Introduction to equilibrium fluctuations:
A simple stochastic OLG approach

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Outline

Recap

Definition

Given k_0, a RCE in the simple OLG model is a price system $\{w_t(k_t), r_t(k_t)\}_0^\infty$ and allocation $\{k_{t+1}(k_t)\}_{t=0}^\infty$ that satisfies:

1. Consumer's lifetime utility maximization, for each generation $t = 0, 1, ...$
2. Firm's profit maximization:

$$\begin{aligned} f'(k_t) &= r_t + \delta \\ f(k_t) - k_t.f'(k_t) &= w_t \end{aligned}$$

3. Market clearing in the credit/capital market:*

$$K_{t+1} = N_t s_t \Rightarrow k_{t+1} = \frac{N_t}{N_{t+1}} s_t$$

What we do today

We want to introduce stochastic elements into the model.

1. A first baby-step we can take to think about models of business cycles that have nice microeconomic foundations.
2. Why do we care about micro-founded models?
3. They are often built on stylized assumptions but can potential replicate realistic business cycle features.
4. The microeconomics then help us discipline our understanding of a very complex real economy.
5. People who build large complex models can generate (fudge?) “good numbers” for their bosses or clients.
6. But the models are so complex that we don’t know what goes on inside! The same problem we faced, when we tried to look into the real world, in the first place!

Preview: What's new?

We will show ...

Definition

Given k_0, and stochastic process for TFP, $\{A_t\}_{t=0}^{\infty}$, a RCE in the stochastic OLG model is a stochastic process for prices $\{w_t(k_t, A_t), r_t(k_t, A_t)\}_0^{\infty}$ and allocation $\{k_{t+1}(k_t, A_t)\}_{t=0}^{\infty}$ that satisfies:

1. Consumer's *expected* lifetime utility maximization, for each generation $t = 0, 1, ...$
2. Firm's profit maximization:

$$\begin{aligned} A_t f'(k_t) &= r_t + \delta \\ A_t \left[f(k_t) - k_t . f'(k_t) \right] &= w_t \end{aligned}$$

3. Market clearing: $k_{t+1} = \frac{N_t}{N_{t+1}} s_t$

Preview: What's new?

1. In practical terms, we can generate random sequences such as $\{A_t\}_{t=0}^{\infty}$.
2. That feeds into the RCE system of equations.
3. Since A_t is random, then equilibrium choices and prices will also be induced stochastic processes – i.e. functions of random variables over time.
4. We can study these random sequences generated by the equilibrium of the model and interpret them as equilibrium prices and quantities over the business cycle!
5. Practical/quantitative applications will involve making the models richer to be able to match business cycle facts of an economy.
6. This approach of marrying stochastic modeling and economic theory allows us to form a coherent (not necessarily correct) viewpoint of the world.

Stochastic OLG model with production

Sectors:

1. Consumer/Hoseholds
2. Perfectly competitive firm
3. No government for now ...

Equilibrium concept:

Definition of *recursive competitive equilibrium* ... .

Assumptions

- Simplest case: two adjacent generations overlapping — i.e. an agent born in time i is old in $i+1$ and dies. For example, a "period" corresponds to 30 years.
- $i = 0, 1, 2, \ldots$ indexes individual i as well as the time period t. The only heterogeneity is between generations.
- At $t \geq 0$, N_t is the number of "newborns". Deterministic process for population growth $N_{t+1} = (1+n)N_t$. Thus, given N_0, $N_t = (1+n)^t N_0$.
- At time t, N_{t-1} old agents consume c_t^{t-1} and N_t young agents consume c_t^t.

New Notation

1. Random TFP level A_t
2. Stochastic TFP process $\{A_t\}_{t=0}^{\infty} = \{A_0, A_1, ...\}$
3. Time t state variables (k_t, A_t) pin down current position of economy, and
4. (k_t, A_t) also condition the *distribution* of future random states (k_{t+1}, A_{t+1}), on which current expectation of future outcomes are calculated.
5. Note though, k_{t+1} is actually known at time t. So it's only A_{t+1} that is random from time-t perspective.
6. Let $\mu(dA_{t+1}; A_t) := \Pr\{A_{t+1} \in B | A_t\}$ denote probability measure on the event B conditional on a realized observation of A_t.

Assumptions

Timing.

1. "Period t":
   - born
   - Realization of random TFP A_t observed.
   - supply 1 unit of labor inelastically for real wage w_t
   - Consume c_t^t and save s_t
   - s_t yields a stochastic return of $r_{t+1}(k_{t+1}, A_{t+1})$.
2. "Period $t+1$":
   - Realization of random TFP A_{t+1} observed.
   - realized $(1 + r_{t+1})$ return on s_t pays for consumption c_{t+1}^t.
   - then die – no bequests.

Random TFP level

Now we assume random shifts A_t in the production function of firms.

Let's assume the stochastic process $\{A_t\}_{t=0}^{\infty}$ is generated by a first-order Markov process of the form:

$$A_{t+1} = A^{1-\gamma} A_t^{\gamma} \omega_{t+1}, \qquad 0 < \gamma < 1, A > 0 \tag{1}$$

where $\omega_{t+1} \sim \Phi([1/A^{1-\gamma}, M])$, where $M < +\infty$.

- That is, the shocks ω_{t+1} are drawn from a fixed distribution Φ defined on a bounded domain $[1/A^{1-\gamma}, M]$.
- The latter assumption keeps the random realizations of $A_t > 0$ and bounded so later expectations (integrals) are "well-defined".
- γ governs the degree of persistence/memory in the process $\{A_t\}_{t=0}^{\infty}$.

Inducing the probability measure μ

Remarks:

1. Note that at the beginning of each t, A_t is realized.
2. So given A_t, at beginning of each $t+1$, nature draws $\omega_{t+1} \sim \Phi([1/A^{1-\gamma}, M])$.
3. That is, nature would effectively draw $A_{t+1} \in B$ with probability $\mu(dA_{t+1}; A_t) := \Pr\{A_{t+1} \in B | A_t\}$.
4. $\mu(dA_{t+1}; A_t)$ is induced by the TFP model (1) together with the distribution function Φ.

Conveniently, $\ln(A_t)$ follows a linear first-order Markov or AR(1) process,

$$\ln(A_{t+1}) = (1-\gamma)\ln(A) + \gamma\ln(A_t) + \epsilon_{t+1}.$$

where

- $\epsilon_{t+1} = \ln(\omega_{t+1})$
- and since $A_t > 0$, $\ln(A_t)$ is well-defined everywhere.

Firm

Technology:

$$Y_t = A_t F(K_t, N_t)$$

is constant returns to scale (homogeneous of degree 1). Thus,

$$Y = AF\left(\frac{K}{N}, 1\right) N = Af(k)N.$$

Competitive firms choose inputs K_t and N_t to maximize per-period profits

$$\pi(K_t, N_t) = A_t F(K_t, N_t) - w_t N_t - (r_t + \delta) K_t.$$

The FONC for profit maximization (in per-worker variable terms) is

$$\begin{aligned} A_t f'(k_t) &= r_t + \delta \\ A_t \left[f(k_t) - k_t f'(k_t) \right] &= w_t \end{aligned}$$

Note that although A_t is a random variable, each period, the firm's problem is static (does not involve intertemporal choice) and A_t is already known as time t unfolds.

So the firm's optimization problem is still trivial in the sense that no expectations of future random outcomes are involved.

Example

The profit maximizing conditions for the case where

$$A_t F(K_t, L_t) = A_t K_t^{\alpha} N_t^{1-\alpha},$$

$\alpha \in (0,1)$ and $A > 0$ are:

$$r_t = \alpha A_t k_t^{\alpha-1} - \delta \tag{2}$$

$$w_t = (1-\alpha) A_t k_t^{\alpha} \tag{3}$$

Consumers

Let $\mathbb{R}_{++}$ be set of strictly positive real numbers.

Assume per-period utility function $u : \mathbb{R}_{++} \to \mathbb{R}$ is

- strictly increasing and concave,
- twice continuously differentiable.

Example (Iso-elastic utility functions)

Suppose $u : \mathbb{R}_{++} \to \mathbb{R}$ is given by the family of functions,

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta},$$

where $\theta > 0$.

Note that

$$\lim_{\theta \to 1} \frac{c^{1-\theta} - 1}{1 - \theta} = \ln(c).$$

Consumers

Since future income and consumption are uncertain, we need a concept for representing people's choice under uncertainty.

For an agent born in t,

1. $c_t^t(k_t, A_t)$, i.e. c_t^t choice function of realized/known random variable at t.
2. $c_{t+1}^t(k_{t+1}, A_{t+1})$, is random from time-$t$ perspective. $\Rightarrow u[c_{t+1}^t(k_{t+1}, A_{t+1})]$ is random too.
3. We use a standard tool from decision theory: von-Neumann-Morgernstern expected utility theory.

Each agent born in each $t \geq 0$, observing A_t, has an expected utility function:

$$U(c_t^t, c_{t+1}^t) = \int_{[A_t^\gamma, M]} \Big\{ u[c_t^t(k_t, A_t)] + \beta u[c_{t+1}^t(k_{t+1}, A_{t+1})] \Big\} \mu(dA_{t+1}; A_t)$$

More compactly, we write the conditional expectations operator as $\mathbb{E}_{\mu,t}$, so then,

$$\begin{aligned} U(c_t^t, c_{t+1}^t) =& \mathbb{E}_{\mu,t} \Big\{ u[c_t^t(k_t, A_t)] + \beta u[c_{t+1}^t(k_{t+1}, A_{t+1})] \Big\} \\ =& u[c_t^t(k_t, A_t)] + \beta \mathbb{E}_{\mu,t} \Big\{ u[c_{t+1}^t(k_{t+1}, A_{t+1})] \Big\} \end{aligned}$$

where subjective discount factor is $\beta = \frac{1}{1+\rho}$, $\rho > 0$.

Remark

1. *μ is the objective probability measure governing the stochastic process underlying the agents' economy.*
2. *μ is also a component of the subjective belief system that agents' use to make forecasts of future events and to calculate their expected utilities.*
3. *An example of the rational expectations hypothesis: the objective or equilibrium-consistent probability measure coincides with subjective beliefs on probable events.*

Short hand notation:

$$
\begin{aligned}
s_t &:= s_t(k_t, A_t), \\
c_t^t &:= c_t^t(k_t, A_t), \\
w_t &:= w_t(k_t, A_t), \\
c_{t+1}^t &:= c_{t+1}^t(k_{t+1}, A_{t+1}).
\end{aligned}
$$

So for each given (k_t, A_t) and each "measurable" A_{t+1}, youth and old-age budget constraints, respectively, are

$$
\begin{aligned}
s_t &= w_t \cdot 1 - c_t^t \\
c_{t+1}^t &= (1 + r_{t+1}) s_t
\end{aligned}
$$

Collapsing these together gives IBC for consumer (born in t):

$$w_t = c_t^t + \frac{c_{t+1}^t}{1 + r_{t+1}}$$

so that lifetime income = stochastic lifetime consumption (in PV terms).

Remark:

1. There is a continuum of such an IBC, each indexed by A_{t+1}. (That's a lot!)
2. Not to worry! Agents can calculate the expectations of their utilities over these (uncountably) many constraints.

Each agent born in each t with state (k_t, A_t) solve the following problem:

$$\max_{c_t^t, c_{t+1}^t} u[c_t^t(k_t, A_t)] + \beta \mathbb{E}_{\mu,t}\left\{u[c_{t+1}^t(k_{t+1}, A_{t+1})]\right\}$$

subject to

$$w_t(k_t, A_t) = c_t^t(k_t, A_t) + \frac{c_{t+1}^t(k_{t+1}, A_{t+1})}{1 + r_{t+1}(k_{t+1}, A_{t+1})}.$$

In general, the optimality conditions characterizing the optimal choice (c_t^t, c_{t+1}^t) are

$$u'(c_t^t) = \beta(1 + r_{t+1})u'(c_{t+1}^t)\mu(A_{t+1}; A_t), \tag{4}$$

and

$$w_t = c_t^t + \frac{c_{t+1}^t}{1 + r_{t+1}}$$

for all $t = 0, 1, 2, ...$, and all measurable A_{t+1}. Or rearrange (4) and taking expectations to get:

$$\beta \mathbb{E}_{\mu,t} \left\{ \frac{u'(c_{t+1}^t)}{u'(c_t^t)}(1 + r_{t+1}) \right\} = 1$$

So in expected terms, the agent equates the $MRS(c, c')$ with the stochastic intertemporal relative price.

Example

Suppose $u(c) = \ln(c)$. Then the Euler equation (4) is

$$\frac{1}{c_t^t} = \beta \left[\frac{1 + r_{t+1}}{c_{t+1}^t} \right] \mu(A_{t+1}; A_t)$$

Taking conditional expectations,

$$\frac{1}{c_t^t} = \beta \mathbb{E}_{\mu,t} \left\{ \left[\frac{1 + r_{t+1}}{c_{t+1}^t} \right] \right\}$$

Recursive competitive equilibrium

Definition

Given k_0, and stochastic process for TFP, $\{A_t\}_{t=0}^{\infty}$, a RCE in the stochastic OLG model is a stochastic process for prices $\{w_t(k_t, A_t), r_t(k_t, A_t)\}_0^{\infty}$ and allocation $\{k_{t+1}(k_t, A_t)\}_{t=0}^{\infty}$ that satisfies:

1. Consumer's *expected* lifetime utility maximization, for each generation $t = 0, 1, ...$:

$$\beta \mathbb{E}_{\mu,t} \left\{ \frac{u'(c_{t+1}^t)}{u'(c_t^t)} (1 + r_{t+1}) \right\} = 1$$

2. Firm's profit maximization:

$$\begin{aligned} A_t f'(k_t) &= r_t + \delta \\ A_t \left[f(k_t) - k_t . f'(k_t) \right] &= w_t \end{aligned}$$

3. Market clearing: $k_{t+1} = \frac{N_t}{N_{t+1}} s_t$

Remark

$$K_{t+1} = s_t N_t.$$

1. *Saving of young equals next period's installed capital stock.*
2. *This result comes for free from goods market clearing each period. Why?*
3. *The old in each period t do not want to leave resources/assets behind.*
4. *So at the end of t, current old sell their existing capital holdings plus any net increase in capital stock, K_{t+1}, to all the current young, who pay from their savings – in aggregate is $s_t N_t$.*
5. *The circle of life goes on ...*

Now we consider the RCE in a specific example with closed-form solution.

Example

Assume that the capital depreciation rate $\delta = 0$. Let, as before,

$$A_t F(K_t, L_t) = A_t K_t^{\alpha} N_t^{1-\alpha}, \quad \alpha \in (0,1),$$

and

$$u(c) = \ln(c),$$

and

$$\ln(A_{t+1}) = (1-\gamma)\ln(A) + \gamma \ln(A_t) + \epsilon_{t+1}, \quad A > 0, \gamma \in (0,1).$$

Example (continued)

We showed, consumer's optimal intertemporal consumption *contingent plans* satisfy:

$$\frac{1}{c_t^t} = \beta \mathbb{E}_{\mu,t}\left\{ \left[\frac{1+r_{t+1}}{c_{t+1}^t} \right] \right\}$$

Using consumer's budget constraints, and using capital market clearing condition, this can be re-written as

$$\frac{1}{w_t - (1+n)k_{t+1}} = \beta \mathbb{E}_{\mu,t}\left\{ \left[\frac{1+r_{t+1}}{(1+r_{t+1})(1+n)k_{t+1}} \right] \right\}$$

Example (continued)

Simplify RHS to get

$$\frac{1}{w_t - (1+n)k_{t+1}} = \beta \mathbb{E}_{\mu,t}\left\{\left[\frac{1}{(1+n)k_{t+1}}\right]\right\}$$

so the only stochastic element $r_{t+1} := r_{t+1}(k_{t+1}, A_{t+1})$ dropped out from the first-order condition.

Since k_{t+1} is known at time t, the condition also holds "within" the expectations operator:

$$\frac{1}{w_t - (1+n)k_{t+1}} = \beta\left[\frac{1}{(1+n)k_{t+1}}\right]$$

Example (continued)

Re-arrange for k_{t+1}, we have

$$k_{t+1} = \frac{\beta}{(1+n)(1+\beta)} w_t.$$

Looks just like its deterministic cousin we derived earlier, hey?

Not quite! Now, from firm's optimal labor demand (3), we have

$$w_t = (1-\alpha) A_t k_t^{\alpha}$$

which suffers from random perturbations by A_t! So we have a stochastic difference equation solution to the RCE of this example:

$$k_{t+1} = \frac{\beta}{(1+n)(1+\beta)} (1-\alpha) A_t k_t^{\alpha}.$$

Example

The solution to this economy's RCE beginning from (k_0, A_0) is a contingent allocation plan (sequence of decision functions), $(k_{t+1}, c_t, c^t_{t+1}, y_t)(k_t, A_t)$ supported by state-contingent prices $(w_t, r_t)(k_t, A_t)$ that satisfies

1. $k_{t+1} = \frac{\beta}{(1+n)(1+\beta)}(1-\alpha)A_t k_t^\alpha$,
2. $w_t = (1-\alpha)A_t k_t^\alpha$,
3. $r_t = \alpha A_t k_t^{\alpha-1} - \delta$,
4. $c_t = w_t - (1+n)k_{t+1}$,
5. $c^t_{t+1} = (1+r_{t+1})k_{t+1}$,
6. $y_t = A_t k_t^\alpha$,
7. $i_t = s_t = (1+n)k_{t+1}$ (investment flow).

for every possible random history of TFP levels, $\{A_t\}_{t=0}^\infty$.

Exercise

- *Choose your parameter values* $(\alpha, \beta, \delta, A, \gamma, M)$.
- *Generate a random sequence* $\{A_t\}_{t=0}^{\infty}$ *according to the law of motion*

$$\ln(A_{t+1}) = (1-\gamma)\ln(A) + \gamma\ln(A_t) + \epsilon_{t+1} \equiv H(A_t, \epsilon_{t+1}).$$

  For example, assume $\epsilon_{t+1} \sim U[-(1-\gamma)\ln(A), \ln(M)]$.

- *Calculate a sample of the stochastic RCE using these equations:*
  1. $k_{t+1} = \frac{\beta}{(1+n)(1+\beta)}(1-\alpha)A_t k_t^{\alpha} \equiv G(A_t, k_t)$,
  2. $w_t = (1-\alpha)A_t k_t^{\alpha}$,
  3. $r_t = \alpha A_t k_t^{\alpha-1} - \delta$,
  4. $c_t = w_t - (1+n)k_{t+1}$,
  5. $c_{t+1}^t = (1+r_{t+1})k_{t+1}$,
  6. $y_t = A_t k_t^{\alpha}$,
  7. $i_t = s_t = (1+n)k_{t+1}$ *(investment flow)*.

Pseudocode

Algorithm 1. Simulating sample RCE outcomes

Input: (A_0, k_0), Equilibrium system: (H, G), random sample $\omega \leftarrow \{\exp(\epsilon_t)\}_{t=0}^{T}$, Null vectors $\mathbf{A}(:) = \mathbf{k}(:) = \mathbf{0}_{(T+1)\times 1}$.

```
set
    A(1) ← A_0
    k(1) ← k_0
    t ← 1
end
while t ≤ T do
    A(t+1) ← H[A(t), ω(t+1)]
    k(t+1) ← G[A(t), k(t)]
    set
        t ← t+1
    end
```

Output: Random sample RCE path $\{k_{t+1}\}_{t=0}^{T} \leftarrow \mathbf{k}$