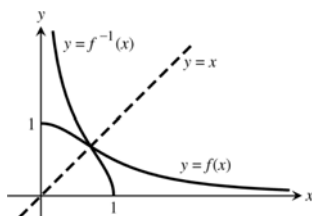


CHAPTER 7 TRANSCENDENTAL FUNCTIONS

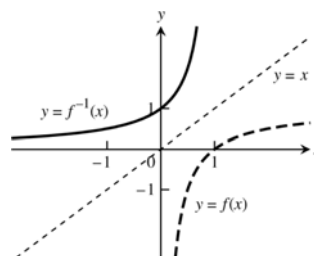
7.1 INVERSE FUNCTIONS AND THEIR DERIVATIVES

1. Yes one-to-one, the graph passes the horizontal line test.
2. Not one-to-one, the graph fails the horizontal line test.
3. Not one-to-one since (for example) the horizontal line $y = 2$ intersects the graph twice.
4. Not one-to-one, the graph fails the horizontal line test.
5. Yes one-to-one, the graph passes the horizontal line test.
6. Yes one-to-one, the graph passes the horizontal line test.
7. Not one-to one since the horizontal line $y = 3$ intersects the graph an infinite number of times.
8. Yes one-to-one, the graph passes the horizontal line test.
9. Yes one-to-one, the graph passes the horizontal line test.
10. Not one-to one since (for example) the horizontal line $y = 1$ intersects the graph twice.

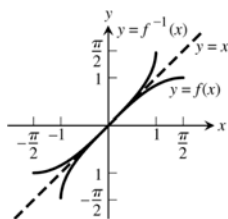
11. Domain: $0 < x \leq 1$, Range: $0 \leq y$



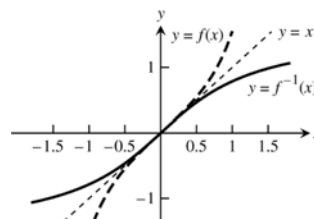
12. Domain: $x < 1$, Range: $y > 0$



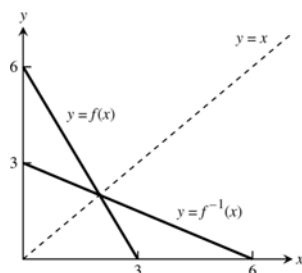
13. Domain: $-1 \leq x \leq 1$, Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



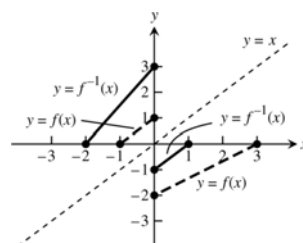
14. Domain: $-\infty < x < \infty$, Range: $-\frac{\pi}{2} < y \leq \frac{\pi}{2}$



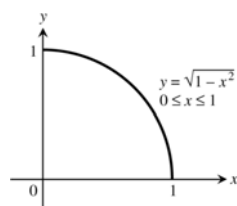
15. Domain:
- $0 \leq x \leq 6$
- , Range:
- $0 \leq y \leq 3$



16. Domain:
- $-2 \leq x \leq 1$
- , Range:
- $-1 \leq y < 3$

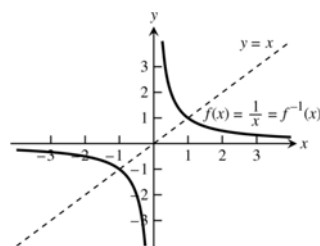


17. The graph is symmetric about
- $y = x$
- .



$$(b) \quad y = \sqrt{1 - x^2} \Rightarrow y^2 = 1 - x^2 \Rightarrow x^2 = 1 - y^2 \Rightarrow x = \sqrt{1 - y^2} \Rightarrow y = \sqrt{1 - x^2} = f^{-1}(x)$$

18. The graph is symmetric about
- $y = x$
- .



$$y = \frac{1}{x} \Rightarrow x = \frac{1}{y} \Rightarrow y = \frac{1}{x} = f^{-1}(x)$$

19. Step 1:
- $y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \sqrt{y - 1}$

$$\text{Step 2: } y = \sqrt{x - 1} = f^{-1}(x)$$

20. Step 1:
- $y = x^2 \Rightarrow x = -\sqrt{y}$
- , since
- $x \leq 0$
- .

$$\text{Step 2: } y = -\sqrt{x} = f^{-1}(x)$$

21. Step 1:
- $y = x^3 - 1 \Rightarrow x^3 = y + 1 \Rightarrow x = (y + 1)^{1/3}$

$$\text{Step 2: } y = \sqrt[3]{x + 1} = f^{-1}(x)$$

22. Step 1:
- $y = x^2 - 2x + 1 \Rightarrow y = (x - 1)^2 \Rightarrow \sqrt{y} = x - 1$
- , since
- $x \geq 1 \Rightarrow x = 1 + \sqrt{y}$

$$\text{Step 2: } y = 1 + \sqrt{x} = f^{-1}(x)$$

23. Step 1:
- $y = (x + 1)^2 \Rightarrow \sqrt{y} = x + 1$
- , since
- $x \geq -1 \Rightarrow x = \sqrt{y} - 1$

$$\text{Step 2: } y = \sqrt{x} - 1 = f^{-1}(x)$$

24. Step 1: $y = x^{2/3} \Rightarrow x = y^{3/2}$

Step 2: $y = x^{3/2} = f^{-1}(x)$

25. Step 1: $y = x^5 \Rightarrow x = y^{1/5}$

Step 2: $y = \sqrt[5]{x} = f^{-1}(x);$

Domain and Range of f^{-1} : all reals; $f(f^{-1}(x)) = (x^{1/5})^5 = x$ and $f^{-1}(f(x)) = (x^5)^{1/5} = x$

26. Step 1: $y = x^4 \Rightarrow x = y^{1/4}$

Step 2: $y = \sqrt[4]{x} = f^{-1}(x);$

Domain of f^{-1} : $x \geq 0$, Range of f^{-1} : $y \geq 0$; $f(f^{-1}(x)) = (x^{1/4})^4 = x$ and $f^{-1}(f(x)) = (x^4)^{1/4} = x$

27. Step 1: $y = x^3 + 1 \Rightarrow x^3 = y - 1 \Rightarrow x = (y - 1)^{1/3}$

Step 2: $y = \sqrt[3]{x-1} = f^{-1}(x);$

Domain and Range of f^{-1} : all reals;

$f(f^{-1}(x)) = ((x-1)^{1/3})^3 + 1 = (x-1) + 1 = x$ and $f^{-1}(f(x)) = ((x^3 + 1) - 1)^{1/3} = (x^3)^{1/3} = x$

28. Step 1: $y = \frac{1}{2}x - \frac{7}{2} \Rightarrow \frac{1}{2}x = y + \frac{7}{2} \Rightarrow x = 2y + 7$

Step 2: $y = 2x + 7 = f^{-1}(x);$

Domain and Range of f^{-1} : all reals;

$f(f^{-1}(x)) = \frac{1}{2}(2x + 7) - \frac{7}{2} = (x + \frac{7}{2}) - \frac{7}{2} = x$ and $f^{-1}(f(x)) = 2(\frac{1}{2}x - \frac{7}{2}) + 7 = (x - 7) + 7 = x$

29. Step 1: $y = \frac{1}{x^2} \Rightarrow x^2 = \frac{1}{y} \Rightarrow x = \frac{1}{\sqrt{y}}$

Step 2: $y = \frac{1}{\sqrt{x}} = f^{-1}(x)$

Domain of f^{-1} : $x > 0$, Range of f^{-1} : $y > 0$; $f(f^{-1}(x)) = \frac{1}{(\frac{1}{\sqrt{x}})^2} = \frac{1}{(\frac{1}{x})} = x$ and $f^{-1}(f(x)) = \frac{1}{\sqrt{\frac{1}{x^2}}} = \frac{1}{(\frac{1}{x})} = x$

since $x > 0$

30. Step 1: $y = \frac{1}{x^3} \Rightarrow x^3 = \frac{1}{y} \Rightarrow x = \frac{1}{y^{1/3}}$

Step 2: $y = \frac{1}{x^{1/3}} = \sqrt[3]{\frac{1}{x}} = f^{-1}(x);$

Domain of f^{-1} : $x \neq 0$, Range of f^{-1} : $y \neq 0$; $f(f^{-1}(x)) = \frac{1}{(\frac{1}{x^{1/3}})^3} = \frac{1}{x^{-1}} = x$ and

$f^{-1}(f(x)) = \left(\frac{1}{x^3}\right)^{-1/3} = \left(\frac{1}{x}\right)^{-1} = x$

31. Step 1: $y = \frac{x+3}{x-2} \Rightarrow y(x-2) = x+3 \Rightarrow xy - 2y = x+3 \Rightarrow xy - x = 2y+3 \Rightarrow x = \frac{2y+3}{y-1}$

Step 2: $y = \frac{2x+3}{x-1} = f^{-1}(x);$

Domain of $f^{-1}: x \neq 1$, Range of $f^{-1}: y \neq 2; f(f^{-1}(x)) = \frac{\left(\frac{2x+3}{x-1}\right)+3}{\left(\frac{2x+3}{x-1}\right)-2} = \frac{(2x+3)+3(x-1)}{(2x+3)-2(x-1)} = \frac{5x}{5} = x$ and

$$f^{-1}(f(x)) = \frac{2\left(\frac{x+3}{x-2}\right)+3}{\left(\frac{x+3}{x-2}\right)-1} = \frac{2(x+3)+3(x-2)}{(x+3)-(x-2)} = \frac{5x}{5} = x$$

32. Step 1: $y = \frac{\sqrt{x}}{\sqrt{x}-3} \Rightarrow y(\sqrt{x}-3) = \sqrt{x} \Rightarrow y\sqrt{x}-3y = \sqrt{x} \Rightarrow y\sqrt{x}-\sqrt{x} = 3y \Rightarrow x = \left(\frac{3y}{y-1}\right)^2$

Step 2: $y = \left(\frac{3x}{x-1}\right)^2 = f^{-1}(x);$

Domain of $f^{-1}: (-\infty, 0] \cup (1, \infty)$, Range of $f^{-1}: [0, 9) \cup (9, \infty); f(f^{-1}(x)) = \frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2}-3};$ If $x > 1$ or

$$x \leq 0 \Rightarrow \frac{3x}{x-1} \geq 0 \Rightarrow \frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2}-3} = \frac{\frac{3x}{x-1}}{\frac{3x}{x-1}-3} = \frac{3x}{3x-3(x-1)} = \frac{3x}{3} = x \text{ and } f^{-1}(f(x)) = \left(\frac{3\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right)}{\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right)-1}\right)^2 = \frac{9x}{(\sqrt{x}-(\sqrt{x}-3))^2} = \frac{9x}{9} = x$$

33. Step 1: $y = x^2 - 2x, x \leq 1 \Rightarrow y+1 = (x-1)^2, x \leq 1 \Rightarrow -\sqrt{y+1} = x-1, x \leq 1 \Rightarrow x = 1 - \sqrt{y+1}$

Step 2: $y = 1 - \sqrt{x+1} = f^{-1}(x);$

Domain of $f^{-1}: [-1, \infty)$, Range of $f^{-1}: (-\infty, 1];$

$$f(f^{-1}(x)) = (1 - \sqrt{x+1})^2 - 2(1 - \sqrt{x+1}) = 1 - 2\sqrt{x+1} + x + 1 - 2 + 2\sqrt{x+1} = x \text{ and}$$

$$f^{-1}(f(x)) = 1 - \sqrt{(x^2 - 2x) + 1}, x \leq 1 = 1 - \sqrt{(x-1)^2}, x \leq 1 = 1 - |x-1| = 1 - (1-x) = x$$

34. Step 1: $y = (2x^3 + 1)^{1/5} \Rightarrow y^5 = 2x^3 + 1 \Rightarrow y^5 - 1 \Rightarrow 2x^3 \Rightarrow \frac{y^5-1}{2} = x^3 \Rightarrow x = \sqrt[3]{\frac{y^5-1}{2}}$

Step 2: $y = \sqrt[3]{\frac{x^5-1}{2}} = f^{-1}(x);$

Domain of $f^{-1}: (-\infty, \infty)$, Range of $f^{-1}: (-\infty, \infty); f(f^{-1}(x)) = \left(2\left(\sqrt[3]{\frac{x^5-1}{2}}\right)^3 + 1\right)^{1/5} = \left(2\left(\frac{x^5-1}{2}\right) + 1\right)^{1/5}$

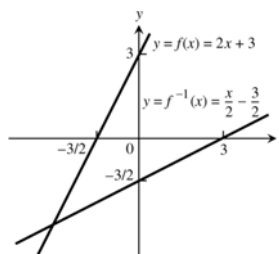
$$= \left((x^5 - 1) + 1\right)^{1/5} = (x^5)^{1/5} = x \text{ and } f^{-1}(f(x)) = \sqrt[3]{\left[\frac{(2x^3+1)^{1/5}-1}{2}\right]^5} = \sqrt[3]{\frac{(2x^3+1)-1}{2}} = \sqrt[3]{\frac{2x^3}{2}} = x$$

35. (a) $y = 2x + 3 \Rightarrow 2x = y - 3$

$$\Rightarrow x = \frac{y}{2} - \frac{3}{2} \Rightarrow f^{-1}(x) = \frac{x}{2} - \frac{3}{2}$$

(c) $\left.\frac{df}{dx}\right|_{x=-1} = 2, \left.\frac{df^{-1}}{dx}\right|_{x=1} = \frac{1}{2}$

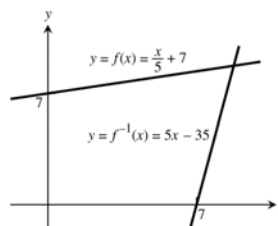
(b)



36. (a) $y = \frac{1}{5}x + 7 \Rightarrow \frac{1}{5}x = y - 7$
 $\Rightarrow x = 5y - 35 \Rightarrow f^{-1}(x) = 5x - 35$

(c) $\left. \frac{df}{dx} \right|_{x=-1} = \frac{1}{5}, \left. \frac{df^{-1}}{dx} \right|_{x=34/5} = 5$

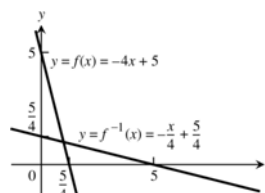
(b)



37. (a) $y = 5 - 4x \Rightarrow 4x = 5 - y$
 $\Rightarrow x = \frac{5}{4} - \frac{y}{4} \Rightarrow f^{-1}(x) = \frac{5}{4} - \frac{x}{4}$

(c) $\left. \frac{df}{dx} \right|_{x=1/2} = -4, \left. \frac{df^{-1}}{dx} \right|_{x=3} = -\frac{1}{4}$

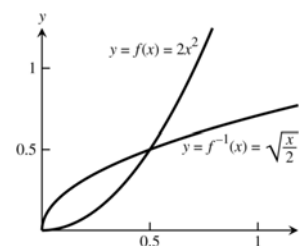
(b)



38. (a) $y = 2x^2 \Rightarrow x^2 = \frac{1}{2}y$
 $\Rightarrow x = \frac{1}{\sqrt{2}}\sqrt{y} \Rightarrow f^{-1}(x) = \sqrt{\frac{x}{2}}$

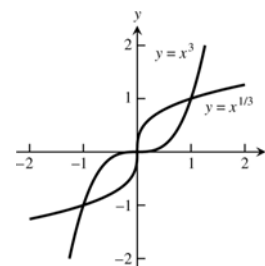
(c) $\left. \frac{df}{dx} \right|_{x=5} = 4x|_{x=5} = 20,$
 $\left. \frac{df^{-1}}{dx} \right|_{x=50} = \frac{1}{2\sqrt{2}}x^{-1/2}|_{x=50} = \frac{1}{20}$

(b)



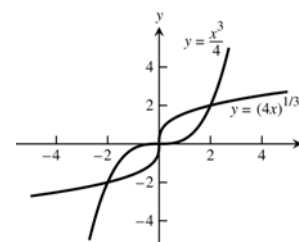
39. (a) $f(g(x)) = (\sqrt[3]{x})^3 = x, g(f(x)) = \sqrt[3]{x^3} = x$
 (c) $f'(x) = 3x^2 \Rightarrow f'(1) = 3, f'(-1) = 3;$
 $g'(x) = \frac{1}{3}x^{-2/3} \Rightarrow g'(1) = \frac{1}{3}, g'(-1) = \frac{1}{3}$
 (d) The line $y = 0$ is tangent to $f(x) = x^3$ at $(0, 0)$; the line $x = 0$ is tangent to $g(x) = \sqrt[3]{x}$ at $(0, 0)$

(b)



40. (a) $h(k(x)) = \frac{1}{4}((4x)^{1/3})^3 = x,$
 $k(h(x)) = \left(4 \cdot \frac{x^3}{4}\right)^{1/3} = x$
 (c) $h'(x) = \frac{3x^2}{4} \Rightarrow h'(2) = 3, h'(-2) = 3;$
 $k'(x) = \frac{4}{3}(4x)^{-2/3} \Rightarrow k'(2) = \frac{1}{3}, k'(-2) = \frac{1}{3}$
 (d) The line $y = 0$ is tangent to $h(x) = \frac{x^3}{4}$ at $(0, 0)$;
 the line $x = 0$ is tangent to $k(x) = (4x)^{1/3}$ at $(0, 0)$

(b)



41. $\frac{df}{dx} = 3x^2 - 6x \Rightarrow \left. \frac{df^{-1}}{dx} \right|_{x=f(3)} = \frac{1}{\left. \frac{df}{dx} \right|_{x=3}} = \frac{1}{9}$

42. $\frac{df}{dx} = 2x - 4 \Rightarrow \left. \frac{df^{-1}}{dx} \right|_{x=f(5)} = \frac{1}{\left. \frac{df}{dx} \right|_{x=5}} = \frac{1}{6}$

$$43. \left. \frac{df^{-1}}{dx} \right|_{x=4} = \left. \frac{df^{-1}}{dx} \right|_{x=f(2)} = \left. \frac{1}{df} \right|_{x=2} = \left(\frac{1}{3} \right) = 3$$

$$44. \left. \frac{dg^{-1}}{dx} \right|_{x=0} = \left. \frac{dg^{-1}}{dx} \right|_{x=f(0)} = \left. \frac{1}{dg} \right|_{x=0} = \frac{1}{2}$$

$$45. (a) y = mx \Rightarrow x = \frac{1}{m}y \Rightarrow f^{-1}(x) = \frac{1}{m}x$$

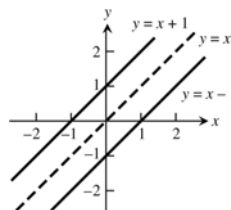
(b) The graph of $y = f^{-1}(x)$ is a line through the origin with slope $\frac{1}{m}$.

$$46. y = mx + b \Rightarrow x = \frac{y}{m} - \frac{b}{m} \Rightarrow f^{-1}(x) = \frac{1}{m}x - \frac{b}{m}; \text{ the graph of } f^{-1}(x) \text{ is a line with slope } \frac{1}{m} \text{ and } y\text{-intercept } -\frac{b}{m}.$$

$$47. (a) y = x + 1 \Rightarrow x = y - 1 \Rightarrow f^{-1}(x) = x - 1$$

$$(b) y = x + b \Rightarrow x = y - b \Rightarrow f^{-1}(x) = x - b$$

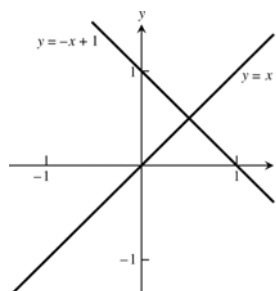
(c) Their graphs will be parallel to one another and lie on opposite sides of the line $y = x$ equidistant from that line.



$$48. (a) y = -x + 1 \Rightarrow x = -y + 1 \Rightarrow f^{-1}(x) = 1 - x; \text{ the lines intersect at a right angle}$$

$$(b) y = -x + b \Rightarrow x = -y + b \Rightarrow f^{-1}(x) = b - x; \text{ the lines intersect at a right angle}$$

(c) Such a function is its own inverse.



49. Let $x_1 \neq x_2$ be two numbers in the domain of an increasing function f . Then, either $x_1 < x_2$ or $x_1 > x_2$ which implies $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, since $f(x)$ is increasing. In either case, $f(x_1) \neq f(x_2)$ and f is one-to-one. Similar arguments hold if f is decreasing.

$$50. f(x) \text{ is increasing since } x_2 > x_1 \Rightarrow \frac{1}{3}x_2 + \frac{5}{6} > \frac{1}{3}x_1 + \frac{5}{6}; \frac{df}{dx} = \frac{1}{3} \Rightarrow \frac{df^{-1}}{dx} = \left(\frac{1}{\frac{1}{3}} \right) = 3$$

$$51. f(x) \text{ is increasing since } x_2 > x_1 \Rightarrow 27x_2^3 > 27x_1^3; y = 27x^3 \Rightarrow x = \frac{1}{3}y^{1/3} \Rightarrow f^{-1}(x) = \frac{1}{3}x^{1/3};$$

$$\frac{df}{dx} = 81x^2 \Rightarrow \frac{df^{-1}}{dx} = \left. \frac{1}{81x^2} \right|_{\frac{1}{3}x^{1/3}} = \frac{1}{9x^{2/3}} = \frac{1}{9}x^{-2/3}$$

$$52. f(x) \text{ is decreasing since } x_2 > x_1 \Rightarrow 1 - 8x_2^3 < 1 - 8x_1^3; y = 1 - 8x^3 \Rightarrow x = \frac{1}{2}(1 - y)^{1/3} \Rightarrow f^{-1}(x) = \frac{1}{2}(1 - x)^{1/3};$$

$$\frac{df}{dx} = -24x^2 \Rightarrow \frac{df^{-1}}{dx} = \left. \frac{1}{-24x^2} \right|_{\frac{1}{2}(1-x)^{1/3}} = \frac{-1}{6(1-x)^{2/3}} = -\frac{1}{6}(1-x)^{-2/3}$$

$$53. f(x) \text{ is decreasing since } x_2 > x_1 \Rightarrow (1 - x_2)^3 < (1 - x_1)^3; y = (1 - x)^3 \Rightarrow x = 1 - y^{1/3} \Rightarrow f^{-1}(x) = 1 - x^{1/3};$$

$$\frac{df}{dx} = -3(1 - x)^2 \Rightarrow \frac{df^{-1}}{dx} = \left. \frac{1}{-3(1-x)^2} \right|_{1-x^{1/3}} = \frac{-1}{3x^{2/3}} = -\frac{1}{3}x^{-2/3}$$

54. $f(x)$ is increasing since $x_2 > x_1 \Rightarrow x_2^{5/3} > x_1^{5/3}$; $y = x^{5/3} \Rightarrow x = y^{3/5} \Rightarrow f^{-1}(x) = x^{3/5}$;
 $\frac{df}{dx} = \frac{5}{3}x^{2/3} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\frac{5}{3}x^{2/3}} \bigg|_{x^{3/5}} = \frac{3}{5x^{2/5}} = \frac{3}{5}x^{-2/5}$
55. The function $g(x)$ is also one-to-one. The reasoning: $f(x)$ is one-to-one means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, so $-f(x_1) \neq -f(x_2)$ and therefore $g(x_1) \neq g(x_2)$. Therefore $g(x)$ is one-to-one as well.
56. The function $h(x)$ is also one-to-one. The reasoning: $f(x)$ is one-to-one means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, so $\frac{1}{f(x_1)} \neq \frac{1}{f(x_2)}$, and therefore $h(x_1) \neq h(x_2)$.
57. The composite is one-to-one also. The reasoning: If $x_1 \neq x_2$ then $g(x_1) \neq g(x_2)$ because g is one-to-one. Since $g(x_1) \neq g(x_2)$, we also have $f(g(x_1)) \neq f(g(x_2))$ because f is one-to-one. Thus, $f \circ g$ is one-to-one because $x_1 \neq x_2 \Rightarrow f(g(x_1)) \neq f(g(x_2))$.
58. Yes, g must be one-to-one. If g were not one-to-one, there would exist numbers $x_1 \neq x_2$ in the domain of g with $g(x_1) = g(x_2)$. For these numbers we would also have $f(g(x_1)) = f(g(x_2))$, contradicting the assumption that $f \circ g$ is one-to-one.
59. $(g \circ f)(x) = x \Rightarrow g(f(x)) = x \Rightarrow g'(f(x))f'(x) = 1$
60. $W(a) = \int_{f(a)}^{f(a)} \pi \left[\left(f^{-1}(y) \right)^2 - a^2 \right] dy = 0 = \int_a^a 2\pi x [f(a) - f(x)] dx = S(a)$;
 $W'(t) = \pi \left[\left(f^{-1}(f(t)) \right)^2 - a^2 \right] f'(t) = \pi (t^2 - a^2) f'(t)$; also
 $S(t) = 2\pi f(t) \int_a^t x dx - 2\pi \int_a^t x f(x) dx = \left[\pi f(t)t^2 - \pi f(t)a^2 \right] - 2\pi \int_a^t x f(x) dx$
 $\Rightarrow S'(t) = \pi t^2 f'(t) + 2\pi t f(t) - \pi a^2 f'(t) - 2\pi t f(t) = \pi (t^2 - a^2) f'(t) \Rightarrow W'(t) = S'(t)$. Therefore, $W(t) = S(t)$ for all $t \in [a, b]$.
- 61-68. Example CAS commands:
Maple:
`with(plots); #61
f := x -> sqrt(3*x-2);
domain:= 2/3..4;
x0:= 3;
Df := D(f); # (a)
plot([f(x), Df(x)], x=domain, color=[red,blue], linestyle=[1,3], legend=["y=f(x)", "y=f'(x)"],
title="#61(a) (Section 7.1)");
q1:= solve(y=f(x), x); # (b)
g := unapply(q1, y);
m1:= Df(x0); # (c)`

```

t1 := f(x0)+m1*(x-x0);
y=t1;
m2 := 1/Df(x0);                                # (d)
t2 := g(f(x0)) + m2*(x-f(x0));
y=t2;
domaing := map(f, domain);                        # (e)
p1 := plot( [f(x), x], x=domain, color=[pink,green], linestyle=[1,9], thickness=[3,0] ):
p2 := plot( g(x), x=domaing, color=cyan, linestyle=3, thickness=4 ):
p3 := plot( t1, x=x0-1..x0+1, color=red, linestyle=4, thickness=0 ):
p4 := plot( t2, x=f(x0)-1..f(x0)+1, color=blue, linestyle=7, thickness=1 ):
p5 := plot([x0,f(x0)], [f(x0),x0], color=green ):
display( [p1,p2,p3,p4,p5], scaling=constrained, title="#61(e) (Section 7.1)" );

```

Mathematica: (assigned function and values for a, b, and x0 may vary)

If a function requires the odd root of a negative number, begin by loading the RealOnly package that allows Mathematica to do this.

```

<<Miscellaneous`RealOnly`
Clear[x, y]
{a,b} = {-2, 1}; x0 = 1/2;
f[x_] = (3x + 2)/(2x - 1)
Plot[{f[x], f'[x]}, {x, a, b}]
sol x = Solve[y == f[x], x]
g[y_] = x /. sol x[[1]]
y0 = f[x0]
f tan[x_] = y0 + f'[x0] (x-x0)
g tan[y_] = x0 + 1/f'[x0] (y-y0)
Plot[{f[x], f tan[x], g[x], g tan[x], Identity[x]}, {x, a, b},
Epilog -> Line[{x0, y0}, {y0, x0}], PlotRange -> {{a,b},{a,b}}, AspectRatio -> Automatic]

```

69-70. Example CAS commands:

Maple:

```

with(plots);
eq := cos(y) = x^(1/5);
domain:= 0..1;
x0:= 1/2;
f := unapply( solve( eq, y ), x);                # (a)
Df := D(f);
plot( [f(x),Df(x), x=domain, color=[red,blue], linestyle=[1,3], legend=["y=f(x)", "y=f'(x)"],
title="#70(a) (Section 7.1)" );
q1 := solve(eq, x );                             # (b)
g := unapply( q1, y );

```



```

m1 := Df(x0);                                # (c)
t1 := f(x0)+m1*(x-x0);
y=t1;
m2 := 1/Df(x0);                                # (d)
t2 := g(f(x0))+m2*(x-f(x0));
y=t2;
domaing := map(f, domain);                    # (e)
p1:=plot( [f(x), x], x=domain, color=[pink,green], linestyle=[1,9], thickness=[3,0] ):
p2:=plot( g(x), x=domaing, color=cyan, linestyle=3, thickness=4 ):
p3:=plot( t1, x=x0-1..x0+1, color=red, linestyle=4, thickness=0 ):
p4:=plot( t2, x=f(x0)-1..f(x0)+1, color=blue, linestyle=7, thickness=1 ):
p5:=plot( [[x0,f(x0)], [f(x0),x0]], color=green ):
display( [p1,p2,p3,p4,p5], scaling=constrained, title="#70(e) (Section 7.1)" );

```

Mathematica: (assigned function and values for a, b, and x0 may vary)

For problems 69 and 70, the code is just slightly altered. At times, different "parts" of solutions need to be used, as in the definitions of f[x] and g[y]

```

Clear[x, y]
{a,b} = {0, 1}; x0 = 1/2;
eqn = Cos[y] == x1/5
soly = Solve[eqn, y]
f[x_] = y /. soly[[2]]
Plot[{f[x], f'[x]}, {x, a, b}]
solx = Solve[eqn, x]
g[y_] = x /. solx[[1]]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x - x0)
gtan[y_] = x0 + 1/f'[x0] (y - y0)
Plot[{f[x], ftan[x], g[x], gtan[x], Identity[x]}, {x, a, b},
Epilog -> Line[{x0, y0}, {y0, x0}], PlotRange -> {{a, b}, {a, b}}, AspectRatio -> Automatic]

```

7.2 NATURAL LOGARITHMS

1. (a) $\ln 0.75 = \ln \frac{3}{4} = \ln 3 - \ln 4 = \ln 3 - \ln 2^2 = \ln 3 - 2 \ln 2$
 (b) $\ln \frac{4}{9} = \ln 4 - \ln 9 = \ln 2^2 - \ln 3^2 = 2 \ln 2 - 2 \ln 3$
 (c) $\ln \frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$ (d) $\ln \sqrt[3]{9} = \frac{1}{3} \ln 9 = \frac{1}{3} \ln 3^2 = \frac{2}{3} \ln 3$
 (e) $\ln 3\sqrt{2} = \ln 3 + \ln 2^{1/2} = \ln 3 + \frac{1}{2} \ln 2$
 (f) $\ln \sqrt{13.5} = \frac{1}{2} \ln 13.5 = \frac{1}{2} \ln \frac{27}{2} = \frac{1}{2} (\ln 3^3 - \ln 2) = \frac{1}{2} (3 \ln 3 - \ln 2)$
2. (a) $\ln \frac{1}{125} = \ln 1 - 3 \ln 5 = -3 \ln 5$ (b) $\ln 9.8 = \ln \frac{49}{5} = \ln 7^2 - \ln 5 = 2 \ln 7 - \ln 5$

(c) $\ln 7\sqrt{7} = \ln 7^{3/2} = \frac{3}{2} \ln 7$

(d) $\ln 1225 = \ln 35^2 = 2 \ln 35 = 2 \ln 5 + 2 \ln 7$

(e) $\ln 0.056 = \ln \frac{7}{125} = \ln 7 - \ln 5^3 = \ln 7 - 3 \ln 5$

(f) $\frac{\ln 35 + \ln \frac{1}{7}}{\ln 25} = \frac{\ln 5 + \ln 7 - \ln 7}{2 \ln 5} = \frac{1}{2}$

3. (a) $\ln \sin \theta - \ln \left(\frac{\sin \theta}{5} \right) = \ln \left(\frac{\sin \theta}{\left(\frac{\sin \theta}{5} \right)} \right) = \ln 5$

(b) $\ln (3x^2 - 9x) + \ln \left(\frac{1}{3x} \right) = \ln \left(\frac{3x^2 - 9x}{3x} \right) = \ln (x - 3)$

(c) $\frac{1}{2} \ln (4t^4) - \ln 2 = \ln \sqrt{4t^4} - \ln 2 = \ln 2t^2 - \ln 2 = \ln \left(\frac{2t^2}{2} \right) = \ln (t^2)$

4. (a) $\ln \sec \theta + \ln \cos \theta = \ln [(\sec \theta)(\cos \theta)] = \ln 1 = 0$

(b) $\ln (8x + 4) - \ln 2^2 = \ln (8x + 4) - \ln 4 = \ln \left(\frac{8x+4}{4} \right) = \ln (2x + 1)$

(c) $3 \ln \sqrt[3]{t^2 - 1} - \ln (t + 1) = 3 \ln (t^2 - 1)^{1/3} - \ln (t + 1) = 3 \left(\frac{1}{3} \right) \ln (t^2 - 1) - \ln (t + 1) = \ln \left(\frac{(t+1)(t-1)}{(t+1)} \right) = \ln (t - 1)$

5. $y = \ln 3x \Rightarrow y' = \left(\frac{1}{3x} \right)(3) = \frac{1}{x}$

6. $y = \ln kx \Rightarrow y' = \left(\frac{1}{kx} \right)(k) = \frac{1}{x}$

7. $y = \ln (t^2) \Rightarrow \frac{dy}{dt} = \left(\frac{1}{t^2} \right)(2t) = \frac{2}{t}$

8. $y = \ln (t^{3/2}) \Rightarrow \frac{dy}{dt} = \left(\frac{1}{t^{3/2}} \right) \left(\frac{3}{2} t^{1/2} \right) = \frac{3}{2t}$

9. $y = \ln \frac{3}{x} = \ln 3x^{-1} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{3x^{-1}} \right) (-3x^{-2}) = -\frac{1}{x}$

10. $y = \ln \frac{10}{x} = \ln 10x^{-1} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{10x^{-1}} \right) (-10x^{-2}) = -\frac{1}{x}$

11. $y = \ln (\theta + 1) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\theta + 1} \right)(1) = \frac{1}{\theta + 1}$

12. $y = \ln (2\theta + 2) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{2\theta + 2} \right)(2) = \frac{1}{\theta + 1}$

13. $y = \ln x^3 \Rightarrow \frac{dy}{dx} = \left(\frac{1}{x^3} \right)(3x^2) = \frac{3}{x}$

14. $y = (\ln x)^3 \Rightarrow \frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{d}{dx} (\ln x) = \frac{3(\ln x)^2}{x}$

15. $y = t(\ln t)^2 \Rightarrow \frac{dy}{dt} = (\ln t)^2 + 2t(\ln t) \cdot \frac{d}{dt} (\ln t) = (\ln t)^2 + \frac{2t \ln t}{t} = (\ln t)^2 + 2 \ln t$

16. $y = t\sqrt{\ln t} = t(\ln t)^{1/2} \Rightarrow \frac{dy}{dt} = (\ln t)^{1/2} + \frac{1}{2} t(\ln t)^{-1/2} \cdot \frac{d}{dt} (\ln t) = (\ln t)^{1/2} + \frac{t(\ln t)^{-1/2}}{2t} = (\ln t)^{1/2} + \frac{1}{2(\ln t)^{1/2}}$

17. $y = \frac{x^4}{4} \ln x - \frac{x^4}{16} \Rightarrow \frac{dy}{dx} = x^3 \ln x + \frac{x^4}{4} \cdot \frac{1}{x} - \frac{4x^3}{16} = x^3 \ln x$

18. $y = (x^2 \ln x)^4 \Rightarrow \frac{dy}{dx} = 4(x^2 \ln x)^3 \left(x^2 \cdot \frac{1}{x} + 2x \ln x \right) = 4x^6 (\ln x)^3 (x + 2x \ln x) = 4x^7 (\ln x)^3 + 8x^7 (\ln x)^4$

19. $y = \frac{\ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (\ln t)(1)}{t^2} = \frac{1 - \ln t}{t^2}$

$$20. \quad y = \frac{1+\ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (1+\ln t)(1)}{t^2} = \frac{1-1-\ln t}{t^2} = -\frac{\ln t}{t^2}$$

$$21. \quad y = \frac{\ln x}{1+\ln x} \Rightarrow y' = \frac{(1+\ln x)\left(\frac{1}{x}\right) - (\ln x)\left(\frac{1}{x}\right)}{(1+\ln x)^2} = \frac{\frac{1}{x} + \frac{\ln x}{x} - \frac{\ln x}{x}}{(1+\ln x)^2} = \frac{1}{x(1+\ln x)^2}$$

$$22. \quad y = \frac{x \ln x}{1+\ln x} \Rightarrow y' = \frac{(1+\ln x)\left(\ln x + x \cdot \frac{1}{x}\right) - (x \ln x)\left(\frac{1}{x}\right)}{(1+\ln x)^2} = \frac{(1+\ln x)^2 - \ln x}{(1+\ln x)^2} = 1 - \frac{\ln x}{(1+\ln x)^2}$$

$$23. \quad y = \ln(\ln x) \Rightarrow y' = \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) = \frac{1}{x \ln x}$$

$$24. \quad y = \ln(\ln(\ln x)) \Rightarrow y' = \frac{1}{\ln(\ln x)} \cdot \frac{d}{dx}(\ln(\ln x)) = \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{d}{dx}(\ln x) = \frac{1}{x(\ln x) \ln(\ln x)}$$

$$25. \quad y = \theta[\sin(\ln \theta) + \cos(\ln \theta)] \Rightarrow \frac{dy}{d\theta} = [\sin(\ln \theta) + \cos(\ln \theta)] + \theta \left[\cos(\ln \theta) \cdot \frac{1}{\theta} - \sin(\ln \theta) \cdot \frac{1}{\theta} \right] \\ = \sin(\ln \theta) + \cos(\ln \theta) + \cos(\ln \theta) - \sin(\ln \theta) = 2 \cos(\ln \theta)$$

$$26. \quad y = \ln(\sec \theta + \tan \theta) \Rightarrow \frac{dy}{d\theta} = \frac{\sec \theta \tan \theta + \sec^2 \theta}{\sec \theta + \tan \theta} = \frac{\sec \theta (\tan \theta + \sec \theta)}{\tan \theta + \sec \theta} = \sec \theta$$

$$27. \quad y = \ln \frac{1}{x\sqrt{x+1}} = -\ln x - \frac{1}{2} \ln(x+1) \Rightarrow y' = -\frac{1}{x} - \frac{1}{2} \left(\frac{1}{x+1} \right) = -\frac{2(x+1)+x}{2x(x+1)} = -\frac{3x+2}{2x(x+1)}$$

$$28. \quad y = \frac{1}{2} \ln \frac{1+x}{1-x} = \frac{1}{2} [\ln(1+x) - \ln(1-x)] \Rightarrow y' = \frac{1}{2} \left[\frac{1}{1+x} - \left(\frac{1}{1-x} \right) (-1) \right] = \frac{1}{2} \left[\frac{1-x+1+x}{(1+x)(1-x)} \right] = \frac{1}{1-x^2}$$

$$29. \quad y = \frac{1+\ln t}{1-\ln t} \Rightarrow \frac{dy}{dt} = \frac{(1-\ln t)\left(\frac{1}{t}\right) - (1+\ln t)\left(\frac{-1}{t}\right)}{(1-\ln t)^2} = \frac{\frac{1}{t} - \frac{\ln t}{t} + \frac{1}{t} + \frac{\ln t}{t}}{(1-\ln t)^2} = \frac{2}{t(1-\ln t)^2}$$

$$30. \quad y = \sqrt{\ln \sqrt{t}} = (\ln t^{1/2})^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2} (\ln t^{1/2})^{-1/2} \cdot \frac{d}{dt}(\ln t^{1/2}) = \frac{1}{2} (\ln t^{1/2})^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{d}{dt}(t^{1/2}) \\ = \frac{1}{2} (\ln t^{1/2})^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{1}{2} t^{-1/2} = \frac{1}{4t\sqrt{\ln \sqrt{t}}}$$

$$31. \quad y = \ln(\sec(\ln \theta)) \Rightarrow \frac{dy}{d\theta} = \frac{1}{\sec(\ln \theta)} \cdot \frac{d}{d\theta}(\sec(\ln \theta)) = \frac{\sec(\ln \theta) \tan(\ln \theta)}{\sec(\ln \theta)} \cdot \frac{d}{d\theta}(\ln \theta) = \frac{\tan(\ln \theta)}{\theta}$$

$$32. \quad y = \ln \frac{\sqrt{\sin \theta \cos \theta}}{1+2 \ln \theta} = \frac{1}{2} (\ln \sin \theta + \ln \cos \theta) - \ln(1+2 \ln \theta) \Rightarrow \frac{dy}{d\theta} = \frac{1}{2} \left(\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} \right) - \frac{\frac{2}{\theta}}{1+2 \ln \theta} \\ = \frac{1}{2} \left[\cot \theta - \tan \theta - \frac{4}{\theta(1+2 \ln \theta)} \right]$$

$$33. \quad y = \ln \left(\frac{(x^2+1)^5}{\sqrt{1-x}} \right) = 5 \ln(x^2+1) - \frac{1}{2} \ln(1-x) \Rightarrow y' = \frac{5 \cdot 2x}{x^2+1} - \frac{1}{2} \left(\frac{1}{1-x} \right) (-1) = \frac{10x}{x^2+1} + \frac{1}{2(1-x)}$$

$$34. \quad y = \ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}} = \frac{1}{2} [5 \ln(x+1) - 20 \ln(x+2)] \Rightarrow y' = \frac{1}{2} \left(\frac{5}{x+1} - \frac{20}{x+2} \right) = \frac{5}{2} \left[\frac{(x+2)-4(x+1)}{(x+1)(x+2)} \right] = -\frac{5}{2} \left[\frac{3x+2}{(x+1)(x+2)} \right]$$

$$35. \quad y = \int_{x^2/2}^{x^2} \ln \sqrt{t} \, dt \Rightarrow \frac{dy}{dx} = \left(\ln \sqrt{x^2} \right) \cdot \frac{d}{dx} (x^2) - \left(\ln \sqrt{\frac{x^2}{2}} \right) \cdot \frac{d}{dx} \left(\frac{x^2}{2} \right) = 2x \ln |x| - x \ln \frac{|x|}{\sqrt{2}}$$

$$36. \quad y = \int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t \, dt \Rightarrow \frac{dy}{dx} = \left(\ln \sqrt[3]{x} \right) \cdot \frac{d}{dx} (\sqrt[3]{x}) - \left(\ln \sqrt{x} \right) \cdot \frac{d}{dx} (\sqrt{x}) = \left(\ln \sqrt[3]{x} \right) \left(\frac{1}{3} x^{-2/3} \right) - \left(\ln \sqrt{x} \right) \left(\frac{1}{2} x^{-1/2} \right) \\ = \frac{\ln \sqrt[3]{x}}{3\sqrt[3]{x^2}} - \frac{\ln \sqrt{x}}{2\sqrt{x}}$$

$$37. \quad \int_{-3}^{-2} \frac{1}{x} \, dx = \left[\ln |x| \right]_{-3}^{-2} = \ln 2 - \ln 3 = \ln \frac{2}{3}$$

$$38. \quad \int_{-1}^0 \frac{3}{3x-2} \, dx = \left[\ln |3x-2| \right]_{-1}^0 = \ln 2 - \ln 5 = \ln \frac{2}{5}$$

$$39. \quad \int \frac{2y}{y^2-25} \, dy = \ln |y^2-25| + C$$

$$40. \quad \int \frac{8r}{4r^2-5} \, dr = \ln |4r^2-5| + C$$

$$41. \quad \int_0^{\pi} \frac{\sin t}{2-\cos t} \, dt = \left[\ln |2-\cos t| \right]_0^{\pi} = \ln 3 - \ln 1 = \ln 3; \text{ or let } u = 2 - \cos t \Rightarrow du = \sin t \, dt \text{ with } t = 0 \Rightarrow u = 1 \text{ and}$$

$$t = \pi \Rightarrow u = 3 \Rightarrow \int_0^{\pi} \frac{\sin t}{2-\cos t} \, dt = \int_1^3 \frac{1}{u} \, du = \left[\ln |u| \right]_1^3 = \ln 3 - \ln 1 = \ln 3$$

$$42. \quad \int_0^{\pi/3} \frac{4 \sin \theta}{1-4 \cos \theta} \, d\theta = \left[\ln |1-4 \cos \theta| \right]_0^{\pi/3} = \ln |1-2| = -\ln 3 = \ln \frac{1}{3}; \text{ or let } u = 1 - 4 \cos \theta \Rightarrow du = 4 \sin \theta \, d\theta \text{ with}$$

$$\theta = 0 \Rightarrow u = -3 \text{ and } \theta = \frac{\pi}{3} \Rightarrow u = -1 \Rightarrow \int_0^{\pi/3} \frac{4 \sin \theta}{1-4 \cos \theta} \, d\theta = \int_{-3}^{-1} \frac{1}{u} \, du = \left[\ln |u| \right]_{-3}^{-1} = -\ln 3 = \ln \frac{1}{3}$$

$$43. \quad \text{Let } u = \ln x \Rightarrow du = \frac{1}{x} \, dx; x = 1 \Rightarrow u = 0 \text{ and } x = 2 \Rightarrow u = \ln 2; \int_1^2 \frac{2 \ln x}{x} \, dx = \int_0^{\ln 2} 2u \, du = \left[u^2 \right]_0^{\ln 2} = (\ln 2)^2$$

$$44. \quad \text{Let } u = \ln x \Rightarrow du = \frac{1}{x} \, dx; x = 2 \Rightarrow u = \ln 2 \text{ and } x = 4 \Rightarrow u = \ln 4;$$

$$\int_2^4 \frac{dx}{x \ln x} = \int_{\ln 2}^{\ln 4} \frac{1}{u} \, du = \left[\ln |u| \right]_{\ln 2}^{\ln 4} = \ln (\ln 4) - \ln (\ln 2) = \ln \left(\frac{\ln 4}{\ln 2} \right) = \ln \left(\frac{\ln 2^2}{\ln 2} \right) = \ln \left(\frac{2 \ln 2}{\ln 2} \right) = \ln 2$$

$$45. \quad \text{Let } u = \ln x \Rightarrow du = \frac{1}{x} \, dx; x = 2 \Rightarrow u = \ln 2 \text{ and } x = 4 \Rightarrow u = \ln 4;$$

$$\int_2^4 \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\ln 4} u^{-2} \, du = \left[-\frac{1}{u} \right]_{\ln 2}^{\ln 4} = -\frac{1}{\ln 4} + \frac{1}{\ln 2} = -\frac{1}{\ln 2^2} + \frac{1}{\ln 2} = -\frac{1}{2 \ln 2} + \frac{1}{\ln 2} = \frac{1}{2 \ln 2} = \frac{1}{\ln 4}$$

$$46. \quad \text{Let } u = \ln x \Rightarrow du = \frac{1}{x} \, dx; x = 2 \Rightarrow u = \ln 2 \text{ and } x = 16 \Rightarrow u = \ln 16;$$

$$\int_2^{16} \frac{dx}{2x\sqrt{\ln x}} = \frac{1}{2} \int_{\ln 2}^{\ln 16} u^{-1/2} \, du = \left[u^{1/2} \right]_{\ln 2}^{\ln 16} = \sqrt{\ln 16} - \sqrt{\ln 2} = \sqrt{4 \ln 2} - \sqrt{\ln 2} = 2\sqrt{\ln 2} - \sqrt{\ln 2} = \sqrt{\ln 2}$$

$$47. \quad \text{Let } u = 6 + 3 \tan t \Rightarrow du = 3 \sec^2 t \, dt; \int \frac{3 \sec^2 t}{6 + 3 \tan t} \, dt = \int \frac{du}{u} = \ln |u| + C = \ln |6 + 3 \tan t| + C$$

$$48. \quad \text{Let } u = 2 + \sec y \Rightarrow du = \sec y \tan y \, dy; \int \frac{\sec y \tan y}{2 + \sec y} \, dy = \int \frac{du}{u} = \ln |u| + C = \ln |2 + \sec y| + C$$

$$49. \quad \text{Let } u = \cos \frac{x}{2} \Rightarrow du = -\frac{1}{2} \sin \frac{x}{2} \, dx \Rightarrow -2 \, du = \sin \frac{x}{2} \, dx; x = 0 \Rightarrow u = 1 \text{ and } x = \frac{\pi}{2} \Rightarrow u = \frac{1}{\sqrt{2}};$$

$$\int_0^{\pi/2} \tan \frac{x}{2} \, dx = \int_0^{\pi/2} \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \, dx = -2 \int_1^{1/\sqrt{2}} \frac{du}{u} = \left[-2 \ln |u| \right]_1^{1/\sqrt{2}} = -2 \ln \frac{1}{\sqrt{2}} = 2 \ln \sqrt{2} = \ln 2$$

50. Let $u = \sin t \Rightarrow du = \cos t \, dt$; $t = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$ and $t = \frac{\pi}{2} \Rightarrow u = 1$;

$$\int_{\pi/4}^{\pi/2} \cot t \, dt = \int_{\pi/4}^{\pi/2} \frac{\cos t}{\sin t} \, dt = \int_{1/\sqrt{2}}^1 \frac{du}{u} = [\ln |u|]_{1/\sqrt{2}}^1 = -\ln \frac{1}{\sqrt{2}} = \ln \sqrt{2}$$

51. Let $u = \sin \frac{\theta}{3} \Rightarrow du = \frac{1}{3} \cos \frac{\theta}{3} d\theta \Rightarrow 6 du = 2 \cos \frac{\theta}{3} d\theta$; $\theta = \frac{\pi}{2} \Rightarrow u = \frac{1}{2}$ and $\theta = \pi \Rightarrow u = \frac{\sqrt{3}}{2}$;

$$\int_{\pi/2}^{\pi} 2 \cot \frac{\theta}{3} d\theta = \int_{\pi/2}^{\pi} \frac{2 \cos \frac{\theta}{3}}{\sin \frac{\theta}{3}} d\theta = 6 \int_{1/2}^{\sqrt{3}/2} \frac{du}{u} = 6 [\ln |u|]_{1/2}^{\sqrt{3}/2} = 6 \left(\ln \frac{\sqrt{3}}{2} - \ln \frac{1}{2} \right) = 6 \ln \sqrt{3} = \ln 27$$

52. Let $u = \cos 3x \Rightarrow du = -3 \sin 3x \, dx \Rightarrow -2 du = 6 \sin 3x \, dx$; $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{12} \Rightarrow u = \frac{1}{\sqrt{2}}$;

$$\int_0^{\pi/12} 6 \tan 3x \, dx = \int_0^{\pi/12} \frac{6 \sin 3x}{\cos 3x} \, dx = -2 \int_1^{1/\sqrt{2}} \frac{du}{u} = -2 [\ln |u|]_1^{1/\sqrt{2}} = -2 \ln \frac{1}{\sqrt{2}} - \ln 1 = 2 \ln \sqrt{2} = \ln 2$$

53. $\int \frac{dx}{2\sqrt{x+2x}} = \int \frac{dx}{2\sqrt{x}(1+\sqrt{x})}$; let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx$; $\int \frac{dx}{2\sqrt{x}(1+\sqrt{x})} = \int \frac{du}{u} = \ln |u| + C$
 $= \ln |1 + \sqrt{x}| + C = \ln (1 + \sqrt{x}) + C$

54. Let $u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x) dx = (\sec x)(\tan x + \sec x) dx \Rightarrow \sec x \, dx = \frac{du}{u}$;
 $\int \frac{\sec x \, dx}{\sqrt{\ln(\sec x + \tan x)}} = \int \frac{du}{u \sqrt{\ln u}} = \int (\ln u)^{-1/2} \cdot \frac{1}{u} du = 2(\ln u)^{1/2} + C = 2\sqrt{\ln(\sec x + \tan x)} + C$

55. $y = \sqrt{x(x+1)} = (x(x+1))^{1/2} \Rightarrow \ln y = \frac{1}{2} \ln(x(x+1)) \Rightarrow 2 \ln y = \ln(x) + \ln(x+1) \Rightarrow \frac{2y'}{y} = \frac{1}{x} + \frac{1}{x+1}$
 $\Rightarrow y' = \left(\frac{1}{2}\right) \sqrt{x(x+1)} \left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{\sqrt{x(x+1)}(2x+1)}{2x(x+1)} = \frac{2x+1}{2\sqrt{x(x+1)}}$

56. $y = \sqrt{(x^2+1)(x-1)^2} \Rightarrow \ln y = \frac{1}{2} [\ln(x^2+1) + 2 \ln(x-1)] \Rightarrow \frac{y'}{y} = \frac{1}{2} \left(\frac{2x}{x^2+1} + \frac{2}{x-1} \right)$
 $\Rightarrow y' = \sqrt{(x^2+1)(x-1)^2} \left(\frac{x}{x^2+1} + \frac{1}{x-1} \right) = \sqrt{(x^2+1)(x-1)^2} \left[\frac{x^2-x+x^2+1}{(x^2+1)(x-1)} \right] = \frac{(2x^2-x+1)|x-1|}{\sqrt{x^2+1}(x-1)}$

57. $y = \sqrt{\frac{t}{t+1}} = \left(\frac{t}{t+1}\right)^{1/2} \Rightarrow \ln y = \frac{1}{2} [\ln t - \ln(t+1)] \Rightarrow \frac{1}{y} \frac{dy}{dt} = \frac{1}{2} \left(\frac{1}{t} - \frac{1}{t+1} \right)$
 $\Rightarrow \frac{dy}{dt} = \frac{1}{2} \sqrt{\frac{t}{t+1}} \left(\frac{1}{t} - \frac{1}{t+1} \right) = \frac{1}{2} \sqrt{\frac{t}{t+1}} \left[\frac{1}{t(t+1)} \right] = \frac{1}{2\sqrt{t}(t+1)^{3/2}}$

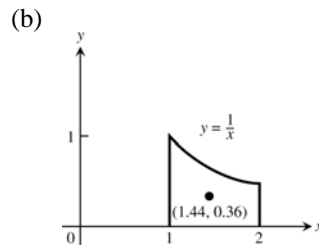
58. $y = \sqrt{\frac{1}{t(t+1)}} = [t(t+1)]^{-1/2} \Rightarrow \ln y = -\frac{1}{2} [\ln t + \ln(t+1)] \Rightarrow \frac{1}{y} \frac{dy}{dt} = -\frac{1}{2} \left(\frac{1}{t} + \frac{1}{t+1} \right)$
 $\Rightarrow \frac{dy}{dt} = -\frac{1}{2} \sqrt{\frac{1}{t(t+1)}} \left[\frac{2t+1}{t(t+1)} \right] = -\frac{2t+1}{2(t^2+t)^{3/2}}$

59. $y = \sqrt{\theta+3} (\sin \theta) = (\theta+3)^{1/2} \sin \theta \Rightarrow \ln y = \frac{1}{2} \ln(\theta+3) + \ln(\sin \theta) \Rightarrow \frac{1}{y} \frac{dy}{d\theta} = \frac{1}{2(\theta+3)} + \frac{\cos \theta}{\sin \theta}$
 $\Rightarrow \frac{dy}{d\theta} = \sqrt{\theta+3} (\sin \theta) \left[\frac{1}{2(\theta+3)} + \cot \theta \right]$

60. $y = (\tan \theta)\sqrt{2\theta+1} = (\tan \theta)(2\theta+1)^{1/2} \Rightarrow \ln y = \ln (\tan \theta) + \frac{1}{2} \ln (2\theta+1) \Rightarrow \frac{1}{y} \frac{dy}{d\theta} = \frac{\sec^2 \theta}{\tan \theta} + \left(\frac{1}{2}\right)\left(\frac{2}{2\theta+1}\right)$
 $\Rightarrow \frac{dy}{d\theta} = (\tan \theta)\sqrt{2\theta+1} \left(\frac{\sec^2 \theta}{\tan \theta} + \frac{1}{2\theta+1} \right) = (\sec^2 \theta) \sqrt{2\theta+1} + \frac{\tan \theta}{\sqrt{2\theta+1}}$
61. $y = t(t+1)(t+2) \Rightarrow \ln y = \ln t + \ln (t+1) + \ln (t+2) \Rightarrow \frac{1}{y} \frac{dy}{dt} = \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2} \Rightarrow \frac{dy}{dt} = t(t+1)(t+2) \left(\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2} \right)$
 $= t(t+1)(t+2) \left[\frac{(t+1)(t+2)+t(t+2)+t(t+1)}{t(t+1)(t+2)} \right] = 3t^2 + 6t + 2$
62. $y = \frac{1}{t(t+1)(t+2)} \Rightarrow \ln y = \ln 1 - \ln t - \ln (t+1) - \ln (t+2) \Rightarrow \frac{1}{y} \frac{dy}{dt} = -\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2}$
 $\Rightarrow \frac{dy}{dt} = \frac{1}{t(t+1)(t+2)} \left[-\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2} \right] = \frac{-1}{t(t+1)(t+2)} \left[\frac{(t+1)(t+2)+t(t+2)+t(t+1)}{t(t+1)(t+2)} \right] = -\frac{3t^2+6t+2}{(t^3+3t^2+2t)^2}$
63. $y = \frac{\theta+5}{\theta \cos \theta} \Rightarrow \ln y = \ln (\theta+5) - \ln \theta - \ln (\cos \theta) \Rightarrow \frac{1}{y} \frac{dy}{d\theta} = \frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin \theta}{\cos \theta} \Rightarrow \frac{dy}{d\theta} = \left(\frac{\theta+5}{\theta \cos \theta} \right) \left(\frac{1}{\theta+5} - \frac{1}{\theta} + \tan \theta \right)$
64. $y = \frac{\theta \sin \theta}{\sqrt{\sec \theta}} \Rightarrow \ln y = \ln \theta + \ln (\sin \theta) - \frac{1}{2} \ln (\sec \theta) \Rightarrow \frac{1}{y} \frac{dy}{d\theta} = \left[\frac{1}{\theta} + \frac{\cos \theta}{\sin \theta} - \frac{(\sec \theta)(\tan \theta)}{2 \sec \theta} \right]$
 $\Rightarrow \frac{dy}{d\theta} = \frac{\theta \sin \theta}{\sqrt{\sec \theta}} \left(\frac{1}{\theta} + \cot \theta - \frac{1}{2} \tan \theta \right)$
65. $y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} \Rightarrow \ln y = \ln x + \frac{1}{2} \ln (x^2+1) - \frac{2}{3} \ln (x+1) \Rightarrow \frac{y'}{y} = \frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}$
 $\Rightarrow y' = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} \left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)} \right]$
66. $y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}} \Rightarrow \ln y = \frac{1}{2} [10 \ln (x+1) - 5 \ln (2x+1)] \Rightarrow \frac{y'}{y} = \frac{5}{x+1} - \frac{5}{2x+1} \Rightarrow y' = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}} \left(\frac{5}{x+1} - \frac{5}{2x+1} \right)$
67. $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}} \Rightarrow \ln y = \frac{1}{3} [\ln x + \ln (x-2) - \ln (x^2+1)] \Rightarrow \frac{y'}{y} = \frac{1}{3} \left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1} \right)$
 $\Rightarrow y' = \frac{1}{3} \sqrt[3]{\frac{x(x-2)}{x^2+1}} \left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1} \right)$
68. $y = 3 \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \Rightarrow \ln y = \frac{1}{3} [\ln x + \ln (x+1) + \ln (x-2) - \ln (x^2+1) - \ln (2x+3)]$
 $\Rightarrow y' = \frac{1}{3} \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3} \right)$
69. (a) $f(x) = \ln (\cos x) \Rightarrow f'(x) = -\frac{\sin x}{\cos x} = -\tan x = 0 \Rightarrow x = 0$; $f'(x) > 0$ for $-\frac{\pi}{4} \leq x < 0$ and $f'(x) < 0$ for $0 < x \leq \frac{\pi}{3} \Rightarrow$ there is a relative maximum at $x = 0$ with $f(0) = \ln (\cos 0) = \ln 1 = 0$; $f\left(-\frac{\pi}{4}\right) = \ln \left(\cos\left(-\frac{\pi}{4}\right)\right) = \ln \left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2} \ln 2$ and $f\left(\frac{\pi}{3}\right) = \ln \left(\cos\left(\frac{\pi}{3}\right)\right) = \ln \frac{1}{2} = -\ln 2$. Therefore, the absolute minimum occurs at $x = \frac{\pi}{3}$ with $f\left(\frac{\pi}{3}\right) = -\ln 2$ and the absolute maximum occurs at $x = 0$ with $f(0) = 0$.

- (b) $f(x) = \cos(\ln x) \Rightarrow f'(x) = \frac{-\sin(\ln x)}{x} = 0 \Rightarrow x = 1$; $f'(x) > 0$ for $\frac{1}{2} \leq x < 1$ and $f'(x) < 0$ for $1 < x \leq 2$
 \Rightarrow there is a relative maximum at $x = 1$ with $f(1) = \cos(\ln 1) = \cos 0 = 1$; $f\left(\frac{1}{2}\right) = \cos\left(\ln\left(\frac{1}{2}\right)\right)$
 $= \cos(-\ln 2) = \cos(\ln 2)$ and $f(2) = \cos(\ln 2)$. Therefore, the absolute minimum occurs at $x = \frac{1}{2}$ and
 $x = 2$ with $f\left(\frac{1}{2}\right) = f(2) = \cos(\ln 2)$, and the absolute maximum occurs at $x = 1$ with $f(1) = 1$.
70. (a) $f(x) = x - \ln x \Rightarrow f'(x) = 1 - \frac{1}{x}$; if $x > 1$, then $f'(x) > 0$ which means that $f(x)$ is increasing
 (b) $f(1) = 1 - \ln 1 = 1 \Rightarrow f(x) = x - \ln x > 0$, if $x > 1$ by part (a) $\Rightarrow x > \ln x$ if $x > 1$
71. $\int_1^5 (\ln 2x - \ln x) dx = \int_1^5 (-\ln x + \ln 2 + \ln x) dx = (\ln 2) \int_1^5 dx = (\ln 2)(5 - 1) = \ln 2^4 = \ln 16$
72. $A = \int_{-\pi/4}^0 (-\tan x) dx + \int_0^{\pi/3} \tan x dx = \int_{-\pi/4}^0 \frac{-\sin x}{\cos x} dx - \int_0^{\pi/3} \frac{\sin x}{\cos x} dx = [\ln |\cos x|]_{-\pi/4}^0 - [\ln |\cos x|]_0^{\pi/3}$
 $= \left(\ln 1 - \ln \frac{1}{\sqrt{2}}\right) - \left(\ln \frac{1}{2} - \ln 1\right) = \ln \sqrt{2} + \ln 2 = \frac{3}{2} \ln 2$
73. $V = \pi \int_0^3 \left(\frac{2}{\sqrt{y+1}}\right)^2 dy = 4\pi \int_0^3 \frac{1}{y+1} dy = 4\pi [\ln |y+1|]_0^3 = 4\pi (\ln 4 - \ln 1) = 4\pi \ln 4$
74. $V = \pi \int_{\pi/6}^{\pi/2} \cot x dx = \pi \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x} dx = \pi [\ln (\sin x)]_{\pi/6}^{\pi/2} = \pi \left(\ln 1 - \ln \frac{1}{2}\right) = \pi \ln 2$
75. $V = 2\pi \int_{1/2}^2 x \left(\frac{1}{x^2}\right) dx = 2\pi \int_{1/2}^2 \frac{1}{x} dx = 2\pi [\ln |x|]_{1/2}^2 = 2\pi \left(\ln 2 - \ln \frac{1}{2}\right) = 2\pi (2 \ln 2) = \pi \ln 2^4 = \pi \ln 16$
76. $V = \pi \int_0^3 \left(\frac{9x}{\sqrt{x^3+9}}\right)^2 dx = 27\pi \int_0^3 \frac{3x^2}{x^3+9} dx = 27\pi \left[\ln (x^3+9)\right]_0^3 = 27\pi (\ln 36 - \ln 9) = 27\pi (\ln 4 + \ln 9 - \ln 9)$
 $= 27\pi \ln 4 = 54\pi \ln 2$
77. (a) $y = \frac{x^2}{8} - \ln x \Rightarrow 1 + (y')^2 = 1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2 = 1 + \left(\frac{x^2-4}{4x}\right)^2 = \left(\frac{x^2+4}{4x}\right)^2 \Rightarrow L = \int_4^8 \sqrt{1+(y')^2} dx$
 $= \int_4^8 \frac{x^2+4}{4x} dx = \int_4^8 \left(\frac{x}{4} + \frac{1}{x}\right) dx = \left[\frac{x^2}{8} + \ln |x|\right]_4^8 = (8 + \ln 8) - (2 + \ln 4) = 6 + \ln 2$
 (b) $x = \left(\frac{y}{4}\right)^2 - 2 \ln \left(\frac{y}{4}\right) \Rightarrow \frac{dx}{dy} = \frac{y}{8} - \frac{2}{y} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{y}{8} - \frac{2}{y}\right)^2 = 1 + \left(\frac{y^2-16}{8y}\right)^2 = \left(\frac{y^2+16}{8y}\right)^2$
 $\Rightarrow L = \int_4^{12} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_4^{12} \frac{y^2+16}{8y} dy = \int_4^{12} \left(\frac{y}{8} + \frac{2}{y}\right) dy = \left[\frac{y^2}{16} + 2 \ln y\right]_4^{12} = (9 + 2 \ln 12) - (1 + 2 \ln 4)$
 $= 8 + 2 \ln 3 = 8 + \ln 9$
78. $L = \int_1^2 \sqrt{1 + \frac{1}{x^2}} dx \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow y = \ln |x| + C = \ln x + C$ since $x > 0 \Rightarrow 0 = \ln 1 + C \Rightarrow C = 0 \Rightarrow y = \ln x$

79. (a) $M_y = \int_1^2 x \left(\frac{1}{x} \right) dx = 1$, $M_x = \int_1^2 \left(\frac{1}{2x} \right) \left(\frac{1}{x} \right) dx$
 $= \frac{1}{2} \int_1^2 \frac{1}{x^2} dx = \left[-\frac{1}{2x} \right]_1^2 = \frac{1}{4}$, $M = \int_1^2 \frac{1}{x} dx$
 $= [\ln |x|]_1^2 = \ln 2 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{1}{\ln 2} \approx 1.44$ and
 $\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{1}{4} \right)}{\ln 2} \approx 0.36$



80. (a) $M_y = \int_1^{16} x \left(\frac{1}{\sqrt{x}} \right) dx = \int_1^{16} x^{1/2} dx = \frac{2}{3} \left[x^{3/2} \right]_1^{16} = 42$; $M_x = \int_1^{16} \left(\frac{1}{2\sqrt{x}} \right) \left(\frac{1}{\sqrt{x}} \right) dx = \frac{1}{2} \int_1^{16} \frac{1}{x} dx$
 $= \frac{1}{2} [\ln |x|]_1^{16} = \ln 4$; $M = \int_1^{16} \frac{1}{\sqrt{x}} dx = \left[2x^{1/2} \right]_1^{16} = 6 \Rightarrow \bar{x} = \frac{M_y}{M} = 7$ and $\bar{y} = \frac{M_x}{M} = \frac{\ln 4}{6}$

(b) $M_y = \int_1^{16} x \left(\frac{1}{\sqrt{x}} \right) \left(\frac{4}{\sqrt{x}} \right) dx = 4 \int_1^{16} dx = 60$, $M_x = \int_1^{16} \left(\frac{1}{2\sqrt{x}} \right) \left(\frac{1}{\sqrt{x}} \right) \left(\frac{4}{\sqrt{x}} \right) dx = 2 \int_1^{16} x^{-3/2} dx$
 $= -4 \left[x^{-1/2} \right]_1^{16} = 3$; $M = \int_1^{16} \left(\frac{1}{\sqrt{x}} \right) \left(\frac{4}{\sqrt{x}} \right) dx = 4 \int_1^{16} \frac{1}{x} dx = [4 \ln |x|]_1^{16} = 4 \ln 16 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{15}{\ln 16}$ and
 $\bar{y} = \frac{M_x}{M} = \frac{3}{4 \ln 16}$

81. $f(x) = \ln(x^3 - 1)$, domain of f : $(1, \infty) \Rightarrow f'(x) = \frac{3x^2}{x^3 - 1}$; $f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$, not in the domain:
 $f'(x) = \text{undefined} \Rightarrow x^3 - 1 = 0 \Rightarrow x = 1$, not a domain. On $(1, \infty)$, $f'(x) > 0 \Rightarrow f$ is increasing on $(1, \infty) \Rightarrow f$ is one-to-one

82. $g(x) = \sqrt{x^2 + \ln x}$, domain of g : $x > 0.652919 \Rightarrow g'(x) = \frac{2x + \frac{1}{x}}{2\sqrt{x^2 + \ln x}} = \frac{2x^2 + 1}{2x\sqrt{x^2 + \ln x}}$; $g'(x) = 0 \Rightarrow 2x^2 + 1 = 0$
 \Rightarrow no real solutions; $g'(x) = \text{undefined} \Rightarrow 2x\sqrt{x^2 + \ln x} = 0 \Rightarrow x = 0$ or $x \approx 0.652919$, neither in domain. On $x > 0.652919$, $g'(x) > 0 \Rightarrow g$ is increasing for $x > 0.652919 \Rightarrow g$ is one-to-one

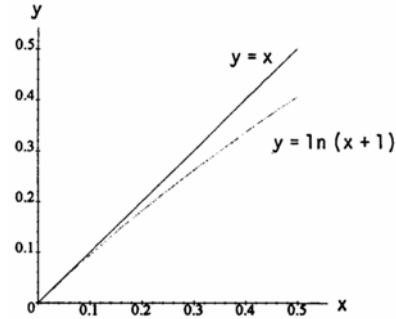
83. $\frac{dy}{dx} = 1 + \frac{1}{x}$ at $(1, 3) \Rightarrow y = x + \ln |x| + C$; $y = 3$ at $x = 1 \Rightarrow C = 2 \Rightarrow y = x + \ln |x| + 2$

84. $\frac{d^2y}{dx^2} = \sec^2 x \Rightarrow \frac{dy}{dx} = \tan x + C$ and $1 = \tan 0 + C \Rightarrow \frac{dy}{dx} = \tan x + 1 \Rightarrow y = \int (\tan x + 1) dx = \ln |\sec x| + x + C_1$ and
 $0 = \ln |\sec 0| + 0 + C_1 \Rightarrow C_1 = 0 \Rightarrow y = \ln |\sec x| + x$

85. (a) $L(x) = f(0) + f'(0) \cdot x$, and $f(x) = \ln(1+x) \Rightarrow f'(x) \Big|_{x=0} = \frac{1}{1+x} \Big|_{x=0} = 1 \Rightarrow L(x) = \ln 1 + 1 \cdot x \Rightarrow L(x) = x$

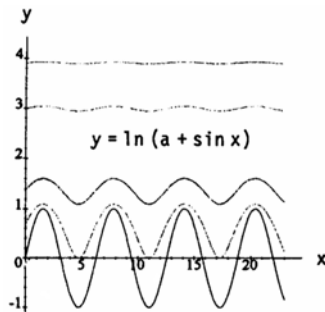
(b) Let $f(x) = \ln(x+1)$. Since $f''(x) = -\frac{1}{(x+1)^2} < 0$ on $[0, 0.1]$, the graph of f is concave down on this interval and the largest error in the linear approximation will occur when $x = 0.1$. This error is $0.1 - \ln(1.1) \approx 0.00469$ to five decimal places.

- (c) The approximation $y = x$ for $\ln(1+x)$ is best for smaller positive values of x ; in particular for $0 \leq x \leq 0.1$ in the graph. As x increases, so does the error $x - \ln(1+x)$. From the graph an upper bound for the error is $0.5 - \ln(1+0.5) \approx 0.095$; i.e., $|E(x)| \leq 0.095$ for $0 \leq x \leq 0.5$. Note from the graph that $0.1 - \ln(1+0.1) \approx 0.00469$ estimates the error in replacing $\ln(1+x)$ by x over $0 \leq x \leq 0.1$. This is consistent with the estimate given in part (b) above.



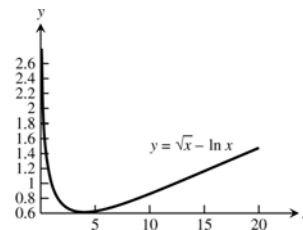
86. For all positive values of x , $\frac{d}{dx} \left[\ln \frac{a}{x} \right] = \frac{1}{\frac{a}{x}} \cdot -\frac{a}{x^2} = -\frac{1}{x}$ and $\frac{d}{dx} [\ln a - \ln x] = 0 - \frac{1}{x} = -\frac{1}{x}$. Since in $\frac{a}{x}$ and $\ln a - \ln x$ have the same derivative, then $\ln \frac{a}{x} = \ln a - \ln x + C$ for some constant C . Since this equation holds for all positive values of x , it must be true for $x = 1 \Rightarrow \ln \frac{a}{1} = \ln a - \ln 1 + C = \ln a - 0 + C \Rightarrow \ln \frac{a}{1} = \ln a + C$. Thus $\ln a = \ln a + C \Rightarrow C = 0 \Rightarrow \ln \frac{a}{x} = \ln a - \ln x$.

87. (a)



- (b) $y' = \frac{\cos x}{a + \sin x}$. Since $|\sin x|$ and $|\cos x|$ are less than or equal to 1, we have for $a > 1$
 $\frac{-1}{a-1} \leq y' \leq \frac{1}{a-1}$ for all x .
 Thus, $\lim_{a \rightarrow +\infty} y' = 0$ for all $x \Rightarrow$ the graph of y looks more and more horizontal as $a \rightarrow +\infty$.

88. (a) The graph of $y = \sqrt{x} - \ln x$ appears to be concave upward for all $x > 0$.



- (b) $y = \sqrt{x} - \ln x \Rightarrow y' = \frac{1}{2\sqrt{x}} - \frac{1}{x} \Rightarrow y'' = -\frac{1}{4x^{3/2}} + \frac{1}{x^2} = \frac{1}{x^2} \left(-\frac{\sqrt{x}}{4} + 1 \right) = 0 \Rightarrow \sqrt{x} = 4 \Rightarrow x = 16$. Thus $y'' > 0$ if $0 < x < 16$ and $y'' < 0$ if $x > 16$ so a point of inflection exists at $x = 16$. The graph of $y = \sqrt{x} - \ln x$ closely resembles a straight line $x \geq 10$ and it is impossible to discuss the point of inflection visually from the graph.

7.3 EXPONENTIAL FUNCTIONS

1. (a) $e^{-0.3t} = 27 \Rightarrow \ln e^{-0.3t} = \ln 3^3 \Rightarrow (-0.3t) \ln e = 3 \ln 3 \Rightarrow -0.3t = 3 \ln 3 \Rightarrow t = -10 \ln 3$
- (b) $e^{kt} = \frac{1}{2} \Rightarrow \ln e^{kt} = \ln 2^{-1} = kt \ln e = -\ln 2 \Rightarrow t = -\frac{\ln 2}{k}$
- (c) $e^{(\ln 0.2)t} = 0.4 \Rightarrow \left(e^{\ln 0.2} \right)^t = 0.4 \Rightarrow 0.2^t = 0.4 \Rightarrow \ln 0.2^t = \ln 0.4 \Rightarrow t \ln 0.2 = \ln 0.4 \Rightarrow t = \frac{\ln 0.4}{\ln 0.2}$

2. (a) $e^{-0.01t} = 1000 \Rightarrow \ln e^{-0.01t} = \ln 1000 \Rightarrow (-0.01t) \ln e = \ln 1000 \Rightarrow -0.01t = \ln 1000 \Rightarrow t = -100 \ln 1000$
 (b) $e^{kt} = \frac{1}{10} \Rightarrow \ln e^{kt} = \ln 10^{-1} \Rightarrow kt \ln e = -\ln 10 \Rightarrow kt = -\ln 10 \Rightarrow t = -\frac{\ln 10}{k}$
 (c) $e^{(\ln 2)t} = \frac{1}{2} \Rightarrow \left(e^{\ln 2}\right)^t = 2^{-1} \Rightarrow 2^t = 2^{-1} \Rightarrow t = -1$
3. $e^{\sqrt{t}} = x^2 \Rightarrow \ln e^{\sqrt{t}} = \ln x^2 \Rightarrow \sqrt{t} = 2 \ln x \Rightarrow t = 4(\ln x)^2$
4. $e^{x^2} e^{2x+1} = e^t \Rightarrow e^{x^2+2x+1} = e^t \Rightarrow \ln e^{x^2+2x+1} = \ln e^t \Rightarrow t = x^2 + 2x + 1$
5. $y = e^{-5x} \Rightarrow y' = e^{-5x} \frac{d}{dx}(-5x) \Rightarrow y' = -5e^{-5x}$
6. $y = e^{2x/3} \Rightarrow y' = e^{2x/3} \frac{d}{dx}\left(\frac{2x}{3}\right) \Rightarrow y' = \frac{2}{3} e^{2x/3}$
7. $y = e^{5-7x} \Rightarrow y' = e^{5-7x} \frac{d}{dx}(5-7x) \Rightarrow y' = -7e^{5-7x}$
8. $y = e^{(4\sqrt{x}+x^2)} \Rightarrow y' = e^{(4\sqrt{x}+x^2)} \frac{d}{dx}(4\sqrt{x}+x^2) \Rightarrow y' = \left(\frac{2}{\sqrt{x}} + 2x\right) e^{(4\sqrt{x}+x^2)}$
9. $y = xe^x - e^x \Rightarrow y' = (e^x + xe^x) - e^x = xe^x$
10. $y = (1+2x)e^{-2x} \Rightarrow y' = 2e^{-2x} + (1+2x)e^{-2x} \frac{d}{dx}(-2x) \Rightarrow y' = 2e^{-2x} - 2(1+2x)e^{-2x} = -4xe^{-2x}$
11. $y = (x^2 - 2x + 2)e^x \Rightarrow y' = (2x - 2)e^x + (x^2 - 2x + 2)e^x = x^2 e^x$
12. $y = (9x^2 - 6x + 2)e^{3x} \Rightarrow y' = (18x - 6)e^{3x} + (9x^2 - 6x + 2)e^{3x} \frac{d}{dx}(3x)$
 $\Rightarrow y' = (18x - 6)e^{3x} + 3(9x^2 - 6x + 2)e^{3x} = 27x^2 e^{3x}$
13. $y = e^\theta (\sin \theta + \cos \theta) \Rightarrow y' = e^\theta (\sin \theta + \cos \theta) + e^\theta (\cos \theta - \sin \theta) = 2e^\theta \cos \theta$
14. $y = \ln(3\theta e^{-\theta}) = \ln 3 + \ln \theta + \ln e^{-\theta} = \ln 3 + \ln \theta - \theta \Rightarrow \frac{dy}{d\theta} = \frac{1}{\theta} - 1$
15. $y = \cos(e^{-\theta^2}) \Rightarrow \frac{dy}{d\theta} = -\sin(e^{-\theta^2}) \frac{d}{d\theta}(e^{-\theta^2}) = \left(-\sin(e^{-\theta^2})\right) \left(e^{-\theta^2}\right) \frac{d}{d\theta}(-\theta^2) = 2\theta e^{-\theta^2} \sin(e^{-\theta^2})$
16. $y = \theta^3 e^{-2\theta} \cos 5\theta \Rightarrow \frac{dy}{d\theta} = (3\theta^2) \left(e^{-2\theta} \cos 5\theta\right) + (\theta^3 \cos 5\theta) e^{-2\theta} \frac{d}{d\theta}(-2\theta) - 5(\sin 5\theta) (\theta^3 e^{-2\theta})$
 $= \theta^2 e^{-2\theta} (3 \cos 5\theta - 2\theta \cos 5\theta - 5\theta \sin 5\theta)$
17. $y = \ln(3te^{-t}) = \ln 3 + \ln t + \ln e^{-t} = \ln 3 + \ln t - t \Rightarrow \frac{dy}{dt} = \frac{1}{t} - 1 = \frac{1-t}{t}$

$$18. \quad y = \ln(2e^{-t} \sin t) = \ln 2 + \ln e^{-t} + \ln \sin t = \ln 2 - t + \ln \sin t \Rightarrow \frac{dy}{dt} = -1 + \left(\frac{1}{\sin t}\right) \frac{d}{dt}(\sin t) = -1 + \frac{\cos t}{\sin t} = \frac{\cos t - \sin t}{\sin t}$$

$$19. \quad y = \ln \frac{e^\theta}{1+e^\theta} = \ln e^\theta - \ln(1+e^\theta) = \theta - \ln(1+e^\theta) \Rightarrow \frac{dy}{d\theta} = 1 - \left(\frac{1}{1+e^\theta}\right) \frac{d}{d\theta}(1+e^\theta) = 1 - \frac{e^\theta}{1+e^\theta} = \frac{1}{1+e^\theta}$$

$$20. \quad y = \ln \frac{\sqrt{\theta}}{1+\sqrt{\theta}} = \ln \sqrt{\theta} - \ln(1+\sqrt{\theta}) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\sqrt{\theta}}\right) \frac{d}{d\theta}(\sqrt{\theta}) - \left(\frac{1}{1+\sqrt{\theta}}\right) \frac{d}{d\theta}(1+\sqrt{\theta}) = \frac{d}{d\theta}(1+\sqrt{\theta}) \\ = \left(\frac{1}{\sqrt{\theta}}\right) \left(\frac{1}{2\sqrt{\theta}}\right) - \left(\frac{1}{1+\sqrt{\theta}}\right) \left(\frac{1}{2\sqrt{\theta}}\right) = \frac{(1+\sqrt{\theta}) - \sqrt{\theta}}{2\theta(1+\sqrt{\theta})} = \frac{1}{2\theta(1+\sqrt{\theta})} = \frac{1}{2\theta(1+\theta^{1/2})}$$

$$21. \quad y = e^{(\cos t + \ln t)} = e^{\cos t} e^{\ln t} = te^{\cos t} \Rightarrow \frac{dy}{dt} = e^{\cos t} + te^{\cos t} \frac{d}{dt}(\cos t) = (1 - t \sin t)e^{\cos t}$$

$$22. \quad y = e^{\sin t} (\ln t^2 + 1) \Rightarrow \frac{dy}{dt} = e^{\sin t} (\cos t) (\ln t^2 + 1) + \frac{2}{t} e^{\sin t} = e^{\sin t} \left[(\ln t^2 + 1)(\cos t) + \frac{2}{t} \right]$$

$$23. \quad \int_0^{\ln x} \sin e^t dt \Rightarrow y' = (\sin e^{\ln x}) \cdot \frac{d}{dx}(\ln x) = \frac{\sin x}{x}$$

$$24. \quad y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t dt \Rightarrow y' = (\ln e^{2x}) \cdot \frac{d}{dx}(e^{2x}) - (\ln e^{4\sqrt{x}}) \cdot \frac{d}{dx}(e^{4\sqrt{x}}) = (2x)(2e^{2x}) - (4\sqrt{x})(e^{4\sqrt{x}}) \cdot \frac{d}{dx}(4\sqrt{x}) \\ = 4xe^{2x} - 4\sqrt{x}e^{4\sqrt{x}} \left(\frac{2}{\sqrt{x}}\right) = 4xe^{2x} - 8e^{4\sqrt{x}}$$

$$25. \quad \ln y = e^y \sin x \Rightarrow \left(\frac{1}{y}\right) y' = (y' e^y) (\sin x) + e^y \cos x \Rightarrow y' \left(\frac{1}{y} - e^y \sin x\right) = e^y \cos x \\ \Rightarrow y' \left(\frac{1 - ye^y \sin x}{y}\right) = e^y \cos x \Rightarrow y' = \frac{ye^y \cos x}{1 - ye^y \sin x}$$

$$26. \quad \ln xy = e^{x+y} \Rightarrow \ln x + \ln y = e^{x+y} \Rightarrow \frac{1}{x} + \left(\frac{1}{y}\right) y' = (1 + y') e^{x+y} \Rightarrow y' \left(\frac{1}{y} - e^{x+y}\right) = e^{x+y} - \frac{1}{x} \\ \Rightarrow y' \left(\frac{1 - ye^{x+y}}{y}\right) = \frac{xe^{x+y} - 1}{x} \Rightarrow y' = \frac{y(xe^{x+y} - 1)}{x(1 - ye^{x+y})}$$

$$27. \quad e^{2x} = \sin(x + 3y) \Rightarrow 2e^{2x} = (1 + 3y') \cos(x + 3y) \Rightarrow 1 + 3y' = \frac{2e^{2x}}{\cos(x+3y)} \Rightarrow 3y' = \frac{2e^{2x}}{\cos(x+3y)} - 1 \\ \Rightarrow y' = \frac{2e^{2x} - \cos(x+3y)}{3 \cos(x+3y)}$$

$$28. \quad \tan y = e^x + \ln x \Rightarrow (\sec^2 y) y' = e^x + \frac{1}{x} \Rightarrow y' = \frac{(xe^x + 1) \cos^2 y}{x}$$

$$29. \quad \int (e^{3x} + 5e^{-x}) dx = \frac{e^{3x}}{3} - 5e^{-x} + C$$

$$30. \quad \int (2e^x - 3e^{-2x}) dx = 2e^x + \frac{3}{2}e^{-2x} + C$$

$$31. \quad \int_{\ln 2}^{\ln 3} e^x dx = \left[e^x \right]_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2} = 3 - 2 = 1$$

$$32. \quad \int_{-\ln 2}^0 e^{-x} dx = \left[-e^{-x} \right]_{-\ln 2}^0 = -e^0 + e^{\ln 2} = -1 + 2 = 1$$

33. $\int 8e^{(x+1)} dx = 8e^{(x+1)} + C$

34. $\int 2e^{(2x-1)} dx = e^{(2x-1)} + C$

35. $\int_{\ln 4}^{\ln 9} e^{x/2} dx = \left[2e^{x/2} \right]_{\ln 4}^{\ln 9} = 2 \left[e^{(\ln 9)/2} - e^{(\ln 4)/2} \right] = 2(e^{\ln 3} - e^{\ln 2}) = 2(3 - 2) = 2$

36. $\int_0^{\ln 16} e^{x/4} dx = \left[4e^{x/4} \right]_0^{\ln 16} = 4(e^{(\ln 16)/4} - e^0) = 4(e^{\ln 2} - 1) = 4(2 - 1) = 4$

37. Let $u = r^{1/2} \Rightarrow du = \frac{1}{2}r^{-1/2} dr \Rightarrow 2 du = r^{-1/2} dr$;

$$\int \frac{e^{\sqrt{r}}}{\sqrt{r}} dr = \int e^{r^{1/2}} \cdot r^{-1/2} dr = 2 \int e^u du = 2e^u + C = 2e^{r^{1/2}} + C = 2e^{\sqrt{r}} + C$$

38. Let $u = -r^{1/2} \Rightarrow du = -\frac{1}{2}r^{-1/2} dr \Rightarrow -2 du = r^{-1/2} dr$;

$$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} dr = \int e^{-r^{1/2}} \cdot r^{-1/2} dr = -2 \int e^u du = -2e^{-r^{1/2}} + C = -2e^{-\sqrt{r}} + C$$

39. Let $u = -t^2 \Rightarrow du = -2t dt \Rightarrow -du = 2t dt$; $\int 2te^{-t^2} dt = -\int e^u du = -e^u + C = -e^{-t^2} + C$

40. Let $u = t^4 \Rightarrow du = 4t^3 dt \Rightarrow \frac{1}{4} du = t^3 dt$; $\int t^3 e^{t^4} dt = \frac{1}{4} \int e^u du = \frac{1}{4} e^{t^4} + C$

41. Let $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx \Rightarrow -du = \frac{1}{x^2} dx$; $\int \frac{e^{1/x}}{x^2} dx = \int -e^u du = -e^u + C = -e^{1/x} + C$

42. Let $u = -x^{-2} \Rightarrow du = 2x^{-3} dx \Rightarrow \frac{1}{2} du = x^{-3} dx$;

$$\int \frac{e^{-1/x^2}}{x^3} dx = \int e^{-x^{-2}} \cdot x^{-3} dx = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{-x^{-2}} + C = \frac{1}{2} e^{-1/x^2} + C$$

43. Let $u = \tan \theta \Rightarrow du = \sec^2 \theta d\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \frac{\pi}{4} \Rightarrow u = 1$;

$$\begin{aligned} \int_0^{\pi/4} (1 + e^{\tan \theta}) \sec^2 \theta d\theta &= \int_0^{\pi/4} \sec^2 \theta d\theta + \int_0^1 e^u du = [\tan \theta]_0^{\pi/4} + [e^u]_0^1 = \left[\tan\left(\frac{\pi}{4}\right) - \tan(0) \right] + (e^1 - e^0) \\ &= (1 - 0) + (e - 1) = e \end{aligned}$$

44. Let $u = \cot \theta \Rightarrow du = -\csc^2 \theta d\theta$; $\theta = \frac{\pi}{4} \Rightarrow u = 1$, $\theta = \frac{\pi}{2} \Rightarrow u = 0$;

$$\begin{aligned} \int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta d\theta &= \int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta - \int_1^0 e^u du = [-\cot \theta]_{\pi/4}^{\pi/2} - [e^u]_1^0 = \left[-\cot\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{4}\right) \right] - (e^0 - e^1) \\ &= (0 + 1) - (1 - e) = e \end{aligned}$$

45. Let $u = \sec \pi t \Rightarrow du = \pi \sec \pi t \tan \pi t dt \Rightarrow \frac{du}{\pi} = \sec \pi t \tan \pi t dt$;

$$\int e^{\sec(\pi t)} \sec(\pi t) \tan(\pi t) dt = \frac{1}{\pi} \int e^u du = \frac{e^u}{\pi} + C = \frac{e^{\sec(\pi t)}}{\pi} + C$$

46. Let $u = \csc(\pi + t) \Rightarrow du = -\csc(\pi + t) \cot(\pi + t) dt$;

$$\int e^{\csc(\pi+t)} \csc(\pi+t) \cot(\pi+t) dt = -\int e^u du = -e^u + C = -e^{\csc(\pi+t)} + C$$

47. Let $u = e^v \Rightarrow du = e^v dv \Rightarrow 2 du = 2e^v dv$; $v = \ln \frac{\pi}{6} \Rightarrow u = \frac{\pi}{6}$, $v = \ln \frac{\pi}{2} \Rightarrow u = \frac{\pi}{2}$;
 $\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v dv = 2 \int_{\pi/6}^{\pi/2} \cos u du = [2 \sin u]_{\pi/6}^{\pi/2} = 2 \left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right) \right] = 2\left(1 - \frac{1}{2}\right) = 1$
48. Let $u = e^{x^2} \Rightarrow du = 2xe^{x^2} dx$; $x = 0 \Rightarrow u = 1$, $x = \sqrt{\ln \pi} \Rightarrow u = e^{\ln \pi} = \pi$;
 $\int_0^{\sqrt{\ln \pi}} 2xe^{x^2} \cos(e^{x^2}) dx = \int_1^{\pi} \cos u du = [\sin u]_1^{\pi} = \sin(\pi) - \sin(1) = -\sin(1) \approx -0.84147$
49. Let $u = 1 + e^r \Rightarrow du = e^r dr$; $\int \frac{e^r}{1+e^r} dr = \int \frac{1}{u} du = \ln |u| + C = \ln(1 + e^r) + C$
50. $\int \frac{1}{1+e^x} dx = \int \frac{e^{-x}}{e^{-x}+1} dx$; let $u = e^{-x} + 1 \Rightarrow du = -e^{-x} dx \Rightarrow -du = e^{-x} dx$;
 $\int \frac{e^{-x}}{e^{-x}+1} dx = -\int \frac{1}{u} du = -\ln |u| + C = -\ln(e^{-x} + 1) + C$
51. $\frac{dy}{dt} = e^t \sin(e^t - 2) \Rightarrow y = \int e^t \sin(e^t - 2) dt$;
 let $u = e^t - 2 \Rightarrow du = e^t dt \Rightarrow y = \int \sin u du = -\cos u + C = -\cos(e^t - 2) + C$; $y(\ln 2) = 0$
 $\Rightarrow -\cos(e^{\ln 2} - 2) + C = 0 \Rightarrow -\cos(2 - 2) + C = 0 \Rightarrow C = \cos 0 = 1$; thus, $y = 1 - \cos(e^t - 2)$
52. $\frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t}) \Rightarrow y = \int e^{-t} \sec^2(\pi e^{-t}) dt$;
 let $u = \pi e^{-t} \Rightarrow du = -\pi e^{-t} dt \Rightarrow -\frac{1}{\pi} du = e^{-t} dt \Rightarrow y = -\frac{1}{\pi} \int \sec^2 u du = -\frac{1}{\pi} \tan u + C$
 $= -\frac{1}{\pi} \tan(\pi e^{-t}) + C$; $y(\ln 4) = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi} \tan(\pi e^{-\ln 4}) + C = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi} \tan\left(\pi \cdot \frac{1}{4}\right) + C = \frac{2}{\pi}$
 $\Rightarrow -\frac{1}{\pi}(1) + C = \frac{2}{\pi} \Rightarrow C = \frac{3}{\pi}$; thus, $y = \frac{3}{\pi} - \frac{1}{\pi} \tan(\pi e^{-t})$
53. $\frac{d^2 y}{dx^2} = 2e^{-x} \Rightarrow \frac{dy}{dx} = -2e^{-x} + C$; $x = 0$ and $\frac{dy}{dx} = 0 \Rightarrow 0 = -2e^0 + C \Rightarrow C = 2$; thus $\frac{dy}{dx} = -2e^{-x} + 2$
 $\Rightarrow y = 2e^{-x} + 2x + C_1$; $x = 0$ and $y = 1 \Rightarrow 1 = 2e^0 + C_1 \Rightarrow C_1 = -1 \Rightarrow y = 2e^{-x} + 2x - 1 = 2(e^{-x} + x) - 1$
54. $\frac{d^2 y}{dt^2} = 1 - e^{2t} \Rightarrow \frac{dy}{dt} = t - \frac{1}{2}e^{2t} + C$; $t = 1$ and $\frac{dy}{dt} = 0 \Rightarrow 0 = 1 - \frac{1}{2}e^2 + C \Rightarrow C = \frac{1}{2}e^2 - 1$; thus
 $\frac{dy}{dt} = t - \frac{1}{2}e^{2t} + \frac{1}{2}e^2 - 1 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t + C_1$; $t = 1$ and $y = -1 \Rightarrow -1 = \frac{1}{2} - \frac{1}{4}e^2 + \frac{1}{2}e^2 - 1 + C_1$
 $\Rightarrow C_1 = -\frac{1}{2} - \frac{1}{4}e^2 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t - \left(\frac{1}{2} + \frac{1}{4}e^2\right)$
55. $y = 2^x \Rightarrow y' = 2^x \ln 2$
56. $y = 3^{-x} \Rightarrow y' = 3^{-x} (\ln 3)(-1) = -3^{-x} \ln 3$
57. $y = 5^{\sqrt{s}} \Rightarrow \frac{dy}{ds} = 5^{\sqrt{s}} (\ln 5) \left(\frac{1}{2} s^{-1/2}\right) = \left(\frac{\ln 5}{2\sqrt{s}}\right) 5^{\sqrt{s}}$
58. $y = 2^{s^2} \Rightarrow \frac{dy}{ds} = 2^{s^2} (\ln 2) 2s = (\ln 2^2) (s 2^{s^2}) = (\ln 4) s 2^{s^2}$

59. $y = x^\pi \Rightarrow y' = \pi x^{(\pi-1)}$

60. $y = t^{1-e} \Rightarrow \frac{dy}{dt} = (1-e)t^{-e}$

61. $y = (\cos \theta)^{\sqrt{2}} \Rightarrow \frac{dy}{d\theta} = -\sqrt{2}(\cos \theta)^{(\sqrt{2}-1)}(\sin \theta)$

62. $y = (\ln \theta)^\pi \Rightarrow \frac{dy}{d\theta} = \pi(\ln \theta)^{(\pi-1)}\left(\frac{1}{\theta}\right) = \frac{\pi(\ln \theta)^{(\pi-1)}}{\theta}$

63. $y = 7^{\sec \theta} \ln 7 \Rightarrow \frac{dy}{d\theta} = \left(7^{\sec \theta} \ln 7\right)(\ln 7)(\sec \theta \tan \theta) = 7^{\sec \theta}(\ln 7)^2(\sec \theta \tan \theta)$

64. $y = 3^{\tan \theta} \ln 3 \Rightarrow \frac{dy}{d\theta} = \left(3^{\tan \theta} \ln 3\right)(\ln 3)\sec^2 \theta = 3^{\tan \theta}(\ln 3)^2 \sec^2 \theta$

65. $y = 2^{\sin 3t} \Rightarrow \frac{dy}{dt} = \left(2^{\sin 3t} \ln 2\right)(\cos 3t)(3) = (3 \cos 3t)\left(2^{\sin 3t}\right)(\ln 2)$

66. $y = 5^{-\cos 2t} \Rightarrow \frac{dy}{dt} = \left(5^{-\cos 2t} \ln 5\right)(\sin 2t)(2) = (2 \sin 2t)\left(5^{-\cos 2t}\right)(\ln 5)$

67. $y = \log_2 5\theta = \frac{\ln 5\theta}{\ln 2} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 2}\right)\left(\frac{1}{5\theta}\right)(5) = \frac{1}{\theta \ln 2}$

68. $y = \log_3(1 + \theta \ln 3) = \frac{\ln(1 + \theta \ln 3)}{\ln 3} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 3}\right)\left(\frac{1}{1 + \theta \ln 3}\right)(\ln 3) = \frac{1}{1 + \theta \ln 3}$

69. $y = \frac{\ln x}{\ln 4} + \frac{\ln x^2}{\ln 4} = \frac{\ln x}{\ln 4} + 2\frac{\ln x}{\ln 4} = 3\frac{\ln x}{\ln 4} \Rightarrow y' = \frac{3}{x \ln 4}$

70. $y = \frac{x \ln e}{\ln 25} - \frac{\ln x}{2 \ln 5} = \frac{x}{2 \ln 5} - \frac{\ln x}{2 \ln 5} = \left(\frac{1}{2 \ln 5}\right)(x - \ln x) \Rightarrow y' = \left(\frac{1}{2 \ln 5}\right)\left(1 - \frac{1}{x}\right) = \frac{x-1}{2x \ln 5}$

71. $y = x^3 \log_{10} x = x^3 \left(\frac{\ln x}{\ln 10}\right) = \frac{1}{\ln 10} x^3 \ln x \Rightarrow y' = \frac{1}{\ln 10} \left(x^3 \cdot \frac{1}{x} + 3x^2 \ln x\right) = \frac{1}{\ln 10} x^2 + 3x^2 \frac{\ln x}{\ln 10}$
 $= \frac{1}{\ln 10} x^2 + 3x^2 \log_{10} x$

72. $y = \log_3 r \cdot \log_9 r = \left(\frac{\ln r}{\ln 3}\right)\left(\frac{\ln r}{\ln 9}\right) = \frac{\ln^2 r}{(\ln 3)(\ln 9)} \Rightarrow \frac{dy}{dr} = \left[\frac{1}{(\ln 3)(\ln 9)}\right](2 \ln r)\left(\frac{1}{r}\right) = \frac{2 \ln r}{r(\ln 3)(\ln 9)}$

73. $y = \log_3 \left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right) = \frac{\ln \left(\frac{x+1}{x-1}\right)^{\ln 3}}{\ln 3} = \frac{(\ln 3) \ln \left(\frac{x+1}{x-1}\right)}{\ln 3} = \ln \left(\frac{x+1}{x-1}\right) = \ln(x+1) - \ln(x-1) \Rightarrow \frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$

74. $y = \log_5 \sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}} = \log_5 \left(\frac{7x}{3x+2}\right)^{(\ln 5)/2} = \frac{\ln \left(\frac{7x}{3x+2}\right)^{(\ln 5)/2}}{\ln 5} = \left(\frac{\ln 5}{2}\right) \left[\frac{\ln \left(\frac{7x}{3x+2}\right)}{\ln 5}\right] = \frac{1}{2} \ln \left(\frac{7x}{3x+2}\right)$
 $= \frac{1}{2} \ln 7x - \frac{1}{2} \ln(3x+2) \Rightarrow \frac{dy}{dx} = \frac{7}{2 \cdot 7x} - \frac{3}{2 \cdot (3x+2)} = \frac{(3x+2)-3x}{2x(3x+2)} = \frac{1}{x(3x+2)}$

75. $y = \theta \sin(\log_7 \theta) = \theta \sin\left(\frac{\ln \theta}{\ln 7}\right) \Rightarrow \frac{dy}{d\theta} = \sin\left(\frac{\ln \theta}{\ln 7}\right) + \theta \left[\cos\left(\frac{\ln \theta}{\ln 7}\right)\right] \left(\frac{1}{\theta \ln 7}\right) = \sin(\log_7 \theta) + \frac{1}{\ln 7} \cos(\log_7 \theta)$

$$76. \quad y = \log_7 \left(\frac{\sin \theta \cos \theta}{e^\theta 2^\theta} \right) = \frac{\ln(\sin \theta) + \ln(\cos \theta) - \ln e^\theta - \ln 2^\theta}{\ln 7} = \frac{\ln(\sin \theta) + \ln(\cos \theta) - \theta - \theta \ln 2}{\ln 7}$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{\cos \theta}{(\sin \theta)(\ln 7)} - \frac{\sin \theta}{(\cos \theta)(\ln 7)} - \frac{1}{\ln 7} - \frac{\ln 2}{\ln 7} = \left(\frac{1}{\ln 7} \right) (\cot \theta - \tan \theta - 1 - \ln 2)$$

$$77. \quad y = \log_{10} e^x = \frac{\ln e^x}{\ln 10} = \frac{x}{\ln 10} \Rightarrow y' = \frac{1}{\ln 10}$$

$$78. \quad y = \frac{\theta \cdot 5^\theta}{2 - \log_5 \theta} = \frac{\theta \cdot 5^\theta}{2 - \frac{\ln \theta}{\ln 5}} \Rightarrow y' = \frac{\left(2 - \frac{\ln \theta}{\ln 5}\right) (\theta \cdot 5^\theta \ln 5 + 5^\theta (1)) - (\theta \cdot 5^\theta) \left(-\frac{1}{\theta \ln 5}\right)}{\left(2 - \frac{\ln \theta}{\ln 5}\right)^2} = \frac{5^\theta \ln 5 (2 - \log_5 \theta) (\theta \ln 5 + 1) + 5^\theta}{\ln 5 (2 - \log_5 \theta)^2}$$

$$79. \quad y = 3^{\log_2 t} = 3^{(\ln t)/(\ln 2)} \Rightarrow \frac{dy}{dt} = \left[3^{(\ln t)/(\ln 2)} (\ln 3) \right] \left(\frac{1}{t \ln 2} \right) = \frac{1}{t} (\log_2 3) 3^{\log_2 t}$$

$$80. \quad y = 3 \log_8 (\log_2 t) = \frac{3 \ln (\log_2 t)}{\ln 8} = \frac{3 \ln \left(\frac{\ln t}{\ln 2} \right)}{\ln 8} \Rightarrow \frac{dy}{dt} = \left(\frac{3}{\ln 8} \right) \left[\frac{1}{(\ln t)/(\ln 2)} \right] \left(\frac{1}{t \ln 2} \right) = \frac{3}{t (\ln t) (\ln 8)} = \frac{1}{t (\ln t) (\ln 2)}$$

$$81. \quad y = \log_2 (8t^{\ln 2}) = \frac{\ln 8 + \ln (t^{\ln 2})}{\ln 2} = \frac{3 \ln 2 + (\ln 2)(\ln t)}{\ln 2} = 3 + \ln t \Rightarrow \frac{dy}{dt} = \frac{1}{t}$$

$$82. \quad y = \frac{t \ln \left((e^{\ln 3})^{\sin t} \right)}{\ln 3} = \frac{t \ln (3^{\sin t})}{\ln 3} = \frac{t (\sin t) (\ln 3)}{\ln 3} = t \sin t \Rightarrow \frac{dy}{dt} = \sin t + t \cos t$$

$$83. \quad \int 5^x dx = \frac{5^x}{\ln 5} + C$$

$$84. \quad \text{Let } u = 3 - 3^x \Rightarrow du = -3^x \ln 3 dx \Rightarrow -\frac{1}{\ln 3} du = 3^x dx; \int \frac{3^x}{3 - 3^x} dx = -\frac{1}{\ln 3} \int \frac{1}{u} du = -\frac{1}{\ln 3} \ln |u| + C = -\frac{\ln |3 - 3^x|}{\ln 3} + C$$

$$85. \quad \int_0^1 2^{-\theta} d\theta = \int_0^1 \left(\frac{1}{2} \right)^\theta d\theta = \left[\frac{\left(\frac{1}{2} \right)^\theta}{\ln \left(\frac{1}{2} \right)} \right]_0^1 = \frac{\frac{1}{2}}{\ln \left(\frac{1}{2} \right)} - \frac{1}{\ln \left(\frac{1}{2} \right)} = -\frac{\frac{1}{2}}{\ln \left(\frac{1}{2} \right)} = \frac{-1}{2(\ln 1 - \ln 2)} = \frac{1}{2 \ln 2}$$

$$86. \quad \int_{-2}^0 5^{-\theta} d\theta = \int_{-2}^0 \left(\frac{1}{5} \right)^\theta d\theta = \left[\frac{\left(\frac{1}{5} \right)^\theta}{\ln \left(\frac{1}{5} \right)} \right]_{-2}^0 = \frac{1}{\ln \left(\frac{1}{5} \right)} - \frac{\left(\frac{1}{5} \right)^{-2}}{\ln \left(\frac{1}{5} \right)} = \frac{1}{\ln \left(\frac{1}{5} \right)} (1 - 25) = \frac{-24}{\ln 1 - \ln 5} = \frac{24}{\ln 5}$$

$$87. \quad \text{Let } u = x^2 \Rightarrow du = 2x dx \Rightarrow \frac{1}{2} du = x dx; x = 1 \Rightarrow u = 1, x = \sqrt{2} \Rightarrow u = 2;$$

$$\int_1^{\sqrt{2}} x 2^{(x^2)} dx = \int_1^2 \left(\frac{1}{2} \right) 2^u du = \frac{1}{2} \left[\frac{2^u}{\ln 2} \right]_1^2 = \left(\frac{1}{2 \ln 2} \right) (2^2 - 2^1) = \frac{1}{\ln 2}$$

$$88. \quad \text{Let } u = x^{1/2} \Rightarrow du = \frac{1}{2} x^{-1/2} dx \Rightarrow 2 du = \frac{dx}{\sqrt{x}}; x = 1 \Rightarrow u = 1, x = 4 \Rightarrow u = 2;$$

$$\int_1^4 \frac{2\sqrt{x}}{\sqrt{x}} dx = \int_1^4 2x^{1/2} \cdot x^{-1/2} dx = 2 \int_1^2 2^u du = \left[\frac{2^{(u+1)}}{\ln 2} \right]_1^2 = \left(\frac{1}{\ln 2} \right) (2^3 - 2^2) = \frac{4}{\ln 2}$$

89. Let
- $u = \cos t \Rightarrow du = -\sin t \, dt \Rightarrow -du = \sin t \, dt; t = 0 \Rightarrow u = 1, t = \frac{\pi}{2} \Rightarrow u = 0$
- ;

$$\int_0^{\pi/2} 7^{\cos t} \sin t \, dt = -\int_1^0 7^u \, du = \left[-\frac{7^u}{\ln 7} \right]_1^0 = \left(\frac{-1}{\ln 7} \right) (7^0 - 7) = \frac{6}{\ln 7}$$

90. Let
- $u = \tan t \Rightarrow du = \sec^2 t \, dt; t = 0 \Rightarrow u = 0, t = \frac{\pi}{4} \Rightarrow u = 1$
- ;

$$\int_0^{\pi/4} \left(\frac{1}{3} \right)^{\tan t} \sec^2 t \, dt = \int_0^1 \left(\frac{1}{3} \right)^u \, du = \left[\frac{\left(\frac{1}{3} \right)^u}{\ln \left(\frac{1}{3} \right)} \right]_0^1 = \left(-\frac{1}{\ln 3} \right) \left[\left(\frac{1}{3} \right)^1 - \left(\frac{1}{3} \right)^0 \right] = \frac{2}{3 \ln 3}$$

91. Let
- $u = x^{2x} \Rightarrow \ln u = 2x \ln x \Rightarrow \frac{1}{u} \frac{du}{dx} = 2 \ln x + (2x) \left(\frac{1}{x} \right) \Rightarrow \frac{du}{dx} = 2u(\ln x + 1) \Rightarrow \frac{1}{2} du = x^{2x} (1 + \ln x) dx$
- ;

$$x = 2 \Rightarrow u = 2^4 = 16, x = 4 \Rightarrow u = 4^8 = 65,536;$$

$$\int_2^4 x^{2x} (1 + \ln x) \, dx = \frac{1}{2} \int_{16}^{65,536} du = \frac{1}{2} [u]_{16}^{65,536} = \frac{1}{2} (65,536 - 16) = \frac{65,520}{2} = 32,760$$

92. Let
- $u = 1 + 2^{x^2} \Rightarrow du = 2x^2 (2x) \ln 2 \, dx \Rightarrow \frac{1}{2 \ln 2} du = 2^{x^2} x \, dx$

$$\int \frac{x 2^{x^2}}{1 + 2^{x^2}} \, dx = \frac{1}{2 \ln 2} \int \frac{1}{u} \, du = \frac{1}{2 \ln 2} \ln |u| + C = \frac{\ln(1 + 2^{x^2})}{2 \ln 2} + C$$

93. $\int 3x^{\sqrt{3}} \, dx = \frac{3x^{(\sqrt{3}+1)}}{\sqrt{3}+1} + C$

94. $\int x^{(\sqrt{2}-1)} \, dx = \frac{x^{\sqrt{2}}}{\sqrt{2}} + C$

95. $\int_0^3 (\sqrt{2} + 1) x^{\sqrt{2}} \, dx = \left[x^{(\sqrt{2}+1)} \right]_0^3 = 3^{(\sqrt{2}+1)}$

96. $\int_1^e x^{(\ln 2)-1} \, dx = \left[\frac{x^{\ln 2}}{\ln 2} \right]_1^e = \frac{e^{\ln 2} - 1^{\ln 2}}{\ln 2} = \frac{2-1}{\ln 2} = \frac{1}{\ln 2}$

97. $\int \frac{\log_{10} x}{x} \, dx = \int \left(\frac{\ln x}{\ln 10} \right) \left(\frac{1}{x} \right) dx; \left[u = \ln x \Rightarrow du = \frac{1}{x} dx \right]$
 $\rightarrow \int \left(\frac{\ln x}{\ln 10} \right) \left(\frac{1}{x} \right) dx = \frac{1}{\ln 10} \int u \, du = \left(\frac{1}{\ln 10} \right) \left(\frac{1}{2} u^2 \right) + C = \frac{(\ln x)^2}{2 \ln 10} + C$

98. $\int_1^4 \frac{\log_2 x}{x} \, dx = \int_1^4 \left(\frac{\ln x}{\ln 2} \right) \left(\frac{1}{x} \right) dx; \left[u = \ln x \Rightarrow du = \frac{1}{x} dx; x = 1 \Rightarrow u = 0, x = 4 \Rightarrow u = \ln 4 \right]$
 $\rightarrow \int_1^4 \left(\frac{\ln x}{\ln 2} \right) \left(\frac{1}{x} \right) dx = \int_0^{\ln 4} \left(\frac{1}{\ln 2} \right) u \, du = \left(\frac{1}{\ln 2} \right) \left[\frac{1}{2} u^2 \right]_0^{\ln 4} = \left(\frac{1}{\ln 2} \right) \left[\frac{1}{2} (\ln 4)^2 \right] = \frac{(\ln 4)^2}{2 \ln 2} = \frac{(\ln 4)^2}{\ln 4} = \ln 4$

99. $\int_1^4 \frac{\ln 2 \log_2 x}{x} \, dx = \int_1^4 \left(\frac{\ln 2}{x} \right) \left(\frac{\ln x}{\ln 2} \right) dx = \int_1^4 \frac{\ln x}{x} \, dx = \left[\frac{1}{2} (\ln x)^2 \right]_1^4 = \frac{1}{2} [(\ln 4)^2 - (\ln 1)^2] = \frac{1}{2} (\ln 4)^2$
 $= \frac{1}{2} (2 \ln 2)^2 = 2 (\ln 2)^2$

100. $\int_1^e \frac{2 \ln 10 (\log_{10} x)}{x} \, dx = \int_1^e \frac{(\ln 10)(2 \ln x)}{(\ln 10)} \left(\frac{1}{x} \right) dx = \left[(\ln x)^2 \right]_1^e = (\ln e)^2 - (\ln 1)^2 = 1$

$$101. \int_0^2 \frac{\log_2(x+2)}{x+2} dx = \frac{1}{\ln 2} \int_0^2 [\ln(x+2)] \left(\frac{1}{x+2}\right) dx = \left(\frac{1}{\ln 2}\right) \left[\frac{(\ln(x+2))^2}{2} \right]_0^2 = \left(\frac{1}{\ln 2}\right) \left[\frac{(\ln 4)^2}{2} - \frac{(\ln 2)^2}{2} \right]$$

$$= \left(\frac{1}{\ln 2}\right) \left[\frac{4(\ln 2)^2}{2} - \frac{(\ln 2)^2}{2} \right] = \frac{3}{2} \ln 2$$

$$102. \int_{1/10}^{10} \frac{\log_{10}(10x)}{x} dx = \frac{10}{\ln 10} \int_{1/10}^{10} [\ln(10x)] \left(\frac{1}{10x}\right) dx = \left(\frac{10}{\ln 10}\right) \left[\frac{(\ln(10x))^2}{20} \right]_{1/10}^{10} = \left(\frac{10}{\ln 10}\right) \left[\frac{(\ln 100)^2}{20} - \frac{(\ln 1)^2}{2} \right]$$

$$= \left(\frac{10}{\ln 10}\right) \left[\frac{4(\ln 10)^2}{20} \right] = 2 \ln 10$$

$$103. \int_0^9 \frac{2 \log_{10}(x+1)}{x+1} dx = \frac{2}{\ln 10} \int_0^9 \ln(x+1) \left(\frac{1}{x+1}\right) dx = \left(\frac{2}{\ln 10}\right) \left[\frac{(\ln(x+1))^2}{2} \right]_0^9 = \left(\frac{2}{\ln 10}\right) \left[\frac{(\ln 10)^2}{2} - \frac{(\ln 1)^2}{2} \right] = \ln 10$$

$$104. \int_2^3 \frac{2 \log_2(x-1)}{x-1} dx = \frac{2}{\ln 2} \int_2^3 \ln(x-1) \left(\frac{1}{x-1}\right) dx = \left(\frac{2}{\ln 2}\right) \left[\frac{(\ln(x-1))^2}{2} \right]_2^3 = \left(\frac{2}{\ln 2}\right) \left[\frac{(\ln 2)^2}{2} - \frac{(\ln 1)^2}{2} \right] = \ln 2$$

$$105. \int \frac{dx}{x \log_{10} x} = \int \left(\frac{\ln 10}{\ln x}\right) \left(\frac{1}{x}\right) dx = (\ln 10) \int \left(\frac{1}{\ln x}\right) \left(\frac{1}{x}\right) dx; \left[u = \ln x \Rightarrow du = \frac{1}{x} dx\right]$$

$$\rightarrow (\ln 10) \int \left(\frac{1}{\ln x}\right) \left(\frac{1}{x}\right) dx = (\ln 10) \int \frac{1}{u} du = (\ln 10) \ln |u| + C = (\ln 10) \ln |\ln x| + C$$

$$106. \int \frac{dx}{x(\log_8 x)^2} = \int \frac{dx}{x \left(\frac{\ln x}{\ln 8}\right)^2} = (\ln 8)^2 \int \frac{(\ln x)^{-2}}{x} dx = (\ln 8)^2 \frac{(\ln x)^{-1}}{-1} + C = -\frac{(\ln 8)^2}{\ln x} + C$$

$$107. \int_1^{\ln x} \frac{1}{t} dt = [\ln |t|]_1^{\ln x} = \ln |\ln x| - \ln 1 = \ln (\ln x), x > 1$$

$$108. \int_1^{e^x} \frac{1}{t} dt = [\ln |t|]_1^{e^x} = \ln e^x - \ln 1 = x \ln e = x$$

$$109. \int_1^{1/x} \frac{1}{t} dt = [\ln |t|]_1^{1/x} = \ln \left|\frac{1}{x}\right| - \ln 1 = (\ln 1 - \ln |x|) - \ln 1 = -\ln x, x > 0$$

$$110. \frac{1}{\ln a} \int_1^x \frac{1}{t} dt = \left[\frac{1}{\ln a} \ln |t| \right]_1^x = \frac{\ln x}{\ln a} - \frac{\ln 1}{\ln a} = \log_a x, x > 0$$

$$111. y = (x+1)^x \Rightarrow \ln y = \ln(x+1)^x = x \ln(x+1) \Rightarrow \frac{y'}{y} = \ln(x+1) + x \cdot \frac{1}{(x+1)} \Rightarrow y' = (x+1)^x \left[\frac{x}{x+1} + \ln(x+1) \right]$$

$$112. y = x^2 + x^{2x} \Rightarrow y - x^2 = x^{2x} \Rightarrow \ln(y - x^2) = \ln x^{2x} = 2x \ln x \Rightarrow \frac{1}{y - x^2} (y' - 2x) = 2x \cdot \frac{1}{x} + 2 \cdot \ln x = 2 + 2 \ln x$$

$$\Rightarrow y' - 2x = (y - x^2)(2 + 2 \ln x) \Rightarrow y' = \left((x^2 + x^{2x}) - x^2 \right) (2 + 2 \ln x) + 2x = 2(x + x^{2x} + x^{2x} \ln x)$$

$$113. y = (\sqrt{t})^t = (t^{1/2})^t = t^{t/2} \Rightarrow \ln y = \ln t^{t/2} = \left(\frac{t}{2}\right) \ln t \Rightarrow \frac{1}{y} \frac{dy}{dt} = \left(\frac{1}{2}\right) (\ln t) + \left(\frac{t}{2}\right) \left(\frac{1}{t}\right) = \frac{\ln t}{2} + \frac{1}{2} \Rightarrow \frac{dy}{dt} = (\sqrt{t})^t \left(\frac{\ln t}{2} + \frac{1}{2} \right)$$

$$114. y = t^{\sqrt{t}} = t^{(t^{1/2})} \Rightarrow \ln y = \ln t^{(t^{1/2})} = (t^{1/2})(\ln t) \Rightarrow \frac{1}{y} \frac{dy}{dt} = \left(\frac{1}{2} t^{-1/2}\right)(\ln t) + t^{1/2} \left(\frac{1}{t}\right) = \frac{\ln t + 2}{2\sqrt{t}} \Rightarrow \frac{dy}{dt} = \left(\frac{\ln t + 2}{2\sqrt{t}}\right) t^{\sqrt{t}}$$

$$115. y = (\sin x)^x \Rightarrow \ln y = \ln (\sin x)^x = x \ln (\sin x) \Rightarrow \frac{y'}{y} = \ln (\sin x) + x \left(\frac{\cos x}{\sin x}\right) \Rightarrow y' = (\sin x)^x [\ln (\sin x) + x \cot x]$$

$$116. y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} = (\sin x)(\ln x) \Rightarrow \frac{y'}{y} = (\cos x)(\ln x) + (\sin x) \left(\frac{1}{x}\right) = \frac{\sin x + x(\ln x)(\cos x)}{x} \\ \Rightarrow y' = x^{\sin x} \left[\frac{\sin x + x(\ln x)(\cos x)}{x} \right]$$

$$117. y = \sin x^x \Rightarrow y' = \cos x^x \frac{d}{dx} (x^x); \text{ if } u = x^x \Rightarrow \ln u = \ln x^x = x \ln x \Rightarrow \frac{u'}{u} = x \cdot \frac{1}{x} + 1 \cdot \ln x = 1 + \ln x \\ \Rightarrow u' = x^x (1 + \ln x) \Rightarrow y' = \cos x^x \cdot x^x (1 + \ln x) = x^x \cos x^x (1 + \ln x)$$

$$118. y = (\ln x)^{\ln x} \Rightarrow \ln y = (\ln x) \ln (\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{x}\right) \ln (\ln x) + (\ln x) \left(\frac{1}{\ln x}\right) \frac{d}{dx} (\ln x) = \frac{\ln(\ln x)}{x} + \frac{1}{x} \\ \Rightarrow y' = \left(\frac{\ln(\ln x) + 1}{x}\right) (\ln x)^{\ln x}$$

$$119. f(x) = e^x - 2x \Rightarrow f'(x) = e^x - 2; f'(x) = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2; f(0) = 1, \text{ the absolute maximum;} \\ f(\ln 2) = 2 - 2 \ln 2 \approx 0.613706, \text{ the absolute minimum;} f(1) = e - 2 \approx 0.71828, \text{ a relative or local maximum} \\ \text{since } f''(x) = e^x \text{ is always positive.}$$

$$120. \text{ The function } f(x) = 2e^{\sin(x/2)} \text{ has a maximum whenever } \sin \frac{x}{2} = 1 \text{ and a minimum whenever } \sin \frac{x}{2} = -1. \\ \text{Therefore the maximums occur at } x = \pi + 2k(2\pi) \text{ and the minimums occur at } x = 3\pi + 2k(2\pi), \text{ where } k \text{ is} \\ \text{any integer. The maximum is } 2e \approx 5.43656 \text{ and the minimum is } \frac{2}{e} \approx 0.73576.$$

$$121. f(x) = xe^{-x} \Rightarrow f'(x) = xe^{-x}(-1) + e^{-x} = e^{-x} - xe^{-x} \Rightarrow f''(x) = -e^{-x} - (xe^{-x}(-1) + e^{-x}) = xe^{-x} - 2e^{-x} \\ \text{(a) } f'(x) = 0 \Rightarrow e^{-x} - xe^{-x} = e^{-x}(1-x) = 0 \Rightarrow e^{-x} = 0 \text{ or } 1-x = 0 \Rightarrow x = 1, f(1) = (1)e^{-1} = \frac{1}{e}; \text{ using second} \\ \text{derivative test, } f''(1) = (1)e^{-1} - 2e^{-1} = -\frac{1}{e} < 0 \Rightarrow \text{absolute maximum at } \left(1, \frac{1}{e}\right) \\ \text{(b) } f''(x) = 0 \Rightarrow xe^{-x} - 2e^{-x} = e^{-x}(x-2) = 0 \Rightarrow e^{-x} = 0 \text{ or } x-2 = 0 \Rightarrow x = 2, f(2) = (2)e^{-2} = \frac{2}{e^2}; \text{ since} \\ f''(1) < 0 \text{ and } f''(3) = e^{-3}(3-2) = \frac{1}{e^3} > 0 \Rightarrow \text{point of inflection at } \left(2, \frac{2}{e^2}\right)$$

$$122. f(x) = \frac{e^x}{1+e^{2x}} \Rightarrow f'(x) = \frac{(1+e^{2x})e^x - e^x(2e^{2x})}{(1+e^{2x})^2} = \frac{e^x - e^{3x}}{(1+e^{2x})^2} \Rightarrow f''(x) = \frac{(1+e^{2x})^2(e^x - 3e^{3x}) - (e^x - e^{3x})2(1+e^{2x})(2e^{2x})}{\left[(1+e^{2x})^2\right]^2} \\ = \frac{e^x(1-6e^{2x}+e^{4x})}{(1+e^{2x})^3}$$

$$(a) \quad f'(x) = 0 \Rightarrow e^x - e^{3x} = 0 \Rightarrow e^x(1 - e^{2x}) = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0; \quad f(0) = \frac{e^0}{1+e^{2(0)}} = \frac{1}{2}; \quad f'(x) = \text{undefined}$$

$$\Rightarrow (1 + e^{2x})^2 = 0 \Rightarrow e^{2x} = -1 \Rightarrow \text{no real solutions. Using the second derivative test,}$$

$$f''(0) = \frac{e^0(1 - 6e^{2(0)} + e^{4(0)})}{(1 + e^{2(0)})^3} = \frac{-4}{8} < 0 \Rightarrow \text{absolute maximum at } \left(0, \frac{1}{2}\right)$$

$$(b) \quad f''(x) = 0 \Rightarrow e^x(1 - 6e^{2x} + e^{4x}) \Rightarrow e^x = 0 \text{ or } 1 - 6e^{2x} + e^{4x} = 0 \Rightarrow e^{2x} = \frac{-(-6) \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2},$$

$$\Rightarrow x = \frac{\ln(3+2\sqrt{2})}{2} \text{ or } x = \frac{\ln(3-2\sqrt{2})}{2}. \quad f\left(\frac{\ln(3+2\sqrt{2})}{2}\right) = \frac{\sqrt{3+2\sqrt{2}}}{4+2\sqrt{2}} \text{ and } f\left(\frac{\ln(3-2\sqrt{2})}{2}\right) = \frac{\sqrt{3-2\sqrt{2}}}{4-2\sqrt{2}}; \text{ since}$$

$$f''(-1) > 0, f''(0) < 0, \text{ and } f''(1) > 0 \Rightarrow \text{points of inflection at } \left(\frac{\ln(3+2\sqrt{2})}{2}, \frac{\sqrt{3+2\sqrt{2}}}{4+2\sqrt{2}}\right) \text{ and}$$

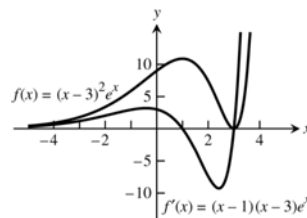
$$\left(\frac{\ln(3-2\sqrt{2})}{2}, \frac{\sqrt{3-2\sqrt{2}}}{4-2\sqrt{2}}\right).$$

$$123. \quad f(x) = x^2 \ln \frac{1}{x} \Rightarrow f'(x) = 2x \ln \frac{1}{x} + x^2 \left(\frac{1}{x}\right) \left(-x^{-2}\right) = 2x \ln \frac{1}{x} - x = -x(2 \ln x + 1); \quad f'(x) = 0 \Rightarrow x = 0 \text{ or}$$

$$\ln x = -\frac{1}{2}. \text{ Since } x = 0 \text{ is not in the domain of } f, \quad x = e^{-1/2} = \frac{1}{\sqrt{e}}. \text{ Also, } f'(x) > 0 \text{ for } 0 < x < \frac{1}{\sqrt{e}} \text{ and}$$

$$f'(x) < 0 \text{ for } x > \frac{1}{\sqrt{e}}. \text{ Therefore, } f\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{e} \ln \sqrt{e} = \frac{1}{e} \ln e^{1/2} = \frac{1}{2e} \ln e = \frac{1}{2e} \text{ is the absolute maximum value of } f \text{ assumed at } x = \frac{1}{\sqrt{e}}.$$

$$124. \quad f(x) = (x-3)^2 e^x \Rightarrow f'(x) = 2(x-3)e^x + (x-3)^2 e^x \\ = (x-3)e^x(2+x-3) = (x-1)(x-3)e^x; \text{ thus} \\ f'(x) > 0 \text{ for } x < 1 \text{ or } x > 3, \text{ and } f'(x) < 0 \text{ for} \\ 1 < x < 3 \Rightarrow f(1) = 4e \approx 10.87 \text{ is a local maximum} \\ \text{and } f(3) = 0 \text{ is a local minimum. Since } f(x) \geq 0 \text{ for} \\ \text{all } x, \quad f(3) = 0 \text{ is also an absolute minimum.}$$



$$125. \quad \int_0^{\ln 3} (e^{2x} - e^x) dx = \left[\frac{e^{2x}}{2} - e^x \right]_0^{\ln 3} = \left(\frac{e^{2 \ln 3}}{2} - e^{\ln 3} \right) - \left(\frac{e^0}{2} - e^0 \right) = \left(\frac{9}{2} - 3 \right) - \left(\frac{1}{2} - 1 \right) = \frac{8}{2} - 2 = 2$$

$$126. \quad \int_0^{2 \ln 2} (e^{x/2} - e^{-x/2}) dx = \left[2e^{x/2} + 2e^{-x/2} \right]_0^{2 \ln 2} = (2e^{\ln 2} + 2e^{-\ln 2}) - (2e^0 + 2e^0) = (4 + 1) - (2 + 2) = 5 - 4 = 1$$

$$127. \quad L = \int_0^1 \sqrt{1 + \frac{e^x}{4}} dx \Rightarrow \frac{dy}{dx} = \frac{e^{x/2}}{2} \Rightarrow y = e^{x/2} + C; \quad y(0) = 0 \Rightarrow 0 = e^0 + C \Rightarrow C = -1 \Rightarrow y = e^{x/2} - 1$$

$$128. \quad S = 2\pi \int_0^{\ln 2} \left(\frac{e^y + e^{-y}}{2} \right) \sqrt{1 + \left(\frac{e^y - e^{-y}}{2} \right)^2} dy = 2\pi \int_0^{\ln 2} \left(\frac{e^y + e^{-y}}{2} \right) \sqrt{1 + \frac{1}{4}(e^{2y} - 2 + e^{-2y})} dy \\ = 2\pi \int_0^{\ln 2} \left(\frac{e^y + e^{-y}}{2} \right) \sqrt{\left(\frac{e^y + e^{-y}}{2} \right)^2} dy = 2\pi \int_0^{\ln 2} \left(\frac{e^y + e^{-y}}{2} \right)^2 dy = \frac{\pi}{2} \int_0^{\ln 2} (e^{2y} + 2 + e^{-2y}) dy$$

$$\begin{aligned}
&= \frac{\pi}{2} \left[\frac{1}{2} e^{2y} + 2y - \frac{1}{2} e^{-2y} \right]_0^{\ln 2} = \frac{\pi}{2} \left[\left(\frac{1}{2} e^{2 \ln 2} + 2 \ln 2 - \frac{1}{2} e^{-2 \ln 2} \right) - \left(\frac{1}{2} + 0 - \frac{1}{2} \right) \right] \\
&= \frac{\pi}{2} \left(\frac{1}{2} \cdot 4 + 2 \ln 2 - \frac{1}{2} \cdot \frac{1}{4} \right) = \frac{\pi}{2} \left(2 - \frac{1}{8} + 2 \ln 2 \right) = \pi \left(\frac{15}{16} + \ln 2 \right)
\end{aligned}$$

$$\begin{aligned}
129. \quad y = \frac{1}{2}(e^x + e^{-x}) &\Rightarrow \frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x}); L = \int_0^1 \sqrt{1 + \left(\frac{1}{2}(e^x - e^{-x}) \right)^2} dx = \int_0^1 \sqrt{1 + \frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}} dx \\
&= \int_0^1 \sqrt{\frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4}} dx = \int_0^1 \sqrt{\left(\frac{1}{2}(e^x + e^{-x}) \right)^2} dx = \int_0^1 \frac{1}{2}(e^x + e^{-x}) dx = \frac{1}{2} \left[e^x - e^{-x} \right]_0^1 = \frac{1}{2} \left(e - \frac{1}{e} \right) - 0 = \frac{e^2 - 1}{2e}
\end{aligned}$$

$$\begin{aligned}
130. \quad y = \ln(e^x - 1) - \ln(e^x + 1) &\Rightarrow \frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1} = \frac{2e^x}{e^{2x} - 1}; L = \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{2e^x}{e^{2x} - 1} \right)^2} dx = \int_{\ln 2}^{\ln 3} \sqrt{1 + \frac{4e^{2x}}{(e^{2x} - 1)^2}} dx \\
&= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{(e^{2x} - 1)^2}} dx = \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{(e^{2x} - 1)^2}} dx = \int_{\ln 2}^{\ln 3} \sqrt{\frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}} dx = \int_{\ln 2}^{\ln 3} \frac{e^{2x} + 1}{e^{2x} - 1} dx = \int_{\ln 2}^{\ln 3} \frac{e^{2x} + 1}{e^{2x} - 1} dx \\
&= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}} dx; \quad [\text{let } u = e^x - e^{-x} \Rightarrow du = (e^x + e^{-x}) dx, x = \ln 2 \Rightarrow u = e^{\ln 2} - e^{-\ln 2} = 2 - \frac{1}{2} = \frac{3}{2}, \\
&x = \ln 3 \Rightarrow u = e^{\ln 3} - e^{-\ln 3} = 3 - \frac{1}{3} = \frac{8}{3}] \rightarrow \int_{3/2}^{8/3} \frac{1}{u} du = [\ln |u|]_{3/2}^{8/3} = \ln\left(\frac{8}{3}\right) - \ln\left(\frac{3}{2}\right) = \ln\left(\frac{16}{9}\right)
\end{aligned}$$

$$\begin{aligned}
131. \quad y = \ln \cos x &\Rightarrow \frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x; L = \int_0^{\pi/4} \sqrt{1 + (-\tan x)^2} dx = \int_0^{\pi/4} \sqrt{1 + \tan^2 x} dx = \int_0^{\pi/4} \sqrt{\sec^2 x} dx \\
&= \int_0^{\pi/4} \sec x dx = [\ln |\sec x + \tan x|]_0^{\pi/4} = \left(\ln \left| \sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right) \right| \right) - (0) = \ln(\sqrt{2} + 1)
\end{aligned}$$

$$\begin{aligned}
132. \quad y = \ln \csc x &\Rightarrow \frac{dy}{dx} = \frac{-\cos x \cot x}{\csc x} = -\cot x; L = \int_{\pi/6}^{\pi/4} \sqrt{1 + (-\cot x)^2} dx = \int_{\pi/6}^{\pi/4} \sqrt{1 + \cot^2 x} dx = \int_{\pi/6}^{\pi/4} \sqrt{\csc^2 x} dx \\
&= \int_{\pi/6}^{\pi/4} \csc x dx = [-\ln |\csc x + \cot x|]_{\pi/6}^{\pi/4} = \left(-\ln \left| \csc\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right) \right| \right) + \left(\ln \left| \csc\left(\frac{\pi}{6}\right) + \cot\left(\frac{\pi}{6}\right) \right| \right) \\
&= -\ln(\sqrt{2} + 1) + \ln(2 + \sqrt{3}) = \ln\left(\frac{2 + \sqrt{3}}{\sqrt{2} + 1}\right)
\end{aligned}$$

$$133. \quad (a) \quad \frac{d}{dx}(x \ln x - x + C) = x \cdot \frac{1}{x} + \ln x - 1 + 0 = \ln x$$

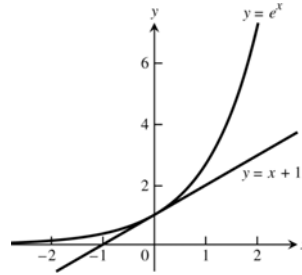
$$(b) \quad \text{average value} = \frac{1}{e-1} \int_1^e \ln x dx = \frac{1}{e-1} [x \ln x - x]_1^e = \frac{1}{e-1} [(e \ln e - e) - (1 \ln 1 - 1)] = \frac{1}{e-1} (e - e + 1) = \frac{1}{e-1}$$

$$134. \quad \text{average value} = \frac{1}{2-1} \int_1^2 \frac{1}{x} dx = [\ln |x|]_1^2 = \ln 2 - \ln 1 = \ln 2$$

$$135. \quad (a) \quad f(x) = e^x \Rightarrow f'(x) = e^x; L(x) = f(0) + f'(0)(x-0) \Rightarrow L(x) = 1 + x$$

$$(b) \quad f(0) = 1 \text{ and } L(0) = 1 \Rightarrow \text{error} = 0; f(0.2) = e^{0.2} \approx 1.22140 \text{ and } L(0.2) = 1.2 \Rightarrow \text{error} \approx 0.02140$$

- (c) Since $y'' = e^x > 0$, the tangent line approximation always lies below the curve $y = e^x$. Thus $L(x) = x + 1$ never overestimates e^x .



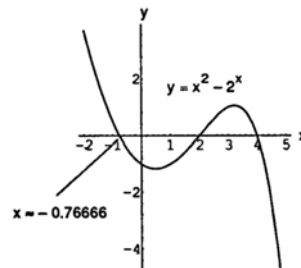
136. (a) $y = e^x \Rightarrow y'' = e^x > 0$ for all $x \Rightarrow$ the graph of $y = e^x$ is always concave upward
- (b) area of the trapezoid $ABCD < \int_{\ln a}^{\ln b} e^x dx < \text{area of the trapezoid } AEFD$
- $\Rightarrow \frac{1}{2}(AB + CD)(\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$. Now $\frac{1}{2}(AB + CD)$ is the height of the midpoint $M = e^{(\ln a + \ln b)/2}$ since the curve containing the points B and C is linear
- $\Rightarrow e^{(\ln a + \ln b)/2}(\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$
- (c) $\int_{\ln a}^{\ln b} e^x dx = [e^x]_{\ln a}^{\ln b} = e^{\ln b} - e^{\ln a} = b - a$, so part (b) implies that
- $e^{(\ln a + \ln b)/2}(\ln b - \ln a) < b - a < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a) \Rightarrow e^{(\ln a + \ln b)/2} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}$
- $\Rightarrow e^{\ln a/2} \cdot e^{\ln b/2} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2} \Rightarrow \sqrt{e^{\ln a}} \sqrt{e^{\ln b}} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2} \Rightarrow \sqrt{ab} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}$

137. $A = \int_{-2}^2 \frac{2x}{1+x^2} dx = 2 \int_0^2 \frac{2x}{1+x^2} dx; [u = 1 + x^2 \Rightarrow du = 2x dx; x = 0 \Rightarrow u = 1, x = 2 \Rightarrow u = 5]$

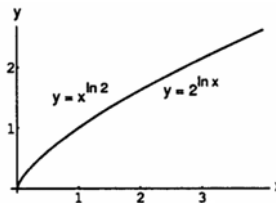
$\rightarrow A = 2 \int_1^5 \frac{1}{u} du = 2 [\ln |u|]_1^5 = 2(\ln 5 - \ln 1) = 2 \ln 5$

138. $A = \int_{-1}^1 2^{(1-x)} dx = 2 \int_{-1}^1 \left(\frac{1}{2}\right)^x dx = 2 \left[\frac{\left(\frac{1}{2}\right)^x}{\ln\left(\frac{1}{2}\right)} \right]_{-1}^1 = -\frac{2}{\ln 2} \left(\frac{1}{2} - 2\right) = \left(-\frac{2}{\ln 2}\right) \left(-\frac{3}{2}\right) = \frac{3}{\ln 2}$

139. From zooming in on the graph at the right, we estimate the third root to be $x \approx -0.76666$

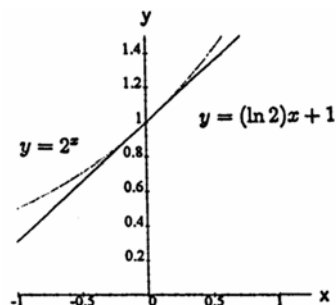
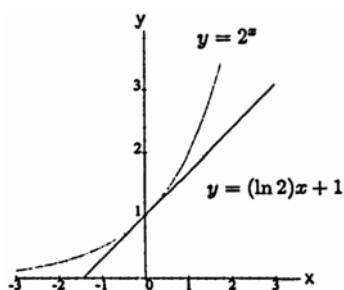


140. The functions $f(x) = x^{\ln 2}$ and $g(x) = 2^{\ln x}$ appear to have identical graphs for $x > 0$. This is no accident, because $x^{\ln 2} = e^{\ln 2 \cdot \ln x} = (e^{\ln 2})^{\ln x} = 2^{\ln x}$.



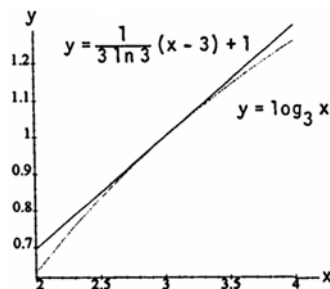
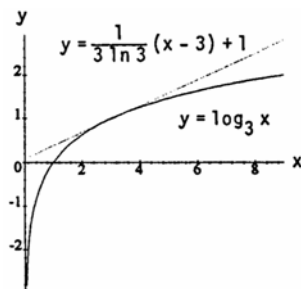
141. (a) $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2; L(x) = (2^0 \ln 2)x + 2^0 = x \ln 2 + 1 \approx 0.69x + 1$

(b)



142. (a) $f(x) = \log_3 x \Rightarrow f'(x) = \frac{1}{x \ln 3}$, and $f(3) = \frac{\ln 3}{\ln 3} \Rightarrow L(x) = \frac{1}{3 \ln 3}(x - 3) + \frac{\ln 3}{\ln 3} = \frac{x}{3 \ln 3} - \frac{1}{\ln 3} + 1 \approx 0.30x + 0.09$

(b)



143. (a) The point of tangency is $(p, \ln p)$ and $m_{\text{tangent}} = \frac{1}{p}$ since $\frac{dy}{dx} = \frac{1}{x}$. The tangent line passes through $(0, 0)$

\Rightarrow the equation of the tangent line is $y = \frac{1}{p}x$. The tangent line also passes through $(p, \ln p)$

$\Rightarrow \ln p = \frac{1}{p}p = 1 \Rightarrow p = e$, and the tangent line equation is $y = \frac{1}{e}x$.

(b) $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$ for $x \neq 0 \Rightarrow y = \ln x$ is concave downward over its domain. Therefore, $y = \ln x$ lies below the graph of $y = \frac{1}{e}x$ for all $x > 0$, $x \neq e$, and $\ln x < \frac{x}{e}$ for $x > 0$, $x \neq e$.

(c) Multiplying by e , $e \ln x < x$ or $\ln x^e < x$.

(d) Exponentiating both sides of $\ln x^e < x$, we have $e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

(e) Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, e^π is bigger.

144. Using Newton's Method: $f(x) = \ln(x) - 1 \Rightarrow f'(x) = \frac{1}{x} \Rightarrow x_{n+1} = x_n - \frac{\ln(x_n) - 1}{\frac{1}{x_n}} \Rightarrow x_{n+1} = x_n [2 - \ln(x_n)]$. Then,

$x_1 = 2$, $x_2 = 2.61370564$, $x_3 = 2.71624393$, and $x_5 = 2.71828183$. Many other methods may be used. For example, graph $y = \ln x - 1$ and determine the zero of y .

7.4 EXPONENTIAL CHANGE AND SEPARABLE DIFFERENTIAL EQUATIONS

1. (a) $y = e^{-x} \Rightarrow y' = -e^{-x} \Rightarrow 2y' + 3y = 2(-e^{-x}) + 3e^{-x} = e^{-x}$

(b) $y = e^{-x} + e^{-3x/2} \Rightarrow y' = -e^{-x} - \frac{3}{2}e^{-3x/2} \Rightarrow 2y' + 3y = 2(-e^{-x} - \frac{3}{2}e^{-3x/2}) + 3(e^{-x} + e^{-3x/2}) = e^{-x}$

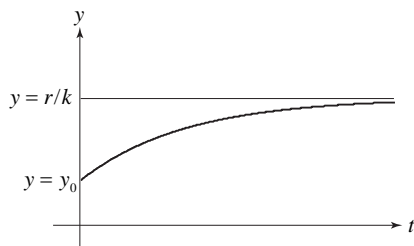
(c) $y = e^{-x} + Ce^{-3x/2} \Rightarrow y' = -e^{-x} - \frac{3}{2}Ce^{-3x/2} \Rightarrow 2y' + 3y = 2(-e^{-x} - \frac{3}{2}Ce^{-3x/2}) + 3(e^{-x} + Ce^{-3x/2}) = e^{-x}$

2. (a) $y = -\frac{1}{x} \Rightarrow y' = \frac{1}{x^2} = \left(-\frac{1}{x}\right)^2 = y^2$
 (b) $y = -\frac{1}{x+3} \Rightarrow y' = \frac{1}{(x+3)^2} = \left[-\frac{1}{(x+3)}\right]^2 = y^2$
 (c) $y = \frac{1}{x+C} \Rightarrow y' = \frac{1}{(x+C)^2} = \left[-\frac{1}{(x+C)}\right]^2 = y^2$
3. $y = \frac{1}{x} \int_1^x \frac{e^t}{t} dt \Rightarrow y' = -\frac{1}{x^2} \int_1^x \frac{e^t}{t} dt + \left(\frac{1}{x}\right)\left(\frac{e^x}{x}\right) \Rightarrow x^2 y' = -\int_1^x \frac{e^t}{t} dt + e^x = -x \left(\frac{1}{x} \int_1^x \frac{e^t}{t} dt\right) + e^x = -xy + e^x$
 $\Rightarrow x^2 y' + xy = e^x$
4. $y = \frac{1}{\sqrt{1+x^4}} \int_1^x \sqrt{1+t^4} dt \Rightarrow y' = -\frac{1}{2} \left[\frac{4x^3}{(\sqrt{1+x^4})^3} \right] \int_1^x \sqrt{1+t^4} dt + \frac{1}{\sqrt{1+x^4}} (\sqrt{1+x^4})$
 $\Rightarrow y' = \left(\frac{-2x^3}{1+x^4}\right) \left(\frac{1}{\sqrt{1+x^4}} \int_1^x \sqrt{1+t^4} dt\right) + 1 \Rightarrow y' = \left(\frac{-2x^3}{1+x^4}\right) y + 1 \Rightarrow y' + \frac{2x^3}{1+x^4} \cdot y = 1$
5. $y = e^{-x} \tan^{-1}(2e^x) \Rightarrow y' = -e^{-x} \tan^{-1}(2e^x) + e^{-x} \left[\frac{1}{1+(2e^x)^2} \right] (2e^x) = -e^{-x} \tan^{-1}(2e^x) + \frac{2}{1+4e^{2x}}$
 $\Rightarrow y' = -y + \frac{2}{1+4e^{2x}} \Rightarrow y' + y = \frac{2}{1+4e^{2x}}; y(-\ln 2) = e^{-(-\ln 2)} \tan^{-1}(2e^{-\ln 2}) = 2 \tan^{-1} 1 = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$
6. $y = (x-2)e^{-x^2} \Rightarrow y' = e^{-x^2} + (-2xe^{-x^2})(x-2) \Rightarrow y' = e^{-x^2} - 2xy; y(2) = (2-2)e^{-2^2} = 0$
7. $y = \frac{\cos x}{x} \Rightarrow y' = \frac{-x \sin x - \cos x}{x^2} \Rightarrow y' = -\frac{\sin x}{x} - \frac{1}{x} \left(\frac{\cos x}{x}\right) \Rightarrow y' = -\frac{\sin x}{x} - \frac{y}{x} \Rightarrow xy' = -\sin x - y$
 $\Rightarrow xy' + y = -\sin x; y\left(\frac{\pi}{2}\right) = \frac{\cos(\pi/2)}{(\pi/2)} = 0$
8. $y = \frac{x}{\ln x} \Rightarrow y' = \frac{\ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} \Rightarrow y' = \frac{1}{\ln x} - \frac{1}{(\ln x)^2} \Rightarrow x^2 y' = \frac{x^2}{\ln x} - \frac{x^2}{(\ln x)^2} \Rightarrow x^2 y' = xy - y^2; y(e) = \frac{e}{\ln e} = e.$
9. $2\sqrt{xy} \frac{dy}{dx} = 1 \Rightarrow 2x^{1/2} y^{1/2} dy = dx \Rightarrow 2y^{1/2} dy = x^{-1/2} dx \Rightarrow \int 2y^{1/2} dy = \int x^{-1/2} dx$
 $\Rightarrow 2\left(\frac{2}{3} y^{3/2}\right) = 2x^{1/2} + C_1 \Rightarrow \frac{2}{3} y^{3/2} - x^{1/2} = C, \text{ where } C = \frac{1}{2} C_1$
10. $\frac{dy}{dx} = x^2 \sqrt{y} \Rightarrow dy = x^2 y^{1/2} dx \Rightarrow y^{-1/2} dy = x^2 dx \Rightarrow \int y^{-1/2} dy = \int x^2 dx \Rightarrow 2y^{1/2} = \frac{x^3}{3} + C \Rightarrow 2y^{1/2} - \frac{1}{3} x^3 = C$
11. $\frac{dy}{dx} = e^{x-y} \Rightarrow dy = e^x e^{-y} dx \Rightarrow e^y dy = e^x dx \Rightarrow \int e^y dy = \int e^x dx \Rightarrow e^y = e^x + C \Rightarrow e^y - e^x = C$
12. $\frac{dy}{dx} = 3x^2 e^{-y} \Rightarrow dy = 3x^2 e^{-y} dx \Rightarrow e^y dy = 3x^2 dx \Rightarrow \int e^y dy = \int 3x^2 dx \Rightarrow e^y = x^3 + C \Rightarrow e^y - x^3 = C$

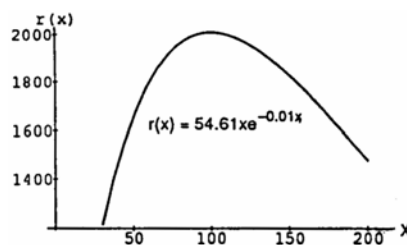
13. $\frac{dy}{dx} = \sqrt{y} \cos^2 \sqrt{y} \Rightarrow dy = (\sqrt{y} \cos^2 \sqrt{y}) dx \Rightarrow \frac{\sec^2 \sqrt{y}}{\sqrt{y}} dy = dx \Rightarrow \int \frac{\sec^2 \sqrt{y}}{\sqrt{y}} dy = \int dx$. In the integral on the left-hand side, substitute $u = \sqrt{y} \Rightarrow du = \frac{1}{2\sqrt{y}} dy \Rightarrow 2 du = \frac{1}{\sqrt{y}} dy$, and we have
 $\int \sec^2 u du = \int dx \Rightarrow 2 \tan u = x + C \Rightarrow -x + 2 \tan \sqrt{y} = C$
14. $\sqrt{2xy} \frac{dy}{dx} = 1 \Rightarrow dy = \frac{1}{\sqrt{2xy}} dx \Rightarrow \sqrt{2} \sqrt{y} dy = \frac{1}{\sqrt{x}} dx \Rightarrow \sqrt{2} y^{1/2} dy = x^{-1/2} dx \Rightarrow \sqrt{2} \int y^{1/2} dy = \int x^{-1/2} dx$
 $\Rightarrow \sqrt{2} \frac{y^{3/2}}{\frac{3}{2}} = \frac{x^{1/2}}{\frac{1}{2}} + C_1 \Rightarrow \sqrt{2} y^{3/2} = 3\sqrt{x} + \frac{3}{2} C_1 \Rightarrow \sqrt{2} (\sqrt{y})^3 - 3\sqrt{x} = C$, where $C = \frac{3}{2} C_1$
15. $\sqrt{x} \frac{dy}{dx} = e^{y+\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{e^y e^{\sqrt{x}}}{\sqrt{x}} \Rightarrow dy = \frac{e^y e^{\sqrt{x}}}{\sqrt{x}} dx \Rightarrow e^{-y} dy = \frac{e^{\sqrt{x}}}{\sqrt{x}} dx \Rightarrow \int e^{-y} dy = \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$. In the integral on the right-hand side, substitute $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2 du = \frac{1}{\sqrt{x}} dx$, and we have
 $\int e^{-y} dy = 2 \int e^u du \Rightarrow -e^{-y} = 2e^u + C_1 \Rightarrow -e^{-y} = 2e^{\sqrt{x}} + C$, where $C = -C_1$
16. $(\sec x) \frac{dy}{dx} = e^{y+\sin x} \Rightarrow \frac{dy}{dx} = e^{y+\sin x} \cos x \Rightarrow dy = (e^y e^{\sin x} \cos x) dx \Rightarrow e^{-y} dy = e^{\sin x} \cos x dx$
 $\Rightarrow \int e^{-y} dy = \int e^{\sin x} \cos x dx \Rightarrow -e^{-y} = e^{\sin x} + C_1 \Rightarrow e^{-y} + e^{\sin x} = C$, where $C = -C_1$
17. $\frac{dy}{dx} = 2x\sqrt{1-y^2} \Rightarrow dy = 2x\sqrt{1-y^2} dx \Rightarrow \frac{dy}{\sqrt{1-y^2}} = 2x dx \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int 2x dx \Rightarrow \sin^{-1} y = x^2 + C$ since
 $|y| < 1 \Rightarrow y = \sin(x^2 + C)$
18. $\frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}} \Rightarrow dy = \frac{e^{2x-y}}{e^{x+y}} dx \Rightarrow dy = \frac{e^{2x} e^{-y}}{e^x e^y} dx = \frac{e^x}{e^{2y}} dx \Rightarrow e^{2y} dy = e^x dx \Rightarrow \int e^{2y} dy = \int e^x dx \Rightarrow \frac{e^{2y}}{2} = e^x + C_1$
 $\Rightarrow e^{2y} - 2e^x = C$ where $C = 2C_1$
19. $y^2 \frac{dy}{dx} = 3x^2 y^3 - 6x^2 \Rightarrow y^2 dy = 3x^2 (y^3 - 2) dx \Rightarrow \frac{y^2}{y^3 - 2} dy = 3x^2 dx \Rightarrow \int \frac{y^2}{y^3 - 2} dy = \int 3x^2 dx$
 $\Rightarrow \frac{1}{3} \ln |y^3 - 2| = x^3 + C$
20. $\frac{dy}{dx} = xy + 3x - 2y - 6 = (y+3)(x-2) \Rightarrow \frac{1}{y+3} dy = (x-2) dx \Rightarrow \int \frac{1}{y+3} dy = \int (x-2) dx$
 $\Rightarrow \ln |y+3| = \frac{1}{2} x^2 - 2x + C$
21. $\frac{1}{x} \frac{dy}{dx} = ye^{x^2} + 2\sqrt{y}e^{x^2} = e^{x^2} (y + 2\sqrt{y}) \Rightarrow \frac{1}{y+2\sqrt{y}} dy = xe^{x^2} dx \Rightarrow \int \frac{1}{y+2\sqrt{y}} dy = \int xe^{x^2} dx$
 $\Rightarrow \int \frac{1}{\sqrt{y}(\sqrt{y}+2)} dy = \int xe^{x^2} dx \Rightarrow 2 \ln |\sqrt{y} + 2| = \frac{1}{2} e^{x^2} + C \Rightarrow 4 \ln |\sqrt{y} + 2| = e^{x^2} + C \Rightarrow 4 \ln (\sqrt{y} + 2) = e^{x^2} + C$
22. $\frac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1 = (e^{-y} + 1)(e^x + 1) \Rightarrow \frac{1}{e^{-y} + 1} dy = (e^x + 1) dx \Rightarrow \int \frac{1}{e^{-y} + 1} dy = \int (e^x + 1) dx$
 $\Rightarrow \int \frac{e^y}{1+e^y} dy = \int (e^x + 1) dx \Rightarrow \ln |1+e^y| = e^x + x + C \Rightarrow \ln (1+e^y) = e^x + x + C$

23. (a) $y = y_0 e^{kt} \Rightarrow 0.99 y_0 = y_0 e^{1000k} \Rightarrow k = \frac{\ln 0.99}{1000} \approx -0.00001$
 (b) $0.9 = e^{(-0.00001)t} \Rightarrow (-0.00001)t = \ln(0.9) \Rightarrow t = \frac{\ln(0.9)}{-0.00001} \approx 10,536$ years
 (c) $y = y_0 e^{(20,000)k} \approx y_0 e^{-0.2} = y_0(0.82) \Rightarrow 82\%$
24. (a) $\frac{dp}{dh} = kp \Rightarrow p = p_0 e^{kh}$ where $p_0 = 1013$; $90 = 1013 e^{20k} \Rightarrow k = \frac{\ln(90) - \ln(1013)}{20} \approx -0.121$
 (b) $p = 1013 e^{-6.05} \approx 2.389$ hectopascals
 (c) $900 = 1013 e^{(-0.121)h} \Rightarrow -0.121h = \ln\left(\frac{900}{1013}\right) \Rightarrow h = \frac{\ln(1013) - \ln(900)}{0.121} \approx 0.9777$ km
25. $\frac{dy}{dt} = -0.6y \Rightarrow y = y_0 e^{-0.6t}$; $y_0 = 100 \Rightarrow y = 100 e^{-0.6t} \Rightarrow y = 100 e^{-0.6} \approx 54.88$ grams when $t = 1$ h
26. $A = A_0 e^{kt} \Rightarrow 800 = 1000 e^{10k} \Rightarrow k = \frac{\ln(0.8)}{10} \Rightarrow A = 1000 e^{(\ln(0.8)/10)t}$, where A represents the amount of sugar that remains after time t . Thus after another 14 hours, $A = 1000 e^{(\ln(0.8)/10)24} \approx 585.35$ kg
27. $L(x) = L_0 e^{-kx} \Rightarrow \frac{L_0}{2} = L_0 e^{-6k} \Rightarrow \ln \frac{1}{2} = -6k \Rightarrow k = \frac{\ln 2}{6} \approx 0.1155 \Rightarrow L(x) = L_0 e^{-0.1155x}$; when the intensity is one-tenth of the surface value, $\frac{L_0}{10} = L_0 e^{-0.1155x} \Rightarrow \ln 10 = 0.1155x \Rightarrow x \approx 19.9$ m
28. $V(t) = V_0 e^{-t/40} \Rightarrow 0.1V_0 = V_0 e^{-t/40}$ when the voltage is 10% of its original value $\Rightarrow t = -40 \ln(0.1) \approx 92.1$ s
29. $y = y_0 e^{kt}$ and $y_0 = 1 \Rightarrow y = e^{kt} \Rightarrow$ at $y = 2$ and $t = 0.5$ we have $2 = e^{0.5k} \Rightarrow \ln 2 = 0.5k \Rightarrow k = \frac{\ln 2}{0.5} = \ln 4$.
 Therefore, $y = e^{(\ln 4)t} \Rightarrow y = e^{24 \ln 4} = 4^{24} = 2.81474978 \times 10^{14}$ at the end of 24 hours
30. $y = y_0 e^{kt}$ and $y(3) = 10,000 \Rightarrow 10,000 = y_0 e^{3k}$; also $y(5) = 40,000 = y_0 e^{5k}$. Therefore
 $y_0 e^{5k} = 4 y_0 e^{3k} \Rightarrow e^{5k} = 4 e^{3k} \Rightarrow e^{2k} = 4 \Rightarrow k = \ln 2$. Thus, $y = y_0 e^{(\ln 2)t} \Rightarrow 10,000 = y_0 e^{3 \ln 2} = y_0 e^{\ln 8}$
 $\Rightarrow 10,000 = 8 y_0 \Rightarrow y_0 = \frac{10,000}{8} = 1250$
31. (a) $10,000 e^{k(1)} = 7500 \Rightarrow e^k = 0.75 \Rightarrow k = \ln 0.75$ and $y = 10,000 e^{(\ln 0.75)t}$. Now $1000 = 10,000 e^{(\ln 0.75)t}$
 $\Rightarrow \ln 0.1 = (\ln 0.75)t \Rightarrow t = \frac{\ln 0.1}{\ln 0.75} \approx 8.00$ years (to the nearest hundredth of a year)
 (b) $1 = 10,000 e^{(\ln 0.75)t} \Rightarrow \ln 0.0001 = (\ln 0.75)t \Rightarrow t = \frac{\ln 0.0001}{\ln 0.75} \approx 32.02$ years (to the nearest hundredth of a year)
32. Let $z = r - ky$. Then $\frac{dz}{dt} = -k \frac{dy}{dt} = -k(r - ky) = -kz$. The equation $dz/dt = -kz$ has solution $z = c e^{-kt}$, so
 $r - ky = c e^{-kt}$ and $y = \frac{1}{k} (r - c e^{-kt})$.
 (a) Since $y(0) = y_0$, we have $y_0 = \frac{1}{k} (r - c)$ and thus $c = r - k y_0$. So
 $y = \frac{1}{k} (r - [r - k y_0] e^{-kt}) = \left(y_0 - \frac{r}{k} \right) e^{-kt} + \frac{r}{k}$.

(b) Since $k > 0$, $\lim_{t \rightarrow \infty} \left[\left(y_0 - \frac{r}{k} \right) e^{-kt} + \frac{r}{k} \right] = \frac{r}{k}$.



33. Let $y(t)$ be the population at time t , so $t(0) = 1147$ and we are interested in $t(20)$. If the population continues to decline at 39% per year, the population in 20 years would be $1147 \cdot (0.61)^{20} \approx 0.06 < 1$, so the species would be extinct.
34. (a) We will ignore leap years. There are $(60)(60)(24)(365) = 31,536,000$ seconds in a year. Thus, assuming exponential growth, $P = 314,419,198e^{kt}$, with t in years, and
- $$314,419,199 = 314,419,198e^{12k/31,536,000} \Rightarrow k = \frac{31,536,000}{12} \ln \left(\frac{314,419,199}{314,419,198} \right) \approx 0.0083583.$$
- (You don't really need to compute that logarithm: it will be very nearly equal to 1 over the denominator of the fraction.)
- (b) In seven years, $P = 314,419,198e^{(0.0083583)(7)} \approx 333,664,000$. (We certainly can't estimate this population to better than six significant digits.)
35. $0.9P_0 = P_0e^k \Rightarrow k = \ln 0.9$; when the well's output falls to one-fifth of its present value $P = 0.2P_0$
- $$\Rightarrow 0.2P_0 = P_0e^{(\ln 0.9)t} \Rightarrow 0.2 = e^{(\ln 0.9)t} \Rightarrow \ln(0.2) = (\ln 0.9)t \Rightarrow t = \frac{\ln 0.2}{\ln 0.9} \approx 15.28 \text{ years}$$
36. (a) $\frac{dp}{dx} = -\frac{1}{100}p \Rightarrow \frac{dp}{p} = -\frac{1}{100}dx \Rightarrow \ln p = -\frac{1}{100}x + C \Rightarrow p = e^{(-0.01x+C)} = e^C e^{-0.01x} = C_1 e^{-0.01x}$;
- $$p(100) = 20.09 \Rightarrow 20.09 = C_1 e^{(-0.01)(100)} \Rightarrow C_1 = 20.09e \approx 54.61 \Rightarrow p(x) = 54.61e^{-0.01x} \text{ (in dollars)}$$
- (b) $p(10) = 54.61e^{(-0.01)(10)} = \49.41 , and $p(90) = 54.61e^{(-0.01)(90)} = \22.20
- (c) $r(x) = xp(x) \Rightarrow r'(x) = p(x) + xp'(x)$;
- $$p'(x) = -0.5461e^{-0.01x}$$
- $$\Rightarrow r'(x) = (54.61 - 0.5461x)e^{-0.01x}.$$
- Thus,
- $$r'(x) = 0 \Rightarrow 54.61 = 0.5461x \Rightarrow x = 100.$$
- Since $r' > 0$ for any $x < 100$ and $r' < 0$ for $x > 100$, then $r(x)$ must be a maximum at $x = 100$.



37. $A = A_0e^{kt}$ and $A_0 = 10 \Rightarrow A = 10e^{kt}$, $5 = 10e^{k(24360)} \Rightarrow k = \frac{\ln(0.5)}{24360} \approx -0.000028454 \Rightarrow A = 10e^{-0.000028454t}$, then $0.2(10) = 10e^{-0.000028454t} \Rightarrow t = \frac{\ln 0.2}{-0.000028454} \approx 56563$ years
38. $A = A_0e^{kt}$ and $\frac{1}{2}A_0 = A_0e^{139k} \Rightarrow \frac{1}{2} = e^{139k} \Rightarrow k = \frac{\ln(0.5)}{139} \approx -0.00499$; then $0.05A_0 = A_0e^{-0.00499t} \Rightarrow t = \frac{\ln 0.05}{-0.00499} \approx 600$ days

$$39. \quad y = y_0 e^{-kt} = y_0 e^{-(k)(3/k)} = y_0 e^{-3} = \frac{y_0}{e^3} < \frac{y_0}{20} = (0.05)(y_0) \Rightarrow \text{after three mean lifetimes less than 5\% remains}$$

$$40. \quad (a) \quad A = A_0 e^{-kt} \Rightarrow \frac{1}{2} = e^{-2.645k} \Rightarrow k = \frac{\ln 2}{2.645} \approx 0.262$$

$$(b) \quad \frac{1}{k} \approx 3.816 \text{ years}$$

$$(c) \quad (0.05)A = A \exp\left(-\frac{\ln 2}{2.645}t\right) \Rightarrow -\ln 20 = \left(-\frac{\ln 2}{2.645}\right)t \Rightarrow t = \frac{2.645 \ln 20}{\ln 2} \approx 11.431 \text{ years}$$

$$41. \quad T - T_s = (T_0 - T_s)e^{-kt}, T_0 = 90^\circ\text{C}, T_s = 20^\circ\text{C}, T = 60^\circ\text{C} \Rightarrow 60 - 20 = 70e^{-10k} \Rightarrow \frac{4}{7} = e^{-10k}$$

$$\Rightarrow k = \frac{\ln\left(\frac{7}{4}\right)}{10} \approx 0.05596$$

$$(a) \quad 35 - 20 = 70e^{-0.05596t} \Rightarrow t \approx 27.5 \text{ min is the total time} \Rightarrow \text{it will take } 27.5 - 10 = 17.5 \text{ minutes longer to reach } 35^\circ\text{C}$$

$$(b) \quad T - T_s = (T_0 - T_s)e^{-kt}, T_0 = 90^\circ\text{C}, T_s = -15^\circ\text{C} \Rightarrow 35 + 15 = 105e^{-0.05596t} \Rightarrow t \approx 13.26 \text{ min}$$

$$42. \quad T - 18^\circ = (T_0 - 18^\circ)e^{-kt} \Rightarrow 2^\circ - 18^\circ = (T_0 - 18^\circ)e^{-10k} \text{ and } 10^\circ - 18^\circ = (T_0 - 18^\circ)e^{-20k}. \text{ Solving } \\ -16^\circ = (T_0 - 18^\circ)e^{-10k} \text{ and } -8^\circ = (T_0 - 18^\circ)e^{-20k} \text{ simultaneously} \Rightarrow (T_0 - 18^\circ)e^{-10k} = 2(T_0 - 18^\circ)e^{-20k} \\ \Rightarrow e^{10k} = 2 \Rightarrow k = \frac{\ln 2}{10} \text{ and } -16^\circ = \frac{T_0 - 18^\circ}{e^{10k}} \Rightarrow -16^\circ \left[e^{10\left(\frac{\ln 2}{10}\right)} \right] = T_0 - 18^\circ \Rightarrow T_0 = 18^\circ - 16^\circ(e^{\ln 2}) \\ = 18^\circ - 32^\circ = -14^\circ$$

$$43. \quad T - T_s = (T_0 - T_s)e^{-kt} \Rightarrow 39 - T_s = (46 - T_s)e^{-10k} \text{ and } 33 - T_s = (46 - T_s)e^{-20k} \Rightarrow \frac{39 - T_s}{46 - T_s} = e^{-10k} \text{ and}$$

$$\frac{33 - T_s}{46 - T_s} = e^{-20k} = \left(e^{-10k}\right)^2 \Rightarrow \frac{33 - T_s}{46 - T_s} = \left(\frac{39 - T_s}{46 - T_s}\right)^2 \Rightarrow (33 - T_s)(46 - T_s) = (39 - T_s)^2$$

$$\Rightarrow 1518 - 79T_s + T_s^2 = 1521 - 78T_s + T_s^2 \Rightarrow -T_s = 3 \Rightarrow T_s = -3^\circ\text{C}$$

44. Let x represent how far above room temperature the silver will be 15 min from now, y how far above room temperature the silver will be 120 min from now, and t_0 the time the silver will be 10°C above room temperature. We then have the following time-temperature table:

time in min.	0	20 (Now)	35	140	t_0
temperature	$T_s + 70^\circ$	$T_s + 60^\circ$	$T_s + x$	$T_s + y$	$T_s + 10^\circ$

$$T - T_s = (T_0 - T_s)e^{-kt} \Rightarrow (60 + T_s) - T_s = [(70 + T_s) - T_s]e^{-20k} \Rightarrow 60 = 70e^{-20k} \Rightarrow k = \left(-\frac{1}{20}\right)\ln\left(\frac{6}{7}\right) \approx 0.00771$$

$$(a) \quad T - T_s = (T_0 - T_s)e^{-0.00771t} \Rightarrow (T_s + x) - T_s = [(70 + T_s) - T_s]e^{-(0.00771)(35)} \Rightarrow x = 70e^{-0.26985} \approx 53.44^\circ\text{C}$$

$$(b) \quad T - T_s = (T_0 - T_s)e^{-0.00771t} \Rightarrow (T_s + y) - T_s = [(70 + T_s) - T_s]e^{-(0.00771)(140)}$$

$$\Rightarrow y = 70e^{-1.0794} \approx 23.79^\circ\text{C}$$

$$(c) \quad T - T_s = (T_0 - T_s)e^{-0.00771t} \Rightarrow (T_s + 10) - T_s = [(70 + T_s) - T_s]e^{-(0.00771)t_0} \Rightarrow 10 = 70e^{-0.00771t_0}$$

$$\Rightarrow \ln\left(\frac{1}{7}\right) = -0.00771t_0 \Rightarrow t_0 = \left(-\frac{1}{0.00771}\right)\ln\left(\frac{1}{7}\right) = 252.39 \Rightarrow 252.39 - 20 \approx 232 \text{ minutes from now the silver will be } 10^\circ\text{C above room temperature}$$

45. From Example 4, the half-life of carbon-14 is 5700 yr $\Rightarrow \frac{1}{2}c_0 = c_0e^{-k(5700)} \Rightarrow k = \frac{\ln 2}{5700} \approx 0.0001216$
 $\Rightarrow c = c_0e^{-0.0001216t} \Rightarrow (0.445)c_0 = c_0e^{-0.0001216t} \Rightarrow t = \frac{\ln(0.445)}{-0.0001216} \approx 6659$ years
46. From Exercise 45, $k \approx 0.0001216$ for carbon-14.
 (a) $c = c_0e^{-0.0001216t} \Rightarrow (0.17)c_0 = c_0e^{-0.0001216t} \Rightarrow t \approx 14,571.44$ years $\Rightarrow 12,571$ BC
 (b) $(0.18)c_0 = c_0e^{-0.0001216t} \Rightarrow t \approx 14,101.41$ years $\Rightarrow 12,101$ BC
 (c) $(0.16)c_0 = c_0e^{-0.0001216t} \Rightarrow t \approx 15,069.98$ years $\Rightarrow 13,070$ BC
47. From Exercise 45, $k \approx 0.0001216$ for carbon-14 $\Rightarrow y = y_0e^{-0.0001216t}$. When $t = 5000$
 $\Rightarrow y = y_0e^{-0.0001216(5000)} \approx 0.5444y_0 \Rightarrow \frac{y}{y_0} \approx 0.5444 \Rightarrow$ approximately 54.44% remains
48. From Exercise 45, $k \approx 0.0001216$ for carbon-14. Thus, $c = c_0e^{-0.0001216t} \Rightarrow (0.995)c_0 = c_0e^{-0.0001216t}$
 $\Rightarrow t = \frac{\ln(0.995)}{-0.0001216} \approx 41$ years old
49. $e^{-(\ln 2/5730)t} = 0.15 \Rightarrow -\frac{\ln 2}{5730}t = \ln(0.15) \Rightarrow t = -\frac{5730 \ln(0.15)}{\ln 2} \approx 15,683$ years
50. (a) $e^{-(\ln 2/5730)(500)} \approx 0.94131$, or about 94%.
 (b) We'll assume that the error could be 1% of the original amount. If the percentage of carbon-14 remaining were 0.93131, the Ice Maiden's actual age would be $-\frac{5730 \ln(0.93131)}{\ln 2} \approx 588$ years.

7.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

- l'Hôpital: $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \frac{1}{2x} \Big|_{x=2} = \frac{1}{4}$ or $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{4}$
- l'Hôpital: $\lim_{x \rightarrow 0} \frac{\sin 5x}{x} = \frac{5 \cos 5x}{1} \Big|_{x=0} = 5$ or $\lim_{x \rightarrow 0} \frac{\sin 5x}{x} = 5 \left[\lim_{5x \rightarrow 0} \frac{\sin 5x}{5x} \right] = 5 \cdot 1 = 5$
- l'Hôpital: $\lim_{x \rightarrow \infty} \frac{5x^2-3x}{7x^2+1} = \lim_{x \rightarrow \infty} \frac{10x-3}{14x} = \lim_{x \rightarrow \infty} \frac{10}{14} = \frac{5}{7}$ or $\lim_{x \rightarrow \infty} \frac{5x^2-3x}{7x^2+1} = \lim_{x \rightarrow \infty} \frac{5-\frac{3}{x}}{7+\frac{1}{x^2}} = \frac{5}{7}$
- l'Hôpital: $\lim_{x \rightarrow 1} \frac{x^3-1}{4x^3-x-3} = \lim_{x \rightarrow 1} \frac{3x^2}{12x^2-1} = \frac{3}{11}$ or $\lim_{x \rightarrow 1} \frac{x^3-1}{4x^3-x-3} = \lim_{x \rightarrow 1} \frac{(x-1)(x^2+x+1)}{(x-1)(4x^2+4x+3)} = \lim_{x \rightarrow 1} \frac{x^2+x+1}{4x^2+4x+3} = \frac{3}{11}$
- l'Hôpital: $\lim_{x \rightarrow 0} \frac{1-\cos x}{x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{2x} = \lim_{x \rightarrow 0} \frac{\cos x}{2} = \frac{1}{2}$ or $\lim_{x \rightarrow 0} \frac{1-\cos x}{x^2} = \lim_{x \rightarrow 0} \left[\frac{(1-\cos x)}{x^2} \left(\frac{1+\cos x}{1+\cos x} \right) \right]$
 $= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim_{x \rightarrow 0} \left[\left(\frac{\sin x}{x} \right) \left(\frac{\sin x}{x} \right) \left(\frac{1}{1+\cos x} \right) \right] = \frac{1}{2}$

6. l'Hôpital: $\lim_{x \rightarrow \infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x \rightarrow \infty} \frac{4x+3}{3x^2+1} = \lim_{x \rightarrow \infty} \frac{4}{6x} = 0$ or $\lim_{x \rightarrow \infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x \rightarrow \infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}+\frac{1}{x^3}} = \frac{0}{1} = 0$
7. $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{1}{2x} = \frac{1}{4}$
8. $\lim_{x \rightarrow -5} \frac{x^2-25}{x+5} = \lim_{x \rightarrow -5} \frac{2x}{1} = -10$
9. $\lim_{t \rightarrow -3} \frac{t^3-4t+15}{t^2-t-12} = \lim_{t \rightarrow -3} \frac{3t^2-4}{2t-1} = \frac{3(-3)^2-4}{2(-3)-1} = -\frac{23}{7}$
10. $\lim_{t \rightarrow -1} \frac{3t^3+3}{4t^3-t+3} = \lim_{t \rightarrow -1} \frac{9t^2}{12t^2-1} = \frac{9}{11}$
11. $\lim_{x \rightarrow \infty} \frac{5x^3-2x}{7x^3+3} = \lim_{x \rightarrow \infty} \frac{15x^2-2}{21x^2} = \lim_{x \rightarrow \infty} \frac{30x}{42x} = \lim_{x \rightarrow \infty} \frac{30}{42} = \frac{5}{7}$
12. $\lim_{x \rightarrow \infty} \frac{x-8x^2}{12x^2+5x} = \lim_{x \rightarrow \infty} \frac{1-16x}{24x+5} = \lim_{x \rightarrow \infty} \frac{-16}{24} = -\frac{2}{3}$
13. $\lim_{t \rightarrow 0} \frac{\sin t^2}{t} = \lim_{t \rightarrow 0} \frac{(\cos t^2)(2t)}{1} = 0$
14. $\lim_{t \rightarrow 0} \frac{\sin 5t}{2t} = \lim_{t \rightarrow 0} \frac{5 \cos 5t}{2} = \frac{5}{2}$
15. $\lim_{x \rightarrow 0} \frac{8x^2}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{16x}{-\sin x} = \lim_{x \rightarrow 0} \frac{16}{-\cos x} = \frac{16}{-1} = -16$
16. $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{-\sin x}{6x} = \lim_{x \rightarrow 0} \frac{-\cos x}{6} = -\frac{1}{6}$
17. $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{2\theta - \pi}{\cos(2\pi - \theta)} = \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{2}{\sin(2\pi - \theta)} = \frac{2}{\sin(\frac{3\pi}{2})} = -2$
18. $\lim_{\theta \rightarrow \frac{\pi}{3}} \frac{3\theta + \pi}{\sin(\theta + \frac{\pi}{3})} = \lim_{\theta \rightarrow \frac{\pi}{3}} \frac{3}{\cos(\theta + \frac{\pi}{3})} = 3$
19. $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{1 - \sin \theta}{1 + \cos 2\theta} = \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{-\cos \theta}{-2 \sin 2\theta} = \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sin \theta}{-4 \cos 2\theta} = \frac{1}{(-4)(-1)} = \frac{1}{4}$
20. $\lim_{x \rightarrow 1} \frac{x-1}{\ln x - \sin(\pi x)} = \lim_{x \rightarrow 1} \frac{1}{\frac{1}{x} - \pi \cos(\pi x)} = \frac{1}{1+\pi}$
21. $\lim_{x \rightarrow 0} \frac{x^2}{\ln(\sec x)} = \lim_{x \rightarrow 0} \frac{2x}{\frac{\sec x \tan x}{\sec x}} = \lim_{x \rightarrow 0} \frac{2x}{\tan x} = \lim_{x \rightarrow 0} \frac{2}{\sec^2 x} = \frac{2}{1^2} = 2$
22. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln(\csc x)}{x - (\frac{\pi}{2})^2} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\frac{(\csc x \cot x)}{\csc x}}{2(x - (\frac{\pi}{2}))} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\cot x}{2(x - (\frac{\pi}{2}))} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\csc^2 x}{2} = \frac{1^2}{2} = \frac{1}{2}$
23. $\lim_{t \rightarrow 0} \frac{t(1 - \cos t)}{t - \sin t} = \lim_{t \rightarrow 0} \frac{(1 - \cos t) + t(\sin t)}{1 - \cos t} = \lim_{t \rightarrow 0} \frac{\sin t + (\sin t + t \cos t)}{\sin t} = \lim_{t \rightarrow 0} \frac{\cos t + \cos t + \cos t - t \sin t}{\cos t} = \frac{1+1+1-0}{1} = 3$
24. $\lim_{t \rightarrow 0} \frac{t \sin t}{1 - \cos t} = \lim_{t \rightarrow 0} \frac{\sin t + t \cos t}{\sin t} = \lim_{t \rightarrow 0} \frac{\cos t + (\cos t - t \sin t)}{\cos t} = \frac{1+(1-0)}{1} = 2$

$$25. \quad \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \left(x - \frac{\pi}{2}\right) \sec x = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\left(x - \frac{\pi}{2}\right)}{\cos x} = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \left(\frac{1}{-\sin x}\right) = \frac{1}{-1} = -1$$

$$26. \quad \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \left(\frac{\pi}{2} - x\right) \tan x = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\left(\frac{\pi}{2} - x\right)}{\cot x} = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \left(\frac{-1}{-\csc^2 x}\right) = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \sin^2 x = 1$$

$$27. \quad \lim_{\theta \rightarrow 0} \frac{3^{\sin \theta} - 1}{\theta} = \lim_{\theta \rightarrow 0} \frac{3^{\sin \theta} (\ln 3)(\cos \theta)}{1} = \frac{(3^0)(\ln 3)(1)}{1} = \ln 3$$

$$28. \quad \lim_{\theta \rightarrow 0} \frac{\left(\frac{1}{2}\right)^\theta - 1}{\theta} = \lim_{\theta \rightarrow 0} \frac{\left(\ln\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)^\theta}{1} = \ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2$$

$$29. \quad \lim_{x \rightarrow 0} \frac{x 2^x}{2^x - 1} = \lim_{x \rightarrow 0} \frac{(1)(2^x) + (x)(\ln 2)(2^x)}{(\ln 2)(2^x)} = \frac{1 \cdot 2^0 + 0}{(\ln 2) \cdot 2^0} = \frac{1}{\ln 2}$$

$$30. \quad \lim_{x \rightarrow 0} \frac{3^x - 1}{2^x - 1} = \lim_{x \rightarrow 0} \frac{3^x \ln 3}{2^x \ln 2} = \frac{3^0 \cdot \ln 3}{2^0 \cdot \ln 2} = \frac{\ln 3}{\ln 2}$$

$$31. \quad \lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\log_2 x} = \lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\left(\frac{\ln x}{\ln 2}\right)} = (\ln 2) \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x+1}\right)}{\left(\frac{1}{x}\right)} = (\ln 2) \lim_{x \rightarrow \infty} \frac{x}{x+1} = (\ln 2) \lim_{x \rightarrow \infty} \frac{1}{1} = \ln 2$$

$$32. \quad \lim_{x \rightarrow \infty} \frac{\log_2 x}{\log_3(x+3)} = \lim_{x \rightarrow \infty} \frac{\left(\frac{\ln x}{\ln 2}\right)}{\left(\frac{\ln(x+3)}{\ln 3}\right)} = \left(\frac{\ln 3}{\ln 2}\right) \lim_{x \rightarrow \infty} \frac{\ln x}{\ln(x+3)} = \left(\frac{\ln 3}{\ln 2}\right) \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x+3}\right)} = \left(\frac{\ln 3}{\ln 2}\right) \lim_{x \rightarrow \infty} \frac{x+3}{x} = \left(\frac{\ln 3}{\ln 2}\right) \lim_{x \rightarrow \infty} \frac{1}{1} = \frac{\ln 3}{\ln 2}$$

$$33. \quad \lim_{x \rightarrow 0^+} \frac{\ln(x^2 + 2x)}{\ln x} = \lim_{x \rightarrow 0^+} \frac{\left(\frac{2x+2}{x^2+2x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow 0^+} \frac{2x^2+2x}{x^2+2x} = \lim_{x \rightarrow 0^+} \frac{4x+2}{2x+2} = \lim_{x \rightarrow 0^+} \frac{2}{2} = 1$$

$$34. \quad \lim_{x \rightarrow 0^+} \frac{\ln(e^x - 1)}{\ln x} = \lim_{x \rightarrow 0^+} \frac{\left(\frac{e^x}{e^x - 1}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow 0^+} \frac{x e^x}{e^x - 1} = \lim_{x \rightarrow 0^+} \frac{e^x + x e^x}{e^x} = \frac{1+0}{1} = 1$$

$$35. \quad \lim_{y \rightarrow 0} \frac{\sqrt{5y+25} - 5}{y} = \lim_{y \rightarrow 0} \frac{(5y+25)^{1/2} - 5}{y} = \lim_{y \rightarrow 0} \frac{\left(\frac{1}{2}\right)(5y+25)^{-1/2}(5)}{1} = \lim_{y \rightarrow 0} \frac{5}{2\sqrt{5y+25}} = \frac{1}{2}$$

$$36. \quad \lim_{y \rightarrow 0} \frac{\sqrt{ay+a^2} - a}{y} = \lim_{y \rightarrow 0} \frac{(ay+a^2)^{1/2} - a}{y} = \lim_{y \rightarrow 0} \frac{\left(\frac{1}{2}\right)(ay+a^2)^{-1/2}(a)}{1} = \lim_{y \rightarrow 0} \frac{a}{2\sqrt{ay+a^2}} = \frac{1}{2}, a > 0$$

$$37. \quad \lim_{x \rightarrow \infty} [\ln 2x - \ln(x+1)] = \lim_{x \rightarrow \infty} \ln\left(\frac{2x}{x+1}\right) = \ln\left(\lim_{x \rightarrow \infty} \frac{2x}{x+1}\right) = \ln\left(\lim_{x \rightarrow \infty} \frac{2}{1}\right) = \ln 2$$

$$38. \quad \lim_{x \rightarrow 0^+} (\ln x - \ln \sin x) = \lim_{x \rightarrow 0^+} \ln\left(\frac{x}{\sin x}\right) = \ln\left(\lim_{x \rightarrow 0^+} \frac{x}{\sin x}\right) = \ln\left(\lim_{x \rightarrow 0^+} \frac{1}{\cos x}\right) = \ln 1 = 0$$

39. $\lim_{x \rightarrow 0^+} \frac{(\ln x)^2}{\ln(\sin x)} = \lim_{x \rightarrow 0^+} \frac{2(\ln x)\left(\frac{1}{x}\right)}{\frac{\cos x}{\sin x}} = \lim_{x \rightarrow 0^+} \frac{2(\ln x)(\sin x)}{x \cos x} = \lim_{x \rightarrow 0^+} \left[\frac{2(\ln x)}{\cos x} \cdot \frac{\sin x}{x} \right] = -\infty \cdot 1 = -\infty$
40. $\lim_{x \rightarrow 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x} \right) = \lim_{x \rightarrow 0^+} \left(\frac{(3x+1)(\sin x) - x}{x \sin x} \right) = \lim_{x \rightarrow 0^+} \frac{3 \sin x + (3x+1)(\cos x) - 1}{\sin x + x \cos x} = \lim_{x \rightarrow 0^+} \left(\frac{3 \cos x + 3 \cos x + (3x+1)(-\sin x)}{\cos x + \cos x - x \sin x} \right)$
 $= \frac{3+3+(1)(0)}{1+1-0} = \frac{6}{2} = 3$
41. $\lim_{x \rightarrow 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right) = \lim_{x \rightarrow 1^+} \left(\frac{\ln x - (x-1)}{(x-1)(\ln x)} \right) = \lim_{x \rightarrow 1^+} \left(\frac{\frac{1}{x} - 1}{(\ln x) + (x-1)\left(\frac{1}{x}\right)} \right) = \lim_{x \rightarrow 1^+} \left(\frac{1-x}{(x \ln x) + x - 1} \right)$
 $= \lim_{x \rightarrow 1^+} \left(\frac{-1}{(\ln x + 1) + 1} \right) = \frac{-1}{(0+1)+1} = -\frac{1}{2}$
42. $\lim_{x \rightarrow 0^+} (\csc x - \cot x + \cos x) = \lim_{x \rightarrow 0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} + \cos x \right) = \lim_{x \rightarrow 0^+} \left(\frac{(1-\cos x) + (\sin x)(\cos x)}{\sin x} \right)$
 $= \lim_{x \rightarrow 0^+} \left(\frac{\sin x + \cos^2 x - \sin^2 x}{\cos x} \right) = \frac{0+1-0}{1} = 1$
43. $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{e^\theta - \theta - 1} = \lim_{\theta \rightarrow 0} \frac{-\sin \theta}{e^\theta - 1} = \lim_{\theta \rightarrow 0} \frac{-\cos \theta}{e^\theta} = -1$
44. $\lim_{h \rightarrow 0} \frac{e^h - (1+h)}{h^2} = \lim_{h \rightarrow 0} \frac{e^h - 1}{2h} = \lim_{h \rightarrow 0} \frac{e^h}{2} = \frac{1}{2}$
45. $\lim_{t \rightarrow \infty} \frac{e^t + t^2}{e^t - 1} = \lim_{t \rightarrow \infty} \frac{e^t + 2t}{e^t} = \lim_{t \rightarrow \infty} \frac{e^t + 2}{e^t} = \lim_{t \rightarrow \infty} \frac{e^t}{e^t} = 1$
46. $\lim_{x \rightarrow \infty} x^2 e^{-x} = \lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$
47. $\lim_{x \rightarrow 0} \frac{x - \sin x}{x \tan x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sec^2 x + \tan x} = \lim_{x \rightarrow 0} \frac{\sin x}{2x \sec^2 x \tan x + 2 \sec^2 x} = \frac{0}{2} = 0$
48. $\lim_{x \rightarrow 0} \frac{(e^x - 1)^2}{x \sin x} = \lim_{x \rightarrow 0} \frac{2(e^x - 1)e^x}{x \cos x + \sin x} = \lim_{x \rightarrow 0} \frac{2e^{2x} - 2e^x}{x \cos x + \sin x} = \lim_{x \rightarrow 0} \frac{4e^{2x} - 2e^x}{-x \sin x + 2 \cos x} = \frac{2}{2} = 1$
49. $\lim_{\theta \rightarrow 0} \frac{\theta - \sin \theta \cos \theta}{\tan \theta - \theta} = \lim_{\theta \rightarrow 0} \frac{1 + \sin^2 \theta - \cos^2 \theta}{\sec^2 \theta - 1} = \lim_{\theta \rightarrow 0} \frac{2 \sin^2 \theta}{\tan^2 \theta} = \lim_{\theta \rightarrow 0} 2 \cos^2 \theta = 2$
50. $\lim_{x \rightarrow 0} \frac{\sin 3x - 3x + x^2}{\sin x \sin 2x} = \lim_{x \rightarrow 0} \frac{3 \cos 3x - 3 + 2x}{2 \sin x \cos 2x + \cos x \sin 2x} = \lim_{x \rightarrow 0} \frac{3 \cos 3x - 3 + 2x}{\sin x \cos 2x + \sin 3x} = \lim_{x \rightarrow 0} \frac{-9 \sin 3x + 2}{-2 \sin x \sin 2x + \cos x \cos 2x + 3 \cos 3x}$
 $= \frac{2}{4} = \frac{1}{2}$
51. The limit leads to the indeterminate form 1^∞ . Let $f(x) = x^{1/(1-x)} \Rightarrow \ln f(x) = \ln \left(x^{1/(1-x)} \right) = \frac{\ln x}{1-x}$. Now
- $\lim_{x \rightarrow 1^+} \ln f(x) = \lim_{x \rightarrow 1^+} \frac{\ln x}{1-x} = \lim_{x \rightarrow 1^+} \frac{\left(\frac{1}{x}\right)}{-1} = -1$. Therefore $\lim_{x \rightarrow 1^+} x^{1/(1-x)} = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$

52. The limit leads to the indeterminate form 1^∞ . Let $f(x) = x^{1/(x-1)} \Rightarrow \ln f(x) = \ln \left(x^{1/(x-1)} \right) = \frac{\ln x}{x-1}$. Now

$$\lim_{x \rightarrow 1^+} \ln f(x) = \lim_{x \rightarrow 1^+} \frac{\ln x}{x-1} = \lim_{x \rightarrow 1^+} \frac{\left(\frac{1}{x}\right)}{1} = 1. \text{ Therefore } \lim_{x \rightarrow 1^+} x^{1/(x-1)} = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} e^{\ln f(x)} = e^1 = e$$

53. The limit leads to the indeterminate form ∞^0 . Let $f(x) = (\ln x)^{1/x} \Rightarrow \ln f(x) = \ln(\ln x)^{1/x} = \frac{\ln(\ln x)}{x}$. Now

$$\lim_{x \rightarrow \infty} \ln f(x) = \lim_{x \rightarrow \infty} \frac{\ln(\ln x)}{x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x \ln x}\right)}{1} = 0. \text{ Therefore } \lim_{x \rightarrow \infty} (\ln x)^{1/x} = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^0 = 1$$

54. The limit leads to the indeterminate form 1^∞ . Let $f(x) = (\ln x)^{1/(x-e)} \Rightarrow \ln f(x) = \frac{\ln(\ln x)}{x-e} = \lim_{x \rightarrow e^+} \ln f(x)$

$$= \lim_{x \rightarrow e^+} \frac{\ln(\ln x)}{x-e} = \lim_{x \rightarrow e^+} \frac{\left(\frac{1}{x \ln x}\right)}{1} = \frac{1}{e}. \text{ Therefore } (\ln x)^{1/(x-e)} = \lim_{x \rightarrow e^+} f(x) = \lim_{x \rightarrow e^+} e^{\ln f(x)} = e^{1/e}$$

55. The limit leads to the indeterminate form 0^0 . Let $f(x) = x^{-1/\ln x} \Rightarrow \ln f(x) = -\frac{\ln x}{\ln x} = -1$. Therefore

$$\lim_{x \rightarrow 0^+} x^{-1/\ln x} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$$

56. The limit leads to the indeterminate form ∞^0 . Let $f(x) = x^{1/\ln x} \Rightarrow \ln f(x) = \frac{\ln x}{\ln x} = 1$. Therefore

$$\lim_{x \rightarrow \infty} x^{1/\ln x} = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^1 = e$$

57. The limit leads to the indeterminate form ∞^0 . Let $f(x) = (1+2x)^{1/(2\ln x)} \Rightarrow \ln f(x) = \frac{\ln(1+2x)}{2\ln x}$

$$\Rightarrow \lim_{x \rightarrow \infty} \ln f(x) = \lim_{x \rightarrow \infty} \frac{\ln(1+2x)}{2\ln x} = \lim_{x \rightarrow \infty} \frac{x}{1+2x} = \lim_{x \rightarrow \infty} \frac{1}{2} = \frac{1}{2}. \text{ Therefore } \lim_{x \rightarrow \infty} (1+2x)^{1/(2\ln x)} = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^{1/2}$$

58. The limit leads to the indeterminate form 1^∞ . Let $f(x) = (e^x + x)^{1/x} \Rightarrow \ln f(x) = \frac{\ln(e^x + x)}{x}$

$$\Rightarrow \lim_{x \rightarrow 0} \ln f(x) = \lim_{x \rightarrow 0} \frac{\ln(e^x + x)}{x} = \lim_{x \rightarrow 0} \frac{e^x + 1}{e^x + x} = 2. \text{ Therefore } \lim_{x \rightarrow 0} (e^x + x)^{1/x} = \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} e^{\ln f(x)} = e^2$$

59. The limit leads to the indeterminate form 0^0 . Let $f(x) = x^x \Rightarrow \ln f(x) = x \ln x \Rightarrow \ln f(x) = \frac{\ln x}{\left(\frac{1}{x}\right)}$

$$= \lim_{x \rightarrow 0^+} \ln f(x) = \lim_{x \rightarrow 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x \rightarrow 0^+} (-x) = 0. \text{ Therefore } \lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^{\ln f(x)} = e^0 = 1$$

60. The limit leads to the indeterminate form ∞^0 . Let $f(x) = \left(1 + \frac{1}{x}\right)^x \Rightarrow \ln f(x) = \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \Rightarrow \lim_{x \rightarrow 0^+} \ln f(x)$

$$= \lim_{x \rightarrow 0^+} \frac{\left(\frac{-x^{-2}}{1+x^{-1}}\right)}{-x^{-2}} = \lim_{x \rightarrow 0^+} \frac{1}{1+x^{-1}} = \lim_{x \rightarrow 0^+} \frac{x}{x+1} = 0. \text{ Therefore } \lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^{\ln f(x)} = e^0 = 1$$

61. The limit leads to the indeterminate form 1^∞ . Let $f(x) = \left(\frac{x+2}{x-1}\right)^x \Rightarrow \ln f(x) = \ln\left(\frac{x+2}{x-1}\right)^x = x \ln\left(\frac{x+2}{x-1}\right)$
- $$\Rightarrow \lim_{x \rightarrow \infty} \ln f(x) = \lim_{x \rightarrow \infty} x \ln\left(\frac{x+2}{x-1}\right) = \lim_{x \rightarrow \infty} \left(\frac{\ln\left(\frac{x+2}{x-1}\right)}{\frac{1}{x}}\right) = \lim_{x \rightarrow \infty} \left(\frac{\ln(x+2) - \ln(x-1)}{\frac{1}{x}}\right) = \lim_{x \rightarrow \infty} \left(\frac{\frac{1}{x+2} - \frac{1}{x-1}}{-\frac{1}{x^2}}\right) = \lim_{x \rightarrow \infty} \left(\frac{\frac{-3}{(x+2)(x-1)}}{-\frac{1}{x^2}}\right)$$
- $$= \lim_{x \rightarrow \infty} \left(\frac{3x^2}{(x+2)(x-1)}\right) = \lim_{x \rightarrow \infty} \left(\frac{6x}{2x+1}\right) = \lim_{x \rightarrow \infty} \left(\frac{6}{2}\right) = 3. \text{ Therefore, } \lim_{x \rightarrow \infty} \left(\frac{x+2}{x-1}\right)^x = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^3$$
62. The limit leads to the indeterminate form ∞^0 . Let $f(x) = \left(\frac{x^2+1}{x+2}\right)^{1/x} \Rightarrow \ln f(x) = \ln\left(\frac{x^2+1}{x+2}\right)^{1/x} = \frac{1}{x} \ln\left(\frac{x^2+1}{x+2}\right)$
- $$\Rightarrow \lim_{x \rightarrow \infty} \ln f(x) = \lim_{x \rightarrow \infty} \frac{1}{x} \ln\left(\frac{x^2+1}{x+2}\right) = \lim_{x \rightarrow \infty} \frac{\ln\left(\frac{x^2+1}{x+2}\right)}{x} = \lim_{x \rightarrow \infty} \frac{\ln(x^2+1) - \ln(x+2)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{2x}{x^2+1} - \frac{1}{x+2}}{1} = \lim_{x \rightarrow \infty} \frac{x^2+4x-1}{(x^2+1)(x+2)}$$
- $$= \lim_{x \rightarrow \infty} \frac{x^2+4x-1}{x^3+2x^2+x+2} = \lim_{x \rightarrow \infty} \frac{2x+4}{3x^2+4x+1} = \lim_{x \rightarrow \infty} \frac{2}{6x+4} = 0. \text{ Therefore, } \lim_{x \rightarrow \infty} \left(\frac{x^2+1}{x+2}\right)^{1/x} = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^0 = 1$$
63. $\lim_{x \rightarrow 0^+} x^2 \ln x = \lim_{x \rightarrow 0^+} \left(\frac{\ln x}{\frac{1}{x^2}}\right) = \lim_{x \rightarrow 0^+} \left(\frac{\frac{1}{x}}{-\frac{2}{x^3}}\right) = \lim_{x \rightarrow 0^+} \left(-\frac{x^3}{2x}\right) = \lim_{x \rightarrow 0^+} \left(-\frac{3x^2}{2}\right) = 0$
64. $\lim_{x \rightarrow 0^+} x(\ln x)^2 = \lim_{x \rightarrow 0^+} \left(\frac{(\ln x)^2}{\frac{1}{x}}\right) = \lim_{x \rightarrow 0^+} \left(\frac{2(\ln x)\frac{1}{x}}{-\frac{1}{x^2}}\right) = \lim_{x \rightarrow 0^+} \left(\frac{2\ln x}{-\frac{1}{x}}\right) = \lim_{x \rightarrow 0^+} \left(\frac{\frac{2}{x}}{\frac{1}{x^2}}\right) = \lim_{x \rightarrow 0^+} \left(\frac{2x^2}{x}\right) = \lim_{x \rightarrow 0^+} (2x) = 0$
65. $\lim_{x \rightarrow 0^+} x \tan\left(\frac{\pi}{2} - x\right) = \lim_{x \rightarrow 0^+} \left(\frac{x}{\cot\left(\frac{\pi}{2} - x\right)}\right) = \lim_{x \rightarrow 0^+} \left(\frac{1}{\csc\left(\frac{\pi}{2} - x\right)}\right) = \frac{1}{1} = 1$
66. $\lim_{x \rightarrow 0^+} \sin x \cdot \ln x = \lim_{x \rightarrow 0^+} \left(\frac{\ln x}{\csc x}\right) = \lim_{x \rightarrow 0^+} \left(\frac{\frac{1}{x}}{-\csc x \cot x}\right) = \lim_{x \rightarrow 0^+} \left(-\frac{\sin x \tan x}{x}\right) = \lim_{x \rightarrow 0^+} \left(-\frac{\sin x \sec^2 x + \cos x \tan x}{1}\right) = \frac{0}{1} = 0$
67. $\lim_{x \rightarrow \infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}} = \sqrt{\lim_{x \rightarrow \infty} \frac{9x+1}{x+1}} = \sqrt{\lim_{x \rightarrow \infty} \frac{9}{1}} = \sqrt{9} = 3$
68. $\lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}} = \sqrt{\lim_{x \rightarrow 0^+} \frac{1}{\frac{\sin x}{x}}} = \sqrt{\frac{1}{1}} = 1$
69. $\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\sec x}{\tan x} = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \left(\frac{1}{\cos x}\right) \left(\frac{\cos x}{\sin x}\right) = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{1}{\sin x} = 1$
70. $\lim_{x \rightarrow 0^+} \frac{\cot x}{\csc x} = \lim_{x \rightarrow 0^+} \frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)} = \lim_{x \rightarrow 0^+} \cos x = 1$
71. $\lim_{x \rightarrow \infty} \frac{2^x - 3^x}{3^x + 4^x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{3}\right)^x - 1}{1 + \left(\frac{4}{3}\right)^x} = 0$

$$72. \lim_{x \rightarrow -\infty} \frac{2^x + 4^x}{5^x - 2^x} = \lim_{x \rightarrow -\infty} \frac{1 + \left(\frac{4}{2}\right)^x}{\left(\frac{5}{2}\right)^x - 1} = \lim_{x \rightarrow -\infty} \frac{1 + 2^x}{\left(\frac{5}{2}\right)^x - 1} = \frac{1+0}{0-1} = -1$$

$$73. \lim_{x \rightarrow \infty} \frac{e^{x^2}}{xe^x} = \lim_{x \rightarrow \infty} \frac{e^{x^2-x}}{x} = \lim_{x \rightarrow \infty} \frac{e^{x(x-1)}}{x} = \lim_{x \rightarrow \infty} \frac{e^{x(x-1)}(2x-1)}{1} = \infty$$

$$74. \lim_{x \rightarrow 0^+} \frac{x}{e^{-1/x}} = \lim_{x \rightarrow 0^+} \frac{e^{1/x}}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{e^{1/x} \left(-\frac{1}{x^2} \right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} e^{1/x} = \infty$$

75. Part (b) is correct because part (a) is neither in the $\frac{0}{0}$ nor $\frac{\infty}{\infty}$ form and so l'Hôpital's rule may not be used.

76. Part (b) is correct; the step $\lim_{x \rightarrow 0} \frac{2x-2}{2x-\cos x} = \lim_{x \rightarrow 0} \frac{2}{2+\sin x}$ in part (a) is false because $\lim_{x \rightarrow 0} \frac{2x-2}{2x-\cos x}$ is not an indeterminate quotient form.

77. Part (d) is correct, the other parts are indeterminate forms and cannot be calculated by the incorrect arithmetic

78. (a) We seek c in $(-2, 0)$ so that $\frac{f'(c)}{g'(c)} = \frac{f(0)-f(-2)}{g(0)-g(-2)} = \frac{0+2}{0-4} = -\frac{1}{2}$. Since $f'(c) = 1$ and $g'(c) = 2c$ we have that $\frac{1}{2c} = -\frac{1}{2} \Rightarrow c = -1$.

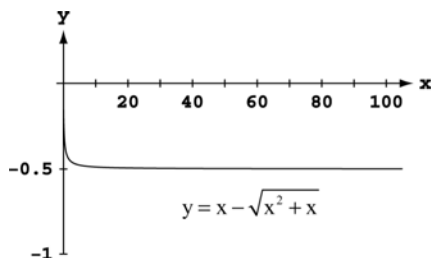
(b) We seek c in (a, b) so that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{b-a}{b^2-a^2} = \frac{1}{b+a}$. Since $f'(c) = 1$ and $g'(c) = 2c$ we have that $\frac{1}{2c} = \frac{1}{b+a} \Rightarrow c = \frac{b+a}{2}$.

(c) We seek c in $(0, 3)$ so that $\frac{f'(c)}{g'(c)} = \frac{f(3)-f(0)}{g(3)-g(0)} = -\frac{3-0}{9-0} = -\frac{1}{3}$. Since $f'(c) = c^2 - 4$ and $g'(c) = 2c$ we have that $\frac{c^2-4}{2c} = -\frac{1}{3} \Rightarrow c = \frac{-1 \pm \sqrt{37}}{3} \Rightarrow c = \frac{-1 + \sqrt{37}}{3}$.

79. If $f(x)$ is to be continuous at $x = 0$, then $\lim_{x \rightarrow 0} f(x) = f(0) \Rightarrow c = f(0) = \lim_{x \rightarrow 0} \frac{9x-3\sin 3x}{5x^3} = \lim_{x \rightarrow 0} \frac{9-9\cos 3x}{15x^2}$
 $= \lim_{x \rightarrow 0} \frac{27\sin 3x}{30x} = \lim_{x \rightarrow 0} \frac{81\cos 3x}{30} = \frac{27}{10}$.

80. $\lim_{x \rightarrow 0} \left(\frac{\tan 2x}{x^3} + \frac{a}{x^2} + \frac{\sin bx}{x} \right) = \lim_{x \rightarrow 0} \left(\frac{\tan 2x + ax + x^2 \sin bx}{x^3} \right) = \lim_{x \rightarrow 0} \left(\frac{2\sec^2 2x + a + bx^2 \cos bx + 2x \sin bx}{3x^2} \right)$ will be in $\frac{0}{0}$ form if $\lim_{x \rightarrow 0} (2\sec^2 2x + a + bx^2 \cos bx + 2x \sin bx) = a + 2 = 0 \Rightarrow a = -2$; $\lim_{x \rightarrow 0} \left(\frac{2\sec^2 2x - 2 + bx^2 \cos bx + 2x \sin bx}{3x^2} \right)$
 $= \lim_{x \rightarrow 0} \left(\frac{8\sec^2 2x \tan 2x - b^2 x^2 \sin bx + 4bx \cos bx + 2 \sin bx}{6x} \right) = \lim_{x \rightarrow 0} \left(\frac{32\sec^2 2x \tan^2 2x + 16\sec^4 2x - b^3 x^2 \cos bx - 6b^2 x \sin bx + 6b \cos bx}{6} \right)$
 $= \frac{16+6b}{6} = 0 \Rightarrow 16+6b = 0 \Rightarrow b = -\frac{8}{3}$

81. (a)



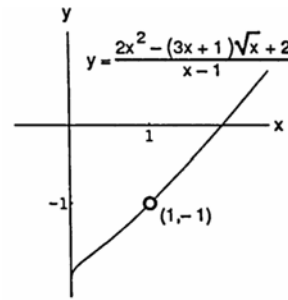
(b) The limit leads to the indeterminate form $\infty - \infty$:

$$\begin{aligned}\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right) &= \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right) \left(\frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}} \right) = \lim_{x \rightarrow \infty} \left(\frac{x^2 - (x^2 + x)}{x + \sqrt{x^2 + x}} \right) = \lim_{x \rightarrow \infty} \frac{-x}{x + \sqrt{x^2 + x}} \\ &= \lim_{x \rightarrow \infty} \frac{-1}{1 + \sqrt{1 + \frac{1}{x}}} = \frac{-1}{1 + \sqrt{1 + 0}} = -\frac{1}{2}\end{aligned}$$

$$82. \quad \lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 1} - \sqrt{x} \right) = \lim_{x \rightarrow \infty} x \left(\frac{\sqrt{x^2 + 1}}{x} - \frac{\sqrt{x}}{x} \right) = \lim_{x \rightarrow \infty} x \left(\sqrt{\frac{x^2 + 1}{x^2}} - \sqrt{\frac{x}{x^2}} \right) = \lim_{x \rightarrow \infty} x \left(\sqrt{1 + \frac{1}{x^2}} - \sqrt{\frac{1}{x}} \right) = \infty$$

83. The graph indicates a limit near -1 . The limit leads

$$\begin{aligned}\text{to the indeterminate form } \frac{0}{0}: \quad & \lim_{x \rightarrow 1} \frac{2x^2 - (3x+1)\sqrt{x} + 2}{x-1} \\ &= \lim_{x \rightarrow 1} \frac{2x^2 - 3x^{3/2} - x^{1/2} + 2}{x-1} = \lim_{x \rightarrow 1} \frac{4x - \frac{9}{2}x^{1/2} - \frac{1}{2}x^{-1/2}}{1} \\ &= \frac{4 - \frac{9}{2} - \frac{1}{2}}{1} = \frac{4-5}{1} = -1\end{aligned}$$

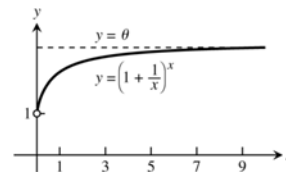


84. (a) The limit leads to the indeterminate form 1^∞ . Let $f(x) = \left(1 + \frac{1}{x}\right)^x \Rightarrow \ln f(x) = x \ln \left(1 + \frac{1}{x}\right) \Rightarrow \lim_{x \rightarrow \infty} \ln f(x)$

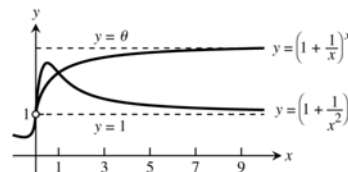
$$\begin{aligned}&= \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{\ln(1+x^{-1})}{x^{-1}} = \lim_{x \rightarrow \infty} \frac{\left(\frac{-x^{-2}}{1+x^{-1}}\right)}{-x^{-2}} = \lim_{x \rightarrow \infty} \frac{1}{1+\left(\frac{1}{x}\right)} = \frac{1}{1+0} = 1 \Rightarrow \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow \infty} f(x) \\ &= \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^1 = e\end{aligned}$$

(b) $x \quad \left(1 + \frac{1}{x}\right)^x$

10	2.5937424601
100	2.70481382942
1000	2.71692393224
10,000	2.71814592683
100,000	2.71826823717



Both functions have limits as x approaches infinity. The function f has a maximum but no minimum while g has no extrema. The limit of $f(x)$ leads to the indeterminate form 1^∞ .



(c) Let $f(x) = \left(1 + \frac{1}{x^2}\right)^x \Rightarrow \ln f(x) = x \ln \left(1 + x^{-2}\right)$

$$\Rightarrow \lim_{x \rightarrow \infty} \ln f(x) = \lim_{x \rightarrow \infty} \frac{\ln(1+x^{-2})}{x^{-1}} = \lim_{x \rightarrow \infty} \frac{\left(\frac{-2x^{-3}}{1+x^{-2}}\right)}{-x^{-2}} = \lim_{x \rightarrow \infty} \frac{2x^2}{x^3+x} = \lim_{x \rightarrow \infty} \frac{4x}{3x^2+1} = \lim_{x \rightarrow \infty} \frac{4}{6x} = 0.$$

$$\text{Therefore } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x^2}\right)^x = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^0 = 1$$

85. Let $f(k) = \left(1 + \frac{r}{k}\right)^k \Rightarrow \ln f(k) = \frac{\ln(1 + rk^{-1})}{k^{-1}} \Rightarrow \lim_{k \rightarrow \infty} \frac{\ln(1 + rk^{-1})}{k^{-1}} = \lim_{k \rightarrow \infty} \frac{\left(\frac{-rk^{-2}}{1 + rk^{-1}}\right)}{-k^{-2}} = \lim_{k \rightarrow \infty} \frac{r}{1 + rk^{-1}} = \lim_{k \rightarrow \infty} \frac{rk}{k + r}$
 $= \lim_{k \rightarrow \infty} \frac{r}{1} = r$. Therefore $\lim_{k \rightarrow \infty} \left(1 + \frac{r}{k}\right)^k = \lim_{k \rightarrow \infty} f(k) = \lim_{k \rightarrow \infty} e^{\ln f(k)} = e^r$.

86. (a) $y = x^{1/x} \Rightarrow \ln y = \frac{\ln x}{x} \Rightarrow \frac{y'}{y} = \frac{\left(\frac{1}{x}\right)(x) - \ln x}{x^2} \Rightarrow y' = \left(\frac{1 - \ln x}{x^2}\right)(x^{1/x})$. The sign pattern is $y' = \begin{array}{c} | + + + + | - - - - \\ 0 \qquad \qquad \qquad e \end{array}$

which indicates a maximum value of $y = e^{1/e}$ when $x = e$

(b) $y = x^{1/x^2} \Rightarrow \ln y = \frac{\ln x}{x^2} \Rightarrow \frac{y'}{y} = \frac{\left(\frac{1}{x}\right)(x^2) - 2x \ln x}{x^4} \Rightarrow y' = \left(\frac{1 - 2 \ln x}{x^3}\right)(x^{1/x^2})$. The sign pattern is

$y' = \begin{array}{c} | + + + | - - - - \\ 0 \qquad \qquad \sqrt{e} \end{array}$ which indicates a maximum of $y = e^{1/(2e)}$ when $x = \sqrt{e}$

(c) $y = x^{1/x^n} \Rightarrow \ln y = \frac{\ln x}{x^n} = \frac{\left(\frac{1}{x}\right)(x^n) - (\ln x)(nx^{n-1})}{x^{2n}} \Rightarrow y' = \frac{x^{n-1}(1 - n \ln x)}{x^{2n}} \cdot x^{1/x^n}$. The sign pattern is

$y' = \begin{array}{c} | + + + | - - - - \\ 0 \qquad \qquad \sqrt[n]{e} \end{array}$ which indicates a maximum of $y = e^{1/(ne)}$ when $x = \sqrt[n]{e}$

(d) $\lim_{x \rightarrow \infty} x^{1/x^n} = \lim_{x \rightarrow \infty} \left(e^{\ln x}\right)^{1/x^n} = \lim_{x \rightarrow \infty} e^{(\ln x)/x^n} = \exp\left(\lim_{x \rightarrow \infty} \frac{\ln x}{x^n}\right) = \exp\left(\lim_{x \rightarrow \infty} \left(\frac{1}{nx^n}\right)\right) = e^0 = 1$

87. (a) $y = x \tan\left(\frac{1}{x}\right)$, $\lim_{x \rightarrow \infty} \left(x \tan\left(\frac{1}{x}\right)\right) = \lim_{x \rightarrow \infty} \left(\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}\right) = \lim_{x \rightarrow \infty} \left(\frac{\sec^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right) = \lim_{x \rightarrow \infty} \sec^2\left(\frac{1}{x}\right) = 1$; $\lim_{x \rightarrow -\infty} \left(x \tan\left(\frac{1}{x}\right)\right)$

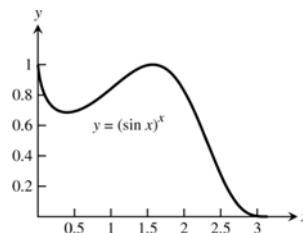
$= \lim_{x \rightarrow -\infty} \left(\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}\right) = \lim_{x \rightarrow -\infty} \left(\frac{\sec^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right) = \lim_{x \rightarrow -\infty} \sec^2\left(\frac{1}{x}\right) = 1 \Rightarrow$ the horizontal asymptote is $y = 1$ as

$x \rightarrow \infty$ and as $x \rightarrow -\infty$.

(b) $y = \frac{3x + e^{2x}}{2x + e^{3x}}$, $\lim_{x \rightarrow \infty} \left(\frac{3x + e^{2x}}{2x + e^{3x}}\right) = \lim_{x \rightarrow \infty} \left(\frac{3 + 2e^{2x}}{2 + 3e^{3x}}\right) = \lim_{x \rightarrow \infty} \left(\frac{4e^{2x}}{9e^{3x}}\right) = \lim_{x \rightarrow \infty} \left(\frac{4}{9e^x}\right) = 0$; $\lim_{x \rightarrow -\infty} \left(\frac{3x + e^{2x}}{2x + e^{3x}}\right) = \lim_{x \rightarrow -\infty} \left(\frac{3 + e^{2x}}{2 + 3e^{3x}}\right) = \lim_{x \rightarrow -\infty} \left(\frac{3 + e^{2x}}{2 + 3e^{3x}}\right) = \frac{3}{2} \Rightarrow$ the horizontal asymptotes are $y = 0$ as $x \rightarrow \infty$ and $y = \frac{3}{2}$ as $x \rightarrow -\infty$.

88. $f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{e^{-1/h^2} - 0}{h} = \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h} = \lim_{h \rightarrow 0} \left(\frac{\frac{1}{h}}{e^{1/h^2}}\right) = \lim_{h \rightarrow 0} \left(\frac{-\frac{1}{h^2}}{e^{1/h^2}\left(\frac{2}{h^3}\right)}\right) = \lim_{h \rightarrow 0} \left(\frac{h}{2e^{1/h^2}}\right)$
 $= \lim_{h \rightarrow 0} \left(\frac{h}{2} e^{-1/h^2}\right) = 0$

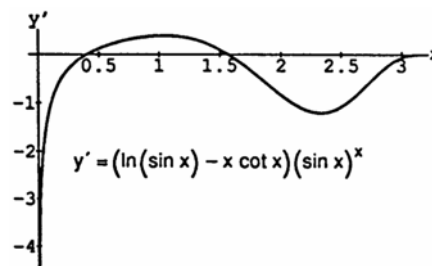
89. (a) We should assign the value 1 to $f(x) = (\sin x)^x$ to make it continuous at $x = 0$.



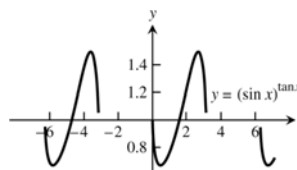
$$\begin{aligned}
 \text{(b)} \quad \ln f(x) &= x \ln(\sin x) = \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} \Rightarrow \lim_{x \rightarrow 0^+} \ln f(x) = \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow 0^+} \frac{\left(\frac{1}{\sin x}\right)(\cos x)}{\left(-\frac{1}{x^2}\right)} = \lim_{x \rightarrow 0} \frac{-x^2}{\tan x} \\
 &= \lim_{x \rightarrow 0} \frac{-2x}{\sec^2 x} = 0 \Rightarrow \lim_{x \rightarrow 0} f(x) = e^0 = 1
 \end{aligned}$$

(c) The maximum value of $f(x)$ is close to 1 near the point $x \approx 1.55$ (see the graph in part (a)).

(d) The root in question is near 1.57.



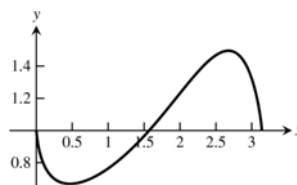
90. (a) When $\sin x < 0$ there are gaps in the sketch. The width of each gap is π .



$$\begin{aligned}
 \text{(b)} \quad \text{Let } f(x) &= (\sin x)^{\tan x} \\
 \Rightarrow \ln f(x) &= (\tan x) \ln(\sin x) \Rightarrow \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \ln f(x) \\
 &= \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\ln(\sin x)}{\cot x} = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\left(\frac{1}{\sin x}\right)(\cos x)}{-\csc^2 x} \\
 &= \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\cos x}{(-\csc x)} = 0 \Rightarrow \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} f(x) = e^0 = 1.
 \end{aligned}$$

Similarly, $\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^+} f(x) = e^0 = 1$. Therefore,

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = 1.$$



- (c) From the graph in part (b) we have a minimum of about 0.665 at $x \approx 0.47$ and the maximum is about 1.491 at $x \approx 2.66$.

7.6 INVERSE TRIGONOMETRIC FUNCTIONS

- | | | | | | |
|-------------------------|----------------------|----------------------|-------------------------|----------------------|----------------------|
| 1. (a) $\frac{\pi}{4}$ | (b) $-\frac{\pi}{3}$ | (c) $\frac{\pi}{6}$ | 2. (a) $-\frac{\pi}{4}$ | (b) $\frac{\pi}{3}$ | (c) $-\frac{\pi}{6}$ |
| 3. (a) $-\frac{\pi}{6}$ | (b) $\frac{\pi}{4}$ | (c) $-\frac{\pi}{3}$ | 4. (a) $\frac{\pi}{6}$ | (b) $-\frac{\pi}{4}$ | (c) $\frac{\pi}{3}$ |
| 5. (a) $\frac{\pi}{3}$ | (b) $\frac{3\pi}{4}$ | (c) $\frac{\pi}{6}$ | 6. (a) $\frac{\pi}{4}$ | (b) $-\frac{\pi}{3}$ | (c) $\frac{\pi}{6}$ |
| 7. (a) $\frac{3\pi}{4}$ | (b) $\frac{\pi}{6}$ | (c) $\frac{2\pi}{3}$ | 8. (a) $\frac{3\pi}{4}$ | (b) $\frac{\pi}{6}$ | (c) $\frac{2\pi}{3}$ |

9. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
10. $\sec\left(\cos^{-1}\frac{1}{2}\right) = \sec\left(\frac{\pi}{3}\right) = 2$
11. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$
12. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cot\left(-\frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}}$
13. $\lim_{x \rightarrow 1^-} \sin^{-1} x = \frac{\pi}{2}$
14. $\lim_{x \rightarrow -1^+} \cos^{-1} x = \pi$
15. $\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$
16. $\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$
17. $\lim_{x \rightarrow \infty} \sec^{-1} x = \frac{\pi}{2}$
18. $\lim_{x \rightarrow -\infty} \sec^{-1} x = \lim_{x \rightarrow -\infty} \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$
19. $\lim_{x \rightarrow \infty} \csc^{-1} x = \lim_{x \rightarrow \infty} \sin^{-1}\left(\frac{1}{x}\right) = 0$
20. $\lim_{x \rightarrow -\infty} \csc^{-1} x = \lim_{x \rightarrow -\infty} \sin^{-1}\left(\frac{1}{x}\right) = 0$
21. $y = \cos^{-1}(x^2) \Rightarrow \frac{dy}{dx} = -\frac{2x}{\sqrt{1-(x^2)^2}} = \frac{-2x}{\sqrt{1-x^4}}$
22. $y = \cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$
23. $y = \sin^{-1}\sqrt{2t} \Rightarrow \frac{dy}{dt} = \frac{\sqrt{2}}{\sqrt{1-(\sqrt{2t})^2}} = \frac{\sqrt{2}}{\sqrt{1-2t^2}}$
24. $y = \sin^{-1}(1-t) \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-(1-t)^2}} = \frac{-1}{\sqrt{2t-t^2}}$
25. $y = \sec^{-1}(2s+1) \Rightarrow \frac{dy}{ds} = \frac{2}{|2s+1|\sqrt{(2s+1)^2-1}} = \frac{2}{|2s+1|\sqrt{4s^2+4s}} = \frac{1}{|2s+1|\sqrt{s^2+s}}$
26. $y = \sec^{-1} 5s \Rightarrow \frac{dy}{ds} = \frac{5}{|5s|\sqrt{(5s)^2-1}} = \frac{1}{|s|\sqrt{25s^2-1}}$
27. $y = \csc^{-1}(x^2+1) \Rightarrow \frac{dy}{dx} = -\frac{2x}{|x^2+1|\sqrt{(x^2+1)^2-1}} = \frac{-2x}{(x^2+1)\sqrt{x^4+2x^2}}$
28. $y = \csc^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = -\frac{\left(\frac{1}{2}\right)}{\left|\frac{x}{2}\right|\sqrt{\left(\frac{x}{2}\right)^2-1}} = \frac{-1}{|x|\sqrt{\frac{x^2-4}{4}}} = \frac{-2}{|x|\sqrt{x^2-4}}$
29. $y = \sec^{-1}\left(\frac{1}{t}\right) = \cos^{-1} t \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$
30. $y = \sin^{-1}\left(\frac{3}{t^2}\right) = \csc^{-1}\left(\frac{t^2}{3}\right) \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{2t}{3}\right)}{\left|\frac{t^2}{3}\right|\sqrt{\left(\frac{t^2}{3}\right)^2-1}} = \frac{-2t}{t^2\sqrt{\frac{t^4-9}{9}}} = \frac{-6}{t\sqrt{t^4-9}}$
31. $y = \cot^{-1}\sqrt{t} = \cot^{-1}(t^{1/2}) \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1+(t^{1/2})^2} = \frac{-1}{2\sqrt{t}(1+t)}$

$$32. \quad y = \cot^{-1} \sqrt{t-1} = \cot^{-1} (t-1)^{1/2} \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)(t-1)^{-1/2}}{1+\left[(t-1)^{1/2}\right]^2} = \frac{-1}{2\sqrt{t-1}(1+t-1)} = \frac{-1}{2t\sqrt{t-1}}$$

$$33. \quad y = \ln(\tan^{-1} x) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{1+x^2}\right)}{\tan^{-1} x} = \frac{1}{(\tan^{-1} x)(1+x^2)}$$

$$34. \quad y = \tan^{-1}(\ln x) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)}{1+(\ln x)^2} = \frac{1}{x[1+(\ln x)^2]}$$

$$35. \quad y = \csc^{-1}(e^t) \Rightarrow \frac{dy}{dt} = -\frac{e^t}{|e^t|\sqrt{(e^t)^2-1}} = \frac{-1}{\sqrt{e^{2t}-1}}$$

$$36. \quad y = \cos^{-1}(e^{-t}) \Rightarrow \frac{dy}{dt} = -\frac{-e^{-t}}{\sqrt{1-(e^{-t})^2}} = \frac{e^{-t}}{\sqrt{1-e^{-2t}}}$$

$$37. \quad y = s\sqrt{1-s^2} + \cos^{-1} s = s(1-s^2)^{1/2} + \cos^{-1} s \Rightarrow \frac{dy}{ds} = (1-s^2)^{1/2} + s\left(\frac{1}{2}\right)(1-s^2)^{-1/2}(-2s) - \frac{1}{\sqrt{1-s^2}} \\ = \sqrt{1-s^2} - \frac{s^2}{\sqrt{1-s^2}} - \frac{1}{\sqrt{1-s^2}} = \sqrt{1-s^2} - \frac{s^2+1}{\sqrt{1-s^2}} = \frac{1-s^2-s^2-1}{\sqrt{1-s^2}} = \frac{-2s^2}{\sqrt{1-s^2}}$$

$$38. \quad y = \sqrt{s^2-1} - \sec^{-1} s = (s^2-1)^{1/2} - \sec^{-1} s \Rightarrow \frac{dy}{dx} = \left(\frac{1}{2}\right)(s^2-1)^{-1/2}(2s) - \frac{1}{|s|\sqrt{s^2-1}} = \frac{s}{\sqrt{s^2-1}} - \frac{1}{|s|\sqrt{s^2-1}} = \frac{s|s|-1}{|s|\sqrt{s^2-1}}$$

$$39. \quad y = \tan^{-1} \sqrt{x^2-1} + \csc^{-1} x = \tan^{-1} (x^2-1)^{1/2} + \csc^{-1} x \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)(x^2-1)^{-1/2}(2x)}{1+\left[(x^2-1)^{1/2}\right]^2} - \frac{1}{|x|\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}} - \frac{1}{|x|\sqrt{x^2-1}} = 0,$$

for $x > 1$

$$40. \quad y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1} x = \frac{\pi}{2} - \tan^{-1}(x^{-1}) - \tan^{-1} x \Rightarrow \frac{dy}{dx} = 0 - \frac{-x^{-2}}{1+(x^{-1})^2} - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0$$

$$41. \quad y = x \sin^{-1} x + \sqrt{1-x^2} = x \sin^{-1} x + (1-x^2)^{1/2} \Rightarrow \frac{dy}{dx} = \sin^{-1} x + x\left(\frac{1}{\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x) \\ = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \sin^{-1} x$$

$$42. \quad y = \ln(x^2+4) - x \tan^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - x\left[\frac{\left(\frac{1}{2}\right)}{1+\left(\frac{x}{2}\right)^2}\right] = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{2x}{4+x^2} = -\tan^{-1}\left(\frac{x}{2}\right)$$

$$43. \quad \int \frac{1}{\sqrt{9-x^2}} dx = \sin^{-1}\left(\frac{x}{3}\right) + C$$

$$44. \int \frac{1}{\sqrt{1-4x^2}} dx = \frac{1}{2} \int \frac{2}{\sqrt{1-(2x)^2}} dx = \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}}, \text{ where } u = 2x \text{ and } du = 2dx$$

$$= \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1}(2x) + C$$

$$45. \int \frac{1}{17+x^2} dx = \int \frac{1}{(\sqrt{17})^2 + x^2} dx = \frac{1}{\sqrt{17}} \tan^{-1} \frac{x}{\sqrt{17}} + C$$

$$46. \int \frac{1}{9+3x^2} dx = \frac{1}{3} \int \frac{1}{(\sqrt{3})^2 + x^2} dx = \frac{1}{3\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + C = \frac{\sqrt{3}}{9} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + C$$

$$47. \int \frac{dx}{x\sqrt{25x^2-2}} = \int \frac{du}{u\sqrt{u^2-2}}, \text{ where } u = 5x \text{ and } du = 5dx$$

$$= \frac{1}{\sqrt{2}} \sec^{-1} \left| \frac{u}{\sqrt{2}} \right| + C = \frac{1}{\sqrt{2}} \sec^{-1} \left| \frac{5x}{\sqrt{2}} \right| + C$$

$$48. \int \frac{dx}{x\sqrt{5x^2-4}} = \int \frac{du}{u\sqrt{u^2-4}}, \text{ where } u = \sqrt{5}x \text{ and } du = \sqrt{5}dx$$

$$= \frac{1}{2} \sec^{-1} \left| \frac{u}{2} \right| + C = \frac{1}{2} \sec^{-1} \left| \frac{\sqrt{5}x}{2} \right| + C$$

$$49. \int_0^1 \frac{4ds}{\sqrt{4-s^2}} = \left[4 \sin^{-1} \frac{s}{2} \right]_0^1 = 4 \left(\sin^{-1} \frac{1}{2} - \sin^{-1} 0 \right) = 4 \left(\frac{\pi}{6} - 0 \right) = \frac{2\pi}{3}$$

$$50. \int_0^{3\sqrt{2}/4} \frac{ds}{\sqrt{9-4s^2}} = \frac{1}{2} \int_0^{3\sqrt{2}/4} \frac{du}{\sqrt{9-u^2}}, \text{ where } u = 2s \text{ and } du = 2ds; \quad s = 0 \Rightarrow u = 0, s = \frac{3\sqrt{2}}{4} \Rightarrow u = \frac{3\sqrt{2}}{2}$$

$$= \left[\frac{1}{2} \sin^{-1} \frac{u}{3} \right]_0^{3\sqrt{2}/2} = \frac{1}{2} \left(\sin^{-1} \frac{\sqrt{2}}{2} - \sin^{-1} 0 \right) = \frac{1}{2} \left(\frac{\pi}{4} - 0 \right) = \frac{\pi}{8}$$

$$51. \int_0^2 \frac{dt}{8+2t^2} = \frac{1}{\sqrt{2}} \int_0^{2\sqrt{2}} \frac{du}{8+u^2}, \text{ where } u = \sqrt{2}t \text{ and } du = \sqrt{2}dt; \quad t = 0 \Rightarrow u = 0, t = 2 \Rightarrow u = 2\sqrt{2}$$

$$= \left[\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{8}} \tan^{-1} \frac{u}{\sqrt{8}} \right]_0^{2\sqrt{2}} = \frac{1}{4} \left(\tan^{-1} \frac{2\sqrt{2}}{\sqrt{8}} - \tan^{-1} 0 \right) = \frac{1}{4} \left(\tan^{-1} 1 - \tan^{-1} 0 \right) = \frac{1}{4} \left(\frac{\pi}{4} - 0 \right) = \frac{\pi}{16}$$

$$52. \int_{-2}^2 \frac{dt}{4+3t^2} = \frac{1}{\sqrt{3}} \int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{du}{4+u^2}, \text{ where } u = \sqrt{3}t \text{ and } du = \sqrt{3}dt; \quad t = -2 \Rightarrow u = -2\sqrt{3}, t = 2 \Rightarrow u = 2\sqrt{3}$$

$$= \left[\frac{1}{\sqrt{3}} \cdot \frac{1}{2} \tan^{-1} \frac{u}{2} \right]_{-2\sqrt{3}}^{2\sqrt{3}} = \frac{1}{2\sqrt{3}} \left[\tan^{-1} \sqrt{3} - \tan^{-1} (-\sqrt{3}) \right] = \frac{1}{2\sqrt{3}} \left[\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right] = \frac{\pi}{3\sqrt{3}}$$

$$53. \int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2-1}} = \int_{-2}^{-\sqrt{2}} \frac{du}{u\sqrt{u^2-1}}, \text{ where } u = 2y \text{ and } du = 2dy; \quad y = -1 \Rightarrow u = -2, y = -\frac{\sqrt{2}}{2} \Rightarrow u = -\sqrt{2}$$

$$= \left[\sec^{-1} |u| \right]_{-2}^{-\sqrt{2}} = \sec^{-1} |-\sqrt{2}| - \sec^{-1} |-2| = \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$$

$$54. \int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}} = \int_{-2}^{-\sqrt{2}} \frac{du}{u\sqrt{u^2-1}}, \text{ where } u = 3y \text{ and } du = 3dy; \quad y = -\frac{2}{3} \Rightarrow u = -2, y = -\frac{\sqrt{2}}{3} \Rightarrow u = -\sqrt{2}$$

$$= \left[\sec^{-1} |u| \right]_{-2}^{-\sqrt{2}} = \sec^{-1} |-\sqrt{2}| - \sec^{-1} |-2| = \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$$

55. $\int \frac{3dr}{\sqrt{1-4(r-1)^2}} = \frac{3}{2} \int \frac{du}{\sqrt{1-u^2}}$, where $u = 2(r-1)$ and $du = 2dr$
 $= \frac{3}{2} \sin^{-1} u + C = \frac{3}{2} \sin^{-1} 2(r-1) + C$
56. $\int \frac{6dr}{\sqrt{4-(r+1)^2}} = 6 \int \frac{du}{\sqrt{4-u^2}}$, where $u = r+1$ and $du = dr$
 $= 6 \sin^{-1} \frac{u}{2} + C = 6 \sin^{-1} \left(\frac{r+1}{2} \right) + C$
57. $\int \frac{dx}{2+(x-1)^2} = \int \frac{du}{2+u^2}$, where $u = x-1$ and $du = dx$
 $= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + C = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x-1}{\sqrt{2}} \right) + C$
58. $\int \frac{dx}{1+(3x+1)^2} = \frac{1}{3} \int \frac{du}{1+u^2}$, where $u = 3x+1$ and $du = 3dx$
 $= \frac{1}{3} \tan^{-1} u + C = \frac{1}{3} \tan^{-1} (3x+1) + C$
59. $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}} = \frac{1}{2} \int \frac{du}{u\sqrt{u^2-4}}$, where $u = 2x-1$ and $du = 2dx$
 $= \frac{1}{2} \cdot \frac{1}{2} \sec^{-1} \left| \frac{u}{2} \right| + C = \frac{1}{4} \sec^{-1} \left| \frac{2x-1}{2} \right| + C$
60. $\int \frac{dx}{(x+3)\sqrt{(x+3)^2-25}} = \int \frac{du}{u\sqrt{u^2-25}}$, where $u = x+3$ and $du = dx$
 $= \frac{1}{5} \sec^{-1} \left| \frac{u}{5} \right| + C = \frac{1}{5} \sec^{-1} \left| \frac{x+3}{5} \right| + C$
61. $\int_{-\pi/2}^{\pi/2} \frac{2 \cos \theta d\theta}{1+(\sin \theta)^2} = 2 \int_{-1}^1 \frac{du}{1+u^2}$, where $u = \sin \theta$ and $du = \cos \theta d\theta$; $\theta = -\frac{\pi}{2} \Rightarrow u = -1, \theta = \frac{\pi}{2} \Rightarrow u = 1$
 $= \left[2 \tan^{-1} u \right]_{-1}^1 = 2 \left(\tan^{-1} 1 - \tan^{-1} (-1) \right) = 2 \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = \pi$
62. $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x dx}{1+(\cot x)^2} = - \int_{\sqrt{3}}^1 \frac{du}{\sqrt{3}+u^2}$, where $u = \cot x$ and $du = -\csc^2 x dx$; $x = \frac{\pi}{6} \Rightarrow u = \sqrt{3}, x = \frac{\pi}{4} \Rightarrow u = 1$
 $= \left[-\tan^{-1} u \right]_{\sqrt{3}}^1 = -\tan^{-1} 1 + \tan^{-1} \sqrt{3} = -\frac{\pi}{4} + \frac{\pi}{3} = \frac{\pi}{12}$
63. $\int_0^{\ln \sqrt{3}} \frac{e^x dx}{1+e^{2x}} = \int_1^{\sqrt{3}} \frac{du}{1+u^2}$, where $u = e^x$ and $du = e^x dx$; $x = 0 \Rightarrow u = 1, x = \ln \sqrt{3} \Rightarrow u = \sqrt{3}$
 $= \left[\tan^{-1} u \right]_1^{\sqrt{3}} = \tan^{-1} \sqrt{3} - \tan^{-1} 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$
64. $\int_1^{e^{\pi/4}} \frac{4dt}{t(1+\ln^2 t)} = 4 \int_0^{\pi/4} \frac{du}{1+u^2}$, where $u = \ln t$ and $du = \frac{1}{t} dt$; $t = 1 \Rightarrow u = 0, t = e^{\pi/4} \Rightarrow u = \frac{\pi}{4}$
 $= \left[4 \tan^{-1} u \right]_0^{\pi/4} = 4 \left(\tan^{-1} \frac{\pi}{4} - \tan^{-1} 0 \right) = 4 \tan^{-1} \frac{\pi}{4}$
65. $\int \frac{y dy}{\sqrt{1-y^4}} = \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}}$, where $u = y^2$ and $du = 2y dy$
 $= \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1} y^2 + C$

66. $\int \frac{\sec^2 y \, dy}{\sqrt{1-\tan^2 y}} = \int \frac{du}{\sqrt{1-u^2}}$, where $u = \tan y$ and $du = \sec^2 y \, dy$
 $= \sin^{-1} u + C = \sin^{-1}(\tan y) + C$
67. $\int \frac{dx}{\sqrt{-x^2+4x-3}} = \int \frac{dx}{\sqrt{1-(x^2-4x+4)}} = \int \frac{dx}{\sqrt{1-(x-2)^2}} = \sin^{-1}(x-2) + C$
68. $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x^2-2x+1)}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}(x-1) + C$
69. $\int_{-1}^0 \frac{6dt}{\sqrt{3-2t-t^2}} = 6 \int_{-1}^0 \frac{dt}{\sqrt{4-(t^2+2t+1)}} = 6 \int_{-1}^0 \frac{dt}{\sqrt{2^2-(t+1)^2}} = 6 \left[\sin^{-1} \left(\frac{t+1}{2} \right) \right]_{-1}^0 = 6 \left[\sin^{-1} \left(\frac{1}{2} \right) - \sin^{-1} 0 \right] = 6 \left(\frac{\pi}{6} - 0 \right) = \pi$
70. $\int_{1/2}^1 \frac{6dt}{\sqrt{3+4t-4t^2}} = 3 \int_{1/2}^1 \frac{2dt}{\sqrt{4-(4t^2-4t+1)}} = 3 \int_{1/2}^1 \frac{2dt}{\sqrt{2^2-(2t-1)^2}} = 3 \left[\sin^{-1} \left(\frac{2t-1}{2} \right) \right]_{1/2}^1 = 3 \left[\sin^{-1} \left(\frac{1}{2} \right) - \sin^{-1} 0 \right]$
 $= 3 \left(\frac{\pi}{6} - 0 \right) = \frac{\pi}{2}$
71. $\int \frac{dy}{y^2-2y+5} = \int \frac{dy}{4+y^2-2y+1} = \int \frac{dy}{2^2+(y-1)^2} = \frac{1}{2} \tan^{-1} \left(\frac{y-1}{2} \right) + C$
72. $\int \frac{dy}{y^2+6y+10} = \int \frac{dy}{1+(y^2+6y+9)} = \int \frac{dy}{1+(y+3)^2} = \tan^{-1}(y+3) + C$
73. $\int_1^2 \frac{8dx}{x^2-2x+2} = 8 \int_1^2 \frac{dx}{1+(x^2-2x+1)} = 8 \int_1^2 \frac{dx}{1+(x-1)^2} = 8 \left[\tan^{-1}(x-1) \right]_1^2 = 8 \left(\tan^{-1} 1 - \tan^{-1} 0 \right) = 8 \left(\frac{\pi}{4} - 0 \right) = 2\pi$
74. $\int_2^4 \frac{2dx}{x^2-6x+10} = 2 \int_2^4 \frac{dx}{1+(x^2-6x+9)} = 2 \int_2^4 \frac{dx}{1+(x-3)^2} = 2 \left[\tan^{-1}(x-3) \right]_2^4 = 2 \left[\tan^{-1} 1 - \tan^{-1}(-1) \right] = 2 \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = \pi$
75. $\int \frac{x+4}{x^2+4} dx = \int \frac{x}{x^2+4} dx + \int \frac{4}{x^2+4} dx$; $\int \frac{x}{x^2+4} dx = \frac{1}{2} \int \frac{1}{u} du$ where $u = x^2 + 4 \Rightarrow du = 2x dx \Rightarrow \frac{1}{2} du = x dx$
 $\Rightarrow \int \frac{x+4}{x^2+4} dx = \frac{1}{2} \ln(x^2 + 4) + 2 \tan^{-1} \left(\frac{x}{2} \right) + C$
76. $\int \frac{t-2}{t^2-6t+10} dt = \int \frac{t-2}{(t-3)^2+1} dt$ [Let $w = t-3 \Rightarrow w+3 = t \Rightarrow dw = dt$] $\rightarrow \int \frac{w+1}{w^2+1} dw = \int \frac{w}{w^2+1} dw + \int \frac{1}{w^2+1} dw$;
 $\int \frac{w}{w^2+1} dw = \frac{1}{2} \int \frac{1}{u} du$ where $u = w^2 + 1 \Rightarrow du = 2w dw \Rightarrow \frac{1}{2} du = w dw \Rightarrow \int \frac{w}{w^2+1} dw = \int \frac{1}{w^2+1} dw$
 $= \frac{1}{2} \ln(w^2 + 1) + \tan^{-1}(w) + C = \frac{1}{2} \ln((t-3)^2 + 1) + \tan^{-1}(t-3) + C = \frac{1}{2} \ln(t^2 - 6t + 10) + \tan^{-1}(t-3) + C$
77. $\int \frac{x^2+2x-1}{x^2+9} dx = \int \left(1 + \frac{2x-10}{x^2+9} \right) dx = \int dx + \int \frac{2x}{x^2+9} dx - 10 \int \frac{1}{x^2+9} dx$; $\int \frac{2x}{x^2+9} dx = \int \frac{1}{u} du$ where
 $u = x^2 + 9 \Rightarrow du = 2x dx \Rightarrow \int dx + \int \frac{2x}{x^2+9} dx - 10 \int \frac{1}{x^2+9} dx = x + \ln(x^2 + 9) - \frac{10}{3} \tan^{-1} \left(\frac{x}{3} \right) + C$

$$78. \int \frac{t^3 - 2t^2 + 3t - 4}{t^2 + 1} dt = \int \left(t - 2 + \frac{2t - 2}{t^2 + 1} \right) dt = \int (t - 2) dt + \int \frac{2t}{t^2 + 1} dt - 2 \int \frac{1}{t^2 + 1} dt; \int \frac{2t}{t^2 + 1} dt = \int \frac{1}{u} du \text{ where } u = t^2 + 1 \Rightarrow du = 2t dt \Rightarrow \int (t - 2) dt + \int \frac{2t}{t^2 + 1} dt - 2 \int \frac{1}{t^2 + 1} dt = \frac{1}{2} t^2 - 2t + \ln(t^2 + 1) - 2 \tan^{-1}(t) + C$$

$$79. \int \frac{dx}{(x+1)\sqrt{x^2 + 2x}} = \int \frac{dx}{(x+1)\sqrt{x^2 + 2x + 1 - 1}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2 - 1}} = \int \frac{du}{u\sqrt{u^2 - 1}}, \text{ where } u = x + 1 \text{ and } du = dx \\ = \sec^{-1} |u| + C = \sec^{-1} |x + 1| + C$$

$$80. \int \frac{dx}{(x-2)\sqrt{x^2 - 4x + 3}} = \int \frac{dx}{(x-2)\sqrt{x^2 - 4x + 4 - 1}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2 - 1}} = \int \frac{1}{u\sqrt{u^2 - 1}} du, \text{ where } u = x - 2 \text{ and } du = dx \\ = \sec^{-1} |u| + C = \sec^{-1} |x - 2| + C$$

$$81. \int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} dx = \int e^u du, \text{ where } u = \sin^{-1} x \text{ and } du = \frac{dx}{\sqrt{1-x^2}} \\ = e^u + C = e^{\sin^{-1} x} + C$$

$$82. \int \frac{e^{\cos^{-1} x}}{\sqrt{1-x^2}} dx = -\int e^u du, \text{ where } u = \cos^{-1} x \text{ and } du = \frac{-dx}{\sqrt{1-x^2}} \\ = -e^u + C = -e^{\cos^{-1} x} + C$$

$$83. \int \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}} dx = \int u^2 du, \text{ where } u = \sin^{-1} x \text{ and } du = \frac{dx}{\sqrt{1-x^2}} \\ = \frac{u^3}{3} + C = \frac{(\sin^{-1} x)^3}{3} + C$$

$$84. \int \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx = \int u^{1/2} du, \text{ where } u = \tan^{-1} x \text{ and } du = \frac{dx}{1+x^2} \\ = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (\tan^{-1} x)^{3/2} + C = \frac{2}{3} \sqrt{(\tan^{-1} x)^3} + C$$

$$85. \int \frac{1}{(\tan^{-1} y)(1+y^2)} dy = \int \frac{\left(\frac{1}{1+y^2}\right)}{\tan^{-1} y} dy = \int \frac{1}{u} du, \text{ where } u = \tan^{-1} y \text{ and } du = \frac{dy}{1+y^2} \\ = \ln |u| + C = \ln |\tan^{-1} y| + C$$

$$86. \int \frac{1}{(\sin^{-1} y)\sqrt{1-y^2}} dy = \int \frac{\left(\frac{1}{\sqrt{1-y^2}}\right)}{\sin^{-1} y} dy = \int \frac{1}{u} du, \text{ where } u = \sin^{-1} y \text{ and } du = \frac{dy}{\sqrt{1-y^2}} \\ = \ln |u| + C = \ln |\sin^{-1} y| + C$$

$$87. \int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1} x)}{x\sqrt{x^2 - 1}} dx = \int_{\pi/4}^{\pi/3} \sec^2 u du, \text{ where } u = \sec^{-1} x \text{ and } du = \frac{dx}{x\sqrt{x^2 - 1}}; x = \sqrt{2} \Rightarrow u = \frac{\pi}{4}, x = 2 \Rightarrow u = \frac{\pi}{3} \\ = [\tan u]_{\pi/4}^{\pi/3} = \tan \frac{\pi}{3} - \tan \frac{\pi}{4} = \sqrt{3} - 1$$

$$88. \int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x)}{x\sqrt{x^2-1}} dx = \int_{\pi/6}^{\pi/3} \cos u \, du, \text{ where } u = \sec^{-1} x \text{ and } du = \frac{dx}{x\sqrt{x^2-1}}; \quad x = \frac{2}{\sqrt{3}} \Rightarrow u = \frac{\pi}{6}, x = 2 \Rightarrow u = \frac{\pi}{3}$$

$$= [\sin u]_{\pi/6}^{\pi/3} = \sin \frac{\pi}{3} - \sin \frac{\pi}{6} = \frac{\sqrt{3}-1}{2}$$

$$89. \int \frac{1}{\sqrt{x}(x+1)\left[(\tan^{-1}\sqrt{x})^2+9\right]} dx = 2 \int \frac{1}{u^2+9} du \text{ where } u = \tan^{-1}\sqrt{x} \Rightarrow du = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} dx \Rightarrow 2du = \frac{1}{(1+x)\sqrt{x}} dx$$

$$= \frac{2}{3} \tan^{-1}\left(\frac{\tan^{-1}\sqrt{x}}{3}\right) + C$$

$$90. \int \frac{e^x \sin^{-1} e^x}{\sqrt{1-e^{2x}}} dx = \int u \, du \text{ where } u = \sin^{-1} e^x \Rightarrow du = \frac{1}{\sqrt{1-e^{2x}}} e^x dx$$

$$= \frac{1}{2} (\sin^{-1} e^x)^2 + C$$

$$91. \lim_{x \rightarrow 0} \frac{\sin^{-1} 5x}{x} = \lim_{x \rightarrow 0} \frac{\left(\frac{5}{\sqrt{1-25x^2}}\right)}{1} = 5$$

$$92. \lim_{x \rightarrow 1^+} \frac{\sqrt{x^2-1}}{\sec^{-1} x} = \lim_{x \rightarrow 1^+} \frac{(x^2-1)^{1/2}}{\sec^{-1} x} = \lim_{x \rightarrow 1^+} \frac{\left(\frac{1}{2}\right)(x^2-1)^{-1/2} (2x)}{\left(\frac{1}{|x|\sqrt{x^2-1}}\right)} = \lim_{x \rightarrow 1^+} x|x| = 1$$

$$93. \lim_{x \rightarrow \infty} x \tan^{-1}\left(\frac{2}{x}\right) = \lim_{x \rightarrow \infty} \frac{\tan^{-1}(2x^{-1})}{x^{-1}} = \lim_{x \rightarrow \infty} \frac{\left(\frac{-2x^{-2}}{1+4x^{-2}}\right)}{-x^{-2}} = \lim_{x \rightarrow \infty} \frac{2}{1+4x^{-2}} = 2$$

$$94. \lim_{x \rightarrow 0} \frac{2 \tan^{-1} 3x^2}{7x^2} = \lim_{x \rightarrow 0} \frac{\left(\frac{12x}{1+9x^4}\right)}{14x} = \lim_{x \rightarrow 0} \frac{6}{7(1+9x^4)} = \frac{6}{7}$$

$$95. \lim_{x \rightarrow 0} \frac{\tan^{-1} x^2}{x \sin^{-1} x} = \lim_{x \rightarrow 0} \left(\frac{\frac{2x}{1+x^4}}{x - \frac{1}{\sqrt{1-x^2}} + \sin^{-1} x} \right) = \lim_{x \rightarrow 0} \left(\frac{\frac{-2(3x^4-1)}{(1+x^4)^2}}{\frac{-x^2+2}{(1-x^2)^{3/2}}} \right) = \frac{\frac{-2(0-1)}{1^2}}{\frac{-0+2}{(1-0)^{3/2}}} = \frac{2}{2} = 1$$

$$96. \lim_{x \rightarrow \infty} \frac{e^x \tan^{-1} e^x}{e^{2x} + x} = \lim_{x \rightarrow \infty} \frac{e^x \tan^{-1} e^x + \frac{e^{2x}}{e^{2x}+1}}{2e^{2x}+1} = \lim_{x \rightarrow \infty} \frac{e^x \tan^{-1} e^x + \frac{e^{2x}}{e^{2x}+1} + \frac{2e^{2x}}{(e^{2x}+1)^2}}{4e^{2x}} = \lim_{x \rightarrow \infty} \frac{e^x \tan^{-1} e^x + \frac{e^{2x}(e^{2x}+3)}{(e^{2x}+1)^2}}{4e^{2x}}$$

$$= \lim_{x \rightarrow \infty} \left[\frac{\tan^{-1} e^x}{4e^x} + \frac{(e^{2x}+3)}{4(e^{2x}+1)^2} \right] = \lim_{x \rightarrow \infty} \left[\frac{\tan^{-1} e^x}{4e^x} + \frac{(1+3e^{-2x})}{4(e^x+e^{-x})^2} \right] = 0 + 0 = 0$$

$$\begin{aligned}
 97. \quad \lim_{x \rightarrow 0^+} \frac{[\tan^{-1}(\sqrt{x})]^2}{x\sqrt{x+1}} &= \lim_{x \rightarrow 0^+} \frac{\tan^{-1}(\sqrt{x}) \frac{1}{\sqrt{x(1+x)}}}{\frac{x}{2\sqrt{x+1}} + \sqrt{x+1}} = \lim_{x \rightarrow 0^+} \frac{\frac{\tan^{-1}(\sqrt{x})}{\sqrt{x(1+x)}}}{\frac{3x+2}{2\sqrt{x+1}}} = \lim_{x \rightarrow 0^+} \left(\frac{2 \tan^{-1}(\sqrt{x})}{(3x+2)\sqrt{x}\sqrt{x+1}} \right) = \lim_{x \rightarrow 0^+} \left(\frac{\frac{1}{\sqrt{x(1+x)}}}{\frac{12x^2+13x+2}{2\sqrt{x}\sqrt{x+1}}} \right) \\
 &= \lim_{x \rightarrow 0^+} \left(\frac{2}{(12x^2+13x+2)\sqrt{x+1}} \right) = \frac{2}{2} = 1
 \end{aligned}$$

$$\begin{aligned}
 98. \quad \lim_{x \rightarrow 0^+} \frac{\sin^{-1}(x^2)}{(\sin^{-1} x)^2} &= \lim_{x \rightarrow 0^+} \left(\frac{\frac{2x}{\sqrt{1-x^4}}}{2(\sin^{-1} x) \frac{1}{\sqrt{1-x^2}}} \right) = \lim_{x \rightarrow 0^+} \left(\frac{x}{\sin^{-1} x \sqrt{1+x^2}} \right) = \lim_{x \rightarrow 0^+} \left(\frac{1}{\sin^{-1} x \cdot \frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1-x^2}} \sqrt{1+x^2}} \right) \\
 &= \lim_{x \rightarrow 0^+} \left(\frac{\sqrt{1+x^2} \sqrt{1-x^2}}{1+x^2+x\sqrt{1-x^2} \sin^{-1} x} \right) = \frac{1}{1} = 1
 \end{aligned}$$

$$\begin{aligned}
 99. \quad \text{If } y = \ln x - \frac{1}{2} \ln(1+x^2) - \frac{\tan^{-1} x}{x} + C, \text{ then } dy &= \left[\frac{1}{x} - \frac{x}{1+x^2} - \frac{\left(\frac{x}{1+x^2}\right) \tan^{-1} x}{x^2} \right] dx \\
 &= \left(\frac{1}{x} - \frac{x}{1+x^2} - \frac{1}{x(1+x^2)} + \frac{\tan^{-1} x}{x^2} \right) dx = \frac{x(1+x^2) - x^3 - x + (\tan^{-1} x)(1+x^2)}{x^2(1+x^2)} dx = \frac{\tan^{-1} x}{x^2} dx, \text{ which verifies the formula}
 \end{aligned}$$

$$\begin{aligned}
 100. \quad \text{If } y = \frac{x^4}{4} \cos^{-1} 5x + \frac{5}{4} \int \frac{x^4}{\sqrt{1-25x^2}} dx, \text{ then} \\
 dy = \left[x^3 \cos^{-1} 5x + \left(\frac{x^4}{4} \right) \left(\frac{-5}{\sqrt{1-25x^2}} \right) + \frac{5}{4} \left(\frac{x^4}{\sqrt{1-25x^2}} \right) \right] dx = (x^3 \cos^{-1} 5x) dx, \text{ which verifies the formula}
 \end{aligned}$$

$$\begin{aligned}
 101. \quad \text{If } y = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1-x^2} \sin^{-1} x + C, \text{ then} \\
 dy = \left[(\sin^{-1} x)^2 + \frac{2x(\sin^{-1} x)}{\sqrt{1-x^2}} - 2 + \frac{-2x}{\sqrt{1-x^2}} \sin^{-1} x + 2\sqrt{1-x^2} \left(\frac{1}{\sqrt{1-x^2}} \right) \right] dx = (\sin^{-1} x)^2 dx, \text{ which verifies the formula}
 \end{aligned}$$

$$\begin{aligned}
 102. \quad \text{If } y = x \ln(a^2 + x^2) - 2x + 2a \tan^{-1} \left(\frac{x}{a} \right) + C, \text{ then } dy &= \left[\ln(a^2 + x^2) + \frac{2x^2}{a^2 + x^2} - 2 + \frac{2}{1 + \left(\frac{x^2}{a^2}\right)} \right] dx \\
 &= \left[\ln(a^2 + x^2) + 2 \left(\frac{a^2 + x^2}{a^2 + x^2} \right) - 2 \right] dx = \ln(a^2 + x^2) dx, \text{ which verifies the formula}
 \end{aligned}$$

$$103. \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow dy = \frac{dx}{\sqrt{1-x^2}} \Rightarrow y = \sin^{-1} x + C; x = 0 \text{ and } y = 0 \Rightarrow 0 = \sin^{-1} 0 + C \Rightarrow C = 0 \Rightarrow y = \sin^{-1} x$$

$$\begin{aligned}
 104. \quad \frac{dy}{dx} = \frac{1}{x^2+1} - 1 \Rightarrow dy &= \left(\frac{1}{1+x^2} - 1 \right) dx \Rightarrow y = \tan^{-1}(x) - x + C; x = 0 \text{ and } y = 1 \Rightarrow 1 = \tan^{-1} 0 - 0 + C \Rightarrow C = 1 \\
 &\Rightarrow y = \tan^{-1}(x) - x + 1
 \end{aligned}$$

$$\begin{aligned}
 105. \quad \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}} \Rightarrow dy &= \frac{dx}{x\sqrt{x^2-1}} \Rightarrow y = \sec^{-1} |x| + C; x = 2 \text{ and } y = \pi \Rightarrow \pi = \sec^{-1} 2 + C \Rightarrow C = \pi - \sec^{-1} 2 \\
 &= \pi - \frac{\pi}{3} = \frac{2\pi}{3} \Rightarrow y = \sec^{-1}(x) + \frac{2\pi}{3}, x > 1
 \end{aligned}$$

$$106. \frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} \Rightarrow dy = \left(\frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} \right) dx \Rightarrow y = \tan^{-1} x - 2 \sin^{-1} x + C; x=0 \text{ and } y=2 \\ \Rightarrow 2 = \tan^{-1} 0 - 2 \sin^{-1} 0 + C \Rightarrow C=2 \Rightarrow y = \tan^{-1} x - 2 \sin^{-1} x + 2$$

107. (a) The angle α is the large angle between the wall and the right end of the blackboard minus the small angle between the left end of the blackboard and the wall $\Rightarrow \alpha = \cot^{-1}\left(\frac{x}{5}\right) - \cot^{-1}(x)$.

(b) $\frac{d\alpha}{dx} = -\frac{\frac{1}{5}}{1+(\frac{x}{5})^2} + \frac{1}{1+(x)^2} = -\frac{5}{25+x^2} + \frac{1}{1+x^2} = \frac{20-4x^2}{(25+x^2)(1+x^2)}$; $\frac{d\alpha}{dx} = 0 \Rightarrow 20-4x^2 = 0 \Rightarrow x = \pm\sqrt{5}$. Since $x > 0$, consider only $x = \sqrt{5} \Rightarrow \alpha(\sqrt{5}) = \cot^{-1}\left(\frac{\sqrt{5}}{5}\right) - \cot^{-1}(\sqrt{5}) \approx 0.729728 \approx 41.8103^\circ$. Using the first derivative test, $\frac{d\alpha}{dx}\big|_{x=1} = \frac{16}{52} > 0$ and $\frac{d\alpha}{dx}\big|_{x=10} = -\frac{380}{1375} < 0 \Rightarrow$ local maximum of 41.8103° when $x = \sqrt{5} \approx 2.24$ m.

$$108. V = \pi \int_0^{\pi/3} [2^2 - (\sec y)^2] dy = \pi [4y - \tan y]_0^{\pi/3} = \pi \left(\frac{4\pi}{3} - \sqrt{3} \right)$$

109. $V = \left(\frac{1}{3}\right)\pi r^2 h = \left(\frac{1}{3}\right)\pi (3 \sin \theta)^2 (3 \cos \theta) = 9\pi (\cos \theta - \cos^3 \theta)$, where $0 \leq \theta \leq \frac{\pi}{2}$
 $\Rightarrow \frac{dV}{d\theta} = -9\pi (\sin \theta) (1 - 3 \cos^2 \theta) = 0 \Rightarrow \sin \theta = 0$ or $\cos \theta = \pm \frac{1}{\sqrt{3}} \Rightarrow$ the critical points are: 0 , $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, and $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$; but $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is not in the domain. When $\theta = 0$, we have a minimum and when $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7^\circ$, we have a maximum volume.

$$110. 65^\circ + (90^\circ - \beta) + (90^\circ - \alpha) = 180^\circ \Rightarrow \alpha = 65^\circ - \beta = 65^\circ - \tan^{-1}\left(\frac{21}{50}\right) \approx 65^\circ - 22.78^\circ \approx 42.22^\circ$$

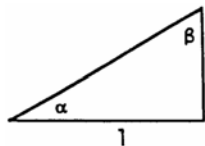
111. Take each square as a unit square. From the diagram we have the following: the smallest angle α has a tangent of 1 $\Rightarrow \alpha = \tan^{-1} 1$; the middle angle β has a tangent of 2 $\Rightarrow \beta = \tan^{-1} 2$; and the largest angle γ has a tangent of 3 $\Rightarrow \gamma = \tan^{-1} 3$. The sum of these three angles is $\pi \Rightarrow \alpha + \beta + \gamma = \pi \Rightarrow \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$.

112. (a) From the symmetry of the diagram, we see that $\pi - \sec^{-1} x$ is the vertical distance from the graph of $y = \sec^{-1} x$ to the line $y = \pi$ and this distance is the same as the height of $y = \sec^{-1} x$ above the x -axis at $-x$; i.e., $\pi - \sec^{-1} x = \sec^{-1}(-x)$.

(b) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, where $-1 \leq x \leq 1 \Rightarrow \cos^{-1}\left(-\frac{1}{x}\right) = \pi - \cos^{-1}\left(\frac{1}{x}\right)$, where $x \geq 1$ or $x \leq -1$
 $\Rightarrow \sec^{-1}(-x) = \pi - \sec^{-1} x$

113. $\sin^{-1}(1) + \cos^{-1}(1) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$; $\sin^{-1}(0) + \cos^{-1}(0) = 0 + \frac{\pi}{2} = \frac{\pi}{2}$; and $\sin^{-1}(-1) + \cos^{-1}(-1) = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$. If $x \in (-1, 0)$ and $x = -a$, then $\sin^{-1}(x) + \cos^{-1}(x) = \sin^{-1}(-a) + \cos^{-1}(-a) = -\sin^{-1} a + \left(\pi - \cos^{-1} a\right)$
 $= \pi - \left(\sin^{-1} a + \cos^{-1} a\right) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$ from Equations (3) and (4) in the text.

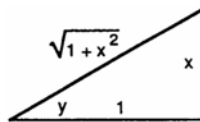
114.



$$x \Rightarrow \tan \alpha = x \text{ and } \tan \beta = \frac{1}{x} \Rightarrow \frac{\pi}{2} = \alpha + \beta = \tan^{-1} x + \tan^{-1} \frac{1}{x}.$$

$$115. \csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u \Rightarrow \frac{d}{dx}(\csc^{-1} u) = \frac{d}{dx}\left(\frac{\pi}{2} - \sec^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}} = -\frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}}, |u| > 1$$

$$116. y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow \frac{d}{dx}(\tan y) = \frac{d}{dx}(x) \\ \Rightarrow (\sec^2 y) \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}, \text{ as}$$



indicated by the triangle

$$117. f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x \Rightarrow \left. \frac{df^{-1}}{dx} \right|_{x=b} = \frac{1}{\left. \frac{df}{dx} \right|_{x=f^{-1}(b)}} = \frac{1}{\sec(\sec^{-1} b) \tan(\sec^{-1} b)} = \frac{1}{b(\pm\sqrt{b^2-1})}. \text{ Since the slope}$$

of $\sec^{-1} x$ is always positive, we obtain the right sign by writing $\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}.$

$$118. \cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u \Rightarrow \frac{d}{dx}(\cot^{-1} u) = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{1+u^2} = -\frac{\frac{du}{dx}}{1+u^2}$$

119. The function f and g have the same derivative (for $x \geq 0$), namely $\frac{1}{\sqrt{x(x+1)}}$. The functions therefore differ by a constant. To identify the constant we can set x equal to 0 in the equation $f(x) = g(x) + C$, obtaining $\sin^{-1}(-1) = 2 \tan^{-1}(0) + C \Rightarrow -\frac{\pi}{2} = 0 + C \Rightarrow C = -\frac{\pi}{2}$. For $x \geq 0$, we have $\sin^{-1}\left(\frac{x-1}{x+1}\right) = 2 \tan^{-1} \sqrt{x} - \frac{\pi}{2}$.

120. The functions f and g have the same derivative for $x > 0$, namely $\frac{-1}{1+x^2}$. The functions therefore differ by a constant for $x > 0$. To identify the constant we can set x equal to 1 in the equation $f(x) = g(x) + C$, obtaining $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \tan^{-1} 1 + C \Rightarrow \frac{\pi}{4} = \frac{\pi}{4} + C \Rightarrow C = 0$. For $x > 0$, we have $\sin^{-1} \frac{1}{\sqrt{x^2+1}} = \tan^{-1} \frac{1}{x}$.

$$121. V = \pi \int_{-\sqrt{3}/3}^{\sqrt{3}} \left(\frac{1}{\sqrt{1+x^2}} \right)^2 dx = \pi \int_{-\sqrt{3}/3}^{\sqrt{3}} \frac{1}{1+x^2} dx = \pi \left[\tan^{-1} x \right]_{-\sqrt{3}/3}^{\sqrt{3}} = \pi \left[\tan^{-1} \sqrt{3} - \tan^{-1} \left(-\frac{\sqrt{3}}{3} \right) \right] \\ = \pi \left[\frac{\pi}{3} - \left(-\frac{\pi}{6} \right) \right] = \frac{\pi^2}{2}$$

122. Consider $y = \sqrt{r^2 - x^2} \Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}}$; Since $\frac{dy}{dx}$ is undefined at $x = r$ and $x = -r$, we will find the length from

$$x = 0 \text{ to } x = \frac{r}{\sqrt{2}} \text{ (in other words, the length of } \frac{1}{8} \text{ of a circle)} \Rightarrow L = \int_0^{r/\sqrt{2}} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}} \right)^2} dx \\ = \int_0^{r/\sqrt{2}} \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx = \int_0^{r/\sqrt{2}} \sqrt{\frac{r^2}{r^2 - x^2}} dx = \int_0^{r/\sqrt{2}} \frac{r}{\sqrt{r^2 - x^2}} dx = \left[r \sin^{-1} \left(\frac{x}{r} \right) \right]_0^{r/\sqrt{2}} = r \sin^{-1} \left(\frac{r/\sqrt{2}}{r} \right) - r \sin^{-1}(0) \\ = r \sin^{-1} \left(\frac{1}{\sqrt{2}} \right) - 0 = r \left(\frac{\pi}{4} \right) = \frac{\pi r}{4}. \text{ The total circumference of the circle is } C = 8L = 8 \left(\frac{\pi r}{4} \right) = 2\pi r.$$

$$123. (a) \quad A(x) = \frac{\pi}{4} (\text{diameter})^2 = \frac{\pi}{4} \left[\frac{1}{\sqrt{1+x^2}} - \left(-\frac{1}{\sqrt{1+x^2}} \right) \right]^2 = \frac{\pi}{1+x^2} \Rightarrow V = \int_a^b A(x) dx = \int_{-1}^1 \frac{\pi dx}{1+x^2} = \pi \left[\tan^{-1} x \right]_{-1}^1 \\ = (\pi)(2) \left(\frac{\pi}{4} \right) = \frac{\pi^2}{2}$$

$$(b) \quad A(x) = (\text{edge})^2 = \left[\frac{1}{\sqrt{1+x^2}} - \left(-\frac{1}{\sqrt{1+x^2}} \right) \right]^2 = \frac{4}{1+x^2} \Rightarrow V = \int_a^b A(x) dx = \int_{-1}^1 \frac{4 dx}{1+x^2} = 4 \left[\tan^{-1} x \right]_{-1}^1 \\ = 4[\tan^{-1}(1) - \tan^{-1}(-1)] = 4 \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = 2\pi$$

$$124. (a) \quad A(x) = \frac{\pi}{4} (\text{diameter})^2 = \frac{\pi}{4} \left(\frac{2}{\sqrt[4]{1-x^2}} - 0 \right)^2 = \frac{\pi}{4} \left(\frac{4}{\sqrt{1-x^2}} \right) = \frac{\pi}{\sqrt{1-x^2}} \Rightarrow V = \int_a^b A(x) dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{\pi}{\sqrt{1-x^2}} dx \\ = \pi \left[\sin^{-1} x \right]_{-\sqrt{2}/2}^{\sqrt{2}/2} = \pi \left[\sin^{-1} \left(\frac{\sqrt{2}}{2} \right) - \sin^{-1} \left(-\frac{\sqrt{2}}{2} \right) \right] = \pi \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = \frac{\pi^2}{2}$$

$$(b) \quad A(x) = \frac{(\text{diagonal})^2}{2} = \frac{1}{2} \left(\frac{2}{\sqrt[4]{1-x^2}} - 0 \right)^2 = \frac{2}{\sqrt{1-x^2}} \Rightarrow V = \int_a^b A(x) dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{2}{\sqrt{1-x^2}} dx = 2 \left[\sin^{-1} x \right]_{-\sqrt{2}/2}^{\sqrt{2}/2} \\ = 2 \left(\frac{\pi}{4} \cdot 2 \right) = \pi$$

$$125. (a) \quad \sec^{-1} 1.5 = \cos^{-1} \frac{1}{1.5} \approx 0.84107$$

$$(b) \quad \csc^{-1}(-1.5) = \sin^{-1} \left(-\frac{1}{1.5} \right) \approx -0.72973$$

$$(c) \quad \cot^{-1} 2 = \frac{\pi}{2} - \tan^{-1} 2 \approx 0.46365$$

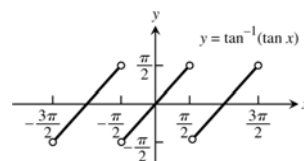
$$126. (a) \quad \sec^{-1}(-3) = \cos^{-1} \left(-\frac{1}{3} \right) \approx 1.91063$$

$$(b) \quad \csc^{-1} 1.7 = \sin^{-1} \left(\frac{1}{1.7} \right) \approx 0.62887$$

$$(c) \quad \cot^{-1}(-2) = \frac{\pi}{2} - \tan^{-1}(-2) \approx 2.67795$$

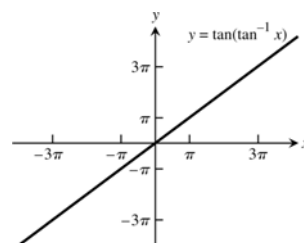
127. (a) Domain: all real numbers except those having the form $\frac{\pi}{2} + k\pi$ where k is an integer.

Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

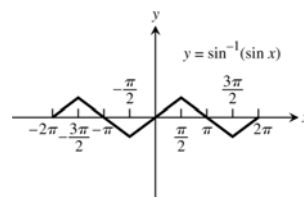


(b) Domain: $-\infty < x < \infty$; Range: $-\infty < y < \infty$

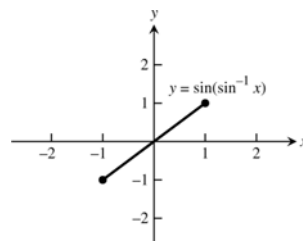
The graph of $y = \tan^{-1}(\tan x)$ is periodic, the graph of $y = \tan(\tan^{-1} x) = x$ for $-\infty < x < \infty$.



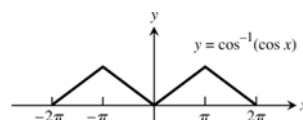
128. (a) Domain: $-\infty < x < \infty$; Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



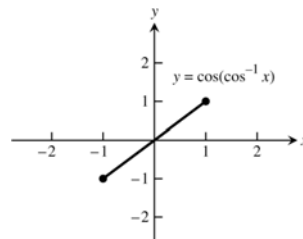
- (b) Domain: $-1 \leq x \leq 1$; Range: $-1 \leq y \leq 1$
 The graph of $y = \sin^{-1}(\sin x)$ is periodic; the graph of $y = \sin(\sin^{-1} x) = x$ for $-1 \leq x \leq 1$.



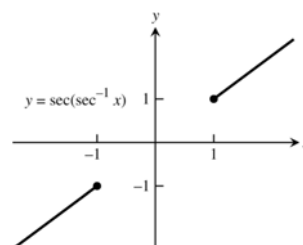
129. (a) Domain: $-\infty < x < \infty$; Range: $0 \leq y \leq \pi$



- (b) Domain: $-1 \leq x \leq 1$; Range: $-1 \leq y \leq 1$
 The graph of $y = \cos^{-1}(\cos x)$ is periodic; the graph of $y = \cos(\cos^{-1} x) = x$ for $-1 \leq x \leq 1$.



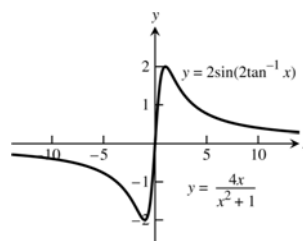
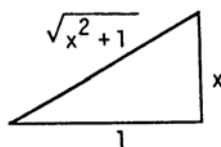
130. Since the domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$, we have $\sec(\sec^{-1} x) = x$ for $|x| \geq 1$. The graph of $y = \sec(\sec^{-1} x)$ is the line $y = x$ with the open line segment from $(-1, -1)$ to $(1, 1)$ removed.



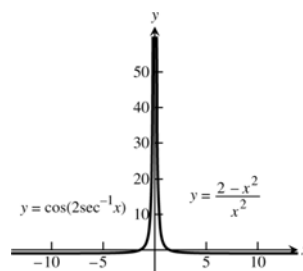
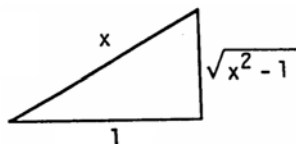
131. The graphs are identical for $y = 2 \sin(2 \tan^{-1} x)$

$$= 4 \left[\sin(\tan^{-1} x) \right] \left[\cos(\tan^{-1} x) \right]$$

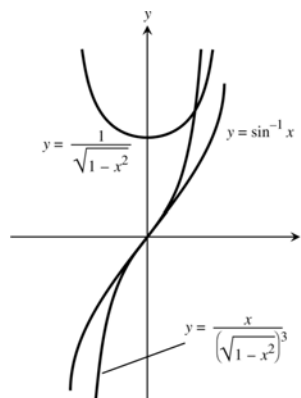
$$= 4 \left(\frac{x}{\sqrt{x^2 + 1}} \right) \left(\frac{1}{\sqrt{x^2 + 1}} \right) = \frac{4x}{x^2 + 1} \text{ from the triangle}$$



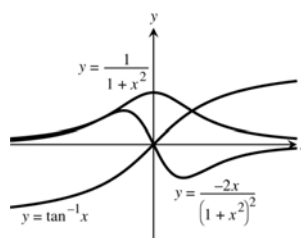
132. The graphs are identical for $y = \cos(2 \sec^{-1} x)$
 $\cos^2(\sec^{-1} x) - \sin^2(\sec^{-1} x) = \frac{1}{x^2} - \frac{x^2 - 1}{x^2} = \frac{2 - x^2}{x^2}$
 from the triangle



133. The values of f increase over the interval $[-1, 1]$ because $f' > 0$, and the graph of f steepens as the values of f' increase toward the ends of the interval. The graph of f is concave down to the left of the origin where $f'' < 0$, and concave up to the right of the origin where $f'' > 0$. There is an inflection point at $x = 0$ where $f'' = 0$ and f' has a local minimum value.



134. The values of f increase throughout the interval $(-\infty, \infty)$ because $f' > 0$, and they increase most rapidly near the origin where the values of f' are relatively large. The graph of f is concave up to the left of the origin where $f'' > 0$, and concave down to the right of the origin where $f'' < 0$. There is an inflection point at $x = 0$ where $f'' = 0$ and f' has a local maximum value.



7.7 HYPERBOLIC FUNCTIONS

- $\sinh x = -\frac{3}{4} \Rightarrow \cosh x = \sqrt{1 + \sinh^2 x} = \sqrt{1 + \left(-\frac{3}{4}\right)^2} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(-\frac{3}{4}\right)}{\left(\frac{5}{4}\right)} = -\frac{3}{5},$
 $\coth x = \frac{1}{\tanh x} = -\frac{5}{3}, \quad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{4}{5}, \quad \text{and} \quad \operatorname{csch} x = \frac{1}{\sinh x} = -\frac{4}{3}$
- $\sinh x = \frac{4}{3} \Rightarrow \cosh x = \sqrt{1 + \sinh^2 x} = \sqrt{1 + \frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{4}{3}\right)}{\left(\frac{5}{3}\right)} = \frac{4}{5}, \quad \coth x = \frac{1}{\tanh x} = \frac{5}{4},$
 $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{3}{5}, \quad \text{and} \quad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{3}{4}$
- $\cosh x = \frac{17}{15}, x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\left(\frac{17}{15}\right)^2 - 1} = \sqrt{\frac{289}{225} - 1} = \sqrt{\frac{64}{225}} = \frac{8}{15}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{8}{15}\right)}{\left(\frac{17}{15}\right)} = \frac{8}{17},$
 $\coth x = \frac{1}{\tanh x} = \frac{17}{8}, \quad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{15}{17}, \quad \text{and} \quad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{15}{8}$
- $\cosh x = \frac{13}{5}, x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\frac{169}{25} - 1} = \sqrt{\frac{144}{25}} = \frac{12}{5}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{12}{5}\right)}{\left(\frac{13}{5}\right)} = \frac{12}{13},$
 $\coth x = \frac{1}{\tanh x} = \frac{13}{12}, \quad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{5}{13}, \quad \text{and} \quad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{5}{12}$
- $2 \cosh(\ln x) = 2 \left(\frac{e^{\ln x} + e^{-\ln x}}{2} \right) = e^{\ln x} + \frac{1}{e^{\ln x}} = x + \frac{1}{x}$

$$6. \quad \sinh(2 \ln x) = \frac{e^{2 \ln x} - e^{-2 \ln x}}{2} = \frac{e^{\ln x^2} - e^{\ln x^{-2}}}{2} = \frac{\left(x^2 - \frac{1}{x^2}\right)}{2} = \frac{x^4 - 1}{2x^2}$$

$$7. \quad \cosh 5x + \sinh 5x = \frac{e^{5x} + e^{-5x}}{2} + \frac{e^{5x} - e^{-5x}}{2} = e^{5x} \quad 8. \quad \cosh 3x - \sinh 3x = \frac{e^{3x} + e^{-3x}}{2} - \frac{e^{3x} - e^{-3x}}{2} = e^{-3x}$$

$$9. \quad (\sinh x + \cosh x)^4 = \left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}\right)^4 = (e^x)^4 = e^{4x}$$

$$10. \quad \ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) = \ln(\cosh^2 x - \sinh^2 x) = \ln 1 = 0$$

$$11. \quad (a) \quad \sinh 2x = \sinh(x + x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x$$

$$(b) \quad \cosh 2x = \cosh(x + x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x$$

$$12. \quad \cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4} \left[(e^x + e^{-x}) + (e^x - e^{-x}) \right] \left[(e^x + e^{-x}) - (e^x - e^{-x}) \right] \\ = \frac{1}{4} (2e^x) (2e^{-x}) = \frac{1}{4} (4e^0) = \frac{1}{4} (4) = 1$$

$$13. \quad y = 6 \sinh \frac{x}{3} \Rightarrow \frac{dy}{dx} = 6 \left(\cosh \frac{x}{3} \right) \left(\frac{1}{3} \right) = 2 \cosh \frac{x}{3}$$

$$14. \quad y = \frac{1}{2} \sinh(2x + 1) \Rightarrow \frac{dy}{dx} = \frac{1}{2} [\cosh(2x + 1)] (2) = \cosh(2x + 1)$$

$$15. \quad y = 2\sqrt{t} \tanh \sqrt{t} = 2t^{1/2} \tanh t^{1/2} \Rightarrow \frac{dy}{dt} = \left[\operatorname{sech}^2(t^{1/2}) \left(\frac{1}{2} t^{-1/2} \right) \right] (2t^{1/2}) + (\tanh t^{1/2}) (t^{-1/2}) = \operatorname{sech}^2 \sqrt{t} + \frac{\tanh \sqrt{t}}{\sqrt{t}}$$

$$16. \quad y = t^2 \tanh \frac{1}{t} = t^2 \tanh t^{-1} \Rightarrow \frac{dy}{dt} = \left[\operatorname{sech}^2(t^{-1}) (-t^{-2}) \right] (t^2) + (2t) (\tanh t^{-1}) = -\operatorname{sech}^2 \frac{1}{t} + 2t \tanh \frac{1}{t}$$

$$17. \quad y = \ln(\sinh z) \Rightarrow \frac{dy}{dz} = \frac{\cosh z}{\sinh z} = \coth z$$

$$18. \quad y = \ln(\cosh z) \Rightarrow \frac{dy}{dz} = \frac{\sinh z}{\cosh z} = \tanh z$$

$$19. \quad y = (\operatorname{sech} \theta)(1 - \ln \operatorname{sech} \theta) \Rightarrow \frac{dy}{d\theta} = \left(-\frac{\operatorname{sech} \theta \tanh \theta}{\operatorname{sech} \theta} \right) (\operatorname{sech} \theta) + (-\operatorname{sech} \theta \tanh \theta) (1 - \ln \operatorname{sech} \theta) \\ = \operatorname{sech} \theta \tanh \theta - (\operatorname{sech} \theta \tanh \theta) (1 - \ln \operatorname{sech} \theta) = (\operatorname{sech} \theta \tanh \theta) [1 - (1 - \ln \operatorname{sech} \theta)] \\ = (\operatorname{sech} \theta \tanh \theta) (\ln \operatorname{sech} \theta)$$

$$20. \quad y = (\operatorname{csch} \theta)(1 - \ln \operatorname{csch} \theta) \Rightarrow \frac{dy}{d\theta} = (\operatorname{csch} \theta) \left(-\frac{-\operatorname{csch} \theta \coth \theta}{\operatorname{csch} \theta} \right) + (1 - \ln \operatorname{csch} \theta) (-\operatorname{csch} \theta \coth \theta) \\ = \operatorname{csch} \theta \coth \theta - (1 - \ln \operatorname{csch} \theta) (\operatorname{csch} \theta \coth \theta) = (\operatorname{csch} \theta \coth \theta) (1 - 1 + \ln \operatorname{csch} \theta) \\ = (\operatorname{csch} \theta \coth \theta) (\ln \operatorname{csch} \theta)$$

21. $y = \ln \cosh v - \frac{1}{2} \tanh^2 v \Rightarrow \frac{dy}{dv} = \frac{\sinh v}{\cosh v} - \left(\frac{1}{2}\right)(2 \tanh v) \left(\operatorname{sech}^2 v\right) = \tanh v - (\tanh v) \left(\operatorname{sech}^2 v\right)$
 $= (\tanh v) \left(1 - \operatorname{sech}^2 v\right) = (\tanh v) \left(\tanh^2 v\right) = \tanh^3 v$
22. $y = \ln \sinh v - \frac{1}{2} \coth^2 v \Rightarrow \frac{dy}{dv} = \frac{\cosh v}{\sinh v} - \left(\frac{1}{2}\right)(2 \coth v) \left(-\operatorname{csch}^2 v\right) = \coth v + (\coth v) \left(\operatorname{csch}^2 v\right)$
 $= (\coth v) \left(1 + \operatorname{csch}^2 v\right) = (\coth v) \left(\coth^2 v\right) = \coth^3 v$
23. $y = (x^2 + 1) \operatorname{sech}(\ln x) = (x^2 + 1) \left(\frac{2}{e^{\ln x} + e^{-\ln x}}\right) = (x^2 + 1) \left(\frac{2}{x + x^{-1}}\right) = (x^2 + 1) \left(\frac{2x}{x^2 + 1}\right) = 2x \Rightarrow \frac{dy}{dx} = 2$
24. $y = (4x^2 - 1) \operatorname{csch}(\ln 2x) = (4x^2 - 1) \left(\frac{2}{e^{\ln 2x} - e^{-\ln 2x}}\right) = (4x^2 - 1) \left(\frac{2}{2x - (2x)^{-1}}\right) = (4x^2 - 1) \left(\frac{4x}{4x^2 - 1}\right) = 4x \Rightarrow \frac{dy}{dx} = 4$
25. $y = \sinh^{-1} \sqrt{x} = \sinh^{-1} \left(x^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)x^{-1/2}}{\sqrt{1 + \left(x^{1/2}\right)^2}} = \frac{1}{2\sqrt{x}\sqrt{1+x}} = \frac{1}{2\sqrt{x(1+x)}}$
26. $y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1} \left(2(x+1)^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{(2)\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{\left[2(x+1)^{1/2}\right]^2 - 1}} = \frac{1}{\sqrt{x+1}\sqrt{4x+3}} = \frac{1}{\sqrt{4x^2+7x+3}}$
27. $y = (1 - \theta) \tanh^{-1} \theta \Rightarrow \frac{dy}{d\theta} = (1 - \theta) \left(\frac{1}{1 - \theta^2}\right) + (-1) \tanh^{-1} \theta = \frac{1}{1 + \theta} - \tanh^{-1} \theta$
28. $y = (\theta^2 + 2\theta) \tanh^{-1}(\theta + 1) \Rightarrow \frac{dy}{d\theta} = (\theta^2 + 2\theta) \left[\frac{1}{1 - (\theta + 1)^2}\right] + (2\theta + 2) \tanh^{-1}(\theta + 1)$
 $= \frac{\theta^2 + 2\theta}{-\theta^2 - 2\theta} + (2\theta + 2) \tanh^{-1}(\theta + 1) = (2\theta + 2) \tanh^{-1}(\theta + 1) - 1$
29. $y = (1 - t) \coth^{-1} \sqrt{t} = (1 - t) \coth^{-1} \left(t^{1/2}\right) \Rightarrow \frac{dy}{dt} = (1 - t) \left[\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1 - \left(t^{1/2}\right)^2}\right] + (-1) \coth^{-1} \left(t^{1/2}\right) = \frac{1}{2\sqrt{t}} - \coth^{-1} \sqrt{t}$
30. $y = (1 - t^2) \coth^{-1} t \Rightarrow \frac{dy}{dt} = (1 - t^2) \left(\frac{1}{1 - t^2}\right) + (-2t) \coth^{-1} t = 1 - 2t \coth^{-1} t$
31. $y = \cos^{-1} x - x \operatorname{sech}^{-1} x \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}} - \left[x \left(\frac{-1}{x\sqrt{1 - x^2}}\right) + (1) \operatorname{sech}^{-1} x\right] = \frac{-1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - x^2}} - \operatorname{sech}^{-1} x = -\operatorname{sech}^{-1} x$
32. $y = \ln x + \sqrt{1 - x^2} \operatorname{sech}^{-1} x = \ln x + (1 - x^2)^{1/2} \operatorname{sech}^{-1} x$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{x} + (1 - x^2)^{1/2} \left(\frac{-1}{x\sqrt{1 - x^2}}\right) + \left(\frac{1}{2}\right)(1 - x^2)^{-1/2} (-2x) \operatorname{sech}^{-1} x = \frac{1}{x} - \frac{1}{x} - \frac{x}{\sqrt{1 - x^2}} \operatorname{sech}^{-1} x = \frac{-x}{\sqrt{1 - x^2}} \operatorname{sech}^{-1} x$

$$33. \quad y = \operatorname{csch}^{-1}\left(\frac{1}{2}\right)^{\theta} \Rightarrow \frac{dy}{d\theta} = -\frac{\left[\ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^{\theta}}{\left(\frac{1}{2}\right)^{\theta} \sqrt{1+\left[\left(\frac{1}{2}\right)^{\theta}\right]^2}} = -\frac{\ln(1)-\ln(2)}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}} = \frac{\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}$$

$$34. \quad y = \operatorname{csch}^{-1} 2^{\theta} \Rightarrow \frac{dy}{d\theta} = -\frac{(\ln 2)2^{\theta}}{2^{\theta} \sqrt{1+(2^{\theta})^2}} = \frac{-\ln 2}{\sqrt{1+2^{2\theta}}}$$

$$35. \quad y = \sinh^{-1}(\tan x) \Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{1+(\tan x)^2}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{|\sec x|} = \frac{|\sec x||\sec x|}{|\sec x|} = |\sec x|$$

$$36. \quad y = \cosh^{-1}(\sec x) \Rightarrow \frac{dy}{dx} = \frac{(\sec x)(\tan x)}{\sqrt{\sec^2 x - 1}} = \frac{(\sec x)(\tan x)}{\sqrt{\tan^2 x}} = \frac{(\sec x)(\tan x)}{|\tan x|} = \sec x, \quad 0 < x < \frac{\pi}{2}$$

$$37. \quad (a) \quad \text{If } y = \tan^{-1}(\sinh x) + C, \text{ then } \frac{dy}{dx} = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x, \text{ which verifies the formula}$$

$$(b) \quad \text{If } y = \sin^{-1}(\tanh x) + C, \text{ then } \frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\operatorname{sech} x} = \operatorname{sech} x, \text{ which verifies the formula}$$

$$38. \quad \text{If } y = \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{2} \sqrt{1-x^2} + C, \text{ then } \frac{dy}{dx} = x \operatorname{sech}^{-1} x + \frac{x^2}{2} \left(\frac{-1}{x\sqrt{1-x^2}} \right) + \frac{2x}{4\sqrt{1-x^2}} = x \operatorname{sech}^{-1} x, \text{ which verifies the formula}$$

$$39. \quad \text{If } y = \frac{x^2-1}{2} \coth^{-1} x + \frac{x}{2} + C, \text{ then } \frac{dy}{dx} = x \coth^{-1} x + \left(\frac{x^2-1}{2} \right) \left(\frac{1}{1-x^2} \right) + \frac{1}{2} = x \coth^{-1} x, \text{ which verifies the formula}$$

$$40. \quad \text{If } y = x \tanh^{-1} x + \frac{1}{2} \ln(1-x^2) + C, \text{ then } \frac{dy}{dx} = \tanh^{-1} x + x \left(\frac{1}{1-x^2} \right) + \frac{1}{2} \left(\frac{-2x}{1-x^2} \right) = \tanh^{-1} x, \text{ which verifies the formula}$$

$$41. \quad \int \sinh 2x \, dx = \frac{1}{2} \int \sinh u \, du, \text{ where } u = 2x \text{ and } du = 2 \, dx \\ = \frac{\cosh u}{2} + C = \frac{\cosh 2x}{2} + C$$

$$42. \quad \int \sinh \frac{x}{5} \, dx = 5 \int \sinh u \, du, \text{ where } u = \frac{x}{5} \text{ and } du = \frac{1}{5} \, dx \\ = 5 \cosh u + C = 5 \cosh \frac{x}{5} + C$$

$$43. \quad \int 6 \cosh \left(\frac{x}{2} - \ln 3 \right) dx = 12 \int \cosh u \, du, \text{ where } u = \frac{x}{2} - \ln 3 \text{ and } du = \frac{1}{2} \, dx \\ = 12 \sinh u + C = 12 \sinh \left(\frac{x}{2} - \ln 3 \right) + C$$

$$44. \quad \int 4 \cosh (3x - \ln 2) \, dx = \frac{4}{3} \int \cosh u \, du, \text{ where } u = 3x - \ln 2 \text{ and } du = 3 \, dx \\ = \frac{4}{3} \sinh u + C = \frac{4}{3} \sinh(3x - \ln 2) + C$$

$$\begin{aligned}
45. \quad \int \tanh \frac{x}{7} dx &= 7 \int \frac{\sinh u}{\cosh u} du, \text{ where } u = \frac{x}{7} \text{ and } du = \frac{1}{7} dx \\
&= 7 \ln |\cosh u| + C_1 = 7 \ln \left| \cosh \frac{x}{7} \right| + C_1 = 7 \ln \left| \frac{e^{x/7} + e^{-x/7}}{2} \right| + C_1 = 7 \ln |e^{x/7} + e^{-x/7}| - 7 \ln 2 + C_1 \\
&= 7 \ln |e^{x/7} + e^{-x/7}| + C
\end{aligned}$$

$$\begin{aligned}
46. \quad \int \coth \frac{\theta}{\sqrt{3}} d\theta &= \sqrt{3} \int \frac{\cosh u}{\sinh u} du, \text{ where } u = \frac{\theta}{\sqrt{3}} \text{ and } du = \frac{d\theta}{\sqrt{3}} \\
&= \sqrt{3} \ln |\sinh u| + C_1 = \sqrt{3} \ln \left| \sinh \frac{\theta}{\sqrt{3}} \right| + C_1 = \sqrt{3} \ln \left| \frac{e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}}{2} \right| + C_1 \\
&= \sqrt{3} \ln |e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}| - \sqrt{3} \ln 2 + C_1 = \sqrt{3} \ln |e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}| + C
\end{aligned}$$

$$\begin{aligned}
47. \quad \int \operatorname{sech}^2 \left(x - \frac{1}{2} \right) dx &= \int \operatorname{sech}^2 u du, \text{ where } u = \left(x - \frac{1}{2} \right) \text{ and } du = dx \\
&= \tanh u + C = \tanh \left(x - \frac{1}{2} \right) + C
\end{aligned}$$

$$\begin{aligned}
48. \quad \int \operatorname{csch}^2 (5-x) dx &= - \int \operatorname{csch}^2 u du, \text{ where } u = (5-x) \text{ and } du = -dx \\
&= -(-\coth u) + C = \coth u + C = \coth (5-x) + C
\end{aligned}$$

$$\begin{aligned}
49. \quad \int \frac{\operatorname{sech} \sqrt{t} \tanh \sqrt{t}}{\sqrt{t}} dt &= 2 \int \operatorname{sech} u \tanh u du, \text{ where } u = \sqrt{t} = t^{1/2} \text{ and } du = \frac{dt}{2\sqrt{t}} \\
&= 2(-\operatorname{sech} u) + C = -2 \operatorname{sech} \sqrt{t} + C
\end{aligned}$$

$$\begin{aligned}
50. \quad \int \frac{\operatorname{csch}(\ln t) \coth(\ln t)}{t} dt &= \int \operatorname{csch} u \coth u du, \text{ where } u = \ln t \text{ and } du = \frac{dt}{t} \\
&= -\operatorname{csch} u + C = -\operatorname{csch}(\ln t) + C
\end{aligned}$$

$$\begin{aligned}
51. \quad \int_{\ln 2}^{\ln 4} \coth x dx &= \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x} dx = \int_{3/4}^{15/8} \frac{1}{u} du \text{ where } u = \sinh x, du = \cosh x dx; \\
x = \ln 2 \Rightarrow u &= \sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - (\frac{1}{2})}{2} = \frac{3}{4}, x = \ln 4 \Rightarrow u = \sinh(\ln 4) = \frac{e^{\ln 4} - e^{-\ln 4}}{2} = \frac{4 - (\frac{1}{4})}{2} = \frac{15}{8} \\
&= [\ln |u|]_{3/4}^{15/8} = \ln \left| \frac{15}{8} \right| - \ln \left| \frac{3}{4} \right| = \ln \left| \frac{15}{8} \cdot \frac{4}{3} \right| = \ln \frac{5}{2}
\end{aligned}$$

$$\begin{aligned}
52. \quad \int_0^{\ln 2} \tanh 2x dx &= \int_0^{\ln 2} \frac{\sinh 2x}{\cosh 2x} dx = \frac{1}{2} \int_1^{17/8} \frac{1}{u} du \text{ where } u = \cosh 2x, du = 2 \sinh(2x) dx, \\
x = 0 \Rightarrow u &= \cosh 0 = 1, x = \ln 2 \Rightarrow u = \cosh(2 \ln 2) = \cosh(\ln 4) = \frac{e^{\ln 4} + e^{-\ln 4}}{2} = \frac{4 + (\frac{1}{4})}{2} = \frac{17}{8} \\
&= \frac{1}{2} [\ln |u|]_1^{17/8} = \frac{1}{2} \left[\ln \left(\frac{17}{8} \right) - \ln 1 \right] = \frac{1}{2} \ln \frac{17}{8}
\end{aligned}$$

$$\begin{aligned}
53. \quad \int_{-\ln 4}^{-\ln 2} 2e^\theta \cosh \theta d\theta &= \int_{-\ln 4}^{-\ln 2} 2e^\theta \left(\frac{e^\theta + e^{-\theta}}{2} \right) d\theta = \int_{-\ln 4}^{-\ln 2} (e^{2\theta} + 1) d\theta = \left[\frac{e^{2\theta}}{2} + \theta \right]_{-\ln 4}^{-\ln 2} \\
&= \left(\frac{e^{-2 \ln 2}}{2} - \ln 2 \right) - \left(\frac{e^{-2 \ln 4}}{2} - \ln 4 \right) = \left(\frac{1}{8} - \ln 2 \right) - \left(\frac{1}{32} - \ln 4 \right) = \frac{3}{32} - \ln 2 + 2 \ln 2 = \frac{3}{32} + \ln 2
\end{aligned}$$

54. $\int_0^{\ln 2} 4e^{-\theta} \sinh \theta d\theta = \int_0^{\ln 2} 4e^{-\theta} \left(\frac{e^{\theta} - e^{-\theta}}{2} \right) d\theta = 2 \int_0^{\ln 2} (1 - e^{-2\theta}) d\theta = 2 \left[\theta + \frac{e^{-2\theta}}{2} \right]_0^{\ln 2}$
 $= 2 \left[\left(\ln 2 + \frac{e^{-2 \ln 2}}{2} \right) - \left(0 + \frac{e^0}{2} \right) \right] = 2 \left(\ln 2 + \frac{1}{8} - \frac{1}{2} \right) = 2 \ln 2 + \frac{1}{4} - 1 = \ln 4 - \frac{3}{4}$
55. $\int_{-\pi/4}^{\pi/4} \cosh(\tan \theta) \sec^2 \theta d\theta = \int_{-1}^1 \cosh u du$ where $u = \tan \theta$, $du = \sec^2 \theta d\theta$, $x = -\frac{\pi}{4} \Rightarrow u = -1$, $x = \frac{\pi}{4} \Rightarrow u = 1$,
 $= [\sinh u]_{-1}^1 = \sinh(1) - \sinh(-1) = \left(\frac{e^1 - e^{-1}}{2} \right) - \left(\frac{e^{-1} - e^1}{2} \right) = \frac{e - e^{-1} - e^{-1} + e}{2} = e - e^{-1}$
56. $\int_0^{\pi/2} 2 \sinh(\sin \theta) \cos \theta d\theta = 2 \int_0^1 \sinh u du$ where $u = \sin \theta$, $du = \cos \theta d\theta$, $x = 0 \Rightarrow u = 0$, $x = \frac{\pi}{2} \Rightarrow u = 1$
 $= 2 [\cosh u]_0^1 = 2(\cosh 1 - \cosh 0) = 2 \left(\frac{e + e^{-1}}{2} - 1 \right) = e + e^{-1} - 2$
57. $\int_1^2 \frac{\cosh(\ln t)}{t} dt = \int_0^{\ln 2} \cosh u du$ where $u = \ln t$, $du = \frac{1}{t} dt$, $x = 1 \Rightarrow u = 0$, $x = 2 \Rightarrow u = \ln 2$
 $= [\sinh u]_0^{\ln 2} = \sinh(\ln 2) - \sinh(0) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} - 0 = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$
58. $\int_1^4 \frac{8 \cosh \sqrt{x}}{\sqrt{x}} dx = 16 \int_1^2 \cosh u du$ where $u = \sqrt{x} = x^{1/2}$, $du = \frac{1}{2} x^{-1/2} dx = \frac{dx}{2\sqrt{x}}$, $x = 1 \Rightarrow u = 1$, $x = 4 \Rightarrow u = 2$
 $= 16 [\sinh u]_1^2 = 16(\sinh 2 - \sinh 1) = 16 \left[\left(\frac{e^2 - e^{-2}}{2} \right) - \left(\frac{e - e^{-1}}{2} \right) \right] = 8(e^2 - e^{-2} - e + e^{-1})$
59. $\int_{-\ln 2}^0 \cosh^2 \left(\frac{x}{2} \right) dx = \int_{-\ln 2}^0 \frac{\cosh x + 1}{2} dx = \frac{1}{2} \int_{-\ln 2}^0 (\cosh x + 1) dx = \frac{1}{2} [\sinh x + x]_{-\ln 2}^0$
 $= \frac{1}{2} [(\sinh 0 + 0) - (\sinh(-\ln 2) - \ln 2)] = \frac{1}{2} \left[(0 + 0) - \left(\frac{e^{-\ln 2} - e^{\ln 2}}{2} - \ln 2 \right) \right] = \frac{1}{2} \left[-\frac{\left(\frac{1}{2} \right) - 2}{2} + \ln 2 \right] = \frac{1}{2} \left(1 - \frac{1}{4} + \ln 2 \right)$
 $= \frac{3}{8} + \frac{1}{2} \ln 2 = \frac{3}{8} + \ln \sqrt{2}$
60. $\int_0^{\ln 10} 4 \sinh^2 \left(\frac{x}{2} \right) dx = \int_0^{\ln 10} 4 \left(\frac{\cosh x - 1}{2} \right) dx = 2 \int_0^{\ln 10} (\cosh x - 1) dx = 2 [\sinh x - x]_0^{\ln 10}$
 $= 2 [(\sinh(\ln 10) - \ln 10) - (\sinh 0 - 0)] = e^{\ln 10} - e^{-\ln 10} - 2 \ln 10 = 10 - \frac{1}{10} - 2 \ln 10 = 9.9 - 2 \ln 10$
61. $\sinh^{-1} \left(\frac{-5}{12} \right) = \ln \left(-\frac{5}{12} + \sqrt{\frac{25}{144} + 1} \right) = \ln \left(\frac{2}{3} \right)$
62. $\cosh^{-1} \left(\frac{5}{3} \right) = \ln \left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1} \right) = \ln 3$
63. $\tanh^{-1} \left(-\frac{1}{2} \right) = \frac{1}{2} \ln \left(\frac{1 - (1/2)}{1 + (1/2)} \right) = -\frac{\ln 3}{3}$
64. $\coth^{-1} \left(\frac{5}{4} \right) = \frac{1}{2} \ln \left(\frac{(9/4)}{(1/4)} \right) = \frac{1}{2} \ln 9 = \ln 3$
65. $\operatorname{sech}^{-1} \left(\frac{3}{5} \right) = \ln \left(\frac{1 + \sqrt{1 - (9/25)}}{(3/5)} \right) = \ln 3$
66. $\operatorname{csch}^{-1} \left(-\frac{1}{\sqrt{3}} \right) = \ln \left(-\sqrt{3} + \frac{\sqrt{4/3}}{(1/\sqrt{3})} \right) = \ln(-\sqrt{3} + 2)$
67. (a) $\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}} = \left[\sinh^{-1} \frac{x}{2} \right]_0^{2\sqrt{3}} = \sinh^{-1} \sqrt{3} - \sinh^{-1} 0 = \sinh^{-1} \sqrt{3}$
 (b) $\sinh^{-1} \sqrt{3} = \ln(\sqrt{3} + \sqrt{3+1}) = \ln(\sqrt{3} + 2)$

68. (a) $\int_0^{1/3} \frac{6dx}{\sqrt{1+9x^2}} = 2 \int_0^1 \frac{dx}{\sqrt{a^2+u^2}}$, where $u = 3x, du = 3 dx, a = 1$

$$= \left[2 \sinh^{-1} u \right]_0^1 = 2 \left(\sinh^{-1} 1 - \sinh^{-1} 0 \right) = 2 \sinh^{-1} 1$$

 (b) $2 \sinh^{-1} 1 = 2 \ln \left(1 + \sqrt{1^2 + 1} \right) = 2 \ln \left(1 + \sqrt{2} \right)$
69. (a) $\int_{5/4}^2 \frac{1}{1-x^2} dx = \left[\coth^{-1} x \right]_{5/4}^2 = \coth^{-1} 2 - \coth^{-1} \frac{5}{4}$
 (b) $\coth^{-1} 2 - \coth^{-1} \frac{5}{4} = \frac{1}{2} \left[\ln 3 - \ln \left(\frac{9/4}{1/4} \right) \right] = \frac{1}{2} \ln \frac{1}{3}$
70. (a) $\int_0^{1/2} \frac{1}{1-x^2} dx = \left[\tanh^{-1} x \right]_0^{1/2} = \tanh^{-1} \frac{1}{2} - \tanh^{-1} 0 = \tanh^{-1} \frac{1}{2}$
 (b) $\tanh^{-1} \frac{1}{2} = \frac{1}{2} \ln \left(\frac{1+(1/2)}{1-(1/2)} \right) = \frac{1}{2} \ln 3$
71. (a) $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}} = \int_{4/5}^{12/13} \frac{du}{u\sqrt{a^2-u^2}}, u = 4x, du = 4 dx, a = 1$

$$= \left[-\operatorname{sech}^{-1} u \right]_{4/5}^{12/13} = -\operatorname{sech}^{-1} \frac{12}{13} + \operatorname{sech}^{-1} \frac{4}{5}$$

 (b) $-\operatorname{sech}^{-1} \frac{12}{13} + \operatorname{sech}^{-1} \frac{4}{5} = -\ln \left(\frac{1+\sqrt{1-(12/13)^2}}{(12/13)} \right) + \ln \left(\frac{1+\sqrt{1-(4/5)^2}}{(4/5)} \right) = -\ln \left(\frac{13+\sqrt{169-144}}{12} \right) + \ln \left(\frac{5+\sqrt{25-16}}{4} \right)$

$$= \ln \left(\frac{5+3}{4} \right) - \ln \left(\frac{13+5}{12} \right) = \ln 2 - \ln \frac{3}{2} = \ln \left(2 \cdot \frac{2}{3} \right) = \ln \frac{4}{3}$$
72. (a) $\int_1^2 \frac{dx}{x\sqrt{4+x^2}} = \left[-\frac{1}{2} \operatorname{csch}^{-1} \left| \frac{x}{2} \right| \right]_1^2 = -\frac{1}{2} \left(\operatorname{csch}^{-1} 1 - \operatorname{csch}^{-1} \frac{1}{2} \right) = \frac{1}{2} \left(\operatorname{csch}^{-1} \frac{1}{2} - \operatorname{csch}^{-1} 1 \right)$
 (b) $\frac{1}{2} \left(\operatorname{csch}^{-1} \frac{1}{2} - \operatorname{csch}^{-1} 1 \right) = \frac{1}{2} \left[\ln \left(2 + \frac{\sqrt{5/4}}{(1/2)} \right) - \ln \left(1 + \sqrt{2} \right) \right] = \frac{1}{2} \ln \left(\frac{2+\sqrt{5}}{1+\sqrt{2}} \right)$
73. (a) $\int_0^\pi \frac{\cos x}{\sqrt{1+\sin^2 x}} dx = \int_0^0 \frac{1}{\sqrt{1+u^2}} du$ where $u = \sin x, du = \cos x dx$;

$$= \left[\sinh^{-1} u \right]_0^0 = \sinh^{-1} 0 - \sinh^{-1} 0 = 0$$

 (b) $\sinh^{-1} 0 - \sinh^{-1} 0 = \ln \left(0 + \sqrt{0+1} \right) - \ln \left(0 + \sqrt{0+1} \right) = 0$
74. (a) $\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}} = \int_0^1 \frac{du}{\sqrt{a^2+u^2}},$ where $u = \ln x, du = \frac{1}{x} dx, a = 1$

$$= \left[\sinh^{-1} u \right]_0^1 = \sinh^{-1} 1 - \sinh^{-1} 0 = \sinh^{-1} 1$$

 (b) $\sinh^{-1} 1 - \sinh^{-1} 0 = \ln \left(1 + \sqrt{1^2 + 1} \right) - \ln \left(0 + \sqrt{0^2 + 1} \right) = \ln \left(1 + \sqrt{2} \right)$
75. Let $E(x) = \frac{f(x)+f(-x)}{2}$ and $O(x) = \frac{f(x)-f(-x)}{2}$. Then $E(x) + O(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = \frac{2f(x)}{2} = f(x)$.
 Also, $E(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(x)+f(-x)}{2} = E(x) \Rightarrow E(x)$ is even, and $O(-x) = \frac{f(-x)-f(-(-x))}{2}$

$= -\frac{f(x)-f(-x)}{2} = -O(x) \Rightarrow O(x)$ is odd. Consequently, $f(x)$ can be written as a sum of an even and an odd function. $f(x) = \frac{f(x)+f(-x)}{2}$ because $\frac{f(x)-f(-x)}{2} = 0$ if f is even, and $f(x) = \frac{f(x)-f(-x)}{2}$ because $\frac{f(x)+f(-x)}{2} = 0$ if f is odd. Thus, if f is even $f(x) = \frac{2f(x)}{2} + 0$ and if f is odd, $f(x) = 0 + \frac{2f(x)}{2}$

76. $y = \sinh^{-1} x \Rightarrow x = \sinh y \Rightarrow x = \frac{e^y - e^{-y}}{2} \Rightarrow 2x = e^y - \frac{1}{e^y} \Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow e^{2y} - 2xe^y - 1 = 0$
 $\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \Rightarrow e^y = x + \sqrt{x^2 + 1} \Rightarrow \sinh^{-1} x = y = \ln(x + \sqrt{x^2 + 1})$. Since $e^y > 0$, we cannot choose $e^y = x - \sqrt{x^2 + 1}$ because $x - \sqrt{x^2 + 1} < 0$.

77. (a) $v = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}} t\right) \Rightarrow \frac{dv}{dt} = \sqrt{\frac{mg}{k}} \left[\operatorname{sech}^2\left(\sqrt{\frac{gk}{m}} t\right) \right] \left(\sqrt{\frac{gk}{m}}\right) = g \operatorname{sech}^2\left(\sqrt{\frac{gk}{m}} t\right)$. Thus
 $m \frac{dv}{dt} = mg \operatorname{sech}^2\left(\sqrt{\frac{gk}{m}} t\right) = mg \left(1 - \tanh^2\left(\sqrt{\frac{gk}{m}} t\right)\right) = mg - kv^2$. Also, since $\tanh x = 0$ when $x = 0$, $v = 0$ when $t = 0$.

(b) $\lim_{t \rightarrow \infty} v = \lim_{t \rightarrow \infty} \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}} t\right) = \sqrt{\frac{mg}{k}} \lim_{t \rightarrow \infty} \tanh\left(\sqrt{\frac{gk}{m}} t\right) = \sqrt{\frac{mg}{k}} (1) = \sqrt{\frac{mg}{k}}$

(c) $\sqrt{\frac{735}{0.235}} = \sqrt{\frac{735,000}{235}} \approx 55.93$ s

78. (a) $s(t) = a \cos kt + b \sin kt \Rightarrow \frac{ds}{dt} = -ak \sin kt + bk \cos kt \Rightarrow \frac{d^2s}{dt^2} = -ak^2 \cos kt - bk^2 \sin kt$
 $= -k^2(a \cos kt + b \sin kt) = -k^2 s(t) \Rightarrow$ acceleration is proportional to s . The negative constant $-k^2$ implies that the acceleration is directed toward the origin.

(b) $s(t) = a \cosh kt + b \sinh kt \Rightarrow \frac{ds}{dt} = ak \sinh kt + bk \cosh kt \Rightarrow \frac{d^2s}{dt^2} = ak^2 \cosh kt + bk^2 \sinh kt$
 $= k^2(a \cosh kt + b \sinh kt) = k^2 s(t) \Rightarrow$ acceleration is proportional to s . The positive constant k^2 implies that the acceleration is directed away from the origin.

79. $V = \pi \int_0^2 (\cosh^2 x - \sinh^2 x) dx = \pi \int_0^2 1 dx = 2\pi$

80. $V = 2\pi \int_0^{\ln \sqrt{3}} \operatorname{sech}^2 x dx = 2\pi [\tanh x]_0^{\ln \sqrt{3}} = 2\pi \left[\frac{\sqrt{3} - (1/\sqrt{3})}{\sqrt{3} + (1/\sqrt{3})} \right] = \pi$

81. $y = \frac{1}{2} \cosh 2x \Rightarrow y' = \sinh 2x \Rightarrow L = \int_0^{\ln \sqrt{5}} \sqrt{1 + (\sinh 2x)^2} dx = \int_0^{\ln \sqrt{5}} \cosh 2x dx = \left[\frac{1}{2} \sinh 2x \right]_0^{\ln \sqrt{5}}$
 $= \left[\frac{1}{2} \left(\frac{e^{2x} - e^{-2x}}{2} \right) \right]_0^{\ln \sqrt{5}} = \frac{1}{4} \left(5 - \frac{1}{5} \right) = \frac{6}{5}$

82. (a) $\lim_{x \rightarrow \infty} \tanh x = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow \infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim_{x \rightarrow \infty} \frac{\left(\frac{e^x - \frac{1}{e^x}}{e^x} \right) \cdot \frac{1}{e^x}}{\left(\frac{e^x + \frac{1}{e^x}}{e^x} \right) \cdot \frac{1}{e^x}} = \lim_{x \rightarrow \infty} \frac{1 - \frac{1}{e^{2x}}}{1 + \frac{1}{e^{2x}}} = \frac{1-0}{1+0} = 1$

(b) $\lim_{x \rightarrow -\infty} \tanh x = \lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow -\infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim_{x \rightarrow -\infty} \frac{\left(\frac{e^x - \frac{1}{e^x}}{e^x} \right) \cdot e^x}{\left(\frac{e^x + \frac{1}{e^x}}{e^x} \right) \cdot e^x} = \lim_{x \rightarrow -\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0-1}{0+1} = -1$

$$(c) \quad \lim_{x \rightarrow \infty} \sinh x = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{2} = \lim_{x \rightarrow \infty} \frac{e^x - \frac{1}{e^x}}{2} = \lim_{x \rightarrow \infty} \left(\frac{e^x}{2} - \frac{1}{2e^x} \right) = \infty - 0 = \infty$$

$$(d) \quad \lim_{x \rightarrow -\infty} \sinh x = \lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{2} = \lim_{x \rightarrow -\infty} \left(\frac{e^x}{2} - \frac{e^{-x}}{2} \right) = 0 - \infty = -\infty$$

$$(e) \quad \lim_{x \rightarrow \infty} \operatorname{sech} x = \lim_{x \rightarrow \infty} \frac{2}{e^x + e^{-x}} = \lim_{x \rightarrow \infty} \frac{2}{e^x + \frac{1}{e^x}} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim_{x \rightarrow \infty} \frac{\frac{2}{e^x}}{1 + \frac{1}{e^{2x}}} = \frac{0}{1+0} = 0$$

$$(f) \quad \lim_{x \rightarrow \infty} \coth x = \lim_{x \rightarrow \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x \rightarrow \infty} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim_{x \rightarrow \infty} \frac{\frac{1 + \frac{1}{e^{2x}}}{e^x}}{\frac{1 - \frac{1}{e^{2x}}}{e^x}} = \frac{1+0}{1-0} = 1$$

$$(g) \quad \lim_{x \rightarrow 0^+} \coth x = \lim_{x \rightarrow 0^+} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x \rightarrow 0^+} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} = \lim_{x \rightarrow 0^+} \frac{e^{2x} + 1}{e^{2x} - 1} = +\infty$$

$$(h) \quad \lim_{x \rightarrow 0^-} \coth x = \lim_{x \rightarrow 0^-} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x \rightarrow 0^-} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} = \lim_{x \rightarrow 0^-} \frac{e^{2x} + 1}{e^{2x} - 1} = -\infty$$

$$(i) \quad \lim_{x \rightarrow -\infty} \operatorname{csch} x = \lim_{x \rightarrow -\infty} \frac{2}{e^x - e^{-x}} = \lim_{x \rightarrow -\infty} \frac{2}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} = \lim_{x \rightarrow -\infty} \frac{2e^x}{e^{2x} - 1} = \frac{0}{0-1} = 0$$

$$83. (a) \quad y = \frac{H}{w} \cosh\left(\frac{w}{H}x\right) \Rightarrow \tan \phi = \frac{dy}{dx} = \left(\frac{H}{w}\right) \left[\frac{w}{H} \sinh\left(\frac{w}{H}x\right) \right] = \sinh\left(\frac{w}{H}x\right)$$

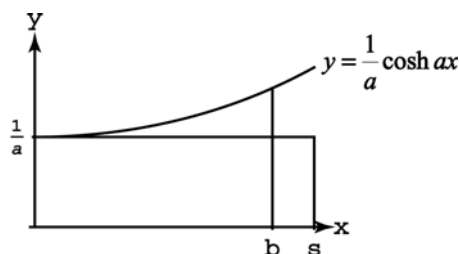
$$(b) \quad \text{The tension at } P \text{ is given by } T \cos \phi = H \Rightarrow T = H \sec \phi = H \sqrt{1 + \tan^2 \phi} = H \sqrt{1 + \left(\sinh \frac{w}{H}x\right)^2} \\ = H \cosh\left(\frac{w}{H}x\right) = w \left(\frac{H}{w}\right) \cosh\left(\frac{w}{H}x\right) = wy$$

$$84. \quad s = \frac{1}{a} \sinh ax \Rightarrow \sinh ax = as \Rightarrow ax = \sinh^{-1} as \Rightarrow x = \frac{1}{a} \sinh^{-1} as; \quad y = \frac{1}{a} \cosh ax = \frac{1}{a} \sqrt{\cosh^2 ax} \\ = \frac{1}{a} \sqrt{\sinh^2 ax + 1} = \frac{1}{a} \sqrt{a^2 s^2 + 1} = \sqrt{s^2 + \frac{1}{a^2}}$$

$$85. \quad \text{To find the length of the curve: } y = \frac{1}{a} \cosh ax \Rightarrow y' = \sinh ax \Rightarrow L = \int_0^b \sqrt{1 + (\sinh ax)^2} dx \Rightarrow L = \int_0^b \cosh ax dx$$

$$= \left[\frac{1}{a} \sinh ax \right]_0^b = \frac{1}{a} \sinh ab. \quad \text{The area under the curve is } A = \int_0^b \frac{1}{a} \cosh ax dx = \left[\frac{1}{a^2} \sinh ax \right]_0^b = \frac{1}{a^2} \sinh ab$$

$= \left(\frac{1}{a}\right) \left(\frac{1}{a} \sinh ab\right)$ which is the area of the rectangle of height $\frac{1}{a}$ and length L as claimed, and which is illustrated below.



86. (a) Let the point located at $(\cosh u, 0)$ be called T . Then $A(u)$ = area of the triangle $\triangle OTP$ minus the area under the curve $y = \sqrt{x^2 - 1}$ from A to $T \Rightarrow A(u) = \frac{1}{2} \cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1} \, dx$.
- (b) $A(u) = \frac{1}{2} \cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1} \, dx \Rightarrow A'(u) = \frac{1}{2} (\cosh^2 u + \sinh^2 u) - \left(\sqrt{\cosh^2 u - 1} \right) (\sinh u)$
 $= \frac{1}{2} \cosh^2 u + \frac{1}{2} \sinh^2 u - \sinh^2 u = \frac{1}{2} (\cosh^2 u - \sinh^2 u) = \left(\frac{1}{2} \right) (1) = \frac{1}{2}$
- (c) $A'(u) = \frac{1}{2} \Rightarrow A(u) = \frac{u}{2} + C$, and from part (a) we have $A(0) = 0 \Rightarrow C = 0 \Rightarrow A(u) = \frac{u}{2} \Rightarrow u = 2A$

7.8 RELATIVE RATES OF GROWTH

- (a) slower, $\lim_{x \rightarrow \infty} \frac{x+3}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$

(b) slower, $\lim_{x \rightarrow \infty} \frac{x^3 + \sin^2 x}{e^x} = \lim_{x \rightarrow \infty} \frac{3x^2 + 2 \sin x \cos x}{e^x} = \lim_{x \rightarrow \infty} \frac{6x + 2 \cos 2x}{e^x} = \lim_{x \rightarrow \infty} \frac{6 - 4 \sin 2x}{e^x} = 0$ by the Sandwich Theorem because $\frac{2}{e^x} \leq \frac{6 - 4 \sin 2x}{e^x} \leq \frac{10}{e^x}$ for all reals, and $\lim_{x \rightarrow \infty} \frac{2}{e^x} = 0 = \lim_{x \rightarrow \infty} \frac{10}{e^x}$

(c) slower, $\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{e^x} = \lim_{x \rightarrow \infty} \frac{x^{1/2}}{e^x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{2}\right)x^{-1/2}}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}e^x} = 0$

(d) faster, $\lim_{x \rightarrow \infty} \frac{4^x}{e^x} = \lim_{x \rightarrow \infty} \left(\frac{4}{e}\right)^x = \infty$ since $\frac{4}{e} > 1$

(e) slower, $\lim_{x \rightarrow \infty} \frac{\left(\frac{3}{2}\right)^x}{e^x} = \lim_{x \rightarrow \infty} \left(\frac{3}{2e}\right)^x = 0$ since $\frac{3}{2e} < 1$

(f) slower, $\lim_{x \rightarrow \infty} \frac{e^{x/2}}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^{x/2}} = 0$

(g) same, $\lim_{x \rightarrow \infty} \frac{\left(\frac{e^x}{2}\right)}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{2} = \frac{1}{2}$

(h) slower, $\lim_{x \rightarrow \infty} \frac{\log_{10} x}{e^x} = \lim_{x \rightarrow \infty} \frac{\ln x}{(\ln 10)e^x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{(\ln 10)e^x} = \lim_{x \rightarrow \infty} \frac{1}{(\ln 10)xe^x} = 0$
- (a) slower, $\lim_{x \rightarrow \infty} \frac{10x^4 + 30x + 1}{e^x} = \lim_{x \rightarrow \infty} \frac{40x^3 + 30}{e^x} = \lim_{x \rightarrow \infty} \frac{120x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{240x}{e^x} = \lim_{x \rightarrow \infty} \frac{240}{e^x} = 0$

(b) slower, $\lim_{x \rightarrow \infty} \frac{x \ln x - x}{e^x} = \lim_{x \rightarrow \infty} \frac{x(\ln x - 1)}{e^x} = \lim_{x \rightarrow \infty} \frac{\ln x - 1 + x\left(\frac{1}{x}\right)}{e^x} = \lim_{x \rightarrow \infty} \frac{\ln x - 1 + 1}{e^x} = \lim_{x \rightarrow \infty} \frac{\ln x}{e^x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{xe^x} = 0$

(c) slower, $\lim_{x \rightarrow \infty} \frac{\sqrt{1+x^4}}{e^x} = \sqrt{\lim_{x \rightarrow \infty} \frac{1+x^4}{e^{2x}}} = \sqrt{\lim_{x \rightarrow \infty} \frac{4x^3}{2e^{2x}}} = \sqrt{\lim_{x \rightarrow \infty} \frac{12x^2}{4e^{2x}}} = \sqrt{\lim_{x \rightarrow \infty} \frac{24x}{8e^{2x}}} = \sqrt{\lim_{x \rightarrow \infty} \frac{24}{16e^{2x}}} = \sqrt{0} = 0$

(d) slower, $\lim_{x \rightarrow \infty} \frac{\left(\frac{5}{2}\right)^x}{e^x} = \lim_{x \rightarrow \infty} \left(\frac{5}{2e}\right)^x = 0$ since $\frac{5}{2e} < 1$

(e) slower, $\lim_{x \rightarrow \infty} \frac{e^{-x}}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^{2x}} = 0$

(f) faster, $\lim_{x \rightarrow \infty} \frac{xe^x}{e^x} = \lim_{x \rightarrow \infty} x = \infty$

(g) slower, since for all reals we have $-1 \leq \cos x \leq 1 \Rightarrow e^{-1} \leq e^{\cos x} \leq e^1 \Rightarrow \frac{e^{-1}}{e^x} \leq \frac{e^{\cos x}}{e^x} \leq \frac{e^1}{e^x}$ and also $\lim_{x \rightarrow \infty} \frac{e^{-1}}{e^x} = 0 = \lim_{x \rightarrow \infty} \frac{e^1}{e^x}$, so by the Sandwich Theorem we conclude that $\lim_{x \rightarrow \infty} \frac{e^{\cos x}}{e^x} = 0$

- (h) same, $\lim_{x \rightarrow \infty} \frac{e^{x-1}}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^{(x-x+1)}} = \lim_{x \rightarrow \infty} \frac{1}{e} = \frac{1}{e}$
3. (a) same, $\lim_{x \rightarrow \infty} \frac{x^2+4x}{x^2} = \lim_{x \rightarrow \infty} \frac{2x+4}{2x} = \lim_{x \rightarrow \infty} \frac{2}{2} = 1$
- (b) slower, $\lim_{x \rightarrow \infty} \frac{x^5-x^2}{x^2} = \lim_{x \rightarrow \infty} (x^3-1) = \infty$
- (c) same, $\lim_{x \rightarrow \infty} \frac{\sqrt{x^4+x^3}}{x^2} = \sqrt{\lim_{x \rightarrow \infty} \frac{x^4+x^3}{x^4}} = \sqrt{\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)} = \sqrt{1} = 1$
- (d) same, $\lim_{x \rightarrow \infty} \frac{(x+3)^2}{x^2} = \lim_{x \rightarrow \infty} \frac{2(x+3)}{2x} = \lim_{x \rightarrow \infty} \frac{2}{2} = 1$
- (e) slower, $\lim_{x \rightarrow \infty} \frac{x \ln x}{x^2} = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$
- (f) slower, $\lim_{x \rightarrow \infty} \frac{2^x}{x^2} = \lim_{x \rightarrow \infty} \frac{(\ln 2)2^x}{2x} = \lim_{x \rightarrow \infty} \frac{(\ln 2)^2 2^x}{2} = \infty$
- (g) slower, $\lim_{x \rightarrow \infty} \frac{x^3 e^{-x}}{x^2} = \lim_{x \rightarrow \infty} \frac{x}{e^x} \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$
- (h) same, $\lim_{x \rightarrow \infty} \frac{8x^2}{x^2} = \lim_{x \rightarrow \infty} 8 = 8$
4. (a) same, $\lim_{x \rightarrow \infty} \frac{x^2+\sqrt{x}}{x^2} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x^{3/2}}\right) = 1$
- (b) same, $\lim_{x \rightarrow \infty} \frac{10x^2}{x^2} = \lim_{x \rightarrow \infty} 10 = 10$
- (c) slower, $\lim_{x \rightarrow \infty} \frac{x^2 e^{-x}}{x^2} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$
- (d) slower, $\lim_{x \rightarrow \infty} \frac{\log_{10} x^2}{x^2} = \lim_{x \rightarrow \infty} \frac{\left(\frac{\ln x^2}{\ln 10}\right)}{x^2} = \frac{1}{\ln 10} \lim_{x \rightarrow \infty} \frac{2 \ln x}{x^2} = \frac{2}{\ln 10} \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{2x} = \frac{1}{\ln 10} \lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$
- (e) faster, $\lim_{x \rightarrow \infty} \frac{x^3-x^2}{x^2} = \lim_{x \rightarrow \infty} (x-1) = \infty$
- (f) slower, $\lim_{x \rightarrow \infty} \frac{\left(\frac{1}{10}\right)^x}{x^2} = \lim_{x \rightarrow \infty} \frac{1}{10^x x^2} = 0$
- (g) faster, $\lim_{x \rightarrow \infty} \frac{(1.1)^x}{x^2} = \lim_{x \rightarrow \infty} \frac{(\ln 1.1)(1.1)^x}{2x} = \lim_{x \rightarrow \infty} \frac{(\ln 1.1)^2 (1.1)^x}{2} = \infty$
- (h) same, $\lim_{x \rightarrow \infty} \frac{x^2+100x}{x^2} = \lim_{x \rightarrow \infty} \left(1 + \frac{100}{x}\right) = 1$
5. (a) same, $\lim_{x \rightarrow \infty} \frac{\log_3 x}{\ln x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{\ln x}{\ln 3}\right)}{\ln x} = \lim_{x \rightarrow \infty} \frac{1}{\ln 3} = \frac{1}{\ln 3}$
- (b) same, $\lim_{x \rightarrow \infty} \frac{\ln 2x}{\ln x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{2x}\right)}{\left(\frac{1}{x}\right)} = 1$
- (c) same, $\lim_{x \rightarrow \infty} \frac{\ln \sqrt{x}}{\ln x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{2}\right) \ln x}{\ln x} = \lim_{x \rightarrow \infty} \frac{1}{2} = \frac{1}{2}$
- (d) faster, $\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln x} = \lim_{x \rightarrow \infty} \frac{x^{1/2}}{\ln x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{2}\right)x^{-1/2}}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{x}{2\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{2} = \infty$

- (e) faster, $\lim_{x \rightarrow \infty} \frac{x}{\ln x} = \lim_{x \rightarrow \infty} \frac{1}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} x = \infty$
- (f) same, $\lim_{x \rightarrow \infty} \frac{5 \ln x}{\ln x} = \lim_{x \rightarrow \infty} 5 = 5$
- (g) slower, $\lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{\ln x} = \lim_{x \rightarrow \infty} \frac{1}{x \ln x} = 0$
- (h) faster, $\lim_{x \rightarrow \infty} \frac{e^x}{\ln x} = \lim_{x \rightarrow \infty} \frac{e^x}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} x e^x = \infty$
6. (a) same, $\lim_{x \rightarrow \infty} \frac{\log_2 x^2}{\ln x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{\ln x^2}{\ln 2}\right)}{\ln x} = \frac{1}{\ln 2} \lim_{x \rightarrow \infty} \frac{\ln x^2}{\ln x} = \frac{1}{\ln 2} \lim_{x \rightarrow \infty} \frac{2 \ln x}{\ln x} = \frac{1}{\ln 2} \lim_{x \rightarrow \infty} 2 = \frac{2}{\ln 2}$
- (b) same, $\lim_{x \rightarrow \infty} \frac{\log_{10} 10x}{\ln x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{\ln 10x}{\ln 10}\right)}{\ln x} = \frac{1}{\ln 10} \lim_{x \rightarrow \infty} \frac{\ln 10x}{\ln x} = \frac{1}{\ln 10} \lim_{x \rightarrow \infty} \frac{\left(\frac{10}{x}\right)}{\left(\frac{1}{x}\right)} = \frac{1}{\ln 10} \lim_{x \rightarrow \infty} 1 = \frac{1}{\ln 10}$
- (c) slower, $\lim_{x \rightarrow \infty} \frac{\left(\frac{1}{\sqrt{x}}\right)}{\ln x} = \lim_{x \rightarrow \infty} \frac{1}{(\sqrt{x})(\ln x)} = 0$
- (d) slower, $\lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x^2}\right)}{\ln x} = \lim_{x \rightarrow \infty} \frac{1}{x^2 \ln x} = 0$
- (e) faster, $\lim_{x \rightarrow \infty} \frac{x-2 \ln x}{\ln x} = \lim_{x \rightarrow \infty} \left(\frac{x}{\ln x} - 2\right) = \left(\lim_{x \rightarrow \infty} \frac{x}{\ln x}\right) - 2 = \left(\lim_{x \rightarrow \infty} \frac{1}{\left(\frac{1}{x}\right)}\right) - 2 = \left(\lim_{x \rightarrow \infty} x\right) - 2 = \infty$
- (f) slower, $\lim_{x \rightarrow \infty} \frac{e^{-x}}{\ln x} = \lim_{x \rightarrow \infty} \frac{1}{e^x \ln x} = 0$
- (g) slower, $\lim_{x \rightarrow \infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1/\ln x}{\ln x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{1}{\ln x} = 0$
- (h) same, $\lim_{x \rightarrow \infty} \frac{\ln(2x+5)}{\ln x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{2x+5}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{2x}{2x+5} = \lim_{x \rightarrow \infty} \frac{2}{2} = \lim_{x \rightarrow \infty} 1 = 1$
7. $\lim_{x \rightarrow \infty} \frac{e^x}{e^{x/2}} = \lim_{x \rightarrow \infty} e^{x/2} = \infty \Rightarrow e^x$ grows faster than $e^{x/2}$; since for $x > e^e$ we have $\ln x > e$ and
- $$\lim_{x \rightarrow \infty} \frac{(\ln x)^x}{e^x} = \lim_{x \rightarrow \infty} \left(\frac{\ln x}{e}\right)^x = \infty \Rightarrow (\ln x)^x \text{ grows faster than } e^x; \text{ since } x > \ln x \text{ for all } x > 0 \text{ and}$$
- $$\lim_{x \rightarrow \infty} \frac{x^x}{(\ln x)^x} = \lim_{x \rightarrow \infty} \left(\frac{x}{\ln x}\right)^x = \infty \Rightarrow x^x \text{ grows faster than } (\ln x)^x. \text{ Therefore, slowest to fastest are:}$$
- $e^{x/2}, e^x, (\ln x)^x, x^x$ so the order is d, a, c, b
8. $\lim_{x \rightarrow \infty} \frac{(\ln 2)^x}{x^2} = \lim_{x \rightarrow \infty} \frac{(\ln(\ln 2))(\ln 2)^x}{2x} = \lim_{x \rightarrow \infty} \frac{(\ln(\ln 2))^2 (\ln 2)^x}{2} = \frac{(\ln(\ln 2))^2}{2} \lim_{x \rightarrow \infty} (\ln 2)^x = 0 \Rightarrow (\ln 2)^x$ grows slower than
- x^2 ; $\lim_{x \rightarrow \infty} \frac{x^2}{2^x} = \lim_{x \rightarrow \infty} \frac{2x}{(\ln 2)2^x} = \lim_{x \rightarrow \infty} \frac{2}{(\ln 2)^2 2^x} = 0 \Rightarrow x^2$ grows slower than 2^x ; $\lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \left(\frac{2}{e}\right)^x = 0 \Rightarrow 2^x$ grows slower than e^x . Therefore, the slowest to the fastest is: $(\ln 2)^x, x^2, 2^x$ and e^x so the order is c, b, a, d
9. (a) false; $\lim_{x \rightarrow \infty} \frac{x}{x} = 1$
- (b) false; $\lim_{x \rightarrow \infty} \frac{x}{x+5} = \frac{1}{1} = 1$

- (c) true; $x < x+5 \Rightarrow \frac{x}{x+5} < 1$ if $x > 1$ (or sufficiently large)
- (d) true; $x < 2x \Rightarrow \frac{x}{2x} < 1$ if $x > 1$ (or sufficiently large)
- (e) true; $\lim_{x \rightarrow \infty} \frac{e^x}{e^{2x}} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$
- (f) true; $\frac{x+\ln x}{x} = 1 + \frac{\ln x}{x} < 1 + \frac{\sqrt{x}}{x} = 1 + \frac{1}{\sqrt{x}} < 2$ if $x > 1$ (or sufficiently large)
- (g) false; $\lim_{x \rightarrow \infty} \frac{\ln x}{\ln 2x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{2}{2x}\right)} = \lim_{x \rightarrow \infty} 1 = 1$
- (h) true; $\frac{\sqrt{x^2+5}}{x} < \frac{\sqrt{(x+5)^2}}{x} < \frac{x+5}{x} = 1 + \frac{5}{x} < 6$ if $x > 1$ (or sufficiently large)
10. (a) true; $\frac{\left(\frac{1}{x+3}\right)}{\left(\frac{1}{x}\right)} = \frac{x}{x+3} < 1$ if $x > 1$ (or sufficiently large)
- (b) true; $\frac{\left(\frac{1+\frac{1}{x^2}}{x}\right)}{\left(\frac{1}{x}\right)} = 1 + \frac{1}{x} < 2$ if $x > 1$ (or sufficiently large)
- (c) false; $\lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x} - \frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right) = 1$
- (d) true; $2 + \cos x \leq 3 \Rightarrow \frac{2+\cos x}{2} \leq \frac{3}{2}$ if x is sufficiently large
- (e) true; $\frac{e^x+x}{e^x} = 1 + \frac{x}{e^x}$ and $\frac{x}{e^x} \rightarrow 0$ as $x \rightarrow \infty \Rightarrow 1 + \frac{x}{e^x} < 2$ if x is sufficiently large
- (f) true; $\lim_{x \rightarrow \infty} \frac{x \ln x}{x^2} = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$
- (g) true; $\frac{\ln(\ln x)}{\ln x} < \frac{\ln x}{\ln x} = 1$ if x is sufficiently large
- (h) false; $\lim_{x \rightarrow \infty} \frac{\ln x}{\ln(x^2+1)} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{2x}{x^2+1}\right)} = \lim_{x \rightarrow \infty} \frac{x^2+1}{2x^2} = \lim_{x \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{2x^2}\right) = \frac{1}{2}$
11. If $f(x)$ and $g(x)$ grow at the same rate, then $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L \neq 0 \Rightarrow \lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = \frac{1}{L} \neq 0$. Then $\left| \frac{f(x)}{g(x)} - L \right| < 1$ if x is sufficiently large $\Rightarrow L-1 < \frac{f(x)}{g(x)} < L+1 \Rightarrow \frac{f(x)}{g(x)} \leq |L|+1$ if x is sufficiently large $\Rightarrow f = O(g)$. Similarly, $\frac{g(x)}{f(x)} \leq \left| \frac{1}{L} \right| + 1 \Rightarrow g = O(f)$.
12. When the degree of f is less than the degree of g since in that case $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$.
13. When the degree of f is less than or equal to the degree of g since $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$ when the degree of f is smaller than the degree of g , and $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{a}{b}$ (the ratio of the leading coefficients) when the degrees are the same.
14. Polynomials of a greater degree grow at a greater rate than polynomials of a lesser degree. Polynomials of the same degree grow at the same rate.
15. $\lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\ln x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x+1}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{x}{x+1} = \lim_{x \rightarrow \infty} \frac{1}{1} = 1$ and $\lim_{x \rightarrow \infty} \frac{\ln(x+999)}{\ln x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x+999}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{x}{x+999} = 1$

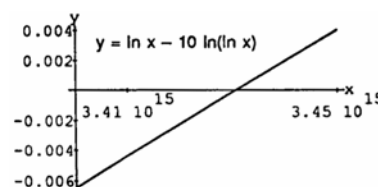
16. $\lim_{x \rightarrow \infty} \frac{\ln(x+a)}{\ln x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x+a}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{x}{x+a} = \lim_{x \rightarrow \infty} \frac{1}{1} = 1$. Therefore, the relative rates are the same.
17. $\lim_{x \rightarrow \infty} \frac{\sqrt{10x+1}}{\sqrt{x}} = \sqrt{\lim_{x \rightarrow \infty} \frac{10x+1}{x}} = \sqrt{10}$ and $\lim_{x \rightarrow \infty} \frac{\sqrt{x+1}}{\sqrt{x}} = \sqrt{\lim_{x \rightarrow \infty} \frac{x+1}{x}} = \sqrt{1} = 1$. Since the growth rate is transitive, we conclude that $\sqrt{10x+1}$ and $\sqrt{x+1}$ have the same growth rate (that of \sqrt{x}).
18. $\lim_{x \rightarrow \infty} \frac{\sqrt{x^4+x}}{x^2} = \sqrt{\lim_{x \rightarrow \infty} \frac{x^4+x}{x^4}} = 1$ and $\lim_{x \rightarrow \infty} \frac{\sqrt{x^4-x^3}}{x^2} = \sqrt{\lim_{x \rightarrow \infty} \frac{x^4-x^3}{x^4}} = 1$. Since the growth rate is transitive, we conclude that $\sqrt{x^4+x}$ and $\sqrt{x^4-x^3}$ have the same growth rate (that of x^2).
19. $\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = \lim_{x \rightarrow \infty} \frac{nx^{n-1}}{e^x} = \dots = \lim_{x \rightarrow \infty} \frac{n!}{e^x} = 0 \Rightarrow x^n = o(e^x)$ for any non-negative integer n
20. If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, then
 $\lim_{x \rightarrow \infty} \frac{p(x)}{e^x} = a_n \lim_{x \rightarrow \infty} \frac{x^n}{e^x} + a_{n-1} \lim_{x \rightarrow \infty} \frac{x^{n-1}}{e^x} + \dots + a_1 \lim_{x \rightarrow \infty} \frac{x}{e^x} + a_0 \lim_{x \rightarrow \infty} \frac{1}{e^x}$ where each limit is zero (from Exercise 19). Therefore, $\lim_{x \rightarrow \infty} \frac{p(x)}{e^x} = 0 \Rightarrow e^x$ grows faster than any polynomial.

21. (a) $\lim_{x \rightarrow \infty} \frac{x^{1/n}}{\ln x} = \lim_{x \rightarrow \infty} \frac{x^{(1-n)/n}}{n\left(\frac{1}{n}\right)} = \left(\frac{1}{n}\right) \lim_{x \rightarrow \infty} x^{1/n} = \infty \Rightarrow \ln x = o(x^{1/n})$ for any positive integer n

(b) $\ln(e^{17,000,000}) = 17,000,000 < \left(e^{17 \times 10^6}\right)^{1/10^6} = e^{17} \approx 24,154,952.75$

(c) $x \approx 3.430631121 \times 10^{15}$

(d) In the interval $[3.41 \times 10^{15}, 3.45 \times 10^{15}]$ we have
 $\ln x = 10 \ln(\ln x)$. The graphs cross at about
 3.4306311×10^{15} .



22. $\lim_{x \rightarrow \infty} \frac{\ln x}{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0} = \frac{\lim_{x \rightarrow \infty} \left(\frac{\ln x}{x^n}\right)}{\lim_{x \rightarrow \infty} \left(a_n + \frac{a_{n-1}}{x} + \dots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)} = \frac{\lim_{x \rightarrow \infty} \left[\frac{1/x}{nx^{n-1}}\right]}{a_n} = \lim_{x \rightarrow \infty} \frac{1}{(a_n)(nx^n)} = 0 \Rightarrow \ln x$ grows slower than any non-constant polynomial ($n \geq 1$)

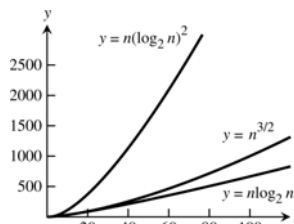
23. (a) $\lim_{n \rightarrow \infty} \frac{n \log_2 n}{n(\log_2 n)^2} = \lim_{n \rightarrow \infty} \frac{1}{\log_2 n} = 0 \Rightarrow n \log_2 n$

grows slower than $n(\log_2 n)^2$;

$$\lim_{n \rightarrow \infty} \frac{n \log_2 n}{n^{3/2}} = \lim_{n \rightarrow \infty} \frac{\left(\frac{\ln n}{\ln 2}\right)}{n^{1/2}} = \frac{1}{\ln 2} \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)^{1/2} = \frac{2}{\ln 2} \lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} = 0 \Rightarrow n \log_2 n$$

grows slower than $n^{3/2}$. Therefore, $n \log_2 n$ grows at the slowest rate \Rightarrow the algorithm that takes $O(n \log_2 n)$ steps is the most efficient in the long run.

(b)



$$24. \quad (a) \quad \lim_{n \rightarrow \infty} \frac{(\log_2 n)^2}{n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{\ln n}{\ln 2}\right)^2}{n} = \lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n(\ln 2)^2} = \lim_{n \rightarrow \infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{(\ln 2)^2} = \frac{2}{(\ln 2)^2} \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \frac{2}{(\ln 2)^2} \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n}\right)}{1} = 0$$

$$\Rightarrow (\log_2 n)^2 \text{ grows slower than } n; \quad \lim_{n \rightarrow \infty} \frac{(\log_2 n)^2}{\sqrt{n} \log_2 n} = \lim_{n \rightarrow \infty} \frac{\log_2 n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\left(\frac{\ln n}{\ln 2}\right)}{n^{1/2}} = \frac{1}{\ln 2} \lim_{n \rightarrow \infty} \frac{\ln n}{n^{1/2}}$$

$$= \frac{1}{\ln 2} \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{2}\right)x^{-1/2}} = \frac{2}{\ln 2} \lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} = 0$$

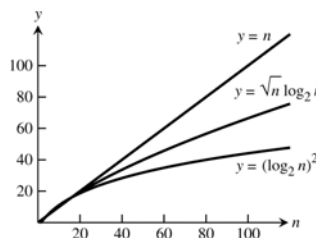
$$\Rightarrow (\log_2 n)^2 \text{ grows slower than } \sqrt{n} \log_2 n.$$

Therefore $(\log_2 n)^2$ grows at the slowest rate

\Rightarrow the algorithm that takes $O((\log_2 n)^2)$ steps

is the most efficient in the long run.

(b)



25. It could take one million steps for a sequential search, but at most 20 steps for a binary search because $2^{19} = 524,288 < 1,000,000 < 1,048,576 = 2^{20}$.

26. It could take 450,000 steps for a sequential search, but at most 19 steps for a binary search because $2^{18} = 262,144 < 450,000 < 524,288 = 2^{19}$.

CHAPTER 7 PRACTICE EXERCISES

$$1. \quad y = 10e^{-x/5} \Rightarrow \frac{dy}{dx} = (10)\left(-\frac{1}{5}\right)e^{-x/5} = -2e^{-x/5} \quad 2. \quad y = \sqrt{2}e^{\sqrt{2}x} \Rightarrow \frac{dy}{dx} = (\sqrt{2})(\sqrt{2})e^{\sqrt{2}x} = 2e^{\sqrt{2}x}$$

$$3. \quad y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} \Rightarrow \frac{dy}{dx} = \frac{1}{4}\left[x(4e^{4x}) + e^{4x}(1)\right] - \frac{1}{16}(4e^{4x}) = xe^{4x} + \frac{1}{4}e^{4x} - \frac{1}{4}e^{4x} = xe^{4x}$$

$$4. \quad y = x^2e^{-2/x} = x^2e^{-2x^{-1}} \Rightarrow \frac{dy}{dx} = x^2\left[(2x^{-2})e^{-2x^{-1}}\right] + e^{-2x^{-1}}(2x) = (2 + 2x)e^{-2x^{-1}} = 2e^{-2/x}(1 + x)$$

$$5. \quad y = \ln(\sin^2 \theta) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sin \theta)(\cos \theta)}{\sin^2 \theta} = \frac{2\cos \theta}{\sin \theta} = 2\cot \theta$$

$$6. \quad y = \ln(\sec^2 \theta) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sec \theta)(\sec \theta \tan \theta)}{\sec^2 \theta} = 2\tan \theta$$

$$7. \quad y = \log_2\left(\frac{x^2}{2}\right) = \frac{\ln\left(\frac{x^2}{2}\right)}{\ln 2} \Rightarrow \frac{dy}{dx} = \frac{1}{\ln 2} \left(\frac{x}{\left(\frac{x^2}{2}\right)}\right) = \frac{2}{(\ln 2)x}$$

$$8. \quad y = \log_5(3x-7) = \frac{\ln(3x-7)}{\ln 5} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\ln 5}\right)\left(\frac{3}{3x-7}\right) = \frac{3}{(\ln 5)(3x-7)}$$

$$9. \quad y = 8^{-t} \Rightarrow \frac{dy}{dt} = 8^{-t}(\ln 8)(-1) = -8^{-t}(\ln 8)$$

10. $y = 9^{2t} \Rightarrow \frac{dy}{dt} = 9^{2t} (\ln 9)(2) = 9^{2t} (2 \ln 9)$
11. $y = 5x^{3.6} \Rightarrow \frac{dy}{dx} = 5(3.6)x^{2.6} = 18x^{2.6}$
12. $y = \sqrt{2}x^{-\sqrt{2}} \Rightarrow \frac{dy}{dx} = (\sqrt{2})(-\sqrt{2})x^{(-\sqrt{2}-1)} = -2x^{(-\sqrt{2}-1)}$
13. $y = (x+2)^{x+2} \Rightarrow \ln y = \ln(x+2)^{x+2} = (x+2)\ln(x+2) \Rightarrow \frac{y'}{y} = (x+2)\left(\frac{1}{x+2}\right) + (1)\ln(x+2)$
 $\Rightarrow \frac{dy}{dx} = (x+2)^{x+2} [\ln(x+2) + 1]$
14. $y = 2(\ln x)^{x/2} \Rightarrow \ln y = \ln [2(\ln x)^{x/2}] = \ln(2) + \left(\frac{x}{2}\right)\ln(\ln x) \Rightarrow \frac{y'}{y} = 0 + \left(\frac{x}{2}\right)\left[\frac{\left(\frac{1}{x}\right)}{\ln x}\right] + (\ln(\ln x))\left(\frac{1}{2}\right)$
 $\Rightarrow y' = \left[\frac{1}{2\ln x} + \left(\frac{1}{2}\right)\ln(\ln x)\right] 2(\ln x)^{x/2} = (\ln x)^{x/2} \left[\ln(\ln x) + \frac{1}{\ln x}\right]$
15. $y = \sin^{-1} \sqrt{1-u^2} = \sin^{-1} (1-u^2)^{1/2} \Rightarrow \frac{dy}{du} = \frac{\frac{1}{2}(1-u^2)^{-1/2}(-2u)}{\sqrt{1-\left[(1-u^2)^{1/2}\right]^2}} = \frac{-u}{\sqrt{1-u^2}\sqrt{1-(1-u^2)}} = \frac{-u}{|u|\sqrt{1-u^2}}$
 $= \frac{-u}{u\sqrt{1-u^2}} = \frac{-1}{\sqrt{1-u^2}}, 0 < u < 1$
16. $y = \sin^{-1} \left(\frac{1}{\sqrt{v}}\right) = \sin^{-1} (v^{-1/2}) \Rightarrow \frac{dy}{dv} = \frac{-\frac{1}{2}v^{-3/2}}{\sqrt{1-(v^{-1/2})^2}} = \frac{-1}{2v^{3/2}\sqrt{1-v^{-1}}} = \frac{-1}{2v^{3/2}\sqrt{\frac{v-1}{v}}} = \frac{-\sqrt{v}}{2v^{3/2}\sqrt{v-1}} = \frac{-1}{2v\sqrt{v-1}}$
17. $y = \ln(\cos^{-1} x) \Rightarrow y' = \frac{\left(\frac{-1}{\sqrt{1-x^2}}\right)}{\cos^{-1} x} = \frac{-1}{\sqrt{1-x^2} \cos^{-1} x}$
18. $y = z \cos^{-1} z - \sqrt{1-z^2} = z \cos^{-1} z - (1-z^2)^{1/2} \Rightarrow \frac{dy}{dz} = \cos^{-1} z - \frac{z}{\sqrt{1-z^2}} - \left(\frac{1}{2}\right)(1-z^2)^{-1/2}(-2z)$
 $= \cos^{-1} z - \frac{z}{\sqrt{1-z^2}} + \frac{z}{\sqrt{1-z^2}} = \cos^{-1} z$
19. $y = t \tan^{-1} t - \left(\frac{1}{2}\right) \ln t \Rightarrow \frac{dy}{dt} = \tan^{-1} t + t \left(\frac{1}{1+t^2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{t}\right) = \tan^{-1} t + \frac{t}{1+t^2} - \frac{1}{2t}$
20. $y = (1+t^2) \cot^{-1} 2t \Rightarrow \frac{dy}{dx} = 2t \cot^{-1} 2t + (1+t^2) \left(\frac{-2}{1+4t^2}\right)$
21. $y = z \sec^{-1} z - \sqrt{z^2-1} = z \sec^{-1} z - (z^2-1)^{1/2} \Rightarrow \frac{dy}{dz} = z \left(\frac{1}{|z|\sqrt{z^2-1}}\right) + (\sec^{-1} z)(1) - \frac{1}{2}(z^2-1)^{-1/2}(2z)$
 $= \frac{z}{|z|\sqrt{z^2-1}} - \frac{z}{\sqrt{z^2-1}} + \sec^{-1} z = \frac{1-z}{\sqrt{z^2-1}} + \sec^{-1} z, z > 1$

$$22. \quad y = 2\sqrt{x-1} \sec^{-1} \sqrt{x} = 2(x-1)^{1/2} \sec^{-1}(x^{1/2}) \Rightarrow \frac{dy}{dx} = 2 \left[\left(\frac{1}{2} \right) (x-1)^{-1/2} \sec^{-1}(x^{1/2}) + (x-1)^{1/2} \left(\frac{\frac{1}{2} x^{-1/2}}{\sqrt{x} \sqrt{x-1}} \right) \right]$$

$$= 2 \left(\frac{\sec^{-1} \sqrt{x}}{2\sqrt{x-1}} + \frac{1}{2x} \right) = \frac{\sec^{-1} \sqrt{x}}{\sqrt{x-1}} + \frac{1}{x}$$

$$23. \quad y = \csc^{-1}(\sec \theta) \Rightarrow \frac{dy}{d\theta} = \frac{-\sec \theta \tan \theta}{|\sec \theta| \sqrt{\sec^2 \theta - 1}} = -\frac{\tan \theta}{|\tan \theta|} = -1, 0 < \theta < \frac{\pi}{2}$$

$$24. \quad y = (1+x^2)e^{\tan^{-1} x} \Rightarrow y' = 2xe^{\tan^{-1} x} + (1+x^2) \left(\frac{e^{\tan^{-1} x}}{1+x^2} \right) = 2xe^{\tan^{-1} x} + e^{\tan^{-1} x}$$

$$25. \quad y = \frac{2(x^2+1)}{\sqrt{\cos 2x}} \Rightarrow \ln y = \ln \left(\frac{2(x^2+1)}{\sqrt{\cos 2x}} \right) = \ln(2) + \ln(x^2+1) - \frac{1}{2} \ln(\cos 2x) \Rightarrow \frac{y'}{y} = 0 + \frac{2x}{x^2+1} - \left(\frac{1}{2} \right) \left(\frac{-2 \sin 2x}{\cos 2x} \right)$$

$$\Rightarrow y' = \left(\frac{2x}{x^2+1} + \tan 2x \right) y = \frac{2(x^2+1)}{\sqrt{\cos 2x}} \left(\frac{2x}{x^2+1} + \tan 2x \right)$$

$$26. \quad y = 10\sqrt{\frac{3x+4}{2x-4}} \Rightarrow \ln y = \ln 10\sqrt{\frac{3x+4}{2x-4}} = \frac{1}{10} [\ln(3x+4) - \ln(2x-4)] \Rightarrow \frac{y'}{y} = \frac{1}{10} \left(\frac{3}{3x+4} - \frac{2}{2x-4} \right)$$

$$\Rightarrow y' = \frac{1}{10} \left(\frac{3}{3x+4} - \frac{1}{x-2} \right) y = 10\sqrt{\frac{3x+4}{2x-4}} \left(\frac{1}{10} \right) \left(\frac{3}{3x+4} - \frac{1}{x-2} \right)$$

$$27. \quad y = \left[\frac{(t+1)(t-1)}{(t-2)(t+3)} \right]^5 \Rightarrow \ln y = 5[\ln(t+1) + \ln(t-1) - \ln(t-2) - \ln(t+3)] \Rightarrow \left(\frac{1}{y} \right) \left(\frac{dy}{dt} \right) = 5 \left(\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t+3} \right)$$

$$\Rightarrow \frac{dy}{dt} = 5 \left[\frac{(t+1)(t-1)}{(t-2)(t+3)} \right]^5 \left(\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t+3} \right)$$

$$28. \quad y = \frac{2u2^u}{\sqrt{u^2+1}} \Rightarrow \ln y = \ln 2 + \ln u + u \ln 2 - \frac{1}{2} \ln(u^2+1) \Rightarrow \left(\frac{1}{y} \right) \left(\frac{dy}{du} \right) = \frac{1}{u} + \ln 2 - \frac{1}{2} \left(\frac{2u}{u^2+1} \right)$$

$$\Rightarrow \frac{dy}{du} = \frac{2u2^u}{\sqrt{u^2+1}} \left(\frac{1}{u} + \ln 2 - \frac{u}{u^2+1} \right)$$

$$29. \quad y = (\sin \theta)^{\sqrt{\theta}} \Rightarrow \ln y = \sqrt{\theta} \ln y(\sin \theta) \Rightarrow \left(\frac{1}{y} \right) \left(\frac{dy}{d\theta} \right) = \sqrt{\theta} \left(\frac{\cos \theta}{\sin \theta} \right) + \frac{1}{2} \theta^{-1/2} \ln(\sin \theta)$$

$$\Rightarrow \frac{dy}{d\theta} = (\sin \theta)^{\sqrt{\theta}} \left(\sqrt{\theta} \cot \theta + \frac{\ln(\sin \theta)}{2\sqrt{\theta}} \right)$$

$$30. \quad y = (\ln x)^{1/\ln x} \Rightarrow \ln y = \left(\frac{1}{\ln x} \right) \ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{\ln x} \right) \left(\frac{1}{\ln x} \right) \left(\frac{1}{x} \right) + \ln(\ln x) \left[\frac{-1}{(\ln x)^2} \right] \left(\frac{1}{x} \right) \Rightarrow y' = (\ln x)^{1/\ln x} \left[\frac{1 - \ln(\ln x)}{x(\ln x)^2} \right]$$

$$31. \quad \int e^x \cos(e^x) dx = \int \cos t \, dt, \text{ where } e^x = t \text{ and } e^x dx = dt$$

$$= \sin t + C = \sin(e^x) + C$$

$$32. \quad \int e^x \sin(5e^x - 7) dx = \frac{1}{5} \int \sin t \, dt, \text{ where } 5e^x - 7 = t \text{ and } e^x dx = \frac{dt}{5}$$

$$= -\cos t + C = -\frac{1}{5} \cos(5e^x - 7) + C$$

33. $\int e^x \sec^2(e^x - 7) dx = \int \sec^2 u \, du$, where $u = e^x - 7$ and $du = e^x dx$
 $= \tan u + C = \tan(e^x - 7) + C$
34. $\int e^y \csc(e^y + 1) \cot(e^y + 1) dy = \int \csc u \cot u \, du$, where $u = e^y + 1$ and $du = e^y dy$
 $= -\csc u + C = -\csc(e^y + 1) + C$
35. $\int (\sec^2 x) e^{\tan x} dx = \int e^u du$, where $u = \tan x$ and $du = \sec^2 x \, dx$
 $= e^u + C = e^{\tan x} + C$
36. $\int (\csc^2 x) e^{\cot x} dx = -\int e^u du$, where $u = \cot x$ and $du = -\csc^2 x \, dx$
 $= -e^u + C = -e^{\cot x} + C$
37. $\int_{-1}^1 \frac{1}{3x-4} dx = \frac{1}{3} \int_{-7}^{-1} \frac{1}{u} du$, where $u = 3x-4, du = 3 \, dx; \quad x = -1 \Rightarrow u = -7, x = 1 \Rightarrow u = -1$
 $= \frac{1}{3} [\ln |u|]_{-7}^{-1} = \frac{1}{3} [\ln |-1| - \ln |-7|] = \frac{1}{3} [0 - \ln 7] = -\frac{\ln 7}{3}$
38. $\int_1^e \frac{\sqrt{\ln x}}{x} dx = \int_0^1 u^{1/2} du$, where $u = \ln x, du = \frac{1}{x} dx; \quad x = 1 \Rightarrow u = 0, x = e \Rightarrow u = 1$
 $= \left[\frac{2}{3} u^{3/2} \right]_0^1 = \left[\frac{2}{3} 1^{3/2} - \frac{2}{3} 0^{3/2} \right] = \frac{2}{3}$
39. $\int_0^\pi \tan\left(\frac{x}{3}\right) dx = \int_0^\pi \frac{\sin(\frac{x}{3})}{\cos(\frac{x}{3})} dx = -3 \int_1^{1/2} \frac{1}{u} du$, where $u = \cos\left(\frac{x}{3}\right), du = -\frac{1}{3} \sin\left(\frac{x}{3}\right) dx; \quad x = 0 \Rightarrow u = 1, x = \pi \Rightarrow u = \frac{1}{2}$
 $= -3 [\ln |u|]_1^{1/2} = -3 \left[\ln \left| \frac{1}{2} \right| - \ln |1| \right] = -3 \ln \frac{1}{2} = \ln 2^3 = \ln 8$
40. $\int_{1/6}^{1/4} 2 \cot \pi x \, dx = 2 \int_{1/6}^{1/4} \frac{\cos \pi x}{\sin \pi x} dx = \frac{2}{\pi} \int_{1/2}^{1/\sqrt{2}} \frac{1}{u} du$, where $u = \sin \pi x, du = \pi \cos \pi x \, dx;$
 $x = \frac{1}{6} \Rightarrow u = \frac{1}{2}, x = \frac{1}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$
 $= \frac{2}{\pi} [\ln |u|]_{1/2}^{1/\sqrt{2}} = \frac{2}{\pi} \left[\ln \left| \frac{1}{\sqrt{2}} \right| - \ln \left| \frac{1}{2} \right| \right] = \frac{2}{\pi} \left[\ln 1 - \frac{1}{2} \ln 2 - \ln 1 + \ln 2 \right] = \frac{2}{\pi} \left[\frac{1}{2} \ln 2 \right] = \frac{\ln 2}{\pi}$
41. $\int_0^4 \frac{2t}{t^2-25} dt = \int_{-25}^{-9} \frac{1}{u} du$, where $u = t^2 - 25, du = 2t \, dt; \quad t = 0 \Rightarrow u = -25, t = 4 \Rightarrow u = -9$
 $= [\ln |u|]_{-25}^{-9} = \ln |-9| - \ln |-25| = \ln 9 - \ln 25 = \ln \frac{9}{25}$
42. $\int_{-\pi/2}^{\pi/6} \frac{\cos t}{1-\sin t} dt = -\int_2^{1/2} \frac{1}{u} du$, where $u = 1 - \sin t, du = -\cos t \, dt; \quad t = -\frac{\pi}{2} \Rightarrow u = 2, t = \frac{\pi}{6} \Rightarrow u = \frac{1}{2}$
 $= -[\ln |u|]_2^{1/2} = -\left[\ln \left| \frac{1}{2} \right| - \ln |2| \right] = -\ln 1 + \ln 2 + \ln 2 = 2 \ln 2 = \ln 4$

$$43. \int \frac{\tan(\ln v)}{v} dv = \int \tan u \, du = \int \frac{\sin u}{\cos u} du, \quad u = \ln v \text{ and } du = \frac{1}{v} dv$$

$$= -\ln |\cos u| + C = -\ln |\cos(\ln v)| + C$$

$$44. \int \frac{1}{v \ln v} dv = \int \frac{1}{u} du, \text{ where } u = \ln v \text{ and } du = \frac{1}{v} dv$$

$$= \ln |u| + C = \ln |\ln v| + C$$

$$45. \int \frac{(\ln x)^{-3}}{x} dx = \int u^{-3} du, \text{ where } u = \ln x \text{ and } du = \frac{1}{x} dx$$

$$= \frac{u^{-2}}{-2} + C = -\frac{1}{2}(\ln x)^{-2} + C$$

$$46. \int \frac{\ln(x+9)}{x+9} dx = \int p \, dp, \text{ where } \ln(x+9) = p \text{ and } \frac{1}{x+9} dx = dp$$

$$= \frac{p^2}{2} + C = \frac{1}{2}[\ln(x+9)]^2 + C$$

$$47. \int \frac{1}{r} \csc^2(1 + \ln r) dr = \int \csc^2 u \, du, \text{ where } u = 1 + \ln r \text{ and } du = \frac{1}{r} dr$$

$$= -\cot u + C = -\cot(1 + \ln r) + C$$

$$48. \int \frac{\sin(2 + \ln x)}{x} dx = \int \sin t \, dt, \text{ where } 2 + \ln x = t \text{ and } \frac{1}{x} dx = dt$$

$$= -\cos t + C = -\cos(2 + \ln x) + C$$

$$49. \int x 3^{x^2} dx = \frac{1}{2} \int 3^u \, du, \text{ where } u = x^2 \text{ and } du = 2x \, dx$$

$$= \frac{1}{2 \ln 3} (3^u) + C = \frac{1}{2 \ln 3} (3^{x^2}) + C$$

$$50. \int 2^{\tan x} \sec^2 x \, dx = \int 2^u \, du, \text{ where } u = \tan x \text{ and } du = \sec^2 x \, dx$$

$$= \frac{1}{\ln 2} (2^u) + C = \frac{2^{\tan x}}{\ln 2} + C$$

$$51. \int_1^9 \frac{5}{x} dx = [5 \ln x]_1^9 = 5(\ln 9 - \ln 1) = 5 \ln 9$$

$$52. \int_1^{81} \frac{1}{4x} dx = \frac{1}{4} [\ln x]_1^{81} = \frac{1}{4} [\ln 81 - \ln 1] = \frac{1}{4} (\ln 3^4 - 0) = \frac{1}{4} 4 \ln 3 = \ln 3$$

$$53. \int_1^4 \left(\frac{x}{8} + \frac{1}{2x} \right) dx = \frac{1}{2} \int_1^4 \left(\frac{1}{4} x + \frac{1}{x} \right) dx = \frac{1}{2} \left[\frac{1}{8} x^2 + \ln |x| \right]_1^4 = \frac{1}{2} \left[\left(\frac{16}{8} + \ln 4 \right) - \left(\frac{1}{8} + \ln 1 \right) \right] = \frac{15}{16} + \frac{1}{2} \ln 4$$

$$= \frac{15}{16} + \ln \sqrt{4} = \frac{15}{16} + \ln 2$$

$$54. \int_1^8 \left(\frac{2}{3x} - \frac{8}{x^2} \right) dx = \frac{2}{3} \int_1^8 \left(\frac{1}{x} - 12x^{-2} \right) dx = \frac{2}{3} \left[\ln |x| + 12x^{-1} \right]_1^8 = \frac{2}{3} \left[\left(\ln 8 + \frac{12}{8} \right) - (\ln 1 + 12) \right]$$

$$= \frac{2}{3} \left(\ln 8 + \frac{3}{2} - 12 \right) = \frac{2}{3} \left(\ln 8 - \frac{21}{2} \right) = \frac{2}{3} (\ln 8) - 7 = \ln(8^{2/3}) - 7 = \ln 4 - 7$$

55. $\int_{-2}^{-1} e^{-(x+1)} dx = -\int_1^0 e^u du$, where $u = -(x+1)$, $du = -dx$; $x = -2 \Rightarrow u = 1$, $x = -1 \Rightarrow u = 0$
 $= -\left[e^u \right]_1^0 = -(e^0 - e^1) = e - 1$
56. $\int_{-\ln 2}^0 e^{2w} dw = \frac{1}{2} \int_{\ln(1/4)}^0 e^u du$, where $u = 2w$, $du = 2dw$; $w = -\ln 2 \Rightarrow u = \ln \frac{1}{4}$, $w = 0 \Rightarrow u = 0$
 $= \frac{1}{2} \left[e^u \right]_{\ln(1/4)}^0 = \frac{1}{2} \left[e^0 - e^{\ln(1/4)} \right] = \frac{1}{2} \left(1 - \frac{1}{4} \right) = \frac{3}{8}$
57. $\int_1^{\ln 5} e^r (3e^r + 1)^{-3/2} dr = \frac{1}{3} \int_4^{16} u^{-3/2} du$, where $u = 3e^r + 1$, $du = 3e^r dr$; $r = 0 \Rightarrow u = 4$, $r = \ln 5 \Rightarrow u = 16$
 $= -\frac{2}{3} \left[u^{-1/2} \right]_4^{16} = -\frac{2}{3} \left(16^{-1/2} - 4^{-1/2} \right) = \left(-\frac{2}{3} \right) \left(\frac{1}{4} - \frac{1}{2} \right) = \left(-\frac{2}{3} \right) \left(-\frac{1}{4} \right) = \frac{1}{6}$
58. $\int_0^{\ln 9} e^\theta (e^\theta - 1)^{1/2} d\theta = \int_0^8 u^{1/2} du$, where $u = e^\theta - 1$, $du = e^\theta d\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \ln 9 \Rightarrow u = 8$
 $= \frac{2}{3} \left[u^{3/2} \right]_0^8 = \frac{2}{3} \left(8^{3/2} - 0^{3/2} \right) = \frac{2}{3} \left(2^{9/2} - 0 \right) = \frac{2^{11/2}}{3} = \frac{32\sqrt{2}}{3}$
59. $\int_1^e \frac{1}{x} (1 + 7 \ln x)^{-1/3} dx = \frac{1}{7} \int_1^8 u^{-1/3} du$, where $u = 1 + 7 \ln x$, $du = \frac{7}{x} dx$; $x = 1 \Rightarrow u = 1$, $x = e \Rightarrow u = 8$
 $= \frac{3}{14} \left[u^{2/3} \right]_1^8 = \frac{3}{14} \left(8^{2/3} - 1^{2/3} \right) = \left(\frac{3}{14} \right) (4 - 1) = \frac{9}{14}$
60. $\int \frac{5}{x\sqrt{\ln x}} dx = \int \frac{5}{\sqrt{t}} dt = 5 \frac{t^{1/2}}{1/2} = 10\sqrt{t} + C = 10\sqrt{\ln x} + C$, where $\ln x = t$, $\frac{1}{x} dx = dt$;
 $\int_{e^2}^{e^3} \frac{5}{x\sqrt{\ln x}} dx = \left[10\sqrt{\ln x} \right]_{e^2}^{e^3} = 10\sqrt{\ln e^3} - 10\sqrt{\ln e^2} = 10(\sqrt{3} - \sqrt{2})$
61. $\int_1^3 \frac{[\ln(v+1)]^2}{v+1} dv = \int_1^3 [\ln(v+1)]^2 \frac{1}{v+1} dv = \int_{\ln 2}^{\ln 4} u^2 du$, where $u = \ln(v+1)$, $du = \frac{1}{v+1} dv$;
 $v = 1 \Rightarrow u = \ln 2$, $v = 3 \Rightarrow u = \ln 4$
 $= \frac{1}{3} \left[u^3 \right]_{\ln 2}^{\ln 4} = \frac{1}{3} \left[(\ln 4)^3 - (\ln 2)^3 \right] = \frac{1}{3} \left[(2 \ln 2)^3 - (\ln 2)^3 \right] = \frac{(\ln 2)^3}{3} (8 - 1) = \frac{7}{3} (\ln 2)^3$
62. $\int (1 + \ln t)(t \ln t) dt = \int p dp = \frac{p^2}{2} + C = \frac{(t \ln t)^2}{2} + C$ where $t \ln t = p$, $\left(t \left(\frac{1}{t} \right) + (\ln t) \right) dt = dp$, $(1 + \ln t) dt = dp$;
 $\int_3^9 (1 + \ln t)(t \ln t) dt = \left[\frac{(t \ln t)^2}{2} \right]_3^9 = \frac{(9 \ln 9)^2}{2} - \frac{(3 \ln 3)^2}{2} = \frac{1}{2} \left[(18 \ln 3)^2 - (3 \ln 3)^2 \right] = \frac{1}{2} 315 (\ln 3)^2$
63. $\int_1^8 \frac{\log_4 \theta}{\theta} d\theta = \frac{1}{\ln 4} \int_1^8 (\ln \theta) \left(\frac{1}{\theta} \right) d\theta = \frac{1}{\ln 4} \int_0^{\ln 8} u du$, where $u = \ln \theta$, $du = \frac{1}{\theta} d\theta$; $\theta = 1 \Rightarrow u = 0$, $\theta = 8 \Rightarrow u = \ln 8$
 $= \frac{1}{2 \ln 4} \left[u^2 \right]_0^{\ln 8} = \frac{1}{\ln 16} \left[(\ln 8)^2 - 0^2 \right] = \frac{(3 \ln 2)^2}{4 \ln 2} = \frac{9 \ln 2}{4}$
64. $\int_1^e \frac{8(\ln 3)(\log_3 \theta)}{\theta} d\theta = \int_1^e \frac{8(\ln 3)(\ln \theta)}{\theta(\ln 3)} d\theta = 8 \int_1^e (\ln \theta) \left(\frac{1}{\theta} \right) d\theta = 8 \int_0^1 u du$, where $u = \ln \theta$, $du = \frac{1}{\theta} d\theta$

$$\theta = 1 \Rightarrow u = 0, \theta = e \Rightarrow u = 1$$

$$= 4 \left[u^2 \right]_0^1 = 4(1^2 - 0^2) = 4$$

$$65. \int_{-3/4}^{3/4} \frac{6}{\sqrt{9-4x^2}} dx = 3 \int_{-3/4}^{3/4} \frac{2}{\sqrt{3^2-(2x)^2}} dx = 3 \int_{-3/2}^{3/2} \frac{1}{\sqrt{3^2-u^2}} du, \text{ where } u = 2x, du = 2 dx;$$

$$x = -\frac{3}{4} \Rightarrow u = -\frac{3}{2}, x = \frac{3}{4} \Rightarrow u = \frac{3}{2}$$

$$= 3 \left[\sin^{-1} \left(\frac{u}{3} \right) \right]_{-3/2}^{3/2} = 3 \left[\sin^{-1} \left(\frac{1}{2} \right) - \sin^{-1} \left(-\frac{1}{2} \right) \right] = 3 \left[\frac{\pi}{6} - \left(-\frac{\pi}{6} \right) \right] = 3 \left(\frac{\pi}{3} \right) = \pi$$

$$66. \int_{-1/5}^{1/5} \frac{6}{\sqrt{4-25x^2}} dx = \frac{6}{5} \int_{-1/5}^{1/5} \frac{5}{\sqrt{2^2-(5x)^2}} dx = \frac{6}{5} \int_{-1}^1 \frac{1}{\sqrt{2^2-u^2}} du, \text{ where } u = 5x, du = 5dx;$$

$$x = -\frac{1}{5} \Rightarrow u = -1, x = \frac{1}{5} \Rightarrow u = 1$$

$$= \frac{6}{5} \left[\sin^{-1} \left(\frac{u}{2} \right) \right]_{-1}^1 = \frac{6}{5} \left[\sin^{-1} \left(\frac{1}{2} \right) - \sin^{-1} \left(-\frac{1}{2} \right) \right] = \frac{6}{5} \left[\frac{\pi}{6} - \left(-\frac{\pi}{6} \right) \right] = \frac{6}{5} \left(\frac{\pi}{3} \right) = \frac{2\pi}{5}$$

$$67. \int_{-2}^2 \frac{3}{4+3t^2} dt = \sqrt{3} \int_{-2}^2 \frac{\sqrt{3}}{2^2+(\sqrt{3}t)^2} dt = \sqrt{3} \int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{1}{2^2+u^2} du, \text{ where } u = \sqrt{3}t, du = \sqrt{3}dt;$$

$$t = -2 \Rightarrow u = -2\sqrt{3}, t = 2 \Rightarrow u = 2\sqrt{3}$$

$$= \sqrt{3} \left[\frac{1}{2} \tan^{-1} \left(\frac{u}{2} \right) \right]_{-2\sqrt{3}}^{2\sqrt{3}} = \frac{\sqrt{3}}{2} \left[\tan^{-1} (\sqrt{3}) - \tan^{-1} (-\sqrt{3}) \right] = \frac{\sqrt{3}}{2} \left[\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right] = \frac{\pi}{\sqrt{3}}$$

$$68. \int_{\sqrt{3}}^3 \frac{1}{3+t^2} dt = \int_{\sqrt{3}}^3 \frac{1}{(\sqrt{3})^2+t^2} dt = \left[\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) \right]_{\sqrt{3}}^3 = \frac{1}{\sqrt{3}} \left(\tan^{-1} \sqrt{3} - \tan^{-1} 1 \right) = \frac{1}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \frac{\sqrt{3}\pi}{36}$$

$$69. \int \frac{1}{y\sqrt{4y^2-1}} dy = \int \frac{2}{(2y)\sqrt{(2y)^2-1}} dy = \int \frac{1}{u\sqrt{u^2-1}} du \text{ where } u = 2y \text{ and } du = 2 dy$$

$$= \sec^{-1} |u| + C = \sec^{-1} |2y| + C$$

$$70. \int \frac{24}{y\sqrt{y^2-16}} dy = 24 \int \frac{1}{y\sqrt{y^2-4^2}} dy = 24 \left(\frac{1}{2} \sec^{-1} \left| \frac{y}{4} \right| \right) + C = 6 \sec^{-1} \left| \frac{y}{4} \right| + C$$

$$71. \int_{\sqrt{2}/3}^{2/3} \frac{1}{|y|\sqrt{9y^2-1}} dy = \int_{\sqrt{2}/3}^{2/3} \frac{3}{|3y|\sqrt{(3y)^2-1}} dy = \int_{\sqrt{2}}^2 \frac{1}{|u|\sqrt{u^2-1}} du, \text{ where } u = 3y, du = 3 dy;$$

$$y = \frac{\sqrt{2}}{3} \Rightarrow u = \sqrt{2}, y = \frac{2}{3} \Rightarrow u = 2$$

$$= \left[\sec^{-1} u \right]_{\sqrt{2}}^2 = \left[\sec^{-1} 2 - \sec^{-1} \sqrt{2} \right] = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

$$72. \int_{-2\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{1}{|y|\sqrt{5y^2-3}} dy = \int_{-2\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{\sqrt{5}}{-\sqrt{5}\sqrt{(\sqrt{5}y)^2-(\sqrt{3})^2}} dy = \int_{-2}^{-\sqrt{6}} \frac{1}{-u\sqrt{u^2-(\sqrt{3})^2}} du, \text{ where } u = \sqrt{5}y, du = \sqrt{5}dy;$$

$$y = -\frac{2}{\sqrt{5}} \Rightarrow u = -2, y = -\frac{\sqrt{6}}{\sqrt{5}} \Rightarrow u = -\sqrt{6}$$

$$= \left[-\frac{1}{\sqrt{3}} \sec^{-1} \left| \frac{u}{\sqrt{3}} \right| \right]_{-2}^{-\sqrt{6}} = \frac{-1}{\sqrt{3}} \left[\sec^{-1} \sqrt{2} - \sec^{-1} \frac{2}{\sqrt{3}} \right] = \frac{-1}{\sqrt{3}} \left(\frac{\pi}{4} - \frac{\pi}{6} \right) = \frac{-1}{\sqrt{3}} \left[\frac{3\pi}{12} - \frac{2\pi}{12} \right] = \frac{-\pi}{12\sqrt{3}} = \frac{-\sqrt{3}\pi}{36}$$

$$\begin{aligned} 73. \quad \int \frac{1}{\sqrt{-2x-x^2}} dx &= \int \frac{1}{\sqrt{1-(x^2+2x+1)}} dx = \int \frac{1}{\sqrt{1-(x+1)^2}} dx = \int \frac{1}{\sqrt{1-u^2}} du, \text{ where } u = x+1 \text{ and } du = dx \\ &= \sin^{-1} u + C = \sin^{-1}(x+1) + C \end{aligned}$$

$$\begin{aligned} 74. \quad \int \frac{1}{\sqrt{-x^2+4x-1}} dx &= \int \frac{1}{\sqrt{3-(x^2-4x+4)}} dx = \int \frac{1}{\sqrt{(\sqrt{3})^2-(x-2)^2}} dx = \int \frac{1}{\sqrt{(\sqrt{3})^2-u^2}} du \text{ where } u = x-2 \text{ and } du = dx \\ &= \sin^{-1} \left(\frac{u}{\sqrt{3}} \right) + C = \sin^{-1} \left(\frac{x-2}{\sqrt{3}} \right) + C \end{aligned}$$

$$\begin{aligned} 75. \quad \int_{-2}^{-1} \frac{2}{v^2+4v+5} dv &= 2 \int_{-2}^{-1} \frac{1}{1+(v^2+4v+4)} dv = 2 \int_{-2}^{-1} \frac{1}{1+(v+2)^2} dv = 2 \int_0^1 \frac{1}{1+u^2} du, \text{ where } u = v+2, du = dv; \\ &v = -2 \Rightarrow u = 0, v = -1 \Rightarrow u = 1 \\ &= 2 \left[\tan^{-1} u \right]_0^1 = 2 \left(\tan^{-1} 1 - \tan^{-1} 0 \right) = 2 \left(\frac{\pi}{4} - 0 \right) = \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} 76. \quad \int_{-1}^1 \frac{3}{4v^2+4v+4} dv &= \frac{3}{4} \int_{-1}^1 \frac{1}{\frac{3}{4}+(v^2+v+\frac{1}{4})} dv = \frac{3}{4} \int_{-1}^1 \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+(v+\frac{1}{2})^2} dv = \frac{3}{4} \int_{-1/2}^{3/2} \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+u^2} du \text{ where } u = v+\frac{1}{2}, du = dv; \\ &v = -1 \Rightarrow u = -\frac{1}{2}, v = 1 \Rightarrow u = \frac{3}{2} \\ &= \frac{3}{4} \left[\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2u}{\sqrt{3}} \right) \right]_{-1/2}^{3/2} = \frac{\sqrt{3}}{2} \left[\tan^{-1} \sqrt{3} - \tan^{-1} \left(-\frac{1}{\sqrt{3}} \right) \right] = \frac{\sqrt{3}}{2} \left[\frac{\pi}{3} - \left(-\frac{\pi}{6} \right) \right] = \frac{\sqrt{3}}{2} \left(\frac{2\pi}{6} + \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2} \cdot \frac{\pi}{2} = \frac{\sqrt{3}\pi}{4} \end{aligned}$$

$$\begin{aligned} 77. \quad \int \frac{1}{(t+1)\sqrt{t^2+2t-8}} dt &= \int \frac{1}{(t+1)\sqrt{(t^2+2t+1)-9}} dt = \int \frac{1}{(t+1)\sqrt{(t+1)^2-3^2}} dt = \int \frac{1}{u\sqrt{u^2-3^2}} du, \text{ where } u = t+1 \text{ and } du = dt \\ &= \frac{1}{3} \sec^{-1} \left| \frac{u}{3} \right| + C = \frac{1}{3} \sec^{-1} \left| \frac{t+1}{3} \right| + C \end{aligned}$$

$$\begin{aligned} 78. \quad \int \frac{1}{(3t+1)\sqrt{9t^2+6t}} dt &= \int \frac{1}{(3t+1)\sqrt{(9t^2+6t+1)-1}} dt = \int \frac{1}{(3t+1)\sqrt{(3t+1)^2-1^2}} dt = \frac{1}{3} \int \frac{1}{u\sqrt{u^2-1}} du, \text{ where } u = 3t+1 \text{ and } du = 3dt \\ &= \frac{1}{3} \sec^{-1} |u| + C = \frac{1}{3} \sec^{-1} |3t+1| + C \end{aligned}$$

$$79. \quad 3^y = 2^{y+1} \Rightarrow \ln 3^y = \ln 2^{y+1} \Rightarrow y(\ln 3) = (y+1) \ln 2 \Rightarrow (\ln 3 - \ln 2)y = \ln 2 \Rightarrow \left(\ln \frac{3}{2} \right) y = \ln 2 \Rightarrow y = \frac{\ln 2}{\ln \left(\frac{3}{2} \right)}$$

$$\begin{aligned} 80. \quad 4^{-y} = 3^{y+2} &\Rightarrow \ln 4^{-y} = \ln 3^{y+2} \Rightarrow -y \ln 4 = (y+2) \ln 3 \Rightarrow -2 \ln 3 = (\ln 3 + \ln 4)y \\ &\Rightarrow (\ln 12)y = -2 \ln 3 \Rightarrow y = -\frac{\ln 9}{\ln 12} \end{aligned}$$

$$81. \quad 9e^{2y} = x^2 \Rightarrow e^{2y} = \frac{x^2}{9} \Rightarrow \ln e^{2y} = \ln \left(\frac{x^2}{9} \right) \Rightarrow 2y(\ln e) = \ln \left(\frac{x^2}{9} \right) \Rightarrow y = \frac{1}{2} \ln \left(\frac{x^2}{9} \right) = \ln \sqrt{\frac{x^2}{9}} = \ln \left| \frac{x}{3} \right| = \ln |x| - \ln 3$$

$$82. \quad 3^y = 3 \ln x \Rightarrow \ln 3^y = \ln(3 \ln x) \Rightarrow y \ln 3 = \ln(3 \ln x) \Rightarrow y = \frac{\ln(3 \ln x)}{\ln 3} = \frac{\ln 3 + \ln(\ln x)}{\ln 3}$$

$$83. \ln(y-1) = x + \ln y \Rightarrow e^{\ln(y-1)} = e^{(x+\ln y)} = e^x e^{\ln y} \Rightarrow y-1 = ye^x \Rightarrow y - ye^x = 1 \Rightarrow y(1-e^x) = 1 \Rightarrow y = \frac{1}{1-e^x}$$

$$84. \ln(10 \ln y) = \ln 5x \Rightarrow e^{\ln(10 \ln y)} = e^{\ln 5x} \Rightarrow 10 \ln y = 5x \Rightarrow \ln y = \frac{x}{2} \Rightarrow e^{\ln y} = e^{x/2} \Rightarrow y = e^{x/2}$$

$$85. \lim_{x \rightarrow 1} \frac{x^2+3x-4}{x-1} = \lim_{x \rightarrow 1} \frac{2x+3}{1} = 5$$

$$86. \lim_{x \rightarrow 1} \frac{x^a-1}{x^b-1} = \lim_{x \rightarrow 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$$

$$87. \lim_{x \rightarrow \pi} \frac{\tan x}{x} = \frac{\tan \pi}{\pi} = 0$$

$$88. \lim_{x \rightarrow 0} \frac{\tan x}{x + \sin x} = \lim_{x \rightarrow 0} \frac{\sec^2 x}{1 + \cos x} = \frac{1}{1+1} = \frac{1}{2}$$

$$89. \lim_{x \rightarrow 0} \frac{\sin^2 x}{\tan(x^2)} = \lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{2x \sec^2(x^2)} = \lim_{x \rightarrow 0} \frac{\sin(2x)}{2x \sec^2(x^2)} = \lim_{x \rightarrow 0} \frac{2 \cos(2x)}{2x(2 \sec^2(x^2) \tan(x^2) \cdot 2x) + 2 \sec^2(x^2)} = \frac{2}{0+2 \cdot 1} = 1$$

$$90. \lim_{x \rightarrow 0} \frac{\sin(mx)}{\sin(nx)} = \lim_{x \rightarrow 0} \frac{m \cos(mx)}{n \cos(nx)} = \frac{m}{n}$$

$$91. \lim_{x \rightarrow (\frac{\pi}{2})^-} \sec(7x) \cos(3x) = \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{\cos(3x)}{\cos(7x)} = \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{-3 \sin(3x)}{-7 \sin(7x)} = \frac{3}{7}$$

$$92. \lim_{x \rightarrow 0^+} \sqrt{x} \sec x = \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\cos x} = \frac{0}{1} = 0$$

$$93. \lim_{x \rightarrow 0} (\csc x - \cot x) = \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x} = \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0$$

$$94. \lim_{x \rightarrow 0} \left(\frac{1}{x^4} - \frac{1}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{1-x^2}{x^4} \right) = \lim_{x \rightarrow 0} (1-x^2) \cdot \frac{1}{x^4} = \lim_{x \rightarrow 0} (1-x^2) \cdot \lim_{x \rightarrow 0} \frac{1}{x^4} = 1 \cdot \infty = \infty$$

$$95. \lim_{x \rightarrow \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x} \right) = \lim_{x \rightarrow \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x} \right) \cdot \frac{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} = \lim_{x \rightarrow \infty} \frac{2x+1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}$$

Notice that $x = \sqrt{x^2}$ for $x > 0$ so this is equivalent to

$$= \lim_{x \rightarrow \infty} \frac{\frac{2x+1}{x}}{\sqrt{\frac{x^2+x+1}{x^2}} + \sqrt{\frac{x^2-x}{x^2}}} = \lim_{x \rightarrow \infty} \frac{2+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x}}} = \frac{2}{\sqrt{1}+\sqrt{1}} = 1$$

$$96. \lim_{x \rightarrow \infty} \left(\frac{x^3}{x^2-1} - \frac{x^3}{x^2+1} \right) = \lim_{x \rightarrow \infty} \frac{x^3(x^2+1) - x^3(x^2-1)}{(x^2-1)(x^2+1)} = \lim_{x \rightarrow \infty} \frac{2x^3}{x^4-1} = \lim_{x \rightarrow \infty} \frac{6x^2}{4x^3} = \lim_{x \rightarrow \infty} \frac{12x}{12x^2} = \lim_{x \rightarrow \infty} \frac{12}{24x} = \lim_{x \rightarrow \infty} \frac{1}{2x} = 0$$

$$97. \text{The limit leads to the indeterminate form } \frac{0}{0}: \lim_{x \rightarrow 0} \frac{10^x-1}{x} = \lim_{x \rightarrow 0} \frac{(\ln 10)10^x}{1} = \ln 10$$

$$98. \text{The limit leads to the indeterminate form } \frac{0}{0}: \lim_{\theta \rightarrow 0} \frac{3^\theta-1}{\theta} = \lim_{\theta \rightarrow 0} \frac{(\ln 3)3^\theta}{1} = \ln 3$$

$$99. \text{The limit leads to the indeterminate form } \frac{0}{0}: \lim_{x \rightarrow 0} \frac{2^{\sin x}-1}{e^x-1} = \lim_{x \rightarrow 0} \frac{2^{\sin x}(\ln 2)(\cos x)}{e^x} = \ln 2$$

$$100. \text{The limit leads to the indeterminate form } \frac{0}{0}: \lim_{x \rightarrow 0} \frac{2^{-\sin x}-1}{e^x-1} = \lim_{x \rightarrow 0} \frac{2^{-\sin x}(\ln 2)(-\cos x)}{e^x} = -\ln 2$$

101. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x \rightarrow 0} \frac{5-5\cos x}{e^x-x-1} = \lim_{x \rightarrow 0} \frac{5\sin x}{e^x-1} = \lim_{x \rightarrow 0} \frac{5\cos x}{e^x} = 5$

102. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x \rightarrow 0} \frac{x \sin x^2}{\tan^3 x} = \lim_{x \rightarrow 0} \frac{2x^2 \cos x^2 + \sin x^2}{3 \tan^2 x \sec^2 x} = \lim_{x \rightarrow 0} \frac{2x^2 \cos x^2 + \sin x^2}{3 \tan^4 x + 3 \tan^2 x}$
 $= \lim_{x \rightarrow 0} \frac{6x \cos x^2 - 4x^2 \sin x^2}{12 \tan^3 x \sec^2 x + 6 \tan x \sec^2 x} = \lim_{x \rightarrow 0} \frac{6x \cos x^2 - 4x^3 \sin x^2}{12 \tan^5 x + 18 \tan^3 x + 6 \tan x} = \lim_{x \rightarrow 0} \frac{(6-8x^4) \cos x^2 - 24x^2 \sin x^2}{60 \tan^4 x \sec^2 x + 54 \tan^2 x \sec^2 x + 6 \sec^2 x} = \frac{6}{6} = 1$

103. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{t \rightarrow 0^+} \frac{t - \ln(1+2t)}{t^2} = \lim_{t \rightarrow 0^+} \frac{(1-\frac{2}{1+2t})}{2t} = -\infty$

104. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x \rightarrow 4} \frac{\sin^2(\pi x)}{e^{x-4} + 3 - x} = \lim_{x \rightarrow 4} \frac{2\pi(\sin \pi x)(\cos \pi x)}{e^{x-4} - 1} = \lim_{x \rightarrow 4} \frac{\pi \sin(2\pi x)}{e^{x-4} - 1}$
 $= \lim_{x \rightarrow 4} \frac{2\pi^2 \cos(2\pi x)}{e^{x-4}} = 2\pi^2$

105. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{t \rightarrow 0^+} \left(\frac{e^t}{t} - \frac{1}{t} \right) = \lim_{t \rightarrow 0^+} \left(\frac{e^t - 1}{t} \right) = \lim_{t \rightarrow 0^+} \frac{e^t}{1} = 1$

106. The limit leads to the indeterminate form $\frac{\infty}{\infty}$: $\lim_{y \rightarrow 0^+} e^{-1/y} \ln y = \lim_{y \rightarrow 0^+} \frac{\ln y}{e^{y^{-1}}} = \lim_{y \rightarrow 0^+} \frac{y^{-1}}{-e^{y^{-1}}(y^{-2})} = \lim_{y \rightarrow 0^+} \left(-\frac{y}{e^{y^{-1}}} \right) = 0$

107. Let $f(x) = \left(\frac{e^x+1}{e^x-1} \right)^{\ln x} \Rightarrow \ln f(x) = \ln x \ln \left(\frac{e^x+1}{e^x-1} \right) \Rightarrow \lim_{x \rightarrow \infty} \ln f(x) = \lim_{x \rightarrow \infty} \ln x \ln \left(\frac{e^x+1}{e^x-1} \right)$; this limit is currently of the form $0 \cdot \infty$. Before we put in one of the indeterminate forms, we rewrite $\frac{e^x+1}{e^x-1} = \frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} = \coth \left(\frac{x}{2} \right)$; the limit is $\lim_{x \rightarrow \infty} \ln x \ln \coth \left(\frac{x}{2} \right) = \lim_{x \rightarrow \infty} \frac{\ln \coth \left(\frac{x}{2} \right)}{\frac{1}{\ln x}}$; the limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x \rightarrow \infty} \frac{\ln \coth \left(\frac{x}{2} \right)}{\frac{1}{\ln x}}$

$$= \lim_{x \rightarrow \infty} \left(\frac{\frac{\operatorname{csch}^2 \left(\frac{x}{2} \right)}{\coth \left(\frac{x}{2} \right)} \left(-\frac{1}{2} \right)}{\frac{1}{(\ln x)^2} \left(\frac{1}{x} \right)} \right) = \lim_{x \rightarrow \infty} \left(\frac{x(\ln x)^2}{2 \sinh \left(\frac{x}{2} \right) \cosh \left(\frac{x}{2} \right)} \right) = \lim_{x \rightarrow \infty} \left(\frac{x(\ln x)^2}{\sinh x} \right) = \lim_{x \rightarrow \infty} \left(\frac{2x(\ln x) \left(\frac{1}{x} \right) + (\ln x)^2}{\cosh x} \right) = \lim_{x \rightarrow \infty} \left(\frac{2 \ln x + (\ln x)^2}{\cosh x} \right)$$

$$= \lim_{x \rightarrow \infty} \left(\frac{2 \left(\frac{1}{x} \right) + 2(\ln x) \left(\frac{1}{x} \right)}{\sinh x} \right) = \lim_{x \rightarrow \infty} \left(\frac{2+2 \ln x}{x \sinh x} \right) = \lim_{x \rightarrow \infty} \left(\frac{\frac{2}{x}}{x^2 \cosh x + x \sinh x} \right) = 0 \Rightarrow \lim_{x \rightarrow \infty} \left(\frac{e^x+1}{e^x-1} \right)^{\ln x}$$

$$= \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^0 = 1$$

108. Let $f(x) = \left(1 + \frac{3}{x} \right)^x \Rightarrow \ln f(x) = x \ln \left(1 + \frac{3}{x} \right) \Rightarrow \lim_{x \rightarrow 0^+} \ln f(x) = \lim_{x \rightarrow 0^+} \frac{\ln(1+3x^{-1})}{x^{-1}}$; the limit leads to the

indeterminate form $\frac{\infty}{\infty}$: $\lim_{x \rightarrow 0^+} \frac{\frac{-3x^{-2}}{1+3x^{-1}}}{-x^{-2}} = \lim_{x \rightarrow 0^+} \frac{3x}{x+3} = 0 \Rightarrow \lim_{x \rightarrow 0^+} \left(1 + \frac{3}{x} \right)^x = \lim_{x \rightarrow 0^+} e^{\ln f(x)} = e^0 = 1$

109. (a) $\lim_{x \rightarrow \infty} \frac{\log_2 x}{\log_3 x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{\ln x}{\ln 2} \right)}{\left(\frac{\ln x}{\ln 3} \right)} = \lim_{x \rightarrow \infty} \frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2} \Rightarrow$ same rate

- (b) $\lim_{x \rightarrow \infty} \frac{x}{x + (\frac{1}{x})} = \lim_{x \rightarrow \infty} \frac{x^2}{x^2 + 1} = \lim_{x \rightarrow \infty} \frac{2x}{2x} = \lim_{x \rightarrow \infty} 1 = 1 \Rightarrow$ same rate
- (c) $\lim_{x \rightarrow \infty} \frac{(\frac{x}{100})}{xe^{-x}} = \lim_{x \rightarrow \infty} \frac{xe^x}{100x} = \lim_{x \rightarrow \infty} \frac{e^x}{100} = \infty \Rightarrow$ faster
- (d) $\lim_{x \rightarrow \infty} \frac{x}{\tan^{-1} x} = \infty \Rightarrow$ faster
- (e) $\lim_{x \rightarrow \infty} \frac{\csc^{-1} x}{(\frac{1}{x})} = \lim_{x \rightarrow \infty} \frac{\sin^{-1}(x^{-1})}{x^{-1}} = \lim_{x \rightarrow \infty} \frac{\frac{(-x^{-2})}{\sqrt{1-(x^{-1})^2}}}{-x^{-2}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1-(\frac{1}{x^2})}} = 1 \Rightarrow$ same rate
- (f) $\lim_{x \rightarrow \infty} \frac{\sinh x}{e^x} = \lim_{x \rightarrow \infty} \frac{(e^x - e^{-x})}{2e^x} = \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{2} = \frac{1}{2} \Rightarrow$ same rate

110. (a) $\lim_{x \rightarrow \infty} \frac{3^{-x}}{2^{-x}} = \lim_{x \rightarrow \infty} \left(\frac{2}{3}\right)^x = 0 \Rightarrow$ slower
- (b) $\lim_{x \rightarrow \infty} \frac{\ln 2x}{\ln x^2} = \lim_{x \rightarrow \infty} \frac{\ln 2 + \ln x}{2 \ln x} = \lim_{x \rightarrow \infty} \left(\frac{\ln 2}{2 \ln x} + \frac{1}{2}\right) = \frac{1}{2} \Rightarrow$ same rate
- (c) $\lim_{x \rightarrow \infty} \frac{10x^3 + 2x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{30x^2 + 4x}{e^x} = \lim_{x \rightarrow \infty} \frac{60x + 4}{e^x} = \lim_{x \rightarrow \infty} \frac{60}{e^x} = 0 \Rightarrow$ slower
- (d) $\lim_{x \rightarrow \infty} \frac{\tan^{-1}(\frac{1}{x})}{(\frac{1}{x})} = \lim_{x \rightarrow \infty} \frac{\tan^{-1}(x^{-1})}{x^{-1}} = \lim_{x \rightarrow \infty} \frac{\frac{(-x^{-2})}{1+x^{-2}}}{-x^{-2}} = \lim_{x \rightarrow \infty} \frac{1}{1+\frac{1}{x^2}} = 1 \Rightarrow$ same rate
- (e) $\lim_{x \rightarrow \infty} \frac{\sin^{-1}(\frac{1}{x})}{(\frac{1}{x^2})} = \lim_{x \rightarrow \infty} \frac{\sin^{-1}(x^{-1})}{x^{-2}} = \lim_{x \rightarrow \infty} \frac{\frac{(-x^{-2})}{\sqrt{1-(x^{-1})^2}}}{-2x^{-3}} = \lim_{x \rightarrow \infty} \frac{x}{2\sqrt{1-\frac{1}{x^2}}} = \infty \Rightarrow$ faster
- (f) $\lim_{x \rightarrow \infty} \frac{\operatorname{sech} x}{e^{-x}} = \lim_{x \rightarrow \infty} \frac{(\frac{2}{e^x + e^{-x}})}{e^{-x}} = \lim_{x \rightarrow \infty} \frac{2}{e^{-x}(e^x + e^{-x})} = \lim_{x \rightarrow \infty} \left(\frac{2}{1+e^{-2x}}\right) = 2 \Rightarrow$ same rate

111. (a) $\frac{(\frac{1}{x^2} + \frac{1}{x^4})}{(\frac{1}{x^2})} = 1 + \frac{1}{x^2} \leq 2$ for x sufficiently large \Rightarrow true
- (b) $\frac{(\frac{1}{x^2} + \frac{1}{x^4})}{(\frac{1}{x^4})} = x^2 + 1 > M$ for any positive integer M whenever $x > \sqrt{M} \Rightarrow$ false
- (c) $\lim_{x \rightarrow \infty} \frac{x}{x + \ln x} = \lim_{x \rightarrow \infty} \frac{1}{1 + \frac{1}{x}} = 1 \Rightarrow$ the same growth rate \Rightarrow false
- (d) $\lim_{x \rightarrow \infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x \rightarrow \infty} \frac{\left[\frac{\ln x}{\ln x}\right]}{(\frac{1}{x})} = \lim_{x \rightarrow \infty} \frac{1}{\ln x} = 0 \Rightarrow$ grows slower \Rightarrow true
- (e) $\frac{\tan^{-1} x}{1} \leq \frac{\pi}{2}$ for all $x \Rightarrow$ true
- (f) $\frac{\cosh x}{e^x} = \frac{1}{2}(1 + e^{-2x}) \leq \frac{1}{2}(1 + 1) = 1$ if $x > 0 \Rightarrow$ true

112. (a) $\frac{(\frac{1}{x^4})}{(\frac{1}{x^2} + \frac{1}{x^4})} = \frac{1}{x^2 + 1} \leq 1$ if $x > 0 \Rightarrow$ true

- (b) $\lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^2} + \frac{1}{x^4}\right)} = \lim_{x \rightarrow \infty} \left(\frac{1}{x^2 + 1}\right) = 0 \Rightarrow \text{true}$
- (c) $\lim_{x \rightarrow \infty} \frac{\ln x}{x+1} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{1} = 0 \Rightarrow \text{true}$
- (d) $\frac{\ln 2x}{\ln x} = \frac{\ln 2}{\ln x} + 1 \leq 1 + 1 = 2$ if $x \geq 2 \Rightarrow \text{true}$
- (e) $\frac{\sec^{-1} x}{1} = \frac{\cos^{-1}\left(\frac{1}{x}\right)}{1} \leq \frac{\left(\frac{\pi}{2}\right)}{1} = \frac{\pi}{2}$ if $x > 1 \Rightarrow \text{true}$
- (f) $\frac{\sinh x}{e^x} = \frac{1}{2}(1 - e^{-2x}) \leq \frac{1}{2}$ if $x > 0 \Rightarrow \text{true}$

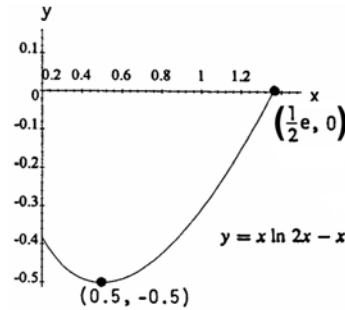
$$113. \frac{df}{dx} = e^x + 1 \Rightarrow \left(\frac{df^{-1}}{dx}\right)_{x=f(\ln 2)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=\ln 2}} \Rightarrow \left(\frac{df^{-1}}{dx}\right)_{x=f(\ln 2)} = \frac{1}{(e^x + 1)_{x=\ln 2}} = \frac{1}{2+1} = \frac{1}{3}$$

$$114. y = f(x) \Rightarrow y = 1 + \frac{1}{x} \Rightarrow \frac{1}{x} = y - 1 \Rightarrow x = \frac{1}{y-1} \Rightarrow f^{-1}(x) = \frac{1}{x-1}; f^{-1}(f(x)) = \frac{1}{\left(1+\frac{1}{x}\right)-1} = \frac{1}{\left(\frac{1}{x}\right)} = x \text{ and}$$

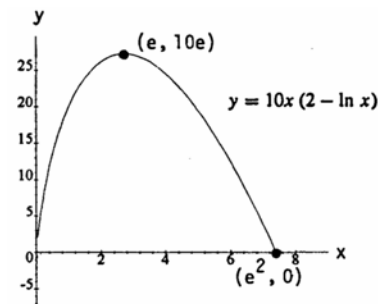
$$f\left(f^{-1}(x)\right) = 1 + \frac{1}{\left(\frac{1}{x-1}\right)} = 1 + (x-1) = x; \left.\frac{df^{-1}}{dx}\right|_{f(x)} = \frac{-1}{(x-1)^2} \Big|_{f(x)} = \frac{-1}{\left[\left(1+\frac{1}{x}\right)-1\right]^2} = -x^2;$$

$$f'(x) = -\frac{1}{x^2} \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{f(x)} = \frac{1}{f'(x)}$$

115. $y = x \ln 2x - x \Rightarrow y' = x\left(\frac{2}{2x}\right) + \ln(2x) - 1 = \ln 2x$;
 solving $y' = 0 \Rightarrow x = \frac{1}{2}$; $y' > 0$ for $x > \frac{1}{2}$ and $y' < 0$ for $x < \frac{1}{2} \Rightarrow$ relative minimum of $-\frac{1}{2}$ at $x = \frac{1}{2}$;
 $f\left(\frac{1}{2e}\right) = -\frac{1}{e}$ and $f\left(\frac{e}{2}\right) = 0 \Rightarrow$ absolute minimum is $-\frac{1}{2}$ at $x = \frac{1}{2}$ and the absolute maximum is 0 at $x = \frac{e}{2}$



116. $y = 10x(2 - \ln x) \Rightarrow y' = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right) = 20 - 10\ln x - 10 = 10(1 - \ln x)$; solving $y' = 0 \Rightarrow x = e$; $y' < 0$ for $x > e$ and $y' > 0$ for $x < e \Rightarrow$ relative maximum at $x = e$ of $10e$; $y \geq 0$ on $(0, e^2]$ and $y(e^2) = 10e^2(2 - 2\ln e) = 0 \Rightarrow$ absolute minimum is 0 at $x = e^2$ and the absolute maximum is $10e$ at $x = e$



$$117. A = \int_1^e \frac{2\ln x}{x} dx = \int_0^1 2u du = \left[u^2\right]_0^1 = 1, \text{ where } u = \ln x \text{ and } du = \frac{1}{x} dx; x = 1 \Rightarrow u = 0, x = e \Rightarrow u = 1$$

$$118. (a) A_1 = \int_{10}^{20} \frac{1}{x} dx = [\ln |x|]_{10}^{20} = \ln 20 - \ln 10 = \ln \frac{20}{10} = \ln 2, \text{ and } A_2 = \int_1^2 \frac{1}{x} dx = [\ln |x|]_1^2 = \ln 2 - \ln 1 = \ln 2$$

$$(b) \quad A_1 = \int_{ka}^{kb} \frac{1}{x} dx = [\ln |x|]_{ka}^{kb} = \ln kb - \ln ka = \ln \frac{kb}{ka} = \ln \frac{b}{a} = \ln b - \ln a, \text{ and } A_2 = \int_a^b \frac{1}{x} dx = [\ln |x|]_a^b = \ln b - \ln a$$

$$119. \quad y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}; \quad \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \left(\frac{1}{x}\right) \sqrt{x} = \frac{1}{\sqrt{x}} \Rightarrow \frac{dy}{dt} \Big|_e = \frac{1}{e} \text{ m/s}$$

$$120. \quad y = 3e^{-x/3} \Rightarrow \frac{dy}{dx} = -e^{-x/3}; \quad \frac{dx}{dt} = \frac{(dy/dt)}{(dy/dx)} \Rightarrow \frac{dx}{dt} = \frac{\left(-\frac{1}{4}\right)\sqrt{3-y}}{-e^{-x/3}}; \quad x = 3 \Rightarrow y = 3e^{-1} \Rightarrow \frac{dx}{dt} \Big|_{x=3} = \frac{\left(-\frac{1}{4}\right)\sqrt{3-\frac{3}{e}}}{\left(-\frac{1}{e}\right)} \\ = \frac{1}{4}\sqrt{e}\sqrt{e-1} \approx 0.54 \text{ m/s}$$

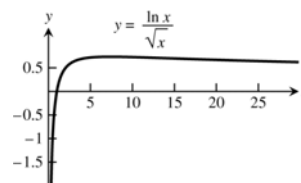
$$121. \quad A = xy = xe^{-x^2} \Rightarrow \frac{dA}{dx} = e^{-x^2} + (x)(-2x)e^{-x^2} = e^{-x^2}(1-2x^2). \text{ Solving } \frac{dA}{dx} = 0 \Rightarrow 1-2x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{2}}; \\ \frac{dA}{dx} < 0 \text{ for } x > \frac{1}{\sqrt{2}} \text{ and } \frac{dA}{dx} > 0 \text{ for } 0 < x < \frac{1}{\sqrt{2}} \Rightarrow \text{absolute maximum of } \frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{\sqrt{2}e} \text{ at } x = \frac{1}{\sqrt{2}} \text{ units long by} \\ y = e^{-1/2} = \frac{1}{\sqrt{e}} \text{ units high.}$$

$$122. \quad A = xy = x\left(\frac{\ln x}{x^2}\right) = \frac{\ln x}{x} \Rightarrow \frac{dA}{dx} = \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1-\ln x}{x^2}. \text{ Solving } \frac{dA}{dx} = 0 \Rightarrow 1-\ln x = 0 \Rightarrow x = e; \frac{dA}{dx} < 0 \text{ for } x > e \text{ and} \\ \frac{dA}{dx} > 0 \text{ for } x < e \Rightarrow \text{absolute maximum of } \frac{\ln e}{e} = \frac{1}{e} \text{ at } x = e \text{ units long and } y = \frac{1}{e^2} \text{ units high.}$$

$$123. (a) \quad y = \frac{\ln x}{\sqrt{x}} \Rightarrow y' = \frac{1}{x\sqrt{x}} - \frac{\ln x}{2x^{3/2}} = \frac{2-\ln x}{2x\sqrt{x}} \\ \Rightarrow y'' = -\frac{3}{4}x^{-5/2}(2-\ln x) - \frac{1}{2}x^{-5/2} = x^{-5/2}\left(\frac{3}{4}\ln x - 2\right);$$

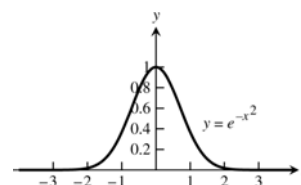
$$\text{solving } y' = 0 \Rightarrow \ln x = 2 \Rightarrow x = e^2; \quad y' < 0 \text{ for } x > e^2 \text{ and } y' > 0 \\ \text{for } x < e^2 \Rightarrow \text{a maximum of } \frac{2}{e}; \quad y'' = 0 \Rightarrow \ln x = \frac{8}{3} \Rightarrow x = e^{8/3};$$

$$\text{the curve is concave down on } (0, e^{8/3}) \text{ and concave up on } (e^{8/3}, \infty); \text{ so there is an inflection point at } \\ \left(e^{8/3}, \frac{8}{3e^{4/3}}\right).$$

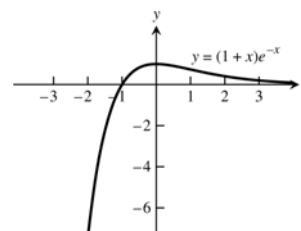


$$(b) \quad y = e^{-x^2} \Rightarrow y' = -2xe^{-x^2} \Rightarrow y'' = -2e^{-x^2} + 4x^2e^{-x^2} \\ = (4x^2 - 2)e^{-x^2}; \text{ solving } y' = 0 \Rightarrow x = 0; \quad y' < 0 \text{ for } x > 0 \text{ and}$$

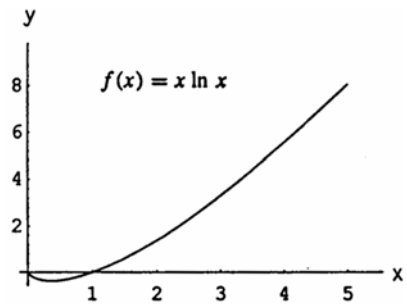
$$y' > 0 \text{ for } x < 0 \Rightarrow \text{a maximum at } x = 0 \text{ of } e^0 = 1; \text{ there are} \\ \text{points of inflection at } x = \pm \frac{1}{\sqrt{2}}; \text{ the curve is concave down} \\ \text{for } -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \text{ and concave up otherwise.}$$



$$(c) \quad y = (1+x)e^{-x} \Rightarrow y' = e^{-x} - (1+x)e^{-x} = -xe^{-x} \\ \Rightarrow y'' = -e^{-x} + xe^{-x} = (x-1)e^{-x}; \text{ solving} \\ y' = 0 \Rightarrow -xe^{-x} = 0 \Rightarrow x = 0; \quad y' < 0 \text{ for } x > 0 \text{ and } y' > 0 \text{ for} \\ x < 0 \Rightarrow \text{a maximum at } x = 0 \text{ of } (1+0)e^0 = 1; \text{ there is a point of} \\ \text{inflection at } x = 1 \text{ and the curve is concave up for } x > 1 \text{ and} \\ \text{concave down for } x < 1.$$



124. $y = x \ln x \Rightarrow y' = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$; solving $y' = 0 \Rightarrow \ln x + 1 = 0$
 $\Rightarrow \ln x = -1 \Rightarrow x = e^{-1}$; $y' > 0$ for $x > e^{-1}$ and $y' < 0$ for $x < e^{-1} \Rightarrow$
 a minimum of $e^{-1} \ln e^{-1} = -\frac{1}{e}$ at $x = e^{-1}$. This minimum is an
 absolute minimum since $y'' = \frac{1}{x}$ is positive for all $x > 0$.



125. $\frac{dy}{dx} = \sqrt{y} \cos^2 \sqrt{y} \Rightarrow \frac{dy}{\sqrt{y} \cos^2 \sqrt{y}} = dx \Rightarrow 2 \tan \sqrt{y} = x + C \Rightarrow y = \left(\tan^{-1} \left(\frac{x+C}{2} \right) \right)^2$
126. $y' = \frac{3y(x+1)^2}{y-1} \Rightarrow \frac{(y-1)}{y} dy = 3(x+1)^2 dx \Rightarrow y - \ln y = (x+1)^3 + C$
127. $yy' = \sec(y^2) \sec^2 x \Rightarrow \frac{y dy}{\sec(y^2)} = \sec^2 x dx \Rightarrow \frac{\sin(y^2)}{2} = \tan x + C \Rightarrow \sin(y^2) = 2 \tan x + C_1$
128. $y \cos^2(x) dy + \sin x dx = 0 \Rightarrow y dy = -\frac{\sin x}{\cos^2(x)} dx \Rightarrow \frac{y^2}{2} = -\frac{1}{\cos(x)} + C \Rightarrow y = \pm \sqrt{\frac{-2}{\cos(x)} + C_1}$
129. $\frac{dy}{dx} = e^{-x-y-2} \Rightarrow e^y dy = e^{-(x+2)} dx \Rightarrow e^y = -e^{-(x+2)} + C$. We have $y(0) = -2$, so $e^{-2} = -e^{-2} + C \Rightarrow C = 2e^{-2}$ and
 $e^y = -e^{-(x+2)} + 2e^{-2} \Rightarrow y = \ln(-e^{-(x+2)} + 2e^{-2})$
130. $\frac{dy}{dx} = \frac{y \ln y}{1+x^2} \Rightarrow \frac{dy}{y \ln y} = \frac{dx}{1+x^2} \Rightarrow \ln(\ln y) = \tan^{-1}(x) + C \Rightarrow y = e^{e^{\tan^{-1}(x)+C}}$. We have $y(0) = e^2 \Rightarrow e^2 = e^{e^{\tan^{-1}(0)+C}}$
 $\Rightarrow e^{\tan^{-1}(0)+C} = 2 \Rightarrow \tan^{-1}(0) + C = \ln 2 \Rightarrow 0 + C = \ln 2 \Rightarrow C = \ln 2 \Rightarrow y = e^{e^{\tan^{-1}(x)+\ln 2}}$
131. $x dy - (y + \sqrt{y}) dx = 0 \Rightarrow \frac{dy}{(y+\sqrt{y})} = \frac{dx}{x} \Rightarrow 2 \ln(\sqrt{y} + 1) = \ln x + C$. We have $y(1) = 1 \Rightarrow 2 \ln(\sqrt{1} + 1) = \ln 1 + C$
 $\Rightarrow 2 \ln 2 = C = \ln 2^2 = \ln 4$. So $2 \ln(\sqrt{y} + 1) = \ln x + \ln 4 = \ln(4x) \Rightarrow \ln(\sqrt{y} + 1) = \frac{1}{2} \ln(4x) = \ln(4x)^{1/2}$
 $\Rightarrow e^{\ln(\sqrt{y}+1)} = e^{\ln(4x)^{1/2}} \Rightarrow \sqrt{y} + 1 = 2\sqrt{x} \Rightarrow y = (2\sqrt{x} - 1)^2$
132. $y^{-2} \frac{dx}{dy} = \frac{e^x}{e^{2x}+1} \Rightarrow \frac{e^{2x}+1}{e^x} dx = \frac{dy}{y^{-2}} \Rightarrow \frac{y^3}{3} = e^x - e^{-x} + C$. We have $y(0) = 1 \Rightarrow \frac{(1)^3}{3} = e^0 - e^0 + C \Rightarrow C = \frac{1}{3}$. So
 $\frac{y^3}{3} = e^x - e^{-x} + \frac{1}{3} \Rightarrow y^3 = 3(e^x - e^{-x}) + 1 \Rightarrow y = \left[3(e^x - e^{-x}) + 1 \right]^{1/3}$
133. Since the half life is 5700 years and $A(t) = A_0 e^{kt}$ we have $\frac{A_0}{2} = A_0 e^{5700k} \Rightarrow \frac{1}{2} = e^{5700k} \Rightarrow \ln(0.5) = 5700k$
 $\Rightarrow k = \frac{\ln(0.5)}{5700}$. With 10% of the original carbon-14 remaining we have $0.1A_0 = A_0 e^{\frac{\ln(0.5)}{5700}t} \Rightarrow 0.1 = e^{\frac{\ln(0.5)}{5700}t}$
 $\Rightarrow \ln(0.1) = \frac{\ln(0.5)}{5700}t \Rightarrow t = \frac{(5700)\ln(0.1)}{\ln(0.5)} \approx 18,935$ years (rounded to the nearest year).

134. $T - T_s = (T_o - T_s)e^{-kt} \Rightarrow 82 - 5 = (104 - 5)e^{-k/4}$, time in hours, $\Rightarrow k = -4 \ln\left(\frac{7}{9}\right) = 4 \ln\left(\frac{9}{7}\right)$
 $\Rightarrow 21 - 5 = (104 - 5)e^{-4 \ln(9/7)t} \Rightarrow t = \frac{\ln 6}{4 \ln(\frac{9}{7})} \approx 1.78 \text{ h} \approx 107 \text{ min}$, the total time \Rightarrow the time it took to cool from
 82°C to 21°C was $107 - 15 = 92 \text{ min}$
135. $\theta = \pi - \cot^{-1}\left(\frac{x}{60}\right) - \cot^{-1}\left(\frac{5}{3} - \frac{x}{30}\right)$, $0 < x < 50 \Rightarrow \frac{d\theta}{dx} = \frac{\left(\frac{1}{60}\right)}{1 + \left(\frac{x}{60}\right)^2} + \frac{\left(-\frac{1}{30}\right)}{1 + \left(\frac{50-x}{30}\right)^2} = 30 \left[\frac{2}{60^2 + x^2} - \frac{1}{30^2 + (50-x)^2} \right]$; solving
 $\frac{d\theta}{dx} = 0 \Rightarrow x^2 - 200x + 3200 = 0 \Rightarrow x = 100 \pm 20\sqrt{17}$, but $100 + 20\sqrt{17}$ is not in the domain; $\frac{d\theta}{dx} > 0$ for
 $x < 20(5 - \sqrt{17})$ and $\frac{d\theta}{dx} < 0$ for $20(5 - \sqrt{17}) < x < 50 \Rightarrow x = 20(5 - \sqrt{17}) \approx 17.54 \text{ m}$ maximizes θ
136. $v = x^2 \ln\left(\frac{1}{x}\right) = x^2(\ln 1 - \ln x) = -x^2 \ln x \Rightarrow \frac{dv}{dx} = -2x \ln x - x^2\left(\frac{1}{x}\right) = -x(2 \ln x + 1)$; solving
 $\frac{dv}{dx} = 0 \Rightarrow 2 \ln x + 1 = 0 \Rightarrow \ln x = -\frac{1}{2} \Rightarrow x = e^{-1/2}$; $\frac{dv}{dx} < 0$ for $x > e^{-1/2}$ and $\frac{dv}{dx} > 0$ for $x < e^{-1/2} \Rightarrow$ a relative
maximum at $x = e^{-1/2}$; $\frac{r}{h} = x$ and $r = 1 \Rightarrow h = e^{1/2} = \sqrt{e} \approx 1.65 \text{ cm}$

CHAPTER 7 ADDITIONAL AND ADVANCED EXERCISES

- $\lim_{b \rightarrow 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}} dx = \lim_{b \rightarrow 1^-} \left[\sin^{-1} x \right]_0^b = \lim_{b \rightarrow 1^-} (\sin^{-1} b - \sin^{-1} 0) = \lim_{b \rightarrow 1^-} (\sin^{-1} b - 0) = \lim_{b \rightarrow 1^-} \sin^{-1} b = \frac{\pi}{2}$
- $\lim_{x \rightarrow \infty} \frac{1}{x} \int_0^x \tan^{-1} t dt = \lim_{x \rightarrow \infty} \frac{\int_0^x \tan^{-1} t dt}{x}$, $\frac{\infty}{\infty}$ form
 $= \lim_{x \rightarrow \infty} \frac{\tan^{-1} x}{1} = \frac{\pi}{2}$
- $y = (\cos \sqrt{x})^{1/x} \Rightarrow \ln y = \frac{1}{x} \ln(\cos \sqrt{x})$ and $\lim_{x \rightarrow 0^+} \frac{\ln(\cos \sqrt{x})}{x} = \lim_{x \rightarrow 0^+} \frac{-\sin \sqrt{x}}{2\sqrt{x} \cos \sqrt{x}} = -\frac{1}{2} \lim_{x \rightarrow 0^+} \frac{\tan \sqrt{x}}{\sqrt{x}}$
 $= -\frac{1}{2} \lim_{x \rightarrow 0^+} \frac{\frac{1}{2} x^{-1/2} \sec^2 \sqrt{x}}{\frac{1}{2} x^{-1/2}} = -\frac{1}{2} \Rightarrow \lim_{x \rightarrow 0^+} (\cos \sqrt{x})^{1/x} = e^{-1/2} = \frac{1}{\sqrt{e}}$
- $y = (x + e^x)^{2/x} \Rightarrow \ln y = \frac{2 \ln(x + e^x)}{x} \Rightarrow \lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{2(1 + e^x)}{x + e^x} = \lim_{x \rightarrow \infty} \frac{2e^x}{1 + e^x} = \lim_{x \rightarrow \infty} \frac{2e^x}{e^x} = 2$
 $\Rightarrow \lim_{x \rightarrow \infty} (x + e^x)^{2/x} = \lim_{x \rightarrow \infty} e^y = e^2$
- $\lim_{x \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} \right) = \lim_{x \rightarrow \infty} \left(\left(\frac{1}{n} \right) \left[\frac{1}{1 + (\frac{1}{n})} \right] + \left(\frac{1}{n} \right) \left[\frac{1}{1 + 2(\frac{1}{n})} \right] + \dots + \left(\frac{1}{n} \right) \left[\frac{1}{1 + n(\frac{1}{n})} \right] \right)$ which can be interpreted as a
Riemann sum with partitioning $\Delta x = \frac{1}{n} \Rightarrow \lim_{x \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} \right) = \int_0^1 \frac{1}{1+x} dx = [\ln(1+x)]_0^1 = \ln 2$

$$6. \lim_{x \rightarrow \infty} \frac{1}{n} [e^{1/n} + e^{2/n} + \dots + e] = \lim_{x \rightarrow \infty} \left[\left(\frac{1}{n} \right) e^{(1/n)} + \left(\frac{1}{n} \right) e^{2(1/n)} + \dots + \left(\frac{1}{n} \right) e^{n(1/n)} \right] \text{ which can be interpreted as a}$$

$$\text{Riemann sum with partitioning } \Delta x = \frac{1}{n} \Rightarrow \lim_{x \rightarrow \infty} \frac{1}{n} [e^{1/n} + e^{2/n} + \dots + e] = \int_0^1 e^x dx = [e^x]_0^1 = e - 1$$

$$7. A(t) = \int_0^t e^{-x} dx = [-e^{-x}]_0^t = 1 - e^{-t}, V(t) = \pi \int_0^t e^{-2x} dx = \left[-\frac{\pi}{2} e^{-2x} \right]_0^t = \frac{\pi}{2} (1 - e^{-2t})$$

$$(a) \lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} (1 - e^{-t}) = 1$$

$$(b) \lim_{t \rightarrow \infty} \frac{V(t)}{A(t)} = \lim_{t \rightarrow \infty} \frac{\frac{\pi}{2} (1 - e^{-2t})}{1 - e^{-t}} = \frac{\pi}{2}$$

$$(c) \lim_{t \rightarrow 0^+} \frac{V(t)}{A(t)} = \lim_{t \rightarrow 0^+} \frac{\frac{\pi}{2} (1 - e^{-2t})}{1 - e^{-t}} = \lim_{t \rightarrow 0^+} \frac{\frac{\pi}{2} (1 - e^{-t})(1 + e^{-t})}{(1 - e^{-t})} = \lim_{t \rightarrow 0^+} \frac{\pi}{2} (1 + e^{-t}) = \pi$$

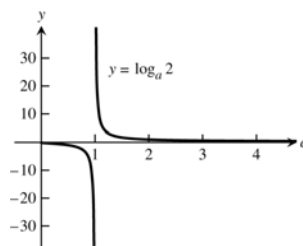
$$8. (a) \lim_{a \rightarrow 0^+} \log_a 2 = \lim_{a \rightarrow 0^+} \frac{\ln 2}{\ln a} = 0;$$

$$\lim_{a \rightarrow 1^-} \log_a 2 = \lim_{a \rightarrow 1^-} \frac{\ln 2}{\ln a} = -\infty;$$

$$\lim_{a \rightarrow 1^+} \log_a 2 = \lim_{a \rightarrow 1^+} \frac{\ln 2}{\ln a} = \infty;$$

$$\lim_{a \rightarrow \infty} \log_a 2 = \lim_{a \rightarrow \infty} \frac{\ln 2}{\ln a} = 0$$

(b)



$$9. A_1 = \int_1^e \frac{2 \log_2 x}{x} dx = \frac{2}{\ln 2} \int_1^e \frac{\ln x}{x} dx = \left[\frac{(\ln x)^2}{\ln 2} \right]_1^e = \frac{1}{\ln 2}; A_2 = \int_1^e \frac{2 \log_4 x}{x} dx = \frac{2}{\ln 4} \int_1^e \frac{\ln x}{x} dx = \left[\frac{(\ln x)^2}{2 \ln 2} \right]_1^e = \frac{1}{2 \ln 2}$$

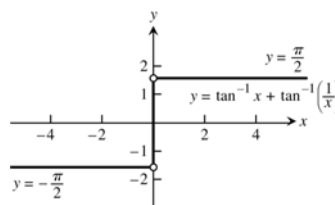
$$\Rightarrow A_1 : A_2 = 2 : 1$$

$$10. y = \tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right) \Rightarrow y' = \frac{1}{1+x^2} + \frac{\left(-\frac{1}{x^2} \right)}{\left(1 + \frac{1}{x^2} \right)} = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \Rightarrow \tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right) \text{ is a constant}$$

and the constant is $\frac{\pi}{2}$ for $x > 0$; it is $-\frac{\pi}{2}$ for $x < 0$

since $\tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right)$ is odd. Next the

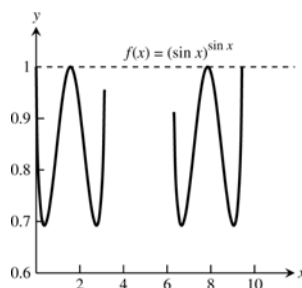
$$\lim_{x \rightarrow 0^+} \left[\tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right) \right] = 0 + \frac{\pi}{2} = \frac{\pi}{2} \text{ and } \lim_{x \rightarrow 0^-} \left(\tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right) \right) = 0 + \left(-\frac{\pi}{2} \right) = -\frac{\pi}{2}$$



$$11. \ln x^{(x^x)} = x^x \ln x \text{ and } \ln(x^x)^x = x \ln x^x = x^2 \ln x; \text{ then, } x^x \ln x = x^2 \ln x \Rightarrow (x^x - x^2) \ln x = 0 \Rightarrow x^x = x^2 \text{ or}$$

$$\ln x = 0. \ln x = 0 \Rightarrow x = 1; x^x = x^2 \Rightarrow x \ln x = 2 \ln x \Rightarrow x = 2. \text{ Therefore, } x^{(x^x)} = (x^x)^x \text{ when } x = 2 \text{ or } x = 1.$$

12. In the interval $\pi < x < 2\pi$ the function $\sin x < 0 \Rightarrow (\sin x)^{\sin x}$ is not defined for all values in that interval or its translation by 2π .



13. $f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)} g'(x)$, where $g'(x) = \frac{x}{1+x^4} \Rightarrow f'(2) = e^0 \left(\frac{2}{1+16} \right) = \frac{2}{17}$
14. (a) $\frac{df}{dx} = \frac{2 \ln e^x}{e^x} \cdot e^x = 2x$ (b) $f(0) = \int_1^1 \frac{2 \ln t}{t} dt = 0$
- (c) $\frac{df}{dx} = 2x \Rightarrow f(x) = x^2 + C; f(0) = 0 \Rightarrow C = 0 \Rightarrow f(x) = x^2 \Rightarrow$ the graph of $f(x)$ is a parabola
15. (a) $g(x) + h(x) = 0 \Rightarrow g(x) = -h(x)$; also $g(x) + h(x) = 0 \Rightarrow g(-x) + h(-x) = 0 \Rightarrow g(x) - h(x) = 0 \Rightarrow g(x) = h(x)$; therefore $-h(x) = h(x) \Rightarrow h(x) = 0 \Rightarrow g(x) = 0$
- (b) $\frac{f(x)+f(-x)}{2} = \frac{[f_E(x)+f_O(x)]+[f_E(-x)+f_O(-x)]}{2} = \frac{f_E(x)+f_O(x)+f_E(x)-f_O(x)}{2} = f_E(x)$;
 $\frac{f(x)-f(-x)}{2} = \frac{[f_E(x)+f_O(x)]-[f_E(-x)+f_O(-x)]}{2} = \frac{f_E(x)+f_O(x)-f_E(x)+f_O(x)}{2} = f_O(x)$
- (c) Part b \Rightarrow such a decomposition is unique.
16. (a) $g(0+0) = \frac{g(0)+g(0)}{1-g(0)g(0)} \Rightarrow [1-g^2(0)]g(0) = 2g(0) \Rightarrow g(0)-g^3(0) = 2g(0) \Rightarrow g^3(0)+g(0) = 0 \Rightarrow g(0)[g^2(0)+1] = 0 \Rightarrow g(0) = 0$
- (b) $g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h)-g(x)}{h} = \lim_{h \rightarrow 0} \frac{\left[\frac{g(x)+g(h)}{1-g(x)g(h)} \right] - g(x)}{h} = \lim_{h \rightarrow 0} \frac{g(x)+g(h)-g(x)+g^2(x)g(h)}{h[1-g(x)g(h)]}$
 $= \lim_{h \rightarrow 0} \left[\frac{g(h)}{h} \right] \left[\frac{1+g^2(x)}{1-g(x)g(h)} \right] = 1 \cdot [1+g^2(x)] = 1+g^2(x) = 1+[g(x)]^2$
- (c) $\frac{dy}{dx} = 1+y^2 \Rightarrow \frac{dy}{1+y^2} = dx \Rightarrow \tan^{-1} y = x+C \Rightarrow \tan^{-1}(g(x)) = x+C; g(0) = 0 \Rightarrow \tan^{-1} 0 = 0+C \Rightarrow C = 0 \Rightarrow \tan^{-1}(g(x)) = x \Rightarrow g(x) = \tan x$
17. $M = \int_0^1 \frac{2}{1+x^2} dx = 2 \left[\tan^{-1} x \right]_0^1 = \frac{\pi}{2}$ and $M_y = \int_0^1 \frac{2x}{1+x^2} dx = \left[\ln(1+x^2) \right]_0^1 = \ln 2 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{\ln 2}{(\frac{\pi}{2})} = \frac{\ln 4}{\pi}; \bar{y} = 0$ by symmetry

$$\begin{aligned}
 18. \quad (a) \quad V &= \pi \int_{1/4}^4 \left(\frac{1}{2\sqrt{x}} \right)^2 dx = \frac{\pi}{4} \int_{1/4}^4 \frac{1}{x} dx = \frac{\pi}{4} [\ln |x|]_{1/4}^4 = \frac{\pi}{4} \left(\ln 4 - \ln \frac{1}{4} \right) = \frac{\pi}{4} \ln 16 = \frac{\pi}{4} \ln (2^4) = \pi \ln 2 \\
 (b) \quad M_y &= \int_{1/4}^4 x \left(\frac{1}{2\sqrt{x}} \right) dx = \frac{1}{2} \int_{1/4}^4 x^{1/2} dx = \left[\frac{1}{3} x^{3/2} \right]_{1/4}^4 = \left(\frac{8}{3} - \frac{1}{24} \right) = \frac{64-1}{24} = \frac{63}{24}; \\
 M_x &= \int_{1/4}^4 \frac{1}{2} \left(\frac{1}{2\sqrt{x}} \right) \left(\frac{1}{2\sqrt{x}} \right) dx = \frac{1}{8} \int_{1/4}^4 \frac{1}{x} dx = \left[\frac{1}{8} \ln |x| \right]_{1/4}^4 = \frac{1}{8} \ln 16 = \frac{1}{2} \ln 2; \\
 M &= \int_{1/4}^4 \frac{1}{2\sqrt{x}} dx = \int_{1/4}^4 \frac{1}{2} x^{-1/2} dx = \left[x^{1/2} \right]_{1/4}^4 = 2 - \frac{1}{2} = \frac{3}{2}; \text{ therefore, } \bar{x} = \frac{M_y}{M} = \left(\frac{63}{24} \right) \left(\frac{2}{3} \right) = \frac{21}{12} = \frac{7}{4} \text{ and} \\
 \bar{y} &= \frac{M_x}{M} = \left(\frac{1}{2} \ln 2 \right) \left(\frac{2}{3} \right) = \frac{\ln 2}{3}
 \end{aligned}$$

$$\begin{aligned}
 19. \quad (a) \quad L &= k \left(\frac{a-b \cot \theta}{R^4} + \frac{b \csc \theta}{r^4} \right) \Rightarrow \frac{dL}{d\theta} = k \left(\frac{b \csc^2 \theta}{R^4} - \frac{b \csc \theta \cot \theta}{r^4} \right); \text{ solving } \frac{dL}{d\theta} = 0 \\
 &\Rightarrow r^4 b \csc^2 \theta - b R^4 \csc \theta \cot \theta = 0 \Rightarrow (b \csc \theta) (r^4 \csc \theta - R^4 \cot \theta) = 0; \text{ but } b \csc \theta \neq 0 \text{ since} \\
 &\theta \neq \frac{\pi}{2} \Rightarrow r^4 \csc \theta - R^4 \cot \theta = 0 \Rightarrow \cos \theta = \frac{r^4}{R^4} \Rightarrow \theta = \cos^{-1} \left(\frac{r^4}{R^4} \right), \text{ the critical value of } \theta \\
 (b) \quad \theta &= \cos^{-1} \left(\frac{5}{6} \right)^4 \approx \cos^{-1}(0.48225) \approx 61^\circ
 \end{aligned}$$

$$\begin{aligned}
 20. \quad &\text{In order to maximize the amount of sunlight, we need to maximize the angle } \theta \text{ formed by extending the two red} \\
 &\text{line segments to their vertex. The angle between the two lines is given by } \theta = \pi - (\theta_1 + (\pi - \theta_2)). \text{ From trig we} \\
 &\text{have } \tan \theta_1 = \frac{105}{135-x} \Rightarrow \theta_1 = \tan^{-1} \left(\frac{105}{135-x} \right) \text{ and } \tan(\pi - \theta_2) = \frac{60}{x} \Rightarrow (\pi - \theta_2) = \tan^{-1} \left(\frac{60}{x} \right) \\
 &\Rightarrow \theta = \pi - (\theta_1 + (\pi - \theta_2)) = \pi - \tan^{-1} \left(\frac{105}{135-x} \right) - \tan^{-1} \left(\frac{60}{x} \right) \\
 &\Rightarrow \frac{d\theta}{dx} = -\frac{1}{1 + \left(\frac{105}{135-x} \right)^2} \cdot \frac{105}{(135-x)^2} - \frac{1}{1 + \left(\frac{60}{x} \right)^2} \cdot \left(-\frac{60}{x^2} \right) = \frac{-105}{(135-x)^2 + 11,025} + \frac{60}{x^2 + 3600} \\
 &\frac{d\theta}{dx} = 0 \Rightarrow \frac{-105}{(135-x)^2 + 11,025} + \frac{60}{x^2 + 3600} = 0 \Rightarrow 60 \left((135-x)^2 + 11,025 \right) = 105(x^2 + 3600) \\
 &\Rightarrow x^2 + 360x - 30,600 = 0 \Rightarrow x = -180 \pm 30\sqrt{70}. \text{ Since } x > 0, \text{ consider only } x = -180 + 30\sqrt{70}. \text{ Using the first} \\
 &\text{derivative test, } \left. \frac{d\theta}{dx} \right|_{x=30} = \frac{9}{1050} > 0 \text{ and } \left. \frac{d\theta}{dx} \right|_{x=120} = \frac{-9}{1500} < 0 \Rightarrow \text{local max when } x = -180 + 30\sqrt{70} \approx 71 \text{ m.}
 \end{aligned}$$

