1. (4pts) With $x_1 = 0$, $x_2 = 1$, and $x_3 = 2$, determine constants w_1, w_2, and w_3, such that $\sum_{i=1}^{3} w_i p(x_i) = \int_0^2 p(t)\,dt$ for all polynomials of degree ≤ 2. Show that this quadrature formula is even exact for all polynomials of degree ≤ 3.

2. (4pts) Compute by whatever means $x \in \mathbb{R}^2$ that minimizes $\left\| \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} x - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\|_2$

3. (6pts)

   (a) Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} -7 & 5 \\ -10 & 8 \end{bmatrix}$.

   (b) Given an explicit expression of the entries of A^n.

4. (4pts) Consider the system of equations $x^2 - y^2 = 1$, $x^3 + y = 6$. Formulate the Newton iteration to approximate a solution of this system, and compute one iteration starting from $(2, 1)$.

5. (4pts) Consider the problem to find the (local) minima of $x^3 + y^3$ under the condition that $x + y = 1$. Set up the Lagrangian, and determine its critical points.

6. (6pts)

   (a) Compute the LU-factorization of $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$. (Note that you can easily verify whether your answer is correct.)

   (b) Explain how an LU factorization of a matrix A is used to solve a system $Ax = b$.

   (c) Let A be $n \times n$. What is the leading term in the operation count for computing its LU factorization (ignore pivoting), and that for subsequently solving one system $Ax = b$?

7. (4pts) For a sufficiently differentiable $f\colon \mathbb{R} \to \mathbb{R}$ and $h > 0$, consider the approximation $\frac{f(x+h)-f(x)}{h}$ for $f'(x)$, where we take $f(x) = e^x$ and $x = 0$ (thus $f(0) = 1$).

   (a) Show that the error in this approximation is $\approx -\frac{h}{2}$.

   Taking h of the form 4^{-k}, using the finite precision data from the second row in the following table, we have computed the total computational error in the aforementioned approximation which is displayed in the third row of this table.

k	1	2	3	4	5
$f(4^{-k})$	1.2840	1.0645	1.0157	1.0039	1.0010
$f'(0) - \frac{f(4^{-k})-f(0)}{4^{-k}}$	–0.1360	-0.0321	-0.0048	0.0016	-0.0240

   (b) Give an upper bound for the total computational absolute error. Using this bound explain that the best results can be expected for $h \approx 0.01$.

1. (6pts)

   (a) Compute the QR factorization of $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$.

   (b) Compute the LU factorization of $\begin{bmatrix} 1 & 4 & -5 \\ 2 & 10 & -4 \\ 3 & 20 & 12 \end{bmatrix}$.

2. (6pts) Let $A := U\Sigma V^\top$, where $U \in \mathbb{R}^{3\times 3}$ is orthogonal, $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$, $V = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$, and $b := U\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.

   (a) Verify that V is orthogonal.

   (b) Find $x \in \mathbb{R}^2$ that minimizes $\|Ax - b\|_2$.

3. (4pts) Let A be a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ for which $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$. For some arbitrary $x^{(0)}$, let $x^{(i+1)} := Ax^{(i)}$. Explain why, in nearly all cases, $\lim_{i\to\infty} \frac{\|x^{(i+1)}\|}{\|x^{(i)}\|} = |\lambda_1|$.

4. (6pts) Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Let x^* be such that $g(x^*) = x^*$ and $|g'(x^*)| < 1$. Given some x_0, for $i = 0, 1, \ldots$ let $x_{i+1} := g(x_i)$.

   (a) Show that for any i, there exists a θ_i between x^* and x_i such that $x^* - x_{i+1} = g'(\theta_i)(x^* - x_i)$. (Hint: apply the mean value theorem).

   (b) Explain that there exists a neighborhood of x^* such that for any x_0 in this neighborhood, $\lim_{i\to\infty} x_i = x^*$, and $\lim_{i\to\infty} \frac{x^* - x_{i+1}}{x^* - x_i} = g'(x^*)$.

5. (4pts) Minimize $x + y + 3$ under the constraint that $x^2 + x + y^2 + y = 4$.

6. (6pts) We equip the space of continuous functions on $(-1, 1)$ with inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ and norm $\|f\| := \langle f, f\rangle^{\frac{1}{2}}$.

   (a) Verify that $\{\frac{1}{2}\sqrt{2}, \sqrt{3/2}\,x\}$ is an orthonomal set of functions on $(-1, 1)$.

   (b) Compute the best approximation on $(-1, 1)$ of x^2 by a linear polynomial.

   (c) Construct an orthonormal basis of the quadratic polynomials on $(-1, 1)$.

7. (4pts) Consider the scalar linear ODE $y'(t) = \lambda y(t)$ ($t \in [0, T]$), $y(0) = y_0$. Taking a uniform step size h with $T/h \in \mathbb{N}$, give the approximations for the solution at $t = T$ provided by the Euler forward and Euler backward methods. Which method do you prefer for $\lambda \ll 0$ and why?

Explain your answers

1. (6 pt) Compute the LU factorization of $\begin{bmatrix} 2 & 4 & -4 \\ 0 & -2 & -2 \\ 1 & 5 & 2 \end{bmatrix}$

2. (9 pt) Suppose you are computing the QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 0 & 1 \\ 3 & 3 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

by Householder transformations.

   (a) How many Householder transformations are required?

   (b) What is the relation between the Householder vector (often denoted by v) and the Householder matrix (often denoted by H)?

   (c) Determine the first Householder transformation. It is sufficient to determine the Householder vector v.

   (d) Consider a general linear least squares problem $Ax \cong b$, where A is an $m \times n$ matrix, $m > n$. How can this problem be solved, assuming that a QR factorization $A = QR$ is given?

3. (6 pt) Let $A = \begin{bmatrix} 3 & 1 & 4 & 7 \\ -2 & 5 & 1 & 0 \\ 1 & 4 & 0 & 4 \\ 2 & 2 & 3 & -1 \end{bmatrix}$.

   (a) In which locations of A does the transformation to Hessenberg form create zeros?

   (b) Determine the first Householder transformation of the transformation to Hessenberg form of A. (It is again sufficient to determine the Householder vector v.)

4. (9 pt) Let $f = 3x^3 - 2x^2 + 4x - 1$. Given are starting points $x_0 = 0$ and $x_1 = 1$.

   (a) Verify that the starting points form a bracket. Execute one step of the bisection method and give the updated bracket.

   (b) Execute one step of the secant method using the given starting points.

   (c) Argue that the bisection method is linearly convergent.

5. (6 pt) Find the interpolating polynomial through the data points $(-1, 1), (0, 3), (1, 2), (3, 1)$. You may use any of the forms covered in the course.

6. (9 pt) Consider the ODE $y' = -10y$ with initial condition $y(0) = 1$. We will solve this ODE numerically with stepsize $h = 0.1$

   (a) Compute the value for the approximate solution at $t = 0.1$ using Euler's method

   (b) Compute the value for the approximate solution at $t = 0.1$ using the backward Euler method

   (c) Is Euler's method stable for this ODE using this stepsize?

Solutions exam 21-1-2019

C.C. Stolk

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1. The first part of this exercise, to find the w_i, refers to the discusion on pages 342-343 of the book, with $a = 0$, $b = 2$, $n = 3$ and $x_1 = 0$, $x_2 = 1$ and $x_2 = 2$.

   The weights must be such that the equality
   $$\int_0^2 f(t)\,dt = w_1 f(0) + w_2 f(1) + w_3 f(2).$$
   holds for $f(t) = 1$, $f(t) = t$ and $f(t) = t^2$. This results in the following equations
   $$\begin{aligned} w_1 \cdot 1 + w_2 \cdot 1 + w_3 \cdot 1 &= \int_0^2 1\,dt = 2 \\ w_1 \cdot 0 + w_2 \cdot 1 + w_3 \cdot 2 &= \int_0^2 t\,dt = \left[\frac{1}{2}t^2\right]_0^2 = 2 \\ w_1 \cdot 0 + w_2 \cdot 1 + w_3 \cdot 4 &= \int_0^2 t^2\,dt = \left[\frac{1}{3}t^3\right]_0^2 = \frac{8}{3} \end{aligned}$$
   It is easily seen that there is a unique solution given by
   $$w_1 = \frac{1}{3} \qquad w_2 = \frac{4}{3} \qquad w_3 = \frac{1}{3}.$$
   As mentioned in example 8.1 this is an instance of Simpson's rule.

   To show that this quadrature formula is exact also for all polynomials of degree ≤ 3 one can use that it is symmetric. The points x_i are symmetric around $x_2 = 1$. The weights have an "even" symmetry, because $w_1 = w_3$.

   The third order monomial t^3 has odd symmetry around $x = 0$.

   The polynomial $q_3 = (t-1)^3$ has odd symmetry around $t = 1$. Applying an "even" quadrature rule to an "odd" function one finds zero. Indeed for the quadrature rule we have $\sum_{i=1}^3 w_i q_3(x_i) = -w_1 + w_3 = 0$. For the integral we also get 0 easily
   $$\int_0^2 (t-1)^3\,dt = \int_{-1}^1 t^3\,dt = 0$$
   using transformation of variables in the first equality sign, and the fact that t^3 is odd and the domain is symmetri in the second equality. All third order polynomials can be written as
   $$p(t) = C(t-1)^3 + D(t)$$
   where $D(t)$ is a polynomial of degree 2. It follows that $\sum_{i=1}^3 w_i p(x_i) = \int_0^2 p(t)\,dt$ for all polynomials of degree 3.

2. This exercise refers to chapter 3 on linear least squares.

   It is relatively easy to write down the normal equations corresponding to this system. These are

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} x = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

   Working this out one finds

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} x = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

   For a 2×2 matrix there is the inversion formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

   Thus the solution is

$$x = \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 11 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/2 \end{bmatrix}$$

   Another methods to solve this question involve the QR decomposition of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$. We refer to book and the materials of the course for this.

3. This exercise is related to section 4.5.1 on power iteration.

   See the formula at the top of page 173.

   (a) The characteristic polynomial is

$$p(\lambda) = (-7 - \lambda)(8 - \lambda) + 50 = \lambda^2 - \lambda - 6$$

   The standard formula for roots of a second order polynomial gives the roots $\lambda_1 = -2$ and $\lambda_2 = 3$. (So the polynomial can be factorized as $p(\lambda) = (\lambda + 2)(\lambda - 3)$.) For the first eigenvalue, we determine a vector $v_1 = [x, y]^T$ such that

$$(A + 2I) \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

   Inserting the numerical values for A the equation becomes

$$\begin{bmatrix} -5 & 5 \\ -10 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

   with solution

$$v_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   For the second eigenvalue, we determine a vector $v_1 = [x, y]^T$ such that

$$(A - 3I) \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Inserting the numerical values for A the equation becomes

$$\begin{bmatrix} -10 & 5 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

with solution

$$v_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(b) We write down the diagonalization of A Let V be the matrix with columns v_1 and v_2, hence $V = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

Using the inversion formula above we find that

$$V^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Then

$$A = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V^{-1}$$

The expression for A^n can be written as

$$\left[A^n = (V\Lambda V^{-1})^n = V\Lambda V^{-1} V\Lambda V^{-1} V ... \Lambda V^{-1} = V \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} V^{-1}. \right]$$

The third equals sign is because the factors $V^{-1}V$ cancel (are equal to the identity matrix).

Thus we have

$$\begin{aligned} A^n &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} (-2)^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (-2)^n & 3^n \\ (-2)^n & 2 \cdot 3^n \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot (-2)^n - 3^n & -(-2)^n + 3^n \\ 2 \cdot (-2)^n - 2 \cdot 3^n & -(-2)^n + 2 \cdot 3^n \end{bmatrix} \end{aligned}$$

4. This exercise refers to chapter 5.

We define $f_1(x, y) = x^2 - y^2 - 1$, $f_2(x, y) = x^3 + y - 6$, and look for a zero of $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

The Jacobian is

$$J_f(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 3x^2 & 1 \end{bmatrix}$$

At $(2, 1)$ it equals

$$J_f(2, 1) = \begin{bmatrix} 4 & -2 \\ 12 & 1 \end{bmatrix}$$

Newton iteration iteratively searches for solutions. If $\begin{bmatrix} x_i \\ y_i \end{bmatrix}$ is the approximate solution after i iterations, the next iterate is (see page 238 of the book)

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} - J_f(x_i, y_i)^{-1} f(x_i, y_i)$$

The function values are $f(2,1) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The Newton step is given by

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} - J_f(2,1)^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

In this case

$$J_f(2,1)^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/5 \\ -3/5 \end{bmatrix}$$

So if $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 9/5 \\ 8/5 \end{bmatrix}$.

5. This question refers to second 6.2, in particular example 6.6 and the preceding theory.

   Let $f(x,y) = x^3 + y^3$ and $g(x,y) = x + y - 1$. We have $J_g(x,y) = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

   The Lagrangian is

   $$x^3 + y^3 + \lambda(x + y - 1)$$

   The system of equations to be solved (see page 265) is

   $$\begin{bmatrix} \nabla f(x,y) + J_g^T(x,y)\lambda \\ g(x,y) \end{bmatrix} = \begin{bmatrix} 3x^2 + \lambda \\ 3y^2 + \lambda \\ x + y - 1 \end{bmatrix} = 0$$

   Note that ∇f here denotes a column vector, $J_g^T(x,y)\lambda$ is a matrix vector product with matrix $J_g^T(x,y)$ of size $n \times m$ and vector λ of size m (notation of page 264), in this case $n = 2$ and $m = 1$.

   It follows from the first two equations that $3x^2 = 3y^2 = -\lambda$, and hence $x = \pm y$.

   From the third equation it follows that the case $x = -y$ is not possible, so $x = y$. Inserting this in the third equation, it follows that $x = y = \frac{1}{2}$. For $\lambda = -\frac{3}{4}$ the first two equations are also satisfied, this is the unique critical point of the Lagrangian.

6. (a) This is standard using Gaussian elimination. A good way of doing this is to follow example 2.13. First put zeros in the first column, positions 2 and 3:

   $$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}}_{\text{matrix } M_1} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}}_{\text{matrix } A} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}}_{\text{matrix } M_1 A}$$

   Next put a zero in the second column, position 3:

   $$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{\text{matrix } M_2} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}}_{\text{matrix } M_1 A} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}}_{\text{matrix } M_2 M_1 A}$$

   Now U is the matrix on the right of this equation:

   $$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

For L we have

$$L = L_1 L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

Note that L_1 is obtained from M_1 by changing the sign of the all the entries below the diagonal, and that the below-the-diagonal entries of L are simply all below-the-diagonal entries of the L_1 and L_2 combined (you don't have to compute a matrix product for the third equality sign).

If you want, you can organize the computation differently and write it down as follows

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \end{aligned}$$

So

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

(b) The system becomes $LUx = b$. Let $y = Ux$. First solve y from the equation $Ly = b$. This is done by back-substitution. Then solve x from the equation $y = Ux$, again by back-substitution (but now in the other direction). (See book page 64 for back-substitution for lower-triangular and for upper-triangular systems.)

(c) We discuss first the LU factorization. For $i = 1, 2, \ldots, n-1$ the coefficients of the submatrix below and to the right of the (i, i) entry is updated. For each entry a multiplication and an addition must be done. To get the leading term of the number of multiplications we must compute the sum

$$\left(1^2 + 2^2 + \ldots + (n-1)^2\right)$$

The leading term in this sum is $\frac{n^3}{3}$. This number is mentioned in the book by Heath.
(For the number of additions the leading term is the same.)
See wikipedia, "Square pyramidal number" for adding consecutive squares.
For backsubstitution, the number of multiplications is given by

$$(1 + 2 + \ldots + (n-1))$$

The leading term is $\frac{n^2}{2}$. Since two backsubstitutions are done to solve the system after LU factorization, the leading term of this part of the computation is n^2.

7. (a) A Taylor expansion gives $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + O(|h|^3)$. It follows that

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{h}{2} f''(x) + O(|h|^2).$$

The error is

$$e(x,h) = \frac{f(x+h) - f(x)}{h} - f'(x) = \frac{h}{2} f''(x) + O(|h|^2)$$

For $f(x) = e^x$, and $x = 0$, a simple computation shows that $f''(x) = 1$. Neglecting the higher order contributions, the error $\approx \frac{h}{2}$.

(b) This exercise refers to example 1.3.

For small h, the formula for the derivative contains the subtraction of two nearby numbers, namely $f(x+h)$ and $f(x)$ which may lead to large errors.

The true difference $f(x+h) - f(x)$ can be estimated by $hf'(x)$. The absolute error in $f(x)$ and in $f(x+h)$ can be estimated on $\epsilon_{\text{mach}} f(x)$, the absolute error in the difference can hence be estimated by $2\epsilon_{\text{mach}} f(x)$. With four digits after the decimal point, $\epsilon_{\text{mach}} = 0.00005$. The relative error due to finite precision arithmetic can be estimated by

$$\frac{2\epsilon_{\text{mach}} f(x)}{hf'(x)}$$

The truncation error, for the given function f, was already estimated by $\frac{h}{2}$. The total error can therefore be estimated by

$$\frac{h}{2} + \frac{2\epsilon_{\text{mach}} f(x)}{hf'(x)} = \frac{h}{2} + \frac{0.0001}{h}$$

where to get the second equality we inserted $f(x) = e^x$ and $x = 0$. To find the minimum of this function, set the derivative w.r.t. h equal to zero and solve. This yields

$$h = \sqrt{0.0002} = 0.0141....$$

For values of $h \approx 0.014$ the best values can be expected.