

## IMPOSSIBILITY RESULTS FOR NONDIFFERENTIABLE FUNCTIONALS

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We examine challenges to estimation and inference when the objects of interest are nondifferentiable functionals of the underlying data distribution. This situation arises in a number of applications of bounds analysis and moment inequality models, and in recent work on estimating optimal dynamic treatment regimes. Drawing on earlier work relating differentiability to the existence of unbiased and regular estimators, we show that if the target object is not differentiable in the parameters of the data distribution, there exist no estimator sequences that are locally asymptotically unbiased or  $\alpha$ -quantile unbiased. This places strong limits on estimators, bias correction methods, and inference procedures, and provides motivation for considering other criteria for evaluating estimators and inference procedures, such as local asymptotic minimaxity and one-sided quantile unbiasedness.

KEYWORDS: Local asymptotics, moment inequality models, bounds, bias-correction.

### 1. INTRODUCTION

IN MOMENT INEQUALITY and related models, certain estimands of interest are nonsmooth functionals of the underlying distribution of the data, and this creates challenges for standard estimation and inference procedures. We examine such cases, and show that nonsmoothness implies sharp limits on the performance of estimators and inference procedures. In particular, if the estimand possesses one-sided derivatives but is not differentiable, then there exist no locally asymptotically unbiased estimators, and there exist no regular estimators, when the underlying set of distributions is a smooth family. Moreover, there exist no locally quantile-unbiased estimators under a mild restriction on their form. Since no locally asymptotically unbiased estimators exist, bias correction procedures cannot completely eliminate local bias, and reducing bias too much will eventually cause the variance of the procedure to diverge. Nonexistence of regular estimators implies that standard arguments for optimality of estimators, such as the convolution theorem for semiparametric estimators, and standard arguments for uniform validity of conventional inference procedures, such as Wald-type confidence intervals, are not valid. While recent work on moment inequality and related models has proposed particular estimators and inference procedures with nonstandard limit distributions, our

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results show that, in many such models, a nonstandard distributional theory is unavoidable for any procedure. These findings provide motivation for considering alternative asymptotic optimality concepts, such as local asymptotic minmaxity, and for considering other estimator criteria, such as the one-sided quantile-unbiasedness criteria considered in Chernozhukov, Lee, and Rosen (2009).

We use Le Cam's limits of experiments approach (Le Cam (1970, 1972, 1986)) to provide a simple and intuitive argument for our impossibility results. To our knowledge, the limits of experiments approach has not been used before in the moment inequality literature. Under local asymptotic normality, the multivariate normal location model serves as a limit experiment, in the sense that any sequence of estimators in the model of interest is matched by some estimator in the normal model, and the target functional of interest is matched by its directional derivative. We show that, in the normal location model, no unbiased or translation equivariant estimator exists unless the estimand is linear. This in turn implies that in the original setting, no locally asymptotically unbiased or regular estimators exist if the functional of interest is not differentiable. A similar argument also applies to the quantile-unbiasedness criterion, provided the cumulative distribution function (CDF) of the estimator satisfies a local monotonicity condition. The result for regular estimators is implied by Theorem 2.1 of van der Vaart (1991b), but by using the Le Cam framework and working initially in a parametric setting, we can simplify the argument and extend it to asymptotically unbiased and quantile-unbiased estimators.

## 2. EXAMPLES

Before developing the theory, we begin with some examples of recent work in economics and biostatistics in which the estimand is a nondifferentiable functional of the data distribution.

### 2.1. *Bounds for an Incomplete Auction Model*

Haile and Tamer (2003) showed that it is possible to obtain useful inference for valuation distributions in auction models without fully specifying the structure of the model. Auctions can have a variable number of bidders  $m \in \{2, \dots, M\}$ . In an auction with  $m$  bidders, suppose that the bidders  $i = 1, \dots, m$  draw valuations  $v_i$  independently from a distribution with CDF  $F(v)$ , which does not depend on  $m$ . Assume that bidders make bids  $b_i$  subject to two restrictions:

- (i)  $b_i \leq v_i$ .
- (ii) Each bidder does not allow an opponent to win the good at a price she is willing to beat.

We observe bids, but not valuations. Let  $G_m(b)$  denote the CDF of bids in an auction with  $m$  bidders. Condition (i) implies  $F(v) \leq G_m(v)$  for all  $v$  and  $m$ , and hence

$$F(v) \leq \min_m G_m(v).$$

The  $G_m(v)$  may differ across  $m$ , if bidders shade their bids differently depending on the number of competing bidders in the auction. By a similar argument, condition (ii) gives a lower bound for  $F(v)$  involving a maximum of estimable quantities.

Haile and Tamer also used the order statistics of the bids (conditional on  $m$ ) to improve their estimator, but even without this additional step, difficulties can arise because the upper bound  $\kappa = \min_m G_m(v)$  is a nondifferentiable functional of the observed bid distribution. The convexity of the minimum function leads to downward bias of sample analog estimators. Simulation results in their paper suggest that the bias can be severe for realistic sample sizes. Haile and Tamer suggested a bias reduction procedure; similar issues in other bounds analyses were noted by [Manski and Pepper \(2000\)](#), and [Kreider and Pepper \(2007\)](#) suggested a bootstrap bias correction. [Chernozhukov, Lee, and Rosen \(2009\)](#) called these settings intersection bounds problems, and developed one-sided bias reduction procedures for estimation and inference on parameters such as  $\kappa$ . In the analysis below, we find that it is impossible to completely eliminate bias, and that reducing bias too much leads to large increases in variance. In addition, estimators cannot be regular or (locally) quantile-unbiased. This provides additional motivation for considering the one-sided bias criteria proposed by [Chernozhukov, Lee, and Rosen \(2009\)](#).

## 2.2. Moment Inequality Models

There is now a large literature on estimation and inference for partially identified and moment inequality models. See, for example, [Imbens and Manski \(2004\)](#), [Chernozhukov, Hong, and Tamer \(2007\)](#), [Andrews and Soares \(2010\)](#), [Andrews and Guggenberger \(2009\)](#), [Pakes, Porter, Ho, and Ishii \(2012\)](#), [Beresteanu and Molinari \(2008\)](#), [Bugni \(2010\)](#), [Canay \(2010\)](#), [Rosen \(2008\)](#), [Stoye \(2009\)](#), [Fan and Park \(2008\)](#), [Galichon and Henry \(2009\)](#), and [Romano and Shaikh \(2010\)](#).

Nondifferentiability can arise in general moment inequality models when a finite set of moment inequality conditions define the identified set of interest. Suppose we have data  $Y_i \stackrel{\text{i.i.d.}}{\sim} P$  for  $i = 1, \dots, n$ , where  $P \in \mathcal{P}$ , and we have moment functions  $m_j(Y_i, \gamma)$ ,  $j = 1, \dots, J$  such that  $E_P[m_j(Y_i, \gamma)] \leq 0$  for some vector of parameters  $\gamma \in \mathbb{R}^L$ . Then the identified set for  $\gamma$  is

$$\Gamma = \{\gamma \in \mathbb{R}^L : E_P[m_j(Y_i, \gamma)] \leq 0 \text{ for } j = 1, \dots, J\}.$$

Suppose that the identified set  $\Gamma$  is nonempty, convex, and closed. For example, [Bontemps, Magnac, and Maurin \(2010\)](#) studied a model with linear moment inequalities and showed that the identified set satisfies these properties, and [Kaido and Santos \(2011\)](#) showed that a convexity condition on the moment conditions leads to a convex identified set. Then  $\Gamma$  can be fully characterized by its support function ([Rockafellar \(1970\)](#)), defined as

$$s(q, \Gamma) = \sup\{q'\gamma : \gamma \in \Gamma\} \quad \text{for } \|q\| = 1.$$

Suppose we fix a direction  $q$  and are interested in estimating the value of the support function at  $q$ . We can regard  $s(q, \Gamma)$  as a functional of  $P$ , the data distribution. In general, however, it may not be smooth in  $P$ . [Beresteanu and Molinari \(2008\)](#), [Bontemps, Magnac, and Maurin \(2010\)](#), and [Kaido and Santos \(2011\)](#) gave examples where this problem arises and leads to nonstandard distribution theory for their proposed procedures. The issue can also be seen in a simple example. Suppose  $Y_i = (X_i, Z_i)'$ ,  $\gamma = (\gamma_1, \gamma_2)'$ , and

$$m_1(Y_i, \gamma) = 2\gamma_1 + \gamma_2 - X_i,$$

$$m_2(Y_i, \gamma) = \gamma_1 + 2\gamma_2 - X_i,$$

$$m_3(Y_i, \gamma) = \gamma_1 + \gamma_2 - Z_i.$$

For  $q = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})'$ , straightforward calculations show that

$$s(q, \Gamma) = \sqrt{2} \min\left\{\frac{2}{3}E[X_i], E[Z_i]\right\}.$$

As noted in the previous example, our results below show that there do not exist regular estimators of  $s(q, \Gamma)$ . As a consequence, standard efficiency bound arguments do not apply, and some inference procedures proposed in the literature, which require uniform asymptotic normality (see [Imbens and Manski \(2004\)](#), [Stoye \(2009\)](#)), cannot be used.

### 2.3. Tests of Superior Predictive Ability

A recent literature has considered the problem of testing whether any of a finite set of forecasting rules outperforms a benchmark; see, for example, [White \(2000\)](#) and [Hansen \(2005\)](#). Suppose there are  $J$  forecasting procedures under consideration, and let  $\delta_j$  denote the relative performance of procedure  $j$  against the benchmark. We wish to test the null hypothesis

$$H_0 : \delta_j \leq 0 \quad \forall j = 1, \dots, J.$$

Suppose there exist statistics  $d_j$  such that, for  $d = (d_1, \dots, d_J)'$  and  $\delta = (\delta_1, \dots, \delta_J)'$ ,

$$\sqrt{n}(d - \delta) \rightsquigarrow N(0, \Omega).$$

White (2000) proposed inference based on the test statistic

$$T = \max_j \{\sqrt{nd_1}, \dots, \sqrt{nd_J}\}.$$

Hansen (2005) suggested an alternative, studentized version of the statistic, and an alternative construction of the null distribution.

#### 2.4. Inference for Expected Outcomes Under the Best Treatment

Consider a randomized experiment comparing outcomes under two treatments,  $T = 0, 1$ . Let  $Y(0)$  and  $Y(1)$  denote potential outcomes under the two treatments, and define

$$\begin{aligned}\pi_0 &= E[Y(0)], \\ \pi_1 &= E[Y(1)].\end{aligned}$$

Interest often focuses on estimating the average treatment effect  $\pi_1 - \pi_0$ . Recent work on optimal treatment assignment rules by Manski (2004), Dehejia (2005), Stoye (2006), Schlag (2006), Tetenov (2012), and Hirano and Porter (2009) has adopted a decision-theoretic approach for choosing the optimal treatment. Another object of interest is

$$\kappa = \max\{\pi_0, \pi_1\}.$$

This can be interpreted as the expected outcome under the best treatment. Clearly, sample analog estimators of this quantity will suffer from bias problems, just as in the previous two examples. Objects of this form play an important role in recent work on treatment assignment problems in dynamic settings initiated by Murphy (2003), where backward induction solutions must take into account the continuation payoffs from choosing the best treatment in later stages. Robins (2004) noted that estimators for many such models will generally suffer from bias and lack of regularity, and developed uniform inference procedures. Moodie and Richardson (2010) and Chakraborty, Strecher, and Murphy (2008) proposed bias-correction procedures. Our results below extend the arguments in Robins (2004) to show that lack of differentiability leads automatically to impossibility of locally asymptotically unbiased or regular estimators.

### 3. SMOOTH PARAMETRIC MODELS

We begin with a standard parametric setting, with data drawn from some member of a family of distributions indexed by a finite-dimensional parameter. We consider estimation of a function of this parameter, and show how lack of differentiability imposes strong restrictions on the asymptotic properties of estimators.

### 3.1. Setup of the Problem

We consider data generated by independent and identically distributed (i.i.d.) sampling from a smooth parametric distribution, although the results also apply to other settings, provided the local asymptotic normality (LAN) condition below holds (see van der Vaart (1998, Lemma 7.6)). Suppose that, for  $i = 1, \dots, n$ , the data  $Y_i$  are i.i.d. with

$$(1) \quad Y_i \sim P_\theta,$$

where  $\theta = (\theta_1, \dots, \theta_k)' \in \Theta \subset \mathbb{R}^k$  and  $\Theta$  is a nonempty open set. Let  $\mathcal{Y}$  denote the support of  $Y_i$ , which could take values in a subset of Euclidean space or in some more general space.

We assume a standard smoothness condition on the parametric statistical model around a centering value  $\theta_0 \in \Theta$ :

**ASSUMPTION 1:** (a) *Differentiability in quadratic mean: there exists a function  $s: \mathcal{Y} \rightarrow \mathbb{R}^m$ , the score function, such that*

$$\begin{aligned} & \int \left[ dP_{\theta_0+h}^{1/2}(y) - dP_{\theta_0}^{1/2}(y) - \frac{1}{2} h' \cdot s(y) dP_{\theta_0}^{1/2}(y) \right]^2 \\ &= o(\|h\|^2) \quad \text{as } h \rightarrow 0. \end{aligned}$$

(b) *The Fisher information matrix  $J_0 = E_{\theta_0}[ss']$  is nonsingular.*

Assumption 1 is a sufficient condition for the model to be locally asymptotically normal at  $\theta_0$  (van der Vaart (1998)). Given this assumption, it is useful to adopt the usual local parametrization around a point  $\theta_0$ ,

$$\theta_{n,h} = \theta_0 + \frac{h}{\sqrt{n}}.$$

Suppose interest centers on some function of the parameters,  $\kappa(\theta)$  where  $\kappa: \Theta \rightarrow \mathbb{R}$ . Under conventional smoothness conditions on the sequence of experiments  $\mathcal{E}^n = \{P_\theta^n: \theta \in \Theta\}$ , the maximum likelihood estimator (MLE)  $\hat{\theta}_{\text{ML}}$  and other estimators such as the Bayes estimator are asymptotically efficient. However, the limit distributions of derived estimators of  $\kappa(\theta)$ , and more generally the feasible limit distributions of any estimators of  $\kappa(\theta)$ , will depend crucially on the smoothness in  $\kappa$  at the point  $\theta_0$ .

Here we want to allow  $\kappa$  to lie in a class of functions that includes certain nondifferentiable functions, such as the min and max functions in the examples. For this purpose, define the *one-sided directional derivative* of  $\kappa$  at  $\theta_0$  in the direction  $h$  as

$$\dot{\kappa}(h) = \lim_{t \downarrow 0} \frac{\kappa(\theta_0 + th) - \kappa(\theta_0)}{t}.$$

(To simplify the notation, we suppress the dependence of  $\kappa$  on  $\theta_0$ .) The following assumption defines the class of functions that we consider.

**ASSUMPTION 2:**  $\kappa$  has one-sided directional derivatives in all directions at  $\theta_0$ .

Under our local parametrization and Assumption 2, a key issue is whether or not  $\kappa$  is differentiable in the ordinary sense at  $\theta_0$ , so that  $\dot{\kappa}(h)$  is linear in  $h$ . In the examples given above,  $\kappa$  satisfies Assumption 2 but not full differentiability for some values of  $\theta_0$ , and we will see that this is enough to rule out many desirable properties of estimators.

In this setting, an estimator (or estimator sequence) is a sequence of functions  $T_n: \mathcal{Y}^n \rightarrow \mathbb{R}$ . We focus on estimators that possess limit distributions in the sense that, for all  $h$ ,

$$(2) \quad \sqrt{n}(T_n - \kappa(\theta_{n,h})) \overset{h}{\rightsquigarrow} L_h,$$

where  $\overset{h}{\rightsquigarrow}$  indicates weak convergence under the drifting parameter sequence  $\theta_{n,h}$ . The  $L_h$  are the limiting laws of the estimator under different local sequences of parameters. These laws could, in general, be degenerate.

The standard definition of a *regular* estimator is one that has  $L_h = L$  for all  $h$ , where  $L$  does not depend on  $h$ . This condition is intended to capture the requirement that the centered limit distributions be invariant to small perturbations of the parameters. Regularity plays an important role in conventional results on optimality of point estimators, such as semiparametric efficiency bounds and convolution theorems, and is also crucial for the uniform validity of standard inference procedures.<sup>2</sup> We also consider the following additional criteria for estimator sequences:

*Local Asymptotic Unbiasedness:*  $L_h$  has expectation 0 for all  $h$ .

*Local Asymptotic  $\alpha$ -Quantile Unbiasedness:* For all  $h$ ,

$$L_h\{(-\infty, 0]\} = \alpha.$$

By the Portmanteau lemma, this last condition is equivalent to

$$\Pr_{\theta_0+h/\sqrt{n}}(T_n \leq \kappa(\theta_0 + h/\sqrt{n})) \rightarrow \alpha.$$

None of these properties implies the other. An implicit assumption of the local asymptotic unbiasedness definition is that expectation of the limit distribution exists for each  $h$ . If a regular estimator sequence exists *and* its limit distribution  $L$  has an expectation, then there exists a locally asymptotically unbiased estimator. (If  $c$  is the expectation of  $L$ , take  $T_n - (c/\sqrt{n})$ .) Similarly,

<sup>2</sup>For instance, regularity is necessary for consistency of the parametric bootstrap in LAN models (Beran (1997)). See also Dümbgen (1993).

if a regular estimator sequence exists *and* its limit distribution has an exact  $\alpha$ -quantile (i.e.,  $c$  such that  $L\{(-\infty, c]\} = \alpha$ ), then  $T_n - (c/\sqrt{n})$  is a locally asymptotically  $\alpha$ -quantile-unbiased estimator.

If, say,  $\kappa(\theta) = \theta_1$  or  $\kappa(\theta)$  is a linear function of  $\theta$ , standard estimators such as the maximum likelihood estimator satisfy regularity, local asymptotic unbiasedness, and local asymptotic median unbiasedness, and it is straightforward to construct statistics that are locally asymptotically  $\alpha$ -quantile-unbiased for any  $\alpha \in (0, 1)$ . Our goal is to examine whether these standard properties can be satisfied for the  $\kappa$  that arise naturally in the examples in Section 2. To this end, Le Cam's limits of experiments theory provides a powerful characterization of the limit distributions  $L_h$  that appear in (2).

### 3.2. Limits of Experiments Analysis

Le Cam (1970, 1972) showed that, under differentiability in quadratic mean, the original statistical *model* can be approximated in large samples by a shifted Gaussian model. Various versions of Le Cam's results are now available, but the asymptotic representation theorem of van der Vaart (Theorem 4.1 of van der Vaart (1991a)) is convenient for our purposes. Specialized to our setting, the result says that for any estimator sequence  $\{T_n\}$  with limits as in (2), the limit laws  $L_h$  correspond to the *exact* distribution of the quantity  $T(Z, U) - \dot{\kappa}(h)$ , where  $T(Z, U)$  is a randomized estimator,

$$Z \sim N(h, J_0^{-1}),$$

and  $U$  is Uniform on  $[0, 1]$  independently of  $Z$ . In this sense, the shifted normal model characterizes the possible limit distributions for estimator sequences in the original model, and we call the shifted normal model the "limit experiment."

Thus, we consider estimation of  $\dot{\kappa}(h)$  in the  $N(h, J_0^{-1})$  model. Each of the estimator sequence properties defined above (regularity, local asymptotic unbiasedness, local asymptotic  $\alpha$ -quantile unbiasedness) implies a corresponding property of the matching estimator in the limit experiment. Let  $T_n$  be an estimator sequence with matching limit experiment estimator  $T$ . If  $T_n$  is regular, then  $T$  is *translation equivariant*. That is, the distribution of  $T - \dot{\kappa}$  under  $N(h, J_0^{-1})$  does not depend on  $h$ . Similarly, local asymptotic unbiasedness of  $T_n$  implies that  $T$  is unbiased, and local asymptotic  $\alpha$ -quantile unbiasedness of  $T_n$  implies  $\alpha$ -quantile unbiasedness of  $T$ .

Assumption 2 also has a key implication in the limit experiment. If  $\kappa$  satisfies Assumption 2, then  $\dot{\kappa}(h)$  is positive homogeneous (of degree 1):

$$\dot{\kappa}(\gamma h) = \gamma \dot{\kappa}(h) \quad \text{for all } \gamma \geq 0 \quad \text{and} \quad h \in \mathbb{R}^k.$$

We show that the existence of an unbiased estimator implies that  $\dot{\kappa}(h)$  is linear in  $h$ . Assumption 2 combined with linearity of the directional derivatives



is equivalent to differentiability of  $\kappa$ . So, we conclude that if  $\kappa$  is not differentiable, then there cannot exist an unbiased estimator for  $\dot{\kappa}$  in the limit experiment. Similar arguments yield the nonexistence of translation equivariance and  $\alpha$ -quantile-unbiased estimators.

To illustrate the argument, consider the quantile-unbiasedness criterion. Assume that  $T(Z, U)$  is an  $\alpha$ -quantile-unbiased estimator of  $\dot{\kappa}(h)$  in the  $N(h, J_0^{-1})$  model. That is, for all  $h$  and any  $\gamma \geq 0$ ,

$$\alpha = \Pr_{\gamma h}(T \leq \dot{\kappa}(\gamma h)) = \Pr_{\gamma h}(T \leq \gamma \dot{\kappa}(h)),$$

where the last equality follows by the homogeneity of  $\dot{\kappa}$ . Let  $F_T$  denote the CDF of  $T$  when  $Z \sim N(0, J_0^{-1})$  (i.e., under  $h = 0$ ). Then

$$\begin{aligned} 0 &= \lim_{\gamma \downarrow 0} \frac{1}{\gamma} [\Pr_{\gamma h}(T \leq \gamma \dot{\kappa}(h)) - \Pr_0(T \leq 0)] \\ &= \lim_{\gamma \downarrow 0} \left\{ \frac{1}{\gamma} [\Pr_{\gamma h}(T \leq \gamma \dot{\kappa}(h)) - \Pr_0(T \leq \gamma \dot{\kappa}(h))] \right. \\ &\quad \left. + \frac{1}{\gamma} [F_T(\gamma \dot{\kappa}(h)) - F_T(0)] \right\}. \end{aligned}$$

If we assume that the CDF is differentiable at zero and denote its derivative by  $f_T$ , then the second term in the last expression has a limit equal to  $f_T(0)\dot{\kappa}(h)$ . For the first term, a uniform integrability condition follows from the exponential tail behavior of the normal distribution, and allows us to pass the limit inside the integral. The limit for the first term can then be expressed as  $c'h$ , where  $c = \int_{[0,1]} \int \mathbf{1}\{T(z, u) \leq 0\} (z'J_0^{-1}) dN(z|0, J_0^{-1}) du$ . Finally, assuming that  $f_T(0) > 0$ , we can solve for  $\dot{\kappa}$  in its linear form explicitly:  $\dot{\kappa}(h) = -(c/f_T(0))'h$ .

Similar arguments can be used for unbiasedness and translation equivariance. Intuitively, the fact that the distribution of  $T$  can depend on  $h$  only through the data  $Z$ , and the positive homogeneity of  $\kappa$ , impose strong restrictions for  $\kappa$  to be attained by any of the criteria we consider. If an unbiased estimator exists, and  $\dot{\kappa}$  is positive homogeneous, a uniform integrability condition leads to the conclusion that  $\dot{\kappa}$  is linear in  $h$ . If a translation equivariant estimator exists, we work with the characteristic function and show linearity. We obtain the following result.

**PROPOSITION 1:** *Let  $Z \sim N(h, J_0^{-1})$ ,  $U \sim \text{Unif}[0, 1]$  independently of  $Z$ , and let  $\dot{\kappa}: \mathbb{R}^k \rightarrow \mathbb{R}$  be positive homogeneous.*

(a) *Suppose there exists an unbiased randomized estimator  $T(Z, U)$ , that is,*

$$\dot{\kappa}(h) = E_h[T(Z, U)] := \int \int T(z, u) dN(z|h, J_0^{-1}) du \quad \text{for all } h \in \mathbb{R}^k.$$

*Then  $\dot{\kappa}(h)$  must be linear in  $h$ .*

- (b) Suppose there exists a randomized estimator  $T(Z, U)$  such that the law of  $T(Z, U) - \dot{\kappa}(h)$  under  $h$  is the same for all  $h$ . Then  $\dot{\kappa}(h)$  is linear in  $h$ .
- (c) Suppose  $T(Z, U)$  is a statistic such that, for some  $\alpha \in (0, 1)$ ,

$$\alpha = \Pr_h(T \leq \dot{\kappa}(h)) \quad \text{for all } h.$$

Let  $F_T$  denote the CDF of  $T$  under  $h = 0$ . Assume that the derivative of  $F_T$  exists at zero and is positive. Then  $\dot{\kappa}(h)$  is linear in  $h$ .

See the [Appendix](#) for the proof.

Thus, unbiasedness, translation equivariance, and quantile unbiasedness imply linearity of the estimand in the shifted normal model. So, for example, there exists no unbiased estimator of the minimum of the elements of  $h$  in the shifted normal model.<sup>3</sup>

We use Proposition 1, which is an exact result for the normal shift experiment, to obtain a large-sample impossibility result in the original smooth parametric model. In particular, a locally asymptotically unbiased estimator for  $\kappa(h)$  in the smooth parametric model is matched by an unbiased estimator for  $\dot{\kappa}(h)$  in the limit experiment. The existence of an unbiased estimator in the limit experiment implies that  $\dot{\kappa}(h)$  is linear in  $h$ , which, combined with Assumption 2, means that  $\kappa(h)$  is differentiable. We state this finding by noting that if  $\kappa(h)$  is not differentiable, then no locally asymptotically unbiased estimator exists. A similar argument yields the analogous result for regularity and local asymptotic  $\alpha$ -quantile unbiasedness.

**THEOREM 2:** Consider the parametric model in Equation (1), and suppose that Assumptions 1 and 2 hold. If  $\kappa$  is not differentiable at  $\theta_0$ , then:

- (a) There exists no locally asymptotically unbiased estimator for  $\kappa$ .
- (b) There exists no regular estimator for  $\kappa$ .
- (c) For any  $\alpha \in (0, 1)$ , there exists no local asymptotic  $\alpha$ -quantile-unbiased estimator  $T_n$  for  $\kappa$  such that  $\sqrt{n}(T_n - \kappa(\theta_0)) \overset{\theta_0}{\rightsquigarrow} L_0$  and  $F_\infty(t) := L_0\{(-\infty, t]\}$  has positive derivative at  $t = 0$ .

### 3.3. Implications

Theorem 2(a) implies that bias correction procedures cannot fully remove the bias of any estimator under our local asymptotic approximation. Moreover, Doss and Sethuraman (1989) showed that in cases where no unbiased estimator exists, a sequence of estimators whose bias approaches zero must

<sup>3</sup>Blumenthal and Cohen (1968) showed this for the case of two independent normals, using a slightly different argument. Fraser (1952) obtained a similar result for impossibility of quantile-unbiased statistics for the maximum of independent normal means, under slightly different restrictions and using a different argument.

have variance approaching infinity. Hence, attempting to make the bias very small may lead to highly variable estimates. This does not preclude modifying procedures to reduce bias, but suggests that one should assess carefully the bias–variance trade-off. One possible approach is to set a bound on the bias that will be permitted, and look for procedures with low variance subject to this restriction. For a discussion of this approach, see [Le Cam \(1993\)](#).

More generally, our results suggest that one should consider other estimator criteria, such as the one-sided quantile unbiasedness condition examined by [Chernozhukov, Lee, and Rosen \(2009\)](#). And while some notions of asymptotic optimality (such as variance bounds corresponding to minimum variance unbiased estimation in the limit experiment) will clearly not lead to useful comparisons of estimators, one can still use concepts such as local asymptotic minmax risk to define asymptotic optimality of estimator sequences. See, for example, [Song \(2010\)](#).

Theorem 2 also has implications for inference. Since no regular estimator exists, the usual arguments for the validity of standard approaches to inference, such as Wald-type procedures, will not be valid. Applying bias-correction before implementing standard inference procedures will not remedy the problem. Hence, one cannot avoid a nonstandard distribution theory for inference in these settings.<sup>4</sup> Part (c) of Theorem 2 also implies that there exist no locally asymptotically similar one-sided confidence intervals (i.e., confidence intervals that are half-lines) for the parameter  $\kappa$  under the assumed conditions. [Andrews and Soares \(2010\)](#) discussed nonsimilarity of tests in moment inequality models, and [Andrews \(2010\)](#) showed that similar-on-the-boundary tests exist, but have poor power.

#### 4. INFINITE-DIMENSIONAL MODELS

In many empirical settings, including many of the examples in Section 2, the set of possible data-generating measures is not finite-dimensional as we assumed in the previous section. However, under suitable conditions, our impossibility results continue to hold. We show this by considering parametric submodels of the larger model, following an approach used in the semiparametric efficiency bound literature (e.g., [Newey \(1990\)](#)).

##### 4.1. *Extension of Impossibility Results*

Let  $\mathcal{P}$  denote a collection of probability measures on  $\mathcal{Y}$  (the sample space as in the previous section). This is a potentially infinite-dimensional statistical model. Now, suppose there is a finite-dimensional submodel  $\mathcal{P}_f \subset \mathcal{P}$ , where

<sup>4</sup> The impossibility theorems of [Gleser and Hwang \(1987\)](#) and [Dufour \(1997\)](#) for weakly identified models are distinct from our results, but they also show that standard distribution theory is not feasible in certain classes of models.

the submodel can be parametrized as  $\mathcal{P}_f = \{P_\theta : \theta \in \Theta\}$  for an open set  $\Theta \subset \mathbb{R}^{k_f}$  with  $k_f$  finite. Fix a “centering” probability measure  $P_0 \in \mathcal{P}$ . The submodel passes through  $P_0$  if  $P_0 \in \mathcal{P}_f$ . That is, for some  $\theta_0 \in \Theta$ ,  $P_0 = P_{\theta_0}$ . A parametric submodel is called *regular* if the submodel passes through  $P_0$  and Assumption 1 is satisfied on  $\mathcal{P}_f$ .

We suppose the object of interest is a real-valued quantity that depends on the underlying probability measure generating the data. We can think of this estimand,  $\kappa[P]$ , as a functional defined on the space  $\mathcal{P}$ . For any regular parametric model  $\mathcal{P}_f$ , we can write this parameter of interest as a function of the submodel parameters by defining the real function  $\kappa_f$  on  $\Theta$  as  $\kappa_f(\theta) = \kappa[P_\theta]$ . In the result below, we require that  $\kappa_f(\theta)$  satisfies Assumption 2 on the parametric submodel, which implicitly limits the analysis to functionals that are estimable at a  $\sqrt{n}$  rate.

Now we can define the different properties of estimators of  $\kappa$  on  $\mathcal{P}$ . The properties given above for a parametric model were defined relative (“local”) to a fixed value  $\theta_0$ . Analogously, the properties of an estimator on  $\mathcal{P}$  will be defined relative to  $P_0$ . An estimator  $T_n$  is *locally asymptotically unbiased* for  $\kappa$  if it is locally asymptotically unbiased for  $\kappa_f$  on every regular parametric submodel. The definitions of locally asymptotically  $\alpha$ -quantile unbiased and regular for an estimator  $T_n$  on  $\mathcal{P}$  are defined analogously by requiring the properties to hold on each regular parametric submodel.

If the assumptions of Theorem 2 are satisfied on the regular parametric submodel  $\mathcal{P}_f$ , then the impossibility of locally asymptotically unbiased, regular, and locally asymptotically  $\alpha$ -quantile-unbiased estimators on  $\mathcal{P}_f$  imply their nonexistence on the infinite-dimensional space  $\mathcal{P}$ . Thus, we have the following corollary.

**COROLLARY 3:** *Suppose the data  $Y_i$  are i.i.d. with  $Y_i \sim P \in \mathcal{P}$ . Fix a  $P_0 \in \mathcal{P}$ . Suppose  $\kappa_f$  satisfies Assumption 2 for a regular parametric submodel  $\mathcal{P}_f$ . If  $\kappa_f$  is not differentiable at  $\theta_0$ , then there does not exist a locally asymptotically unbiased estimator or a regular estimator for  $\kappa$ . Moreover, for any  $\alpha \in (0, 1)$ , there does not exist a locally asymptotically  $\alpha$ -quantile-unbiased estimator  $T_n$  for  $\kappa$  such that  $\sqrt{n}(T_n - \kappa[P_0]) \overset{P_0}{\rightsquigarrow} L_0$  and  $F_\infty(t) := L_0\{(-\infty, t]\}$  has positive derivative at  $t = 0$ .*

#### 4.2. Application to the Haile–Tamer Model

We illustrate Corollary 3 for the example in Section 2.1. For each auction  $i = 1, \dots, n$ , we observe the number of participants,  $m_i \in \{2, \dots, M\}$ , in the auction, and we observe the vector of bids  $b_i \in \mathbb{R}_+^{m_i}$ . We regard  $(m_i, b_i)$  as an i.i.d. draw from a joint distribution  $P$  on  $\{2, \dots, M\} \times \mathcal{B}$ , where the space of bid vectors is  $\mathcal{B} = \mathbb{R}_+^2 \cup \mathbb{R}_+^3 \cup \dots \cup \mathbb{R}_+^M$ . Let  $\pi = (\pi_2, \dots, \pi_M)$  give the marginal probabilities for  $m_i$ , that is,  $\Pr(m_i = m) = \pi_m$ . For the conditional distribution of  $b_i$  given  $m_i = m$ , assume it concentrates on  $\mathbb{R}_+^m$  and is equal to the  $m$ -fold

product of some probability measure  $P_m^b$  with CDF  $G_m$ . This captures the situation where the bid vector  $b_i$  contains  $m_i$  i.i.d. draws from a distribution that depends on the number of bidders in auction  $i$ . Hence the distribution  $P$  can be characterized by  $\pi$  and  $G_2, \dots, G_M$ . The functional of interest,  $\kappa[P]$ , equals  $\min\{G_2(v), \dots, G_M(v)\}$  for some fixed  $v$ .

The statistical model  $\mathcal{P}$  is the set of possible distributions  $P$ . For example, we could take  $\mathcal{P}$  to include all  $P$  with  $\pi$  in the interior of the  $(M-2)$ -dimensional simplex  $\Delta_{M-2}$ , and CDFs  $G_2, \dots, G_M$  with nonnegative support. Suppose we fix a centering  $P_0 \in \mathcal{P}$  such that  $\kappa[P_0] = G_j(v) = G_k(v)$  for some  $j \neq k$ . Then, considering local alternatives around  $P_0$  will capture the situation where two (or more) of the elements in the minimum defining  $\kappa$  are close to each other, which is the source of the bias problem highlighted by [Haile and Tamer \(2003\)](#).

Suppose we have a parametric submodel  $\mathcal{P}_f$  with parameter

$$\theta = (\pi_2, \dots, \pi_M, \mu_2, \sigma_2, \dots, \mu_M, \sigma_M),$$

where  $\pi_m = \Pr(m_i = m)$  (as before), and, given  $\theta$ , the bid CDF  $G_m$  is equal to the lognormal CDF

$$G_m(v) = \Phi\left(\frac{\log v - \mu_m}{\sigma_m}\right).$$

Then

$$\kappa_f(\theta) = \min\left\{\Phi\left(\frac{\log v - \mu_2}{\sigma_2}\right), \dots, \Phi\left(\frac{\log v - \mu_M}{\sigma_M}\right)\right\}.$$

The parameter space  $\Theta$  is some open subset of  $\text{int}(\Delta_{M-2}) \times \{\mathbb{R} \times \mathbb{R}_{++}\}^{M-2}$ . Suppose  $\mathcal{P}_f$  is a regular parametric submodel (at  $P_0$ ) so that there is a  $\theta_0 \in \Theta$  such that  $P_{\theta_0} = P_0$ . This implies that, for  $\theta_0 = (\pi_{02}, \dots, \pi_{0M}, \mu_{02}, \sigma_{02}, \dots, \mu_{0M}, \sigma_{0M})$ , we have

$$\frac{\log v - \mu_{0j}}{\sigma_{0j}} = \frac{\log v - \mu_{0k}}{\sigma_{0k}} \leq \frac{\log v - \mu_{0m}}{\sigma_{0m}} \quad \forall m \neq j, k.$$

Then all the conditions of [Corollary 3](#) are satisfied, and  $\kappa_f$  is nondifferentiable at  $\theta_0$ , so the conclusion of [Corollary 3](#) follows.

We also note that it would be possible to construct regular parametric submodels such that  $\kappa_f$  is in fact differentiable, but we only require nondifferentiability for a single submodel to obtain the impossibility result.

## 5. CONCLUSION

The limits of experiments framework provides a useful tool for characterizing asymptotic statistical problems, by reducing the analysis to consideration

of the multivariate normal shift experiment. We find that unbiasedness, regularity, and quantile unbiasedness impose strict conditions on the estimands, which can be violated in a number of econometric applications of recent interest. This suggests that alternative criteria should be the focus of attention, at least in these settings.

Local asymptotic normality also provides a useful way to devise alternative procedures with good properties, and to compare different procedures. Any sequence of statistics with limit distributions has a matching statistic in the limit experiment. This suggests that we could work directly in the normal model, propose alternative estimators or inference procedures, and compare their distributions under different parameter values. If we find a satisfactory procedure in the normal model, it is usually possible to construct the matching sequence of estimators in the original problem of interest.

#### APPENDIX: PROOFS

Let  $\phi(\cdot|h, J_0^{-1})$  denote the density of the multivariate normal distribution  $N(h, J_0^{-1})$ . We first establish some uniform integrability conditions.

LEMMA 1: (a) For all  $h$ ,

$$\int_{[0,1]} \int \sup_{r \in (0,1]} \left| \frac{1}{r} [\phi(z|rh, J_0^{-1}) - \phi(z|0, J_0^{-1})] \right| dz du < \infty.$$

(b) If  $E_{\tilde{h}}[|T(Z, U)|] < \infty$  for all  $\tilde{h}$ , then, for all  $h$ ,

$$\int_{[0,1]} \int \sup_{r \in (0,1]} \left| T(z, u) \cdot \frac{1}{r} [\phi(z|rh, J_0^{-1}) - \phi(z|0, J_0^{-1})] \right| dz du < \infty.$$

PROOF: Let

$$\mathcal{Z}^+ = \{z : h'J_0(z - h) \geq 0\} \quad \text{and} \quad \mathcal{Z}^- = \{z : h'J_0(z + h) \leq 0\}.$$

For  $z \in \mathcal{Z}^+$  and  $r \in (0, 1]$ ,  $h'J_0(z - rh) = h'J_0z - rh'J_0h \geq h'J_0(z - h) \geq 0$ . Also, for  $z \in \mathcal{Z}^+$  and  $r \in (0, 1]$ ,

$$\begin{aligned} -\frac{1}{2}(z - rh - h)'J_0(z - rh - h) &\leq -\frac{1}{2}(z - h)'J_0(z - h) + h'J_0(z - h) \\ &= -\frac{1}{2}(z - 2h)'J_0(z - 2h) + \frac{1}{2}h'J_0h. \end{aligned}$$

For  $z \in \mathcal{Z}^-$  and  $r \in (0, 1]$ ,  $h'J_0(z - rh) \leq h'J_0(z + h) \leq 0$ . Also, for  $z \in \mathcal{Z}^-$  and  $r \in (0, 1]$ ,

$$-\frac{1}{2}(z - rh + h)'J_0(z - rh + h) \leq -\frac{1}{2}(z + h)'J_0(z + h).$$

For  $z \in (\mathcal{Z}^+ \cup \mathcal{Z}^-)^c$  and  $r \in (0, 1]$ ,  $-h'J_0h < h'J_0z < h'J_0h$ , so  $-2h'J_0h < h'J_0(z - rh) < h'J_0h$ , so  $|h'J_0(z - rh)| < 2h'J_0h$ . Also, for  $z \in (\mathcal{Z}^+ \cup \mathcal{Z}^-)^c$  and  $r \in (0, 1]$ ,

$$-\frac{1}{2}(z - rh)'J_0(z - rh) \leq -\frac{1}{2}z'J_0z + rh'J_0z \leq -\frac{1}{2}z'J_0z + h'J_0h.$$

Consider

$$\begin{aligned} & \sup_{r \in (0, 1]} \frac{1}{r} |\phi(z|rh, J_0^{-1}) - \phi(z|0, J_0^{-1})| \\ & \leq \sup_{r \in (0, 1]} \left| \frac{\partial}{\partial r} \phi(z|rh, J_0^{-1}) \right| \\ & = \sup_{r \in (0, 1]} (2\pi)^{-k/2} \det(J_0)^{1/2} \exp\left(-\frac{1}{2}(z - rh)'J_0(z - rh)\right) \\ & \quad \times |h'J_0(z - rh)| \\ & \leq \begin{cases} \sup_{r \in (0, 1]} (2\pi)^{-k/2} \det(J_0)^{1/2} \exp\left(-\frac{1}{2}(z - rh)'J_0(z - rh)\right) \\ \quad \times \exp(h'J_0(z - rh)), & \text{if } z \in \mathcal{Z}^+, \\ \sup_{r \in (0, 1]} (2\pi)^{-k/2} \det(J_0)^{1/2} \exp\left(-\frac{1}{2}(z - rh)'J_0(z - rh)\right) \\ \quad \times \exp(-h'J_0(z - rh)), & \text{if } z \in \mathcal{Z}^-, \\ \sup_{r \in (0, 1]} (2\pi)^{-k/2} \det(J_0)^{1/2} \exp\left(-\frac{1}{2}(z - rh)'J_0(z - rh)\right) \\ \quad \times 2h'J_0h, & \text{if } z \in (\mathcal{Z}^+ \cup \mathcal{Z}^-)^c \end{cases} \\ & \leq \begin{cases} (2\pi)^{-k/2} \det(J_0)^{1/2} \exp\left(-\frac{1}{2}(z - 2h)'J_0(z - 2h)\right) \\ \quad \times \exp(h'J_0h), & \text{if } z \in \mathcal{Z}^+, \\ (2\pi)^{-k/2} \det(J_0)^{1/2} \exp\left(-\frac{1}{2}(z + h)'J_0(z + h)\right) \\ \quad \times \exp(h'J_0h/2), & \text{if } z \in \mathcal{Z}^-, \\ (2\pi)^{-k/2} \det(J_0)^{1/2} \exp\left(-\frac{1}{2}z'J_0z\right) \\ \quad \times \exp(h'J_0h)2h'J_0h, & \text{if } z \in (\mathcal{Z}^+ \cup \mathcal{Z}^-)^c \end{cases} \\ & = \begin{cases} \phi(z|2h, J_0^{-1}) \exp(h'J_0h), & \text{if } z \in \mathcal{Z}^+, \\ \phi(z|-h, J_0^{-1}) \exp(h'J_0h/2), & \text{if } z \in \mathcal{Z}^-, \\ \phi(z|0, J_0^{-1}) 2 \exp(h'J_0h) h'J_0h, & \text{if } z \in (\mathcal{Z}^+ \cup \mathcal{Z}^-)^c. \end{cases} \end{aligned}$$

The first inequality above follows by  $e^v > v$ . Then,

$$\begin{aligned}
 & \int_{[0,1]} \int \sup_{r \in (0,1]} \left| \frac{1}{r} [\phi(z|rh, J_0^{-1}) - \phi(z|0, J_0^{-1})] \right| dz du \\
 & \leq \int_{[0,1]} \left\{ \int_{\mathcal{Z}^+} \phi(z|2h, J_0^{-1}) \exp(h'J_0h) dz du \right. \\
 & \quad + \int_{\mathcal{Z}^-} \phi(z|-h, J_0^{-1}) \exp(h'J_0h/2) dz du \\
 & \quad \left. + \int_{(\mathcal{Z}^+ \cup \mathcal{Z}^-)^c} \phi(z|0, J_0^{-1}) 2 \exp(h'J_0h) h'J_0h dz du \right\} \\
 & \leq \exp(h'J_0h) + \exp(h'J_0h/2) + 2 \exp(h'J_0h) h'J_0h \\
 & < \infty.
 \end{aligned}$$

And,

$$\begin{aligned}
 & \int_{[0,1]} \int |T(z, u)| \sup_{r \in (0,1]} \left| \frac{1}{r} [\phi(z|rh, J_0^{-1}) - \phi(z|0, J_0^{-1})] \right| dz du \\
 & \leq \int_{[0,1]} \left\{ \int_{\mathcal{Z}^+} |T(z, u)| \phi(z|2h, J_0^{-1}) \exp(h'J_0h) dz du \right. \\
 & \quad + \int_{\mathcal{Z}^-} |T(z, u)| \phi(z|-h, J_0^{-1}) \exp(h'J_0h/2) dz du \\
 & \quad \left. + \int_{(\mathcal{Z}^+ \cup \mathcal{Z}^-)^c} |T(z, u)| \phi(z|0, J_0^{-1}) 2 \exp(h'J_0h) h'J_0h dz du \right\} \\
 & \leq E_{2h}[|T(Z, U)|] \exp(h'J_0h) + E_{-h}[|T(Z, U)|] \exp(h'J_0h/2) \\
 & \quad + 2E_0[|T(Z, U)|] \exp(h'J_0h) h'J_0h \\
 & < \infty. \qquad \qquad \qquad Q.E.D.
 \end{aligned}$$

**PROOF OF PROPOSITION 1:** If  $\dot{\kappa}(h) = 0$  for all  $h$ , then  $\dot{\kappa}$  is trivially linear in  $h$ . So consider the case where  $\dot{\kappa}(h) \neq 0$  for some  $h$ . Positive homogeneity implies  $\dot{\kappa}(0) = 0$  so  $h \neq 0$ .

(a) Assuming that  $T(Z, U)$  is unbiased for  $\dot{\kappa}(h)$  implies existence of the expectation. For  $r \geq 0$ ,

$$r\dot{\kappa}(h) = \dot{\kappa}(rh) = E_{rh}[T(Z, U)] = \int_{[0,1]} \int T(z, u) \phi(z|rh, J_0^{-1}) dz du.$$



By Lemma 1, the limit below passes through the integrals, so

$$\begin{aligned}\dot{\kappa}(h) &= \frac{\partial}{\partial r}[r\dot{\kappa}(h)] \Big|_{r=0} = \lim_{r \downarrow 0} \frac{1}{r}[\dot{\kappa}(rh) - \dot{\kappa}(0)] \\ &= \lim_{r \downarrow 0} \int_{[0,1]} \int T(z, u) \cdot \frac{1}{r}[\phi(z|rh, J_0^{-1}) - \phi(z|0, J_0^{-1})] dz du \\ &= \int_{[0,1]} \int T(z, u) \cdot \lim_{r \downarrow 0} \frac{1}{r}[\phi(z|rh, J_0^{-1}) - \phi(z|0, J_0^{-1})] dz du \\ &= \int_{[0,1]} \int T(z, u) \left( \frac{\partial}{\partial \tilde{h}} \phi(z|\tilde{h}, J_0^{-1}) \right)_{\tilde{h}=0} h dz du \\ &= \left\{ \int_{[0,1]} \int T(z, u)(z'J_0)\phi(z|0, J_0^{-1}) dz du \right\} h.\end{aligned}$$

(b) The characteristic function of the recentered estimator is  $\psi_h(s) = E_h[\exp(is(T - \dot{\kappa}(h)))]$ . By assumption,  $\psi_h(s)$  does not depend on  $h$ . So, for any  $r > 0$ ,

$$\begin{aligned}0 &= \frac{1}{r}(\psi_{rh}(s) - \psi_0(s)) \\ &= \frac{1}{r} \exp(-isr\dot{\kappa}(h)) \\ &\quad \times (E_{rh}[\exp(isT(Z, U))] - E_0[\exp(isT(Z, U))]) \\ &\quad + \frac{1}{r}(\exp(-isr\dot{\kappa}(h)) - 1)E_0[\exp(isT(Z, U))].\end{aligned}$$

Now, we take limits. Notice that  $|\exp(-isr\dot{\kappa}(h))| \leq 1$ , so we can pass the limit inside the integrals representing the expectations in the first term above by Lemma 1. We have

$$\begin{aligned}&\lim_{r \downarrow 0} \frac{1}{r} \exp(-isr\dot{\kappa}(h)) \\ &\quad \times (E_{rh}[\exp(isT(Z, U))] - E_0[\exp(isT(Z, U))]) \\ &= \int_{[0,1]} \int \lim_{r \downarrow 0} \exp(-isr\dot{\kappa}(h)) \exp(isT(z, u)) \\ &\quad \times \frac{1}{r}[\phi(z|rh, J_0^{-1}) - \phi(z|0, J_0^{-1})] dz du \\ &= \left\{ \int_{[0,1]} \int \exp(isT(z, u))(z'J_0)\phi(z|0, J_0^{-1}) dz du \right\} h.\end{aligned}$$

Also,

$$\begin{aligned} \lim_{r \downarrow 0} \frac{1}{r} (\exp(-isr\dot{\kappa}(h)) - 1) E_0[\exp(isT(Z, U))] \\ = \dot{\kappa}(h)(-is) E_0[\exp(isT(Z, U))]. \end{aligned}$$

Note that  $E_0[\exp(isT(Z, U))]$  is the characteristic function for  $T$  under  $h = 0$ . The characteristic function is continuous in  $s$  at  $s = 0$ , and at  $s = 0$ , the characteristic function is equal to 1. So, there exists  $s \neq 0$  such that  $E_0[\exp(isT(Z, U))] \neq 0$ . For this  $s$ ,

$$\dot{\kappa}(h) = \frac{iE_0[\exp(isT(Z, U))(Z'J_0)]}{sE_0[\exp(isT(Z, U))]}h.$$

(c) For  $r \geq 0$ ,

$$\alpha = \Pr_{rh}(T \leq r\dot{\kappa}(h)) = \Pr_{rh}(T \leq r\dot{\kappa}(h)).$$

Evaluating the above expression at some  $r > 0$  and at  $r = 0$ , we have

$$0 = \alpha - \alpha = \Pr_{rh}(T \leq r\dot{\kappa}(h)) - \Pr_0(T \leq 0),$$

and

$$\begin{aligned} (3) \quad 0 &= \lim_{r \downarrow 0} \left\{ \frac{1}{r} [\Pr_{rh}(T \leq r\dot{\kappa}(h)) - \Pr_0(T \leq r\dot{\kappa}(h))] \right. \\ &\quad \left. + \frac{1}{r} [\Pr_0(T \leq r\dot{\kappa}(h)) - \Pr_0(T \leq 0)] \right\}. \end{aligned}$$

The limit applied to each of the last two terms in (3) exists, so we can write the limit of the sum of terms as the sum of the limits. Consider the limit of the first term. Applying the uniform integrability condition shown in Lemma 1, and the dominated convergence theorem, we have

$$\begin{aligned} \lim_{r \downarrow 0} \frac{1}{r} [\Pr_{rh}(T \leq r\dot{\kappa}(h)) - \Pr_0(T \leq r\dot{\kappa}(h))] \\ = \lim_{r \downarrow 0} \int_{[0,1]} \int \mathbf{1}\{T(z, u) \leq r\dot{\kappa}(h)\} \\ \times \frac{1}{r} [\phi(z|rh, J_0^{-1}) - \phi(z|0, J_0^{-1})] dz du \\ = \int_{[0,1]} \int \lim_{r \downarrow 0} \mathbf{1}\{T(z, u) \leq r\dot{\kappa}(h)\} \end{aligned}$$

$$\begin{aligned} & \times \frac{1}{r} [\phi(z|rh, J_0^{-1}) - \phi(z|0, J_0^{-1})] dz du \\ &= \int_{[0,1]} \int \mathbf{1}\{T(z, u) \leq 0\} \left( \frac{\partial}{\partial \tilde{h}} \phi(z|\tilde{h}, J_0^{-1}) \right)_{\tilde{h}=0} h dz du \\ &= \left\{ \int_{[0,1]} \int \mathbf{1}\{T(z, u) \leq 0\} (z' J_0) \phi(z|0, J_0^{-1}) dz du \right\} h. \end{aligned}$$

The limit of the second term in (3) is straightforward due to the assumption of differentiability of  $F_T$  at zero. So, (3) can be rewritten as

$$0 = \left\{ \int_{[0,1]} \int \mathbf{1}\{T(z, u) \leq 0\} (z' J_0) \phi(z|0, J_0^{-1}) dz du \right\} h + f_T(0) \dot{\kappa}(h),$$

and since  $f_T(0) > 0$ ,

$$\dot{\kappa}(h) = -\frac{1}{f_T(0)} \left\{ \int_{[0,1]} \int \mathbf{1}\{T(z, u) \leq 0\} (z' J_0) \phi(z|0, J_0^{-1}) dz du \right\} h. \quad Q.E.D.$$

PROOF OF THEOREM 2: Suppose Assumptions 1 and 2 hold. By van der Vaart (1998, Theorem 7.2), Assumption 1(a) implies, for every converging sequence  $h_n \rightarrow h$ ,

$$\log \prod_{i=1}^n \frac{dP_{\theta_0+h_n/\sqrt{n}}}{dP_{\theta_0}}(Y_i) = \frac{1}{\sqrt{n}} \sum_{i=1}^n h' s(Y_i) - \frac{1}{2} h' J_{\theta_0} h + o_{P_{\theta_0}}(1).$$

By the definition of local asymptotic normality (LAN) (see van der Vaart (1998)) the above likelihood ratio convergence, along with  $n^{-1/2} \sum_i s(Y_i) \overset{\theta_0}{\rightsquigarrow} N(0, J_{\theta_0})$ , implies that the sequence of experiments  $\mathcal{E}_n = \{P_{\theta_0+h/\sqrt{n}}^n : h \in \mathbb{R}^k\}$  is locally asymptotically normal (note that, for  $h$  such that  $\theta_0 + h/\sqrt{n} \notin \Theta$ ,  $P_{\theta_0+h/\sqrt{n}}^n$  can be defined arbitrarily). By van der Vaart (1998, Corollary 9.5), the LAN property, along with the nonsingularity of  $J_{\theta_0}$ , given in Assumption 1(b), implies that the sequence of experiments  $\mathcal{E}_n$  converges to the limit experiment  $\{N(h, J_{\theta_0}^{-1}) : h \in \mathbb{R}^k\}$ .

By Assumption 2,  $\sqrt{n}(\kappa(\theta_0 + h/\sqrt{n}) - \kappa(\theta_0)) \rightarrow \dot{\kappa}(h)$ , with  $\dot{\kappa}(0) = 0$ .

Next, assume that there exists an estimator  $T_n$  that is locally asymptotically unbiased, or regular, or locally asymptotically  $\alpha$ -quantile unbiased for  $\kappa$ . In each case, the assumption will imply that  $\sqrt{n}(T_n - \kappa(\theta_0 + h/\sqrt{n})) \overset{h}{\rightsquigarrow} L_h$  for every  $h \in \mathbb{R}^k$ .

By van der Vaart (1991a, Theorem 4.1), there exists a randomized estimator  $T$  in the limit experiment  $\mathcal{E}$ , where  $Z \sim N(h, J_0^{-1})$  and  $U \sim \text{Unif}[0, 1]$  independent of  $Z$ , such that  $L_h = \mathcal{L}_h(T(Z, U) - \dot{\kappa}(h))$  for every  $h \in \mathbb{R}^k$ .

Finally, Assumption 2 immediately implies positive homogeneity of  $\dot{\kappa}$ .

Now consider the three properties desired for  $T_n$ :

(a) If we assume that  $T_n$  is locally asymptotically unbiased for  $\kappa$ , then  $L_h$  has expectation zero, which implies  $\dot{\kappa}(h) = E_h[T(Z, U)]$ . By Proposition 1(a),  $\dot{\kappa}(h)$  is linear in  $h$ .

(b) If we assume that  $T_n$  is a regular estimator for  $\kappa$ , then  $L_h$  is the same for all  $h$  and  $\mathcal{L}_h(T(Z, U) - \dot{\kappa}(h))$  is the same for all  $h$ . Proposition 1(b) concludes that  $\dot{\kappa}(h)$  is linear in  $h$ .

(c) If we assume that  $T_n$  is locally asymptotically  $\alpha$ -quantile unbiased for  $\kappa$  and  $F_\infty(t)$  has positive derivative at  $t = 0$ , then  $L_h\{(-\infty, 0]\} = \alpha$  for all  $h$ . Hence,  $\Pr_h(T \leq \dot{\kappa}(h))$  for all  $h$ . Also,  $F_\infty$  is the CDF from  $L_0$ , which is the CDF for  $T - \dot{\kappa}(0) = T$  under  $h = 0$ . So, by Proposition 1(c),  $\dot{\kappa}(h)$  is linear in  $h$ .

In each of the cases (a), (b), and (c), the conclusion is that  $\dot{\kappa}$  is linear. Given Assumption 2, linearity implies that  $\kappa$  is differentiable at  $\theta_0$ . This contradicts the supposition of Theorem 2. Q.E.D.

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