Homework 1.

$$U(c;\theta) = \frac{c^{1-\theta}}{1-\theta} - \frac{1}{1-\theta} \quad . \quad \theta > 0$$

1. $\lim\limits_{\theta \to 1} U(c,\theta) = \frac{c^{1-\theta}-1}{1-\theta} = \frac{0}{0} \rightarrow$ L'Hôpital Rule :

$$\lim_{\theta \to 1} \frac{e^{\ln(c^{1-\theta})} - 1}{1-\theta} = \lim_{\theta \to 1} \frac{e^{(1-\theta)\ln c} - 1}{1-\theta} = \lim_{\theta \to 1} \frac{e^{(1-\theta)\ln c} \cdot (-\ln c)}{-1}$$

$$= \ln c$$

2. Verify : $\sigma(c) = \frac{u'(c)}{u''(c) \cdot c} = \sigma$

$$U(c_t;\theta) = \frac{c_t^{1-\theta} - 1}{1-\theta}$$

$$u'(c_t) = c_t^{-\theta} \qquad u''(c_t) = \theta c^{-\theta-1}$$

$$\sigma(c) = \frac{c_t^{-\theta}}{-\theta c^{-\theta-1} \cdot c} = -\frac{1}{\theta} = \sigma$$

The EIS is not constant and it changes with θ.

σ measures how much the consumption in the future A substitudes for present consumption.

A low value of θ (high value of σ) means the consumption growth is very sensitive to changes in the real interest rate.

Homework 2

(a) Young-age budget constraint:

$$W_t - a_t = C_t + S_t$$

Expected t-old budget constraint:

$$d_{t+1} = R^e_{t+1} S_t + Z^e_{t+1}$$

Consumer's desition problem:

$$U(C_t) + \beta U(d_{t+1})$$

$$= U(W_t - a_t - S) + \beta U(Z^e_{t+1} + R^e_{t+1} S)$$

$$\frac{W_t - a_t - S}{Z^e_{t+1} + R^e_{t+1} S} = \frac{1}{\beta R^e_{t+1}}$$

$$(W_t - a_t - S)(\beta R^e_{t+1}) = Z^e_{t+1} + R^e_{t+1} S \qquad W_t - a_t = W_t(1 - T_t)$$

$$[W_t(1-T_t) - S] \cdot \beta R^e_{t+1} = Z^e_{t+1} + R^e_{t+1} S$$

$$W_t(1-T_t) \cdot \beta R^e_{t+1} - Z^e_{t+1} = S \cdot (\beta + 1) \cdot R^e_{t+1}$$

$$\tilde{S} = \frac{W_t(1-T_t) \cdot \beta R^e_{t+1} - Z^e_{t+1}}{(\beta+1) R^e_{t+1}}$$

$$= \frac{\beta}{1+\beta}(1-T_t) W_t - \frac{Z^e_{t+1}}{R^e_{t+1}} \cdot \frac{1}{\beta+1}$$

(b) $\pi(k_t, N_t) = F(k_t, N_t^d) - w_t N_t^d - (R_t + \delta - 1)\, k_t^d N_t^d$

$$= (k_t^d)^\alpha \cdot N_t^d - w_t N_t^d - (R_t - \delta - 1)\, k_t^d N_t^d$$

$$\frac{\partial \pi}{\partial k_t} = \left[\alpha \cdot (k_t^d)^{\alpha-1} - (R_t + \delta - 1)\right] N_t^d = 0$$

$$\alpha (k_t^d)^{\alpha-1} = R_t + \delta - 1$$

$$R_t = \alpha (k^d)^{\alpha-1} - \delta + 1$$

$$\frac{\partial \pi}{\partial N_t^d} = \underline{(k_t^d)^\alpha} - w_t - (R_t + \underline{\delta - 1)\, k_t^d} = 0$$

$$(k_t^d)^\alpha + (1-\delta)\, k_t^d = w_t + R_t \cdot k_t^d$$

we have known that $f(k_t) = k_t^\alpha + (1-\delta)\, k_t$

$$\tilde{f}(k_t) = w_t + R_t \cdot k_t^d$$

$$= w_t + \left[\alpha \cdot (k^d)^{\alpha-1} + (1-\delta)\right] k_t^d$$

$$\tilde{f}'(k_t) = \alpha \cdot k_t^{\alpha-1} + 1 - \delta$$

$$\tilde{f}(k_t) - \tilde{f}'(k_t) \cdot k_t^d = w_t$$

$$\begin{cases} w_t = (k_t^d)^\alpha + (1-\delta)\, k_t^d - \alpha \cdot (k_t^d)^\alpha - k_t^d + \delta k_t^d \\ \quad = (1-\alpha)(k_t^d)^\alpha \\ R_t = \alpha (k^d)^{\alpha-1} - \delta + 1 \end{cases}$$

(c)

$$N_t \tilde{f}(k_t) = N_t c_t + N_{t-1} d_t + N_t s_t \qquad a_t = W_t \gamma_t$$

$$\underline{\tilde{f}(k_t)} = c_t + s_t + \frac{d_t}{n+1} \qquad W_t - a_t = c_t + s_t$$

$$= W_t(1-\gamma_t) + \frac{R_t s_{t-1} + (1+n) a_{t+1}}{1+n} \qquad W_t(1-\gamma_t) = c_t + s_t$$

$$= W_t(1-\gamma_t) + \frac{R_t s_{t-1} + (1+n) W_t \gamma_t}{1+n} \qquad d_{t+1} = R^e_{t+1} s_t + z^e_{t+1}$$

$$= R^e_{t+1} s_t + (1+n) a_{t+1}$$

$$= W_t - W_t \gamma_t + \frac{R_t s_{t-1}}{1+n} + W_t \gamma_t$$

$$= W_t + \frac{R_t s_{t-1}}{1+n}$$

From (b), we know $\begin{cases} W_t = \tilde{f}(k_t) - f'(k_t) \cdot k_t^d \\ R_t = f'(k_t) \end{cases}$ thus

$$\tilde{f}(k_t) = \underline{\tilde{f}(k_t)} - \tilde{f}'(k_t) \cdot k_t^d + \frac{f'(k_t) s_{t-1}}{1+n}$$

$$k_t^d = \frac{s_{t-1}}{1+n}$$

(d)

$$\begin{cases} W_t = f(k_t) - f'(k_t) \cdot k_t \\ R^e_{t+1} = f'(k_{t+1}) + 1 - \delta \\ (1+n) k_{t+1} = \tilde{s}(W_t - a_t, z^e_{t+1}, R^e_{t+1}) \\ z^e_{t+1} = (1+n) a_{t+1} \end{cases}$$

From (a) (b) (c), we know

$$\begin{cases} W_t = (1-\alpha)(k_t^d)^\alpha \\ R_{t+1} = \alpha(k_t^d)^{\alpha-1} - \delta + 1 \\ \tilde{S} = \frac{\beta}{1+\beta}(1-T_t)W_t - \frac{z_{t+1}^e}{R_{t+1}^e} \cdot \frac{1}{\beta+1} \\ z_{t+1}^d = (1+n)\, a_{t+1} \end{cases}$$