PHYS 633 reading 7 - Chaos and Long-term Dynamics of the Solar System

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This week was meant to give a general sense for what chaos is and how it can be applied to our own Solar System dynamics.

1 Before reading, give a definition of what you think chaos is, then update it as you go through the reading.

Chaos is a word used to describe systems whose outcomes are extremely sensitive to initial conditions. Despite this, the system remains deterministic, meaning the future state of a system is completely determined by its current state and the laws governing it (so nothing random!). However any small deviations from initial conditions can lead to exponentially different evolutions. With regards to the Solar System, the text explicitly states: "An object in the solar system can be said to exhibit chaotic motion if its final dynamical state is sensitively dependent on its initial state."

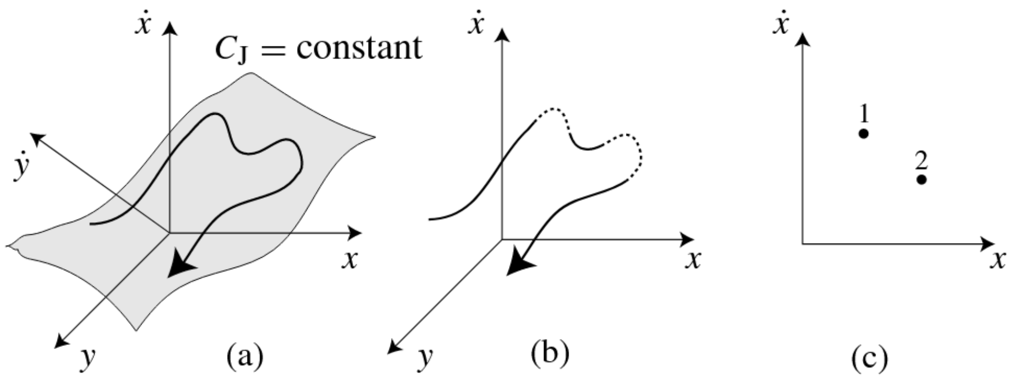
2 (a) How does the existence of the Jacobi constant reduce the dimensionality of phase space in the circular restricted three-body problem, and why is this reduction essential for constructing a Poincaré surface of section? (b) What's the physical meaning of a Jacobi surface?

a) Recall from Chapter 3 of Murray and Dermott for the case of a planar, circular restricted three-body problem, there exists a constant of motion defined by

$$C_J = n^2(x^2 + y^2) + 2\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) - \dot{x}^2 - \dot{y}^2.$$

Because of the existence of the Jacobi constant, which can be fixed, only 3 variables need to be determined, and the 4th can be extracted from the above equation. That is, any permutaion of C_J

with three of $(x, y, \dot{x}, \dot{y})$. This reduction in dimensionality means that the path of the particle in phase space is restricted to live on a surface[1]. Now this reduction allows us to define the Pointcaré surface of section. This is done by choosing a Jacobi constant to set up a surface, and plotting points only when they cross this surface. So we have gone from R^4(phase space path of particle) $\rightarrow$ R^3(Jacobi surface) $\rightarrow$ R^2(Poicaré surface of section, or points). This is useful in illustrating the regular and chaotic regions of an orbit.



b) Mathematically, its the set of all positions and velocities that a test particle is allowed to have for a fixed value of the Jacobi constant in the circular restricted three-body problem. Because the Jacobi constant combines kinetic energy and an effective potential in the rotating frame, the Jacobi surface acts like an energy constraint: the particle's motion is confined to this surface and cannot access regions of phase space that would violate that energy balance. It fixes where the particle can move and how fast it can move.

3 What do the islands in Fig 9.5 symbolize physically and what happens when we place the starting condition further from the center of the island that straddles $\dot{x} = 0$?

The islands are a characteristic of resonant motion. In the case where we have a mean motion resonance of $p+q : p$, we would have q islands. The center of each island would be a stable periodic orbit in exact resonance, while the surrounding closed curves correspond to quasi-periodic librations about that periodic configuration. Physically, the islands represent regions of stable resonant trapping where conjunctions occur in a repeating geometric pattern and the motion is dynamically protected from chaotic diffusion. Its interesting to note that points appear successively at one different island every time, they don't 'trace out' one island, then move to the next.

[1] Think of how we defined the equation for a surface in linear algebra. For the linear case, it is given by $ax+by+cz = d$ where (a,b,c,d) are constants. We have a similar situation here except it's a 3D surface embedded in 4D space.

As alluded to above, placing the particle at the staring condition for the island that straddles $\dot{x} = 0$ yields a trajectory that would appear as a succession of three points, one at the centre of each island in turn. This is because the centre of each island corresponds to a starting condition that places the test particle at the middle of the resonance. By moving the starting location further away from the centre, the islands would get larger, corresponding to larger variations in e and a.

4 What is the maximum Lyapounov characteristic exponent and what does it tell you?

For two orbits separated in phase space by a distance d_0 at time t_0. Let d be their separation at time t. The orbit is chaotic if d is approximately related to d_0 by the **Lyapounov exponent** γ in

$$d = d_0 \ \mathrm{exp} \gamma (t - t_0).$$

The maximum Lyapunov characteristic exponent (LCE) measures the rate at which nearby trajectories in phase space diverge. A positive LCE indicates chaos, meaning small differences in initial conditions grow exponentially over time, while a zero or negative LCE corresponds to regular, stable motion. It provides a quantitative measure of sensitive dependence on initial conditions.

5 What is problematic with modeling long term solar system dynamics with a perturbation expansion?

The problem with the perturbation expansion is that although the expansion is done in powers of small parameters, the existence of resonances between the planets will introduce small divisors into the expansion terms. Such small divisors can make high order ter ms in the power series unexpectedly large and destroy the convergence of the series.

6 This review paper dates from 1993 and states that 'a detailed understanding of Pluto's behavior is likely to be obtained only with the next generation of simulations'. From you knowledge has this been done? (You can also quickly search this up).

One of the many examples: In 2016, Japanese physicists published a paper titles 'The motion of Pluto over the age of the solar system' where they carried out the numerical integration of Pluto over the age of the solar system (5.7 billion years towards the past and 5.5 billion years towards the future). They found that the time evolution of Keplerian elements of a nearby trajectory of Pluto at first grow linearly with the time and then start to increase exponentially. These exponential divergences stop at about 420 Myr and saturate. The exponential divergences are suppressed by three resonances that Pluto has.

7 What is the general consensus of the numerical simulations mentioned in this chapter? Does the solar system exhibit long-term stability?

The genral concensus it that the Solar System exhibits chaotic behavior, particularly in the inner planets, with Lyapunov timescales of a few million years (Laskar 1989, 1990; Sussman & Wisdom 1992). The outer planets and Pluto also show signs of chaos, though on slightly longer timescales, and the chaotic behavior of Pluto appears largely independent of the giant planets.

At the same time, the Solar System has survived for billions of years, suggesting that the chaotic regions are narrow and the system is globally stable, at least over the timescales explored in the simulations. In other words, the Solar System is probabilistically stable: most configurations remain regular, but precise long-term predictions are limited by chaos.

This paper insinuates that this remains an open ended question. but in the past decades, better computing power has allowed us to numerically integrate the solar system for billions of years. For example, Laskar and Gastineau (2009) found that in about 1% of 2501 simulated Solar System trajectories, Mercury's eccentricity grew large enough to allow a collision with Venus or the Sun, and in one case, this triggered instabilities among all the terrestrial planets.