DSE'23 - class 7: Matrices

19/10/2023

📚 Theory

Manipulations with matrices/ vectors

Rule	Comments
$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$	order is reversed, everything is transposed
$(\mathbf{a}^T\mathbf{Bc})^T = \mathbf{c}^T\mathbf{B}^T\mathbf{a}$	as above
$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a}$	(the result is a scalar, and the transpose of a scalar is itself)
$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$	multiplication is distributive
$(\mathbf{a}+\mathbf{b})^T\mathbf{C} = \mathbf{a}^T\mathbf{C}+\mathbf{b}^T\mathbf{C}$	as above, with vectors
$\mathbf{AB} \neq \mathbf{BA}$	multiplication is **not** commutative

Common derivatives

Scalar derivative			Vector derivative		
$f(x)$	$\rightarrow$	$\frac{df}{dx}$	$f(\mathbf{x})$	$\rightarrow$	$\frac{df}{d\mathbf{x}}$
bx	$\rightarrow$	b	$\mathbf{x}^T\mathbf{B}$	$\rightarrow$	$\mathbf{B}$
bx	$\rightarrow$	b	$\mathbf{x}^T\mathbf{b}$	$\rightarrow$	$\mathbf{b}$
x^2	$\rightarrow$	$2x$	$\mathbf{x}^T\mathbf{x}$	$\rightarrow$	$2\mathbf{x}$
bx^2	$\rightarrow$	$2bx$	$\mathbf{x}^T\mathbf{Bx}$	$\rightarrow$	$2\mathbf{Bx}$

📝 Problems

Block 1: training differential muscles 💪

Problem 1: light start

(a) Find dF

$$F = \begin{pmatrix} 5 & 6x_1 \\ x_1x_2 & x_1^2x_2 \end{pmatrix}$$

(b) F is 2x2 matrix with elements f_{ij} placed at row i column j. Write down explicitly expression for $tr(F'dF)$ explicitly (without using matrices)

Problem 2: life gets more interesting

A, B — matrices of constants, R — matrix with variables, r — column vector of variables. Find differentials for each of the cases below and simplify the answers for the case of symmetric A:

(a) $d(ARB)$

(b) $d(r'r)$

(c) $d(r'Ar)$

(d) $d(R^{-1})$

(e) $d(cos(r'r))$

(f) $d(r'Ar/r'r)$

Block 2: applications 🥷

Problem 3: Classics

Let's derive traditional OLS regression in matrix form:

(a) Consider simple regression without constant $y_i = \hat{\beta}_1 z_i + \hat{\beta}_2 x_i$. Place all data in matrix X and vectors y, z, x:

1. Explicitly write down following matrices and calculate their shape: X' , $X'y$, $X'X$, $y'X$, $y'z$
2. Write down first order conditions for $\hat{\beta}_1$, $\hat{\beta}_2$ in two forms: in matrices and as a linear system
3. Simplify the system for the case of $\hat{y}_i = \hat{\beta}_1 + \hat{\beta}_2 x_i$

(b) Now let's generalize the derivations from (a) to the case of general OLS regression

$$Q(\hat{\beta}) = (y - X\hat{\beta})'(y - X\hat{\beta})$$

1. Find $dQ(\hat{\beta})$ and $d^2Q(\hat{\beta})$
2. Write down FOC for OLS problem
3. Find $\hat{\beta}$ assuming that $X'X$ is invertible
4. Show that $\hat{\beta}$ is unbiased
5. Find $Var(\hat{\beta})$
6. Derive in matrix from TSS = ESS + RSS and state assumptions under which this equation is true

Problem 4: Omitted variable

Consider true model $y = x_1'\beta_1 + x_2'\beta_2 + e$, $E[e|x_1, x_2] = 0$, assume random sampling. Suppose that β_1 is estimated by regression of y on x_1 only. Find probability limit of the estimator. What are the conditions when it is consistent for β_1?

🤓 References

1. More problems
2. Yandex handbook on matrix differentials
3. Nice book on Econometrics fundamentals