Homework 0

Exercise 1

$U(c) = \ln(c)$

$$\begin{cases} \beta \dfrac{U_c(C_{t+1}^t)}{U_c(C_t^t)} = \dfrac{1}{1+r_{t+1}} \\ C_t^t + \dfrac{C_{t+1}^t}{1+r_{t+1}} = w_t \end{cases}$$

$$\frac{\frac{1}{c'}}{\frac{1}{c}} = \frac{1}{\beta(1+r_{t+1})}$$

$$c = \frac{c'}{\beta(1+r_{t+1})}$$

$$c\beta = \frac{c'}{(1+r_{t+1})}$$

$$c + \frac{c'}{1+r_{t+1}} = w_t$$

$$c + c\beta = w_t$$

$$c = \frac{w}{1+\beta}$$

saving $= w - c$

r_{t+1} is the interest rate when old, and it depends k_{t+1}^*, but it has nothing to do with saving when young.

$$k_{t+1}^* = \frac{w-c}{1+n} = \frac{w - \frac{w}{1+\beta}}{1+n} = \frac{w\beta}{(1+n)(1+\beta)}$$

Exercise 2: $U(c) = \ln c$ $\quad F(k_t, N_t) = k^\alpha (AN_t)^{1-\alpha}$

1.

$$f'(k) = \alpha A^{1-\alpha} \cdot k^{\alpha-1}$$

$$k_{t+1} = \frac{w_t - c_t^t}{1+n}$$

$$= \frac{w_t - \frac{w_t}{1+\beta}}{1+n}$$

$$w = f(k) - kf'(k)$$

$$= A^{1-\alpha} \cdot k^\alpha - \alpha A^{1-\alpha} k^\alpha$$

$$= A^{1-\alpha} k^\alpha (1-\alpha)$$

$$= \frac{w\beta}{(1+n)(1+\beta)}$$

$$= \frac{A^{1-\alpha}k^{\alpha}(1-\alpha)\beta}{(1+n)(1+\beta)}$$

2. $k_{ss} = k_{t+1} = k_t$

$$= \frac{A^{1-\alpha}k_{ss}^{\alpha}(1-\alpha)\beta}{(1+n)(1+\beta)}$$

$$k_{ss}^{1-\alpha} = \frac{A^{1-\alpha}(1-\alpha)\beta}{(1+n)(1+\beta)}$$

$$k_{ss} = \left(\frac{A^{1-\alpha}(1-\alpha)\beta}{(1+n)(1+\beta)}\right)^{\frac{1}{1-\alpha}}$$

3. β indexes each agent's "patience". From $c_t = \frac{w_t}{1+\beta}$, $c_{t+1} = c_t\,\beta(1+r_{t+1})$, so it means in young age, the larger β is, the less the consumer consumes while in old age, the larger β is, the more the consumer consumes.

4. k_{ss} will decrease.

5. k_{ss} will increase.

Home Work 1

$$k_{t+1} = \frac{1-\alpha}{(1+n)(2+\rho)} k_t^{\alpha} =: g(k_t) \qquad U(c, c') = \ln(c) + \beta \ln(c')$$

$$F(K,N) = K^{\alpha} N^{1-\alpha} \quad \alpha \in (0,1) \qquad \rho = \frac{1}{\beta} - 1 \quad \beta \in (0,1)$$

1. $f'(k) = \alpha k^{\alpha-1}$

$$w_t = f(k) - k f'(k) = k^{\alpha} - k \cdot \alpha k^{\alpha-1} = (1-\alpha) k^{\alpha}$$

$$\frac{U_c(C_{t+1}^t)}{U_c(C_t^t)} = \frac{1}{\beta(1+r_{t+1})} \qquad C_t^t + \frac{C_{t+1}^t}{1+r_{t+1}} = w_t$$

$$\frac{\frac{1}{c'}}{\frac{1}{c}} = \frac{1}{\beta(1+r_{t+1})} \qquad C + C\beta = W_t$$

$$C(1+\beta) = W_t$$

$$C = \frac{c'}{\beta(1+r_{t+1})} \qquad C = \frac{W_t}{1+\beta}$$

$$c\beta = \frac{c'}{1+r_{t+1}} \qquad k_{t+1} = \frac{W_t - C_t^t}{1+n}$$

$$= \frac{(1-\alpha)k^{\alpha} - \frac{(1-\alpha)k^{\alpha}}{1+\beta}}{1+n}$$

$$\rho = \frac{1}{\beta} - 1$$

$$\rho + 1 = \frac{1}{\beta} \qquad = \frac{(1-\alpha)\cdot k^{\alpha}\beta}{(1+n)(1+\beta)}$$

$$\beta = \frac{1}{\rho+1}$$

$$\frac{\beta}{1+\beta} = \frac{\frac{1}{\rho+1}}{1+\frac{1}{\rho+1}} = \frac{1}{2+\rho} \qquad = \frac{(1-\alpha)k^{\alpha}}{(1+n)} \cdot \frac{1}{2+\rho}$$

2. $k_{ss} = k_{t+1}$

$$= \frac{(1-\alpha)k_{ss}^{\alpha}}{(1+n)(2+\rho)}$$

$$k_{ss}^{1-\alpha} = \frac{1-\alpha}{(1+n)(2+\rho)}$$

$$k_{ss} = \left(\frac{1-\alpha}{(1+n)(2+\rho)}\right)^{\frac{1}{1-\alpha}}$$

3. $g(k_0) = \frac{1-\alpha}{(1+n)(2+\rho)} k_t^{\alpha}$

$$g'(k_0) = \frac{\alpha(1-\alpha)}{(1+n)(2+\rho)} k_t^{\alpha-1}$$

$$g(k_{ss}) = k_{ss}$$

$$g(k_t) = k_{ss} + \frac{\frac{\alpha(1-\alpha)}{(1+n)(2+\rho)} k_{ss}^{\alpha-1}}{1!}(k_t - k_{ss})$$

$$= k_{ss} + \frac{\alpha(1-\alpha)}{(1+n)(2+\rho)} \cdot k_{ss}^{\alpha-1}(k_t - k_{ss})$$

$$= \frac{\alpha(1-\alpha)}{(1+n)(2+\rho)}(k_{ss}^{\alpha-1} \cdot k_t - k_{ss}^{\alpha}) + \left(\frac{1-\alpha}{(1+n)(2+\rho)}\right)^{\frac{1}{1-\alpha}}$$

$$= \frac{\alpha(1-\alpha)}{(1+n)(2+\rho)} \cdot \frac{(1+n)(2+\rho)}{(1-\alpha)} \cdot k_t - \frac{\alpha(1-\alpha)}{(1+n)(2+\rho)} \cdot \frac{\alpha(1-\alpha)}{(1+n)(2+\rho)}^{\frac{\alpha}{1-\alpha}} + \left(\frac{1-\alpha}{(1+n)(2+\rho)}\right)^{\frac{1}{1-\alpha}}$$

$$= \alpha k_t$$

4. $R(k_0) = \frac{k_{t+1}}{k_t} - 1$

$= \frac{1-\alpha}{(1+n)(2+\rho)} \cdot k_t^{\alpha-1} - 1$

$R'(k_0) = \frac{(\alpha-1)(1-\alpha)}{(1+n)(2+\rho)} \cdot k_t^{\alpha-2} < 0$

The rate of Covergence is decreasing and the rate depends on n, ρ, α.

5. $Z k_0^{\alpha}$

$w_0 = (1-\alpha) \cdot k_0^{\alpha}$

$= (1-\alpha) \cdot Z \cdot k_0^{\alpha}$

$w_1 = k_1 f'(k_1)$

$k_1 = \frac{\alpha}{(1+n)(2+\rho)} \cdot Z \cdot k_0^{\alpha}$

$w_1 = k_1 \cdot f'(k_1) = \alpha \cdot k_1^{\alpha-1} \cdot \frac{1-\alpha}{(1+n)(2+\rho)} \cdot Z \cdot k_0^{\alpha}$

$= \alpha \cdot k_1^{\alpha-1} \cdot \frac{1-\alpha}{(1+n)(2+\rho)} \cdot k_0^{\alpha} \cdot Z$

$= \alpha \cdot k_1^{\alpha-1} \cdot \frac{1-\alpha}{(1+n)(2+\rho)} \cdot k_0^{\alpha} \cdot Z.$

6. Solow - Swan:

$k_{t+1} = \frac{(1-\delta)k_t + sAk_t^{\alpha}}{1+n}$

$k_{ss} = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$

OLG:

$k_{ss} = k_{ss} = \left(\frac{1-\alpha}{(1+n)(2+\rho)}\right)^{\frac{1}{1-\alpha}}$

Homework 2.

$u(c) + \beta u(c')$

$u(x) : \lim_{\sigma \to 6} \left\{ \frac{x^{1-\sigma} - 1}{x - 6} \right\}$

① Young Age: $\frac{C_t}{W_t}$

$$\lim_{\sigma \to 6} \frac{x^{1-6}-1}{1-6}$$

$$\left(\frac{x^{1-6}-1}{1-6}\right)' = x^{-6}$$

$$\begin{cases} \left(\frac{c'}{c}\right)^{-6} = \frac{1}{\beta(1+r)} \Rightarrow \frac{c}{c'} = [\beta(1+r)]^{\frac{1}{6}} \cdots ① \\ c + \frac{c'}{1+r} = W_t \Rightarrow c' = (W_t - c)(1+r) \cdots ② \end{cases}$$

$$\begin{cases} \beta \frac{U_c(C_{t+1}^t)}{U_c(C_t^t)} = \frac{1}{1+r_{t+1}} \\ C_t^t + \frac{C_{t+1}^t}{1+r_{t+1}} = W_t \end{cases}$$

From ① and ②, we can obtain the equation of C and W, and we can know $\frac{C_t}{W_t}$.

② Old Age: $\frac{c'}{kf'(k)}$

$f'(k_t) = \alpha k_t^{1-\alpha}$ $W_t = (1-\alpha) k_t^{\alpha}$ ...③

$$\frac{c'}{kf'(k)} = \frac{c'}{\alpha k_t^{2-\alpha}}$$

$$\begin{cases} \frac{c}{c'} = [\beta(1+r)]^{\frac{1}{6}} \cdots ① \\ c' = (W_t - c)(1+r) \sim ② \end{cases}$$

From ①, ②, we can draw the relation between c' and W_t, so $\frac{c'}{kf'(k)} = \frac{g(W_t)}{\alpha k^{2-\alpha}}$. From ③, we can also know that $\frac{c'}{kf'(k)} = \frac{g[w_t(k)]}{kf'(k)}$.

In conclution, no matter at young age or at old age, c and c' are always some kinds of equation with σ, so the marginal propensity to consume depends on σ.

I can't solve for an equilibrium by hand cuz it is two complicated, and the numerial proceeding is in the above.