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Entropy

$\boxed{\text{1. Use diff to calculate } E(X) \quad \text{2. How to make money with entropy.}}$

Recap.

$CE(q \,\|\, p)$

true prob. used by the guy who sample the value of X → p

guessing guy → q

x	A	B	C
p	p_A	p_B	p_C
q	q_A	q_B	q_C

guessing guy $\min_{\text{strat}} E(N)$ N - number of questions

$\boxed{\text{from } p \text{ to } q}$

$CE(q \,\|\, p)$ = the expected number of questions using strategy that is tuned to wrong probab-s q while true prob-s are p.

$$CE(q \,\|\, p) = -\sum_x p(x) \cdot \ln q(x)$$

Theorem $\boxed{CE(q \,\|\, p) \geq CE(p \,\|\, p) = H(p)}$

$p(x, a)$

$\boxed{a_T \text{ - true value of param-r.}}$

$\boxed{a \text{ - wrong value used by guessing person}}$

$\forall a$

$$f(a, a_T) = \boxed{CE\left(p(x,a) \,\|\, p(x,a_T)\right)} \geq CE\left(p(x,a_T) \,\|\, p(x,a_T)\right)$$

$q = p(x, a)$ $\quad p = p(x, a_T)$

$$\boxed{\text{If } f \text{ is diff'ble then } \left.\frac{\partial f(a, a_T)}{\partial a}\right|_{a=a_T} = 0}$$

$\boxed{a = a_T \text{ gives the min of } f(a, a_T)}$

$$f(a, a_T) \geqslant f(a_T, a_T)$$

$$\frac{\partial\, CE(p(x,a) \| p(x,a_T))}{\partial a} = 0$$

$$CE = E_{a_T}(-\ln p(x,a))$$

$$E_{a_T}\left(-\left.\frac{\partial}{a} \ln p(x,a)\right|_{a=a_T}\right) = 0$$

$$\boxed{E\left(\frac{\partial \ln p(x, a_T)}{\partial a_T}\right) = 0 \quad !!}$$

$$E\left(\frac{\partial \text{ log - likehood}}{\partial a_T}\right) = 0 \quad [\text{at } a_T]$$

Exercise.

$$X \sim Exp(\lambda)$$

$$pdf = \begin{cases} \lambda \cdot \exp(-\lambda x) & x > 0 \\ 0 & x < 0 \end{cases}$$

way 1:

$$\int_0^\infty x \cdot \lambda \exp(-\lambda x)\, dx =$$

$$= \{\text{by parts}\} = \ldots$$

way 2

$$\ln p = \ln \lambda - \lambda x$$

$$\frac{\partial \ln p}{\partial \lambda} = \frac{1}{\lambda} - x$$

$$E\left(\frac{1}{\lambda} - X\right) = 0$$

$$E(X) = \frac{1}{\lambda}$$

How to make money with entropy?

Example

→ Favorable game.
→ ∞ many times.

$X_1, X_2, X_3, \dots$ iid

x	0	5
$P(X=x)$	0,2	0,8

→ at t you can ~~invest~~ invest any value $\in [0; V_t]$

$V_0 = 100\$$

→ your investments at time t are mult by X_{t+1}

Question: What is the „best" long term strategy?

A1. one half

$E(X) = 5 \cdot 0,8 = 4 > 1$

In the long term „invest all" is a bad strategy:

$$\operatorname{plim}_{t\to\infty} V_t = 0$$

$$\lim_{t} E(V_t) = +\infty$$

Good Question: How to calculate optimal proportion?

$\boxed{b = 0,5}$

$\boxed{V_0} \to \boxed{0,5\, V_0}$

$\boxed{V_0} \to \boxed{0,5\, V_0} + \boxed{5 \cdot 0,5\, V_0}$

$\boxed{b} \in [0;1]$

$V_0 \to$	$(1-b) \cdot V_0$	0,2
$V_0 \to$	$(1+4b) \cdot V_0$	0,8

$V_1 \to (1-b) V_0 \quad \times 0,2$

$V_1 \to (1-b) V_0 + \boxed{5 \cdot b\, V_0} \quad \times 0,8$

$\max_b E(V_1) \quad b = 1$

Bad target !!

Intuition.

LLN: $\frac{\sum_{i=1}^{n} Y_i}{n} \xrightarrow{n\to\infty} E(Y_i)$

$\operatorname{plim}_{n\to\infty} \bar{Y} = E(Y_i) \qquad Y_i \sim iid$

intuition: that's "all" we can stabilize when $n \to \infty$.

$$\boxed{V_t \begin{cases} (1-b)\cdot V_0 & 0.2 \\ (1+4b)\cdot V_0 & 0.8 \end{cases}}$$

$V_1 = M_1^b \cdot V_0 \qquad V_2 = M_2^b \cdot V_1$

$V_t = V_0 \cdot M_1^b \cdot M_2^b \cdot \ldots \cdot M_t^b$ $\quad \downarrow \ln$

$\ln V_t = \ln V_0 + \sum_{i=1}^{t} \ln M_i^b$

$$\frac{\ln V_t - \ln V_0}{t} = \frac{\sum \ln M_i^b}{t} \xrightarrow{LLN} \underbrace{E(\ln M_i^b)}_{= r}$$

We can $\boxed{\max_b E(\ln M_i^b)}$ — long-term interest rate

"Kelly criterion"

[good target]

deterministic case:

$V_0 \cdot \exp(rt) \qquad \ln\left(\frac{V_t}{V_0 \exp(rt)}\right) =$

$\ln V_t - \ln V_0 - rt = 0.$

$\boxed{\operatorname{plim} \frac{S_t}{S_0 \exp(bt)} = 1} \longleftrightarrow V_t$ has long term beh-r as deterministic with rate r

$$\frac{\ln V_t - \ln V_0}{t} \to r$$

$P(M_i=m)$	$0{,}2$	$0{,}8$
m	$(1-b)$	$(1+4b)$

$$\max_b r = \max_b E(\ln M_i) =$$

$$= \max_b \; 0{,}2 \cdot \ln(1-b) + 0{,}8 \cdot \ln(1+4b)$$

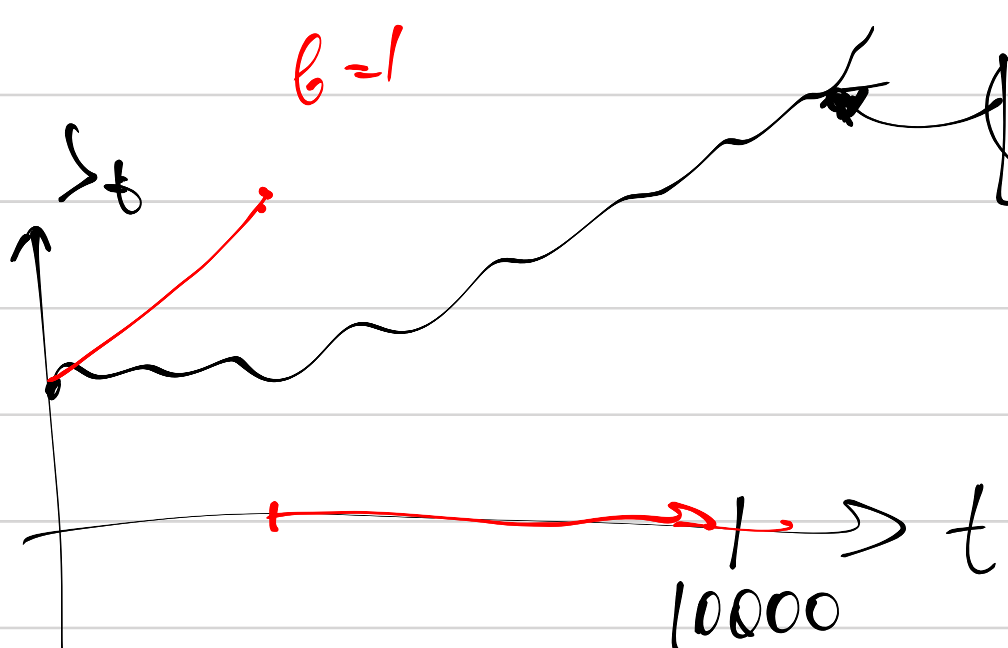
....

$$\frac{-0{,}2}{1-b} + \frac{0{,}8\cdot 4}{1+4b} = 0$$

$$\frac{1}{1-b} = \frac{16}{1+4b} \qquad 1+4b = 16-16b$$

$$20b = 15$$

$$\boxed{b^* = \frac{3}{4}}$$



$$r = \boxed{0{,}2 \cdot \ln\left(1-\frac{3}{4}\right) + 0{,}8 \cdot \ln\left(1+4\cdot\frac{3}{4}\right)}$$

$$\sum_j prob_j \cdot \ln M_j$$

It looks like entropy a little bit

Small fact

Opt: $2 \cdot \ln p_A + 3 \ln p_B + 5 \ln p_C \to \max$

s.t. $p_A + p_B + p_C = 1$

$\ln(p_A^2 \cdot p_B^3 \cdot p_C^5)$

$p_A^2 \cdot p_B^3 \cdot p_C^5 \to \max$

Sol-n? $p_A^* = ?$ $p_B^* = ?$ $p_C^* = ?$

$$\text{const}\cdot p_A^2 \cdot p_B^3 \cdot p_C^5 = \frac{p_A}{2}\cdot\frac{p_A}{2}\cdot\frac{p_B}{3}\cdot\frac{p_B}{3}\cdot\frac{p_C}{5}\cdot\frac{p_C}{5}\cdot\ldots\cdot\frac{p_C}{5}$$

s.t.

$$1 = \underbrace{\frac{p_A}{2}+\frac{p_A}{2}}_{2\text{ terms}} + \underbrace{\frac{p_B}{3}+\frac{p_B}{3}+\frac{p_B}{3}}_{3\text{ terms}} + \underbrace{\frac{p_C}{5}+\frac{p_C}{5}\ldots\frac{p_C}{5}}_{5\text{ terms.}}$$

Opt. point (by symmetry) $\frac{p_A}{2} = \frac{p_B}{3} = \frac{p_C}{5}$

[otherwise : increase the lowest
decrease the highest
to keep the constr]

$a\cdot b \to \max$

$a+b=100$

$\downarrow 60\cdot 40 \uparrow$ that is not optimal.

$$\max\ 2\cdot\ln p_A + 3\ln p_B + 5\cdot\ln p_C$$

s.t. $p_A + p_B + p_C = 1$

Small Fact

Solution:

$$p_A^* = \frac{2}{2+3+5}$$

$$p_B^* = \frac{3}{2+3+5} \qquad p_C^* = \frac{5}{2+3+5}$$

Fair game $\to$ Favorable game.

Horses:	A	B
	0,2	0.8
stake multipl.	$\times 5$	$\times\frac{1}{0{,}8}=\frac{5}{4}$

Fair game

R.Vs S and H are dependent

Signal \ Horses	A	B
Rainy	0,1	0,5
Sunny	0,1	0,2
Cloudy	0	0,1
	0,2	0,8

Favorable game (if we know S)

V_t - value of my portfolio
S_t - signal
H_t = winning horse.

$b(h|s)$ - bid on horse h given signal s

$$b(A|s) + b(B|s) = 1 \quad \forall s$$

$$\sum_h b(h|s) = 1$$

Kelly criterion:

$$\max_{\sum_h b(h|s)=1} \boxed{E\left(\ln \frac{V_1}{V_0}\right)}$$

$$E\left(\ln \frac{V_1}{V_0}\right) = \sum_{h,s} p(h,s) \cdot \ln \frac{V_0 \cdot b(h|s) \cdot \frac{1}{p(h)}}{V_0} =$$

$$= \sum_{h,s} p(h,s) \cdot \left(\ln b(h|s) - \ln p(h)\right) =$$

$$= \boxed{-\sum_{h,s} p(h,s) \cdot \ln p(h)} - \boxed{\left(-\sum p(h,s) \cdot \ln b(h|s)\right)}$$

joint distr
$p(H=A, S=\text{Sunny}) = 0{,}1$

conditional
$\boxed{p(H=A|S=\text{Sunny}) = \frac{1}{3}}$

$\to$ max:
$b^*(h|s) = \boxed{p(h|s)}$
(using Small Fact)

$$\boxed{-\sum_h p(h) \cdot \ln p(h)}$$

$$\boxed{\text{long-term}} = \underbrace{H(\text{Horse}) - H(\text{Horse}|\text{Signal})}_{\text{utility of a signal}}$$

[nat] [nat]