Chaos and long-term behavior questions

Ch. 9 of Murray and Dermott

First half

Q1. Before reading, give a definition of what you think chaos is, then update it as you go through the reading.

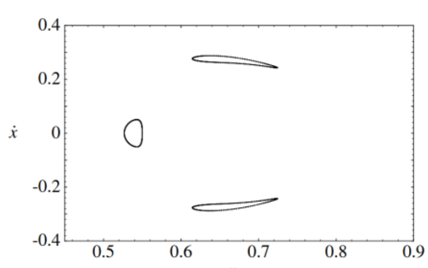
A1. Sensitivity to initial conditions (small differences in initial conditions lead to diverging final states of a solution). From the text: "An object in the solar system can be said to exhibit chaotic motion if its final dynamical state is sensitively dependent on its initial state."

Q2. How does the existence of the Jacobi constant reduce the dimensionality of phase space in the circular restricted three-body problem, and why is this reduction essential for constructing a Poincaré surface of section? What's the physical meaning of a jacobi surface? [p.414-415]

A2. Because only three variables need to be determined, since the Jacobi constant comes from the integral.

The set of all positions and velocities that a test particle is *allowed* to have for a fixed value of the Jacobi constant in the circular restricted three-body problem. Because the Jacobi constant combines kinetic energy and an effective potential in the rotating frame, the Jacobi surface acts like an energy constraint: the particle's motion is confined to this surface and cannot access regions of phase space that would violate that energy balance. It fixes where the particle can move and how fast it can move.

Q3. **What do the islands in Fig 9.5 symbolize physically and what happens when we place the starting condition further from the center of the island that straddles $\dot{x} = 0$? [p.417-418]**



A3. By moving the starting location further away from the centre the islands would get larger, corresponding to larger variations in e and a. Eventually some starting values would lead to trajectories that were not in resonant motion and there would no longer be distinct islands in the section plot.

Q4. Why does the Pointcarré surface in Fig 9.7 tell you that the orbit is chaotic? Why is 'the trajectories having no obvious pattern in the variation of the orbital elements' not enough to deem it as chaotic? [417]

A4. The Poincaré surface in Fig. 9.7 shows the orbit is chaotic because the points fill a two-dimensional area instead of lying on a smooth curve, indicating the motion explores a large region of phase space. Irregular variations in orbital elements alone are not enough to identify chaos, since quasi-periodic or secular motion can also appear irregular; the surface of section reveals whether the motion truly spreads across phase space.

Q5. What is the maximum Lyapounov characteristic exponent and what does it tell you? [418]

$$d = d_0 \exp \gamma (t - t_0),$$

The maximum Lyapunov characteristic exponent (LCE) measures the rate at which nearby trajectories in phase space diverge. A positive LCE indicates chaos, meaning small differences in initial conditions grow exponentially over time, while a zero or negative LCE corresponds to regular, stable motion. It provides a quantitative measure of sensitive dependence on initial conditions.

Q6.

Second half