1}\left(15\sqrt{\left(t^2-1\right)+2}\right)\left(2\sqrt{t^2+1}\right) dt$

$=\int_{-1}^{1}30\left(t^2+1\right) dt=\left[30\left(\frac{t^3}{3}+t\right)\right]_{-1}^{1}=60\left(\frac{1}{3}+1\right)=80;$

$M_{xz}=\int_C y\delta(x, y, z)\, ds=\int_{-1}^{1}\left(t^2-1\right)\left[30\left(t^2+1\right)\right] dt$

$=\int_{-1}^{1}30\left(t^4-1\right) dt=\left[30\left(\frac{t^5}{5}-t\right)\right]_{-1}^{1}=60\left(\frac{1}{5}-1\right)=-48 \Rightarrow \overline{y}=\frac{M_{xz}}{M}=-\frac{48}{80}=-\frac{3}{5};\ M_{yz}=\int_C x\delta(x, y, z)\, ds$

$=\int_C 0\delta\, ds=0 \Rightarrow \overline{x}=0;\ \overline{z}=0$ by symmetry (since δ is independent of z) $\Rightarrow (\overline{x}, \overline{y}, \overline{z})=\left(0, -\frac{3}{5}, 0\right)$

35. $\mathbf{r}(t)=\sqrt{2}t\mathbf{i}+\sqrt{2}t\mathbf{j}+\left(4-t^2\right)\mathbf{k},\ 0\le t\le 1 \Rightarrow \frac{d\mathbf{r}}{dt}=\sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j}-2t\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{2+2+4t^2}=2\sqrt{1+t^2};$

(a) $M=\int_C \delta\, ds=\int_0^1(3t)\left(2\sqrt{1+t^2}\right) dt=\left[2\left(1+t^2\right)^{3/2}\right]_0^1=2\left(2^{3/2}-1\right)=4\sqrt{2}-2$

(b) $M=\int_C \delta\, ds=\int_0^1(1)\left(2\sqrt{1+t^2}\right) dt=\left[t\sqrt{1+t^2}+\ln\left(t+\sqrt{1+t^2}\right)\right]_0^1=\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]-(0+\ln 1)$

$=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$

36. $\mathbf{r}(t)=t\mathbf{i}+2t\mathbf{j}+\frac{2}{3}t^{3/2}\mathbf{k},\ 0\le t\le 2 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{i}+2\mathbf{j}+t^{1/2}\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{1+4+t}=\sqrt{5+t};$

$M=\int_C \delta\, ds=\int_0^2\left(3\sqrt{5+t}\right)\left(\sqrt{5+t}\right) dt=\int_0^2 3(5+t)\, dt=\left[\frac{3}{2}(5+t)^2\right]_0^2=\frac{3}{2}\left(7^2-5^2\right)=\frac{3}{2}(24)=36;$

$M_{yz}=\int_C x\delta\, ds=\int_0^2 t[3(5+t)]\, dt=\int_0^2\left(15t+3t^2\right) dt=\left[\frac{15}{2}t^2+t^3\right]_0^2=30+8=38;$

$M_{xz}=\int_C y\delta\, ds=\int_0^2 2t[3(5+t)]\, dt=2\int_0^2\left(15t+3t^2\right) dt=76;\ M_{xy}=\int_C z\delta\, ds=\int_0^2\frac{2}{3}t^{3/2}[3(5+t)]\, dt$

$=\int_0^2\left(10t^{3/2}+2t^{5/2}\right) dt=\left[4t^{5/2}+\frac{4}{7}t^{7/2}\right]_0^2=4(2)^{5/2}+\frac{4}{7}(2)^{7/2}=16\sqrt{2}+\frac{32}{7}\sqrt{2}=\frac{144}{7}\sqrt{2}$

$\Rightarrow \overline{x}=\frac{M_{yz}}{M}=\frac{38}{36}=\frac{19}{18},\ \overline{y}=\frac{M_{xz}}{M}=\frac{76}{36}=\frac{19}{9}$, and $\overline{z}=\frac{M_{xy}}{M}=\frac{144\sqrt{2}}{7\cdot 36}=\frac{4}{7}\sqrt{2}$

37. Let $x=a\cos t$ and $y=a\sin t,\ 0\le t\le 2\pi$. Then $\frac{dx}{dt}=-a\sin t,\ \frac{dy}{dt}=a\cos t,\ \frac{dz}{dt}=0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\, dt=a\, dt;\ I_z=\int_C\left(x^2+y^2\right)\delta\, ds=\int_0^{2\pi}\left(a^2\sin^2 t+a^2\cos^2 t\right)a\delta\, dt$

$=\int_0^{2\pi}a^3\delta\, dt=2\pi\delta a^3.$

38. $\mathbf{r}(t)=t\mathbf{j}+(2-2t)\mathbf{k},\ 0\le t\le 1 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{j}-2\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{5};\ M=\int_C \delta\, ds=\int_0^1\delta\sqrt{5}\, dt=\delta\sqrt{5};$

$I_x=\int_C\left(y^2+z^2\right)\delta\, ds=\int_0^1\left[t^2+(2-2t)^2\right]\delta\sqrt{5}\, dt=\int_0^1\left(5t^2-8t+4\right)\delta\sqrt{5}\, dt=\delta\sqrt{5}\left[\frac{5}{3}t^3-4t^2+4t\right]_0^1=\frac{5}{3}\delta\sqrt{5};$

$$I_y = \int_C \left(x^2+z^2\right)\delta\, ds = \int_0^1 \left[0^2+(2-2t)^2\right]\delta\sqrt{5}\, dt = \int_0^1 \left(4t^2-8t+4\right)\delta\sqrt{5}\, dt = \delta\sqrt{5}\left[\tfrac{4}{3}t^3-4t^2+4t\right]_0^1 = \tfrac{4}{3}\delta\sqrt{5};$$

$$I_z = \int_C \left(x^2+y^2\right)\delta\, ds = \int_0^1 \left(0^2+t^2\right)\delta\sqrt{5}\, dt = \delta\sqrt{5}\left[\tfrac{t^3}{3}\right]_0^1 = \tfrac{1}{3}\delta\sqrt{5}$$

39. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2};$

(a) $I_z = \int_C \left(x^2+y^2\right)\delta\, ds = \int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right)\delta\sqrt{2}\, dt = 2\pi\delta\sqrt{2}$

(b) $I_z = \int_C \left(x^2+y^2\right)\delta\, ds = \int_0^{4\pi} \delta\sqrt{2}\, dt = 4\pi\delta\sqrt{2}$

40. $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \sqrt{2t}\,\mathbf{k}$

$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(t+1)^2} = t+1$ for $0 \le t \le 1$; $M = \int_C \delta\, ds = \int_0^1 (t+1)\, dt = \left[\frac{1}{2}(t+1)^2\right]_0^1 = \frac{1}{2}\left(2^2-1^2\right) = \frac{3}{2};$

$M_{xy} = \int_C z\delta\, ds = \int_0^1 \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)(t+1)\, dt = \frac{2\sqrt{2}}{3}\int_0^1 \left(t^{5/2}+t^{3/2}\right) dt = \frac{2\sqrt{2}}{3}\left[\frac{2}{7}t^{7/2}+\frac{2}{5}t^{5/2}\right]_0^1$

$= \frac{2\sqrt{2}}{3}\left(\frac{2}{7}+\frac{2}{5}\right) = \frac{2\sqrt{2}}{3}\left(\frac{24}{35}\right) = \frac{16\sqrt{2}}{35} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{16\sqrt{2}}{35}\right)\left(\frac{2}{3}\right) = \frac{32\sqrt{2}}{105};$

$I_z = \int_C \left(x^2+y^2\right)\delta\, ds = \int_0^1 \left(t^2\cos^2 t + t^2\sin^2 t\right)(t+1)\, dt = \int_0^1 \left(t^3+t^2\right) dt = \left[\frac{t^4}{4}+\frac{t^3}{3}\right]_0^1 = \frac{1}{4}+\frac{1}{3} = \frac{7}{12}$

41. $\delta(x,y,z) = 2-z$ and $\mathbf{r}(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k},\ 0 \le t \le \pi \Rightarrow M = 2\pi - 2$ as found in Example 3 of the text; also $\left|\frac{d\mathbf{r}}{dt}\right| = 1$; $I_x = \int_C \left(y^2+z^2\right)\delta\, ds = \int_0^{\pi} \left(\cos^2 t + \sin^2 t\right)(2-\sin t)\, dt = \int_0^{\pi} (2-\sin t)\, dt = 2\pi - 2$

42. $\mathbf{r}(t) = t\mathbf{i} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{j} + \frac{t^2}{2}\mathbf{k},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \sqrt{2}\,t^{1/2}\mathbf{j} + t\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+2t+t^2} = \sqrt{(1+t)^2} = 1+t$ for

$0 \le t \le 2$; $M = \int_C \delta\, ds = \int_0^2 \left(\frac{1}{t+1}\right)(1+t)\, dt = \int_0^2 dt = 2$; $M_{yz} = \int_C x\delta\, ds = \int_0^2 t\left(\frac{1}{t+1}\right)(1+t)\, dt = \left[\frac{t^2}{2}\right]_0^2 = 2;$

$M_{xz} = \int_C y\delta\, ds = \int_0^2 \frac{2\sqrt{2}}{3}t^{3/2}\, dt = \left[\frac{4\sqrt{2}}{15}t^{5/2}\right]_0^2 = \frac{32}{15}$; $M_{xy} = \int_C z\delta\, ds = \int_0^2 \frac{t^2}{2}\, dt = \left[\frac{t^3}{6}\right]_0^2 = \frac{4}{3} \Rightarrow \bar{x} = \frac{M_{yz}}{M} = 1,$

$\bar{y} = \frac{M_{xz}}{M} = \frac{16}{15}$, and $\bar{z} = \frac{M_{xy}}{M} = \frac{2}{3}$; $I_x = \int_C \left(y^2+z^2\right)\delta\, ds = \int_0^2 \left(\frac{8}{9}t^3+\frac{1}{4}t^4\right) dt = \left[\frac{2}{9}t^4+\frac{t^5}{20}\right]_0^2 = \frac{32}{9}+\frac{32}{20} = \frac{232}{45};$

$I_y = \int_C \left(x^2+z^2\right)\delta\, ds = \int_0^2 \left(t^2+\frac{1}{4}t^4\right) dt = \left[\frac{t^3}{3}+\frac{t^5}{20}\right]_0^2 = \frac{8}{3}+\frac{32}{20} = \frac{64}{15};$

$I_z = \int_C \left(x^2+y^2\right)\delta\, ds = \int_0^2 \left(t^2+\frac{8}{9}t^3\right) dt = \left[\frac{t^3}{3}+\frac{2}{9}t^4\right]_0^2 = \frac{8}{3}+\frac{32}{9} = \frac{56}{9}$

43-46. Example CAS commands:

Maple:

```
f:= (x,y,z) -> sqrt(1+30*x^2+10*y);
g:= t -> t;
h:= t -> t^2;
k:= t -> 3*t^2;
a,b:= 0.2;
```

```
ds:= ( D(g)^2 + D(h)^2 + D(k)^2)^(1/2):        #(a)
'ds'=ds(t)*'dt';
F:= f(g,h,k):                                   #(b)
'F(t)'=F(t);
Int( f, s=C..NULL ) = Int( simplify(F(t)*ds(t)), t = a..b );   #(c)
``= value(rhs(%));
```

Mathematica: (functions and domains may vary)

```
Clear[x, y, z, r, t, f]
f[x_,y_,z_]:=Sqrt(1+30x^2 +10y]
{a, b} = {0, 2};
x[t_]:=t
y[t_]:=t^2
z[t_]:=3t^2
r[t_]:={x[t], y[t], z[t]}
v[t_]:= D[r[t],t]
mag[vector_]:=Sqrt[vector.vector]
Integrate[f[x(t),y(t),z[t]] mag[v[t]], {t, a, b}]
N[%]
```

16.2 VECTOR FIELDS AND LINE INTEGRALS: WORK, CIRCULATION, AND FLUX

1. $f(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-1/2} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2x) = -x\left(x^2 + y^2 + z^2\right)^{-3/2}$; similarly, $\frac{\partial f}{\partial y} = -y\left(x^2 + y^2 + z^2\right)^{-3/2}$ and $\frac{\partial f}{\partial z} = -z\left(x^2 + y^2 + z^2\right)^{-3/2} \Rightarrow \nabla f = \frac{-x\mathbf{i} - y\mathbf{j} - z\mathbf{k}}{\left(x^2+y^2+z^2\right)^{3/2}}$

2. $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\ln\left(x^2 + y^2 + z^2\right) \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{2}\left(\frac{1}{x^2+y^2+z^2}\right)(2x) = \frac{x}{x^2+y^2+z^2}$; similarly, $\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2+z^2}$ and $\frac{\partial f}{\partial z} = \frac{z}{x^2+y^2+z^2} \Rightarrow \nabla f = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2}$

3. $g(x, y, z) = e^z - \ln\left(x^2 + y^2\right) \Rightarrow \frac{\partial g}{\partial x} = -\frac{2x}{x^2+y^2}$, $\frac{\partial g}{\partial y} = -\frac{2y}{x^2+y^2}$ and $\frac{\partial g}{\partial z} = e^z \Rightarrow \nabla g = \left(\frac{-2x}{x^2+y^2}\right)\mathbf{i} - \left(\frac{2y}{x^2+y^2}\right)\mathbf{j} + e^z\mathbf{k}$

4. $g(x, y, z) = xy + yz + xz \Rightarrow \frac{\partial g}{\partial x} = y + z$, $\frac{\partial g}{\partial y} = x + z$, and $\frac{\partial g}{\partial z} = y + x \Rightarrow \nabla g = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$

5. $|\mathbf{F}|$ inversely proportional to the square of the distance from (x, y) to the origin $\Rightarrow \sqrt{\left(M(x, y)\right)^2 + \left(N(x, y)\right)^2} = \frac{k}{x^2+y^2}$, $k > 0$; $\mathbf{F}$ points toward the origin $\Rightarrow \mathbf{F}$ is in the direction of $\mathbf{n} = \frac{-x}{\sqrt{x^2+y^2}}\mathbf{i} - \frac{y}{\sqrt{x^2+y^2}}\mathbf{j} \Rightarrow \mathbf{F} = a\,\mathbf{n}$, for some constant $a > 0$. Then $M(x, y) = \frac{-ax}{\sqrt{x^2+y^2}}$ and $N(x, y) = \frac{-ay}{\sqrt{x^2+y^2}} \Rightarrow \sqrt{\left(M(x, y)\right)^2 + \left(N(x, y)\right)^2} = a$ $\Rightarrow a = \frac{k}{x^2+y^2} \Rightarrow \mathbf{F} = \frac{-kx}{\left(x^2+y^2\right)^{3/2}}\mathbf{i} - \frac{ky}{\left(x^2+y^2\right)^{3/2}}\mathbf{j}$, for any constant $k > 0$

6. Given $x^2+y^2=a^2+b^2$, let $x=\sqrt{a^2+b^2}\cos t$ and $y=-\sqrt{a^2+b^2}\sin t$. Then $\mathbf{r}=\left(\sqrt{a^2+b^2}\cos t\right)\mathbf{i}-\left(\sqrt{a^2+b^2}\sin t\right)\mathbf{j}$ traces the circle in a clockwise direction as t goes from 0 to 2π $\Rightarrow \mathbf{v}=\left(-\sqrt{a^2+b^2}\sin t\right)\mathbf{i}-\left(\sqrt{a^2+b^2}\cos t\right)\mathbf{j}$ is tangent to the circle in a clockwise direction. Thus, let $\mathbf{F}=\mathbf{v}\Rightarrow\mathbf{F}=y\mathbf{i}-x\mathbf{j}$ and $\mathbf{F}(0,0)=\mathbf{0}$.

7. Substitute the parametric representations for $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt$.

(a) $\mathbf{F}=3t\mathbf{i}+2t\mathbf{j}+4t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt}=\mathbf{i}+\mathbf{j}+\mathbf{k}\Rightarrow\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}=9t\Rightarrow\int_0^1 9t\,dt=\frac{9}{2}$

(b) $\mathbf{F}=3t^2\mathbf{i}+2t\mathbf{j}+4t^4\mathbf{k}$ and $\frac{d\mathbf{r}}{dt}=\mathbf{i}+2t\mathbf{j}+4t^3\mathbf{k}\Rightarrow\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}=7t^2+16t^7\Rightarrow\int_0^1\left(7t^2+16t^7\right)dt=\left[\frac{7}{3}t^3+2t^8\right]_0^1$ $=\frac{7}{3}+2=\frac{13}{3}$

(c) $\mathbf{r}_1=t\mathbf{i}+t\mathbf{j}$ and $\mathbf{r}_2=\mathbf{i}+\mathbf{j}+t\mathbf{k}$; $\mathbf{F}_1=3t\mathbf{i}+2t\mathbf{j}$ and $\frac{d\mathbf{r}_1}{dt}=\mathbf{i}+\mathbf{j}\Rightarrow\mathbf{F}_1\cdot\frac{d\mathbf{r}_1}{dt}=5t\Rightarrow\int_0^1 5t\,dt=\frac{5}{2}$;

$\mathbf{F}_2=3\mathbf{i}+2\mathbf{j}+4t\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt}=\mathbf{k}\Rightarrow\mathbf{F}_2\cdot\frac{d\mathbf{r}_2}{dt}=4t\Rightarrow\int_0^1 4t\,dt=2\Rightarrow\frac{5}{2}+2=\frac{9}{2}$

8. Substitute the parametric representation for $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt$.

(a) $\mathbf{F}=\left(\frac{1}{t^2+1}\right)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt}=\mathbf{i}+\mathbf{j}+\mathbf{k}+\Rightarrow\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}=\frac{1}{t^2+1}\Rightarrow\int_0^1\frac{1}{t^2+1}\,dt=\left[\tan^{-1}t\right]_0^1=\frac{\pi}{4}$

(b) $\mathbf{F}=\left(\frac{1}{t^2+1}\right)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt}=\mathbf{i}+2t\mathbf{j}+4t^3\mathbf{k}\Rightarrow\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}=\frac{2t}{t^2+1}\Rightarrow\int_0^1\frac{2t}{t^2+1}\,dt=\left[\ln\left(t^2+1\right)\right]_0^1=\ln 2$

(c) $\mathbf{r}_1=t\mathbf{i}+t\mathbf{j}$ and $\mathbf{r}_2=\mathbf{i}+\mathbf{j}+t\mathbf{k}$; $\mathbf{F}_1=\left(\frac{1}{t^2+1}\right)\mathbf{j}$ and $\frac{d\mathbf{r}_1}{dt}=\mathbf{i}+\mathbf{j}\Rightarrow\mathbf{F}_1\cdot\frac{d\mathbf{r}_1}{dt}=\frac{1}{t^2+1}$; $\mathbf{F}_2=\frac{1}{2}\mathbf{j}$ and $\frac{d\mathbf{r}_2}{dt}=\mathbf{k}$

$\Rightarrow\mathbf{F}_2\cdot\frac{dr_2}{dt}=0\Rightarrow\int_0^1\frac{1}{t^2+1}\,dt=\frac{\pi}{4}$

9. Substitute the parametric representation for $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt$.

(a) $\mathbf{F}=\sqrt{t}\mathbf{i}-2t\mathbf{j}+\sqrt{t}\mathbf{k}$ and $\frac{d\mathbf{r}}{dt}=\mathbf{i}+\mathbf{j}+\mathbf{k}\Rightarrow\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}=2\sqrt{t}-2t\Rightarrow\int_0^1\left(2\sqrt{t}-2t\right)dt=\left[\frac{4}{3}t^{3/2}-t^2\right]_0^1=\frac{1}{3}$

(b) $\mathbf{F}=t^2\mathbf{i}-2t\mathbf{j}+t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt}=\mathbf{i}+2t\mathbf{j}+4t^3\mathbf{k}\Rightarrow\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}=4t^4-3t^2\Rightarrow\int_0^1\left(4t^4-3t^2\right)dt=\left[\frac{4}{5}t^5-t^3\right]_0^1=-\frac{1}{5}$

(c) $\mathbf{r}_1=t\mathbf{i}+t\mathbf{j}$ and $\mathbf{r}_2=\mathbf{i}+\mathbf{j}+t\mathbf{k}$; $\mathbf{F}_1=-2t\mathbf{j}+\sqrt{t}\mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt}=\mathbf{i}+\mathbf{j}\Rightarrow\mathbf{F}_1\cdot\frac{d\mathbf{r}_1}{dt}=-2t\Rightarrow\int_0^1-2t\,dt=-1$;

$\mathbf{F}_2=\sqrt{t}\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt}=\mathbf{k}\Rightarrow\mathbf{F}_2\cdot\frac{d\mathbf{r}_2}{dt}=1\Rightarrow\int_0^1 dt=1\Rightarrow-1+1=0$

10. Substitute the parametric representation for $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt$.

(a) $\mathbf{F}=t^2\mathbf{i}+t^2\mathbf{j}+t^2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt}=\mathbf{i}+\mathbf{j}+\mathbf{k}\Rightarrow\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}=3t^2\Rightarrow\int_0^1 3t^2\,dt=1$

(b) $\mathbf{F} = t^3\mathbf{i} - t^6\mathbf{j} + t^5\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^7 + 4t^8 \Rightarrow \int_0^1 \left(t^3 + 2t^7 + 4t^8\right) dt$

$= \left[\frac{t^4}{4} + \frac{t^8}{4} + \frac{4}{9}t^9\right]_0^1 = \frac{17}{18}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = t^2\mathbf{i}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = t^2 \Rightarrow \int_0^1 t^2 dt = \frac{1}{3}$;

$\mathbf{F}_2 = \mathbf{i} + t\mathbf{j} + t\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow \int_0^1 t\, dt = \frac{1}{2} \Rightarrow \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

11. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$.

(a) $\mathbf{F} = \left(3t^2 - 3t\right)\mathbf{i} + 3t\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 + 1 \Rightarrow \int_0^1 \left(3t^2 + 1\right) dt = \left[t^3 + t\right]_0^1 = 2$

(b) $\mathbf{F} = \left(3t^2 - 3t\right)\mathbf{i} + 3t^4\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 4t^3 + 3t^2 - 3t$

$\Rightarrow \int_0^1 \left(6t^5 + 4t^3 + 3t^2 - 3t\right) dt = \left[t^6 + t^4 + t^3 - \frac{3}{2}t^2\right]_0^1 = \frac{3}{2}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = \left(3t^2 - 3t\right)\mathbf{i} + \mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 3t^2 - 3t$

$\Rightarrow \int_0^1 \left(3t^2 - 3t\right) dt = \left[t^3 - \frac{3}{2}t^2\right]_0^1 = -\frac{1}{2}$; $\mathbf{F}_2 = 3t\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow \int_0^1 dt = 1 \Rightarrow -\frac{1}{2} + 1 = \frac{1}{2}$

12. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$.

(a) $\mathbf{F} = 2t\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t \Rightarrow \int_0^1 6t\, dt = \left[3t^2\right]_0^1 = 3$

(b) $\mathbf{F} = \left(t^2 + t^4\right)\mathbf{i} + \left(t^4 + t\right)\mathbf{j} + \left(t + t^2\right)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 5t^4 + 3t^2$

$\Rightarrow \int_0^1 \left(6t^5 + 5t^4 + 3t^2\right) dt = \left[t^6 + t^5 + t^3\right]_0^1 = 3$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 2t \Rightarrow \int_0^1 2t\, dt = 1$;

$\mathbf{F}_2 = (1+t)\mathbf{i} + (t+1)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 2 \Rightarrow \int_0^1 2\, dt = 2 \Rightarrow 1 + 2 = 3$

13. $x = t,\ y = 2t + 1,\ 0 \le t \le 3 \Rightarrow dx = dt \Rightarrow \int_C (x - y)\, dx = \int_0^3 \left(t - (2t+1)\right) dt = \int_0^3 (-t - 1)\, dt = \left[-\frac{1}{2}t^2 - t\right]_0^3 = -\frac{15}{2}$

14. $x = t,\ y = t^2,\ 1 \le t \le 2 \Rightarrow dy = 2t\, dt \Rightarrow \int_C \frac{x}{y}\, dy = \int_1^2 \frac{t}{t^2}(2t)\, dt = \int_1^2 2\, dt = \left[2t\right]_1^2 = 2$

15. $C_1: x = t,\ y = 0,\ 0 \le t \le 3 \Rightarrow dy = 0$; $C_2: x = 3,\ y = t,\ 0 \le t \le 3 \Rightarrow dy = dt \Rightarrow \int_C \left(x^2 + y^2\right) dy$

$= \int_{C_1} \left(x^2 + y^2\right) dx + \int_{C_2} \left(x^2 + y^2\right) dx = \int_0^3 \left(t^2 + 0^2\right) \cdot 0 + \int_0^3 \left(3^2 + t^2\right) dt = \int_0^3 \left(9 + t^2\right) dt = \left[9t + \frac{1}{3}t^3\right]_0^3 = 36$

16. $C_1: x=t,\ y=3t,\ 0\le t\le 1 \Rightarrow dx=dt;\ C_2: x=1-t,\ y=3,\ 0\le t\le 1 \Rightarrow dx=-dt;\ C_3: x=0,\ y=3-t,\ 0\le t\le 3$
$\Rightarrow dx=0 \Rightarrow \int_C \sqrt{x+y}\,dx = \int_{C_1}\sqrt{x+y}\,dx + \int_{C_2}\sqrt{x+y}\,dx + \int_{C_3}\sqrt{x+y}\,dx$
$= \int_0^1 \sqrt{t+3t}\,dt + \int_0^1 \sqrt{(1-t)+3}(-1)\,dt + \int_0^3 \sqrt{0+(3-t)}\cdot 0 = \int_0^1 2\sqrt{t}\,dt - \int_0^1 \sqrt{4-t}\,dt$
$= \left[\frac{4}{3}t^{2/3}\right]_0^1 + \left[\frac{2}{3}(4-t)^{2/3}\right]_0^1 = \frac{4}{3} + \left(2\sqrt{3} - \frac{16}{3}\right) = 2\sqrt{3}-4$

17. $\mathbf{r}(t) = t\mathbf{i} - \mathbf{j} + t^2\mathbf{k},\ 0\le t\le 1 \Rightarrow dx=dt,\ dy=0,\ dz=2t\,dt$
(a) $\int_C (x+y-z)\,dx = \int_0^1 \left(t-1-t^2\right)dt = \left[\frac{1}{2}t^2 - t - \frac{1}{3}t^3\right]_0^1 = -\frac{5}{6}$
(b) $\int_C (x+y-z)\,dy = \int_0^1 \left(t-1-t^2\right)\cdot 0 = 0$
(c) $\int_C (x+y-z)\,dz = \int_0^1 \left(t-1-t^2\right)2t\,dt = \int_0^1 \left(2t^2-2t-2t^3\right)dt = \left[\frac{2}{3}t^3 - t^2 - \frac{1}{2}t^4\right]_0^1 = -\frac{5}{6}$

18. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - (\cos t)\mathbf{k},\ 0\le t\le \pi \Rightarrow dx = -\sin t\,dt,\ dy=\cos t\,dt,\ dz=\sin t\,dt$
(a) $\int_C x\,z\,dx = \int_0^\pi (\cos t)(-\cos t)(-\sin t)dt = \int_0^\pi \cos^2 t\sin t\,dt = \left[-\frac{1}{3}(\cos t)^3\right]_0^\pi = \frac{2}{3}$
(b) $\int_C x\,z\,dy = \int_0^\pi (\cos t)(-\cos t)(\cos t)dt = -\int_0^\pi \cos^3 t\,dt = -\int_0^\pi \left(1-\sin^2 t\right)\cos t\,dt = \left[\frac{1}{3}(\sin t)^3 - \sin t\right]_0^\pi = 0$
(c) $\int_C x\,y\,z\,dz = \int_0^\pi (\cos t)(\sin t)(-\cos t)(\sin t)dt = -\int_0^\pi \cos^2 t\sin^2 t\,dt = -\frac{1}{4}\int_0^\pi \sin^2 2t\,dt = -\frac{1}{4}\int_0^\pi \frac{1-\cos 4t}{2}\,dt$
$= -\frac{1}{8}\int_0^\pi (1-\cos 4t)\,dt = \left[-\frac{1}{8}t + \frac{1}{32}\sin 4t\right]_0^\pi = -\frac{\pi}{8}$

19. $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k},\ 0\le t\le 1$, and $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} \Rightarrow \mathbf{F} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$
$\Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 2t^3 \Rightarrow \text{work} = \int_0^1 2t^3\,dt = \frac{1}{2}$

20. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \frac{t}{6}\mathbf{k},\ 0\le t\le 2\pi$, and $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x+y)\mathbf{k}$
$\Rightarrow \mathbf{F} = (2\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + (\cos t + \sin t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{1}{6}\mathbf{k}$
$\Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 3\cos^2 t - 2\sin^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t \Rightarrow \text{work} = \int_0^{2\pi}\left(3\cos^2 t - 2\sin^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right)dt$
$= \left[\frac{3}{2}t + \frac{3}{4}\sin 2t - t + \frac{\sin 2t}{2} + \frac{1}{6}\sin t - \frac{1}{6}\cos t\right]_0^{2\pi} = \pi$

21. $\mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k},\ 0\le t\le 2\pi$, and $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k} \Rightarrow \mathbf{F} = t\mathbf{i} + (\sin t)\mathbf{j} + (\cos t)\mathbf{k}$ and
$\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = t\cos t - \sin^2 t + \cos t \Rightarrow \text{work} = \int_0^{2\pi}\left(t\cos t - \sin^2 t + \cos t\right)dt$
$= \left[\cos t + t\sin t - \frac{t}{2} + \frac{\sin 2t}{4} + \sin t\right]_0^{2\pi} = -\pi$

22. $\mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{t}{6}\mathbf{k},\ 0 \le t \le 2\pi$, and $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k} \Rightarrow \mathbf{F} = t\mathbf{i} + \left(\cos^2 t\right)\mathbf{j} + (12 \sin t)\mathbf{k}$ and
$\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \frac{1}{6}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t \cos t - \sin t \cos^2 t + 2 \sin t$
$\Rightarrow \text{work} = \int_0^{2\pi} \left(t \cos t - \sin t \cos^2 t + 2 \sin t\right) dt = \left[\cos t + t \sin t + \frac{1}{3}\cos^3 t - 2 \cos t\right]_0^{2\pi} = 0$

23. $x = t$ and $y = x^2 = t^2 \Rightarrow \mathbf{r} = t\mathbf{i} + t^2\mathbf{j},\ -1 \le t \le 2$, and $\mathbf{F} = xy\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = t^3\mathbf{i} + \left(t + t^2\right)\mathbf{j}$ and
$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + \left(2t^2 + 2t^3\right) = 3t^3 + 2t^2 \Rightarrow \int_C xy\, dx + (x+y)\, dy = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_{-1}^{2} \left(3t^3 + 2t^2\right) dt$
$= \left[\frac{3}{4}t^4 + \frac{2}{3}t^3\right]_{-1}^{2} = \left(12 + \frac{16}{3}\right) - \left(\frac{3}{4} - \frac{2}{3}\right) = \frac{45}{4} + \frac{18}{3} = \frac{69}{4}$

24. Along $(0,0)$ to $(1,0)$: $\mathbf{r} = t\mathbf{i},\ 0 \le t \le 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = t\mathbf{i} + t\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t$;
Along $(1,0)$ to $(0,1)$: $\mathbf{r} = (1-t)\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = (1-2t)\mathbf{i} + \mathbf{j}$ and
$\frac{d\mathbf{r}}{dt} = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t$;
Along $(0,1)$ to $(0,0)$: $\mathbf{r} = (1-t)\mathbf{j},\ 0 \le t \le 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = (t-1)\mathbf{i} + (1-t)\mathbf{j}$ and
$\frac{d\mathbf{r}}{dr} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t - 1 \Rightarrow \int_C (x-y)\, dx + (x+y)\, dy = \int_0^1 t\, dt + \int_0^1 2t\, dt + \int_0^1 (t-1)\, dt = \int_0^1 (4t-1)\, dt$
$= \left[2t^2 - t\right]_0^1 = 2 - 1 = 1$

25. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = y^2\mathbf{i} + y\mathbf{j},\ 2 \ge y \ge -1$, and $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} = y^4\mathbf{i} - y\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dy} = 2y\mathbf{i} + \mathbf{j}$ and $\mathbf{F} \cdot \frac{d\mathbf{r}}{dy} = 2y^5 - y$
$\Rightarrow \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_2^{-1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dy}\, dy = \int_2^{-1} \left(2y^5 - y\right) dy = \left[\frac{1}{3}y^6 - \frac{1}{2}y^2\right]_2^{-1} = \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{64}{3} - \frac{4}{2}\right) = \frac{3}{2} - \frac{63}{3} = -\frac{39}{2}$

26. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$, and $\mathbf{F} = y\mathbf{i} - x\mathbf{j} \Rightarrow \mathbf{F} = (\sin t)\mathbf{i} - (\cos t)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$
$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin^2 t - \cos^2 t = -1 \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (-1)\, dt = -\frac{\pi}{2}$

27. $\mathbf{r} = (\mathbf{i} + \mathbf{j}) + t(\mathbf{i} + 2\mathbf{j}) = (1+t)\mathbf{i} + (1+2t)\mathbf{j},\ 0 \le t \le 1$, and $\mathbf{F} = xy\mathbf{i} + (y-x)\mathbf{j} \Rightarrow \mathbf{F} = \left(1 + 3t + 2t^2\right)\mathbf{i} + t\mathbf{j}$ and
$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 1 + 5t + 2t^2 \Rightarrow \text{work} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^1 \left(1 + 5t + 2t^2\right) dt = \left[t + \frac{5}{2}t^2 + \frac{2}{3}t^3\right]_0^1 = \frac{25}{6}$

28. $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j},\ 0 \le t \le 2\pi$, and $\mathbf{F} = \nabla f = 2(x+y)\mathbf{i} + 2(x+y)\mathbf{j}$
$\Rightarrow \mathbf{F} = 4(\cos t + \sin t)\mathbf{i} + 4(\cos t + \sin t)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j}$
$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -8\left(\sin t \cos t + \sin^2 t\right) + 8\left(\cos^2 t + \cos t \sin t\right) = 8\left(\cos^2 t - \sin^2 t\right) = 8 \cos 2t$
$\Rightarrow \text{work} = \int_C \nabla f \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{2\pi} 8 \cos 2t\, dt = \left[4 \sin 2t\right]_0^{2\pi} = 0$

29. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi,\ \mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$, and $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$,
$\mathbf{F}_1 = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, and $\mathbf{F}_2 = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 0$ and $\mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = \sin^2 t + \cos^2 t = 1$

$\Rightarrow \text{Circ}_1 = \int_0^{2\pi} 0\, dt = 0$ and $\text{Circ}_2 = \int_0^{2\pi} dt = 2\pi$; $\mathbf{n} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n} = \cos^2 t + \sin^2 t = 1$ and

$\mathbf{F}_2 \cdot \mathbf{n} = 0 \Rightarrow \text{Flux}_1 = \int_0^{2\pi} dt = 2\pi$ and $\text{Flux}_2 = \int_0^{2\pi} 0\, dt = 0$

(b) $\mathbf{r} = (\cos t)\mathbf{i} + (4\sin t)\mathbf{j},\ 0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (4\cos t)\mathbf{j},\ \mathbf{F}_1 = (\cos t)\mathbf{i} + (4\sin t)\mathbf{j}$, and

$\mathbf{F}_2 = (-4\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 15\sin t\cos t$ and $\mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = 4 \Rightarrow \text{Circ}_1 = \int_0^{2\pi} 15\sin t\cos t\, dt$

$= \left[\frac{15}{2}\sin^2 t\right]_0^{2\pi} = 0$ and $\text{Circ}_2 = \int_0^{2\pi} 4\, dt = 8\pi$; $\mathbf{n} = \left(\frac{4}{\sqrt{17}}\cos t\right)\mathbf{i} + \left(\frac{1}{\sqrt{17}}\sin t\right)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n}$

$= \frac{4}{\sqrt{17}}\cos^2 t + \frac{4}{\sqrt{17}}\sin^2 t$ and $\mathbf{F}_2 \cdot \mathbf{n} = -\frac{15}{\sqrt{17}}\sin t\cos t \Rightarrow \text{Flux}_1 = \int_0^{2\pi} (\mathbf{F}_1 \cdot \mathbf{n})\,|\mathbf{v}|\, dt = \int_0^{2\pi} \left(\frac{4}{\sqrt{17}}\right)\sqrt{17}\, dt$

$= 8\pi$ and $\text{Flux}_2 = \int_0^{2\pi} (\mathbf{F}_2 \cdot \mathbf{n})\,|\mathbf{v}|\, dt = \int_0^{2\pi} \left(-\frac{15}{\sqrt{17}}\sin t\cos t\right)\sqrt{17}\, dt = \left[-\frac{15}{2}\sin^2 t\right]_0^{2\pi} = 0$

30. $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ 0 \le t \le 2\pi,\ \mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j}$, and $\mathbf{F}_2 = 2x\mathbf{i} + (x-y)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}$,
$\mathbf{F}_1 = (2a\cos t)\mathbf{i} - (3a\sin t)\mathbf{j}$, and $\mathbf{F}_2 = (2a\cos t)\mathbf{i} + (a\cos t - a\sin t)\mathbf{j} \Rightarrow \mathbf{n}\,|\mathbf{v}| = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$,
$\mathbf{F}_1 \cdot \mathbf{n}\,|\mathbf{v}| = 2a^2\cos^2 t - 3a^2\sin^2 t$, and $\mathbf{F}_2 \cdot \mathbf{n}\,|\mathbf{v}| = 2a^2\cos^2 t + a^2\sin t\cos t - a^2\sin^2 t$
$\Rightarrow \text{Flux}_1 = \int_0^{2\pi} \left(2a^2\cos^2 t - 3a^2\sin^2 t\right) dt = 2a^2\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} - 3a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = -\pi a^2$, and
$\text{Flux}_2 = \int_0^{2\pi} \left(2a^2\cos^2 t - a^2\sin t\cos t - a^2\sin^2 t\right) dt$
$= 2a^2\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} + \frac{a^2}{2}\left[\sin^2 t\right]_0^{2\pi} - a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi a^2$

31. $\mathbf{F}_1 = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ \frac{d\mathbf{r}_1}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 0 \Rightarrow \text{Circ}_1 = 0;\ M_1 = a\cos t$,
$N_1 = a\sin t,\ dx = -a\sin t\, dt,\ dy = a\cos t\, dt \Rightarrow \text{Flux}_1 = \int_C M_1\, dy - N_1\, dx = \int_0^{\pi} \left(a^2\cos^2 t + a^2\sin^2 t\right) dt$
$= \int_0^{\pi} a^2 dt = a^2\pi$;
$\mathbf{F}_2 = t\mathbf{i},\ \frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow \text{Circ}_2 = \int_{-a}^{a} t\, dt = 0;\ M_2 = t,\ N_2 = 0,\ dx = dt,\ dy = 0$
$\Rightarrow \text{Flux}_2 = \int_C M_2\, dy - N_2\, dx = \int_{-a}^{a} 0\, dt = 0$;
therefore, $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = a^2\pi$

32. $\mathbf{F}_1 = \left(a^2\cos^2 t\right)\mathbf{i} + \left(a^2\sin^2 t\right)\mathbf{j},\ \frac{d\mathbf{r}_1}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -a^3\sin t\cos^2 t + a^3\cos t\sin^2 t$
$\Rightarrow \text{Circ}_1 = \int_0^{\pi} \left(-a^3\sin t\cos^2 t + a^3\cos t\sin^2 t\right) dt = -\frac{2a^3}{3};\ M_1 = a^2\cos^2 t,\ N_1 = a^2\sin^2 t,\ dy = a\cos t\, dt$,
$dx = -a\sin t\, dt \Rightarrow \text{Flux}_1 = \int_C M_1\, dy - N_1\, dx = \int_0^{\pi} \left(a^3\cos^3 t + a^3\sin^3 t\right) dt = \frac{4}{3}a^3$;
$\mathbf{F}_2 = t^2\mathbf{i},\ \frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t^2 \Rightarrow \text{Circ}_2 = \int_{-a}^{a} t^2\, dt = \frac{2a^3}{3};\ M_2 = t^2,\ N_2 = 0,\ dy = 0,\ dx = dt$
$\Rightarrow \text{Flux}_2 = \int_C M_2\, dy - N_2\, dx = 0$; therefore, $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = \frac{4}{3}a^3$

33. $\mathbf{F}_1 = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^2 \sin^2 t + a^2 \cos^2 t = a^2$

$\Rightarrow \text{Circ}_1 = \int_0^{\pi} a^2\, dt = a^2\pi$; $M_1 = -a \sin t$, $N_1 = a \cos t$, $dx = -a \sin t\, dt$, $dy = a \cos t\, dt$

$\Rightarrow \text{Flux}_1 = \int_C M_1\, dy - N_1\, dx = \int_0^{\pi} \left(-a^2 \sin t \cos t + a^2 \sin t \cos t\right) dt = 0$; $\mathbf{F}_2 = t\mathbf{j}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0$

$\Rightarrow \text{Circ}_2 = 0$; $M_2 = 0$, $N_2 = t$, $dx = dt$, $dy = 0 \Rightarrow \text{Flux}_2 = \int_C M_2\, dy - N_2\, dx = \int_{-a}^{a} -t\, dt = 0$; therefore,

$\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = a^2\pi$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$

34. $\mathbf{F}_1 = \left(-a^2 \sin^2 t\right)\mathbf{i} + \left(a^2 \cos^2 t\right)\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^3 \sin^3 t + a^3 \cos^3 t$

$\Rightarrow \text{Circ}_1 = \int_0^{\pi} \left(a^3 \sin^3 t + a^3 \cos^3 t\right) dt = \frac{4}{3}a^3$; $M_1 = -a^2 \sin^2 t$, $N_1 = a^2 \cos^2 t$, $dy = a \cos t\, dt$, $dx = -a \sin t\, dt$

$\Rightarrow \text{Flux}_1 = \int_C M_1\, dy - N_1\, dx = \int_0^{\pi} \left(-a^3 \cos t \sin^2 t + a^3 \sin t \cos^2 t\right) dt = \frac{2}{3}a^3$; $\mathbf{F}_2 = t^2\mathbf{j}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0$

$\Rightarrow \text{Circ}_2 = 0$; $M_2 = 0$, $N_2 = t^2$, $dy = 0$, $dx = dt \Rightarrow \text{Flux}_2 = \int_C M_2 dy - N_2\, dx = \int_{-a}^{a} -t^2 dt = -\frac{2}{3}a^3$; therefore,

$\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = \frac{4}{3}a^3$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$

35. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le \pi$, and $\mathbf{F} = (x+y)\mathbf{i} - \left(x^2 + y^2\right)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ and

$\mathbf{F} = (\cos t + \sin t)\mathbf{i} - \left(\cos^2 t + \sin^2 t\right)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t - \sin^2 t - \cos t \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T}\, ds$

$= \int_0^{\pi} \left(-\sin t \cos t - \sin^2 t - \cos t\right) dt = \left[-\frac{1}{2} \sin^2 t - \frac{t}{2} + \frac{\sin 2t}{4} - \sin t\right]_0^{\pi} = -\frac{\pi}{2}$

(b) $\mathbf{r} = (1-2t)\mathbf{i}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - \left(x^2 + y^2\right)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = -2\mathbf{i}$ and $\mathbf{F} = (1-2t)\mathbf{i} - (1-2t)^2\,\mathbf{j} \Rightarrow$

$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 4t - 2 \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^1 (4t-2)\, dt = \left[2t^2 - 2t\right]_0^1 = 0$

(c) $\mathbf{r}_1 = (1-t)\mathbf{i} - t\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - \left(x^2 + y^2\right)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{F} = (1-2t)\mathbf{i} - \left(1 - 2t + 2t^2\right)\mathbf{j}$

$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = (2t-1) + \left(1 - 2t + 2t^2\right) = 2t^2 \Rightarrow \text{Flow}_1 = \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = \int_0^1 2t^2\, dt = \frac{2}{3}$; $\mathbf{r}_2 = -t\mathbf{i} + (t-1)\mathbf{j}$,

$0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - \left(x^2 + y^2\right)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} + \mathbf{j}$ and $\mathbf{F} = -\mathbf{i} - \left(t^2 + t^2 - 2t + 1\right)\mathbf{j}$

$= -\mathbf{i} - \left(2t^2 - 2t + 1\right)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = 1 - \left(2t^2 - 2t + 1\right) = 2t - 2t^2 \Rightarrow \text{Flow}_2 = \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = \int_0^1 \left(2t - 2t^2\right) dt$

$= \left[t^2 - \frac{2}{3}t^3\right]_0^1 = \frac{1}{3} \Rightarrow \text{Flow} = \text{Flow}_1 + \text{Flow}_2 = \frac{2}{3} + \frac{1}{3} = 1$

36. From $(1,0)$ to $(0,1)$: $\mathbf{r}_1 = (1-t)\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - \left(x^2 + y^2\right)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} + \mathbf{j}$,

$\mathbf{F} = \mathbf{i} - \left(1 - 2t + 2t^2\right)\mathbf{j}$, and $\mathbf{n}_1\, |\mathbf{v}_1| = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_1\, |\mathbf{v}_1| = 2t - 2t^2 \Rightarrow \text{Flux}_1 = \int_0^1 \left(2t - 2t^2\right) dt = \left[t^2 - \frac{2}{3}t^3\right]_0^1 = \frac{1}{3}$;

From $(0, 1)$ to $(-1, 0)$: $\mathbf{r}_2 = -t\mathbf{i} + (1-t)\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - \left(x^2 + y^2\right)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} - \mathbf{j}$,

$\mathbf{F} = (1-2t)\mathbf{i} - \left(1 - 2t + 2t^2\right)\mathbf{j}$, and $\mathbf{n}_2\, |\mathbf{v}_2| = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_2\, |\mathbf{v}_2| = (2t-1) + \left(-1 + 2t - 2t^2\right) = -2 + 4t - 2t^2$

$\Rightarrow \text{Flux}_2 = \int_0^1 \left(-2 + 4t - 2t^2\right) dt = \left[-2t + 2t^2 - \frac{2}{3}t^3\right]_0^1 = -\frac{2}{3}$;

From $(-1, 0)$ to $(1, 0)$: $\mathbf{r}_3 = (-1+2t)\mathbf{i},\ 0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - \left(x^2 + y^2\right)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_3}{dt} = 2\mathbf{i}$,

$\mathbf{F} = (-1+2t)\mathbf{i} - \left(1-4t+4t^2\right)\mathbf{j}$, and $\mathbf{n}_3\,|\mathbf{v}_3| = -2\mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n}_3\,|\mathbf{v}_3| = 2\left(1-4t+4t^2\right)$

$\Rightarrow \text{Flux}_3 = 2\int_0^1 \left(1-4t+4t^2\right) dt = 2\left[t - 2t^2 + \frac{4}{3}t^3\right]_0^1 = \frac{2}{3} \Rightarrow \text{Flux} = \text{Flux}_1 + \text{Flux}_2 + \text{Flux}_3 = \frac{1}{3} - \frac{2}{3} + \frac{2}{3} = \frac{1}{3}$

37. (a) $y = 2x,\ 0 \le x \le 2 \Rightarrow r(t) = t\mathbf{i} + 2t\mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left((2t)^2\mathbf{i} + 2(t)(2t)\mathbf{j}\right)\cdot(\mathbf{i} + 2\mathbf{j})$

$= 4t^2 + 8t^2 = 12t^2 \Rightarrow \text{Flow} = \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^2 12t^2\,dt = \left[4t^3\right]_0^2 = 32$

(b) $y = x^2,\ 0 \le x \le 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j},\ \ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left(\left(t^2\right)^2\mathbf{i} + 2(t)\left(t^2\right)\mathbf{j}\right)\cdot(\mathbf{i} + 2t\mathbf{j})$

$= t^4 + 4t^4 = 5t^4 \Rightarrow \text{Flow} = \int_c \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^2 5t^4\,dt = \left[t^5\right]_0^2 = 32$

(c) answers will vary, one possible path is $y = \frac{1}{2}x^3,\ 0 \le x \le 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^3\mathbf{j},\ \ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 3t^2\mathbf{j}$

$\Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left(\left(\frac{1}{2}t^3\right)^2\mathbf{i} + 2(t)\left(\frac{1}{2}t^3\right)\mathbf{j}\right)\cdot\left(\mathbf{i} + 3t^2\mathbf{j}\right) = \frac{1}{4}t^6 + \frac{3}{2}t^6 = \frac{7}{4}t^6 \Rightarrow \text{Flow} = \int_c \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^2 \frac{7}{4}t^6\,dt$

$= \left[\frac{1}{4}t^7\right]_0^2 = 32$

38. (a) $C_1 : \mathbf{r}(t) = (1-t)\mathbf{i} + \mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left((1)\mathbf{i} + \left((1-t) + 2(1)\right)\mathbf{j}\right)\cdot(-\mathbf{i}) = -1;$

$C_2 : \mathbf{r}(t) = -\mathbf{i} + (1-t)\mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left((1-t)\mathbf{i} + \left((-1) + 2(1-t)\right)\mathbf{j}\right)\cdot(-\mathbf{j}) = 2t - 1;$

$C_3 : \mathbf{r}(t) = (t-1)\mathbf{i} - \mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left((-1)\mathbf{i} + \left((t-1) + 2(-1)\right)\mathbf{j}\right)\cdot(\mathbf{i}) = -1;$

$C_4 : \mathbf{r}(t) = \mathbf{i} + (t-1)\mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left((t-1)\mathbf{i} + \left((1) + 2(t-1)\right)\mathbf{j}\right)\cdot(\mathbf{j}) = 2t - 1;$

$\Rightarrow \text{Flow} = \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt = \int_{C_1} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt + \int_{C_2} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt + \int_{C_3} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt + \int_{C_4} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt$

$= \int_0^2 (-1)\,dt + \int_0^2 (2t-1)\,dt + \int_0^2 (-1)\,dt + \int_0^2 (2t-1)\,dt = \left[-t\right]_0^2 + \left[t^2 - t\right]_0^2 + \left[-t\right]_0^2 + \left[t^2 - t\right]_0^2$

$= -2 + 2 - 2 + 2 = 0$

(b) $x^2 + y^2 = 4 \Rightarrow \mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j},\ 0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j}$

$\Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left((2\sin t)\mathbf{i} + \left(2\cos t + 2(2\sin t)\right)\mathbf{j}\right)\cdot\left((-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j}\right) = -4\sin^2 t + 4\cos^2 t + 8\sin t\cos t$

$= 4\cos 2t + 4\sin 2t \Rightarrow \text{Flow} = \int_C \mathbf{F}\cdot\frac{dr}{dt}\,dt = \int_0^{2\pi} (4\cos 2t + 4\sin 2t)\,dt = \left[2\sin 2t - 2\cos 2t\right]_0^{2\pi} = 0$

(c) answers will vary, one possible path is:

$C_1 : \mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left((0)\mathbf{i} + \left(t + 2(1)\right)\mathbf{j}\right)\cdot(\mathbf{i}) = 0;$

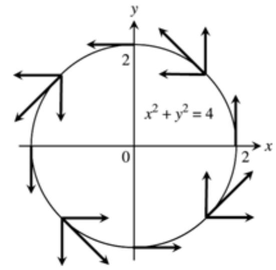
$C_2 : \mathbf{r}(t) = (1-t)\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left(t\mathbf{i} + \left((1-t) + 2t\right)\mathbf{j}\right)\cdot(-\mathbf{i} + \mathbf{j}) = 1;$

$C_3 : \mathbf{r}(t) = (1-t)\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left((1-t)\mathbf{i} + \left(0 + 2(1-t)\right)\mathbf{j}\right)\cdot(-\mathbf{j}) = 2t - 1;$

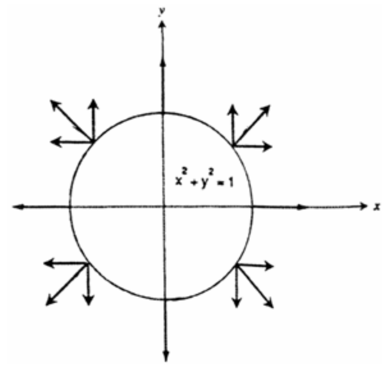
$\Rightarrow \text{Flow} = \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt = \int_{C_1} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt + \int_{C_2} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt + \int_{C_3} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt = \int_0^1 (0)\,dt + \int_0^1 (1)\,dt + \int_0^1 (2t-1)\,dt$

$= 0 + \left[t\right]_0^1 + \left[t^2 - t\right]_0^1 = 1 + (-1) = 0$

39. $\mathbf{F} = -\frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j}$ on $x^2 + y^2 = 4$;
at $(2, 0)$, $\mathbf{F} = \mathbf{j}$; at $(0, 2)$, $\mathbf{F} = -\mathbf{i}$;
at $(-2, 0)$, $\mathbf{F} = -\mathbf{j}$; at $(0, -2)$, $\mathbf{F} = \mathbf{i}$;
at $\left(\sqrt{2}, \sqrt{2}\right)$, $\mathbf{F} = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$;
at $\left(\sqrt{2}, -\sqrt{2}\right)$, $\mathbf{F} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$;
at $\left(-\sqrt{2}, \sqrt{2}\right)$, $\mathbf{F} = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$;
at $\left(-\sqrt{2}, -\sqrt{2}\right)$, $\mathbf{F} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$



40. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ on $x^2 + y^2 = 1$; at $(1, 0)$, $\mathbf{F} = \mathbf{i}$;
at $(-1, 0)$, $\mathbf{F} = -\mathbf{i}$; at $(0, 1)$, $\mathbf{F} = \mathbf{j}$;
at $(0, -1)$, $\mathbf{F} = -\mathbf{j}$; at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\mathbf{F} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$;
at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\mathbf{F} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$;
at $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\mathbf{F} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$;
at $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\mathbf{F} = -\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$.



41. (a) $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is to have a magnitude $\sqrt{a^2 + b^2}$ and to be tangent to $x^2 + y^2 = a^2 + b^2$ in a counterclockwise direction. Thus $x^2 + y^2 = a^2 + b^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$ is the slope of the tangent line at any point on the circle $\Rightarrow y' = -\frac{a}{b}$ at (a, b). Let $\mathbf{v} = -b\mathbf{i} + a\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{a^2 + b^2}$, with $\mathbf{v}$ in a counterclockwise direction and tangent to the circle. Then let $P(x, y) = -y$ and $Q(x, y) = x$ $\Rightarrow \mathbf{G} = -y\mathbf{i} + x\mathbf{j} \Rightarrow$ for (a,b) on $x^2 + y^2 = a^2 + b^2$ we have $\mathbf{G} = -b\mathbf{i} + a\mathbf{j}$ and $|\mathbf{G}| = \sqrt{a^2 + b^2}$.

(b) $\mathbf{G} = \left(\sqrt{x^2 + y^2}\right)\mathbf{F} = \left(\sqrt{a^2 + b^2}\right)\mathbf{F}$.

42. (a) From Exercise 41, part a, $-y\mathbf{i} + x\mathbf{j}$ is a vector tangent to the circle and pointing in a counterclockwise direction $\Rightarrow y\mathbf{i} - x\mathbf{j}$ is a vector tangent to the circle pointing in a clockwise direction $\Rightarrow \mathbf{G} = \frac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2+y^2}}$ is a unit vector tangent to the circle and pointing in a clockwise direction.

(b) $\mathbf{G} = -\mathbf{F}$

43. The slope of the line through (x, y) and the origin is $\frac{y}{x} \Rightarrow \mathbf{v} = x\mathbf{i} + y\mathbf{j}$ is a vector parallel to that line and pointing away from the origin $\Rightarrow \mathbf{F} = -\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$ is the unit vector pointing toward the origin.

44. (a) From Exercise 43, $-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ is a unit vector through (x, y) pointing toward the origin and we want $|\mathbf{F}|$ to have magnitude $\sqrt{x^2+y^2} \Rightarrow \mathbf{F} = \sqrt{x^2+y^2}\left(-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}\right) = -x\mathbf{i} - y\mathbf{j}.$

(b) We want $|\mathbf{F}| = \frac{C}{\sqrt{x^2+y^2}}$ where $C \neq 0$ is a constant $\Rightarrow \mathbf{F} = \frac{C}{\sqrt{x^2+y^2}}\left(-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}\right) = -C\left(\frac{x\mathbf{i}+y\mathbf{j}}{x^2+y^2}\right).$

45. Yes. The work and area have the same numerical value because work $= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C y\mathbf{i}\cdot d\mathbf{r}$

$= \int_b^a [f(t)\mathbf{i}]\cdot\left[\mathbf{i} + \frac{df}{dt}\mathbf{j}\right] dt$ [On the path, y equals $f(t)$]

$= \int_a^b f(t)\, dt =$ Area under the curve [because $f(t) > 0$]

46. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + f(x)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + f'(x)\mathbf{j}$; $\mathbf{F} = \frac{k}{\sqrt{x^2+y^2}}(x\mathbf{i} + y\mathbf{j})$ has constant magnitude k and points away from the origin $\Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dx} = \frac{kx}{\sqrt{x^2+y^2}} + \frac{k\cdot y\cdot f'(x)}{\sqrt{x^2+y^2}} = \frac{kx + k\cdot f(x)\cdot f'(x)}{\sqrt{x^2+[f(x)]^2}} = k\frac{d}{dx}\sqrt{x^2+[f(x)]^2}$, by the chain rule

$\Rightarrow \int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dx}\, dx = \int_a^b k\frac{d}{dx}\sqrt{x^2+[f(x)]^2}\, dx = k\left[\sqrt{x^2+[f(x)]^2}\right]_a^b$

$= k\left(\sqrt{b^2+[f(b)]^2} - \sqrt{a^2+[f(a)]^2}\right)$, as claimed.

47. $\mathbf{F} = -4t^3\mathbf{i} + 8t^2\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 12t^3 \Rightarrow \text{Flow} = \int_0^2 12t^3 dt = \left[3t^4\right]_0^2 = 48$

48. $\mathbf{F} = 12t^2\mathbf{j} + 9t^2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = 3\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 72t^2 \Rightarrow \text{Flow} = \int_0^1 72t^2 dt = \left[24t^3\right]_0^1 = 24$

49. $\mathbf{F} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -\sin t\cos t + 1$

$\Rightarrow \text{Flow} = \int_0^\pi (-\sin t\cos t + 1)\, dt = \left[\frac{1}{2}\cos^2 t + t\right]_0^\pi = \left(\frac{1}{2} + \pi\right) - \left(\frac{1}{2} + 0\right) = \pi$

50. $\mathbf{F} = (-2\sin t)\mathbf{i} - (2\cos t)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -4\sin^2 t - 4\cos^2 t + 4 = 0$

$\Rightarrow \text{Flow} = 0$

51. C_1: $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le \frac{\pi}{2} \Rightarrow \mathbf{F} = (2\cos t)\mathbf{i} + 2t\mathbf{j} + (2\sin t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$

$\Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -2\cos t\sin t + 2t\cos t + 2\sin t = -\sin 2t + 2t\cos t + 2\sin t$

$\Rightarrow \text{Flow}_1 = \int_0^{\pi/2}(-\sin 2t + 2t\cos t + 2\sin t)\, dt = \left[\frac{1}{2}\cos 2t + 2t\sin t + 2\cos t - 2\cos t\right]_0^{\pi/2} = -1 + \pi;$

C_2: $\mathbf{r} = \mathbf{j} + \frac{\pi}{2}(1-t)\mathbf{k}$, $0 \le t \le 1 \Rightarrow \mathbf{F} = \pi(1-t)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = -\frac{\pi}{2}\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -\pi$

$\Rightarrow \text{Flow}_2 = \int_0^1 -\pi\, dt = \left[-\pi t\right]_0^1 = -\pi;$

C_3: $\mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j},\ 0 \le t \le 1 \Rightarrow \mathbf{F} = 2t\mathbf{i} + 2(1-t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t$

$\Rightarrow \text{Flow}_3 = \int_0^1 2t\, dt = \left[t^2\right]_0^1 = 1 \Rightarrow \text{Circulation} = (-1+\pi) - \pi + 1 = 0$

52. $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$, where $f(x, y, z) = \frac{1}{2}\left(x^2, y^2 + x^2\right) \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(f\left(\mathbf{r}(t)\right)\right)$ by the chain rule $\Rightarrow$ Circulation $= \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_a^b \frac{d}{dt}\left(f\left(\mathbf{r}(t)\right)\right) dt = f\left(\mathbf{r}(b)\right) - f\left(\mathbf{r}(a)\right)$. Since C is an entire ellipse, $\mathbf{r}(b) = \mathbf{r}(a)$, thus the Circulation $= 0$.

53. Let $x = t$ be the parameter $\Rightarrow y = x^2 = t^2$ and $z = x = t \Rightarrow \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k},\ 0 \le t \le 1$ from $(0, 0, 0)$ to $(1, 1, 1)$ $\Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$ and $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^3 - t^3 = 2t^3 \Rightarrow \text{Flow} = \int_0^1 2t^3\, dt = \frac{1}{2}$

54. (a) $\mathbf{F} = \nabla(xy^2z^3) \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial z}\frac{dz}{dt} = \frac{df}{dt}$, where $f(x, y, z) = xy^2z^3 \Rightarrow \oint_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$

$= \int_a^b \frac{d}{dt}\left(f\left(\mathbf{r}(t)\right)\right) dt = f\left(\mathbf{r}(b)\right) - f\left(\mathbf{r}(a)\right) = 0$ since C is an entire ellipse.

(b) $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \int_{(1,1,1)}^{(2,1,-1)} \frac{d}{dt}\left(xy^2z^3\right) dt = \left[xy^2z^3\right]_{(1,1,1)}^{(2,1,-1)} = (2)(1)^2(-1)^3 - (1)(1)^2(1)^3 = -2 - 1 = -3$

55-60. Example CAS commands:

Maple:

```
with( LinearAlgebra );#55
F:= r -> < r[1]*r[2]^6|3*r[1]*(r[1]*r[2]^5+2>;
r:= t-> < 2*cos(t)| sin(t) >;
a,b:= 0.2*Pi;
dr:= map(diff,r(t),t):                          # (a)
F(r(t));                                         # (b)
q1:= simplify( F(r(t)) . dr ) assuming t::real;      # (c)
q2:= Int( q1, t = a..b );
value( q2 );
```

Mathematica: (functions and bounds will vary):

Exercises 55 and 56 use vectors in 2 dimensions

```
Clear[x, y, t, f, r, v]
f[x_,y_]:={x y^6, 3x(x y^5 + 2)}
{a, b}= {0, 2π};
x[t_]:= 2 Cos[t]
y[t_]:= Sin[t]
r[t_]:= {x[t], y[t]}
v[t_]:= r'{t]
integrand=f[x[t], y[t]]. v[t]//Simplify
Integrate[integrand, (t, a, b)]
N[%]
```

If the integration takes too long or cannot be done, use NIntegrate to integrate numerically. This is suggested for Exercises 57 - 60 that use vectors in 3 dimensions. Be certain to leave spaces between variables to be multiplied.

```
Clear[x, y, z, t, f, r, v]
f[x_,y_, z_]:=[y + y z Cos[x, y, z], x^2 +x z Cos[x, y, z], z+ x y Cos[x y z]}
[a, b]= {0, 2π};
x[t_]:= 2 Cos[t]
y[t_]:= 3 Sin[t]
z[t_]:= 1
r[t_]:={x[t], y[t],z[t]}
v[t_]:= r'[t]
integrand = f[x[t], y[t],z[t]]· v[t]/Simplify
NIntegrate[integrand (t, a, b}]
```

16.3 PATH INDEPENDENCE, POTENTIAL FUNCTIONS, AND CONSERVATIVE FIELDS

1. $\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

2. $\frac{\partial P}{\partial y} = x \cos z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y \cos z = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \sin z = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

3. $\frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial N}{\partial z} \Rightarrow$ Not Conservative

4. $\frac{\partial N}{\partial x} = 1 \neq -1 = \frac{\partial M}{\partial y} \Rightarrow$ Not Conservative

5. $\frac{\partial N}{\partial x} = 0 \neq 1 = \frac{\partial M}{\partial y} \Rightarrow$ Not Conservative

6. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

7. $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 3y \Rightarrow g(y, z) = \frac{3y^2}{2} + h(z) \Rightarrow f(x, y, z) = x^2 + \frac{3y^2}{2} + h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 4z \Rightarrow h(z) = 2z^2 + C \Rightarrow f(x, y, z) = x^2 + \frac{3y^2}{2} + 2z^2 + C$

8. $\frac{\partial f}{\partial x} = y + z \Rightarrow f(x, y, z) = (y+z)x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x + z \Rightarrow \frac{\partial g}{\partial y} = z \Rightarrow g(y, z) = zy + h(z)$
$\Rightarrow f(x, y, z) = (y+z)x + zy + h(z) \Rightarrow \frac{\partial f}{\partial z} = x + y + h'(z) = x + y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z)$
$= (y+z)x + zy + C$

9. $\frac{\partial f}{\partial x} = e^{y+2z} \Rightarrow f(x, y, z) = xe^{y+2z} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xe^{y+2z} + \frac{\partial g}{\partial y} = xe^{y+2z} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow f(x, y, z)$
$= xe^{y+2z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2xe^{y+2z} + h'(z) = 2xe^{y+2z} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xe^{y+2z} + C$

10. $\frac{\partial f}{\partial x} = y \sin z \Rightarrow f(x, y, z) = xy \sin z + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y} = x \sin z = \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z)$
$\Rightarrow f(x, y, z) = xy \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = xy \cos z + h'(z) = xy \cos z \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z)$
$= xy \sin z + C$

11. $\frac{\partial f}{\partial z}=\frac{z}{y^2+z^2}\Rightarrow f(x,y,z)=\frac{1}{2}\ln\left(y^2+z^2\right)+g(x,y)\Rightarrow \frac{\partial f}{\partial x}=\frac{\partial g}{\partial x}=\ln x+\sec^2(x+y)\Rightarrow g(x,y)$
$=(x\ln x-x)+\tan(x+y)+h(y)\Rightarrow f(x,y,z)=\frac{1}{2}\ln\left(y^2+z^2\right)+(x\ln x-x)+\tan(x+y)+h(y)$
$\Rightarrow \frac{\partial f}{\partial y}=\frac{y}{y^2+z^2}+\sec^2(x+y)+h'(y)=\sec^2(x+y)+\frac{y}{y^2+z^2}\Rightarrow h'(y)=0\Rightarrow h(y)=C\Rightarrow f(x,y,z)$
$=\frac{1}{2}\ln\left(y^2+z^2\right)+(x\ln x-x)+\tan(x+y)+C$

12. $\frac{\partial f}{\partial x}=\frac{y}{1+x^2y^2}\Rightarrow f(x,y,z)=\tan^{-1}(xy)+g(y,z)\Rightarrow \frac{\partial f}{\partial y}=\frac{x}{1+x^2y^2}+\frac{\partial g}{\partial y}=\frac{x}{1+x^2y^2}+\frac{z}{\sqrt{1-y^2z^2}}$
$\Rightarrow \frac{\partial g}{\partial y}=\frac{z}{\sqrt{1-y^2z^2}}\Rightarrow g(y,z)=\sin^{-1}(yz)+h(z)\Rightarrow f(x,y,z)=\tan^{-1}(xy)+\sin^{-1}(yz)+h(z)$
$\Rightarrow \frac{\partial f}{\partial z}=\frac{y}{\sqrt{1-y^2z^2}}+h'(z)=\frac{y}{\sqrt{1-y^2z^2}}+\frac{1}{z}\Rightarrow h'(z)=\frac{1}{z}\Rightarrow h(z)=\ln|z|+C$
$\Rightarrow f(x,y,z)=\tan^{-1}(xy)+\sin^{-1}(yz)+\ln|z|+C$

13. Let $\mathbf{F}(x,y,z)=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}\Rightarrow \frac{\partial P}{\partial y}=0=\frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z}=0=\frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x}=0=\frac{\partial M}{\partial y}\Rightarrow M\,dx+N\,dy+P\,dz$ is exact;
$\frac{\partial f}{\partial x}=2x\Rightarrow f(x,y,z)=x^2+g(y,z)\Rightarrow \frac{\partial f}{\partial y}=\frac{\partial g}{\partial y}=2y\Rightarrow g(y,z)=y^2+h(z)\Rightarrow f(x,y,z)=x^2+y^2=h(z)$
$\Rightarrow \frac{\partial f}{\partial z}=h'(z)=2z\Rightarrow h(z)=z^2+C\Rightarrow f(x,y,z)=x^2+y^2+z^2+C\Rightarrow \int_{(0,0,0)}^{(2,3,-6)}2x\,dx+2y\,dy+2z\,dz$
$=f(2,3,-6)-f(0,0,0)=2^2+3^2+(-6)^2=49$

14. Let $\mathbf{F}(x,y,z)=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}\Rightarrow \frac{\partial P}{\partial y}=x=\frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z}=y=\frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x}=z=\frac{\partial M}{\partial y}\Rightarrow M\,dx+N\,dy+P\,dz$ is
exact; $\frac{\partial f}{\partial x}=yz\Rightarrow f(x,y,z)=xyz+g(y,z)\Rightarrow \frac{\partial f}{\partial y}=xz+\frac{\partial g}{\partial y}=xz\Rightarrow \frac{\partial g}{\partial y}=0\Rightarrow g(y,z)=h(z)$
$\Rightarrow f(x,y,z)=xyz+h(z)\Rightarrow \frac{\partial f}{\partial z}=xy+h'(z)=xy\Rightarrow h'(z)=0\Rightarrow h(z)=C\Rightarrow f(x,y,z)=xyz+C$
$\Rightarrow \int_{(1,1,2)}^{(3,5,0)}yz\,dx+xz\,dy+xy\,dz=f(3,5,0)-f(1,1,2)=0-2=-2$

15. Let $\mathbf{F}(x,y,z)=2xy\mathbf{i}+\left(x^2-z^2\right)\mathbf{j}-2yz\mathbf{k}\Rightarrow \frac{\partial P}{\partial y}=-2z=\frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z}=0=\frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x}=2x=\frac{\partial M}{\partial y}$
$\Rightarrow M\,dx+N\,dy+P\,dz$ is exact; $\frac{\partial f}{\partial x}=2xy\Rightarrow f(x,y,z)=x^2y+g(y,z)\Rightarrow \frac{\partial f}{\partial y}=x^2+\frac{\partial g}{\partial y}=x^2-z^2\Rightarrow \frac{\partial g}{\partial y}=-z^2$
$\Rightarrow g(y,z)=-yz^2+h(z)\Rightarrow f(x,y,z)=x^2y-yz^2+h(z)\Rightarrow \frac{\partial f}{\partial z}=-2yz+h'(z)=-2yz\Rightarrow h'(z)=0\Rightarrow h(z)=C$
$\Rightarrow f(x,y,z)=x^2y-yz^2+C\Rightarrow \int_{(0,0,0)}^{(1,2,3)}2xy\,dx+\left(x^2-z^2\right)dy-2yz\,dz=f(1,2,3)-f(0,0,0)=2-2(3)^2$
$=-16$

16. Let $\mathbf{F}(x,y,z)=2x\mathbf{i}-y^2\mathbf{j}-\left(\frac{4}{1+z^2}\right)\mathbf{k}\Rightarrow \frac{\partial P}{\partial y}=0=\frac{\partial N}{\partial z},\frac{\partial M}{\partial z}=0=\frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x}=0=\frac{\partial M}{\partial y}$
$\Rightarrow M\,dx+N\,dy+P\,dz$ is exact; $\frac{\partial f}{\partial x}=2x\Rightarrow f(x,y,z)=x^2+g(y,z)\Rightarrow \frac{\partial f}{\partial y}=\frac{\partial g}{\partial y}=-y^2\Rightarrow g(y,z)=-\frac{y^3}{3}+h(z)$
$\Rightarrow f(x,y,z)=x^2-\frac{y^3}{3}+h(z)\Rightarrow \frac{\partial f}{\partial z}=h'(z)=-\frac{4}{1+z^2}\Rightarrow h(z)=-4\tan^{-1}z+C$

$\Rightarrow f(x, y, z) = x^2 - \frac{y^3}{3} - 4\tan^{-1} z + C \Rightarrow \int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2 dy - \frac{4}{1+z^2}\,dz = f(3,3,1) - f(0,0,0)$

$= \left(9 - \frac{27}{3} - 4\cdot\frac{\pi}{4}\right) - (0-0-0) = -\pi$

17. Let $\mathbf{F}(x, y, z) = (\sin y \cos x)\mathbf{i} + (\cos y \sin x)\mathbf{j} + \mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \cos y \cos x = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = \sin y \cos x \Rightarrow f(x, y, z) = \sin y \sin x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \cos y \sin x + \frac{\partial g}{\partial y}$

$= \cos y \sin x \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = \sin y \sin x + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 1 \Rightarrow h(z) = z + C$

$\Rightarrow f(x, y, z) = \sin y \sin x + z + C \Rightarrow \int_{(1,0,0)}^{(0,1,1)} \sin y \cos x\,dx + \cos y \sin x\,dy + dz = f(0,1,1) - f(1,0,0)$

$= (0+1) - (0+0) = 1$

18. Let $\mathbf{F}(x, y, z) = (2\cos y)\mathbf{i} + \left(\frac{1}{y} - 2x\sin y\right)\mathbf{j} + \left(\frac{1}{z}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2\sin y = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2\cos y \Rightarrow f(x, y, z) = 2x\cos y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -2x\sin y + \frac{\partial g}{\partial y}$

$= \frac{1}{y} - 2x\sin y \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y} \Rightarrow g(y, z) = \ln|y| + h(z) \Rightarrow f(x, y, z) = 2x\cos y + \ln|y| + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = \frac{1}{z}$

$\Rightarrow h(z) = \ln|z| + C \Rightarrow f(x, y, z) = 2x\cos y + \ln|y| + \ln|z| + C$

$\Rightarrow \int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right)dy + \frac{1}{z}\,dz = f\left(1, \frac{\pi}{2}, 2\right) - f(0,2,1)$

$= \left(2\cdot 0 + \ln\frac{\pi}{2} + \ln 2\right) - (0\cdot\cos 2 + \ln 2 + \ln 1) = \ln\frac{\pi}{2}$

19. Let $\mathbf{F}(x, y, z) = 3x^2\mathbf{i} + \left(\frac{z^2}{y}\right)\mathbf{j} + (2z\ln y)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = \frac{2z}{y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 3x^2 \Rightarrow f(x, y, z) = x^3 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = \frac{z^2}{y} \Rightarrow g(y, z) = z^2\ln y + h(z)$

$\Rightarrow f(x, y, z) = x^3 + z^2\ln y + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2z\ln y + h'(z) = 2z\ln y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$

$\Rightarrow f(x, y, z) = x^3 + z^2\ln y + C \Rightarrow \int_{(1,1,1)}^{(1,2,3)} 3x^2 dx + \frac{z^2}{y}\,dy + 2z\ln y\,dz = f(1,2,3) - f(1,1,1)$

$= (1 + 9\ln 2 + C) - (1 + 0 + C) = 9\ln 2$

20. Let $\mathbf{F}(x, y, z) = (2x\ln y - yz)\mathbf{i} + \left(\frac{x^2}{y} - xz\right)\mathbf{j} - (xy)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{2x}{y} - z = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2x\ln y - yz \Rightarrow f(x, y, z) = x^2\ln y - xyz + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial g}{\partial y}$

$= \frac{x^2}{y} - xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = x^2\ln y - xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = -xy + h'(z) = -xy \Rightarrow h'(z) = 0$

$\Rightarrow h(z) = C \Rightarrow f(x, y, z) = x^2\ln y - xyz + C \Rightarrow \int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$

$= f(2,1,1) - f(1,2,1) = (4\ln 1 - 2 + C) - (\ln 2 - 2 + C) = -\ln 2$

21. Let $\mathbf{F}(x, y, z) = \left(\frac{1}{y}\right)\mathbf{i} + \left(\frac{1}{z} - \frac{x}{y^2}\right)\mathbf{j} - \left(\frac{y}{z^2}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -\frac{1}{z^2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{1}{y^2} = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = \frac{1}{y} \Rightarrow f(x, y, z) = \frac{x}{y} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{\partial g}{\partial y} = \frac{1}{z} - \frac{x}{y^2}$

$\Rightarrow \frac{\partial g}{\partial y} = \frac{1}{z} \Rightarrow g(y, z) = \frac{y}{z} + h(z) \Rightarrow f(x, y, z) = \frac{x}{y} + \frac{y}{z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = -\frac{y}{z^2} + h'(z) = -\frac{y}{z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$

$\Rightarrow f(x, y, z) = \frac{x}{y} + \frac{y}{z} + C \Rightarrow \int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}\, dx + \left(\frac{1}{z} - \frac{x}{y^2}\right) dy - \frac{y}{z^2}\, dz = f(2, 2, 2) - f(1,1,1) = \left(\frac{2}{2} + \frac{2}{2} + C\right) - \left(\frac{1}{1} + \frac{1}{1} + C\right)$

$= 0$

22. Let $\mathbf{F}(x, y, z) = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{x^2+y^2+z^2}$ $\left(\text{and let } \rho^2 = x^2 + y^2 + z^2 \Rightarrow \frac{\partial \rho}{\partial x} = \frac{x}{\rho}, \frac{\partial \rho}{\partial y} = \frac{y}{\rho}, \frac{\partial \rho}{\partial z} = \frac{z}{\rho}\right)$

$\Rightarrow \frac{\partial P}{\partial y} = -\frac{4yz}{\rho^4} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{4xz}{\rho^4} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{4xy}{\rho^4} = \frac{\partial M}{\partial y} \Rightarrow M\ dx + N\ dy + P\ dz$ is exact;

$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2+z^2} \Rightarrow f(x, y, z) = \ln\left(x^2 + y^2 + z^2\right) + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2+z^2} + \frac{\partial g}{\partial y} = \frac{2y}{x^2+y^2+z^2}$

$\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = \ln\left(x^2 + y^2 + z^2\right) + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{2z}{x^2+y^2+z^2} + h'(z)$

$= \frac{2z}{x^2+y^2+z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = \ln\left(x^2 + y^2 + z^2\right) + C$

$\Rightarrow \int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} = f(2, 2, 2) - f(-1, -1, -1) = \ln 12 - \ln 3 = \ln 4$

23. $\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = (1+t)\mathbf{i} + (1+2t)\mathbf{j} + (1-2t)\mathbf{k},\ 0 \le t \le 1 \Rightarrow dx = dt,\ dy = 2\ dt,\ dz = -2\ dt$

$\Rightarrow \int_{(1,1,1)}^{(2,3,-1)} y\ dx + x\ dy + 4\ dz = \int_0^1 (2t+1)\ dt + (t+1)(2\ dt) + 4(-2)\ dt = \int_0^1 (4t-5)\ dt = \left[2t^2 - 5t\right]_0^1 = -3$

24. $\mathbf{r} = t(3\mathbf{j} + 4\mathbf{k}),\ 0 \le t \le 1 \Rightarrow dx = 0,\ dy = 3\ dt,\ dz = 4\ dt \Rightarrow \int_{(0,0,0)}^{(0,3,4)} x^2 dx + yz\ dy + \left(\frac{y^2}{2}\right) dz$

$= \int_0^1 \left(12t^2\right)(3\ dt) + \left(\frac{9t^2}{2}\right)(4\ dt) = \int_0^1 54t^2 dt = \left[18t^2\right]_0^1 = 18$

25. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M\ dx + N\ dy + P\ dz$ is exact $\Rightarrow \mathbf{F}$ is conservative

$\Rightarrow$ path independence

26. $\frac{\partial P}{\partial y} = -\frac{yz}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{xz}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{xy}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\partial M}{\partial y}$

$\Rightarrow M\ dx + N\ dy + P\ dz$ is exact $\Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ path independence

27. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{2x}{y^2} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;

$\frac{\partial f}{\partial x} = \frac{2x}{y} \Rightarrow f(x, y) = \frac{x^2}{y} + g(y) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + g'(y) = \frac{1-x^2}{y^2} \Rightarrow g'(y) = \frac{1}{y^2} \Rightarrow g(y) = -\frac{1}{y} + C$

$\Rightarrow f(x, y) = \frac{x^2}{y} - \frac{1}{y} + C \Rightarrow \mathbf{F} = \nabla\left(\frac{x^2-1}{y}\right)$

28. $\frac{\partial P}{\partial y} = \cos z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{e^x}{y} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;

$\frac{\partial f}{\partial x} = e^x \ln y \Rightarrow f(x, y, z) = e^x \ln y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{e^x}{y} + \frac{\partial g}{\partial y} = \frac{e^x}{y} + \sin z \Rightarrow \frac{\partial g}{\partial y} = \sin z \Rightarrow g(y, z)$

$= y \sin z + h(z) \Rightarrow f(x, y, z) = e^x \ln y + y \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = y \cos z + h'(z) = y \cos z \Rightarrow h'(z) = 0$

$\Rightarrow h(z) = C \Rightarrow f(x, y, z) = e^x \ln y + y \sin z + C \Rightarrow \mathbf{F} = \nabla\left(e^x \ln y + y \sin z\right)$

29. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial y} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;

$\frac{\partial f}{\partial x} = x^2 + y \Rightarrow f(x, y, z) = \frac{1}{3}x^3 + xy + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = y^2 + x \Rightarrow \frac{\partial g}{\partial y} = y^2 \Rightarrow g(y, z) = \frac{1}{3}y^3 + h(z)$

$\Rightarrow f(x, y, z) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = ze^z \Rightarrow h(z) = ze^z - e^z + C$

$\Rightarrow f(x, y, z) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z + C \Rightarrow \mathbf{F} = \nabla\left(\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right)$

(a) work $= \int_A^B \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = \left(\frac{1}{3} + 0 + 0 + e - e\right) - \left(\frac{1}{3} + 0 + 0 - 1\right) = 1$

(b) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

(c) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

<u>Note</u>: Since $\mathbf{F}$ is conservative, $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from $(1, 0, 0)$ to $(1, 0, 1)$.

30. $\frac{\partial P}{\partial y} = xe^{yz} + xyze^{yz} + \cos y = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = ye^{yz} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = ze^{yz} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$; $\frac{\partial f}{\partial x} = e^{yz} \Rightarrow f(x, y, z) = xe^{yz} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xze^{yz} + \frac{\partial g}{\partial y} = xze^{yz} + z \cos y \Rightarrow \frac{\partial g}{\partial y} = z \cos y$

$\Rightarrow g(y, z) = z \sin y + h(z) \Rightarrow f(x, y, z) = xe^{yz} + z \sin y + h(z) \Rightarrow \frac{\partial f}{\partial z} = xye^{yz} + \sin y + h'(z) = xye^{yz} + \sin y$

$\Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xe^{yz} + z \sin y + C \Rightarrow \mathbf{F} = \nabla\left(xe^{yz} + z \sin y\right)$

(a) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[xe^{yz} + z \sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = (1+0) - (1+0) = 0$

(b) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[xe^{yz} + z \sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = 0$

(c) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[xe^{yz} + z \sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = 0$

<u>Note</u>: Since $\mathbf{F}$ is conservative, $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from $(1, 0, 1)$ to $\left(1, \frac{\pi}{2}, 0\right)$.

31. (a) $\mathbf{F} = \nabla\left(x^3y^2\right) \Rightarrow \mathbf{F} = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$; let C_1 be the path from $(-1, 1)$ to $(0, 0) \Rightarrow x = t - 1$ and $y = -t + 1, 0 \le t \le 1 \Rightarrow \mathbf{F} = 3(t-1)^2(-t+1)^2\mathbf{i} + 2(t-1)^3(-t+1)\mathbf{j} = 3(t-1)^4\mathbf{i} - 2(t-1)^4\mathbf{j}$ and $\mathbf{r}_1 = (t-1)\mathbf{i} + (-t+1)\mathbf{j} \Rightarrow d\mathbf{r}_1 = dt\,\mathbf{i} - dt\,\mathbf{j} \Rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_0^1 \left[3(t-1)^4 + 2(t-1)^4\right] dt$

$= \int_0^1 5(t-1)^4 dt = \left[(t-1)^5\right]_0^1 = 1$; let C_2 be the path from $(0, 0)$ to $(1, 1) \Rightarrow x = t$ and $y = t$, $0 \le t \le 1 \Rightarrow \mathbf{F} = 3t^4\mathbf{i} + 2t^4\mathbf{j}$ and $\mathbf{r}_2 = t\mathbf{i} + t\mathbf{j} \Rightarrow d\mathbf{r}_2 = dt\mathbf{i} + dt\mathbf{j} \Rightarrow \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = \int_0^1 \left(3t^4 + 2t^4\right) dt$

$= \int_0^1 5t^4 dt = 1 \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = 2$

(b) Since $f(x, y) = x^3y^2$ is a potential function for $\mathbf{F}, \int_{(-1,1)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r} = f(1,1) - f(-1,1) = 2$

32. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2x \sin y = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;
$\frac{\partial f}{\partial x} = 2x \cos y \Rightarrow f(x, y, z) = x^2 \cos y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -x^2 \sin y + \frac{\partial g}{\partial y} = -x^2 \sin y \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z)$
$\Rightarrow f(x, y, z) = x^2 \cos y + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = x^2 \cos y + C \Rightarrow \mathbf{F} = \nabla\left(x^2 \cos y\right)$

(a) $\int_C 2x \cos y\, dx - x^2 \sin y\, dy = \left[x^2 \cos y\right]_{(1,0)}^{(0,1)} = 0 - 1 = -1$

(b) $\int_C 2x \cos y\, dx - x^2 \sin y\, dy = \left[x^2 \cos y\right]_{(-1,\pi)}^{(1,0)} = 1 - (-1) = 2$

(c) $\int_C 2x \cos y\, dx - x^2 \sin y\, dy = \left[x^2 \cos y\right]_{(-1,0)}^{(1,0)} = 1 - 1 = 0$

(d) $\int_C 2x \cos y\, dx - x^2 \sin y\, dy = \left[x^2 \cos y\right]_{(1,0)}^{(1,0)} = 1 - 1 = 0$

33. (a) If the differential form is exact, then $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \Rightarrow 2ay = cy$ for all $y \Rightarrow 2a = c$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \Rightarrow 2cx = 2cx$ for all x, and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow by = 2ay$ for all $y \Rightarrow b = 2a$ and $c = 2a$

(b) $\mathbf{F} = \nabla f \Rightarrow$ the differential form with $a = 1$ in part (a) is exact $\Rightarrow b = 2$ and $c = 2$

34. $\mathbf{F} = \nabla f \Rightarrow g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(x,y,z)} \nabla f \cdot d\mathbf{r} = f(x, y, z) - f(0, 0, 0) \Rightarrow \frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} - 0, \frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} - 0$, and $\frac{\partial g}{\partial z} = \frac{\partial f}{\partial z} - 0 \Rightarrow \nabla g = \nabla f = \mathbf{F}$, as claimed

35. The path will not matter; the work along any path will be the same because the field is conservative.

36. The field is not conservative, for otherwise the work would be the same along C_1 and C_2.

37. Let the coordinates of points A and B be (x_A, y_A, z_A) and (x_B, y_B, z_B), respectively. The force $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is conservative because all the partial derivatives of M, N, and P are zero. Therefore, the potential function is $f(x, y, z) = ax + by + cz + C$, and the work done by the force in moving a particle along any path from A to B is $f(B) - f(A) = f(x_B, y_B, z_B) - f(x_A, y_A, z_A)$
$= (ax_B + by_B + cz_B + C) - (ax_A + by_A + cz_A + C) = a(x_B - x_A) + b(y_B - y_A) + c(z_B - z_A) = \mathbf{F} \cdot \overrightarrow{BA}$

38. (a) Let $-GmM = C \Rightarrow \mathbf{F} = C\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}\right]$
$\Rightarrow \frac{\partial P}{\partial y} = \frac{-3yzC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{-3xzC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{-3xyC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} = \nabla f$ for some f;
$\frac{\partial f}{\partial x} = \frac{xC}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f(x, y, z) = -\frac{C}{(x^2+y^2+z^2)^{1/2}} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{yC}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial y}$

$= \frac{yC}{\left(x^2+y^2+z^2\right)^{3/2}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{zC}{\left(x^2+y^2+z^2\right)^{3/2}} + h'(z) = \frac{zC}{\left(x^2+y^2+z^2\right)^{3/2}}$

$\Rightarrow h(z) = C_1 \Rightarrow f(x, y, z) = -\frac{C}{\left(x^2+y^2+z^2\right)^{1/2}} + C_1$. Let $C_1 = 0 \Rightarrow f(x, y, z) = \frac{GmM}{\left(x^2+y^2+z^2\right)^{1/2}}$ is a potential function for $\mathbf{F}$.

(b) If s is the distance of (x, y, z) from the origin, then $s = \sqrt{x^2+y^2+z^2}$. The work done by the gravitational field $\mathbf{F}$ is work $= \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \left[\frac{GmM}{\sqrt{x^2+y^2+z^2}}\right]_{P_1}^{P_2} = \frac{GmM}{s_2} - \frac{GmM}{s_1} = GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right)$, as claimed.

16.4 GREEN'S THEOREM IN THE PLANE

1. $M = -y = -a \sin t$, $N = x = a \cos t$, $dx = -a \sin t\, dt$, $dy = a \cos t\, dt \Rightarrow \frac{\partial M}{\partial y} = 0$, $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$, and $\frac{\partial N}{\partial y} = 0$;

Equation (3): $\oint_C M\, dy - N\, dx = \int_0^{2\pi} \left[(-a \sin t)(a \cos t) - (a \cos t)(-a \sin t)\right] dt = \int_0^{2\pi} 0\, dt = 0$;

$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy = \iint_R 0\, dx\, dy = 0$, Flux

Equation (4): $\oint_C M\, dx + N\, dy = \int_0^{2\pi} \left[(-a \sin t)(-a \sin t) - (a \cos t)(a \cos t)\right] dt = \int_0^{2\pi} a^2 dt = 2\pi a^2$;

$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_{-a}^{a} \int_{-c}^{\sqrt{a^2-x^2}} 2\, dy\, dx = \int_{-a}^{a} 4\sqrt{a^2-x^2}\, dx = 4\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{-a}^{a}$

$= 2a^2\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 2a^2\pi$, Circulation

2. $M = y = a \sin t$, $N = 0$, $dx = -a \sin t\, dt$, $dy = a \cos t\, dt \Rightarrow \frac{\partial M}{\partial x} = 0$, $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 0$, and $\frac{\partial N}{\partial y} = 0$;

Equation (3): $\oint_C M\, dy - N\, dx = \int_0^{2\pi} a^2 \sin t \cos t\, dt = a^2\left[\frac{1}{2}\sin^2 t\right]_0^{2\pi} = 0$; $\iint_R 0\, dx\, dy = 0$, Flux

Equation (4): $\oint_C M\, dx + N\, dy = \int_0^{2\pi} \left(-a^2 \sin^2 t\right) dt = -a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = -\pi a^2$;

$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R -1\, dx\, dy = \int_0^{2\pi}\int_0^{a} -r\, dr\, d\theta = \int_0^{2\pi} -\frac{a^2}{2}\, d\theta = -\pi a^2$, Circulation

3. $M = 2x = 2a \cos t$, $N = -3y = -3a \sin t$, $dx = -a \sin t\, dt$, $dy = a \cos t\, dt \Rightarrow \frac{\partial M}{\partial x} = 2$, $\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = 0$, and $\frac{\partial N}{\partial y} = -3$;

Equation (3): $\oint_C M\, dy - N\, dx = \int_0^{2\pi} \left[(2a \cos t)(a \cos t) + (3a \sin t)(-a \sin t)\right] dt$

$= \int_0^{2\pi} \left(2a^2 \cos^2 t - 3a^2 \sin^2 t\right) dt = 2a^2\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} - 3a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = 2\pi a^2 - 3\pi a^2 = -\pi a^2$;

$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) = \iint_R -1\, dx\, dy = \int_0^{2\pi}\int_0^{a} -r\, dr\, d\theta = \int_0^{2\pi} -\frac{a^2}{2}\, d\theta = -\pi a^2$, Flux

Equation (4): $\oint_C M\, dx + N\, dy = \int_0^{2\pi} \left[(2a \cos t)(-a \sin t) + (-3a \sin t)(a \cos t)\right] dt$

$= \int_0^{2\pi} \left(-2a^2 \sin t \cos t - 3a^2 \sin t \cos t\right) dt = -5a^2\left[\frac{1}{2}\sin^2 t\right]_0^{2\pi} = 0$; $\iint_R 0\, dx\, dy = 0$, Circulation

4. $M=-x^2y=-a^3\cos^2 t,\ N=xy^2=a^3\cos t\sin^2 t,\ dx=-a\sin t\,dt,\ dy=a\cos t\,dt$
$\Rightarrow \frac{\partial M}{\partial x}=-2xy,\ \frac{\partial M}{\partial y}=-x^2,\ \frac{\partial N}{\partial x}=y^2$, and $\frac{\partial N}{\partial y}=2xy;$

Equation (3): $\oint_C M\,dy-N\,dx=\int_0^{2\pi}\left(-a^4\cos^3 t\sin t+a^4\cos t\sin^3 t\right)=\left[\frac{a^4}{4}\cos^4 t+\frac{a^4}{4}\sin^4 t\right]_0^{2\pi}=0;$

$\iint_R\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dx\,dy=\iint_R(-2xy+2xy)\,dx\,dy=0$, Flux

Equation (4): $\oint_C M\,dx+N\,dy=\int_0^{2\pi}\left(a^4\cos^2 t\sin^2 t+a^4\cos^2 t\sin^2 t\right)dt=\int_0^{2\pi}\left(2a^4\cos^2 t\sin^2 t\right)dt$

$=\int_0^{2\pi}\frac{1}{2}a^4\sin^2 2t\,dt=\frac{a^4}{4}\int_0^{4\pi}\sin^2 u\,du=\frac{a^4}{4}\left[\frac{u}{2}-\frac{\sin 2u}{4}\right]_0^{4\pi}=\frac{\pi a^4}{2};\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy=\iint_R\left(y^2+x^2\right)dx\,dy$

$=\int_0^{2\pi}\int_0^a r^2\cdot r\,dr\,d\theta=\int_0^{2\pi}\frac{a^4}{4}\,d\theta==\frac{\pi a^4}{2}$, Circulation

5. $M=x-y,\ N=y-x\Rightarrow\frac{\partial M}{\partial x}=1,\ \frac{\partial M}{\partial y}=-1,\ \frac{\partial N}{\partial x}=-1,\ \frac{\partial N}{\partial y}=1\Rightarrow$ Flux $=\iint_R 2\,dx\,dy=\int_0^1\int_0^1 2\,dx\,dy=2;$

Circ $=\iint_R[-1-(-1)]\,dx\,dy=0$

6. $M=x^2+4y,\ N=x+y^2\Rightarrow\frac{\partial M}{\partial x}=2x,\ \frac{\partial M}{\partial y}=4,\ \frac{\partial N}{\partial x}=1,\ \frac{\partial N}{\partial y}=2y\Rightarrow$ Flux $=\iint_R(2x+2y)\,dx\,dy$

$=\int_0^1\int_0^1(2x+2y)\,dx\,dy=\int_0^1\left[x^2+2xy\right]_0^1 dy=\int_0^1(1+2y)\,dy=\left[y+y^2\right]_0^1=2;$ Circ $=\iint_R(1-4)\,dx\,dy$

$=\int_0^1\int_0^1-3\,dx\,dy=-3$

7. $M=y^2-x^2,\ N=x^2+y^2\Rightarrow\frac{\partial M}{\partial x}=-2x,\ \frac{\partial M}{\partial y}=2y,\ \frac{\partial N}{\partial x}=2x,\ \frac{\partial N}{\partial y}=2y\Rightarrow$ Flux $=\iint_R(-2x+2y)\,dx\,dy$

$=\int_0^3\int_0^x(-2x+2y)\,dy\,dx=\int_0^3\left(-2x^2+x^2\right)dx=\left[-\frac{1}{3}x^3\right]_0^3=-9;$ Circ $=\iint_R(2x-2y)\,dx\,dy$

$=\int_0^3\int_0^x(2x-2y)\,dy\,dx=\int_0^3 x^2dx=9$

8. $M=x+y,\ N=-\left(x^2+y^2\right)\Rightarrow\frac{\partial M}{\partial x}=1,\ \frac{\partial M}{\partial y}=1,\ \frac{\partial N}{\partial x}=-2x,\ \frac{\partial N}{\partial y}=-2y\Rightarrow$ Flux $=\iint_R(1-2y)\,dx\,dy$

$=\int_0^1\int_0^x(1-2y)\,dy\,dx=\int_0^1\left(x-x^2\right)dx=\frac{1}{6};$ Circ $=\iint_R(-2x-1)\,dx\,dy=\int_0^1\int_0^x(-2x-1)\,dy\,dx$

$=\int_0^1\left(-2x^2-x\right)dx=-\frac{7}{6}$

9. $M=xy+y^2,\ N=x-y\Rightarrow\frac{\partial M}{\partial x}=y,\ \frac{\partial M}{\partial y}=x+2y,\ \frac{\partial N}{dx}=1,\ \frac{\partial N}{\partial y}=-1\Rightarrow$ Flux $=\iint_R\left(y+(-1)\right)dy\,dx$

$=\int_0^1\int_{x^2}^{\sqrt{x}}(y-1)\,dy\,dx=\int_0^1\left(\frac{1}{2}x-\sqrt{x}-\frac{1}{2}x^4+x^2\right)dx=-\frac{11}{60};$ Circ $=\iint_R\left(1-(x+2y)\right)dy\,dx$

$=\int_0^1\int_{x^2}^{\sqrt{x}}(1-x-2y)\,dy\,dx=\int_0^1\left(\sqrt{x}-x^{3/2}-x-x^2+x^3+x^4\right)dx=-\frac{7}{60}$

10. $M = x+3y,\ N = 2x-y \Rightarrow \frac{\partial M}{\partial x} = 1,\ \frac{\partial M}{\partial y} = 3,\ \frac{\partial N}{\partial x} = 2,\ \frac{\partial N}{\partial y} = -1 \Rightarrow \text{Flux} = \iint\limits_R \left(1+(-1)\right)\, dy\, dx = 0$

$\text{Circ} = \iint\limits_R (2-3)\, dy\, dx = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{\left(2-x^2\right)/2}}^{\sqrt{\left(2-x^2\right)/2}} (-1)\, dy\, dx = -\frac{2}{\sqrt{2}} \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2}\, dx = -\pi\sqrt{2}$

11. $M = x^3y^2,\ N = \frac{1}{2}x^4 y \Rightarrow \frac{\partial M}{\partial x} = 3x^2y^2,\ \frac{\partial M}{\partial y} = 2x^3 y,\ \frac{\partial N}{\partial x} = 2x^3 y,\ \frac{\partial N}{\partial y} = \frac{1}{2}x^4 \Rightarrow \text{Flux} = \iint\limits_R \left(3x^2y^2 + \frac{1}{2}x^4\right) dy\, dx$

$= \int_0^2 \int_{x^2-x}^{x} \left(3x^2y^2 + \frac{1}{2}x^4\right) dy\, dx = \int_0^2 \left(3x^5 - \frac{7}{2}x^6 + 3x^7 - x^8\right) dx = \frac{64}{9};\ \text{Circ} = \iint\limits_R \left(2x^3y - 2x^3y\right) dy\, dx = 0$

12. $M = \frac{x}{1+y^2},\ N = \tan^{-1} y \Rightarrow \frac{\partial M}{\partial x} = \frac{1}{1+y^2},\ \frac{\partial M}{\partial y} = \frac{-2xy}{\left(1+y^2\right)^2},\ \frac{\partial N}{\partial x} = 0,\ \frac{\partial N}{\partial y} = \frac{1}{1+y^2} \Rightarrow \text{Flux} = \iint\limits_R \left(\frac{1}{1+y^2} + \frac{1}{1+y^2}\right) dx\, dy$

$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{1+y^2} dx\, dy = \int_{-1}^{1} \frac{4\sqrt{1-y^2}}{1+y^2}\, dx = 4\pi\sqrt{2} - 4\pi;\ \text{Circ} = \iint\limits_R \left(0 - \left(\frac{-2xy}{\left(1+y^2\right)^2}\right)\right) dy\, dx$

$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\frac{2xy}{\left(1+y^2\right)^2}\right) dy\, dx = \int_{-1}^{1} (0)\, dx = 0$

13. $M = x + e^x \sin y,\ N = x + e^x \cos y \Rightarrow \frac{\partial M}{\partial x} = 1 + e^x \sin y,\ \frac{\partial M}{\partial y} = e^x \cos y,\ \frac{\partial N}{\partial x} = 1 + e^x \cos y,\ \frac{\partial N}{\partial y} = -e^x \sin y$

$\Rightarrow \text{Flux} = \iint\limits_R dx\, dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2}\cos 2\theta\right) d\theta = \left[\frac{1}{4}\sin 2\theta\right]_{-\pi/4}^{\pi/4} = \frac{1}{2};$

$\text{Circ} = \iint\limits_R \left(1 + e^x \cos y - e^x \cos y\right) dx\, dy = \iint\limits_R dx\, dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2}\cos 2\theta\right) d\theta = \frac{1}{2}$

14. $M = \tan^{-1}\frac{y}{x},\ N = \ln\left(x^2+y^2\right) \Rightarrow \frac{\partial M}{\partial x} = \frac{-y}{x^2+y^2},\ \frac{\partial M}{\partial y} = \frac{x}{x^2+y^2},\ \frac{\partial N}{\partial x} = \frac{2x}{x^2+y^2},\ \frac{\partial N}{\partial y} = \frac{2y}{x^2+y^2}$

$\Rightarrow \text{Flux} = \iint\limits_R \left(\frac{-y}{x^2+y^2} + \frac{2y}{x^2+y^2}\right) dx\, dy = \int_0^{\pi} \int_1^2 \left(\frac{r\sin\theta}{r^2}\right) r\, dr\, d\theta = \int_0^{\pi} \sin\theta\, d\theta = 2;$

$\text{Circ} = \iint\limits_R \left(\frac{2x}{x^2+y^2} - \frac{x}{x^2+y^2}\right) dx\, dy = \int_0^{\pi} \int_1^2 \left(\frac{r\cos\theta}{r^2}\right) r\, dr\, d\theta = \int_0^{\pi} \cos\theta\, d\theta = 0$

15. $M = xy,\ N = y^2 \Rightarrow \frac{\partial M}{\partial x} = y,\ \frac{\partial M}{\partial y} = x,\ \frac{\partial N}{\partial x} = 0,\ \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint\limits_R (y+2y)\, dy\, dx = \int_0^1 \int_{x^2}^{x} 3y\, dy\, dx$

$= \int_0^1 \left(\frac{3x^2}{2} - \frac{3x^4}{2}\right) dx = \frac{1}{5};\ \text{Circ} = \iint\limits_R -x\, dy\, dx = \int_0^1 \int_{x^2}^{x} -x\, dy\, dx = \int_0^1 \left(-x^2 + x^3\right) dx = -\frac{1}{12}$

16. $M = -\sin y,\ N = x\cos y \Rightarrow \frac{\partial M}{\partial x} = 0,\ \frac{\partial M}{\partial y} = -\cos y,\ \frac{\partial N}{\partial x} = \cos y,\ \frac{\partial N}{\partial y} = -x\sin y$

$\Rightarrow \text{Flux} = \iint\limits_R (-x\sin y)\, dx\, dy = \int_0^{\pi/2} \int_0^{\pi/2} (-x\sin y)\, dx\, dy = \int_0^{\pi/2} \left(-\frac{\pi^2}{8}\sin y\right) dy = -\frac{\pi^2}{8};$

$\text{Circ} = \iint\limits_R \left[\cos y - (-\cos y)\right] dx\, dy = \int_0^{\pi/2} \int_0^{\pi/2} 2\cos y\, dx\, dy = \int_0^{\pi/2} \pi\cos y\, dy = \left[\pi\sin y\right]_0^{\pi/2} = \pi$

17. $M = 3xy - \frac{x}{1+y^2},\ N = e^x + \tan^{-1} y \Rightarrow \frac{\partial M}{\partial x} = 3y - \frac{1}{1+y^2},\ \frac{\partial N}{\partial y} = \frac{1}{1+y^2}$

$\Rightarrow \text{Flux} = \iint_R \left(3y - \frac{1}{1+y^2} + \frac{1}{1+y^2}\right) dx\, dy = \iint_R 3y\, dx\, dy = \int_0^{2\pi} \int_0^{a(1+\cos\theta)} (3r \sin\theta) r\, dr\, d\theta$

$= \int_0^{2\pi} a^3 (1+\cos\theta)^3 (\sin\theta)\, d\theta = \left[-\frac{a^3}{4}(1+\cos\theta)^4\right]_0^{2\pi} = -4a^3 - \left(-4a^3\right) = 0$

18. $M = y + e^x \ln y,\ N = \frac{e^x}{y} \Rightarrow \frac{\partial M}{\partial y} = 1 + \frac{e^x}{y},\ \frac{\partial N}{\partial x} = \frac{e^x}{y} \Rightarrow \text{Circ} = \iint_R \left[\frac{e^x}{y} - \left(1 + \frac{e^x}{y}\right)\right] dx\, dy = \iint_R (-1)\, dx\, dy$

$= \int_{-1}^{1} \int_{x^4+1}^{3-x^2} -dy\, dx = -\int_{-1}^{1} \left[\left(3 - x^2\right) - \left(x^4 + 1\right)\right] dx = \int_{-1}^{1} \left(x^4 + x^2 - 2\right) dx = -\frac{44}{15}$

19. $M = 2xy^3,\ N = 4x^2y^2 \Rightarrow \frac{\partial M}{\partial y} = 6xy^2,\ \frac{\partial N}{\partial x} = 8xy^2 \Rightarrow \text{work} = \oint_C 2xy^3 dx + 4x^2y^2 dy = \iint_R \left(8xy^2 - 6xy^2\right) dx\, dy$

$= \int_0^1 \int_0^{x^3} 2xy^2 dy\, dx = \int_0^1 \frac{2}{3} x^{10}\, dx = \frac{2}{33}$

20. $M = 4x - 2y,\ N = 2x - 4y \Rightarrow \frac{\partial M}{\partial y} = -2,\ \frac{\partial N}{\partial x} = 2 \Rightarrow \text{work} = \oint_C (4x - 2y)\, dx + (2x - 4y)\, dy$

$= \iint_R [2 - (-2)]\, dx\, dy = 4\iint_R dx\, dy = 4(\text{Area of the circle}) = 4(\pi \cdot 4) = 16\pi$

21. $M = y^2,\ N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = 2x \Rightarrow \oint_C y^2\, dx + x^2 dy = \iint_R (2x - 2y)\, dy\, dx$

$= \int_0^1 \int_0^{1-x} (2x - 2y)\, dy\, dx = \int_0^1 \left(-3x^2 + 4x - 1\right) dx = \left[-x^3 + 2x^2 - x\right]_0^1 = -1 + 2 - 1 = 0$

22. $M = 3y,\ N = 2x \Rightarrow \frac{\partial M}{\partial y} = 3,\ \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C 3y\, dx + 2x\, dy = \iint_R (2 - 3)\, dx\, dy = \int_0^{\pi} \int_0^{\sin x} (-1)\, dy\, dx$

$= -\int_0^{\pi} \sin x\, dx = -2$

23. $M = 6y + x,\ N = y + 2x \Rightarrow \frac{\partial M}{\partial y} = 6,\ \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C (6y + x)\, dx + (y + 2x)\, dy = \iint_R (2 - 6)\, dy\, dx$

$= -4(\text{Area of the circle}) = -16\pi$

24. $M = 2x + y^2, N = 2xy + 3y \Rightarrow \frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = 2y \Rightarrow \oint_C (2x + y^2)\, dx + (2xy + 3y)\, dy = \iint_R (2y - 2y)\, dx\, dy = 0$

25. $M = x = a\cos t,\ N = y = a\sin t \Rightarrow dx = -a\sin t\, dt,\ dy = a\cos t\, dt \Rightarrow \text{Area} = \frac{1}{2}\oint_C x\, dy - y\, dx$

$= \frac{1}{2}\int_0^{2\pi} \left(a^2 \cos^2 t + a^2 \sin^2 t\right) dt = \frac{1}{2}\int_0^{2\pi} a^2\, dt = \pi a^2$

26. $M = x = a\cos t,\ N = y = b\sin t \Rightarrow dx = -a\sin t\, dt,\ dy = b\cos t\, dt \Rightarrow \text{Area} = \frac{1}{2}\oint_C x\, dy - y\, dx$

$= \frac{1}{2}\int_0^{2\pi} \left(ab\cos^2 t + ab\sin^2 t\right) dt = \frac{1}{2}\int_0^{2\pi} ab\, dt = \pi ab$

27. $M = x = \cos^3 t,\ N = y = \sin^3 t \Rightarrow dx = -3\cos^2 t \sin t\ dt,\ dy = 3\sin^2 t \cos t\ dt \Rightarrow \text{Area} = \frac{1}{2}\oint_C x\ dy - y\ dx$
$= \frac{1}{2}\int_0^{2\pi}\left(3\sin^2 t\cos^2 t\right)\left(\cos^2 t + \sin^2 t\right) dt = \frac{1}{2}\int_0^{2\pi}\left(3\sin^2 t\cos^2 t\right) dt = \frac{3}{8}\int_0^{2\pi}\sin^2 2t\ dt = \frac{3}{16}\int_0^{4\pi}\sin^2 u\ du$
$= \frac{3}{16}\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = \frac{3}{8}\pi$

28. $C_1: M = x = t,\ N = y = 0 \Rightarrow dx = dt,\ dy = 0;\ C_2: M = x = (2\pi - t) - \sin(2\pi - t) = 2\pi - t + \sin t,$
$N = y = 1 - \cos(2\pi - t) = 1 - \cos t \Rightarrow dx = (\cos t - 1)\ dt,\ dy = \sin t\ dt$
$\Rightarrow \text{Area} = \frac{1}{2}\oint_C x\ dy - y\ dx = \frac{1}{2}\oint_{C_1} x\ dy - y\ dx + \frac{1}{2}\oint_{C_2} x\ dy - y\ dx$
$= \frac{1}{2}\int_0^{2\pi}(0)\ dt + \frac{1}{2}\int_0^{2\pi}\left[(2\pi - t + \sin t)(\sin t) - (1 - \cos t)(\cos t - 1)\right] dt = -\frac{1}{2}\int_0^{2\pi}(2\cos t + t\sin t - 2 - 2\pi\sin t)\ dt$
$= -\frac{1}{2}\left[3\sin t - t\cos t - 2t - 2\pi\cos t\right]_0^{2\pi} = 3\pi$

29. (a) $M = f(x),\ N = g(y) \Rightarrow \frac{\partial M}{\partial y} = 0,\ \frac{\partial N}{\partial x} = 0 \Rightarrow \oint_C f(x)\ dx + g(y)\ dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\ dy = \iint_R 0\ dx\ dy = 0$

(b) $M = ky,\ N = hx \Rightarrow \frac{\partial M}{\partial y} = k,\ \frac{\partial N}{\partial x} = h \Rightarrow \oint_C ky\ dx + hx\ dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\ dy$
$= \iint_R (h - k)\ dx\ dy = (h - k)(\text{Area of the region})$

30. $M = xy^2,\ N = x^2 y + 2x \Rightarrow \frac{\partial M}{\partial y} = 2xy,\ \frac{\partial N}{\partial x} = 2xy + 2 \Rightarrow \oint_C xy^2 dx + \left(x^2 y + 2x\right) dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\ dy$
$= \iint_R (2xy + 2 - 2xy)\ dx\ dy = 2\iint_R dx\ dy = 2$ times the area of the square

31. The integral is 0 for any simple closed plane curve C. The reasoning: By the tangential form of Green's Theorem, with $M = 4x^3 y$ and $N = x^4$, $\oint_C 4x^3 y\ dx + x^4 dy = \iint_R \left[\frac{\partial}{\partial x}\left(x^4\right) - \frac{\partial}{\partial y}\left(4x^3 y\right)\right] dx\ dy$
$= \iint_R \underbrace{\left(4x^3 - 4x^3\right)}_{0} dx\ dy = 0.$

32. The integral is 0 for any simple closed curve C. The reasoning: By the normal form of Green's theorem, with
$M = x^3$ and $N = -y^3$, $\oint_C -y^3 dy + x^3 dx = \iint_R \left[\underbrace{\frac{\partial}{\partial x}\left(-y^3\right)}_{0} - \underbrace{\frac{\partial}{\partial y}\left(x^3\right)}_{0}\right] dx\ dy = 0.$

33. Let $M = x$ and $N = 0 \Rightarrow \frac{\partial M}{\partial x} = 1$ and $\frac{\partial N}{\partial y} = 0 \Rightarrow \oint_C M\ dy - N\ dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\ dy \Rightarrow \oint_C x\ dy$
$\iint_R (1 + 0)\ dx\ dy \Rightarrow$ Area of $R = \iint_R dx\ dy = \oint_C x\ dy$; similarly, $M = y$ and $N = 0 \Rightarrow \frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 0$
$\Rightarrow \oint_C M\ dx + N\ dy = \iint_R \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) dy\ dx \Rightarrow \oint_C y\ dx = \iint_R (0 - 1)\ dy\ dx \Rightarrow -\oint_C y\ dx = \iint_R dx\ dy =$ Area of R

34. $\int_a^b f(x)\,dx =$ Area of $R = -\oint_C y\,dx,$ from Exercise 33

35. Let $\delta(x, y) = 1 \Rightarrow \overline{x} = \frac{M_y}{M} = \frac{\iint\limits_R x\,\delta(x, y)\,dA}{\iint\limits_R \delta(x, y)\,dA} = \frac{\iint\limits_R x\,dA}{\iint\limits_R dA} = \frac{\iint\limits_R x\,dA}{A} \Rightarrow A\overline{x} = \iint\limits_R x\,dA = \iint\limits_R (x+0)\,dx\,dy$

$= \oint_C \frac{x^2}{2}\,dy,\ A\overline{x} = \iint\limits_R x\,dA = \iint\limits_R (0+x)\,dx\,dy = -\oint_C xy\,dx,$ and $A\overline{x} = \iint\limits_R x\,dA = \iint\limits_R \left(\frac{2}{3}x + \frac{1}{3}x\right)dx\,dy$

$= \oint_C \frac{1}{3}x^2 dy - \frac{1}{3}xy\,dx \Rightarrow \frac{1}{2}\oint_C x^2 dy = -\oint_C xy\,dx = \frac{1}{3}\oint_C x^2 dy - xy\,dx = A\overline{x}$

36. If $\delta(x, y) = 1$ then $I_y = \iint\limits_R x^2\delta(x, y)\,dA = \iint\limits_R x^2\ dA = \iint\limits_R \left(x^2 + 0\right)dy\,dx = \frac{1}{3}\oint_C x^3\ dy,$

$\iint\limits_R x^2\ dA = \iint\limits_R \left(0 + x^2\right)dy\,dx = -\oint_C x^2 y\,dx,$ and $\iint\limits_R x^2\ dA = \iint\limits_R \left(\frac{3}{4}x^2 + \frac{1}{4}x^2\right)dy\,dx$

$= \oint_C \frac{1}{4}x^3\ dy - \frac{1}{4}x^2 y\,dx = \frac{1}{4}\oint_C x^3\ dy - x^2 y\,dx \Rightarrow \frac{1}{3}\oint_C x^3 dy = -\oint_C x^2 y\,dx = \frac{1}{4}\oint_C x^3\ dy - x^2 y\,dx = I_y$

37. $M = \frac{\partial f}{\partial y},\ N = -\frac{\partial f}{\partial x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y^2},\ \frac{\partial N}{\partial x} = -\frac{\partial^2 f}{\partial x^2} \Rightarrow \oint_C \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = \iint\limits_R \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)dx\,dy = 0$ for such curves C

38. $M = \frac{1}{4}x^2 y + \frac{1}{3}y^3,\ N = x \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{4}x^2 + y^2,\ \frac{\partial N}{\partial x} = 1 \Rightarrow$ Curl $= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - \left(\frac{1}{4}x^2 + y^2\right) > 0$ in the interior of the ellipse $\frac{1}{4}x^2 + y^2 = 1 \Rightarrow$ work $= \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_R \left(1 - \frac{1}{4}x^2 - y^2\right)dx\,dy$ will be maximized on the region $R = \{(x, y)|\text{curl}\,\mathbf{F}\} \ge 0$ or over the region enclosed by $1 = \frac{1}{4}x^2 + y^2$

39. (a) $\nabla f = \left(\frac{2x}{x^2+y^2}\right)\mathbf{i} + \left(\frac{2y}{x^2+y^2}\right)\mathbf{j} \Rightarrow M = \frac{2x}{x^2+y^2},\ N = \frac{2y}{x^2+y^2};$ since M, N are discontinuous at $(0, 0),$ we compute $\int_C \nabla f\cdot\mathbf{n}\,ds$ directly since Green's Theorem does not apply. Let $x = a\cos t,\ y = a\sin t$ $\Rightarrow dx = -a\sin t\,dt,\ \ dy = a\cos t\ dt,\ M = \frac{2}{a}\cos t,\ N = \frac{2}{a}\sin t,\ 0 \le t \le 2\pi,$ so $\int_C \nabla f\cdot\mathbf{n}\,ds = \int_C M\ dy - N\ dx$ $= \int_0^{2x}\left[\left(\frac{2}{a}\cos t\right)(a\cos t) - \left(\frac{2}{a}\sin t\right)(-a\sin t)\right]dt = \int_0^{2\pi} 2\left(\cos^2 t + \sin^2 t\right)dt = 4\pi.$ Note that this holds for any $a > 0,$ so $\int_C \nabla f\cdot\mathbf{n}\,ds = 4\pi$ for any circle C centered at $(0, 0)$ traversed counterclockwise and $\int_C \nabla f\cdot\mathbf{n}\,ds = -4\pi$ if C is traversed clockwise.

(b) If K does not enclose the point $(0, 0)$ we may apply Green's Theorem: $\int_C \nabla f\cdot\mathbf{n}\,ds = \int_C M\ dy - N\ dx$ $= \iint\limits_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy = \iint\limits_R \left(\frac{2\left(y^2 - x^2\right)}{\left(x^2+y^2\right)^2} + \frac{2\left(x^2 - y^2\right)}{\left(x^2+y^2\right)^2}\right)dx\,dy = \iint\limits_R 0\ dx\,dy = 0.$ If K does enclose the point $(0, 0)$ we proceed as follows:
Choose a small enough so that the circle C centered at $(0, 0)$ of radius a lies entirely within K. Green's Theorem applies to the region R that lies between K and C. Thus, as before, $0 = \iint\limits_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy$ $= \int_K M\ dy - N\ dx + \int_C M\ dy - N\ dx$ where K is traversed counterclockwise and C is traversed clockwise.

Hence by part (a) $0=\left[\int_K M\ dy - N\ dx\right] - 4\pi \Rightarrow 4\pi = \int_K M\ dy - N\ dx = \int_K \nabla f \cdot \mathbf{n}\ ds.$ We have shown:

$$\int_K \nabla f \cdot \mathbf{n}\ ds = \begin{cases} 0 & \text{if } (0,0) \text{ lies inside } K \\ 4\pi & \text{if } (0,0) \text{ lies outside } K \end{cases}$$

40. Assume a particle has a closed trajectory in R and let C_1 be the path $\Rightarrow C_1$ encloses a simply connected region $R_1 \Rightarrow C_1$ is a simple closed curve. Then the flux over R_1 is $\oint_{C_1} \mathbf{F}\cdot\mathbf{n}\ ds = 0,$ since the velocity vectors $\mathbf{F}$ are tangent to C_1. But $0 = \oint_{C_1} \mathbf{F}\cdot\mathbf{n}\ ds = \oint_{C_1} M\ dy - N\ dx = \iint\limits_{R_1}\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\ dy \Rightarrow M_x + N_y = 0,$ which is a contradiction. Therefore, C_1 cannot be a closed trajectory.

41. $\int_{g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x} dx\ dy = N(g_2(y), y) - N(g_1(y), y) \Rightarrow \int_c^d \int_{g_1(y)}^{g_2(y)} \left(\frac{\partial N}{\partial x} dx\right) dy = \int_c^d \left[N(g_2(y), y) - N(g_1(y), y)\right] dy$

$= \int_c^d N(g_2(y), y)\ dy - \int_c^d N(g_1(y), y)\ dy = \int_c^d N(g_2(y), y)\ dy + \int_d^c N(g_1(y), y)\ dy = \int_{C_2} N\ dy + \int_{C_1} N\ dy$

$= \oint_C dy \Rightarrow \oint_C N\ dy = \iint\limits_R \frac{\partial N}{\partial x}\ dx\ dy$

42. The curl of a conservative two-dimensional field is zero. The reasoning. A two-dimensional field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ can be considered to be the restriction to the xy-plane of a three-dimensional field whose *k* component, *P*, is zero, and whose **i** and **j** components are independent of *z*. For such a field to be conservative, we must have $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ by the component test in Section 16.3 $\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial x}$, and $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$.

43-46. Example CAS commands:

Maple:

```
with( plots );#43
M:= (x,y) -> 2*x-y;
N:= (x,y) -> x+3*y;
C:= x^2+4*y^2= 4;
implicitplot( C, x=-2..2,y= 2..2, scaling=constrained, title="#43(a)(Section 16.4)");
curlF_k:=D[1](N) - D[2](M):                #(b)
'curlF_k' = curlF_k(x,y);
top,bot:= solve( C, y );                   #(c)
left,right:= -2, 2;
q1:= Int( Int( curlF_k(x,y),y=bot..top ), x=left..right );
value( q1 );
```

Mathematica: (functions and bounds will vary)

The **ImplicitPlot** command will be useful for 43 and 44, but is not needed for 45 and 46. In 46, the equation of the line from $(0, 4)$ to $(2, 0)$ must be determined first.

```
Clear[x, y, f]
<<Graphics `ImplicitPlot`
```

```
f[x_, y_]:= {2x − y, x + 3y}
curve= x^2 + 4y^2 ==4
ImplicitPlot[curve, [x, −3,3}, {y, − 2, 2}, AspectRatio → Automatic, AxesLabel → {x, y}];
ybonds= Solve[curve, y]
{y1, y2]=y/.bounds;
integrand:=D[f(x,y][[2]],x] − D[f[x,y]],y]/Simplify
Integrate[integrand, {x, − 2, 2}, {y, y1, y2}]
N[%]
```

Bounds for y are determined differently in 45 and 46. In 46, note equation of the line from $(0, 4)$ to $(2, 0)$.

```
Clear[x, y, f]
f[x_, y_]:= {x Exp[y], 4x^2Log[y]}
ybound = 4 − 2x
Plot[{0, ybound}, {x, 0, 2.1}, AspectRatio → Automatic, AxesLabel → {x,y}];
integrand:= D[f[x, y][[2]],x] − D[f[x,y][[1]],y]//Simplify
Integrate[integrand, [x, 0, 2], {y, 0, ybound }]
N[%]
```

16.5 SURFACES AND AREA

1. In cylindrical coordinates, let $x = r\cos\theta,\ y = r\sin\theta,\ z = \left(\sqrt{x^2+y^2}\right)^2 = r^2.$

   Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k},\ \ 0 \le r \le 2, 0 \le \theta \le 2\pi.$

2. In cylindrical coordinates, let $x = r\cos\theta,\ y = r\sin\theta,\ z = 9 - x^2 - y^2 = 9 - r^2.$

   Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(9 - r^2\right)\mathbf{k}; z \ge 0 \Rightarrow 9 - r^2 \ge 0 \Rightarrow r^2 \le 9 \Rightarrow -3 \le r \le 3, 0 \le \theta \le 2\pi.$

   But $-3 \le r \le 0$ gives the same points as $0 \le r \le 3,$ so let $0 \le r \le 3.$

3. In cylindrical coordinates, let $x = r\cos\theta,\ y = r\sin\theta,\ z = \frac{\sqrt{x^2+y^2}}{2} \Rightarrow z = \frac{r}{2}.$

   Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{r}{2}\right)\mathbf{k}.$ For $0 \le z \le 3, 0 \le \frac{r}{2} \le 3 \Rightarrow 0 \le r \le 6;$ to get only the first octant, let $0 \le \theta \le \frac{\pi}{2}.$

4. In cylindrical coordinates, let $x = r\cos\theta,\ y = r\sin\theta,\ z = 2\sqrt{x^2+y^2} \Rightarrow z = 2r.$

   Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k}.$ For $2 \le z \le 4, 2 \le 2r \le 4 \Rightarrow 1 \le r \le 2,$ and let $0 \le \theta \le 2\pi.$

5. In cylindrical coordinates, let $x = r\cos\theta,\ y = r\sin\theta;$ since $x^2 + y^2 = r^2 \Rightarrow z^2 = 9 - \left(x^2 + y^2\right) = 9 - r^2$

   $\Rightarrow z = \sqrt{9 - r^2}, z \ge 0.$ Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \sqrt{9 - r^2}\mathbf{k}.$ Let $0 \le \theta \le 2\pi.$ For the domain of r:

$z=\sqrt{x^2+y^2}$ and $x^2+y^2+z^2=9 \Rightarrow x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=9 \Rightarrow 2\left(x^2+y^2\right)=9 \Rightarrow 2r^2=9$
$\Rightarrow r=\frac{3}{\sqrt{2}} \Rightarrow 0 \le r \le \frac{3}{\sqrt{2}}.$

6. In cylindrical coordinates, $\mathbf{r}(r,\theta)=(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}+\sqrt{4-r^2}\,\mathbf{k}$ (see Exercise 5 above with $x^2+y^2+z^2=4$, instead of $x^2+y^2+z^2=9$). For the first octant, let $0\le\theta\le\frac{\pi}{2}$. For the domain of r: $z=\sqrt{x^2+y^2}$ and $x^2+y^2+z^2=4 \Rightarrow x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=4 \Rightarrow 2\left(x^2+y^2\right)=4 \Rightarrow 2r^2=4 \Rightarrow r=\sqrt{2}.$ Thus, let $\sqrt{2}\le r\le 2$ (to get the portion of the sphere between the cone and the xy-plane).

7. In spherical coordinates, $x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ \rho=\sqrt{x^2+y^2+z^2} \Rightarrow \rho^2=3 \Rightarrow \rho=\sqrt{3}$ $\Rightarrow z=\sqrt{3}\cos\phi$ for the sphere; $z=\frac{\sqrt{3}}{2}=\sqrt{3}\cos\phi \Rightarrow \cos\phi=\frac{1}{2} \Rightarrow \phi=\frac{\pi}{3}; z=-\frac{\sqrt{3}}{2} \Rightarrow -\frac{\sqrt{3}}{2}=\sqrt{3}\cos\phi$ $\Rightarrow \cos\phi=-\frac{1}{2} \Rightarrow \phi=\frac{2\pi}{3}.$ Then $\mathbf{r}(\phi,\theta)=\left(\sqrt{3}\sin\phi\cos\theta\right)\mathbf{i}+\left(\sqrt{3}\sin\phi\sin\theta\right)\mathbf{j}+\left(\sqrt{3}\cos\phi\right)\mathbf{k},$ $\frac{\pi}{3}\le\phi\le\frac{2\pi}{3}$ and $0\le\theta\le 2\pi.$

8. In spherical coordinates, $x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ \rho=\sqrt{x^2+y^2+z^2} \Rightarrow \rho^2=8 \Rightarrow \rho=\sqrt{8}=2\sqrt{2}$ $\Rightarrow x=2\sqrt{2}\sin\phi\cos\theta,\ y=2\sqrt{2}\sin\phi\sin\theta,$ and $z=2\sqrt{2}\cos\phi.$ Thus let $\mathbf{r}(\phi,\theta)=\left(2\sqrt{2}\sin\phi\cos\theta\right)\mathbf{i}+\left(2\sqrt{2}\sin\phi\sin\theta\right)\mathbf{j}+\left(2\sqrt{2}\cos\phi\right)\mathbf{k}; z=-2 \Rightarrow -2=2\sqrt{2}\cos\phi$ $\Rightarrow \cos\phi=-\frac{1}{\sqrt{2}} \Rightarrow \phi=\frac{3\pi}{4}; z=2\sqrt{2} \Rightarrow 2\sqrt{2}=2\sqrt{2}\cos\phi \Rightarrow \cos\phi=1 \Rightarrow \phi=0.$ Thus $0\le\phi\le\frac{3\pi}{4}$ and $0\le\theta\le 2\pi.$

9. Since $z=4-y^2$, we can let $\mathbf{r}$ be a function of x and y $\Rightarrow \mathbf{r}(x,y)=x\mathbf{i}+y\mathbf{j}+\left(4-y^2\right)\mathbf{k}.$ Then $z=0$ $\Rightarrow 0=4-y^2 \Rightarrow y=\pm 2.$ Thus, let $-2\le y\le 2$ and $0\le x\le 2.$

10. Since $y=x^2$, we can let $\mathbf{r}$ be a function of x and $z \Rightarrow \mathbf{r}(x,z)=x\mathbf{i}+x^2\mathbf{j}+z\mathbf{k}.$ Then $y=2$ $\Rightarrow x^2=2 \Rightarrow x=\pm\sqrt{2}.$ Thus, let $-\sqrt{2}\le x\le\sqrt{2}$ and $0\le z\le 3.$

11. When $x=0$, let $y^2+z^2=9$ be the circular section in the yz-plane. Use polar coordinates in the yz-plane $\Rightarrow y=3\cos\theta$ and $z=3\sin\theta.$ Thus let $x=u$ and $\theta=v \Rightarrow \mathbf{r}(u,v)=u\mathbf{i}+(3\cos v)\mathbf{j}+(3\sin v)\mathbf{k}$ where $0\le u\le 3,$ and $0\le v\le 2\pi.$

12. When $y=0,$ let $x^2+z^2=4$ be the circular section in the xz-plane. Use polar coordinates in the xz-plane $\Rightarrow x=2\cos\theta$ and $z=2\sin\theta.$ Thus let $y=u$ and $\theta=v \Rightarrow \mathbf{r}(u,v)=(2\cos v)\mathbf{i}+u\mathbf{j}+(3\sin v)\mathbf{k}$ where $-2\le u\le 2,$ and $0\le v\le\pi$ (since we want the portion <u>above</u> the xy-plane).

13. (a) $x+y+z=1 \Rightarrow z=1-x-y.$ In cylindrical coordinates, let $x=r\cos\theta$ and $y=r\sin\theta$ $\Rightarrow z=1-r\cos\theta-r\sin\theta \Rightarrow \mathbf{r}(r,\theta)=(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}+(1-r\cos\theta-r\sin\theta)\mathbf{k}, 0\le\theta\le 2\pi$ and $0\le r\le 3.$

(b) In a fashion similar to cylindrical coordinates, but working in the yz-plane instead of the xy-plane, let $y = u\cos v$, $z = u\sin v$ where $u = \sqrt{y^2+z^2}$ and v is the angle formed by (x, y, z), $(x, 0, 0)$, and $(x, y, 0)$ with $(x, 0, 0)$ as vertex. Since $x + y + z = 1 \Rightarrow x = 1 - y - z \Rightarrow x = 1 - u\cos v - u\sin v$, then $\mathbf{r}$ is a function of u and $v \Rightarrow \mathbf{r}(u, v) = (1 - u\cos v - u\sin v)\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le 3$ and $0 \le v \le 2\pi$.

14. (a) In a fashion similar to cylindrical coordinates, but working in the xz-plane instead of the xy-plane, let $x = u\cos v$, $z = u\sin v$ where $u = \sqrt{x^2+z^2}$ and v is the angle formed by (x, y, z), $(y, 0, 0)$, and $(x, y, 0)$ with vertex $(y, 0, 0)$. Since $x - y + 2z = 2 \Rightarrow y = x + 2z - 2$, then $\mathbf{r}(u, v) = (u\cos v)\mathbf{i} + (u\cos v + 2u\sin v - 2)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le \sqrt{3}$ and $0 \le v \le 2\pi$.

(b) In a fashion similar to cylindrical coordinates, but working in the yz-plane instead of the xy-plane, let $y = u\cos v$, $z = u\sin v$ where $u = \sqrt{y^2+z^2}$ and v is the angle formed by (x, y, z), $(x, 0, 0)$, and $(x, y, 0)$ with vertex $(x, 0, 0)$. Since $x - y + 2z = 2 \Rightarrow x = y - 2z + 2$, then $\mathbf{r}(u, v) = (u\cos v - 2u\sin v + 2)\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le \sqrt{2}$ and $0 \le v \le 2\pi$.

15. Let $x = w\cos v$ and $z = w\sin v$. Then $(x-2)^2 + z^2 = 4 \Rightarrow x^2 - 4x + z^2 = 0$ $\Rightarrow w^2\cos^2 v - 4w\cos v + w^2\sin^2 v = 0 \Rightarrow w^2 - 4w\cos v = 0 \Rightarrow w = 0$ or $w - 4\cos v = 0 \Rightarrow w = 0$ or $w = 4\cos v$. Now $w = 0 \Rightarrow x = 0$ and $y = 0$, which is a line not a cylinder. Therefore, let $w = 4\cos v \Rightarrow x = (4\cos v)(\cos v) = 4\cos^2 v$ and $z = 4\cos v\sin v$. Finally, let $y = u$. Then $\mathbf{r}(u, v) = \left(4\cos^2 v\right)\mathbf{i} + u\mathbf{j} + (4\cos v\sin v)\mathbf{k}, -\frac{\pi}{2} \le v \le \frac{\pi}{2}$ and $0 \le u \le 3$.

16. Let $y = w\cos v$ and $z = w\sin v$. Then $y^2 + (z-5)^2 = 25 \Rightarrow y^2 + z^2 - 10z = 0$ $\Rightarrow w^2\cos^2 v + w^2\sin^2 v - 10w\sin v = 0 \Rightarrow w^2 - 10w\sin v = 0 \Rightarrow w(w - 10\sin v) = 0 \Rightarrow w = 0$ or $w = 10\sin v$. Now $w = 0 \Rightarrow y = 0$ and $z = 0$, which is a line not a cylinder. Therefore, let $w = 10\sin v \Rightarrow y = 10\sin v\cos v$ and $z = 10\sin^2 v$. Finally, let $x = u$. Then $\mathbf{r}(u, v) = u\mathbf{i} + (10\sin v\cos v)\mathbf{j} + \left(10\sin^2 v\right)\mathbf{k}$, $0 \le u \le 10$ and $0 \le v \le \pi$.

17. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{2 - r\sin\theta}{2}\right)\mathbf{k}$, $0 \le r \le 1$ and $0 \le \theta \le 2\pi$ $\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - \left(\frac{\sin\theta}{2}\right)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} - \left(\frac{r\cos\theta}{2}\right)\mathbf{k}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -\frac{\sin\theta}{2} \\ -r\sin\theta & r\cos\theta & -\frac{r\cos\theta}{2} \end{vmatrix}$$

$$= \left(\frac{-r\sin\theta\cos\theta}{2} + \frac{(\sin\theta)(r\cos\theta)}{2}\right)\mathbf{i} + \left(\frac{r\sin^2\theta}{2} + \frac{r\cos^2\theta}{2}\right)\mathbf{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\mathbf{k} = \frac{r}{2}\mathbf{j} + r\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\frac{r^2}{4} + r^2} = \frac{\sqrt{5}r}{2} \Rightarrow A = \int_0^{2\pi}\int_0^1 \frac{\sqrt{5}r}{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{\sqrt{5}r^2}{4}\right]_0^1 d\theta = \int_0^{2\pi} d\theta = \frac{\pi\sqrt{5}}{2}$$

18. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = -x = -r\cos\theta$, $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - (r\cos\theta)\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - (\cos\theta)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} + (r\sin\theta)\mathbf{k}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -\cos\theta \\ -r\sin\theta & r\cos\theta & r\sin\theta \end{vmatrix}$$

$$= \left(r\sin^2\theta + r\cos^2\theta\right)\mathbf{i} + (r\sin\theta\cos\theta - r\sin\theta\cos\theta)\mathbf{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\mathbf{k} = r\mathbf{i} + r\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2 + r^2} = r\sqrt{2} \Rightarrow A = \int_0^{2\pi}\int_0^2 r\sqrt{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{2}}{2}\right]_0^2 d\theta = \int_0^{2\pi} 2\sqrt{2}\,d\theta = 4\pi\sqrt{2}$$

19. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = 2\sqrt{x^2 + y^2} = 2r, 1 \le r \le 3$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r\cos\theta)\mathbf{i} - (2r\sin\theta)\mathbf{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\mathbf{k}$$

$$\Rightarrow (-2r\cos\theta)\mathbf{i} - (2r\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2} = \sqrt{5r^2} = r\sqrt{5}$$

$$\Rightarrow A = \int_0^{2\pi}\int_1^3 r\sqrt{5}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{5}}{2}\right]_1^3 d\theta = \int_0^{2\pi} 4\sqrt{5}\,d\theta = 8\pi\sqrt{5}$$

20. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = \frac{\sqrt{x^2+y^2}}{3} = \frac{r}{3}, 3 \le r \le 4$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{r}{3}\right)\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \left(\frac{1}{3}\right)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & \frac{1}{3} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left(-\tfrac{1}{3}r\cos\theta\right)\mathbf{i} - \left(\tfrac{1}{3}r\sin\theta\right)\mathbf{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\mathbf{k}$$

$$= \left(-\tfrac{1}{3}r\cos\theta\right)\mathbf{i} - \left(\tfrac{1}{3}r\sin\theta\right)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\tfrac{1}{9}r^2\cos^2\theta + \tfrac{1}{9}r^2\sin^2\theta + r^2} = \sqrt{\frac{10r^2}{9}} = \frac{r\sqrt{10}}{3}$$

$$\Rightarrow A = \int_0^{2\pi}\int_3^4 \frac{r\sqrt{10}}{3}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{10}}{6}\right]_3^4 d\theta = \int_0^{2\pi}\frac{7\sqrt{10}}{6}\,d\theta = \frac{7\pi\sqrt{10}}{3}$$

21. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow r^2 = x^2 + y^2 = 1, 1 \le z \le 4$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(z,\theta) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_z = \mathbf{k}$ and $\mathbf{r}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} \Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\Rightarrow A = \int_0^{2\pi}\int_1^4 1\,dr\,d\theta = \int_0^{2\pi} 3\,d\theta = 6\pi$$

22. Let $x = u\cos v$ and $z = u\sin v \Rightarrow u^2 = x^2 + z^2 = 10,\ -1 \le y \le 1, 0 \le v \le 2\pi$. Then $\mathbf{r}(y,v) = (u\cos v)\mathbf{i} + y\mathbf{j} + (u\sin v)\mathbf{k} = \left(\sqrt{10}\cos v\right)\mathbf{i} + y\mathbf{j} + \left(\sqrt{10}\sin v\right)\mathbf{k} \Rightarrow \mathbf{r}_v = \left(-\sqrt{10}\sin v\right)\mathbf{i} + \left(\sqrt{10}\cos v\right)\mathbf{k}$

and $\mathbf{r}_y = \mathbf{j} \Rightarrow \mathbf{r}_v \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{10}\sin v & 0 & \sqrt{10}\cos v \\ 0 & 1 & 0 \end{vmatrix} = \left(-\sqrt{10}\cos v\right)\mathbf{i} - \left(\sqrt{10}\sin v\right)\mathbf{k} \Rightarrow |\mathbf{r}_v \times \mathbf{r}_y| = \sqrt{10}$

$\Rightarrow A = \int_0^{2\pi}\int_{-1}^{1} \sqrt{10}\, du\, dv = \int_0^{2\pi}\left[\sqrt{10}u\right]_{-1}^{1} dv = \int_0^{2\pi} 2\sqrt{10}\, dv = 4\pi\sqrt{10}$

23. $z = 2 - x^2 - y^2$ and $z = \sqrt{x^2+y^2} \Rightarrow z = 2 - z^2 \Rightarrow z^2 + z - 2 = 0 \Rightarrow z = -2$ or $z = 1$. Since $z = \sqrt{x^2+y^2} \ge 0$, we get $z = 1$ where the cone intersects the paraboloid. When $x = 0$ and $y = 0$, $z = 2 \Rightarrow$ the vertex of the paraboloid is $(0, 0, 2)$. Therefore, z ranges from 1 to 2 on the "cap" $\Rightarrow$ r ranges from 1 (when $x^2 + y^2 = 1$) to 0 (when $x = 0$ and $y = 0$ at the vertex). Let $x = r\cos\theta$, $y = r\sin\theta$, and $z = 2 - r^2$. Then $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(2 - r^2\right)\mathbf{k}$, $0 \le r \le 1, 0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - 2r\mathbf{k}$ and

$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= \left(2r^2\cos\theta\right)\mathbf{i} + \left(2r^2\sin\theta\right)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = r\sqrt{4r^2+1}$

$\Rightarrow A = \int_0^{2\pi}\int_0^1 r\sqrt{4r^2+1}\, dr\, d\theta = \int_0^{2\pi}\left[\tfrac{1}{12}\left(4r^2+1\right)^{3/2}\right]_0^1 d\theta = \int_0^{2\pi}\left(\tfrac{5\sqrt{5}-1}{12}\right) d\theta = \tfrac{\pi}{6}\left(5\sqrt{5}-1\right)$

24. Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = x^2 + y^2 = r^2$. Then $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}$, $1 \le r \le 2$, $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left(-2r^2\cos\theta\right)\mathbf{i} - \left(2r^2\sin\theta\right)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta|$

$= \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = r\sqrt{4r^2+1} \Rightarrow A = \int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\, dr\, d\theta = \int_0^{2\pi}\left[\tfrac{1}{12}\left(4r^2+1\right)^{3/2}\right]_1^2 d\theta$

$= \int_0^{2\pi}\left(\tfrac{17\sqrt{17}-5\sqrt{5}}{12}\right) d\theta = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$

25. Let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi \Rightarrow \rho = \sqrt{x^2+y^2+z^2} = \sqrt{2}$ on the sphere. Next, $x^2 + y^2 + z^2 = 2$ and $z = \sqrt{x^2+y^2} \Rightarrow z^2 + z^2 = 2 \Rightarrow z^2 = 1 \Rightarrow z = 1$ since $z \ge 0 \Rightarrow \phi = \tfrac{\pi}{4}$. For the lower portion of the sphere cut by the cone, we get $\phi = \pi$. Then $\mathbf{r}(\phi, \theta) = \left(\sqrt{2}\sin\phi\cos\theta\right)\mathbf{i} + \left(\sqrt{2}\sin\phi\sin\theta\right)\mathbf{j} + \left(\sqrt{2}\cos\phi\right)\mathbf{k}$, $\tfrac{\pi}{4} \le \phi \le \pi, 0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_\phi = \left(\sqrt{2}\sin\phi\cos\theta\right)\mathbf{i} + \left(\sqrt{2}\cos\phi\sin\theta\right)\mathbf{j} - \left(\sqrt{2}\sin\phi\right)\mathbf{k}$ and $\mathbf{r}_\theta = \left(-\sqrt{2}\sin\phi\sin\theta\right)\mathbf{i} + \left(\sqrt{2}\sin\phi\cos\theta\right)\mathbf{j}$

$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2}\cos\phi\cos\theta & \sqrt{2}\cos\phi\sin\theta & -\sqrt{2}\sin\phi \\ -\sqrt{2}\sin\phi\sin\theta & \sqrt{2}\sin\phi\cos\theta & 0 \end{vmatrix} = \left(2\sin^2\phi\cos\theta\right)\mathbf{i} + \left(2\sin^2\phi\sin\theta\right)\mathbf{j} + (2\sin\phi\cos\phi)\mathbf{k}$

$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{4\sin^4\phi\cos^2\theta + 4\sin^4\phi\sin^2\theta + 4\sin^2\phi\cos^2\phi} = \sqrt{4\sin^2\phi} = 2|\sin\phi| = 2\sin\phi$

$\Rightarrow A = \int_0^{2\pi}\int_{\pi/4}^{\pi} 2\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\left(2+\sqrt{2}\right)d\theta = \left(4+2\sqrt{2}\right)\pi$

26. Let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi \Rightarrow \rho = \sqrt{x^2+y^2+z^2} = 2$ on the sphere. Next, $z = -1 \Rightarrow -1 = 2\cos\phi \Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$; $z = \sqrt{3} \Rightarrow \sqrt{3} = 2\cos\phi \Rightarrow \cos\phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}$. Then $\mathbf{r}(\phi,\theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}$, $\frac{\pi}{6} \le \phi \le \frac{2\pi}{3}, 0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_\phi = (2\cos\phi\cos\theta)\mathbf{i} + (2\cos\phi\sin\theta)\mathbf{j} - (2\sin\phi)\mathbf{k}$ and $\mathbf{r}_\theta = (-2\sin\phi\sin\theta)\mathbf{i} + (2\sin\phi\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix} = \left(4\sin^2\phi\cos\theta\right)\mathbf{i} + \left(4\sin^2\phi\sin\theta\right)\mathbf{j} + \left(4\sin\phi\cos\phi\right)\mathbf{k}$$

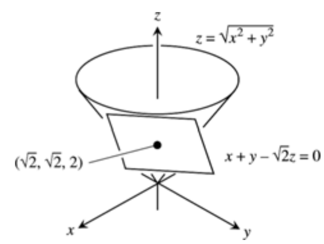
$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi}$

$\Rightarrow A = \int_0^{2\pi}\int_{\pi/6}^{2\pi/3} 4\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\left(2+2\sqrt{3}\right)d\theta = \left(4+4\sqrt{3}\right)\pi$

27. The parametrization $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$ at $P_0 = \left(\sqrt{2},\sqrt{2},2\right) \Rightarrow \theta = \frac{\pi}{4}, r = 2$,

$\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \mathbf{k}$ and

$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

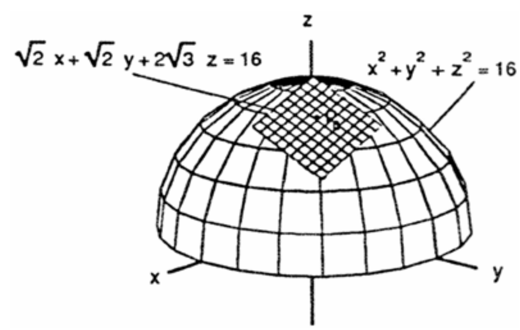


$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2}/2 & \sqrt{2}/2 & 1 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix} = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k}$$

$\Rightarrow$ the tangent plane is $0 = \left(-\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k}\right)\cdot\left[\left(x-\sqrt{2}\right)\mathbf{i} + \left(y-\sqrt{2}\right)\mathbf{j} + (z-2)\mathbf{k}\right] \Rightarrow \sqrt{2}x + \sqrt{2}y - 2z = 0$, or $x + y - \sqrt{2}z = 0$. The parametrization $\mathbf{r}(r,\theta) \Rightarrow x = r\cos\theta$, $y = r\sin\theta$ and $z = r \Rightarrow x^2 + y^2 = r^2 = z^2$ $\Rightarrow$ the surface is $z = \sqrt{x^2+y^2}$.

28. The parametrization $\mathbf{r}(\phi,\theta) = (4\sin\phi\cos\theta)\mathbf{i} + (4\sin\phi\sin\theta)\mathbf{j} + (4\cos\phi)\mathbf{k}$ at $P_0 = \left(\sqrt{2},\sqrt{2},2\sqrt{3}\right) \Rightarrow \rho = 4$ and $z = 2\sqrt{3} = 4\cos\phi \Rightarrow \phi = \frac{\pi}{6}$; also $x = \sqrt{2}$ and $y = \sqrt{2}$ $\Rightarrow \theta = \frac{\pi}{4}$. Then

$\mathbf{r}_\phi = (4\cos\phi\cos\theta)\mathbf{i} + (4\cos\phi\sin\theta)\mathbf{j} - (4\sin\phi)\mathbf{k}$ $= \sqrt{6}\mathbf{i} + \sqrt{6}\mathbf{j} - 2\mathbf{k}$ and



$$\mathbf{r}_\theta = (-4\sin\phi\sin\theta)\mathbf{i} + (4\sin\phi\cos\theta)\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} \text{ at } P_0 \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{6} & \sqrt{6} & -2 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix} = 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k}$$

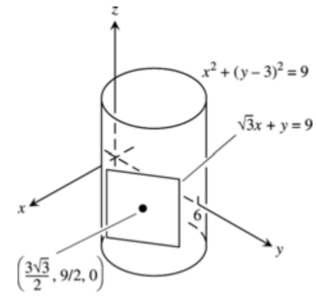
$\Rightarrow$ the tangent plane is $\left(2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k}\right)\cdot\left[\left(x-\sqrt{2}\right)\mathbf{i} + \left(y-\sqrt{2}\right)\mathbf{j} + \left(z-2\sqrt{3}\right)\mathbf{k}\right] = 0$

$\Rightarrow \sqrt{2}x+\sqrt{2}y+2\sqrt{3}z=16$, or $x+y+\sqrt{6}z=8\sqrt{2}$. The parametrization

$\Rightarrow x=4\sin\phi\cos\theta,\ y=4\sin\phi\sin\theta,\ z=4\cos\phi \ \Rightarrow$ the surface is $x^2+y^2+z^2=16,\ z\ge 0$.

29. The parametrization $\mathbf{r}(\theta, z)=(3\sin 2\theta)\mathbf{i}+\left(6\sin^2\theta\right)\mathbf{j}+z\mathbf{k}$ at $P_0=\left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right) \Rightarrow \theta=\frac{\pi}{3}$ and $z=0$. Then $\mathbf{r}_\theta=(6\cos 2\theta)\mathbf{i}+(12\sin\theta\cos\theta)\mathbf{j}=-3\mathbf{i}+3\sqrt{3}\mathbf{j}$ and

$\mathbf{r}_z=\mathbf{k}$ at $P_0 \Rightarrow \mathbf{r}_\theta\times\mathbf{r}_z=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -3 & 3\sqrt{3} & 0\\ 0 & 0 & 1\end{vmatrix}=3\sqrt{3}\mathbf{i}+3\mathbf{j}$

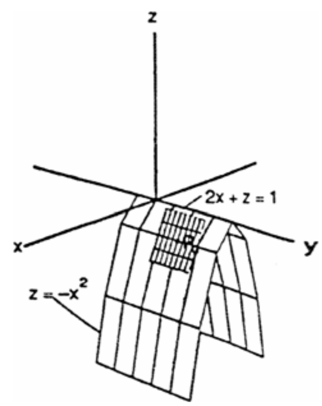


$\Rightarrow$ the tangent plane is $\left(3\sqrt{3}\mathbf{i}+3\mathbf{j}\right)\cdot\left[\left(x-\frac{3\sqrt{3}}{2}\right)\mathbf{i}+\left(y-\frac{9}{2}\right)\mathbf{j}+(z-0)\mathbf{k}\right]=0 \Rightarrow \sqrt{3}x+y=9$. The parametrization

$\Rightarrow x=3\sin 2\theta$ and $y=6\sin^2\theta \Rightarrow x^2+y^2=9\sin^2 2\theta+\left(6\sin^2\theta\right)^2$

$=9\left(4\sin^2\theta\cos^2\theta\right)+36\sin^4\theta=6\left(6\sin^2\theta\right)=6y \Rightarrow x^2+y^2-6y+9=9 \Rightarrow x^2+(y-3)^2=9$

30. The parametrization $\mathbf{r}(x, y)=x\mathbf{i}+y\mathbf{j}-x^2\mathbf{k}$ at $P_0=(1, 2, -1) \Rightarrow \mathbf{r}_x=\mathbf{i}-2x\mathbf{k}=\mathbf{i}-2\mathbf{k}$ and $\mathbf{r}_y=\mathbf{j}$ at $P_0 \Rightarrow \mathbf{r}_x\times\mathbf{r}_y=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -2\\ 0 & 1 & 0\end{vmatrix}=2\mathbf{i}+\mathbf{k} \Rightarrow$ the tangent plane is $\left(2\mathbf{i}+\mathbf{k}\right)\cdot\left[(x-1)\mathbf{i}+(y-2)\mathbf{j}+(z+1)\mathbf{k}\right]=0$ $\Rightarrow 2x+z=1$. The parametrization $\Rightarrow x=x,\ y=y$ and $z=-x^2 \Rightarrow$ the surface is $z=-x^2$



31. (a) An arbitrary point on the circle C is $(x, z)=(R+r\cos u, r\sin u) \Rightarrow (x, y, z)$ is on the torus with $x=(R+r\cos u)\cos v,\ y=(R+r\cos u)\sin v$, and $z=r\sin u,\ 0\le u\le 2\pi,\ 0\le v\le 2\pi$

(b) $\mathbf{r}_u=(-r\sin u\cos v)\mathbf{i}-(r\sin u\sin v)\mathbf{j}+(r\cos u)\mathbf{k}$ and $\mathbf{r}_v=\left(-(R+r\cos u)\sin v\right)\mathbf{i}+\left((R+r\cos u)\cos v\right)\mathbf{j}$

$$\Rightarrow \mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -r\sin u\cos v & -r\sin u\sin v & r\cos u\\ -(R+r\cos u)\sin v & (R+r\cos u)\cos v & 0\end{vmatrix}$$

$$=-(R+r\cos u)(r\cos v\cos u)\mathbf{i}-(R+r\cos u)(r\sin v\cos u)\mathbf{j}+(-r\sin u)(R+r\cos u)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_u\times\mathbf{r}_v|^2=(R+r\cos u)^2\left(r^2\cos^2 v\cos^2 u+r^2\sin^2 v\cos^2 u+r^2\sin^2 u\right) \Rightarrow |\mathbf{r}_u\times\mathbf{r}_v|=r(R+r\cos u)$$

$$\Rightarrow A=\int_0^{2\pi}\int_0^{2\pi}\left(rR+r^2\cos u\right)du\,dv=\int_0^{2\pi}2\pi rR\,dv=4\pi^2 rR$$

32. (a) The point (x, y, z) is on the surface for fixed $x = f(u)$ when $y = g(u)\sin\left(\frac{\pi}{2} - v\right)$ and
$z = g(u)\cos\left(\frac{\pi}{2} - v\right) \Rightarrow x = f(u),\ y = g(u)\cos v,$ and $z = g(u)\sin v \Rightarrow \mathbf{r}(u, v)$
$= f(u)\mathbf{i} + \left(g(u)\cos v\right)\mathbf{j} + \left(g(u)\sin v\right)\mathbf{k},\ \ 0 \le v \le 2\pi,\ a \le u \le b$

(b) Let $u = y$ and $x = u^2 \Rightarrow f(u) = u^2$ and $g(u) = u \Rightarrow \mathbf{r}(u, v) = u^2\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}$,
$0 \le v \le 2\pi,\ 0 \le u$

33. (a) Let $w^2 + \frac{z^2}{c^2} = 1$ where $w = \cos\phi$ and $\frac{z}{c} = \sin\phi \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\phi \Rightarrow \frac{x}{a} = \cos\phi\cos\theta$ and $\frac{y}{b} = \cos\phi\sin\theta$
$\Rightarrow x = a\cos\theta\cos\phi,\ y = b\sin\theta\cos\phi,$ and $z = c\sin\phi$
$\Rightarrow \mathbf{r}(\theta, \phi) = (a\cos\theta\,\cos\phi)\mathbf{i} + (b\sin\theta\,\cos\phi)\mathbf{j} + (c\sin\phi)\mathbf{k}$

(b) $\mathbf{r}_\theta = (-a\sin\theta\cos\phi)\mathbf{i} + (b\cos\theta\cos\phi)\mathbf{j}$ and $\mathbf{r}_\phi = (-a\cos\theta\,\sin\phi)\mathbf{i} - (b\sin\theta\,\sin\phi)\mathbf{j} + (c\cos\phi)\mathbf{k}$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\theta\cos\phi & b\cos\theta\cos\phi & 0 \\ -a\cos\theta\sin\phi & -b\sin\theta\sin\phi & c\cos\phi \end{vmatrix}$$

$= \left(bc\cos\theta\cos^2\phi\right)\mathbf{i} + \left(ac\sin\theta\cos^2\phi\right)\mathbf{j} + \left(ab\sin\phi\cos\phi\right)\mathbf{k}$

$\Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_\phi|^2 = b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi + a^2b^2\sin^2\phi\cos^2\phi$, and the result follows.

$$A = \int_0^{2\pi}\int_0^{2\pi} |\mathbf{r}_\theta \times \mathbf{r}_\phi|\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\left[a^2b^2\sin^2\phi\cos^2\phi + b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi\right]^{1/2} d\phi\,d\theta$$

34. (a) $\mathbf{r}(\theta, u) = (\cosh u\cos\theta)\mathbf{i} + (\cosh u\sin\theta)\mathbf{j} + (\sinh u)\mathbf{k}$

(b) $\mathbf{r}(\theta, u) = (a\cosh u\cos\theta)\mathbf{i} + (b\cosh u\sin\theta)\mathbf{j} + (c\sinh u)\mathbf{k}$

35. $\mathbf{r}(\theta, u) = (5\cosh u\cos\theta)\mathbf{i} + (5\cosh u\sin\theta)\mathbf{j} + (5\sinh u)\mathbf{k} \Rightarrow \mathbf{r}_\theta = (-5\cosh u\sin\theta)\mathbf{i} + (5\cosh u\cos\theta)\mathbf{j}$ and

$$\mathbf{r}_u = (5\sinh u\cos\theta)\mathbf{i} + (5\sinh u\sin\theta)\mathbf{j} + (5\cosh u)\mathbf{k} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_u = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5\cosh u\sin\theta & 5\cosh u\cos\theta & 0 \\ 5\sinh u\cos\theta & 5\sinh u\sin\theta & 5\cosh u \end{vmatrix}$$

$= \left(25\cosh^2 u\cos\theta\right)\mathbf{i} + \left(25\cosh^2 u\sin\theta\right)\mathbf{j} - (25\cosh u\,\sinh u)\mathbf{k}$. At the point $(x_0, y_0, 0)$, where $x_0^2 + y_0^2 = 25$ we have $5\sinh u = 0 \Rightarrow u = 0$ and $x_0 = 25\cos\theta,\ y_0 = 25\sin\theta \ \Rightarrow$ the tangent plane is
$5\left(x_0\mathbf{i} + y_0\mathbf{j}\right)\cdot\left[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + z\mathbf{k}\right] = 0 \Rightarrow x_0x - x_0^2 + y_0y - y_0^2 = 0 \Rightarrow x_0x + y_0y = 25$

36. Let $\frac{z^2}{c^2} - w^2 = 1$ where $\frac{z}{c} = \cosh u$ and $w = \sinh u \Rightarrow w^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow \frac{x}{a} = w\cos\theta$ and $\frac{y}{b} = w\sin\theta$
$\Rightarrow x = a\sinh u\cos\theta,\ y = b\sinh u\sin\theta,$ and $z = c\cosh u$
$\Rightarrow \mathbf{r}(\theta, u) = (a\sinh u\cos\theta)\mathbf{i} + (b\sinh u\sin\theta)\mathbf{j} + (c\cosh u)\mathbf{k},\ \ 0 \le \theta \le 2\pi, -\infty < u < \infty$

37. $\mathbf{p} = \mathbf{k},\ \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (2y)^2 + (-1)^2} = \sqrt{4x^2 + 4y^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1$;
$z = 2 \Rightarrow x^2 + y^2 = 2$; thus $S = \iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = \iint\limits_R \sqrt{4x^2 + 4y^2 + 1}\,dx\,dy = \iint\limits_R \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1}\,r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2 + 1}\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{12}\left(4r^2 + 1\right)^{3/2}\right]_0^{\sqrt{2}} d\theta = \int_0^{2\pi}\frac{13}{6}\,d\theta = \frac{13}{3}\pi$$

38. $\mathbf{p}=\mathbf{k}$, $\nabla f=2x\mathbf{i}+2y\mathbf{j}-\mathbf{k} \Rightarrow |\nabla f|=\sqrt{4x^2+4y^2+1}$ and $|\nabla f\cdot\mathbf{p}|=1$; $2\le x^2+y^2\le 6 \Rightarrow S=\iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA$

$=\iint\limits_R \sqrt{4x^2+4y^2+1}\,dx\,dy=\iint\limits_R \sqrt{4r^2+1}\,r\,dr\,d\theta=\int_0^{2\pi}\int_{\sqrt{2}}^{\sqrt{6}}\sqrt{4r^2+1}\,r\,dr\,d\theta=\int_0^{2\pi}\left[\frac{1}{12}\left(4r^2+1\right)^{3/2}\right]_{\sqrt{2}}^{\sqrt{6}}d\theta$

$=\int_0^{2\pi}\frac{49}{6}\,d\theta=\frac{49}{3}\pi$

39. $\mathbf{p}=\mathbf{k}$, $\nabla f=\mathbf{i}+2\mathbf{j}+2\mathbf{k} \Rightarrow |\nabla f|=3$ and $|\nabla f\cdot\mathbf{p}|=2$; $x=y^2$ and $x=2-y^2$ intersect at $(1,1)$ and $(1,-1)$

$\Rightarrow S=\iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA=\iint\limits_R \frac{3}{2}\,dx\,dy=\int_{-1}^{1}\int_{y^2}^{2-y^2}\frac{3}{2}\,dx\,dy=\int_{-1}^{1}\left(3-3y^2\right)dy=4$

40. $\mathbf{p}=\mathbf{k}$, $\nabla f=2x\mathbf{i}-2\mathbf{k} \Rightarrow |\nabla f|=\sqrt{4x^2+4}=2\sqrt{x^2+1}$ and $|\nabla f\cdot\mathbf{p}|=2 \Rightarrow S=\iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA$

$=\iint\limits_R \frac{2\sqrt{x^2+1}}{2}\,dx\,dy=\int_0^{\sqrt{3}}\int_0^{x}\sqrt{x^2+1}\,dy\,dx=\int_0^{\sqrt{3}}x\sqrt{x^2+1}\,dx=\left[\frac{1}{3}\left(x^2+1\right)^{3/2}\right]_0^{\sqrt{3}}=\frac{1}{3}(4)^{3/2}-\frac{1}{3}=\frac{7}{3}$

41. $\mathbf{p}=\mathbf{k}$, $\nabla f=2x\mathbf{i}-2\mathbf{j}-2\mathbf{k} \Rightarrow |\nabla f|=\sqrt{(2x)^2+(-2)^2+(-2)^2}=\sqrt{4x^2+8}=2\sqrt{x^2+2}$ and $|\nabla f\cdot\mathbf{p}|=2$

$\Rightarrow S=\iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA=\iint\limits_R \frac{2\sqrt{x^2+2}}{2}\,dx\,dy=\int_0^{2}\int_0^{3x}\sqrt{x^2+2}\,dy\,dx=\int_0^{2}3x\sqrt{x^2+2}\,dx=\left[\left(x^2+2\right)^{3/2}\right]_0^{2}=6\sqrt{6}-2\sqrt{2}$

42. $\mathbf{p}=\mathbf{k}$, $\nabla f=2x\mathbf{i}+2y+2z\mathbf{k} \Rightarrow |\nabla f|=\sqrt{4x^2+4y^2+4z^2}=\sqrt{8}=2\sqrt{2}$ and $|\nabla f\cdot\mathbf{p}|=2z$; $x^2+y^2+z^2=2$ and $z=\sqrt{x^2+y^2} \Rightarrow x^2+y^2=1$; thus, $S=\iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA=\iint\limits_R \frac{2\sqrt{2}}{2z}\,dA=\sqrt{2}\iint\limits_R \frac{1}{z}\,dA$

$=\sqrt{2}\iint\limits_R \frac{1}{\sqrt{2-\left(x^2+y^2\right)}}\,dA=\sqrt{2}\int_0^{2\pi}\int_0^{1}\frac{r\,dr\,d\theta}{\sqrt{2-r^2}}=\sqrt{2}\int_0^{2\pi}\left(-1+\sqrt{2}\right)d\theta=2\pi\left(2-\sqrt{2}\right)$

43. $\mathbf{p}=\mathbf{k}$, $\nabla f=c\mathbf{i}-\mathbf{k} \Rightarrow |\nabla f|=\sqrt{c^2+1^2}$ and $|\nabla f\cdot\mathbf{p}|=1 \Rightarrow S=\iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA=\iint\limits_R \sqrt{c^2+1}\,dx\,dy$

$=\int_0^{2\pi}\int_0^{1}\sqrt{c^2+1}\,r\,dr\,d\theta=\int_0^{2\pi}\frac{\sqrt{c^2+1}}{2}\,d\theta=\pi\sqrt{c^2+1}$

44. $\mathbf{p}=\mathbf{k}$, $\nabla f=2x\mathbf{i}+2z\mathbf{j} \Rightarrow |\nabla f|=\sqrt{(2x)^2+(2z)^2}=2$ and $|\nabla f\cdot\mathbf{p}|=2z$ for the upper surface, $z\ge 0$

$\Rightarrow S=\iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA=\iint\limits_R \frac{2}{2z}\,dA=\iint\limits_R \frac{1}{\sqrt{1-x^2}}\,dx\,dy=2\int_{-1/2}^{1/2}\int_0^{1/2}\frac{1}{\sqrt{1-x^2}}\,dy\,dx=\int_{-1/2}^{1/2}\frac{1}{\sqrt{1-x^2}}\,dx$

$=\left[\sin^{-1}x\right]_{-1/2}^{1/2}=\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)=\frac{\pi}{3}$

45. $\mathbf{p}=\mathbf{i}$, $\nabla f=\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \Rightarrow |\nabla f|=\sqrt{1^2+(2y)^2+(2z)^2}=\sqrt{1+4y^2+4z^2}$ and $|\nabla f\cdot\mathbf{p}|=1$; $1\le y^2+z^2\le 4$

$\Rightarrow S=\iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA=\iint\limits_R\sqrt{1+4y^2+4z^2}\,dy\,dz=\int_0^{2\pi}\int_1^2\sqrt{1+4r^2\cos^2\theta+4r^2\sin^2\theta}\,r\,dr\,d\theta$

$=\int_0^{2\pi}\int_1^2\sqrt{1+4r^2}\,r\,dr\,d\theta=\int_0^{2\pi}\left[\frac{1}{12}\left(1+4r^2\right)^{3/2}\right]_1^2 d\theta=\int_0^{2\pi}\frac{1}{12}\left(17\sqrt{17}-5\sqrt{5}\right)d\theta=\frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$

46. $\mathbf{p}=\mathbf{j}$, $\nabla f=2x\mathbf{i}+\mathbf{j}+2z\mathbf{k}\Rightarrow|\nabla f|=\sqrt{4x^2+4z^2+1}$ and $|\nabla f\cdot\mathbf{p}|=1$; $y=0$ and $x^2+y+z^2=2\Rightarrow x^2+z^2=2$;

thus $S=\iint\limits_R\frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA=\iint\limits_R\sqrt{4x^2+4z^2+1}\,dx\,dz=\int_0^{2\pi}\int_0^{\sqrt{2}}\sqrt{4r^2+1}\,r\,dr\,d\theta=\int_0^{2\pi}\frac{13}{6}\,d\theta=\frac{13}{3}\pi$

47. $\mathbf{p}=\mathbf{k}$, $\nabla f=\left(2x-\frac{2}{x}\right)\mathbf{i}+\sqrt{15}\mathbf{j}-\mathbf{k}\Rightarrow|\nabla f|=\sqrt{\left(2x-\frac{2}{x}\right)^2+\left(\sqrt{15}\right)^2+(-1)^2}=\sqrt{4x^2+8+\frac{4}{x^2}}=\sqrt{\left(2x+\frac{2}{x}\right)^2}$

$=2x+\frac{2}{x}$, on $1\le x\le 2$ and $|\nabla f\cdot\mathbf{p}|=1\Rightarrow S=\iint\limits_R\frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA=\iint\limits_R\left(2x+2x^{-1}\right)dx\,dy=\int_0^1\int_1^2\left(2x+2x^{-1}\right)dx\,dy$

$=\int_0^1\left[x^2+2\ln x\right]_1^2 dy=\int_0^1(3+2\ln 2)\,dy=3+2\ln 2$

48. $\mathbf{p}=\mathbf{k}$, $\nabla f=3\sqrt{x}\mathbf{i}+3\sqrt{y}\mathbf{j}-3\mathbf{k}\Rightarrow|\nabla f|=\sqrt{9x+9y+9}=3\sqrt{x+y+1}$ and $|\nabla f\cdot\mathbf{p}|=3$

$\Rightarrow S=\iint\limits_R\frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA=\iint\limits_R\sqrt{x+y+1}\,dx\,dy=\int_0^1\int_0^1\sqrt{x+y+1}\,dx\,dy=\int_0^1\left[\frac{2}{3}(x+y+1)^{3/2}\right]_0^1 dy$

$=\int_0^1\left[\frac{2}{3}(y+2)^{3/2}-\frac{2}{3}(y+1)^{3/2}\right]_0^1 dy=\left[\frac{4}{15}(y+2)^{5/2}-\frac{4}{15}(y+1)^{5/2}\right]_0^1=\frac{4}{15}\left[(3)^{5/2}-(2)^{5/2}-(2)^{5/2}+1\right]$

$=\frac{4}{15}\left(9\sqrt{3}-8\sqrt{2}+1\right)$

49. $f_x(x,y)=2x$, $f_y(x,y)=2y\Rightarrow\sqrt{f_x^2+f_y^2+1}=\sqrt{4x^2+4y^2+1}\Rightarrow$ Area $=\iint\limits_R\sqrt{4x^2+4y^2+1}\,dx\,dy$

$=\int_0^{2\pi}\int_0^{\sqrt{3}}\sqrt{4r^2+1}\,r\,dr\,d\theta=\frac{\pi}{6}\left(13\sqrt{13}-1\right)$

50. $f_x(y,z)=-2y$, $f_z(y,z)=-2z\Rightarrow\sqrt{f_y^2+f_z^2+1}=\sqrt{4y^2+4z^2+1}\Rightarrow$ Area $=\iint\limits_R\sqrt{4y^2+4z^2+1}\,dy\,dz$

$=\int_0^{2\pi}\int_0^1\sqrt{4r^2+1}\,r\,dr\,d\theta=\frac{\pi}{6}\left(5\sqrt{5}-1\right)$

51. $f_x(x,y)=\frac{x}{\sqrt{x^2+y^2}}$, $f_y(x,y)=\frac{x}{\sqrt{x^2+y^2}}\Rightarrow\sqrt{f_x^2+f_y^2+1}=\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}=\sqrt{2}\Rightarrow$ Area $=\iint\limits_{R_{xy}}\sqrt{2}\,dx\,dy$

$=\sqrt{2}$ (Area between the ellipse and the circle) $=\sqrt{2}(6\pi-\pi)=5\pi\sqrt{2}$

52. Over R_{xy}: $z=2-\frac{2}{3}x-2y\Rightarrow f_x(x,y)=-\frac{2}{3}$, $f_y(x,y)=-2\Rightarrow\sqrt{f_x^2+f_y^2+1}=\sqrt{\frac{4}{9}+4+1}=\frac{7}{3}$

$\Rightarrow$ Area $=\iint\limits_{R_{xy}}\frac{7}{3}\,dA=\frac{7}{3}$ (Area of the shadow triangle in the xy-plane) $=\left(\frac{7}{3}\right)\left(\frac{3}{2}\right)=\frac{7}{2}$.

Over R_{xz}: $y = 1 - \frac{1}{3}x - \frac{1}{2}z \Rightarrow f_x(x, z) = -\frac{1}{3},\ f_z(x, z) = -\frac{1}{2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{\frac{1}{9} + \frac{1}{4} + 1} = \frac{7}{6}$

$\Rightarrow$ Area $= \iint\limits_{R_{xz}} \frac{7}{6}\, dA = \frac{7}{6}$ (Area of the shadow triangle in the xz-plane) $= \left(\frac{7}{6}\right)(3) = \frac{7}{2}.$

Over R_{yz}: $x = 3 - 3y - \frac{3}{2}z \Rightarrow f_y(y, z) = -3,\ f_z(y, z) = -\frac{3}{2} \Rightarrow \sqrt{f_y^2 + f_z^2 + 1} = \sqrt{9 + \frac{9}{4} + 1} = \frac{7}{2}$

$\Rightarrow$ Area $= \iint\limits_{R_{yz}} \frac{7}{2}\, dA = \frac{7}{2}$ (Area of the shadow triangle in the yz-plane) $= \left(\frac{7}{2}\right)(1) = \frac{7}{2}.$

53. $y = \frac{2}{3}z^{3/2} \Rightarrow f_x(x, z) = 0,\ f_z(x, z) = z^{1/2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{z+1};\ y = \frac{16}{3} \Rightarrow \frac{16}{3} = \frac{2}{3}z^{3/2} \Rightarrow z = 4$

$\Rightarrow$ Area $= \int_0^4 \int_0^1 \sqrt{z+1}\, dx\, dz = \int_0^4 \sqrt{z+1}\, dz = \frac{2}{3}\left(5\sqrt{5} - 1\right)$

54. $y = 4 - z \Rightarrow f_x(x, z) = 0,\ f_z(x, z) = -1 \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{2} \Rightarrow$ Area $= \iint\limits_{R_{xz}} \sqrt{2}\, dA = \int_0^2 \int_0^{4-z^2} \sqrt{2}\, dx\, dz$

$= \sqrt{2}\int_0^2 \left(4 - z^2\right) dz = \frac{16\sqrt{2}}{3}$

55. $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\,\mathbf{k} \Rightarrow \mathbf{r}_x(x, y) = \mathbf{i} + f_x(x, y)\,\mathbf{k},\ \mathbf{r}_y(x, y) = \mathbf{j} + f_y(x, y)\,\mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x, y) \\ 0 & 1 & f_y(x, y) \end{vmatrix} = -f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k}$$

$$\Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(-f_x(x, y)\right)^2 + \left(-f_y(x, y)\right)^2 + 1^2} = \sqrt{f_x(x, y)^2 + f_y(x, y)^2 + 1}$$

$$\Rightarrow d\sigma = \sqrt{f_x(x, y)^2 + f_y(x, y)^2 + 1}\, dA$$

56. S is obtained by rotating $y = f(x),\ a \le x \le b$ about the x-axis where $f(x) \ge 0$

(a) Let (x, y, z) be a point on S. Consider the cross section when $x = x^*$, the cross section is a circle with radius $r = f(x^*)$. The set of parametric equations for this circle are given by $y(\theta) = r\cos\theta = f(x^*)\cos\theta$ and $z(\theta) = r\sin\theta = f(x^*)\sin\theta$ where $0 \le \theta \le 2\pi$. Since x can take on any value between a and b we have $x(x, \theta) = x,\ y(x, \theta) = f(x)\cos\theta,\ z(x, \theta) = f(x)\sin\theta$ where $a \le x \le b$ and $0 \le \theta \le 2\pi$. Thus $\mathbf{r}(x, \theta) = x\mathbf{i} + f(x)\cos\theta\mathbf{j} + f(x)\sin\theta\mathbf{k}$

(b) $\mathbf{r}_x(x, \theta) = \mathbf{i} + f'(x)\cos\theta\mathbf{j} + f'(x)\sin\theta\mathbf{k}$ and $\mathbf{r}_\theta(x, \theta) = -f(x)\sin\theta\mathbf{j} + f(x)\cos\theta\mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix} = f(x)\cdot f'(x)\mathbf{i} - f(x)\cos\theta\mathbf{j} - f(x)\sin\theta\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_x \times \mathbf{r}_\theta| = \sqrt{\left(f(x)\cdot f'(x)\right)^2 + \left(-f(x)\cos\theta\right)^2 + \left(-f(x)\sin\theta\right)^2} = f(x)\sqrt{1 + \left(f'(x)\right)^2}$$

$$A = \int_a^b \int_0^{2\pi} f(x)\sqrt{1 + \left(f'(x)\right)^2}\, d\theta\, dx = \int_a^b \left[\left(f(x)\sqrt{1 + \left(f'(x)\right)^2}\right)\theta\right]_0^{2\pi} dx = \int_a^b 2\pi\, f(x)\sqrt{1 + \left(f'(x)\right)^2}\, dx$$

16.6 SURFACE INTEGRALS

1. Let the parametrization be $\mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix}$

   $= 2x\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+1} \Rightarrow \iint\limits_S G(x, y, z)\, d\sigma = \int_0^3 \int_0^2 x\sqrt{4x^2+1}\, dx\, dz = \int_0^3 \left[\frac{1}{12}\left(4x^2+1\right)^{3/2} \right]_0^2 dz$

   $= \int_0^3 \frac{1}{12}\left(17\sqrt{17} - 1\right) dz = \frac{17\sqrt{17}-1}{4}$

2. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \sqrt{4-y^2}\mathbf{k}, -2 \le y \le 2 \Rightarrow \mathbf{r}_x = \mathbf{i}$ and $\mathbf{r}_y = \mathbf{j} - \frac{y}{\sqrt{4-y^2}}\mathbf{k}$

   $\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{y}{\sqrt{4-y^2}} \end{vmatrix} = \frac{y}{\sqrt{4-y^2}}\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\frac{y^2}{4-y^2} + 1} = \frac{2}{\sqrt{4-y^2}}$

   $\Rightarrow \iint\limits_S G(x, y, z)\, d\sigma = \int_1^4 \int_{-2}^2 \sqrt{4-y^2} \left(\frac{2}{\sqrt{4-y^2}} \right) dy\, dx = 24$

3. Let the parametrization be $\mathbf{r}(\phi, \theta) = (\sin\phi\cos\theta)\mathbf{i} + (\sin\phi\sin\theta)\mathbf{j} + (\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = 1$ on the sphere), $0 \le \phi \le \pi, 0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_\phi = (\cos\phi\cos\theta)\mathbf{i} + (\cos\phi\sin\theta)\mathbf{j} - (\sin\phi)\mathbf{k}$ and

   $\mathbf{r}_\theta = (-\sin\phi\sin\theta)\mathbf{i} + (\sin\phi\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$

   $= \left(\sin^2\phi\cos\theta\right)\mathbf{i} + \left(\sin^2\phi\sin\theta\right)\mathbf{j} + \left(\sin\phi\cos\theta\right)\mathbf{k} \Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}$

   $= \sin\phi;\ x = \sin\phi\cos\theta \Rightarrow G(x, y, z) = \cos^2\theta\sin^2\phi \Rightarrow \iint\limits_S G(x, y, z)\, d\sigma = \int_0^{2\pi} \int_0^\pi \left(\cos^2\theta\sin^2\phi\right)(\sin\phi)\, d\phi\, d\theta$

   $= \int_0^{2\pi} \int_0^\pi \left(\cos^2\theta\right)\left(1 - \cos^2\phi\right)(\sin\phi)\, d\phi\, d\theta; \begin{bmatrix} u = \cos\phi \\ du = -\sin\phi\, d\phi \end{bmatrix} \to \int_0^{2\pi} \int_1^{-1} \left(\cos^2\theta\right)\left(u^2 - 1\right) du\, d\theta$

   $= \int_0^{2\pi} \left(\cos^2\theta\right) \left[\frac{u^3}{3} - u \right]_1^{-1} d\theta = \frac{4}{3}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{4}{3}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi} = \frac{4\pi}{3}$

4. Let the parametrization be $\mathbf{r}(\phi, \theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = a,\ a \ge 0,$ on the sphere), $0 \le \phi \le \frac{\pi}{2}$ (since $z \ge 0$), $0 \le \theta \le 2\pi$

   $\Rightarrow \mathbf{r}_\phi = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} - (a\sin\phi)\mathbf{k}$ and $\mathbf{r}_\theta = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j}$

   $\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix} = \left(a^2\sin^2\phi\cos\theta\right)\mathbf{i} + \left(a^2\sin^2\phi\sin\theta\right)\mathbf{j} + \left(a^2\sin\phi\cos\phi\right)\mathbf{k}$

$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4 \sin^4\phi \cos^2\theta + a^4 \sin^4\phi \sin^2\theta + a^4 \sin^2\phi \cos^2\phi} = a^2 \sin\phi;\ z = a\cos\phi$

$\Rightarrow G(x, y, z) = a^2\cos^2\phi \Rightarrow \iint\limits_S G(x, y, z)\,d\sigma = \int_0^{2\pi}\int_0^{\pi/2}\left(a^2\cos^2\phi\right)\left(a^2\sin\phi\right)d\phi\, d\theta = \frac{2}{3}\pi a^4$

5. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (4 - x - y)\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - \mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$

$= \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{3} \Rightarrow \iint\limits_S F(x, y, z)\, d\sigma = \int_0^1\int_0^1 (4 - x - y)\sqrt{3}\, dy\, dx = \int_0^1 \sqrt{3}\left[4y - xy - \frac{y^2}{2}\right]_0^1 dx$

$= \int_0^1 \sqrt{3}\left(\frac{7}{2} - x\right) dx = \sqrt{3}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^1 = 3\sqrt{3}$

6. Let the parametrization be $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2} = r\sqrt{2};\ z = r$ and $x = r\cos\theta$

$\Rightarrow F(x, y\ z) = r - r\cos\theta \Rightarrow \iint\limits_S F(x, y, z)\,d\sigma = \int_0^{2\pi}\int_0^1 (r - r\cos\theta)\left(r\sqrt{2}\right)dr\, d\theta$

$= \sqrt{2}\int_0^{2\pi}\int_0^1 (1 - \cos\theta)r^2\, dr\, d\theta = \frac{2\pi\sqrt{2}}{3}$

7. Let the parametrization be $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (1 - r^2)\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left(2r^2\cos\theta\right)\mathbf{i} + \left(2r^2\sin\theta\right)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta|$

$= \sqrt{\left(2r^2\cos\theta\right)^2 + \left(2r^2\sin\theta\right)^2 + r^2} = r\sqrt{1 + 4r^2};\ z = 1 - r^2$ and $x = r\cos\theta \Rightarrow H(x, y\ z) = \left(r^2\cos^2\theta\right)\sqrt{1 + 4r^2}$

$\Rightarrow \iint\limits_S H(x, y, z)\,d\sigma = \int_0^{2\pi}\int_0^1 \left(r^2\cos^2\theta\right)\left(\sqrt{1 + 4r^2}\right)\left(r\sqrt{1 + 4r^2}\right)dr\, d\theta = \int_0^{2\pi}\int_0^1 r^3\left(1 + 4r^2\right)\cos^2\theta\, dr\, d\theta = \frac{11\pi}{12}$

8. Let the parametrization be $\mathbf{r}(\phi, \theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = 2$ on the sphere), $0 \le \phi \le \frac{\pi}{4}$; $x^2 + y^2 + z^2 = 4$ and $z = \sqrt{x^2 + y^2} = z^2 + z^2 = 4 \Rightarrow z^2 = 2 \Rightarrow z = \sqrt{2}$ (since $z \ge 0$) $\Rightarrow 2\cos\phi = \sqrt{2} \Rightarrow \cos\phi = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}, 0 \le \theta \le 2\pi$; $\mathbf{r}_\phi = (2\cos\phi\cos\theta)\mathbf{i} + (2\cos\phi\sin\theta)\mathbf{j} - (2\sin\phi)\mathbf{k}$

and $r_\theta = (-2\sin\phi\sin\theta)\mathbf{i} + (2\sin\phi\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$

$= \left(4\sin^2\phi\cos\theta\right)\mathbf{i} + \left(4\sin^2\phi\sin\theta\right)\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}$

$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi} = 4\sin\phi$; $y = 2\sin\phi\sin\theta$ and

$z = 2\cos\phi \Rightarrow H(x, y, z) = 4\cos\phi\sin\phi\sin\theta \Rightarrow \iint_S H(x, y, z)\,d\sigma = \int_0^{2\pi}\int_0^{\pi/4}(4\cos\phi\sin\phi\sin\theta)(4\sin\phi)\,d\phi\,d\theta$

$= \int_0^{2\pi}\int_0^{\pi/4} 16\sin^2\phi\cos\phi\sin\theta\,d\phi\,d\theta = 0$

9. The bottom face S of the cube is in the xy-plane $\Rightarrow z = 0 \Rightarrow G(x, y, 0) = x + y$ and $f(x, y, z) = z = 0 \Rightarrow \mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx\,dy \Rightarrow \iint_S G\,d\sigma = \iint_R (x + y)\,dx\,dy = \int_0^a\int_0^a (x + y)\,dx\,dy$ $= \int_0^a \left(\frac{a^2}{2} + ay\right) dy = a^3$. Because of symmetry, we also get a^3 over the face of the cube in the xz-plane and a^3 over the face of the cube in the yz-plane. Next, on the top of the cube, $G(x, y, z) = G(x, y, a) = x + y + a$ and $f(x, y, z) = z = a \Rightarrow \mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx\,dy$ $\Rightarrow \iint_S G\,d\sigma = \iint_R (x + y + a)\,dx\,dy = \int_0^a\int_0^a (x + y + a)\,dx\,dy \int_0^a\int_0^a (x + y)\,dx\,dy + \int_0^a\int_0^a a\,dx\,dy = 2a^3$. Because of symmetry, the integral is also $2a^3$ over each of the other two faces. Therefore, $\iint_{\text{cube}} (x + y + z)\,d\sigma = 3\left(a^3 + 2a^3\right) = 9a^3$.

10. On the face S in the xz-plane, we have $y = 0 \Rightarrow f(x, y, z) = y = 0$ and $G(x, y, z) = G(x, 0, z) = z \Rightarrow \mathbf{p} = \mathbf{j}$ and $\nabla f = \mathbf{j} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx\,dz \Rightarrow \iint_S G\,d\sigma = \iint_S (y + z)\,d\sigma = \int_0^1\int_0^2 z\,dx\,dz = \int_0^1 2z\,dz = 1.$

On the face in the xy-plane, we have $z = 0 \Rightarrow f(x, y, z) = z = 0$ and $G(x, y, z) = G(x, y, 0) = y \Rightarrow \mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1$ and $\Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx\,dy \Rightarrow \iint_S G\,d\sigma = \iint_S y\,d\sigma = \int_0^1\int_0^2 y\,dx\,dy = 1.$

On the triangular face in the plane $x = 2$ we have $f(x, y, z) = x = 2$ and $G(x, y, z) = G(2, y, z) = y + z$ $\Rightarrow \mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz\,dy \Rightarrow \iint_S G\,d\sigma = \iint_S (y + z)\,d\sigma$ $= \int_0^1\int_0^{1-y} (y + z)\,dz\,dy = \int_0^1 \frac{1}{2}\left(1 - y^2\right) dy = \frac{1}{3}.$

On the triangular face in the yz-plane we have $x = 0 \Rightarrow f(x, y, z) = x = 0$ and $G(x, y, z) = G(0, y, z) = y + z$ $\Rightarrow \mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz\,dy \Rightarrow \iint_S G\,d\sigma = \iint_S (y + z)\,d\sigma$ $= \int_0^1\int_0^{1-y} (y + z)\,dz\,dy = \frac{1}{3}.$

Finally, on the sloped face, we have $y + z = 1 \Rightarrow f(x, y, z) = y + z = 1$ and $G(x, y, z) = y + z = 1 \Rightarrow \mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{j} + \mathbf{k} \Rightarrow |\nabla f| = \sqrt{2}$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{2}\,dx\,dy \Rightarrow \iint_S G\,d\sigma = \iint_S (y + z)\,d\sigma = \int_0^1\int_0^2 \sqrt{2}\,dx\,dy = 2\sqrt{2}.$

Therefore, $\iint_{\text{wedge}} G(x, y, z)\,d\sigma = 1 + 1 + \frac{1}{3} + \frac{1}{3} + 2\sqrt{2} = \frac{8}{3} + 2\sqrt{2}$

11. On the faces in the coordinate planes, $G(x, y, z) = 0 \Rightarrow$ the integral over these faces is 0.
On the face $x = a$, we have $f(x, y, z) = x = a$ and $G(x, y, z) = G(a, y, z) = ayz \Rightarrow \mathbf{p} = \mathbf{i}$ and
$\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dy\, dz \Rightarrow \iint\limits_S G\, d\sigma = \iint\limits_S ayz\, d\sigma = \int_0^c \int_0^b ayz\, dy\, dz = \frac{ab^2c^2}{4}.$
On the face $y = b$, we have $f(x, y, z) = y = b$ and $G(x, y, z) = G(x, b, z) = bxz \Rightarrow \mathbf{p} = \mathbf{j}$ and
$\nabla f = \mathbf{j} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx\, dz \Rightarrow \iint\limits_S G\, d\sigma = \iint\limits_S bxz\, d\sigma = \int_0^c \int_0^b bxz\, dx\, dz = \frac{a^2bc^2}{4}.$
On the face $z = c$, we have $f(x, y, z) = z = c$ and $G(x, y, z) = G(x, y, c) = cxy \Rightarrow \mathbf{p} = \mathbf{k}$ and
$\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dy\, dx \Rightarrow \iint\limits_S G\, d\sigma = \iint\limits_S cxy\, d\sigma = \int_0^b \int_0^a cxy\, dx\, dy = \frac{a^2b^2c}{4}.$
Therefore, $\iint\limits_S G(x, y, z)\, d\sigma = \frac{abc(ab+ac+bc)}{4}.$

12. On the face $x = a$, we have $f(x, y, z) = x = a$ and $G(x, y, z) = G(a, y, z) = ayz \Rightarrow \mathbf{p} = \mathbf{i}$ and
$\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz\, dy \Rightarrow \iint\limits_S G\, d\sigma = \iint\limits_S ayz\, d\sigma = \int_{-b}^{b} \int_{-c}^{c} ayz\, dz\, dy = 0.$ Because of the symmetry of G on all the other faces, all the integrals are 0, and $\iint\limits_S G(x, y, z)\, d\sigma = 0.$

13. $f(x, y, z) = 2x + 2y + z = 2 \Rightarrow \nabla f = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $G(x, y, z) = x + y + (2 - 2x - 2y) = 2 - x - y \Rightarrow \mathbf{p} = \mathbf{k},$
$|\nabla f| = 3$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = 3\, dy\, dx;$ $z = 0 \Rightarrow 2x + 2y = 2 \Rightarrow y = 1 - x \Rightarrow \iint\limits_S G\, d\sigma = \iint\limits_S (2 - x - y)\, d\sigma$
$= 3\int_0^1 \int_0^{1-x} (2 - x - y)\, dy\, dx = 3\int_0^1 \left[(2 - x)(1 - x) - \frac{1}{2}(1 - x)^2\right] dx = 3\int_0^1 \left(\frac{3}{2} - 2x + \frac{x^2}{2}\right) dx = 2$

14. $f(x, y, z) = y^2 + 4z = 16 \Rightarrow \nabla f = 2y\mathbf{j} + 4\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 16} = 2\sqrt{y^2 + 4}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 4$
$\Rightarrow d\sigma = \frac{2\sqrt{y^2+4}}{4}\, dx\, dy \Rightarrow \iint\limits_S G\, d\sigma = \int_{-4}^{4} \int_0^1 \left(x\sqrt{y^2 + 4}\right)\left(\frac{\sqrt{y^2+4}}{2}\right) dx\, dy = \int_{-4}^{4} \int_0^1 \frac{x\left(y^2+4\right)}{2}\, dx\, dy$
$= \int_{-4}^{4} \frac{1}{4}\left(y^2 + 4\right) dy = \frac{1}{2}\left[\frac{y^3}{3} + 4y\right]_0^4 = \frac{1}{2}\left(\frac{64}{3} + 16\right) = \frac{56}{3}$

15. $f(x, y, z) = x + y^2 - z = 0 \Rightarrow \nabla f = \mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 2} = \sqrt{2}\sqrt{2y^2 + 1}$ and
$\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{2}\sqrt{2y^2+1}}{1}\, dx\, dy \Rightarrow \iint\limits_S G\, d\sigma = \int_0^1 \int_0^y \left(x + y^2 - x\right)\sqrt{2}\sqrt{2y^2 + 1}\, dx\, dy$
$= \sqrt{2}\int_0^1 \int_0^y y^2\sqrt{2y^2 + 1}\, dx\, dy = \sqrt{2}\int_0^1 y^3\sqrt{2y^2 + 1}\, dy = \frac{6\sqrt{6}+\sqrt{2}}{30}$

16. $f(x, y, z) = x^2 + y - z = 0 \Rightarrow \nabla f = 2x\mathbf{i} + \mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 2} = \sqrt{2}\sqrt{2x^2 + 1}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1$
$\Rightarrow d\sigma = \frac{\sqrt{2}\sqrt{2x^2+1}}{1}\, dx\, dy \Rightarrow \iint\limits_S G\, d\sigma = \int_{-1}^{1} \int_0^1 x\sqrt{2}\sqrt{2x^2 + 1}\, dx\, dy = \sqrt{2}\int_{-1}^{1} \int_0^1 x\sqrt{2x^2 + 1}\, dx\, dy$
$= \frac{3\sqrt{6}-\sqrt{2}}{6}\int_0^1 dy = \frac{3\sqrt{6}-\sqrt{2}}{3}$

17. $f(x, y, z) = 2x + y + z = 2 \Rightarrow \nabla f = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\nabla f| = \sqrt{6}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{6}}{1}\,dy\,dx$

$$\Rightarrow \iint_S G\,d\sigma = \int_0^1 \int_{1-x}^{2-2x} xy(2-2x-y)\sqrt{6}\,dy\,dx = \sqrt{6}\int_0^1 \int_{1-x}^{2-2x} \left(2xy - 2x^2y - xy^2\right) dy\,dx$$

$$= \sqrt{6}\int_0^1 \left(\tfrac{2}{3}x - 2x^2 + 2x^3 - \tfrac{2}{3}x^4\right) dx = \tfrac{\sqrt{6}}{30}$$

18. $f(x, y, z) = x + y = 1 \Rightarrow \nabla f = \mathbf{i} + \mathbf{j} \Rightarrow |\nabla f| = \sqrt{2}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{2}}{1}\,dz\,dx$

$$\Rightarrow \iint_S G\,d\sigma = \int_0^1 \int_0^1 \left(x - (1-x) - z\right)\sqrt{2}\,dz\,dx = \sqrt{2}\int_0^1 \int_0^1 (2x - z - 1)\,dz\,dx = \sqrt{2}\int_0^1 \left(2x - \tfrac{3}{2}\right) dx = -\tfrac{\sqrt{2}}{2}$$

19. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \left(4 - y^2\right)\mathbf{k},\ 0 \le x \le 1,\ -2 \le y \le 2;\ z = 0 \Rightarrow 0 = 4 - y^2$

$$\Rightarrow y = \pm 2;\ \mathbf{r}_x = \mathbf{i} \text{ and } \mathbf{r}_y = \mathbf{j} - 2y\mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\tfrac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}\,|\mathbf{r}_x \times \mathbf{r}_y|\,dy\,dx$$

$$= (2xy - 3z)\,dy\,dx = \left[2xy - 3\left(4 - y^2\right)\right] dy\,dx \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^1 \int_{-2}^2 \left(2xy + 3y^2 - 12\right) dy\,dx$$

$$= \int_0^1 \left[xy^2 + y^3 - 12y\right]_{-2}^2 dx = \int_0^1 (-32)\,dx = -32$$

20. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k},\ -1 \le x \le 1,\ 0 \le z \le 2 \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\tfrac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|}\,|\mathbf{r}_x \times \mathbf{r}_z|\,dz\,dx = -x^2\,dz\,dx$$

$$\Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_{-1}^1 \int_0^2 \left(-x^2\right) dz\,dx = -\tfrac{4}{3}$$

21. Let the parametrization be $\mathbf{r}(\phi, \theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = a,\ a \ge 0$, on the sphere), $0 \le \phi \le \frac{\pi}{2}$ (for the first octant), $0 \le \theta \le \frac{\pi}{2}$ (for the first octant)

$$\Rightarrow \mathbf{r}_\phi = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} - (a\sin\phi)\mathbf{k} \text{ and } \mathbf{r}_\theta = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j}$$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix} = \left(a^2\sin^2\phi\cos\theta\right)\mathbf{i} + \left(a^2\sin^2\phi\sin\theta\right)\mathbf{j} + \left(a^2\sin\phi\cos\phi\right)\mathbf{k}$$

$$\Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\tfrac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|}\,|\mathbf{r}_\phi \times \mathbf{r}_\theta|\,d\theta\,d\phi = a^3\cos^2\phi\sin\phi\,d\theta\,d\phi \text{ since } \mathbf{F} = z\mathbf{k} = (a\cos\phi)\mathbf{k}$$

$$\Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} a^3\cos^2\phi\sin\phi\,d\phi\,d\theta = \tfrac{\pi a^3}{6}$$

22. Let the parametrization be $\mathbf{r}(\phi, \theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = a,\ a \ge 0$, on the sphere), $0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$

$$\Rightarrow \mathbf{r}_\phi = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} - (a\sin\phi)\mathbf{k} \text{ and } \mathbf{r}_\theta = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j}$$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix} = \left(a^2\sin^2\phi\cos\theta\right)\mathbf{i} + \left(a^2\sin^2\phi\sin\theta\right)\mathbf{j} + \left(a^2\sin\phi\cos\phi\right)\mathbf{k}$$

$$\Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\tfrac{\mathbf{r}_\phi\times\mathbf{r}_\theta}{|\mathbf{r}_\phi\times\mathbf{r}_\theta|}\,|\mathbf{r}_\phi\times\mathbf{r}_\theta|\,d\theta\,d\phi = \left(a^3\sin^3\phi\cos^2\phi + a^3\sin^3\phi\sin^2\theta + a^3\sin\phi\cos^2\phi\right)d\theta\,d\phi$$

$= a^3\sin\phi\,d\theta\,d\phi$ since $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$

$$\Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^{\pi} a^3\sin\phi\,d\phi\,d\theta = 4\pi a^3$$

23. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (2a - x - y)\mathbf{k}$, $0 \le x \le a$, $0 \le y \le a \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - \mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\tfrac{\mathbf{r}_x\times\mathbf{r}_y}{|\mathbf{r}_x\times\mathbf{r}_y|}\,|\mathbf{r}_x\times\mathbf{r}_y|\,dy\,dx$$

$= \left[2xy + 2y(2a - x - y) + 2x(2a - x - y)\right]dy\,dx$ since $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$

$= 2xy\mathbf{i} + 2y(2a - x - y)\mathbf{j} + 2x(2a - x - y)\mathbf{k} \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$

$$= \int_0^a\int_0^a \left[2xy + 2y(2a - x - y) + 2x(2a - x - y)\right]dy\,dx = \int_0^a\int_0^a \left(4ay - 2y^2 + 4ax - 2x^2 - 2xy\right)dy\,dx$$

$$= \int_0^a \left(\tfrac{4}{3}a^3 + 3a^2x - 2ax^2\right)dx = \left(\tfrac{4}{3} + \tfrac{3}{2} - \tfrac{2}{3}\right)a^4 = \tfrac{13a^4}{6}$$

24. Let the parametrization be $\mathbf{r}(\theta, z) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k}$, $0 \le z \le a$, $0 \le \theta \le 2\pi$ (where $r = \sqrt{x^2 + y^2} = 1$ on the cylinder) $\Rightarrow \mathbf{r}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$

$\Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\tfrac{\mathbf{r}_\theta\times\mathbf{r}_z}{|\mathbf{r}_\theta\times\mathbf{r}_z|}\,|\mathbf{r}_\theta\times\mathbf{r}_z|\,dz\,d\theta = \left(\cos^2\theta + \sin^2\theta\right)dz\,d\theta = dz\,d\theta$, since $\mathbf{F} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k}$

$$\Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^a 1\,dz\,d\theta = 2\pi a$$

25. Let the parametrization be $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k} \text{ and } \mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\tfrac{\mathbf{r}_\theta\times\mathbf{r}_r}{|\mathbf{r}_\theta\times\mathbf{r}_r|}\,|\mathbf{r}_\theta\times\mathbf{r}_r|\,d\theta\,dr = \left(r^3\sin\theta\cos^2\theta + r^2\right)d\theta\,dr$ since

$\mathbf{F} = \left(r^2\sin\theta\cos\theta\right)\mathbf{i} - r\mathbf{k} \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^1 \left(r^3\sin\theta\cos^2\theta + r^2\right)dr\,d\theta = \int_0^{2\pi}\left(\tfrac{1}{4}\sin\theta\cos^2\theta + \tfrac{1}{3}\right)d\theta$

$$= \left[-\tfrac{1}{12}\cos^3\theta + \tfrac{\theta}{3}\right]_0^{2\pi} = \tfrac{2\pi}{3}$$

26. Let the parametrization be $\mathbf{r}(r,\theta)=(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}+2r\mathbf{k}$, $0\le r\le 1$ (since $0\le z\le 2$) and $0\le\theta\le 2\pi$

$$\Rightarrow \mathbf{r}_r=(\cos\theta)\mathbf{i}+(\sin\theta)\mathbf{j}+2\mathbf{k} \text{ and } \mathbf{r}_\theta=(-r\sin\theta)\mathbf{i}+(r\cos\theta)\mathbf{j}\Rightarrow \mathbf{r}_\theta\times\mathbf{r}_r=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ -r\sin\theta & r\cos\theta & 0\\ \cos\theta & \sin\theta & 2\end{vmatrix}$$

$$=(2r\cos\theta)\mathbf{i}+(2r\sin\theta)\mathbf{j}-r\mathbf{k}\Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma=\mathbf{F}\cdot\tfrac{\mathbf{r}_\theta\times\mathbf{r}_r}{|\mathbf{r}_\theta\times\mathbf{r}_r|}\,|\mathbf{r}_\theta\times\mathbf{r}_r|\,d\theta\,dr$$

$$=\left(2r^3\sin^2\theta\cos\theta+4r^3\cos\theta\sin\theta+r\right)d\theta\,dr \text{ since } \mathbf{F}=\left(r^2\sin^2\theta\right)\mathbf{i}+\left(2r^2\cos\theta\right)\mathbf{j}-\mathbf{k}$$

$$\Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma=\int_0^{2\pi}\int_0^1\left(2r^3\sin^2\theta\cos\theta+4r^3\cos\theta\sin\theta+r\right)dr\,d\theta=\int_0^{2\pi}\left(\tfrac{1}{2}\sin^2\theta\cos\theta+\cos\theta\sin\theta+\tfrac{1}{2}\right)d\theta$$

$$=\left[\tfrac{1}{6}\sin^3\theta+\tfrac{1}{2}\sin^2\theta+\tfrac{1}{2}\theta\right]_0^{2\pi}=\pi$$

27. Let the parametrization be $\mathbf{r}(r,\theta)=(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}+r\mathbf{k}$, $1\le r\le 2$ (since $1\le z\le 2$) and $0\le\theta\le 2\pi$

$$\Rightarrow \mathbf{r}_r=(\cos\theta)\mathbf{i}+(\sin\theta)\mathbf{j}+\mathbf{k} \text{ and } \mathbf{r}_\theta=(-r\sin\theta)\mathbf{i}+(r\cos\theta)\mathbf{j}\Rightarrow \mathbf{r}_\theta\times\mathbf{r}_r=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ -r\sin\theta & r\cos\theta & 0\\ \cos\theta & \sin\theta & 1\end{vmatrix}$$

$$=(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}-r\mathbf{k}\Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma=\mathbf{F}\cdot\tfrac{\mathbf{r}_\theta\times\mathbf{r}_r}{|\mathbf{r}_\theta\times\mathbf{r}_r|}\,|\mathbf{r}_\theta\times\mathbf{r}_r|\,d\theta\,dr=\left(-r^2\cos^2\theta-r^2\sin^2\theta-r^3\right)d\theta\,dr$$

$$=\left(-r^2-r^3\right)d\theta\,dr \text{ since } \mathbf{F}=(-r\cos\theta)\mathbf{i}-(r\sin\theta)\mathbf{j}+r^2\mathbf{k}\Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma=\int_0^{2\pi}\int_1^2\left(-r^2-r^3\right)dr\,d\theta=-\tfrac{73\pi}{6}$$

28. Let the parametrization be $\mathbf{r}(r,\theta)=(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}+r^2\mathbf{k}$, $0\le r\le 1$ (since $0\le z\le 1$) and $0\le\theta\le 2\pi$

$$\Rightarrow \mathbf{r}_r=(\cos\theta)\mathbf{i}+(\sin\theta)\mathbf{j}+2r\mathbf{k} \text{ and } \mathbf{r}_\theta=(-r\sin\theta)\mathbf{i}+(r\cos\theta)\mathbf{j}\Rightarrow \mathbf{r}_\theta\times\mathbf{r}_r=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ -r\sin\theta & r\cos\theta & 0\\ \cos\theta & \sin\theta & 2r\end{vmatrix}$$

$$=\left(2r^2\cos\theta\right)\mathbf{i}+\left(2r^2\sin\theta\right)\mathbf{j}-r\mathbf{k}\Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma=\mathbf{F}\cdot\tfrac{\mathbf{r}_\theta\times\mathbf{r}_r}{|\mathbf{r}_\theta\times\mathbf{r}_r|}\,|\mathbf{r}_\theta\times\mathbf{r}_r|\,d\theta\,dr=\left(8r^3\cos^2\theta+8r^3\sin^2\theta-2r\right)d\theta\,dr$$

$$=\left(8r^3-2r\right)d\theta\,dr \text{ since } \mathbf{F}=(4r\cos\theta)\mathbf{i}+(4r\sin\theta)\mathbf{j}+2\mathbf{k}\Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma=\int_0^{2\pi}\int_0^1\left(8r^3-2r\right)dr\,d\theta=2\pi$$

29. $g(x,y,z)=z$, $\mathbf{p}=\mathbf{k}\Rightarrow\nabla g=\mathbf{k}\Rightarrow|\nabla g|=1$ and $|\nabla g\cdot\mathbf{p}|=1\Rightarrow$ Flux $=\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma=\iint_R(\mathbf{F}\cdot\mathbf{k})\,dA$

$$=\int_0^2\int_0^3 3\,dy\,dx=18$$

30. $g(x,y,z)=y$, $\mathbf{p}=-\mathbf{j}\Rightarrow\nabla g=\mathbf{j}\Rightarrow|\nabla g|=1$ and $|\nabla g\cdot\mathbf{p}|=1\Rightarrow$ Flux $=\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma=\iint_R(\mathbf{F}\cdot-\mathbf{j})\,dA$

$$=\int_{-1}^2\int_2^7 2\,dz\,dx=\int_{-1}^2 2(7-2)\,dx=10(2+1)=30$$

31. $\nabla g = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2+4y^2+4z^2} = 2a;\ \mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{z^2}{a};$

$|\nabla g\cdot\mathbf{k}| = 2z \Rightarrow d\sigma = \frac{2a}{2z}\,dA \Rightarrow \text{Flux} = \iint\limits_S \left(\frac{z^2}{a}\right)\left(\frac{a}{z}\right) dA = \iint\limits_S z\ dA = \iint\limits_S \sqrt{a^2-\left(x^2+y^2\right)}\ dx\ dy$

$= \int_0^{\pi/2}\int_0^a \sqrt{a^2-r^2}\,r\ dr\ d\theta = \frac{\pi a^3}{6}$

32. $\nabla g = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2+4y^2+4z^2} = 2a;\ \mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{-xy}{a}+\frac{xy}{a} = 0;$

$|\nabla g\cdot\mathbf{k}| = 2z \Rightarrow d\sigma = \frac{2a}{2z}\,dA \Rightarrow \text{Flux} = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iint\limits_S 0\ d\sigma = 0$

33. From Exercise 31, $\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{xy}{a}-\frac{xy}{a}+\frac{z}{a} = \frac{z}{a} \Rightarrow \text{Flux} = \iint\limits_R \left(\frac{z}{a}\right)\left(\frac{a}{z}\right) dA = \iint\limits_R 1\ dA = \frac{\pi a^2}{4}$

34. From Exercise 31, $\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{zx^2}{a}+\frac{zy^2}{a}+\frac{z^3}{a} = z\left(\frac{x^2+y^2+z^2}{a}\right) = az$

$\Rightarrow \text{Flux} = \iint\limits_R (za)\left(\frac{a}{z}\right) = \iint\limits_R a^2 dx\ dy = a^2(\text{Area of } R) = \frac{1}{4}\pi a^4 dx\ dy$

35. From Exercise 31, $\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{x^2}{a}+\frac{y^2}{a}+\frac{z^2}{a} = a \Rightarrow \text{Flux} = \iint\limits_R a\left(\frac{a}{z}\right) dA = \iint\limits_R \frac{a^2}{z}\,dA$

$= \iint\limits_R \frac{a^2}{\sqrt{a^2-\left(x^2+y^2\right)}}\,dA = \int_0^{\pi/2}\int_0^a \frac{a^2}{\sqrt{a^2-r^2}}\ r\ dr\ d\theta = \int_0^{\pi/2} a^2\left[-\sqrt{a^2-r^2}\right]_0^a d\theta = \frac{\pi a^3}{2}$

36. From Exercise 31, $\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{\left(\frac{x^2}{a}\right)+\left(\frac{y^2}{a}\right)+\left(\frac{z^3}{a}\right)}{\sqrt{x^2+y^2+z^2}} = \frac{\left(\frac{a^2}{a}\right)}{a} = 1$

$\Rightarrow \text{Flux} = \iint\limits_R \frac{a}{z}\,dx\ dy = \iint\limits_R \frac{a}{\sqrt{a^2-\left(x^2+y^2\right)}}\,dx\ dy = \int_0^{\pi/2}\int_0^a \frac{a}{\sqrt{a^2-r^2}}\ r\ dr\ d\theta = \frac{\pi a^2}{2}$

37. $g(x,y,z) = y^2+z = 4 \Rightarrow \nabla g = 2y\mathbf{j}+\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4y^2+1} \Rightarrow \mathbf{n} = \frac{2y\mathbf{j}+\mathbf{k}}{\sqrt{4y^2+1}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{2xy-3z}{\sqrt{4y^2+1}};$

$\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2+1}\ dA \Rightarrow \text{Flux} = \iint\limits_R \left(\frac{2xy-3z}{\sqrt{4y^2+1}}\right)\sqrt{4y^2+1}\ dA = \iint\limits_R (2xy-3z)\ dA;$

$z = 0$ and $z = 4-y^2 \Rightarrow y^2 = 4 \Rightarrow \text{Flux} = \iint\limits_R \left[2xy-3\left(4-y^2\right)\right] dA = \int_0^1\int_{-2}^2 \left(2xy-12+3y^2\right) dy\ dx$

$= \int_0^1 \left[xy^2-12y+y^3\right]_{-2}^2 dx = \int_0^1 (-32)\ dx = -32$

38. $g(x,y,z) = x^2+y^2-z = 0 \Rightarrow \nabla g = 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2+4y^2+1} = \sqrt{4\left(x^2+y^2\right)+1}$

$\Rightarrow \mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}}{\sqrt{4\left(x^2+y^2\right)+1}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{8x^2+8y^2-2}{\sqrt{4\left(x^2+y^2\right)+1}};\ \mathbf{p} = \mathbf{k} \Rightarrow |\nabla g\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4\left(x^2+y^2\right)+1}\ dA$

$\Rightarrow \text{Flux} = \iint\limits_R \left(\frac{8x^2+8y^2-2}{\sqrt{4(x^2+y^2)+1}} \right) \sqrt{4\left(x^2+y^2\right)+1}\, dA = \iint\limits_R \left(8x^2+8y^2-2\right) dA;\ z=1 \text{ and } x^2+y^2=z$

$\Rightarrow x^2+y^2=1 \Rightarrow \text{Flux} = \int_0^{2\pi}\int_0^1 \left(8r^2-2\right) r\, dr\, d\theta = 2\pi$

39. $g(x, y, z) = y - e^x = 0 \Rightarrow \nabla g = -e^x\mathbf{i} + \mathbf{j} \Rightarrow |\nabla g| = \sqrt{e^{2x}+1} \Rightarrow \mathbf{n} = \frac{e^x\mathbf{i}-\mathbf{j}}{\sqrt{e^{2x}+1}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{-2e^x-2y}{\sqrt{e^{2x}+1}};$

$\mathbf{p} = \mathbf{i} \Rightarrow |\nabla g \cdot \mathbf{p}| = e^x \Rightarrow d\sigma = \frac{\sqrt{e^{2x}+1}}{e^x}\, dA \Rightarrow \text{Flux} = \iint\limits_R \left(\frac{-2e^x-2y}{\sqrt{e^{2x}+1}} \right)\left(\frac{\sqrt{e^{2x}+1}}{e^x} \right) dA = \iint\limits_R \frac{-2e^x-2e^x}{e^x}\, dA$

$= \iint\limits_R (-4)\, dA = \int_0^1\int_1^2 (-4)\, dy\, dz = -4$

40. $g(x, y, z) = y - \ln x = 0 \Rightarrow \nabla g = -\frac{1}{x}\mathbf{i} + \mathbf{j} \Rightarrow |\nabla g| = \sqrt{\frac{1}{x^2}+1} = \frac{\sqrt{1+x^2}}{x}$ since $1 \le x \le e \Rightarrow \mathbf{n} = \frac{\left(-\frac{1}{x}\mathbf{i}+\mathbf{j}\right)}{\left(\frac{\sqrt{1+x^2}}{x}\right)} = \frac{-\mathbf{i}+x\mathbf{j}}{\sqrt{1+x^2}}$

$\Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{2xy}{\sqrt{1+x^2}};\ \mathbf{p} = \mathbf{j} \Rightarrow |\nabla g\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{1+x^2}}{x}\, dA \Rightarrow \text{Flux} = \iint\limits_R \left(\frac{2xy}{\sqrt{1+x^2}}\right)\left(\frac{\sqrt{1+x^2}}{x}\right) dA$

$= \int_0^1\int_1^e 2y\, dx\, dz = \int_1^e\int_0^1 2\ln x\, dz\, dx = \int_1^e 2\ln x\, dx = 2\left[x\ln x - x\right]_1^e = 2(e-e) - 2(0-1) = 2$

41. On the face $z = a: g(x, y, z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1;\ \mathbf{n} = \mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n} = 2xz = 2ax$ since $z = a;\ d\sigma = dx\, dy$

$\Rightarrow \text{Flux} = \iint\limits_R 2ax\, dx\, dy = \int_0^a\int_0^a 2ax\, dx\, dy = a^4.$

On the face $z = 0: g(x, y, z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1;\ \mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -2xz = 0$ since $z = 0;$

$d\sigma = dx\, dy \Rightarrow \text{Flux} = \iint\limits_R 0\, dx\, dy = 0.$

On the face $x = a$: $g(x, y, z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1;\ \mathbf{n} = \mathbf{i} \Rightarrow \mathbf{F}\cdot\mathbf{n} = 2xy = 2ay$ since $x = a;$

$d\sigma = dy\, dz \Rightarrow \text{Flux} = \int_0^a\int_0^a 2ay\, dy\, dz = a^4.$

On the face $x = 0$: $g(x, y, z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1;\ \mathbf{n} = -\mathbf{i} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -2xy = 0$ since $x = 0 \Rightarrow \text{Flux} = 0.$

On the face $y = a$: $g(x, y, z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1;\ \mathbf{n} = \mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n} = 2yz = 2az$ since $y = a;$

$d\sigma = dz\, dx \Rightarrow \text{Flux} = \int_0^a\int_0^a 2az\, dz\, dx = a^4.$

On the face $y = 0$: $g(x, y, z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1;\mathbf{n} = -\mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -2yz = 0$ since $y = 0 \Rightarrow \text{Flux} = 0.$

Therefore, Total Flux $= 3a^4$.

42. Across the cap: $g(x, y, z) = x^2+y^2+z^2 = 25 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2+4y^2+4z^2} = 10$

$\Rightarrow \mathbf{n} = \frac{\nabla g}{|\nabla g|} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{5} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{x^2z}{5} + \frac{y^2z}{5} + \frac{z}{5};\ \mathbf{p} = \mathbf{k} \Rightarrow |\nabla g\cdot\mathbf{p}| = 2z$ since $z \ge 0 \Rightarrow d\sigma = \frac{10}{2z}\, dA$

$\Rightarrow \text{Flux}_{\text{cap}} = \iint\limits_{\text{cap}} \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint\limits_R \left(\frac{x^2z}{5}+\frac{y^2z}{5}+\frac{z}{5}\right)\left(\frac{5}{z}\right) dA = \iint\limits_R \left(x^2+y^2+1\right) dx\, dy = \int_0^{2\pi}\int_0^4 \left(r^2+1\right) r\, dr\, d\theta$

$= \int_0^{2\pi} 72\, d\theta = 144\pi.$

Across the bottom: $g(x, y, z) = z = 3 \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1 \Rightarrow \mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -1; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1$ $\Rightarrow d\sigma = dA \Rightarrow \text{Flux}_{\text{bottom}} = \iint\limits_{\text{bottom}} \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint\limits_R -1\, dA = -1(\text{Area ot the circular region}) = -16\pi$. Therefore, $\text{Flux} = \text{Flux}_{\text{cap}} + \text{Flux}_{\text{bottom}} = 128\pi$

43. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0 \Rightarrow d\sigma = \frac{2a}{2z}\, dA$ $= \frac{a}{z}\, dA; M = \iint\limits_S \delta\, d\sigma = \frac{\delta}{8}(\text{surface area of sphere}) = \frac{\delta\pi a^2}{2}; M_{xy} = \iint\limits_S z\delta\, d\sigma = \delta \iint\limits_R z\left(\frac{a}{z}\right) dA = a\delta \iint\limits_R dA$ $= a\delta \int_0^{\pi/2}\int_0^a r\, dr\, d\theta = \frac{\delta\pi a^3}{4} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \left(\frac{\delta\pi a^3}{4}\right)\left(\frac{2}{\delta\pi a^2}\right) = \frac{a}{2}$. Because of symmetry, $\overline{x} = \overline{y} = \frac{a}{2}$ $\Rightarrow$ the centroid is $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$.

44. $\nabla f = 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 4z^2} = \sqrt{4\left(y^2 + z^2\right)} = 6; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{k}| = 2z$ since $z \ge 0 \Rightarrow d\sigma = \frac{6}{2z}\, dA$ $= \frac{3}{z}\, dA; M = \iint\limits_S 1\, d\sigma = \int_{-3}^{3}\int_0^3 \frac{3}{z}\, dx\, dy = \int_{-3}^{3}\int_0^3 \frac{3}{\sqrt{9-y^2}}\, dx\, dy = 9\pi; M_{xy} = \iint\limits_S z\, d\sigma = \int_{-3}^{3}\int_0^3 z\left(\frac{3}{z}\right) dx\, dy = 54;$ $M_{xz} = \iint\limits_S y\, d\sigma = \int_{-3}^{3}\int_0^3 y\left(\frac{3}{z}\right) dx\, dy = \int_{-3}^{3}\int_0^3 \frac{3y}{\sqrt{9-y^2}}\, dx\, dy = 0; M_{yz} = \iint\limits_S x\, d\sigma = \int_{-3}^{3}\int_0^3 \frac{3x}{\sqrt{9-y^2}}\, dx\, dy = \frac{27}{2}\pi.$ Therefore, $\overline{x} = \frac{\left(\frac{27}{2}\pi\right)}{9\pi} = \frac{3}{2}, \overline{y} = 0$, and $\overline{z} = \frac{54}{9\pi} = \frac{6}{\pi}$

45. Because of symmetry, $\overline{x} = \overline{y} = 0; M = \iint\limits_S \delta\, d\sigma = \delta \iint\limits_S d\sigma = (\text{Area of } S)\delta = 3\pi\sqrt{2}\delta; \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$ $\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2}; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{x^2+y^2+z^2}}{2z}\, dA$ $= \frac{\sqrt{x^2+y^2+\left(x^2+y^2\right)}}{z}\, dA = \frac{\sqrt{2}\sqrt{x^2+y^2}}{z}\, dA \Rightarrow M_{xy} = \delta \iint\limits_R z\left(\frac{\sqrt{2}\sqrt{x^2+y^2}}{z}\right) dA = \delta \iint\limits_R \sqrt{2}\sqrt{x^2 + y^2}\, dA$ $= \delta \int_0^{2\pi}\int_1^2 \sqrt{2} r^2\, dr\, d\theta = \frac{14\pi\sqrt{2}}{3}\delta \Rightarrow \overline{z} = \frac{\left(\frac{14\pi\sqrt{2}}{3}\delta\right)}{3\pi\sqrt{2}\delta} = \frac{14}{9} \Rightarrow (\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{14}{9}\right)$. Next, $I_z = \iint\limits_S \left(x^2 + y^2\right)\delta\, d\sigma$ $= \iint\limits_R (x^2 + y^2)\left(\frac{\sqrt{2}\sqrt{x^2+y^2}}{z}\right)\delta\, dA = \delta\sqrt{2} \iint\limits_R \left(x^2 + y^2\right) dA = \delta\sqrt{2}\int_0^{2\pi}\int_1^2 r^3 dr\, d\theta = \frac{15\pi\sqrt{2}}{2}\delta \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{\sqrt{10}}{2}$

46. $f(x, y, z) = 4x^2 + 4y^2 - z^2 = 0 \Rightarrow \nabla f = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{64x^2 + 64y^2 + 4z^2}$ $= 2\sqrt{16x^2 + 16y^2 + z^2} = 2\sqrt{4z^2 + z^2} = 2\sqrt{5}\, z$ since $z \ge 0; \mathbf{P} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{5}z}{2z}\, dA = \sqrt{5}\, dA$ $\Rightarrow I_z = \iint\limits_S \left(x^2 + y^2\right)\delta\, d\sigma = \delta\sqrt{5} \iint\limits_R \left(x^2 + y^2\right) dx\, dy = \delta\sqrt{5}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\, dr\, d\theta = \frac{3\sqrt{5}\pi\delta}{2}$

47. (a) Let the diameter lie on the z-axis and let $f(x, y, z) = x^2 + y^2 + z^2 = a^2, z \ge 0$ be the upper hemisphere $\Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a, a > 0; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0$ $\Rightarrow d\sigma = \frac{a}{z}\, dA \Rightarrow I_z = \iint\limits_S \delta\left(x^2 + y^2\right)\left(\frac{a}{z}\right) d\sigma = a\delta \iint\limits_R \frac{x^2+y^2}{\sqrt{a^2-\left(x^2+y^2\right)}}\, dA = a\delta\int_0^{2\pi}\int_0^a \frac{r^2}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta$

$= a\delta\int_0^{2\pi}\left[-r^2\sqrt{a^2-r^2}-\frac{2}{3}\left(a^2-r^2\right)^{3/2}\right]_0^a d\theta = a\delta\int_0^{2\pi}\frac{2}{3}a^3\ d\theta = \frac{4\pi}{3}a^4\delta \Rightarrow$ the moment of inertia is $\frac{8\pi}{3}a^4\delta$ for the whole sphere

(b) $I_L = I_{\text{c.m.}} + mh^2$, where m is the mass of the body and h is the distance between the parallel lines; now, $I_{\text{c.m.}} = \frac{8\pi}{3}a^4\delta$ (from part a) and $\frac{m}{2} = \iint_S \delta\ d\sigma = \delta\iint_R \left(\frac{a}{z}\right) dA = a\delta\iint_R \frac{1}{\sqrt{a^2-\left(x^2+y^2\right)}}dy\,dx$

$= a\delta\int_0^{2\pi}\int_0^a \frac{1}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = a\delta\int_0^{2\pi}\left[-\sqrt{a^2-r^2}\right]_0^a d\theta = a\delta\int_0^{2\pi} a\,d\theta = 2\pi a^2\delta$ and $h = a$

$\Rightarrow I_L = \frac{8\pi}{3}a^4\delta + 4\pi a^2\delta a^2 = \frac{20\pi}{3}a^4\delta$

48. Let $z = \frac{h}{a}\sqrt{x^2+y^2}$ be the cone from $z = 0$ to $z = h$, $h > 0$. Because of symmetry, $\overline{x} = 0$ and $\overline{y} = 0$; $z = \frac{h}{a}\sqrt{x^2+y^2} \Rightarrow f(x, y, z) = \frac{h^2}{a^2}\left(x^2+y^2\right) - z^2 = 0 \Rightarrow \nabla f = \frac{2xh^2}{a^2}\mathbf{i} + \frac{2yh^2}{a^2}\mathbf{j} - 2z\mathbf{k}$

$\Rightarrow |\nabla f| = \sqrt{\frac{4x^2h^4}{a^4}+\frac{4y^2h^4}{a^4}+4z^2} = 2\sqrt{\frac{h^4}{a^4}\left(x^2+y^2\right)+\frac{h^2}{a^2}\left(x^2+y^2\right)} = 2\sqrt{\left(\frac{h^2}{a^2}\right)\left(x^2+y^2\right)\left(\frac{h^2}{a^2}+1\right)} = 2\sqrt{z^2\left(\frac{h^2+a^2}{a^2}\right)}$

$= \left(\frac{2z}{a}\right)\sqrt{h^2+a^2}$ since $z \ge 0$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{\left(\frac{2z}{a}\right)\sqrt{h^2+a^2}}{2z}\,dA = \frac{\sqrt{h^2+a^2}}{a}\,dA$;

$M = \iint_S d\sigma = \iint_R \frac{\sqrt{h^2+a^2}}{a}\,dA = \frac{\sqrt{h^2+a^2}}{a}\left(\pi a^2\right) = \pi a\sqrt{h^2+a^2}$; $M_{xy} = \iint_S z\,d\sigma = \iint_R z\left(\frac{\sqrt{h^2+a^2}}{a}\right)dA$

$= \frac{\sqrt{h^2+a^2}}{a}\iint_R \frac{h}{a}\sqrt{x^2+y^2}\ dx\,dy = \frac{h\sqrt{h^2+a^2}}{a^2}\int_0^{2\pi}\int_0^a r^2\ dr\,d\theta = \frac{2\pi ah\sqrt{h^2+a^2}}{3} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \frac{2h}{3} \Rightarrow$ the centroid is $\left(0, 0, \frac{2h}{3}\right)$

16.7 STOKES' THEOREM

1. $\text{curl } \mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (2-0)\mathbf{k} = 2\mathbf{k}$ and $\mathbf{n} = \mathbf{k} \Rightarrow \text{curl } \mathbf{F}\cdot\mathbf{n} = 2 \Rightarrow d\sigma = dx\,dy$

$\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R 2\ dA = 2(\text{Area of the ellipse}) = 4\pi$

2. $\text{curl } \mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (3-2)\mathbf{k} = \mathbf{k}$ and $\mathbf{n} = \mathbf{k} \Rightarrow \text{curl } \mathbf{F}\cdot\mathbf{n} = 1 \Rightarrow d\sigma = dx\,dy$

$\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R dx\,dy = \text{Area of circle} = 9\pi$

3. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{vmatrix} = -x\mathbf{i} - 2x\mathbf{j} + (z-1)\mathbf{k}$ and $\mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} \Rightarrow$ curl $\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{3}}(-x-2x+z-1)$

$\Rightarrow d\sigma = \frac{\sqrt{3}}{1}\, dA \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \frac{1}{\sqrt{3}}(-3x+z-1)\sqrt{3}\, dA = \int_0^1 \int_0^{1-x} \left[-3x + (1-x-y) - 1\right] dy\, dx$

$= \int_0^1 \int_0^{1-x} (-4x-y)\, dy\, dx = \int_0^1 -\left[4x(1-x) + \frac{1}{2}(1-x)^2\right] dx = -\int_0^1 \left(\frac{1}{2} + 3x - \frac{7}{2}x^2\right) dx = -\frac{5}{6}$

4. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{vmatrix} = (2y-2z)\mathbf{i} + (2z-2x)\mathbf{j} + (2x-2y)\mathbf{k}$ and $\mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$

$\Rightarrow$ curl $\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{3}}(2y-2z+2z-2x+2x-2y) = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S 0\, d\sigma = 0$

5. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+y^2 & x^2+y^2 \end{vmatrix} = 2y\mathbf{i} + (2z-2x)\mathbf{j} + (2x-2y)\mathbf{k}$ and $\mathbf{n} = \mathbf{k} \Rightarrow$ curl $\mathbf{F} \cdot \mathbf{n} = 2x - 2y$

$\Rightarrow d\sigma = dx\, dy \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^1 \int_{-1}^1 (2x-2y)\, dx\, dy = \int_{-1}^1 \left[x^2 - 2xy\right]_{-1}^1 dy = \int_{-1}^1 -4y\, dy = 0$

6. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 3x^2y^2\mathbf{k}$ and $\mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{4} \Rightarrow$ curl $\mathbf{F} \cdot \mathbf{n} = -\frac{3}{4}x^2y^2z;$

$d\sigma = \frac{4}{z}\, dA$ (Section 16.6, Example 6, with $a = 4$) $\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(-\frac{3}{4}x^2y^2z\right)\left(\frac{4}{z}\right) dA$

$= -3\int_0^{2\pi} \int_0^2 \left(r^2 \cos^2\theta\right)\left(r^2 \sin^2\theta\right) r\, dr\, d\theta = -3\int_0^{2\pi} \left[\frac{r^6}{6}\right]_0^2 (\cos\theta \sin\theta)^2\, d\theta = -32\int_0^{2\pi} \frac{1}{4}\sin^2 2\theta\, d\theta$

$= -4\int_0^{4\pi} \sin^2 u\, du = -4\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = -8\pi$

7. $x = 3\cos t$ and $y = 2\sin t \Rightarrow \mathbf{F} = (2\sin t)\mathbf{i} + \left(9\cos^2 t\right)\mathbf{j} + \left(9\cos^2 t + 16\sin^4 t\right)\sin e^{\sqrt{(6\sin t\cos t)(0)}}\mathbf{k}$ at the base of the shell; $\mathbf{r} = (3\cos t)\mathbf{i} + (2\sin t)\mathbf{j} \Rightarrow d\mathbf{r} = (-3\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -6\sin^2 t + 18\cos^3 t$

$\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \left(-6\sin^2 t + 18\cos^3 t\right) dt = \left[-3t + \frac{3}{2}\sin 2t + 6(\sin t)\left(\cos^2 t + 2\right)\right]_0^{2\pi} = -6\pi$

8. curl $\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z+\frac{1}{2+x} & \tan^{-1} y & x+\frac{1}{4+z} \end{vmatrix} = -2\mathbf{j};\ f(x, y, z) = 4x^2 + y + z^2 \Rightarrow \nabla f = 8x\mathbf{i} + \mathbf{j} + 2z\mathbf{k}$

$\Rightarrow \mathbf{n} = \frac{\nabla f}{|\nabla f|}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla f\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = |\nabla f|\,dA;\ \nabla\times\mathbf{F}\cdot\mathbf{n} = \frac{1}{|\nabla f|}\left(-2\mathbf{j}\cdot\nabla f\right) = \frac{-2}{|\nabla f|}$

$\Rightarrow \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = -2\,dA \Rightarrow \iint\limits_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint\limits_R -2\,dA = -2\,(\text{Area of } R) = -2(\pi\cdot 1\cdot 2) = -4\pi,$ where R is the elliptic region in the xz-plane enclosed by $4x^2 + z^2 = 4$.

9. Flux of $\nabla\times\mathbf{F} = \iint\limits_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \oint_C \mathbf{F}\cdot d\mathbf{r},$ so let C be parametrized by $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ 0 \le t \le 2\pi$

$\Rightarrow \frac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = ay\sin t + ax\cos t = a^2\sin^2 t + a^2\cos^2 t = a^2$

$\Rightarrow$ Flux of $\nabla\times\mathbf{F} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} a^2\,dt = 2\pi a^2$

10. $\nabla\times(y\mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = -\mathbf{k};\ \mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow \nabla\times(y\mathbf{i})\cdot\mathbf{n} = -z;\ d\sigma = \frac{1}{z}\,dA$

(Section 16.6, Example 6, with $a = 1$) $\Rightarrow \iint\limits_S \nabla\times(y\mathbf{i})\cdot\mathbf{n}\,d\sigma = \iint\limits_R (-z)\left(\frac{1}{z}\,dA\right) = -\iint\limits_R dA = -\pi,$ where R is the disk $x^2 + y^2 \le 1$ in the xy-plane.

11. For the upper hemisphere with $z \ge 0$, the boundary C is the unit circle of radius 1 centered at the origin in the xy-plane. An outward normal on the upper hemisphere corresponds to counterclockwise circulation around the boundary, so the boundary can be parametrized as $\mathbf{r}(\theta) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 0\mathbf{k},$ with $0 \le \theta \le 2\pi$. Thus $d\mathbf{r} = (-\sin\theta\,d\theta)\mathbf{i} + (\cos\theta\,d\theta)\mathbf{j}$. For the field $\mathbf{A} = (y + \sqrt{z})\mathbf{i} + e^{xyz}\mathbf{j} + (\cos xz)\mathbf{k}$, the flux of $\mathbf{F} = \nabla\times\mathbf{A}$ across the upper hemisphere is, by Stokes' Theorem, equal to the circulation of $\mathbf{A}$ on the boundary. Since $z = 0$ and $y = \sin\theta$ on the boundary, the field $\mathbf{A}$ on the boundary is $(\sin\theta)\mathbf{i} + \mathbf{j} + \mathbf{k}$. The circulation of $\mathbf{A}$ on C is

$$\oint\limits_C \mathbf{A}\cdot d\mathbf{r} = \oint\limits_C ((\sin\theta)\mathbf{i} + \mathbf{j} + \mathbf{k})\cdot((-\sin\theta\,d\theta)\mathbf{i} + (\cos\theta\,d\theta)\mathbf{j}) = \int_0^{2\pi}\left(\cos\theta - \sin^2\theta\right)d\theta$$

$$= \int_0^{2\pi}\left(\cos\theta + \tfrac{1}{2}\left(\cos 2\theta - 1\right)\right)d\theta = -\pi$$

12. Since the outward normal on the bottom hemisphere corresponds to clockwise circulation on the boundary, the flux of $\mathbf{F}$ through the bottom hemisphere will be π and the total flux through the sphere will be 0.

13. $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k};\ \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$\Rightarrow \mathbf{r}_r\times\mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left(2r^2\cos\theta\right)\mathbf{i} + \left(2r^2\sin\theta\right)\mathbf{j} + r\mathbf{k};\ \mathbf{n} = \frac{\mathbf{r}_r\times\mathbf{r}_\theta}{|\mathbf{r}_r\times\mathbf{r}_\theta|}$ and $d\sigma = |\mathbf{r}_r\times\mathbf{r}_\theta|\,dr\,d\theta$

$\Rightarrow \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = (\nabla\times\mathbf{F})\cdot(\mathbf{r}_r\times\mathbf{r}_\theta)\,dr\,d\theta = \left(10r^2\cos\theta + 4r^2\sin\theta + 3r\right)dr\,d\theta \Rightarrow \iint\limits_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma$

$= \int_0^{2\pi}\int_0^2 \left(10r^2\cos\theta + 4r^2\sin\theta + 3r\right)dr\,d\theta = \int_0^{2\pi}\left[\frac{10}{3}r^3\cos\theta + \frac{4}{3}r^3\sin\theta + \frac{3}{2}r^2\right]_0^2 d\theta$

$= \int_0^{2\pi}\left(\frac{80}{3}\cos\theta + \frac{32}{3}\sin\theta + 6\right)d\theta = 6(2\pi) = 12\pi$

14. $\nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y-z & z-x & x+z\end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k};\ \mathbf{r}_r\times\mathbf{r}_\theta = \left(2r^2\cos\theta\right)\mathbf{i} + \left(2r^2\sin\theta\right)\mathbf{j} + r\mathbf{k}$ and

$\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = (\nabla\times\mathbf{F})\cdot(\mathbf{r}_r\times\mathbf{r}_\theta)\,dr\,d\theta$ (see Exercise 13 above)

$\Rightarrow \iint\limits_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^3\left(-2r^2\cos\theta - 4r^2\sin\theta - 2r\right)dr\,d\theta = \int_0^{2\pi}\left[-\frac{2}{3}r^3\cos\theta - \frac{4}{3}r^3\sin\theta - r^2\right]_0^3 d\theta$

$= \int_0^{2\pi}(-18\cos\theta - 36\sin\theta - 9)\,d\theta = -9(2\pi) = -18\pi$

15. $\nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y & 2y^3z & 3z\end{vmatrix} = -2y^3\mathbf{i} + 0\mathbf{j} - x^2\mathbf{k};\ \mathbf{r}_r\times\mathbf{r}_\theta = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & \sin\theta & 1\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix} = (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k}$ and

$\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = (\nabla\times\mathbf{F})\cdot(\mathbf{r}_r\times\mathbf{r}_\theta)\,dr\,d\theta$ (see Exercise 13 above) $\Rightarrow \iint\limits_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint\limits_R \left(2ry^3\cos\theta - rx^2\right)dr\,d\theta$

$= \int_0^{2\pi}\int_0^1\left(2r^4\sin^3\theta\cos\theta - r^3\cos^2\theta\right)dr\,d\theta = \int_0^{2\pi}\left(\frac{2}{5}\sin^3\theta\cos\theta - \frac{1}{4}\cos^2\theta\right)d\theta = \left[\frac{1}{10}\sin^4\theta - \frac{1}{4}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\right]_0^{2\pi}$

$= -\frac{\pi}{4}$

16. $\nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x-y & y-z & z-x\end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k};\ \mathbf{r}_r\times\mathbf{r}_\theta = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & \sin\theta & -1\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix} = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$ and

$\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = (\nabla\times\mathbf{F})\cdot(\mathbf{r}_r\times\mathbf{r}_\theta)\,dr\,d\theta$ (see Exercise 13 above)

$\Rightarrow \iint\limits_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^5 (r\cos\theta + r\sin\theta + r)\,dr\,d\theta = \int_0^{2\pi}\left[(\cos\theta + \sin\theta + 1)\frac{r^2}{2}\right]_0^5 d\theta = \left(\frac{25}{2}\right)(2\pi) = 25\pi$

17. $\nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3y & 5-2x & z^2-2\end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 5\mathbf{k};\ \mathbf{r}_\phi\times\mathbf{r}_\theta = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \sqrt{3}\cos\phi\cos\theta & \sqrt{3}\cos\phi\sin\theta & -\sqrt{3}\sin\phi\\ -\sqrt{3}\sin\phi\sin\theta & \sqrt{3}\sin\phi\cos\theta & 0\end{vmatrix}$

$= \left(3\sin^2\phi\cos\theta\right)\mathbf{i} + \left(3\sin^2\phi\sin\theta\right)\mathbf{j} + (3\sin\phi\cos\phi)\mathbf{k};\ \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = (\nabla\times\mathbf{F})\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\,d\phi\,d\theta$ (see Exercise 13

above) $\Rightarrow \iint\limits_S \nabla\times\mathbf{F}\cdot\mathbf{n}d\sigma = \int_0^{2\pi}\int_0^{\pi/2}(-15\cos\phi\sin\phi)\,d\phi\,d\theta = \int_0^{2\pi}\left[\frac{15}{2}\cos^2\phi\right]_0^{\pi/2} d\theta = \int_0^{2\pi} -\frac{15}{2}\,d\theta = -15\pi$

18. $\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ y^2&z^2&x\end{vmatrix}=-2z\mathbf{i}-\mathbf{j}-2y\mathbf{k};\ \mathbf{r}_\phi\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ 2\cos\phi\cos\theta&2\cos\phi\sin\theta&-2\sin\phi\\ -2\sin\phi\sin\theta&2\sin\phi\cos\theta&0\end{vmatrix}$

$=\left(4\sin^2\phi\cos\theta\right)\mathbf{i}+\left(4\sin^2\phi\sin\theta\right)\mathbf{j}+\left(4\sin\phi\cos\phi\right)\mathbf{k};\ \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma=(\nabla\times\mathbf{F})\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\,d\phi\,d\theta$ (see Exercise 13 above) $\Rightarrow\iint\limits_S\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma=\iint\limits_R\left(-8z\sin^2\phi\cos\theta-4\sin^2\phi\sin\theta-8y\sin\phi\cos\theta\right)d\phi\,d\theta$

$=\int_0^{2\pi}\int_0^{\pi/2}\left(-16\sin^2\phi\cos\phi\cos\theta-4\sin^2\phi\sin\theta-16\sin^2\phi\sin\theta\cos\theta\right)d\phi\,d\theta$

$=\int_0^{2\pi}\left[-\frac{16}{3}\sin^3\phi\cos\theta-4\left(\frac{\phi}{2}-\frac{\sin 2\phi}{4}\right)(\sin\theta)-16\left(\frac{\phi}{2}-\frac{\sin 2\phi}{4}\right)(\sin\theta\cos\theta)\right]_0^{\pi/2}d\theta$

$=\int_0^{2\pi}\left(-\frac{16}{3}\cos\theta-\pi\sin\theta-4\pi\sin\theta\cos\theta\right)d\theta=\left[-\frac{16}{3}\sin\theta+\pi\cos\theta-2\pi\sin^2\theta\right]_0^{2\pi}=0$

19. We first compute the circulation of $\mathbf{F}=y\mathbf{i}-x\mathbf{j}+x^2\mathbf{k}$ on the curve C given by $\mathbf{r}(t)=(2\cos t)\mathbf{i}+(2\sin t)\mathbf{j}+(3-2\cos^3 t)\mathbf{k}$ for $0\le t\le 2\pi$. On C $\mathbf{F}=(2\sin t)\mathbf{i}-(2\cos t)\mathbf{j}+(4\cos^2 t)\mathbf{k}$, and $d\mathbf{r}=(-2\sin t\,dt)\mathbf{i}+(2\cos t\,dt)\mathbf{j}-(6\sin t\cos^2 t\,dt)\mathbf{k}$.

$\oint\limits_C\mathbf{F}\cdot d\mathbf{r}=\oint\limits_C((2\sin t)\mathbf{i}-(2\cos t)\mathbf{j}+(4\cos^2 t)\mathbf{k})\cdot((-2\sin t\,dt)\mathbf{i}+(2\cos t\,dt)\mathbf{j}-(6\sin t\cos^2 t\,dt)\mathbf{k})$

$=-4\int_0^{2\pi}(\sin^2 t+\cos^2 t+6\sin t\cos^4 t)\,dt=-4\int_0^{2\pi}(1+6\sin t\cos^4 t)\,dt=-8\pi.$

Now we find the flux of $\nabla\times\mathbf{F}$ across the surface S. Note that counterclockwise circulation on C corresponds to inward normals on the cylindrical portion of S and upward normals on the base disk.

For the field $\mathbf{F}=y\mathbf{i}-x\mathbf{j}+x^2\mathbf{k}$, $\nabla\times\mathbf{F}=(-2x)\mathbf{j}-2\mathbf{k}$.

On the base disk, the unit upward normal is $\mathbf{k}$ so $\nabla\times\mathbf{F}\cdot\mathbf{n}=((-2x)\mathbf{j}-2\mathbf{k})\cdot\mathbf{k}=-2$. The integral of the constant -2 over a disk of area 4π is -8π, so to verify Stokes' Theorem in this case it remains to show that the flux across the cylindrical portion of S is 0.

We'll reuse the parameter t and parametrize the cylinder by $\mathbf{s}(t,z)=(2\cos t)\mathbf{i}+(2\sin t)\mathbf{j}+z\mathbf{k}$ with $0\le z\le 3-2\cos^3 t$. An inward unit normal is $(-\cos t)\mathbf{i}+(-\sin t)\mathbf{j}$ and the area element is $d\sigma=2\,dz\,dt$. On the cylinder the field $\nabla\times\mathbf{F}=(-4\cos t)\mathbf{j}-2\mathbf{k}$. Thus

$\iint\limits_S\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma=\iint\limits_S((-4\cos t)\mathbf{j}-2\mathbf{k})\cdot((-\cos t)\mathbf{i}+(-\sin t)\mathbf{j})\,d\sigma$

$=\int_0^{2\pi}\int_0^{3-2\cos^3 t}(4\sin t\cos t)\,2dz\,dt=8\int_0^{2\pi}(\sin t\cos t)(3-2\cos^3 t)\,dt$

$=8\int_0^{2\pi}(3\sin t\cos t-2\sin t\cos^4 t)\,dt=8\left(\frac{3}{2}\sin^2 t+\frac{2}{5}\cos^5 t\right)\Bigg]_0^{2\pi}=0$

20. The boundary C of the paraboloid S given by $z=4-x^2-y^2$ is the circle of radius 2 centered at the origin in the xy-plane. An upward normal on the paraboloid corresponds to counterclockwise circulation around C, so we can parametrize C by $\mathbf{r}(t)=(2\cos t)\mathbf{i}+(2\sin t)\mathbf{j}$ for $0\le t\le 2\pi$, with $d\mathbf{r}=(-2\sin t\,dt)\mathbf{i}+(2\cos t\,dt)\mathbf{j}$. On C the field $\mathbf{F}=2xy\mathbf{i}+x\mathbf{j}+(y+z)\mathbf{k}$ is equal to $(8\cos t\sin t)\mathbf{i}+(2\cos t)\mathbf{j}+(2\sin t)\mathbf{k}$.

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C ((8\cos t\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + (2\sin t)\mathbf{k})\cdot((-2\sin t\,dt)\mathbf{i} + (2\cos t\,dt)\mathbf{j})$$

$$= 4\int_0^{2\pi}(-4\cos t\sin^2 t + \cos^2 t)\,dt = 4\left(-\frac{4}{3}\sin^3 t + \frac{1}{4}\sin 2t + \frac{1}{2}t\right)\Bigg]_0^{2\pi} = 4\pi$$

Now for comparison we integrate $\nabla\times\mathbf{F} = \mathbf{i} + (1-2x)\mathbf{k}$ over the paraboloid S. We can parametrize the paraboloid as $\mathbf{s}(u,t) = (u\cos t)\mathbf{i} + (u\sin t)\mathbf{j} + (4-u^2)\mathbf{k}$ with $0\le u\le 2$ and $0\le t\le 2\pi$. Thus on S the field $\nabla\times\mathbf{F}$ is equal to $\mathbf{i} + (1-2u\cos t)\mathbf{k}$.

First we find the vector area element:

$$\mathbf{s}_u \times \mathbf{s}_t = ((\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (-2u)\mathbf{k}) \times ((-u\sin t)\mathbf{i} + (u\cos t)\mathbf{j} + 0\mathbf{k})$$
$$= (2u^2\cos t)\mathbf{i} + (2u^2\sin t)\mathbf{j} + u\mathbf{k}$$

which is upward, as we require. The integral of the outward component of the field is then

$$\iint_S (\nabla\times\mathbf{F})\cdot(\mathbf{s}_u\times\mathbf{s}_t)\,du\,dt = \int_0^{2\pi}\int_0^2 \left(\mathbf{i} + (1-2u\cos t)\mathbf{k}\right)\cdot\left((2u^2\cos t)\mathbf{i} + (2u^2\sin t)\mathbf{j} + u\mathbf{k}\right)du\,dt$$

$$= \int_0^{2\pi}\int_0^2 u\,du\,dt = \int_0^{2\pi}\left(\frac{u^2}{2}\Bigg]_0^2\right)dt = 4\pi$$

Thus the circulation of **F** around the boundary of the paraboloid is equal to the flux of $\nabla\times\mathbf{F}$ through the paraboloid.

21. (a) $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \text{curl}\,\mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S 0\,d\sigma = 0$

(b) Let $f(x,y,z) = x^2y^2z^3 \Rightarrow \nabla\times\mathbf{F} = \nabla\times\nabla f = \mathbf{0} \Rightarrow \text{curl}\,\mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S 0\,d\sigma = 0$

(c) $\mathbf{F} = \nabla\times(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{0} \Rightarrow \nabla\times\mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S 0\,d\sigma = 0$

(d) $\mathbf{F} = \nabla f \Rightarrow \nabla\times\mathbf{F} = \nabla\times\nabla f = \mathbf{0} \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\;d\sigma = \iint_S 0\,d\sigma = 0$

22. $\mathbf{F} = \nabla f = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2x)\mathbf{i} - \frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2y)\mathbf{j} - \frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2z)\mathbf{k}$

$= -x\left(x^2+y^2+z^2\right)^{-3/2}\mathbf{i} - y\left(x^2+y^2+z^2\right)^{-3/2}\mathbf{j} - z\left(x^2+y^2+z^2\right)^{-3/2}\mathbf{k}$

(a) $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ 0\le t\le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}$

$\Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -x\left(x^2+y^2+z^2\right)^{-3/2}(-a\sin t) - y\left(x^2+y^2+z^2\right)^{-3/2}(a\cos t)$

$= \left(-\frac{a\cos t}{a^3}\right)(-a\sin t) - \left(\frac{a\sin t}{a^3}\right)(a\cos t) = 0 \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = 0$

(b) $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S \nabla\times\nabla f\cdot\mathbf{n}d\sigma = \iint_S \mathbf{0}\cdot\mathbf{n}\,d\sigma = \iint_S 0\,d\sigma = 0$

23. Let $\mathbf{F} = 2y\mathbf{i} + 3z\mathbf{j} - x\mathbf{k} \Rightarrow \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3z & -x \end{vmatrix} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k};\ \mathbf{n} = \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{3} \Rightarrow \nabla\times\mathbf{F}\cdot\mathbf{n} = -2$

$\Rightarrow \oint_C 2y\,dx + 3z\,dy - x\,dz = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S -2\,d\sigma = -2\iint_S d\sigma$, where $\iint_S d\sigma$ is the area of the region enclosed by C on the plane S: $2x + 2y + z = 2$

24. $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$

25. Suppose $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ exists such that $\nabla\times\mathbf{F} = \left(\frac{\partial p}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Then $\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) = \frac{\partial}{\partial x}(x) \Rightarrow \frac{\partial^2 P}{\partial x\partial y} - \frac{\partial^2 N}{\partial x\partial y} = 1$.

Likewise, $\frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial y}(y) \Rightarrow \frac{\partial^2 M}{\partial y\partial z} - \frac{\partial^2 P}{\partial y\partial x} = 1$ and $\frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{\partial}{\partial z}(z) \Rightarrow \frac{\partial^2 N}{\partial z\partial x} - \frac{\partial^2 M}{\partial z\partial y} = 1$.

Summing the calculated equations $\Rightarrow \left(\frac{\partial^2 P}{\partial x\partial y} - \frac{\partial^2 P}{\partial y\partial x}\right) + \left(\frac{\partial^2 N}{\partial z\partial x} - \frac{\partial^2 N}{\partial x\partial z}\right) + \left(\frac{\partial^2 M}{\partial y\partial z} - \frac{\partial^2 M}{\partial z\partial y}\right) = 3$ or $0 = 3$

(assuming the second mixed partials are equal). This result is a contradiction, so there is no field $\mathbf{F}$ such that curl $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

26. Yes: If $\nabla\times\mathbf{F} = \mathbf{0}$, then the circulation of $\mathbf{F}$ around the boundary C of any oriented surface S in the domain of $\mathbf{F}$ is zero. The reason is this: By Stokes' theorem, circulation $= \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S \mathbf{0}\cdot\mathbf{n}\,d\sigma = 0$.

27. $r = \sqrt{x^2+y^2} \Rightarrow r^4 = \left(x^2+y^2\right)^2 \Rightarrow \mathbf{F} = \nabla\left(r^4\right) = 4x\left(x^2+y^2\right)\mathbf{i} + 4y\left(x^2+y^2\right)\mathbf{j} = M\mathbf{i} + N\mathbf{j}$

$\Rightarrow \oint_C \nabla\left(r^4\right)\cdot\mathbf{n}\,ds = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy$

$= \iint_R \left[4\left(x^2+y^2\right) + 8x^2 + 4\left(x^2+y^2\right) + 8y^2\right]dA = \iint_R 16\left(x^2+y^2\right)dA = 16\iint_R x^2\,dA + 16\iint_R y^2\,dA$

$= 16I_y + 16I_x$.

28. $\frac{\partial P}{\partial y} = 0, \frac{\partial N}{\partial z} = 0, \frac{\partial M}{\partial z} = 0, \frac{\partial P}{\partial x} = 0, \frac{\partial N}{\partial x} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}, \frac{\partial M}{\partial y} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2} \Rightarrow$ curl $\mathbf{F} = \left[\frac{y^2-x^2}{\left(x^2+y^2\right)^2} - \frac{y^2-x^2}{\left(x^2+y^2\right)^2}\right]\mathbf{k} = \mathbf{0}$.

However, $x^2 + y^2 = 1 \Rightarrow \mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$

$\Rightarrow \mathbf{F} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \sin^2 t + \cos^2 t = 1 \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_0^{2\pi} 1\,dt = 2\pi$ which is not zero.

16.8 THE DIVERGENCE THEOREM AND A UNIFIED THEORY

1. $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{\sqrt{x^2+y^2}} \Rightarrow \text{div } \mathbf{F} = \frac{xy-xy}{(x^2+y^2)^{3/2}} = 0$

2. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} \Rightarrow \text{div } \mathbf{F} = 1+1 = 2$

3. $\mathbf{F} = -\frac{GM\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{(x^2+y^2+z^2)^{3/2}}$

$$\Rightarrow \text{div } \mathbf{F} = -GM\left[\frac{\left(x^2+y^2+z^2\right)^{3/2}-3x^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^3}\right] - GM\left[\frac{\left(x^2+y^2+z^2\right)^{3/2}-3y^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^3}\right]$$
$$-GM\left[\frac{\left(x^2+y^2+z^2\right)^{3/2}-3z^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^3}\right] = -GM\left[\frac{3\left(x^2+y^2+z^2\right)^2-3\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{7/2}}\right] = 0$$

4. $z = a^2 - r^2$ in cylindrical coordinates $\Rightarrow z = a^2 - \left(x^2+y^2\right) \Rightarrow \mathbf{v} = \left(a^2 - x^2 - y^2\right)\mathbf{k} \Rightarrow \text{div } \mathbf{v} = 0$

5. $\frac{\partial}{\partial x}(y-x) = -1, \frac{\partial}{\partial y}(z-y) = -1, \frac{\partial}{\partial z}(y-x) = 0 \Rightarrow \nabla\cdot\mathbf{F} = -2 \Rightarrow \text{Flux} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} -2\,dx\,dy\,dz = -2\left(2^3\right) = -16$

6. $\frac{\partial}{\partial x}\left(x^2\right) = 2x, \frac{\partial}{\partial y}\left(y^2\right) = 2y, \frac{\partial}{\partial z}\left(z^2\right) = 2z \Rightarrow \nabla\cdot\mathbf{F} = 2x+2y+2z$

(a) $\text{Flux} = \int_0^1\int_0^1\int_0^1 (2x+2y+2z)\,dx\,dy\,dz = \int_0^1\int_0^1\left[x^2+2x(y+z)\right]_0^1 dy\,dz = \int_0^1\int_0^1 (1+2y+2z)\,dy\,dz$

$= \int_0^1\left[y(1+2z)+y^2\right]_0^1 dz = \int_0^1 (2+2z)\,dz = \left[2z+z^2\right]_0^1 = 3$

(b) $\text{Flux} = \int_{-1}^1\int_{-1}^1\int_{-1}^1 (2x+2y+2z)\,dx\,dy\,dz = \int_{-1}^1\int_{-1}^1\left[x^2+2x(y+z)\right]_{-1}^1 dy\,dz = \int_{-1}^1\int_{-1}^1 (4y+4z)\,dy\,dz$

$= \int_{-1}^1\left[2y^2+4yz\right]_{-1}^1 dz = \int_{-1}^1 8z\,dz = \left[4z^2\right]_{-1}^1 = 0$

(c) In cylindrical coordinates, $\text{Flux} = \iiint_D (2x+2y+2z)dx\,dy\,dz$

$= \int_0^1\int_0^{2\pi}\int_0^2 (2r\cos\theta + 2r\sin\theta + 2z)\,r\,dr\,d\theta\,dz = \int_0^1\int_0^{2\pi}\left[\frac{2}{3}r^3\cos\theta + \frac{2}{3}r^3\sin\theta + zr^2\right]_0^2 d\theta\,dz$

$= \int_0^1\int_0^{2\pi}\left(\frac{16}{3}\cos\theta + \frac{16}{3}\sin\theta + 4z\right)d\theta\,dz = \int_0^1\left[\frac{16}{3}\sin\theta - \frac{16}{3}\cos\theta + 4z\theta\right]_0^{2\pi} dz = \int_0^1 8\pi z\,dz = \left[4\pi z^2\right]_0^1 = 4\pi$

7. $\frac{\partial}{\partial x}(y) = 0, \frac{\partial}{\partial y}(xy) = x, \frac{\partial}{\partial z}(-z) = -1 \Rightarrow \nabla\cdot\mathbf{F} = x-1;\ z = x^2+y^2 \Rightarrow z = r^2$ in cylindrical coordinates

$\Rightarrow \text{Flux} = \iiint_D (x-1)dz\,dy\,dx = \int_0^{2\pi}\int_0^2\int_0^{r^2} (r\cos\theta - 1)\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2\left(r^3\cos\theta - r^2\right)r\,dr\,d\theta$

$= \int_0^{2\pi}\left[\frac{r^5}{5}\cos\theta - \frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi}\left(\frac{32}{5}\cos\theta - 4\right)d\theta = \left[\frac{32}{5}\sin\theta - 4\theta\right]_0^{2\pi} = -8\pi$

8. $\frac{\partial}{\partial x}\left(x^2\right)=2x,\ \frac{\partial}{\partial y}(xz)=0,\ \frac{\partial}{\partial z}(3z)=3 \Rightarrow \nabla\cdot\mathbf{F}=2x+3 \Rightarrow \text{Flux}=\iiint\limits_D (2x+3)\,dV$

$=\int_0^{2\pi}\int_0^{\pi}\int_0^{2}(2\rho\sin\phi\cos\theta+3)\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^4}{2}\sin\phi\cos\theta+\rho^3\right]_0^2\sin\phi\,d\phi\,d\theta$

$=\int_0^{2\pi}\int_0^{\pi}(8\sin\phi\cos\theta+8)\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}\left[8\left(\frac{\phi}{2}-\frac{\sin 2\phi}{4}\right)\cos\theta-8\cos\phi\right]_0^{\pi}d\theta=\int_0^{2\pi}(4\pi\cos\theta+16)\,d\theta=32\pi$

9. $\frac{\partial}{\partial x}\left(x^2\right)=2x,\ \frac{\partial}{\partial y}(-2xy)=-2x,\ \frac{\partial}{\partial z}(3xz)=3x \Rightarrow \text{Flux}=\iiint\limits_D 3x\,dx\,dy\,dz$

$=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2}(3\rho\sin\phi\cos\theta)\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta=\int_0^{\pi/2}\int_0^{\pi/2}12\sin^2\phi\cos\theta\,d\phi\,d\theta=\int_0^{\pi/2}3\pi\cos\theta\,d\theta=3\pi$

10. $\frac{\partial}{\partial x}\left(6x^2+2xy\right)=12x+2y,\ \frac{\partial}{\partial y}\left(2y+x^2z\right)=2,\ \frac{\partial}{\partial z}\left(4x^2y^3\right)=0 \Rightarrow \nabla\cdot\mathbf{F}=12x+2y+2$

$\Rightarrow \text{Flux}=\iiint\limits_D(12x+2y+2)dV=\int_0^3\int_0^{\pi/2}\int_0^2(12r\cos\theta+2r\sin\theta+2)\,r\,dr\,d\theta\,dz$

$=\int_0^3\int_0^{\pi/2}\left(32\cos\theta+\frac{16}{3}\sin\theta+4\right)d\theta\,dz=\int_0^3\left(32+2\pi+\frac{16}{3}\right)dz=112+6\pi$

11. $\frac{\partial}{\partial x}(2xz)=2z,\ \frac{\partial}{\partial y}(-xy)=-x,\ \frac{\partial}{\partial z}\left(-z^2\right)=-2z \Rightarrow \nabla\cdot\mathbf{F}=-x \Rightarrow \text{Flux}=\iiint\limits_D -x\,dV$

$=\int_0^2\int_0^{\sqrt{16-4x^2}}\int_0^{4-y}(-x)\,dz\,dy\,dx=\int_0^2\int_0^{\sqrt{16-4x^2}}(xy-4x)\,dy\,dx=\int_0^2\left[\frac{1}{2}x\left(16-4x^2\right)-4x\sqrt{16-4x^2}\right]dx$

$=\left[4x^2-\frac{1}{2}x^4+\frac{1}{3}\left(16-4x^2\right)^{3/2}\right]_0^2=-\frac{40}{3}$

12. $\frac{\partial}{\partial x}\left(x^3\right)=3x^2,\ \frac{\partial}{\partial y}\left(y^3\right)=3y^2,\ \frac{\partial}{\partial z}\left(z^3\right)=3z^2 \Rightarrow \nabla\cdot\mathbf{F}=3x^2+3y^2+3z^2 \Rightarrow \text{Flux}=\iiint\limits_D 3\left(x^2+y^2+z^2\right)dV$

$=3\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\rho^2\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta=3\int_0^{2\pi}\int_0^{\pi}\frac{a^5}{5}\sin\phi\,d\phi\,d\theta=3\int_0^{2\pi}\frac{2a^5}{5}\,d\theta=\frac{12\pi a^5}{5}$

13. Let $\rho=\sqrt{x^2+y^2+z^2}$. Then $\frac{\partial p}{\partial x}=\frac{x}{\rho},\ \frac{\partial\rho}{\partial y}=\frac{y}{\rho},\ \frac{\partial p}{\partial z}=\frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x}(\rho x)=\left(\frac{\partial\rho}{\partial x}\right)x+\rho=\frac{x^2}{\rho}+\rho,$

$\frac{\partial}{\partial y}(\rho y)=\left(\frac{\partial\rho}{\partial y}\right)y+\rho=\frac{y^2}{\rho}+\rho,\ \frac{\partial}{\partial z}(\rho z)=\left(\frac{\partial\rho}{\partial z}\right)z+\rho=\frac{z^2}{\rho}+\rho \Rightarrow \nabla\cdot\mathbf{F}=\frac{x^2+y^2+z^2}{\rho}+3\rho=4\rho,$ since

$\rho=\sqrt{x^2+y^2+z^2} \Rightarrow \text{Flux}=\iiint\limits_D 4\rho dV=\int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}}(4\rho)\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi}3\sin\phi\,d\phi\,d\theta$

$=\int_0^{2\pi}6\,d\theta=12\pi$

14. Let $\rho=\sqrt{x^2+y^2+z^2}$. Then $\frac{\partial\rho}{\partial x}=\frac{x}{\rho}, \frac{\partial\rho}{\partial y}=\frac{y}{\rho}, \frac{\partial\rho}{\partial z}=\frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x}\left(\frac{x}{\rho}\right)=\frac{1}{\rho}-\left(\frac{x}{\rho^2}\right)\frac{\partial\rho}{\partial x}=\frac{1}{\rho}-\frac{x^2}{\rho^3}$. Similarly, $\frac{\partial}{\partial y}\left(\frac{y}{\rho}\right)=\frac{1}{\rho}-\frac{y^2}{\rho^3}$ and $\frac{\partial}{\partial z}\left(\frac{y}{\rho}\right)=\frac{1}{\rho}-\frac{z^2}{\rho^3} \Rightarrow \nabla\cdot\mathbf{F}=\frac{3}{\rho}-\frac{x^2+y^2+z^2}{\rho^3}=\frac{2}{\rho}$

$\Rightarrow$ Flux $=\iiint\limits_D \frac{2}{\rho}\,dV=\int_0^{2\pi}\int_0^{\pi}\int_1^2\left(\frac{2}{\rho}\right)\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi}3\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}6\,d\theta=12\pi$

15. $\frac{\partial}{\partial x}\left(5x^3+12xy^2\right)=15x^2+12y^2, \frac{\partial}{\partial y}\left(y^3+e^y\sin z\right)=3y^2+e^y\sin z, \frac{\partial}{\partial z}\left(5z^3+e^y\cos z\right)=15z^2-e^y\sin z$

$\Rightarrow \nabla\cdot\mathbf{F}=15x^2+15y^2+15z^2=15\rho^2 \Rightarrow$ Flux $=\iiint\limits_D 15\rho^2 dV=\int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}}\left(15\rho^2\right)\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta$

$=\int_0^{2\pi}\int_0^{\pi}\left(12\sqrt{2}-3\right)\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}\left(24\sqrt{2}-6\right)d\theta=\left(48\sqrt{2}-12\right)\pi$

16. $\frac{\partial}{\partial x}\left[\ln\left(x^2+y^2\right)\right]=\frac{2x}{x^2+y^2}, \frac{\partial}{\partial y}\left(-\frac{2z}{x}\tan^{-1}\frac{y}{x}\right)=\left(-\frac{2z}{x}\right)\left[\frac{\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2}\right]=-\frac{2z}{x^2+y^2}, \frac{\partial}{\partial z}\left(z\sqrt{x^2+y^2}\right)=\sqrt{x^2+y^2}$

$\Rightarrow \nabla\cdot\mathbf{F}=\frac{2x}{x^2+y^2}-\frac{2z}{x^2+y^2}+\sqrt{x^2+y^2} \Rightarrow$ Flux $=\iiint\limits_D\left(\frac{2x}{x^2+y^2}-\frac{2z}{x^2+y^2}+\sqrt{x^2+y^2}\right)dz\,dy\,dx$

$=\int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-1}^{2}\left(\frac{2r\cos\theta}{r^2}-\frac{2z}{r^2}+r\right)dz\,r\,dr\,d\theta=\int_0^{2\pi}\int_1^{\sqrt{2}}\left(6\cos\theta-\frac{3}{r}+3r^2\right)dr\,d\theta$

$=\int_0^{2\pi}\left[6\left(\sqrt{2}-1\right)\cos\theta-3\ln\sqrt{2}+2\sqrt{2}-1\right]d\theta=2\pi\left(-\frac{3}{2}\ln 2+2\sqrt{2}-1\right)$

17. (a) $\frac{\partial}{\partial x}(x)=1, \frac{\partial}{\partial y}(y)=1, \frac{\partial}{\partial z}(z)=1 \Rightarrow \nabla\cdot\mathbf{F}=3 \Rightarrow$ Flux $=\iiint\limits_D 3\,dV=3\iiint\limits_D dV=3$ (Volume of the solid)

(b) If $\mathbf{F}$ is orthogonal to $\mathbf{n}$ at every point of S, then $\mathbf{F}\cdot\mathbf{n}=0$ everywhere $\Rightarrow$ Flux $=\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,d\sigma=0$. But the flux is 3 (Volume of the solid) $\neq 0$, so $\mathbf{F}$ is not orthogonal to $\mathbf{n}$ at every point.

18. Yes, the outward flux through the top is 5. The reason is this: Since $\nabla\cdot\mathbf{F}=\nabla\cdot(x\mathbf{i}-2y\mathbf{j}+(z+3)\mathbf{k}$ $=1-2+1=0$, the outward flux across the closed cubelike surface is 0 by the Divergence Theorem. The flux across the top is therefore the negative of the flux across the sides and base. Routine calculations show that the sum of these latter fluxes is -5. (The flux across the sides that lie in the xz-plane and the yz-plane are 0, while the flux across the xy-plane is -3.) Therefore the flux across the top is 5.

19. For the field $\mathbf{F}=(y\cos 2x)\mathbf{i}+(y^2\sin 2x)\mathbf{j}+(x^2y+x)\mathbf{k}$, $\nabla\cdot\mathbf{F}=-2y\sin 2x+2y\sin 2x+1=1$. If $\mathbf{F}$ were the curl of a field $\mathbf{A}$ whose component functions have continuous second partial derivatives, then we would have div $\mathbf{F}=$ div(curl $\mathbf{A}$) $=\nabla\cdot(\nabla\times\mathbf{A})=0$. Since div $\mathbf{F}=1$, $\mathbf{F}$ is not the curl of such a field.

20. From the Divergence Theorem, $\iint\limits_S \nabla f\cdot\mathbf{n}\,d\sigma=\iiint\limits_D \nabla\cdot\nabla f\,dV=\iiint\limits_D\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)dV$. Now,

$f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}=\frac{1}{2}\ln\left(x^2+y^2+z^2\right)\Rightarrow\frac{\partial f}{\partial x}=\frac{x}{x^2+y^2+z^2}, \frac{\partial f}{\partial y}=\frac{y}{x^2+y^2+z^2}, \frac{\partial f}{\partial z}=\frac{z}{x^2+y^2+z^2}$

$\Rightarrow\frac{\partial^2 f}{\partial x^2}=\frac{-x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2}, \frac{\partial^2 f}{\partial y^2}=\frac{x^2-y^2+z^2}{\left(x^2+y^2+z^2\right)^2}, \frac{\partial^2 f}{\partial z^2}=\frac{x^2+y^2-z^2}{\left(x^2+y^2+z^2\right)^2}, \Rightarrow \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=\frac{x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2}=\frac{1}{x^2+y^2+z^2}$

$$\Rightarrow \iint_S \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \frac{dV}{x^2+y^2+z^2} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \frac{\rho^2 \sin\phi}{\rho^2}\, d\rho\, d\phi\, d\theta = \int_0^{\pi/2}\int_0^{\pi/2} a \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{\pi/2} \left[-a\cos\phi\right]_0^{\pi/2} d\theta = \int_0^{\pi/2} a\, d\theta = \frac{\pi a}{2}$$

21. The integral's value never exceeds the surface area of S. Since $|\mathbf{F}| \le 1$, we have $|\mathbf{F}\cdot\mathbf{n}| = |\mathbf{F}||\mathbf{n}| \le (1)(1) = 1$ and

$$\begin{aligned} \iiint_D \nabla\cdot\mathbf{F}\, d\sigma &= \iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma && \text{[Divergence Theorem]} \\ &\le \iint_S |\mathbf{F}\cdot\mathbf{n}|\, d\sigma && \text{[A property of integrals]} \\ &\le \iint_S (1)\, d\sigma && \left[|\mathbf{F}\cdot\mathbf{n}| \le 1\right] \\ &= \text{Area of } S. \end{aligned}$$

22. $\nabla\cdot\mathbf{F} = -2x - 4y - 6z + 12 \Rightarrow \text{Flux} = \int_0^a\int_0^b\int_0^1 (-2x-4y-6z+12)\, dz\, dy\, dx = \int_0^a\int_0^b (-2x-4y+9)\, dy\, dx =$
$\int_0^a \left(-2xb - 2b^2 + 9b\right) dx = -a^2b - 2ab^2 + 9ab = ab(-a-2b+9) = f(a,b);\ \frac{\partial f}{\partial a} = -2ab - 2b^2 + 9b$ and
$\frac{\partial f}{\partial b} = -a^2 - 4ab + 9a$ so that $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0 \Rightarrow b(-2a-2b+9) = 0$ and $a(-a-4b+9) = 0 \Rightarrow b = 0$ or
$-2a - 2b + 9 = 0$, and $a = 0$ or $-a - 4b + 9 = 0$. Now $b = 0$ or $a = 0 \Rightarrow \text{Flux} = 0; -2a - 2b + 9 = 0$ and
$-a - 4b + 9 = 0 \Rightarrow 3a - 9 = 0 \Rightarrow a = 3 \Rightarrow b = \frac{3}{2}$ so that $f\left(3, \frac{3}{2}\right) = \frac{27}{2}$ is the maximum flux.

23. By the Divergence Theorem, the net outward flux of the field $\mathbf{F} = xy\mathbf{i} + (\sin xz + y^2)\mathbf{j} + (e^{xy^2} + x)\mathbf{k}$ over the surface S will be equal to the integral of $\nabla\cdot\mathbf{F} = y + 2y = 3y$ over the region D bounded by S. We will integrate using the area in the zx-plane bounded by $z = 0$ and $z = 1 - x^2$ as the base. The y height at any point (x, z) will be $2 - z$. Thus the integral of div $\mathbf{F}$ over D is

$$\int_{-1}^1\int_0^{1-x^2}\int_0^{2-z} 3y\, dy\, dz\, dx = \int_{-1}^1\int_0^{1-x^2} \left.\frac{3}{2}y^2\right]_0^{2-z} dz\, dx = \int_{-1}^1\int_0^{1-x^2} \frac{3}{2}(2-z)^2\, dz\, dx$$

$$= \int_{-1}^1 \left.-\frac{1}{2}(2-z)^3\right]_0^{1-x^2} dx = \int_{-1}^1 4 - \frac{1}{2}(x^2+1)^3\, dx = \left.\left(\frac{7}{2}x - \frac{1}{3}x^3 - \frac{3}{10}x^5 - \frac{1}{14}x^7\right)\right]_{-1}^1 = \frac{184}{35}$$

24. The field $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) / \left(x^2 + y^2 + z^2\right)^{3/2}$ is discussed in Example 5 in Section 16.8, where we show that the flux of $\mathbf{F}$ across any closed surface enclosing the origin is 4π. Note that the divergence of $\mathbf{F}$ is not defined at the origin, so we need an argument like that shown in Example 5.

25. (a) $\operatorname{div}(g\mathbf{F}) = \nabla\cdot g\mathbf{F} = \frac{\partial}{\partial x}(gM) + \frac{\partial}{\partial y}(gN) + \frac{\partial}{\partial z}(gP) = \left(g\frac{\partial M}{\partial x} + M\frac{\partial g}{\partial x}\right) + \left(g\frac{\partial N}{\partial y} + N\frac{\partial g}{\partial y}\right) + \left(g\frac{\partial P}{\partial z} + P\frac{\partial g}{\partial z}\right)$

$= \left(M\frac{\partial g}{\partial x} + N\frac{\partial g}{\partial y} + P\frac{\partial g}{\partial z}\right) + g\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) = g\nabla\cdot\mathbf{F} + \nabla g\cdot\mathbf{F}$

(b) $\nabla\times(g\mathbf{F})=\left[\frac{\partial}{\partial y}(gP)-\frac{\partial}{\partial z}(gN)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(gM)-\frac{\partial}{\partial x}(gP)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(gN)-\frac{\partial}{\partial y}(gM)\right]\mathbf{k}$

$=\left(P\frac{\partial g}{\partial y}+g\frac{\partial P}{\partial y}-N\frac{\partial g}{\partial z}-g\frac{\partial N}{\partial z}\right)\mathbf{i}+\left(M\frac{\partial g}{\partial z}+g\frac{\partial M}{\partial z}-P\frac{\partial g}{\partial x}-g\frac{\partial P}{\partial x}\right)\mathbf{j}+\left(N\frac{\partial g}{\partial x}+g\frac{\partial N}{\partial x}-M\frac{\partial g}{\partial y}-g\frac{\partial M}{\partial y}\right)\mathbf{k}$

$=\left(P\frac{\partial g}{\partial y}-N\frac{\partial g}{\partial z}\right)\mathbf{i}+\left(g\frac{\partial P}{\partial y}-g\frac{\partial N}{\partial z}\right)\mathbf{i}+\left(M\frac{\partial g}{\partial z}-P\frac{\partial g}{\partial x}\right)\mathbf{j}+\left(g\frac{\partial M}{\partial z}-g\frac{\partial P}{\partial x}\right)\mathbf{j}+\left(N\frac{\partial g}{\partial x}-M\frac{\partial g}{\partial y}\right)\mathbf{k}+\left(g\frac{\partial N}{\partial x}-g\frac{\partial M}{\partial y}\right)\mathbf{k}$

$=g\nabla\times\mathbf{F}+\nabla g\times\mathbf{F}$

26. (a) Let $\mathbf{F}_1=M_1\mathbf{i}+N_1\mathbf{j}+P_1\mathbf{k}$ and $\mathbf{F}_2=M_2\mathbf{i}+N_2\mathbf{j}+P_2\mathbf{k}$

$\Rightarrow a\mathbf{F}_1+b\mathbf{F}_2=(aM_1+bM_2)\mathbf{i}+(aN_1+bN_2)\mathbf{j}+(aP_1+bP_2)\mathbf{k}$

$\Rightarrow\nabla\cdot(a\mathbf{F}_1+b\mathbf{F}_2)=\left(a\frac{\partial M_1}{\partial x}+b\frac{\partial M_2}{\partial x}\right)+\left(a\frac{\partial N_1}{\partial y}+b\frac{\partial N_2}{\partial y}\right)+\left(a\frac{\partial P_1}{\partial z}+b\frac{\partial P_2}{\partial z}\right)$

$=a\left(\frac{\partial M_1}{\partial x}+\frac{\partial N_1}{\partial y}+\frac{\partial P_1}{\partial z}\right)+b\left(\frac{\partial M_2}{\partial x}+\frac{\partial N_2}{\partial y}+\frac{\partial P_2}{\partial z}\right)=a(\nabla\cdot\mathbf{F}_1)+b(\nabla\cdot\mathbf{F}_2)$

(b) Define $\mathbf{F}_1$ and $\mathbf{F}_2$ as in part a

$\Rightarrow\nabla\times(a\mathbf{F}_1+b\mathbf{F}_2)=\left[\left(a\frac{\partial P_1}{\partial y}+b\frac{\partial P_2}{\partial y}\right)-\left(a\frac{\partial N_1}{\partial z}+b\frac{\partial N_2}{\partial z}\right)\right]\mathbf{i}+\left[\left(a\frac{\partial M_1}{\partial z}+b\frac{\partial M_2}{\partial z}\right)-\left(a\frac{\partial P_1}{\partial x}+b\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j}$

$+\left[\left(a\frac{\partial N_1}{\partial x}+b\frac{\partial N_2}{\partial x}\right)-\left(a\frac{\partial M_1}{\partial y}+b\frac{\partial M_2}{\partial y}\right)\right]\mathbf{k}$

$=a\left[\left(\frac{\partial P_1}{\partial y}-\frac{\partial N_1}{\partial z}\right)\mathbf{i}+\left(\frac{\partial M_1}{\partial z}-\frac{\partial P_1}{\partial x}\right)\mathbf{j}+\left(\frac{\partial N_1}{\partial x}-\frac{\partial M_1}{\partial y}\right)\mathbf{k}\right]+b\left[\left(\frac{\partial P_2}{\partial y}-\frac{\partial N_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial M_2}{\partial z}-\frac{\partial P_2}{\partial x}\right)\mathbf{j}+\left(\frac{\partial N_2}{\partial x}-\frac{\partial M_2}{\partial y}\right)\mathbf{k}\right]$

$=a\nabla\times\mathbf{F}_1+b\nabla\times\mathbf{F}_2$

(c) $\mathbf{F}_1\times\mathbf{F}_2=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\M_1&N_1&P_1\\M_2&N_2&P_2\end{vmatrix}=(N_1P_2-P_1N_2)\mathbf{i}-(M_1P_2-P_1M_2)\mathbf{j}+(M_1N_2-N_1M_2)\mathbf{k}$

$\Rightarrow\nabla\cdot(\mathbf{F}_1\times\mathbf{F}_2)=\nabla\cdot\left[(N_1P_2-P_1N_2)\mathbf{i}-(M_1P_2-P_1M_2)\mathbf{j}+(M_1N_2-N_1M_2)\mathbf{k}\right]$

$=\frac{\partial}{\partial x}(N_1P_2-P_1N_2)-\frac{\partial}{\partial y}(M_1P_2-P_1M_2)+\frac{\partial}{\partial z}(M_1N_2-N_1M_2)$

$=\left(P_2\frac{\partial N_1}{\partial x}+N_1\frac{\partial P_2}{\partial x}-N_2\frac{\partial P_1}{\partial x}-P_1\frac{\partial N_2}{\partial x}\right)-\left(M_1\frac{\partial P_2}{\partial y}+P_2\frac{\partial M_1}{\partial y}-P_1\frac{\partial M_2}{\partial y}-M_2\frac{\partial P_1}{\partial y}\right)$

$+\left(M_1\frac{\partial N_2}{\partial z}+N_2\frac{\partial M_1}{\partial z}-N_1\frac{\partial M_2}{\partial z}-M_2\frac{\partial N_1}{\partial z}\right)$

$=M_2\left(\frac{\partial P_1}{\partial y}-\frac{\partial N_1}{\partial z}\right)+N_2\left(\frac{\partial M_1}{\partial z}-\frac{\partial P_1}{\partial x}\right)+P_2\left(\frac{\partial N_1}{\partial x}-\frac{\partial M_1}{\partial y}\right)+M_1\left(\frac{\partial N_2}{\partial z}-\frac{\partial P_2}{\partial y}\right)+N_1\left(\frac{\partial P_2}{\partial x}-\frac{\partial M_2}{\partial z}\right)+P_1\left(\frac{\partial M_2}{\partial y}-\frac{\partial N_2}{\partial x}\right)$

$=\mathbf{F}_2\cdot\nabla\times\mathbf{F}_1-\mathbf{F}_1\cdot\nabla\times\mathbf{F}_2$

27. Let $\mathbf{F}_1=M_1\mathbf{i}+N_1\mathbf{j}+P_1\mathbf{k}$ and $\mathbf{F}_2=M_2\mathbf{i}+N_2\mathbf{j}+P_2\mathbf{k}$.

(a) $\mathbf{F}_1\times\mathbf{F}_2=(N_1P_2-P_1N_2)\mathbf{i}+(P_1M_2-M_1P_2)\mathbf{j}+(M_1N_2-N_1M_2)\mathbf{k}$

$\Rightarrow\nabla\times(\mathbf{F}_1\times\mathbf{F}_2)=\left[\frac{\partial}{\partial y}(M_1N_2-N_1M_2)-\frac{\partial}{\partial z}(P_1M_2-M_1P_2)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(N_1P_2-P_1N_2)-\frac{\partial}{\partial x}(M_1N_2-N_1M_2)\right]\mathbf{j}$

$+\left[\frac{\partial}{\partial x}(P_1M_2-M_1P_2)-\frac{\partial}{\partial y}(N_1P_2-P_1N_2)\right]\mathbf{k}$

consider the **i**-component only: $\frac{\partial}{\partial y}(M_1N_2-N_1M_2)-\frac{\partial}{\partial z}(P_1M_2-M_1P_2)$

$=N_2\frac{\partial M_1}{\partial y}+M_1\frac{\partial N_2}{\partial y}-M_2\frac{\partial N_1}{\partial y}-N_1\frac{\partial M_2}{\partial y}-M_2\frac{\partial P_1}{\partial z}-P_1\frac{\partial M_2}{\partial z}+P_2\frac{\partial M_1}{\partial z}+M_1\frac{\partial P_2}{\partial z}$

$=\left(N_2\frac{\partial M_1}{\partial y}+P_2\frac{\partial M_1}{\partial z}\right)-\left(N_1\frac{\partial M_2}{\partial y}+P_1\frac{\partial M_2}{\partial z}\right)+\left(\frac{\partial N_2}{\partial y}+\frac{\partial P_2}{\partial z}\right)M_1-\left(\frac{\partial N_1}{\partial y}+\frac{\partial P_1}{\partial z}\right)M_2$

$$=\left(M_2\frac{\partial M_1}{\partial x}+N_2\frac{\partial M_1}{\partial y}+P_2\frac{\partial M_1}{\partial z}\right)-\left(M_1\frac{\partial M_2}{\partial x}+N_1\frac{\partial M_2}{\partial y}+P_1\frac{\partial M_2}{\partial z}\right)+\left(\frac{\partial M_2}{\partial x}+\frac{\partial N_2}{\partial y}+\frac{\partial P_2}{\partial z}\right)M_1-\left(\frac{\partial M_1}{\partial x}+\frac{\partial N_1}{\partial y}+\frac{\partial P_1}{\partial z}\right)M_2$$

Now, **i**-comp of $(\mathbf{F}_2\cdot\nabla)\mathbf{F}_1=\left(M_2\frac{\partial}{\partial x}+N_2\frac{\partial}{\partial y}+P_2\frac{\partial}{\partial z}\right)M_1=\left(M_2\frac{\partial M_1}{\partial x}+N_2\frac{\partial M_1}{\partial y}+P_2\frac{\partial M_1}{\partial z}\right)$;

likewise, **i**-comp of $(\mathbf{F}_1\cdot\nabla)\mathbf{F}_2=\left(M_1\frac{\partial M_2}{\partial x}+N_1\frac{\partial M_2}{\partial y}+P_1\frac{\partial M_2}{\partial z}\right)$;

i comp of $(\nabla\cdot\mathbf{F}_2)\mathbf{F}_1=\left(\frac{\partial M_2}{\partial x}+\frac{\partial N_2}{\partial y}+\frac{\partial P_2}{\partial z}\right)M_1$ and **i**-comp of $(\nabla\cdot\mathbf{F}_1)\mathbf{F}_2=\left(\frac{\partial M_1}{\partial x}+\frac{\partial N_1}{\partial y}+\frac{\partial P_1}{\partial z}\right)M_2$.

Similar results hold for the **j** and **k** components of $\nabla\times(\mathbf{F}_1\times\mathbf{F}_2)$. In summary, since the corresponding components are equal, we have the result $\nabla\times(\mathbf{F}_1\times\mathbf{F}_2)=(\mathbf{F}_2\cdot\nabla)\mathbf{F}_1-(\mathbf{F}_1\cdot\nabla)\mathbf{F}_2+(\nabla\cdot\mathbf{F}_2)\mathbf{F}_1-(\nabla\cdot\mathbf{F}_1)\mathbf{F}_2$

(b) Here again we consider only the **i**-component of each expression. Thus, the **i**-comp of $\nabla(\mathbf{F}_1\cdot\mathbf{F}_2)$

$$=\frac{\partial}{\partial x}(M_1M_2+N_1N_2+P_1P_2)=\left(M_1\frac{\partial M_2}{\partial x}+M_2\frac{\partial M_1}{\partial x}+N_1\frac{\partial N_2}{\partial x}+N_2\frac{\partial N_1}{\partial x}+P_1\frac{\partial P_2}{\partial x}+P_2\frac{\partial P_1}{\partial x}\right)$$

i-comp of $(\mathbf{F}_1\cdot\nabla)\mathbf{F}_2=\left(M_1\frac{\partial M_2}{\partial x}+N_1\frac{\partial M_2}{\partial y}+P_1\frac{\partial M_2}{\partial z}\right)$,

i-comp of $(\mathbf{F}_2\cdot\nabla)\mathbf{F}_1=\left(M_2\frac{\partial M_1}{\partial x}+N_2\frac{\partial M_1}{\partial y}+P_2\frac{\partial M_1}{\partial z}\right)$,

i-comp of $\mathbf{F}_1\times(\nabla\times\mathbf{F}_2)=N_1\left(\frac{\partial N_2}{\partial x}-\frac{\partial M_2}{\partial y}\right)-P_1\left(\frac{\partial M_2}{\partial z}-\frac{\partial P_2}{\partial x}\right)$, and

i-comp of $\mathbf{F}_2\times(\nabla\times\mathbf{F}_1)=N_2\left(\frac{\partial N_1}{\partial x}-\frac{\partial M_1}{\partial y}\right)-P_2\left(\frac{\partial M_1}{\partial z}-\frac{\partial P_1}{\partial x}\right)$.

Since corresponding components are equal, we see that
$\nabla(\mathbf{F}_1\cdot\mathbf{F}_2)=(\mathbf{F}_1\cdot\nabla)\mathbf{F}_2+(\mathbf{F}_2\cdot\nabla)\mathbf{F}_1+\mathbf{F}_1\times(\nabla\times\mathbf{F}_2)+\mathbf{F}_2\times(\nabla\times\mathbf{F}_1)$, as claimed.

28. (a) From the Divergence Theorem, $\iint_S \nabla f\cdot\mathbf{n}\,d\sigma=\iiint_D \nabla\cdot\nabla f\;dV=\iiint_D\left(\nabla^2 f\right)dV=\iiint_D 0\;dV=0$

(b) From the Divergence Theorem, $\iint_S f\,\nabla f\cdot\mathbf{n}\;d\sigma=\iiint_D \nabla\cdot f\,\nabla f\;dV$. Now,

$$f\nabla f=\left(f\frac{\partial f}{\partial x}\right)\mathbf{i}+\left(f\frac{\partial f}{\partial y}\right)\mathbf{j}+\left(f\frac{\partial f}{\partial z}\right)\mathbf{k}\Rightarrow\nabla\cdot f\,\nabla f=\left[f\frac{\partial^2 f}{\partial x^2}+\left(\frac{\partial f}{\partial x}\right)^2\right]+\left[f\frac{\partial^2 f}{\partial y^2}+\left(\frac{\partial f}{\partial y}\right)^2\right]+\left[f\frac{\partial^2 f}{\partial z^2}+\left(\frac{\partial f}{\partial x}\right)^2\right]$$

$=f\nabla^2 f+|\nabla f|^2=0+|\nabla f|^2$ since f is harmonic $\Rightarrow\iint_S f\;\nabla f\cdot\mathbf{n}\,d\sigma=\iiint_D|\nabla f|^2\,dV$, as claimed.

29. $$\iint_S f\nabla g\cdot\mathbf{n}\;d\sigma=\iiint_D \nabla\cdot f\;\nabla\;g\;dV=\iiint_D \nabla\cdot\left(f\frac{\partial g}{\partial x}\mathbf{i}+f\frac{\partial g}{\partial y}\mathbf{j}+f\frac{\partial g}{\partial z}\mathbf{k}\right)dV$$

$$=\iiint_D\left(f\frac{\partial^2 g}{\partial x^2}+\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+f\frac{\partial^2 g}{\partial y^2}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+f\frac{\partial^2 g}{\partial z^2}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)dV$$

$$=\iiint_D\left[f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right)+\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)\right]dV=\iiint_D\left(f\nabla^2 g+\nabla\;f\cdot\nabla\;g\right)dV$$

30. By Exercise 29, $\iint_S f\nabla g\cdot\mathbf{n}\,d\sigma=\iiint_D \left(f\nabla^2 g+\nabla f\cdot\nabla g\right)dV$ and by interchanging the roles of f and g, $\iint_S g\nabla f\cdot\mathbf{n}\,d\sigma=\iiint_D\left(g\nabla^2 f+\nabla g\cdot\nabla f\right)dV$. Subtracting the second equation from the first yields: $\iint_S (f\nabla g-g\nabla f)\cdot\mathbf{n}\,d\sigma=\iiint_D \left(f\nabla^2 g-g\nabla^2 f\right)dV$ since $\nabla f\cdot\nabla g=\nabla g\cdot\nabla f$.

31. (a) The integral $\iiint_D p(t,x,y,z)\,dV$ represents the mass of the fluid at any time t. The equation says that the instantaneous rate of change of mass is flux of the fluid through the surface S enclosing the region D: the mass decreases if the flux is outward (so the fluid flows out of D), and increases if the flow is inward (interpreting $\mathbf{n}$ as the outward pointing unit normal to the surface).

(b) $\iiint_D \frac{\partial p}{\partial t}\,dV=\frac{d}{dt}\iiint_D p\,dV=-\iint_S p\mathbf{v}\cdot\mathbf{n}\,d\sigma=-\iiint_D \nabla\cdot p\mathbf{v}\,dV\Rightarrow\frac{\partial\rho}{\partial t}=-\nabla\cdot p\mathbf{v}$

Since the law is to hold for all regions D, $\nabla\cdot p\mathbf{v}+\frac{\partial p}{\partial t}=0$, as claimed

32. (a) ∇T points in the direction of maximum change of the temperature, so if the solid is heating up at the point the temperature is greater in a region surrounding the point $\Rightarrow\nabla T$ points away from the point $\Rightarrow-\nabla T$ points toward the point $\Rightarrow-\nabla T$ points in the direction the heat flows.

(b) Assuming the Law of Conservation of Mass (Exercise 31) with $-k\nabla T=p\mathbf{v}$ and $c\rho T=p$, we have $\frac{d}{dt}\iiint_D c\rho T\,dV=-\iint_S -k\nabla T\cdot\mathbf{n}\,d\sigma\Rightarrow$ the continuity equation, $\nabla\cdot(-k\nabla T)+\frac{\partial}{\partial t}(c\rho T)=0$

$\Rightarrow c\rho\frac{\partial T}{\partial t}=-\nabla\cdot(-k\nabla T)=k\nabla^2 T\Rightarrow\frac{\partial T}{\partial t}=\frac{k}{c\rho}\nabla^2 T=K\nabla^2 T$, as claimed

CHAPTER 16 PRACTICE EXERCISES

1. Path 1: $\mathbf{r}=t\mathbf{i}+t\mathbf{j}+t\mathbf{k}\Rightarrow x=t,\ y=t,\ z=t,\ 0\le t\le 1\Rightarrow f\left(g(t),h(t),k(t)\right)=3-3t^2$ and $\frac{dx}{dt}=1,\ \frac{dy}{dt}=1,$

$\frac{dz}{dt}=1\Rightarrow\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}dt=\sqrt{3}\,dt\Rightarrow\int_C f(x,y,z)\,ds=\int_0^1\sqrt{3}\left(3-3t^2\right)dt=2\sqrt{3}$

Path 2: $\mathbf{r}_1=t\mathbf{i}+t\mathbf{j},\ 0\le t\le 1\Rightarrow x=t,\ y=t,\ z=0\Rightarrow f\left(g(t),h(t),k(t)\right)=2t-3t^2+3$ and $\frac{dx}{dt}=1,\ \frac{dy}{dt}=1,$

$\frac{dz}{dt}=0\Rightarrow\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=\sqrt{2}\,dt\Rightarrow\int_{C_1} f(x,y,z)\,ds=\int_0^1\sqrt{2}\left(2t-3t^2+3\right)dt=3\sqrt{2};$

$\mathbf{r}_2=\mathbf{i}+\mathbf{j}+t\mathbf{k}\Rightarrow x=1,\ y=1,\ z=t\Rightarrow f\left(g(t),h(t),k(t)\right)=2-2t$ and $\frac{dx}{dt}=0,\ \frac{dy}{dt}=0,\ \frac{dz}{dt}=1$

$\Rightarrow\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=dt\Rightarrow\int_{C_2} f(x,y,z)\,ds=\int_0^1(2-2t)\,dt=1$

$\Rightarrow\int_C f(x,y,z)\,ds=\int_{C_1} f(x,y,z)\,ds+\int_{C_2} f(x,y,z)=3\sqrt{2}+1$

2. Path 1: $\mathbf{r}_1=t\mathbf{i}\Rightarrow x=t,\ y=0,\ z=0\Rightarrow f\left(g(t),h(t),k(t)\right)=t^2$ and $\frac{dx}{dt}=1,\ \frac{dy}{dt}=0,\ \frac{dz}{dt}=0$

$\Rightarrow\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}dt=dt\Rightarrow\int_{C_1} f(x,y,z)\,ds=\int_0^1 t^2\,dt=\frac{1}{3};$

$\mathbf{r}_2=\mathbf{i}+t\mathbf{j}\Rightarrow x=1,\ y=t,\ z=0\Rightarrow f\left(g(t),h(t),k(t)\right)=1+t$ and $\frac{dx}{dt}=0,\ \frac{dy}{dt}=1,\ \frac{dz}{dt}=0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_2} f(x, y, z)\,ds = \int_0^1 (1+t)\,dt = \frac{3}{2};$

$\mathbf{r}_3 = \mathbf{i}+\mathbf{j}+t\mathbf{k} \Rightarrow x=1,\ y=1,\ z=t \Rightarrow f\left(g(t), h(t), k(t)\right) = 2-t$ and $\frac{dx}{dt}=0,\ \frac{dy}{dt}=0,\ \frac{dz}{dt}=1$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_3} f(x, y, z)\,ds = \int_0^1 (2-t)\,dt = \frac{3}{2}$

$\Rightarrow \int_{\text{Path 1}} f(x, y. z)\,ds = \int_{C_1} f(x, y, z)\,ds + \int_{C_2} f(x, y, z)\,ds + \int_{C_3} f(x, y, z)\,ds = \frac{10}{3}$

Path2: $\mathbf{r}_4 = t\mathbf{i}+t\mathbf{j} \Rightarrow x=t,\ y=t,\ z=0 \Rightarrow f\left(g(t), h(t), k(t)\right) = t^2+t$ and $\frac{dx}{dt}=1,\ \frac{dy}{dt}=1,\ \frac{dz}{dt}=0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{2}\,dt \Rightarrow \int_{C_4} f(x, y, z)\,ds = \int_0^1 \sqrt{2}\left(t^2+t\right)dt = \frac{5}{6}\sqrt{2};$

$\mathbf{r}_3 = \mathbf{i}+\mathbf{j}+t\mathbf{k}$ (see above) $\Rightarrow \int_{C_3} f(x, y, z)\,ds = \frac{3}{2}$

$\Rightarrow \int_{\text{Path 2}} f(x, y, z)\,ds = \int_{C_3} f(x, y, z)\,ds + \int_{C_4} f(x, y, z)\,ds = \frac{5}{6}\sqrt{2}+\frac{3}{2} = \frac{5\sqrt{2}+9}{6}$

Path 3: $\mathbf{r}_5 = t\mathbf{k} \Rightarrow x=0,\ y=0,\ z=t,\ 0\le t\le 1 \Rightarrow f\left(g(t), h(t), k(t)\right) = -t$ and $\frac{dx}{dt}=0,\ \frac{dy}{dt}=0,\ \frac{dz}{dt}=1$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_5} f(x, y, z)\,ds = \int_0^1 -t\,dt = -\frac{1}{2};$

$\mathbf{r}_6 = t\mathbf{j}+\mathbf{k} \Rightarrow x=0,\ y=t,\ z=1,\ 0\le t\le 1 \Rightarrow f\left(g(t), h(t), k(t)\right) = t-1$ and $\frac{dx}{dt}=0,\ \frac{dy}{dt}=1,\ \frac{dz}{dt}=0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_6} f(x, y, z)\,ds = \int_0^1 (t-1)\,dt = -\frac{1}{2};$

$\mathbf{r}_7 = t\mathbf{i}+\mathbf{j}+\mathbf{k} \Rightarrow x=t,\ y=1,\ z=1,\ 0\le t\le 1 \Rightarrow f\left(g(t), h(t), k(t)\right) = t^2$ and $\frac{dx}{dt}=1,\ \frac{dy}{dt}=0,\ \frac{dz}{dt}=0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_7} f(x, y, z)\,ds = \int_0^1 t^2\,dt = \frac{1}{3}$

$\Rightarrow \int_{\text{Path 3}} f(x, y, z)\,ds = \int_{C_5} f(x, y, z)\,ds + \int_{C_6} f(x, y, z)\,ds + \int_{C_7} f(x, y, z)\,ds = -\frac{1}{2}-\frac{1}{2}+\frac{1}{3} = -\frac{2}{3}$

3. $\mathbf{r} = (a\cos t)\mathbf{j}+(a\sin t)\mathbf{k} \Rightarrow x=0,\ y=a\cos t,\ z=a\sin t \Rightarrow f\left(g(t), h(t), k(t)\right) = \sqrt{a^2\sin^2 t} = a\,|\sin t|$ and

$\frac{dx}{dt}=0,\ \frac{dy}{dt}=-a\sin t,\ \frac{dz}{dt}=a\cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = a\,dt$

$\Rightarrow \int_C f(x, y, z)\,ds = \int_0^{2\pi} a^2|\sin t|\,dt = \int_0^{\pi} a^2\sin t\,dt + \int_{\pi}^{2\pi}\left(-a^2\sin t\right)dt = 4a^2$

4. $\mathbf{r} = (\cos t+t\sin t)\mathbf{i}+(\sin t-t\cos t)\mathbf{j} \Rightarrow x=\cos t+t\sin t,\ y=\sin t-t\cos t,\ z=0$

$\Rightarrow f\left(g(t), h(t), k(t)\right) = \sqrt{(\cos t+t\sin t)^2+(\sin t-t\cos t)^2} = \sqrt{1+t^2}$ and

$\frac{dx}{dt} = -\sin t+\sin t+t\cos t = t\cos t,\ \frac{dy}{dt} = \cos t-\cos t+t\sin t = t\sin t,\ \frac{dz}{dt}=0 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$

$= \sqrt{t^2\cos^2 t+t^2\sin^2 t}\,dt = |t|\,dt = t\,dt$ since $0\le t\le\sqrt{3} \Rightarrow \int_C f(x, y, z)\,ds = \int_0^{\sqrt{3}} t\sqrt{1+t^2}\,dt = \frac{7}{3}$

5. $\frac{\partial P}{\partial y} = -\frac{1}{2}(x+y+z)^{-3/2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{1}{2}(x+y+z)^{-3/2} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{1}{2}(x+y+z)^{-3/2} = \frac{\partial M}{\partial y}$
$\Rightarrow M\ dx + N\ dy + P\ dz$ is exact; $\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x+y+z}} \Rightarrow f(x, y, z) = 2\sqrt{x+y+z} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{1}{\sqrt{x+y+z}} + \frac{\partial g}{\partial y}$
$= \frac{1}{\sqrt{x+y+z}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = 2\sqrt{x+y+z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{1}{\sqrt{x+y+z}} + h'(z)$
$= \frac{1}{\sqrt{x+y+z}} \Rightarrow h'(x) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = 2\sqrt{x+y+z} + C \Rightarrow \int_{(-1,1,1)}^{(4,-3,0)} \frac{dx+dy+dz}{\sqrt{x+y+z}}$
$= f(4, -3, 0) - f(-1, 1, 1) = 2\sqrt{1} - 2\sqrt{1} = 0$

6. $\frac{\partial P}{\partial y} = -\frac{1}{2\sqrt{yz}} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M\ dx + N\ dy + P\ dz$ is exact; $\frac{\partial f}{\partial x} = 1$
$\Rightarrow f(x, y, z) = x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -\sqrt{\frac{z}{y}} \Rightarrow g(y, z) = -2\sqrt{yz} + h(z) \Rightarrow f(x, y, z) = x - 2\sqrt{yz} + h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = -\sqrt{\frac{y}{z}} + h'(z) = -\sqrt{\frac{y}{z}} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = x - 2\sqrt{yz} + C$
$\Rightarrow \int_{(1,1,1)}^{(10,3,3)} dx - \sqrt{\frac{z}{y}}\ dy - \sqrt{\frac{y}{z}}\ dz = f(10, 3, 3) - f(1, 1, 1) = (10 - 2\cdot 3) - (1 - 2\cdot 1) = 4 + 1 = 5$

7. $\frac{\partial M}{\partial z} = -y \cos z \neq y \cos z = \frac{\partial P}{\partial x} \Rightarrow \mathbf{F}$ is not conservative; $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} - \mathbf{k}, 0 \le t \le 2\pi$
$\Rightarrow d\mathbf{r} = (-2\sin t)\mathbf{i} - (2\cos t)\mathbf{j} \Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \left[-(-2\sin t)(\sin(-1))(-2\sin t) + (2\cos t)(\sin(-1))(-2\cos t)\right] dt$
$= 4\sin(1)\int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt = 8\pi \sin(1)$

8. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 3x^2 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = 0$

9. Let $M = 8x \sin y$ and $N = -8y \cos x \Rightarrow \frac{\partial M}{\partial y} = 8x \cos y$ and $\frac{\partial N}{\partial x} = 8y \sin x \Rightarrow \int_C 8x \sin y\ dx - 8y \cos x\ dy$
$= \iint_R (8y \sin x - 8x \cos y)\ dy\ dx = \int_0^{\pi/2}\int_0^{\pi/2} (8y \sin x - 8x \cos y)\ dy\ dx = \int_0^{\pi/2} \left(\pi^2 \sin x - 8x\right) dx = -\pi^2 + \pi^2 = 0$

10. Let $M = y^2$ and $N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = 2x \Rightarrow \int_C y^2 dx + x^2 dy = \iint_R (2x - 2y)\ dx\ dy$
$= \int_0^{2\pi}\int_0^2 (2r\cos\theta - 2r\sin\theta)\ r\ dr\ d\theta = \int_0^{2\pi} \frac{16}{3} (\cos\theta - \sin\theta)\ d\theta = 0$

11. Let $z = 1 - x - y \Rightarrow f_x(x, y) = -1$ and $f_y(x, y) = -1 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{3} \Rightarrow$ Surface Area $= \iint_R \sqrt{3}\ dx\ dy$
$= \sqrt{3}$ (Area of the circular region in the xy-plane) $= \pi\sqrt{3}$

12. $\nabla f = -3\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}, \mathbf{p} = \mathbf{i} \Rightarrow |\nabla f| = \sqrt{9 + 4y^2 + 4z^2}$ and $|\nabla f \cdot \mathbf{p}| = 3$
$\Rightarrow$ Surface Area $= \iint_R \frac{\sqrt{9+4y^2+4z^2}}{3}\ dy\ dz = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{\sqrt{9+4r^2}}{3}\ r\ dr\ d\theta = \frac{1}{3}\int_0^{2\pi} \left(\frac{7}{4}\sqrt{21} - \frac{9}{4}\right) d\theta = \frac{\pi}{6}\left(7\sqrt{21} - 9\right)$

13. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k},\ \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2$ and $|\nabla f \cdot \mathbf{p}| = |2z| = 2z$ since $z \ge 0 \Rightarrow$ Surface Area $= \iint_R \frac{2}{2z}\,dA = \iint_R \frac{1}{z}\,dA = \iint_R \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-r^2}}\,r\,dr\,d\theta$ $= \int_0^{2\pi}\left[-\sqrt{1-r^2}\right]_0^{1/\sqrt{2}} d\theta = \int_0^{2\pi}\left(1-\frac{1}{\sqrt{2}}\right)d\theta = 2\pi\left(1-\frac{1}{\sqrt{2}}\right)$

14. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k},\ \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 4$ and $|\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0 \Rightarrow$ Surface Area $= \iint_R \frac{4}{2z}\,dA = \iint_R \frac{2}{z}\,dA = 2\int_0^{\pi/2}\int_0^{2\cos\theta} \frac{2}{\sqrt{4-r^2}}\,r\,dr\,d\theta = 4\pi - 8$

(b) $\mathbf{r} = 2\cos\theta \Rightarrow d\mathbf{r} = -2\sin\theta\,d\theta;\ ds^2 = r^2 d\theta^2 + dr^2$ (Arc length in polar coordinates) $\Rightarrow ds^2 = (2\cos\theta)^2 d\theta^2 + dr^2 = 4\cos^2\theta\,d\theta^2 + 4\sin^2\theta\,d\theta^2 = 4\,d\theta^2 \Rightarrow ds = 2\,d\theta$; the height of the cylinder is $z = \sqrt{4-r^2} = \sqrt{4-4\cos^2\theta} = 2|\sin\theta| = 2\sin\theta$ if $0 \le \theta \le \frac{\pi}{2}$

$\Rightarrow$ Surface Area $= \int_{-\pi/2}^{\pi/2} h\,ds = 2\int_0^{\pi/2}(2\sin\theta)(2\,d\theta) = 8$

15. $f(x, y, z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow \nabla f = \left(\frac{1}{a}\right)\mathbf{i} + \left(\frac{1}{b}\right)\mathbf{j} + \left(\frac{1}{c}\right)\mathbf{k} \Rightarrow |\nabla f| = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = \frac{1}{c}$ since $c > 0 \Rightarrow$ Surface Area $= \iint_R \frac{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}{\left(\frac{1}{c}\right)}\,dA = c\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}\iint_R dA = \frac{1}{2}abc\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$, since the area of the triangular region R is $\frac{1}{2}ab$. To check this result, let $\mathbf{v} = a\mathbf{i} + c\mathbf{k}$ and $\mathbf{w} = -a\mathbf{i} + b\mathbf{j}$; the area can be found by computing $\frac{1}{2}|\mathbf{v} \times \mathbf{w}|$.

16. (a) $\nabla f = 2y\mathbf{j} - \mathbf{k},\ \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2+1}$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2+1}\,dx\,dy$

$\Rightarrow \iint_S g(x, y, z)\,d\sigma = \iint_R \frac{yz}{\sqrt{4y^2+1}}\sqrt{4y^2+1}\,dx\,dy = \iint_R y\left(y^2-1\right)dx\,dy = \int_{-1}^{1}\int_0^3\left(y^3-y\right)dx\,dy$

$= \int_{-1}^{1} 3\left(y^3-y\right)dy = 3\left[\frac{y^4}{4} - \frac{y^2}{2}\right]_{-1}^{1} = 0$

(b) $\iint_S g(x, y, z)\,d\sigma = \iint_R \frac{z}{\sqrt{4y^2+1}}\sqrt{4y^2+1}\,dx\,dy = \int_{-1}^{1}\int_0^3\left(y^2-1\right)dx\,dy = \int_{-1}^{1}3\left(y^2-1\right)dy = 3\left[\frac{y^3}{3} - y\right]_{-1}^{1} = -4$

17. $\nabla f = 2y\mathbf{j} + 2z\mathbf{k},\ \mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 4z^2} = 2\sqrt{y^2+z^2} = 10$ and $|\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0$ $\Rightarrow d\sigma = \frac{10}{2z}dx\,dy = \frac{5}{z}dx\,dy = \iint_S g(x, y, z)\,d\sigma = \iint_R \left(x^4 y\right)\left(y^2+z^2\right)\left(\frac{5}{z}\right)dx\,dy$

$= \iint_R \left(x^4 y\right)(25)\left(\frac{5}{\sqrt{25-y^2}}\right)dx\,dy = \int_0^4\int_0^1 \frac{125y}{\sqrt{25-y^2}}\,x^4\,dx\,dy = \int_0^4 \frac{25y}{\sqrt{25-y^2}}\,dy = 50$

18. Define the coordinate system so that the origin is at the center of the earth, the z-axis is the earth's axis (north is the positive z direction), and the xz-plane contains the earth's prime meridian. Let S denote the surface

which is Wyoming so then S is part of the surface $z=\left(R^2-x^2-y^2\right)^{1/2}$. Let R_{xy} be the projection of S onto the xy-plane. The surface area of Wyoming is $\iint\limits_{S} 1\,d\sigma=\iint\limits_{R_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$

$=\iint\limits_{R_{xy}}\sqrt{\frac{x^2}{R^2-x^2-y^2}+\frac{y^2}{R^2-x^2-y^2}+1}\,dA=\iint\limits_{R_{xy}}\frac{R}{\left(R^2-x^2-y^2\right)^{1/2}}\,dA=\int_{\theta_1}^{\theta_2}\int_{R\sin 45°}^{R\sin 49°}R\left(R^2-r^2\right)^{-1/2}r\,dr\,d\theta$ (where θ_1 and θ_2 are the radian equivalent to $104°3'$ and $111°3'$, respectively) $=\int_{\theta_1}^{\theta_2}\left[-R\left(R^2-r^2\right)^{1/2}\right]_{R\sin 45°}^{R\sin 49°}d\theta$

$=\int_{\theta_1}^{\theta_2}\left[R\left(R^2-R^2\sin^2 45°\right)^{1/2}-R\left(R^2-R^2\sin^2 49°\right)^{1/2}\right]d\theta=(\theta_2-\theta_1)R^2\left(\cos 45°-\cos 49°\right)$

$=\frac{7\pi}{180}R^2\left(\cos 45°-\cos 49°\right)=\frac{7\pi}{180}(6371)^2\left(\cos 45°-\cos 49°\right)\approx 253{,}143\text{ km}^2.$

19. A possible parametrization is $\mathbf{r}(\phi,\theta)=(6\sin\phi\cos\theta)\mathbf{i}+(6\sin\phi\sin\theta)\mathbf{j}+(6\cos\phi)\mathbf{k}$ (spherical coordinates); now $\rho=6$ and $z=-3\Rightarrow -3=6\cos\phi\Rightarrow\cos\phi=-\frac{1}{2}\Rightarrow\phi=\frac{2\pi}{3}$ and $z=3\sqrt{3}\Rightarrow 3\sqrt{3}=6\cos\phi$ $\Rightarrow\cos\phi=\frac{\sqrt{3}}{2}\Rightarrow\phi=\frac{\pi}{6}\Rightarrow\frac{\pi}{6}\le\phi\le\frac{2\pi}{3}$; also $0\le\theta\le 2\pi$

20. A possible parametrization is $\mathbf{r}(r,\theta)=(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}-\left(\frac{r^2}{2}\right)\mathbf{k}$ (cylindrical coordinates); now $r=\sqrt{x^2+y^2}\Rightarrow z=-\frac{r^2}{2}$ and $-2\le z\le 0\Rightarrow -2\le-\frac{r^2}{2}\le 0\Rightarrow 4\ge r^2\ge 0\Rightarrow 0\le r\le 2$ since $r\ge 0$; also $0\le\theta\le 2\pi$

21. A possible parametrization is $\mathbf{r}(r,\theta)=(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}+(1+r)\mathbf{k}$ (cylindrical coordinates); now $r=\sqrt{x^2+y^2}\Rightarrow z=1+r$ and $1\le z\le 3\Rightarrow 1\le 1+r\le 3\Rightarrow 0\le r\le 2$; also $0\le\theta\le 2\pi$

22. A possible parametrization is $\mathbf{r}(x,y)=x\mathbf{i}+y\mathbf{j}+\left(3-x-\frac{y}{2}\right)\mathbf{k}$ for $0\le x\le 2$ and $0\le y\le 2$

23. Let $x=u\cos v$ and $z=u\sin v$, where $u=\sqrt{x^2+z^2}$ and v is the angle in the xz-plane with the x-axis $\Rightarrow\mathbf{r}(u,v)=(u\cos v)\mathbf{i}+2u^2\mathbf{j}+(u\sin v)\mathbf{k}$ is a possible parametrization; $0\le y\le 2\Rightarrow 2u^2\le 2\Rightarrow u^2\le 1$ $\Rightarrow 0\le u\le 1$ since $u\ge 0$; also, for just the upper half of the paraboloid, $0\le v\le\pi$

24. A possible parametrization is $\left(\sqrt{10}\sin\phi\cos\theta\right)\mathbf{i}+\left(\sqrt{10}\sin\phi\sin\theta\right)\mathbf{j}+\left(\sqrt{10}\cos\phi\right)\mathbf{k}$, $0\le\phi\le\frac{\pi}{2}$ and $0\le\theta\le\frac{\pi}{2}$

25. $\mathbf{r}_u=\mathbf{i}+\mathbf{j},\ \mathbf{r}_v=\mathbf{i}-\mathbf{j}+\mathbf{k}\Rightarrow\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\1&-1&1\end{vmatrix}=\mathbf{i}-\mathbf{j}-2\mathbf{k}\Rightarrow|\mathbf{r}_u\times\mathbf{r}_v|=\sqrt{6}$

$\Rightarrow$ Surface Area $=\iint\limits_{R_{uv}}|\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv=\int_0^1\int_0^1\sqrt{6}\,du\,dv=\sqrt{6}$

26. $\iint_S \left(xy - z^2\right) d\sigma = \int_0^1\int_0^1 \left[(u+v)(u-v) - v^2\right]\sqrt{6}\, du\, dv = \sqrt{6}\int_0^1\int_0^1 \left(u^2 - 2v^2\right) du\, dv$

$= \sqrt{6}\int_0^1 \left[\frac{u^2}{3} - 2uv^2\right]_0^1 dv = \sqrt{6}\int_0^1 \left(\frac{1}{3} - 2v^2\right) dv = \sqrt{6}\left[\frac{1}{3}v - \frac{2}{3}v^3\right]_0^1 = -\frac{\sqrt{6}}{3} = -\sqrt{\frac{2}{3}}$

27. $\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j},\ \mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix}$

$= (\sin\theta)\mathbf{i} - (\cos\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{1+r^2} \Rightarrow$ Surface Area $= \iint_{R_{r\theta}} |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\, dr\, d\theta = \int_0^{2\pi}\left[\frac{r}{2}\sqrt{1+r^2} + \frac{1}{2}\ln\left(r + \sqrt{1+r^2}\right)\right]_0^1 d\theta = \int_0^{2\pi}\left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\left(1+\sqrt{2}\right)\right] d\theta$

$= \pi\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$

28. $\iint_S \sqrt{x^2 + y^2 + 1}\, d\sigma = \int_0^{2\pi}\int_0^1 \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + 1}\sqrt{1+r^2}\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \left(1+r^2\right) dr\, d\theta$

$= \int_0^{2\pi}\left[r + \frac{r^3}{3}\right]_0^1 d\theta = \int_0^{2\pi} \frac{4}{3}\, d\theta = \frac{8}{3}\pi$

29. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

30. $\frac{\partial P}{\partial y} = \frac{-3zy}{\left(x^2+y^2+z^2\right)^{-5/2}} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = \frac{-3xz}{\left(x^2+y^2+z^2\right)^{-5/2}} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = \frac{-3xy}{\left(x^2+y^2+z^2\right)^{-5/2}} = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

31. $\frac{\partial P}{\partial y} = 0 \ne ye^z = \frac{\partial N}{\partial z} \Rightarrow$ Not Conservative

32. $\frac{\partial P}{\partial y} = \frac{x}{(x+yz)^2} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = \frac{-y}{(x+yz)^2} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = \frac{-z}{(x+yz)^2} = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

33. $\frac{\partial f}{\partial x} = 2 \Rightarrow f(x, y, z) = 2x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y + z \Rightarrow g(y, z) = y^2 + zy + h(z) \Rightarrow f(x, y, z)$

$= 2x + y^2 + zy + h(z) \Rightarrow \frac{\partial f}{\partial z} = y + h'(z) = y + 1 \Rightarrow h'(z) = 1 \Rightarrow h(z) = z + C \Rightarrow f(x, y, z) = 2x + y^2 + zy + z + C$

34. $\frac{\partial f}{\partial x} = z\cos xz \Rightarrow f(x, y, z) = \sin xz + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = e^y \Rightarrow g(y, z) = e^y + h(z) \Rightarrow f(x, y, z)$

$= \sin xz + e^y + h(z) \Rightarrow \frac{\partial f}{\partial z} = x\cos xz + h'(z) = x\cos xz \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = \sin xz + e^y + C$

35. Over Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \Rightarrow x = t,\ y = t,\ z = t$ and $d\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k})\, dt \Rightarrow \mathbf{F} = 2t^2\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$

$\Rightarrow \mathbf{F} \cdot d\mathbf{r} = \left(3t^2 + 1\right) dt \Rightarrow$ Work $= \int_0^1 \left(3t^2 + 1\right) dt = 2;$

Over Path 2: $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1 \Rightarrow x = t,\ y = t,\ z = 0$ and $d\mathbf{r}_1 = (\mathbf{i} + \mathbf{j})dt \Rightarrow \mathbf{F}_1 = 2t^2\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$

$\Rightarrow \mathbf{F}_1 \cdot d\mathbf{r}_1 = \left(2t^2+1\right) dt \Rightarrow \text{Work}_1 = \int_0^1 \left(2t^2+1\right) dt = \frac{5}{3}$; $\mathbf{r}_2 = \mathbf{i}+\mathbf{j}+t\mathbf{k},\ 0 \le t \le 1 \Rightarrow x=1,\ y=1,\ z=t$ and $d\mathbf{r}_2 = \mathbf{k}\ dt \Rightarrow \mathbf{F}_2 = 2\mathbf{i}+\mathbf{j}+\mathbf{k} \Rightarrow \mathbf{F}_2 \cdot d\mathbf{r}_2 = dt \Rightarrow \text{Work}_2 = \int_0^1 dt = 1 \Rightarrow \text{Work} = \text{Work}_1 + \text{Work}_2 = \frac{5}{3}+1 = \frac{8}{3}$

36. Over Path 1: $\mathbf{r} = t\mathbf{i}+t\mathbf{j}+t\mathbf{k},\ 0 \le t \le 1 \Rightarrow x=t,\ y=t,\ z=t$ and $d\mathbf{r} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\ dt \Rightarrow \mathbf{F} = 2t^2\mathbf{i}+t^2\mathbf{j}+\mathbf{k}$ $\Rightarrow \mathbf{F}\cdot d\mathbf{r} = \left(3t^2+1\right) dt \Rightarrow \text{Work} = \int_0^1 \left(3t^2+1\right) dt = 2$;
Over Path 2: Since f is conservative, $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ around any simply closed curve C. Thus consider $\int_{\text{curve}} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$, where C_1 is the path from $(0,0,0)$ to $(1,1,0)$ to $(1,1,1)$ and C_2 is the path from $(1,1,1)$ to $(0,0,0)$. Now, from Path 1 above, $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = -2 \Rightarrow 0 = \int_{\text{curve}} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + (-2)$ $\Rightarrow \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 2$

37. (a) $\mathbf{r} = \left(e^t \cos t\right)\mathbf{i} + \left(e^t \sin t\right)\mathbf{j} \Rightarrow x = e^t \cos t,\ y = e^t \sin t$ from $(1,0)$ to $\left(e^{2\pi},0\right) \Rightarrow 0 \le t \le 2\pi$

$$\Rightarrow \frac{d\mathbf{r}}{dt} = \left(e^t \cos t - e^t \sin t\right)\mathbf{i} + \left(e^t \sin t + e^t \cos t\right)\mathbf{j} \text{ and } \mathbf{F} = \frac{x\mathbf{i}+y\mathbf{j}}{\left(x^2+y^2\right)^{3/2}} = \frac{\left(e^t \cos t\right)\mathbf{i}+\left(e^t \sin t\right)\mathbf{j}}{\left(e^{2t}\cos^2 t + e^{2t}\sin^2 t\right)^{3/2}}$$

$$= \left(\frac{\cos t}{e^{2t}}\right)\mathbf{i} + \left(\frac{\sin t}{e^{2t}}\right)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left(\frac{\cos^2 t}{e^t} - \frac{\sin t \cos t}{e^t} + \frac{\sin^2 t}{e^t} + \frac{\sin t \cos t}{e^t}\right) = e^{-t} \Rightarrow \text{Work} = \int_0^{2\pi} e^{-t}\, dt = 1 - e^{-2\pi}$$

(b) $$\mathbf{F} = \frac{x\mathbf{i}+y\mathbf{j}}{\left(x^2+y^2\right)^{3/2}} \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{\left(x^2+y^2\right)^{3/2}} \Rightarrow f(x,y,z) = -\left(x^2+y^2\right)^{-1/2} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{y}{\left(x^2+y^2\right)^{3/2}} + \frac{\partial g}{\partial y}$$

$$= \frac{y}{(x^2+y^2)^{3/2}} \Rightarrow g(y,z) = C \Rightarrow f(x,y,z) = -(x^2+y^2)^{-1/2} \text{ is a potential function for } \mathbf{F}$$

$$\Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = f\left(e^{2\pi},0\right) - f(1,0) = 1 - e^{-2\pi}$$

38. (a) $\mathbf{F} = \nabla\left(x^2 z e^y\right) \Rightarrow \mathbf{F}$ is conservative $\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for <u>any</u> closed path C

(b) $$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{(1,0,0)}^{(1,0,2\pi)} \nabla\left(x^2 z e^y\right)\cdot d\mathbf{r} = \left(x^2 z e^y\right)\Big|_{(1,0,2\pi)} - \left(x^2 z e^y\right)\Big|_{(1,0,0)} = 2\pi - 0 = 2\pi$$

39. $$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -y & 3z^2 \end{vmatrix} = -2y\mathbf{k};\ \text{unit normal to the plane is } \mathbf{n} = \frac{2\mathbf{i}+6\mathbf{j}-3\mathbf{k}}{\sqrt{4+36+9}} = \frac{2}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}$$

$$\Rightarrow \nabla\times\mathbf{F}\cdot\mathbf{n} = \frac{6}{7}y; \mathbf{p} = \mathbf{k} \text{ and } f(x,y,z) = 2x+6y-3z \Rightarrow |\nabla f\cdot\mathbf{p}| = 3 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\, dA = \frac{7}{3}\, dA$$

$$\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \frac{6}{7} y\, d\sigma = \iint_R \left(\frac{6}{7}y\right)\left(\frac{7}{3}dA\right) = \iint_R 2y\, dA = \int_0^{2\pi}\int_0^1 2r\sin\theta\, r\, dr\, d\theta = \int_0^{2\pi} \frac{2}{3}\sin\theta\, d\theta = 0$$

40. $\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2+y & x+y & 4y^2-z\end{vmatrix}=8y\mathbf{i}$; the circle lies in the plane $f(x,y,z)=y+z=0$ with unit normal $\mathbf{n}=\frac{1}{\sqrt{2}}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}\Rightarrow\nabla\times\mathbf{F}\cdot\mathbf{n}=0\Rightarrow\oint_C\mathbf{F}\cdot d\mathbf{r}=\iint\limits_R\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma=\iint\limits_R 0\,d\sigma=0$

41. (a) $\mathbf{r}=\sqrt{2}t\mathbf{i}+\sqrt{2}t\mathbf{j}+\left(4-t^2\right)\mathbf{k},\ 0\le t\le 1\Rightarrow x=\sqrt{2}t,\ y=\sqrt{2}t,\ z=4-t^2\Rightarrow\frac{dx}{dt}=\sqrt{2},\ \frac{dy}{dt}=\sqrt{2},\ \frac{dz}{dt}=-2t$ $\Rightarrow\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=\sqrt{4+4t^2}\,dt\Rightarrow M=\int_C\delta(x,y,z)\,ds=\int_0^1 3t\sqrt{4+4t^2}\,dt=\left[\frac{1}{4}(4+4t)^{3/2}\right]_0^1$ $=4\sqrt{2}-2$

(b) $M=\int_C\delta(x,y,z)\,ds=\int_0^1\sqrt{4+4t^2}\,dt=\left[t\sqrt{1+t^2}+\ln\left(t+\sqrt{1+t^2}\right)\right]_0^1=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$

42. $\mathbf{r}=t\mathbf{i}+2t\mathbf{j}+\frac{2}{3}t^{3/2}\mathbf{k},\ 0\le t\le 2\Rightarrow x=t,\ y=2t,\ z=\frac{2}{3}t^{3/2}\Rightarrow\frac{dx}{dt}=1,\ \frac{dy}{dt}=2,\ \frac{dz}{dt}=t^{1/2}$ $\Rightarrow\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=\sqrt{t+5}\ dt\Rightarrow M=\int_C\delta(x,y,z)\,ds=\int_0^2 3\sqrt{5+t}\sqrt{t+5}\ dt=\int_0^2 3(t+5)\ dt=36;$ $M_{yz}=\int_C x\delta ds=\int_0^2 3t(t+5)\ dt=38;\ M_{xz}=\int_C y\delta\ ds=\int_0^2 6t(t+5)\ dt=76;\ M_{xy}=\int_C z\delta\ ds$ $=\int_0^2 2t^{3/2}(t+5)\ dt=\frac{144}{7}\sqrt{2}\Rightarrow\bar{x}=\frac{M_{yz}}{M}=\frac{38}{36}=\frac{19}{18},\ \bar{y}=\frac{M_{xz}}{M}=\frac{76}{36}=\frac{19}{9},\ \bar{z}=\frac{M_{xy}}{M}=\frac{\left(\frac{144}{7}\sqrt{2}\right)}{36}=\frac{4}{7}\sqrt{2}$

43. $\mathbf{r}=t\mathbf{i}+\left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\mathbf{j}+\left(\frac{t^2}{2}\right)\mathbf{k},\ 0\le t\le 2\Rightarrow x=t,\ y=\frac{2\sqrt{2}}{3}t^{3/2},\ z=\frac{t^2}{2}\Rightarrow\frac{dx}{dt}=1,\ \frac{dy}{dt}=\sqrt{2}t^{1/2},\ \frac{dz}{dt}=t$ $\Rightarrow\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=\sqrt{1+2t+t^2}\,dt=\sqrt{(t+1)^2}\,dt=|t+1|\ dt=(t+1)\ dt$ on the domain given. Then $M=\int_C\delta ds=\int_0^2\left(\frac{1}{t+1}\right)(t+1)\ dt=\int_0^2 dt=2;\ M_{yz}=\int_C x\delta ds=\int_0^2 t\left(\frac{1}{t+1}\right)(t+1)\ dt=\int_0^2 t\,dt=2;$ $M_{xz}=\int_C y\delta\ ds=\int_0^2\left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\left(\frac{1}{t+1}\right)(t+1)\ dt=\int_0^2\frac{2\sqrt{2}}{3}t^{3/2}\ dt=\frac{32}{15};\ M_{xy}=\int_C z\delta\ ds$ $=\int_0^2\left(\frac{t^2}{2}\right)\left(\frac{1}{t+1}\right)(t+1)\ dt=\int_0^2\frac{t^2}{2}\ dt=\frac{4}{3}\Rightarrow\bar{x}=\frac{M_{yz}}{M}=\frac{2}{2}=1;\ \bar{y}=\frac{M_{xz}}{M}=\frac{\left(\frac{32}{15}\right)}{2}=\frac{16}{15};\ \bar{z}=\frac{M_{xy}}{M}=\frac{\left(\frac{4}{3}\right)}{2}=\frac{2}{3};$ $I_x=\int_C\left(y^2+z^2\right)\delta ds=\int_0^2\left(\frac{8}{9}t^3+\frac{t^4}{4}\right)dt=\frac{232}{45};\ I_y=\int_C\left(x^2+z^2\right)\delta ds=\int_0^2\left(t^2+\frac{t^4}{4}\right)dt=\frac{64}{15};$ $I_z=\int_C\left(y^2+x^2\right)\delta\ ds=\int_0^2\left(t^2+\frac{8}{9}t^3\right)dt=\frac{56}{9}$

44. $\bar{z}=0$ because the arch is in the xy-plane, and $\bar{x}=0$ because the mass is distributed symmetrically with respect to the y-axis; $\mathbf{r}(t)=(a\cos t)\mathbf{i}+(a\sin t)\mathbf{j},\ 0\le t\le\pi\Rightarrow ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$ $=\sqrt{(-a\sin t)^2+(a\cos t)^2}\,dt=a\,dt$, since $a\ge 0;\ M=\int_C\delta\ ds=\int_C(2a-y)\ ds=\int_0^\pi(2a-a\sin t)\ a\,dt$

$= 2a^2\pi - 2a^2; M_{xz} = \int_C y\delta\, dt = \int_C y(2a-y)\, ds = \int_0^{\pi} (a \sin t)(2a - a\sin t)\, dt = \int_0^{\pi} \left(2a^2 \sin t - a^2 \sin^2 t\right) dt$

$= \left[-2a^2 \cos t - a^2\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\right]_0^{\pi} = 4a^2 - \frac{a^2\pi}{2} \Rightarrow \overline{y} = \frac{\left(4a^2 - \frac{a^2\pi}{2}\right)}{2a^2\pi - 2a^2} = \frac{8-\pi}{4\pi-4} \Rightarrow (\overline{x}, \overline{y}, \overline{z}) = \left(0, \frac{8-\pi}{4\pi-4}, 0\right)$

45. $\mathbf{r}(t) = \left(e^t \cos t\right)\mathbf{i} + \left(e^t \sin t\right)\mathbf{j} + e^t\mathbf{k},\ 0 \le t \le \ln 2 \Rightarrow x = e^t \cos t,\ y = e^t \sin t, z = e^t \Rightarrow \frac{dx}{dt} = \left(e^t \cos t - e^t \sin t\right),$

$\frac{dy}{dt} = \left(e^t \sin t + e^t \cos t\right), \frac{dz}{dt} = e^t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$= \sqrt{\left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2 + \left(e^t\right)^2}\ dt = \sqrt{3e^{2t}}\, dt = \sqrt{3}e^t dt; M = \int_c \delta\, ds = \int_0^{\ln 2} \sqrt{3}\, e^t dt$

$= \sqrt{3}; M_{xy} = \int_c z\delta\, ds = \int_0^{\ln 2} \left(\sqrt{3}\, e^t\right)\left(e^t\right) dt = \int_0^{\ln 2} \sqrt{3}\, e^{2t} dt = \frac{3\sqrt{3}}{2} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \frac{\left(\frac{3\sqrt{3}}{2}\right)}{\sqrt{3}} = \frac{3}{2};$

$I_z = \int_c \left(x^2 + y^2\right) \delta\, ds = \int_0^{\ln 2} \left(e^{2t} \cos^2 t + e^{2t} \sin^2 t\right)\left(\sqrt{3}\, e^t\right) dt = \int_0^{\ln 2} \sqrt{3}\, e^{3t} dt = \frac{7\sqrt{3}}{3}$

46. $\mathbf{r}(t) = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 3t\mathbf{k},\ 0 \le t \le 2\pi \Rightarrow x = 2 \sin t,\ y = 2\cos t,\ z = 3t \Rightarrow \frac{dx}{dt} = 2 \cos t,\ \frac{dy}{dt} = -2 \sin t,$

$\frac{dz}{dt} = 3 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{4+9}\ dt = \sqrt{13}\ dt; M = \int_c \delta\, ds = \int_0^{2\pi} \delta\sqrt{13}\ dt = 2\pi\delta\sqrt{13};$

$M_{xy} = \int_c z\delta\, ds = \int_0^{2\pi} (3t)\left(\delta\sqrt{13}\right) dt = 6\delta\pi^2\sqrt{13}; M_{yz} = \int_c x\delta\, ds = \int_0^{2\pi} (2 \sin t)\left(\delta\sqrt{13}\right) dt = 0;$

$M_{xz} = \int_c y\delta\, ds = \int_0^{2\pi} (2\cos t)\left(\delta\sqrt{13}\right) dt = 0 \Rightarrow \overline{x} = \overline{y} = 0$ and $\overline{z} = \frac{M_{xy}}{M} = \frac{6\delta\pi^2\sqrt{13}}{2\delta\pi\sqrt{13}} = 3\pi \Rightarrow (0, 0, 3\pi)$ is the center of mass

47. Because of symmetry $\overline{x} = \overline{y} = 0$. Let $f(x, y, z) = x^2 + y^2 + z^2 = 25 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 10$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z,$ since $z \ge 0 \Rightarrow M = \iint_R \delta(x, y, z)\, d\delta$

$= \iint_R z\left(\frac{10}{2z}\right) dA = \iint_R 5\, dA = 5$ (Area of the circular region) $= 80\pi; M_{xy} = \iint_R z\delta\, d\delta = \iint_R 5z\, dA$

$= \iint_R 5\sqrt{25 - x^2 - y^2}\, dx\, dy = \int_0^{2\pi}\int_0^4 \left(5\sqrt{25 - r^2}\right) r\, dr\, d\theta = \int_0^{2\pi} \frac{490}{3}\, d\theta = \frac{980}{3}\pi \Rightarrow \overline{z} = \frac{\left(\frac{980}{3}\pi\right)}{80\pi} = \frac{49}{12}$

$\Rightarrow (\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{49}{12}\right); I_z = \iint_R \left(x^2 + y^2\right)\delta\, d\sigma = \iint_R 5\left(x^2 + y^2\right) dx\, dy = \int_0^{2\pi}\int_0^4 5r^3 dr\, d\theta = \int_0^{2\pi} 320\, d\theta$

$= 640\pi$

48. On the face $z = 1: g(x, y, z) = z = 1$ and $\mathbf{p} = \mathbf{k} \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$

$\Rightarrow I = \iint_R \left(x^2 + y^2\right) dA = 2\int_0^{\pi/4}\int_0^{\sec\theta} r^3 dr\, d\theta = \frac{2}{3};$ On the face $z = 0: g(x, y, z) = z = 0 \Rightarrow \nabla g = \mathbf{k}$ and $\mathbf{p} = \mathbf{k}$

$\Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R \left(x^2 + y^2\right) dA = \frac{2}{3};$ On the face $y = 0: g(x, y, z) = y = 0$

$\Rightarrow \nabla g = \mathbf{j}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R \left(x^2 + 0\right) dA = \int_0^1\int_0^1 x^2 dx\, dz = \frac{1}{3};$ On the face

$y=1$: $g(x,y,z)=y=1 \Rightarrow \nabla g=\mathbf{j}$ and $\mathbf{p}=\mathbf{j} \Rightarrow |\nabla g|=1 \Rightarrow |\nabla g\cdot\mathbf{p}|=1 \Rightarrow d\sigma=dA \Rightarrow I=\iint_R \left(x^2+1^2\right) dA$

$=\int_0^1\int_0^1\left(x^2+1\right) dx\,dz=\frac{4}{3}$; On the face $x=1$: $g(x,y,z)=x=1 \Rightarrow \nabla g=\mathbf{i}$ and $\mathbf{p}=\mathbf{i} \Rightarrow |\nabla g|=1 \Rightarrow |\nabla g\cdot\mathbf{p}|=1$

$\Rightarrow d\sigma=dA \Rightarrow I=\iint_R \left(1^2+y^2\right) dA=\int_0^1\int_0^1\left(1+y^2\right) dy\,dz=\frac{4}{3}$; On the face $x=0$: $g(x,y,z)=x=0 \Rightarrow \nabla g=\mathbf{i}$

and $\mathbf{p}=\mathbf{i} \Rightarrow |\nabla g|=1 \Rightarrow |\nabla g\cdot\mathbf{p}|=1 \Rightarrow d\sigma=dA \Rightarrow I=\iint_R \left(0^2+y^2\right) dA=\int_0^1\int_0^1 y^2 dy\,dz=\frac{1}{3}$

$\Rightarrow I_z=\frac{2}{3}+\frac{2}{3}+\frac{1}{3}+\frac{4}{3}+\frac{4}{3}+\frac{1}{3}=\frac{14}{3}$

49. $M=2xy+x$ and $N=xy-y \Rightarrow \frac{\partial M}{\partial x}=2y+1, \frac{\partial M}{\partial y}=2x, \frac{\partial N}{\partial x}=y, \frac{\partial N}{\partial y}=x-1$

$\Rightarrow \text{Flux}=\iint_R\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right) dx\,dy=\iint_R (2y+1+x-1)\,dy\,dx=\int_0^1\int_0^1(2y+x)\,dy\,dx=\frac{3}{2}$;

$\text{Circ}=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy=\iint_R (y-2x)\,dy\,dx=\int_0^1\int_0^1(y-2x)\,dy\,dx=-\frac{1}{2}$

50. $M=y-6x^2$ and $N=x+y^2 \Rightarrow \frac{\partial M}{\partial x}=-12x, \frac{\partial M}{\partial y}=1, \frac{\partial N}{\partial x}=1, \frac{\partial N}{\partial y}=2y$

$\Rightarrow \text{Flux}=\iint_R\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right) dx\,dy=\iint_R(-12x+2y)\,dx\,dy=\int_0^1\int_y^1(-12x+2y)\,dx\,dy=\int_0^1\left(4y^2+2y-6\right) dy=-\frac{11}{3}$;

$\text{Circ}=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy=\iint_R(1-1)\,dx\,dy=0$

51. $M=-\frac{\cos y}{x}$ and $N=\ln x\sin y \Rightarrow \frac{\partial M}{\partial y}=\frac{\sin y}{x}$ and $\frac{\partial N}{\partial x}=\frac{\sin y}{x} \Rightarrow \oint_C \ln x\sin y\,dy-\frac{\cos y}{x}dx$

$=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy=\iint_R\left(\frac{\sin y}{x}-\frac{\sin y}{x}\right) dx\,dy=0$

52. (a) Let $M=x$ and $N=y \Rightarrow \frac{\partial M}{\partial x}=1, \frac{\partial M}{\partial y}=0, \frac{\partial N}{\partial x}=0, \frac{\partial N}{\partial y}=1$

$\Rightarrow \text{Flux}=\iint_R\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right) dx\,dy=\iint_R(1+1)\,dx\,dy=2\iint_R dx\,dy=2(\text{Area of the region})$

(b) Let C be a closed curve to which Green's Theorem applies and let **n** be the unit normal vector to C. Let $\mathbf{F}=x\mathbf{i}+y\mathbf{j}$ and assume **F** is orthogonal to **n** at every point of C. Then the flux density of **F** at every point of C is 0 since $\mathbf{F}\cdot\mathbf{n}=0$ at every point of $C \Rightarrow \frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}=0$ at every point of C

$\Rightarrow \text{Flux}=\iint_R\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right) dx\,dy=\iint_R 0\,dx\,dy=0.$ But part (a) above states that the flux is 2(Area of the region) $\Rightarrow$ the area of the region would be $0 \Rightarrow$ contradiction. Therefore, **F** cannot be orthogonal to **n** at every point of C.

53. $\frac{\partial}{\partial x}(2xy)=2y, \frac{\partial}{\partial y}(2yz)=2z, \frac{\partial}{\partial z}(2xz)=2x \Rightarrow \nabla\cdot\mathbf{F}=2y+2z+2x \Rightarrow \text{Flux}=\iiint_D(2x+2y+2z)\,dV$

$=\int_0^1\int_0^1\int_0^1(2x+2y+2z)\,dx\,dy\,xz=\int_0^1\int_0^1(1+2y+2z)\,dy\,dz=\int_0^1(2+2z)\,dz=3$

54. $\frac{\partial}{\partial x}(xz) = z, \frac{\partial}{\partial z}(yz) = z, \frac{\partial}{\partial z}(1) = 0 \Rightarrow \nabla \cdot \mathbf{F} = 2z \Rightarrow \text{Flux} = \iiint\limits_D 2z\, r\, dr\, d\theta\, dz$

$\int_0^{2\pi}\int_0^4\int_3^{\sqrt{25-r^2}} 2z\, dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^4 r\left(16-r^2\right)dr\, d\theta = \int_0^{2\pi} 64\, d\theta = 128\pi$

55. $\frac{\partial}{\partial x}(-2x) = -2, \frac{\partial}{\partial y}(-3y) = -3, \frac{\partial}{\partial z}(z) = 1 \Rightarrow \nabla \cdot \mathbf{F} = -4; x^2+y^2+z^2 = 2 \text{ and } x^2+y^2 = z \Rightarrow z = 1$

$\Rightarrow x^2+y^2 = 1 \Rightarrow \text{Flux} = \iiint\limits_D -4\, dV = -4\int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = -4\int_0^{2\pi}\int_0^1\left(r\sqrt{2-r^2}-r^3\right)dr\, d\theta$

$= -4\int_0^{2\pi}\left(-\frac{7}{12}+\frac{2}{3}\sqrt{2}\right)d\theta = \frac{2}{3}\pi\left(7-8\sqrt{2}\right)$

56. $\frac{\partial}{\partial x}(6x+y) = 6, \frac{\partial}{\partial y}(-x-z) = 0, \frac{\partial}{\partial z}(4yz) = 4y \Rightarrow \nabla \cdot \mathbf{F} = 6+4y; z = \sqrt{x^2+y^2} = r$

$\Rightarrow \text{Flux} = \iiint\limits_D (6+4y)\, dV = \int_0^{\pi/2}\int_0^1\int_0^r (6+4r\sin\theta)\, dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^1\left(6r^2+4r^3\sin\theta\right)dr\, d\theta$

$= \int_0^{\pi/2}(2+\sin\theta)\, d\theta = \pi + 1$

57. $\mathbf{F} = y\mathbf{i}+z\mathbf{j}+x\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 0 \Rightarrow \text{Flux} = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iiint\limits_D \nabla\cdot\mathbf{F}\, dV = 0$

58. $\mathbf{F} = 3xz^2\mathbf{i}+y\mathbf{j}-z^3\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 3z^2+1-3z^2 = 1 \Rightarrow \text{Flux} = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iiint\limits_D \nabla\cdot\mathbf{F}\, dV$

$= \int_0^4\int_0^{\sqrt{16-x^2}/2}\int_0^{y/2} 1\, dz\, dy\, dx = \int_0^4\left(\frac{16-x^2}{16}\right)dx = \left[x-\frac{x^3}{48}\right]_0^4 = \frac{8}{3}$

59. $\mathbf{F} = xy^2\mathbf{i}+x^2y\mathbf{j}+y\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = y^2+x^2+0 \Rightarrow \text{Flux} = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iiint\limits_D \nabla\cdot\mathbf{F}\, dV$

$= \iiint\limits_D \left(x^2+y^2\right)dV = \int_0^{2\pi}\int_0^1\int_{-1}^1 r^2 dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 2r^3 dr\, d\theta = \int_0^{2\pi}\frac{1}{2}d\theta = \pi$

60. (a) $\mathbf{F} = (3z+1)\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 3 \Rightarrow$ Flux across the hemisphere $= \iint\limits_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iiint\limits_D \nabla\cdot\mathbf{F}\, dV = \iiint\limits_D 3\, dV$

$= 3\left(\frac{1}{2}\right)\left(\frac{4}{3}\pi a^3\right) = 2\pi a^3$

(b) $f(x,y,z) = x^2+y^2+z^2-a^2 = 0 \Rightarrow \nabla f = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2+4y^2+4z^2} = \sqrt{4a^2} = 2a$ since $a \ge 0 \Rightarrow \mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2a} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a} \Rightarrow \mathbf{F}\cdot\mathbf{n} = (3z+1)\left(\frac{z}{a}\right); \mathbf{p} = \mathbf{k} \Rightarrow \nabla f\cdot\mathbf{p} = \nabla f\cdot\mathbf{k} = 2z \Rightarrow |\nabla f\cdot\mathbf{p}| = 2z$ since $z \ge 0 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|} = \frac{2a}{2z}dA = \frac{a}{z}dA \Rightarrow \iint\limits_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint\limits_{R_{xy}} (3z+1)\left(\frac{z}{a}\right)\left(\frac{a}{z}\right)dA = \iint\limits_{R_{xy}} (3z+1)\, dx\, dy$

$= \iint\limits_{R_{xy}} \left(3\sqrt{a^2-x^2-y^2}+1\right)dx\, dy = \int_0^{2\pi}\int_0^a\left(3\sqrt{a^2-r^2}+1\right)r\, dr\, d\theta = \int_0^{2\pi}\left(\frac{a^2}{2}+a^3\right)d\theta = \pi a^2+2\pi a^3,$

which is the flux across the hemisphere. Across the base we find $\mathbf{F} = [3(0)+1]\mathbf{k} = \mathbf{k}$ since $z = 0$ in the

xy-plane $\Rightarrow \mathbf{n} = -\mathbf{k}$ (outward normal) $\Rightarrow \mathbf{F}\cdot\mathbf{n} = -1 \Rightarrow$ Flux across the base $= \iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma$

$= \iint_{R_{xy}} (-1)\,dx\,dy = -\pi a^2$. Therefore, the total flux across the closed surface is

$\left(\pi a^2 + 2\pi a^3\right) - \pi a^2 = 2\pi a^3$.

CHAPTER 16 ADDITIONAL AND ADVANCED EXERCISES

1. $dx = (-2\sin t + 2\sin 2t)\,dt$ and $dy = (2\cos t - 2\cos 2t)\,dt$; Area $= \frac{1}{2}\oint_C x\,dy - y\,dx$

$= \frac{1}{2}\int_0^{2\pi}\left[(2\cos t - \cos 2t)(2\cos t - 2\cos 2t) - (2\sin t - \sin 2t)(-2\sin t + 2\sin 2t)\right] dt$

$= \frac{1}{2}\int_0^{2\pi}\left[6 - (6\cos t\cos 2t + 6\sin t\sin 2t)\right] dt = \frac{1}{2}\int_0^{2\pi}(6 - 6\cos t)\,dt = 6\pi$

2. $dx = (-2\sin t - 2\sin 2t)\,dt$ and $dy = (2\cos t - 2\cos 2t)\,dt$; Area $= \frac{1}{2}\oint_C x\,dy - y\,dx$

$= \frac{1}{2}\int_0^{2\pi}\left[(2\cos t + \cos 2t)(2\cos t - 2\cos 2t) - (2\sin t - \sin 2t)(-2\sin t - 2\sin 2t)\right] dt$

$= \frac{1}{2}\int_0^{2\pi}\left[2 - 2(\cos t\cos 2t - \sin t\sin 2t)\right] dt = \frac{1}{2}\int_0^{2\pi}(2 - 2\cos 3t)\,dt = \frac{1}{2}\left[2t - \frac{2}{3}\sin 3t\right]_0^{2\pi} = 2\pi$

3. $dx = \cos 2t\,dt$ and $dy = \cos t\,dt$; Area $= \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{\pi}\left(\frac{1}{2}\sin 2t\cos t - \sin t\cos 2t\right) dt$

$= \frac{1}{2}\int_0^{\pi}\left[\sin t\cos^2 t - (\sin t)\left(2\cos^2 t - 1\right)\right] dt = \frac{1}{2}\int_0^{\pi}\left(-\sin t\cos^2 t + \sin t\right) dt = \frac{1}{2}\left[\frac{1}{3}\cos^3 t - \cos t\right]_0^{\pi} = -\frac{1}{3} + 1 = \frac{2}{3}$

4. $dx = (-2a\sin t - 2a\cos 2t)\,dt$ and $dy = (b\cos t)\,dt$; Area$= \frac{1}{2}\oint_C x\,dy - y\,dx$

$= \frac{1}{2}\int_0^{2\pi}\left[\left(2ab\cos^2 t - ab\cos t\sin 2t\right) - \left(-2ab\sin^2 t - 2ab\sin t\cos 2t\right)\right] dt$

$= \frac{1}{2}\int_0^{2\pi}\left[2ab - 2ab\cos^2 t\sin t + 2ab(\sin t)\left(2\cos^2 t - 1\right)\right] dt = \frac{1}{2}\int_0^{2\pi}\left(2ab + 2ab\cos^2 t\sin t - 2ab\sin t\right) dt$

$= \frac{1}{2}\left[2abt - \frac{2}{3}ab\cos^3 t + 2ab\cos t\right]_0^{2\pi} = 2\pi ab$

5. (a) $\mathbf{F}(x, y, z,) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ is $\mathbf{0}$ only at the point $(0, 0, 0)$, and curl $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is never $\mathbf{0}$.
   (b) $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{k}$ is $\mathbf{0}$ only on the line $x = t,\ y = 0,\ z = 0$ and curl $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j}$ is never $\mathbf{0}$.
   (c) $\mathbf{F}(x, y, z) = z\mathbf{i}$ is $\mathbf{0}$ only when $z = 0$ (the xy-plane) and curl $\mathbf{F}(x, y, z) = \mathbf{j}$ is never $\mathbf{0}$.

6. $\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$ and $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{R}$, so $\mathbf{F}$ is parallel to $\mathbf{n}$ when $yz^2 = \frac{cx}{R}$, $xz^2 = \frac{cy}{R}$,

and $2xyz = \frac{cz}{R} \Rightarrow \frac{yz^2}{x} = \frac{xz^2}{y} = 2xy \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$ and $z^2 = \pm\frac{c}{R} = 2x^2 \Rightarrow z = \pm\sqrt{2}x$. Also,

$x^2 + y^2 + z^2 = R^2 \Rightarrow x^2 + x^2 + 2x^2 = R^2 \Rightarrow 4x^2 = R^2 \Rightarrow x = \pm\frac{R}{2}$. Thus the points are: $\left(\frac{R}{2}, \frac{R}{2}, \frac{\sqrt{2}R}{2}\right)$,

$\left(\frac{R}{2},\frac{R}{2},-\frac{\sqrt{2}R}{2}\right),\left(-\frac{R}{2},-\frac{R}{2},\frac{\sqrt{2}R}{2}\right),\left(-\frac{R}{2},-\frac{R}{2},-\frac{\sqrt{2}R}{2}\right),\left(\frac{R}{2},-\frac{R}{2},\frac{\sqrt{2}R}{2}\right),\left(\frac{R}{2},-\frac{R}{2},-\frac{\sqrt{2}R}{2}\right),$
$\left(-\frac{R}{2},\frac{R}{2},\frac{\sqrt{2}R}{2}\right),\left(-\frac{R}{2},\frac{R}{2},-\frac{\sqrt{2}R}{2}\right)$

7. Set up the coordinate system so that $(a, b, c)=(0, R, 0) \Rightarrow \delta(x, y, z)=\sqrt{x^2+(y-R)^2+z^2}$ $=\sqrt{x^2+y^2+z^2-2Ry+R^2}=\sqrt{2R^2-2Ry}$; let $f(x, y, z)=x^2+y^2+z^2-R^2$ and $\mathbf{p}=\mathbf{i}$ $\Rightarrow \nabla f=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \Rightarrow |\nabla f|=2\sqrt{x^2+y^2+z^2}=2R \Rightarrow d\sigma=\frac{|\nabla f|}{|\nabla f\cdot\mathbf{i}|}dz\,dy=\frac{2R}{2x}\,dz\,dy$ $\Rightarrow \text{Mass}=\iint_S \delta(x, y, z)\,d\sigma=\iint_{R_{yz}}\sqrt{2R^2-2Ry}\left(\frac{R}{x}\right)dz\,dy=R\iint_{R_{yz}}\frac{\sqrt{2R^2-2Ry}}{\sqrt{R^2-y^2-z^2}}\,dz\,dy$ $=4R\int_{-R}^{R}\int_0^{\sqrt{R^2-y^2}}\frac{\sqrt{2R^2-2Ry}}{\sqrt{R^2-y^2-z^2}}\,dz\,dy=4R\int_{-R}^{R}\sqrt{2R^2-2Ry}\,\sin^{-1}\left(\frac{z}{\sqrt{R^2-y^2}}\right)\Big|_0^{\sqrt{R^2-y^2}}dy$ $=2\pi R\int_{-R}^{R}\sqrt{2R^2-2Ry}\,dy=2\pi R\left(\frac{-1}{3R}\right)(2R^2-2Ry)^{3/2}\Big|_{-R}^{R}=\frac{16\pi R^3}{3}$

8. $\mathbf{r}(r, \theta)=(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}+\theta\mathbf{k},\ 0\le r\le 1,\ 0\le\theta\le 2\pi \Rightarrow \mathbf{r}_r\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & \sin\theta & 0\\ -r\sin\theta & r\cos\theta & 1\end{vmatrix}$ $=(\sin\theta)\mathbf{i}-(\cos\theta)\mathbf{j}+r\mathbf{k} \Rightarrow |\mathbf{r}_r\times\mathbf{r}_\theta|=\sqrt{1+r^2}$; $\delta=2\sqrt{x^2+y^2}=2\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}=2r$ $\Rightarrow \text{Mass}=\iint_S \delta(x, y, z)\,d\sigma=\int_0^{2\pi}\int_0^1 2r\sqrt{1+r^2}\,dr\,d\theta=\int_0^{2\pi}\left[\frac{2}{3}\left(1+r^2\right)^{3/2}\right]_0^1 d\theta=\int_0^{2\pi}\frac{2}{3}\left(2\sqrt{2}-1\right)d\theta$ $=\frac{4\pi}{3}\left(2\sqrt{2}-1\right)$

9. $M=x^2+4xy$ and $N=-6y \Rightarrow \frac{\partial M}{\partial x}=2x+4y$ and $\frac{\partial N}{\partial y}=-6 \Rightarrow \text{Flux}=\int_0^b\int_0^a(2x+4y-6)\,dx\,dy$ $=\int_0^b\left(a^2+4ay-6a\right)dy=a^2b+2ab^2-6ab.$ We want to minimize $f(a, b)=a^2b+2ab^2-6ab=ab(a+2b-6).$ Thus, $f_a(a,b)=2ab+2b^2-6b=0$ and $f_b(a, b)=a^2+4ab-6a=0 \Rightarrow b(2a+2b-6)=0 \Rightarrow b=0$ or $b=-a+3.$ Now $b=0 \Rightarrow a^2-6a=0 \Rightarrow a=0$ or $a=6 \Rightarrow (0,0)$ and $(6, 0)$ are critical points. On the other hand, $b=-a+3 \Rightarrow a^2+4a(-a+3)-6a=0$ $\Rightarrow -3a^2+6a=0 \Rightarrow a=0$ or $a=2 \Rightarrow (0, 3)$ and $(2, 1)$ are also critical points. The flux at $(0, 0)=0$, the flux at $(6, 0)=0$, the flux at $(0, 3)=0$ and the flux at $(2, 1)=-4.$ Therefore, the flux is minimized at $(2, 1)$ with value -4.

10. A plane through the origin has equation $ax+by+cz=0.$ Consider first the case when $c\ne 0.$ Assume the plane is given by $z=ax+by$ and let $f(x, y, z)=x^2+y^2+z^2=4.$ Let C denote the circle of intersection of the plane with the sphere. By Stokes' Theorem, $\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma,$ where $\mathbf{n}$ is a unit normal to the plane. Let

$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (ax + by)\mathbf{k}$ be a parametrization of the surface. Then $\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & a \\ 0 & 1 & b \end{vmatrix} = -a\mathbf{i} - b\mathbf{j} + \mathbf{k}$

$\Rightarrow d\sigma = \left|\mathbf{r}_x \times \mathbf{r}_y\right| dx\, dy = \sqrt{a^2 + b^2 + 1}\, dx\, dy.$ Also, $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{n} = \frac{a\mathbf{i} + b\mathbf{j} - \mathbf{k}}{\sqrt{a^2+b^2+1}}$

$\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{R_{xy}} \frac{a+b-1}{\sqrt{a^2+b^2+1}} \sqrt{a^2 + b^2 + 1}\, dx\, dy = \iint_{R_{xy}} (a + b - 1)\, dx\, dy = (a + b - 1) \iint_{R_{xy}} dx\, dy.$ Now

$x^2 + y^2 + (ax + by)^2 = 4 \Rightarrow \left(\frac{a^2+1}{4}\right) x^2 + \left(\frac{b^2+1}{4}\right) y^2 + \left(\frac{ab}{2}\right) xy = 1 \Rightarrow$ the region R_{xy} is the interior of the ellipse

$Ax^2 + Bxy + Cy^2 = 1$ in the xy-plane, where $A = \frac{a^2+1}{4}$, $B = \frac{ab}{2}$, and $C = \frac{b^2+1}{4}$. The area of the ellipse is

$\frac{2\pi}{\sqrt{4AC - B^2}} = \frac{4\pi}{\sqrt{a^2+b^2+1}} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = h(a, b) = \frac{4\pi(a+b-1)}{\sqrt{a^2+b^2+1}}$. Thus we optimize $H(a, b) = \frac{(a+b-1)^2}{a^2+b^2+1}$:

$\frac{\partial H}{\partial a} = \frac{2(a+b-1)\left(b^2+1+a-ab\right)}{\left(a^2+b^2+1\right)^2} = 0$ and $\frac{\partial H}{\partial b} = \frac{2(a+b-1)\left(a^2+1+b-ab\right)}{\left(a^2+b^2+1\right)^2} = 0 \Rightarrow a + b - 1 = 0$, or $b^2 + 1 + a - ab = 0$ and

$a^2 + 1 + b - ab = 0 \Rightarrow a + b - 1 = 0$, or $a^2 - b^2 + (b - a) = 0 \Rightarrow a + b - 1 = 0$, or $(a - b)(a + b - 1) = 0$

$\Rightarrow a + b - 1 = 0$ or $a = b$. The critical values $a + b - 1 = 0$ give a saddle. If $a = b$, then $0 = b^2 + 1 + a - ab$

$\Rightarrow a^2 + 1 + a - a^2 = 0 \Rightarrow a = -1 \Rightarrow b = -1$. Thus, the point $(a, b) = (-1, -1)$ gives a local extremum for

$\oint_C \mathbf{F} \cdot d\mathbf{r} \Rightarrow z = -x - y \Rightarrow x + y + z = 0$ is the desired plane, if $c \neq 0$.

<u>Note</u>: Since $h(-1, -1)$ is negative, the circulation about $\mathbf{n}$ is <u>clockwise</u>, so $-\mathbf{n}$ is the correct pointing normal for the counterclockwise circulation. Thus $\iint_S \nabla \times \mathbf{F} \cdot (-\mathbf{n})\, d\sigma$ actually gives the <u>maximum</u> circulation.

If $c = 0$, one can see that the corresponding problem is equivalent to the calculation above when $b = 0$, which does not lead to a local extreme.

11. (a) Partition the string into small pieces. Let $\Delta_i s$ be the length of the i^{th} piece. Let (x_i, y_i) be a point in the i^{th} piece. The work done by gravity in moving the i^{th} piece to the x- axis is approximately $W_i = (g x_i y_i \Delta_i s) y_i$ where $x_i y_i \Delta_i s$ is approximately the mass of the i^{th} piece. The total work done by gravity in moving the string to the x-axis is $\sum_i W_i = \sum_i g x_i y_i^2 \Delta_i s \Rightarrow \text{Work} = \int_C gxy^2\, ds$

(b) $\text{Work} = \int_C gxy^2 ds = \int_0^{\pi/2} g(2\cos t)(4\sin^2 t)\sqrt{4\sin^2 t + 4\cos^2 t}\, dt = 16g\int_0^{\pi/2} \cos t \sin^2 t\, dt$

$= \left[16g\left(\frac{\sin^3 t}{3}\right)\right]_0^{\pi/2} = \frac{16}{3} g$

(c) $\overline{x} = \frac{\int_C x(xy) ds}{\int_C xy\, ds}$ and $\overline{y} = \frac{\int_C y(xy)\, ds}{\int_C xy\, ds}$; the mass of the string is $\int_C xy\, ds$ and the weight of the string is $g\int_C xy\, ds$. Therefore, the work done in moving the point mass at $(\overline{x}, \overline{y})$ to the x-axis is

$W = \left(g\int_C xy\, ds\right)\overline{y} = g\int_C xy^2\, ds = \frac{16}{3} g.$

12. (a) Partition the sheet into small pieces. Let $\Delta_i\sigma$ be the area of the i^{th} piece and select a point (x_i, y_i, z_i) in the i^{th} piece. The mass of the i^{th} piece is approximately $x_i y_i \Delta_i\sigma$. The work done by gravity in moving the i^{th} piece to the xy-plane is approximately $(gx_i y_i \Delta_i\sigma)z_i = gx_i y_i z_i \Delta_i\sigma \Rightarrow$ Work $= \iint_S gxyz\, d\sigma$.

(b) $\iint_S gxyz\, d\sigma = g\iint_{Rxy} xy(1-x-y)\sqrt{1+(-1)^2+(-1)^2}\, dA = \sqrt{3}g \int_0^1\int_0^{1-x}\left(xy - x^2y - xy^2\right) dy\, dx$

$= \sqrt{3}g\int_0^1\left[\frac{1}{2}xy^2 - \frac{1}{2}x^2y^2 - \frac{1}{3}xy^3\right]_0^{1-x} dx = \sqrt{3}g\int_0^1\left[\frac{1}{6}x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4\right]dx$

$= \sqrt{3}g\left[\frac{1}{12}x^2 - \frac{1}{6}x^3 + \frac{1}{6}x^4 - \frac{1}{30}x^5\right]_0^1 = \sqrt{3}g\left(\frac{1}{12} - \frac{1}{30}\right) = \frac{\sqrt{3}g}{20}$

(c) The center of mass of the sheet is the point $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{z} = \frac{M_{xy}}{M}$ with $M_{xy} = \iint_S xyz\, d\sigma$ and $M = \iint_S xy\, d\sigma$. The work done by gravity in moving the point mass at $(\overline{x}, \overline{y}, \overline{z})$ to the xy-plane is $gM\overline{z} = gM\left(\frac{M_{xy}}{M}\right) = gM_{xy} = \iint_S gxyz\, d\sigma = \frac{\sqrt{3}g}{20}$.

13. (a) Partition the sphere $x^2 + y^2 + (z-2)^2 = 1$ into small pieces. Let $\Delta_i\sigma$ be the surface area of the i^{th} piece and let (x_i, y_i, z_i) be a point on the i^{th} piece. The force due to pressure on the i^{th} piece is approximately $w(4-z_i)\Delta_i\sigma$. The total force on S is approximately $\sum_i w(4-z_i)\Delta_i\sigma$. This gives the actual force to be $\iint_S w(4-z)\, d\sigma$.

(b) The upward buoyant force is a result of the $\mathbf{k}$-component of the force on the ball due to liquid pressure. The force on the ball at (x, y, z) is $w(4-z)(-\mathbf{n}) = w(z-4)\mathbf{n}$, where $\mathbf{n}$ is the outer unit normal at (x, y, z). Hence the $\mathbf{k}$-component of this force is $w(z-4)\mathbf{n}\cdot\mathbf{k} = w(z-4)\mathbf{k}\cdot\mathbf{n}$. The (magnitude of the) buoyant force on the ball is obtained by adding up all these $\mathbf{k}$-component s to obtain $\iint_S w(z-4)\mathbf{k}\cdot\mathbf{n}\, d\sigma$.

(c) The Divergence Theorem says $\iint_S w(z-4)\mathbf{k}\cdot\mathbf{n}\, d\sigma = \iiint_D \operatorname{div}\left(w(z-4)\mathbf{k}\right) dV = \iiint_D w\, dV$, where D is $x^2 + y^2 + (z-2)^2 \le 1 \Rightarrow \iint_S w(z-4)\mathbf{k}\cdot\mathbf{n}\, d\sigma = w\iiint_D 1\, dV = \frac{4}{3}\pi w$, the weight of the fluid if it were to occupy the region D.

14. The surface S is $z = \sqrt{x^2+y^2}$ from $z = 1$ to $z = 2$. Partition S into small pieces and let $\Delta_i\sigma$ be the area of the i^{th} piece. Let (x_i, y_i, z_i) be a point on the i^{th} piece. Then the magnitude of the force on the i^{th} piece due to liquid pressure is approximately $F_i = w(2-z_i)\Delta_i\sigma \Rightarrow$ the total force on S is approximately $\sum_i F_i = \sum w(2-z_i)\Delta_i\sigma \Rightarrow$ the actual force is $\iint_S w(2-z)\, d\sigma = \iint_{R_{xy}} w\left(2-\sqrt{x^2+y^2}\right)\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\, dA$

$= \iint_{R_{xy}} \sqrt{2}w\left(2-\sqrt{x^2+y^2}\right)dA = \int_0^{2\pi}\int_1^2 \sqrt{2}w(2-r)r\, dr\, d\theta = \int_0^{2\pi}\sqrt{2}w\left[r^2 - \frac{1}{3}r^3\right]_1^2 d\theta = \int_0^{2\pi}\frac{2\sqrt{2}w}{3}d\theta = \frac{4\sqrt{2}\pi w}{3}$

15. Assume that S is a surface to which Stokes' Theorem applies. Then $\oint_C \mathbf{E}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{E})\cdot\mathbf{n}\,d\sigma$ $= \iint_S \left(-\frac{\partial B}{\partial t}\right)\cdot\mathbf{n}\,d\sigma = -\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot\mathbf{n}\,d\sigma.$ Thus the voltage around a loop equals the negative of the rate of change of magnetic flux through the loop.

16. According to Gauss's Law, $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = 4\pi GmM$ for any surface enclosing the origin. But if $\mathbf{F} = \nabla\times\mathbf{H}$ then the integral over such a closed surface would have to be 0 by the Divergence Theorem since div $\mathbf{F} = 0$.

17. $\oint_C f\,\nabla g\cdot d\mathbf{r} = \iint_S \nabla\times(f\,\nabla\, g)\cdot\mathbf{n}\,d\sigma$ (Stokes' Theorem)

$= \iint_S (f\,\nabla\times\nabla\, g + \nabla f\times\nabla\, g)\cdot\mathbf{n}\,d\sigma$ (Section 16.8,Exercise 19b)

$= \iint_S [(f)(\mathbf{0}) + \nabla\, f\times\nabla\, g]\cdot\mathbf{n}\,d\sigma$ (Section 16.7, Equation 8)

$= \iint_S (\nabla\, f\times\nabla\, g)\cdot\mathbf{n}\,d\sigma$

18. $\nabla\times\mathbf{F}_1 = \nabla\times\mathbf{F}_2 \Rightarrow \nabla\times(\mathbf{F}_2-\mathbf{F}_1) = \mathbf{0} \Rightarrow \mathbf{F}_2-\mathbf{F}_1$ is conservative $\Rightarrow \mathbf{F}_2-\mathbf{F}_1 = \nabla\, f$; also, $\nabla\cdot\mathbf{F}_1 = \nabla\cdot\mathbf{F}_2$ $\Rightarrow \nabla\cdot(\mathbf{F}_2-\mathbf{F}_1) = 0 \Rightarrow \nabla^2 f = 0$ (so f is harmonic). Finally, on the surface S, $\nabla f\cdot\mathbf{n} = (\mathbf{F}_2-\mathbf{F}_1)\cdot\mathbf{n}$ $= \mathbf{F}_2\cdot\mathbf{n} - \mathbf{F}_1\cdot\mathbf{n} = 0$. Now, $\nabla\cdot(f\nabla f) = \nabla f\cdot\nabla f + f\nabla^2 f$ so the Divergence Theorem gives $\iiint_D |\nabla f|^2\,dV + \iiint_D f\nabla^2 f\,dV = \iiint_D \nabla\cdot(f\nabla f)\,dV = \iint_S f\nabla f\cdot\mathbf{n}\,d\sigma = 0,$ and since $\nabla^2 f = 0$ we have $\iiint_D |\nabla f|^2\,dV + 0 = 0 \Rightarrow \iiint_D |\mathbf{F}_2-\mathbf{F}_1|^2\,dV = 0 = \iint_S f\nabla f\cdot\mathbf{n}\,d\sigma = 0,$ as claimed.

19. False; let $\mathbf{F} = y\mathbf{i} + x\mathbf{j} \neq 0 \Rightarrow \nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) = 0$ and $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

20. $|\mathbf{r}_u\times\mathbf{r}_v|^2 = |\mathbf{r}_u|^2|\mathbf{r}_v|^2\sin^2\theta = |\mathbf{r}_u|^2|\mathbf{r}_v|^2(1-\cos^2\theta) = |\mathbf{r}_u|^2|\mathbf{r}_v|^2 - |\mathbf{r}_u|^2|\mathbf{r}_v|^2\cos^2\theta = |\mathbf{r}_u|^2|\mathbf{r}_v|^2 - (\mathbf{r}_u\cdot\mathbf{r}_v)^2$ $\Rightarrow |\mathbf{r}_u\times\mathbf{r}_v|^2 = EG - F^2 \Rightarrow d\sigma = |\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv = \sqrt{EG-F^2}\,du\,dv$

21. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow \nabla\cdot\mathbf{r} = 1+1+1+ = 3 \Rightarrow \iiint_D \nabla\cdot\mathbf{r}\,dV = 3\iiint_D dV = 3V \Rightarrow V = \frac{1}{3}\iiint_D \nabla\cdot\mathbf{r}dV = \frac{1}{3}\iint_S \mathbf{r}\cdot\mathbf{n}\,d\sigma,$ by the Divergence Theorem