Supplementary Material for "Hybrid and Oriented Harmonic Potentials for Safe Task Execution in Unknown Environment"

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Abstract

This document contains the additional definitions, proofs, algorithmic and experiment details that are omitted in the main manuscript of "Hybrid and Oriented Harmonic Potentials for Safe Task Execution in Unknown Environment", due to limited space. References to the main manuscript including equations, definitions, lemmas, theorems are marked by "P[X]".

1. Definition of Scaling Matrix

$$a = \begin{cases} (g_{h,p(d_h)}[0] + \dfrac{a_{h,d_h}}{2}) - q_{h,d_h}[0],\ 0 \leqslant \Theta(q') < \dfrac{\pi}{2} \\ q_{h,d_h}[0] - (g_{h,p(d_h)}[0] - \dfrac{a_{h,d_h}}{2}),\ \dfrac{\pi}{2} \leqslant \Theta(q') < \pi \\ q_{h,d_h}[0] - (g_{h,p(d_h)}[0] - \dfrac{a_{h,d_h}}{2}), \pi \leqslant \Theta(q') < \dfrac{3\pi}{2} \\ (g_{h,p(d_h)}[0] + \dfrac{a_{h,d_h}}{2}) - q_{h,d_h}[0], \dfrac{3\pi}{2} \leqslant \Theta(q') < 2\pi \end{cases} \tag{1}$$

$$b = \begin{cases} (g_{h,p(d_h)}[1] + \dfrac{b_{h,d_h}}{2}) - q_{h,d_h}[1],\ 0 \leqslant \Theta(q') < \dfrac{\pi}{2} \\ q_{h,d_h}[1] - (g_{h,p(d_h)}[1] - \dfrac{b_{h,d_h}}{2}),\ \dfrac{\pi}{2} \leqslant \Theta(q') < \pi \\ q_{h,d_h}[1] - (g_{h,p(d_h)}[1] - \dfrac{b_{h,d_h}}{2}), \pi \leqslant \Theta(q') < \dfrac{3\pi}{2} \\ (g_{h,p(d_h)}[1] + \dfrac{b_{h,d_h}}{2}) - q_{h,d_h}[1], \dfrac{3\pi}{2} \leqslant \Theta(q') < 2\pi \end{cases} \tag{2}$$

where $\Theta(q') = \arctan 2(q') + \pi \in [0, 2\pi]$ represents the angle of q', $g_{h,p(d_h)}$ is the geometric center of the parent of leaf and q_{h,d_h} is the established same center of parent and son obstacle.

2. Online Computation of Harmonic Potentials

2.1. Independent Stars

Lemma 1. *Each time an independent obstacle $\mathcal{O}_{k+1}$ is added to the workspace, the online omitted product $\overline{\beta}_0^{k+1}(q)$ for $\mathcal{O}_0$ can be calculated using $\overline{\beta}_0^{k}(q)$ as:*

$$\overline{\beta}_0^{k+1}(q) = \overline{\beta}_0^{k}(q)\,\beta_{k+1}(q). \tag{3}$$

Proof. By Definition **??**, $\overline{\beta}_0^{k}(q) = \prod_{j=0,j\neq i}^{k}\beta_j(q)$ at step-k and $\overline{\beta}_0^{k+1}(q) = \prod_{j=0,j\neq i}^{k+1}\beta_j(q)$ at step-$(k+1)$. Comparing these two equations, it follows that $\overline{\beta}_0^{k+1}(q) = \overline{\beta}_0^{k}(q)\,\beta_{k+1}(q)$. □

Lemma 2. *Each time an independent obstacle $\mathcal{O}_{k+1}$ is added to the workspace, the online analytic switch $s_0^{k+1}(q)$ for $\mathcal{O}_0$ can be calculated using $s_0^k(q)$ as:*

$$s_0^{k+1}(q) = \frac{s_0^k(q)}{\alpha^k(q)\,[1 - s_0^k(q)] + s_0^k(q)}, \tag{4}$$

with $\alpha^k(q) = \lambda_{k+1}/[\lambda_k\,\beta_{k+1}(q)]$.

Proof. By Definition **??**, $s_0^k(q) = \frac{\gamma_G(q)\,\overline{\beta}_0^k(q)}{\lambda_k\,\beta_0(q)+\gamma_G(q)\,\overline{\beta}_0^k(q)}$, which can be rearranged as $\overline{\beta}_0^k(q) = \frac{\lambda_k\,s_0^k(q)\,\beta_0(q)}{[1-s_0^k(q)]\,\gamma_G(q)}$. Further by Definition **??** and Lemma 1, $s_0^{k+1}(q) = \frac{\gamma_G(q)\,\overline{\beta}_0^{k+1}(q)}{\lambda_{k+1}\,\beta_0(q)+\gamma_G(q)\,\overline{\beta}_0^{k+1}(q)} = \frac{\gamma_G(q)\,\beta_{k+1}(q)\,\overline{\beta}_0^k(q)}{\lambda_{k+1}\,\beta_0(q)+\gamma_G(q)\,\beta_{k+1}(q)\,\overline{\beta}_0^k(q)}$. Substituting the expression for $\overline{\beta}_0^k(q)$ derived earlier, it holds that

$$\begin{aligned} s_0^{k+1}(q) &= \frac{\gamma_G(q)\,\beta_{k+1}(q)\,\frac{\lambda_k\,s_0^k(q)\,\beta_0(q)}{(1-s_0^k(q))\,\gamma_G(q)}}{\lambda_{k+1}\,\beta_0(q) + \gamma_G(q)\,\beta_{k+1}(q)\,\frac{\lambda_k\,s_0^k(q)\,\beta_0(q)}{(1-s_0^k(q))\,\gamma_G(q)}} \\ &= \frac{\lambda_k\,s_0^k(q)\,\beta_{k+1}(q)}{\lambda_{k+1}\,[1 - s_0^k(q)] + \lambda_k\,s_0^k(q)\,\beta_{k+1}(q)}. \end{aligned}$$

Finally, considering $\alpha^k(q) = \lambda_{k+1}/[\lambda_k\,\beta_{k+1}(q)]$, it follows $s_0^{k+1}(q) = \frac{s_0^k(q)}{\alpha^k(q)\,[1-s_0^k(q)]+s_0^k(q)}$. □

Lemma 3. *Each time an independent obstacle $\mathcal{O}_{k+1}$ is added to the workspace, the online omitted product $\overline{\beta}_{i+1}^{k+1}(q)$ for $\mathcal{O}_{i+1}$ can be calculated using $\overline{\beta}_i^{k+1}(q)$ as:*

$$\overline{\beta}_{i+1}^{k+1}(q) = \overline{\beta}_i^{k+1}(q)\,\frac{\beta_i(q)}{\beta_{i+1}(q)}. \tag{5}$$

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Proof. By Definition **??**, $\overline{\beta}_i^{k+1}(q) = \prod_{j=0,j\neq i}^{k+1} \beta_j(q)$ and $\overline{\beta}_{i+1}^{k+1}(q) = \prod_{j=0,j\neq i+1}^{k+1} \beta_j(q)$. Comparing these two equations, it follows that $\overline{\beta}_{i+1}^{k+1}(q) = \overline{\beta}_i^{k+1}(q)\, \frac{\beta_i(q)}{\beta_{i+1}(q)}$. $\square$

Lemma 4. *Each time an independent obstacle $\mathcal{O}_{k+1}$ is added to the workspace, the online analytic switch $s_{i+1}^{k+1}(q)$ for $\mathcal{O}_{i+1}$ can be calculated using $s_i^{k+1}(q)$ as:*

$$s_{i+1}^{k+1}(q) = \frac{s_i^{k+1}(q)}{\alpha_i(q)\,[1 - s_i^{k+1}(q)] + s_i^{k+1}(q)}, \tag{6}$$

with $\alpha_i(q) = [\beta_{i+1}(q)]^2/[\beta_i(q)]^2$.

Proof. By Definition **??**, $s_i^{k+1}(q) = \frac{\gamma_G(q)\,\overline{\beta}_i^{k+1}(q)}{\lambda_{k+1}\,\beta_i(q)+\gamma_G(q)\,\overline{\beta}_i^{k+1}(q)}$,which can be rearranged as $\overline{\beta}_i^{k+1}(q) = \frac{\lambda_{k+1}\,s_i^{k+1}(q)\,\beta_i(q)}{(1-s_i^{k+1}(q))\,\gamma_G(q)}$. Further by Definition **??** and Lemma 3, $s_{i+1}^{k+1}(q) = \frac{\gamma_G(q)\,\overline{\beta}_{i+1}^{k+1}(q)}{\lambda_{k+1}\,\beta_{i+1}(q)+\gamma_G(q)\,\overline{\beta}_{i+1}^{k+1}(q)}$ $= \frac{\gamma_G(q)\,[\beta_i(q)]^2\,\overline{\beta}_i^{k+1}(q)}{\lambda_{k+1}\,[\beta_{i+1}(q)]^2+\gamma_G(q)\,[\beta_i(q)]^2\,\overline{\beta}_i^{k+1}(q)}$. Substituting the expression for $\overline{\beta}_0^k(q)$ derived earlier, it holds that

$$s_{i+1}^{k+1}(q) = \frac{s_i^{k+1}(q)\,[\beta_i(q)]^2}{[1 - s_i^{k+1}(q)]\,[\beta_{i+1}(q)]^2 + s_i^{k+1}(q)\,[\beta_i(q)]^2}.$$

Finally, considering $\alpha_i(q) = [\beta_{i+1}(q)]^2/[\beta_i(q)]^2$, it follows $s_{i+1}^{k+1}(q) = \frac{s_i^{k+1}(q)}{\alpha_i(q)\,[1-s_i^{k+1}(q)]+s_i^{k+1}(q)}$. $\square$

2.2. Overlapping Stars

Lemma 5. *Each time an obstacle $\mathcal{O}_{k+1}$ is added to the workspace and overlapping with an existing obstacle, the online omitted product $\overline{\beta}_{k+1}(q)$ for $\mathcal{O}_{k+1}$ can be calculated using $\overline{\beta}_k(q)$ as:*

$$\overline{\beta}_{k+1}(q) = \overline{\beta}_k(q)\,\frac{[\beta_k(q)]^2\,\tilde{\beta}_{k+1}(q)\,\beta_{k^\star}(q)}{\tilde{\beta}_k(q)\,\beta_{(k+1)^\star}(q)}. \tag{7}$$

Proof. By Definition **??**, $\overline{\beta}_{k+1}(q) \triangleq \left(\prod_{j=0,j\neq(k+1)^\star}^{k} \beta_j(q)\right)\left(\prod_{j\in\mathcal{L}\setminus\{k+1\}} \beta_j(q)\right)\tilde{\beta}_{k+1}(q)$, and $\overline{\beta}_k(q) \triangleq \left(\prod_{j=0,j\neq k^\star}^{k-1} \beta_j(q)\right)\left(\prod_{j\in\mathcal{L}\setminus\{k\}} \beta_j(q)\right)\tilde{\beta}_k(q)$. Comparing these two equations, it follows that $\overline{\beta}_{k+1}(q) = \overline{\beta}_k(q)\,\frac{[\beta_k(q)]^2\,\tilde{\beta}_{k+1}(q)\,\beta_{k^\star}(q)}{\tilde{\beta}_k(q)\,\beta_{(k+1)^\star}(q)}$. $\square$

Lemma 6. *Each time an obstacle $\mathcal{O}_{k+1}$ is added to the workspace and overlapping with an existing obstacle, the online analytic switch $\sigma_{k+1}(q)$ for $\mathcal{O}_{k+1}$ can be calculated using $\sigma_k(q)$ as:*

$$\sigma_{k+1}(q) = \frac{\sigma_k(q)}{\alpha_k(q)\,[1 - \sigma_k(q)] + \sigma_k(q)}, \tag{8}$$

with $\alpha_k(q) = \frac{\xi_{k+1}\,\beta_{k+1}(q)\,\tilde{\beta}_k(q)\,\beta_{(k+1)^\star}(q)}{\xi_k\,[\beta_k(q)]^3\,\tilde{\beta}_{k+1}(q)\,\beta_{k^\star}(q)}$.

Proof. By Definition **??**, $\sigma_k(q) = \frac{\gamma_G(q)\,\overline{\beta}_k(q)}{\xi_k\,\beta_k(q)+\gamma_G(q)\,\overline{\beta}_k(q)}$,which can be rearranged as $\overline{\beta}_k(q) = \frac{\xi_k\,\sigma_k(q)\,\beta_k(q)}{(1-\sigma_k(q))\,\gamma_G(q)}$. Further by Definition **??** and Lemma 5, $\sigma_{k+1}(q) = \frac{\gamma_G(q)\,\overline{\beta}_{k+1}(q)}{\xi_{k+1}\,\beta_{k+1}(q)+\gamma_G(q)\,\overline{\beta}_{k+1}(q)}$ $= \frac{\gamma_G(q)\,\xi_k\,[\beta_k(q)]^3\,\tilde{\beta}_{k+1}(q)\,\beta_{k^\star}(q)\,\overline{\beta}_k(q)}{\xi_{k+1}\,\beta_{k+1}(q)\,\tilde{\beta}_k(q)\,\beta_{(k+1)^\star}(q)+\gamma_G(q)\,\xi_k\,[\beta_k(q)]^3\,\tilde{\beta}_{k+1}(q)\,\beta_{k^\star}(q)\,\overline{\beta}_k(q)}$. Substituting $\overline{\beta}_k(q)$ results that $\sigma_{k+1}(q) = \frac{\sigma_k(q)}{\alpha_k(q)\,[1-\sigma_k(q)]+\sigma_k(q)}$ with $\alpha_k(q) = \frac{\xi_{k+1}\,\beta_{k+1}(q)\,\tilde{\beta}_k(q)\,\beta_{(k+1)^\star}(q)}{\xi_k\,[\beta_k(q)]^3\,\tilde{\beta}_{k+1}(q)\,\beta_{k^\star}(q)}$. $\square$

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Lemma 7. *The iterative transformation $\Phi_{\mathcal{F}\to\mathcal{M}}^{(k)}$ in (**??**) is equivalent to the original transformation in (**??**).*

Proof. The equivalence between (**??**) and (**??**) is shown by induction, for the outer boundary and inner obstacles separately. Namely, the initial value of F_0^k is given by:

$$\begin{aligned} F_0^0 &= (1 - s_0^0)q + s_0^0[v_0(q - q_0) + q_0] \\ &= s_0^0(v_0 - 1)(q - q_0) + q, \end{aligned}$$

which fulfills (**??**). Now assume that $F_0^l = s_0^l(v_0-1)(q-q_0)+q$ is consistent with (**??**) for a particular $l \geqslant 0$. Then, F_0^{l+1} is derived via (**??**) as:

$$\begin{aligned} F_0^{l+1} &= \frac{s_0^l\lambda_l\beta_l}{(1 - s_0^l)\lambda_{l+1}}(v_0 - 1)(q - q_0) + q \\ &= s_0^{l+1}(v_0 - 1)(q - q_0) + q \end{aligned}$$

with $s_0^{l+1} = \gamma\overline{\beta}_0^{l+1}/(\gamma\overline{\beta}_0^{l+1} + \lambda_{l+1}\beta_0)$ and $\overline{\beta}_0^{l+1} = \overline{\beta}_0^l\beta_l$, which is also consistent with (**??**). Similar analyses can be applied to the iterative transformation $F_i^k = s_i^k(v_i - 1)(q - q_i) + q$ for the i-th inner obstacle in (**??**) is consistent with (**??**). Therefore, the transformation (**??**) is simplified to:

$$\begin{aligned} \Phi_{\mathcal{S}\to\mathcal{M}}^k(q) &= \sum_{i=0}^{k} F_i^k(q) \\ &= (1 - \sum_{i=0}^{M} s_i)\,q + \sum_{k=0}^{M} s_i[v_i(q - q_i) + q_i], \end{aligned}$$

which is identical to (**??**). $\square$

2.3. Overall Results

Lemma 8. *The iterative transformation $\Phi_{\mathcal{F}\to\mathcal{M}}^{(k)}$ in (**??**) is deffeomorphic as long as $\lambda_k > \Lambda_k$ and $\xi_k > \Xi_k$ hold for appropriate constants Λ_k and Ξ_k, $\forall k = 1, \cdots, M + N$.*

Proof. Since the original construction of harmonic potentials as discussed in Sec. **??** is proven to be deffeomorphic, it suffices to show that the proposed iterative transformation in (**??**) is equivalent to the static construction after each update.

First, the consistence between (**??**) and (**??**) is shown by induction, for the outer boundary and inner obstacles separately. Namely, the initial value of $T_0^{(k)}$ is given by:

$$\begin{aligned} T_0^{(0)} &= (1 - s_0^{(0)})q + s_0^{(0)}[v_0(q - q_0) + q_0] \\ &= s_0^{(0)}(v_0 - 1)(q - q_0) + q, \end{aligned}$$

which fulfills (**??**). Now assume that $T_0^{(l)} = s_0^{(l)}(v_0 - 1)(q - q_0) + q$ is consistent with (**??**) for a particular $l \geqslant 0$. Then, $T_0^{(l+1)}$ is derived via (**??**) as:

$$\begin{aligned} T_0^{(l+1)} &= \frac{s_0^{(l)} \lambda_l \beta_l}{(1 - s_0^{(l)}) \lambda_{l+1}} (v_0 - 1)(q - q_0) + q \\ &= s_0^{(l+1)} (v_0 - 1)(q - q_0) + q \end{aligned}$$

with $s_0^{(l+1)} = \gamma \overline{\beta}_0^{(l+1)} / (\gamma \overline{\beta}_0^{(l+1)} + \lambda_{l+1} \beta_0)$ and $\overline{\beta}_0^{(l+1)} = \overline{\beta}_0^{(l)} \beta_l$, which is also consistent with (**??**). Similar analyses can be applied to the iterative transformation $T_i^{(k)} = s_i^{(k)}(v_i - 1)(q - q_i) + q$ for the i-th inner obstacle in (**??**) is consistent with (**??**). Therefore, the transformation (**??**) is simplified to:

$$\begin{aligned} \Phi^{(k)}_{\mathcal{S} \to \mathcal{M}}(q) &= \sum_{i=0}^{k} \big(T_i^{(k)}(q) - q \big) + q \\ &= (1 - \sum_{i=0}^{M} s_i)\, q + \sum_{k=0}^{M} s_i [v_i (q - q_i) + q_i], \end{aligned}$$

which is identical to (**??**).

Second, the correctness of (**??**) can be proven in a similar way. Namely, the main difference between (**??**) and (**??**) is that the former method purges all the leaf obstacles at each step while the latter purges one single leaf obstacle, i.e., the new obstacle being added to the workspace. As a result, it suffices to verify that the iterative update f_k in (**??**) is consistent with (**??**). Initially, $f_1 = \sigma_1(v_1 - 1)(q - c_1) + q$ in (**??**) is consistent with (**??**). Consequently, the induction procedure can be done in a similar way as for $T_0^{(l)}$. TODO: I suggest to add two lines of details here for f_k. In summary, the iterative transformation $\Phi^{(k)}_{\mathcal{F} \to \mathcal{M}}$ is identical with the static construction in (**??**), thus deffeomorphic as long as the original conditions in (Li & Tanner, 2018, Theorem 1, 2), i.e., $\lambda_k > \Lambda_k$ and $\xi_k > \Xi_k$ are satisfied. TODO: Should these conditions appear in Lemma 1 also? □

3. Experiment Details

References

Li, C., & Tanner, H. G. (2018). Navigation functions with time-varying destination manifolds in star worlds. *IEEE Transactions on Robotics*, *35*, 35–48.