HA-1 Theory - solution notes

If have any questions, please contact Shuana

Problems

https://github.com/Shuaynat/DSE-23-24/tree/main/02-home-assignments/ha-01-theory

Solutions

Playlist with recording:

https://www.youtube.com/playlist?list=PLyjahhN4Wdd99xaZOCZ99hNcLiM66QR9E

Problem 1 (2 points)

(a) $H(X) - ?$ (1 point)

$$H(X) = -\sum_i P(X = x_i) \log_2 P(X = x_i)$$

x_i	0	1	2	3
$P(X = x_i)$	$0,7^3 = 0,343$	$3 * 0,3 * 0,7^2 = 0,441$	$3 * 0,7 * 0,3^2 = 0,189$	$0,3^3 = 0,027$
$\log_2(P(X = x_i))$	-1,544	-1,181	-2,4	-5,2
$P(X = x_i) \log_2(P(X = x_i))$	-0,529	-0,521	-0,454	-0,141
$\ln(P(X = x_i))$	-1,07	-0,819	-1,667	-3,612
$P(X = x_i) \ln(P(X = x_i))$	-0,367	-0,361	-0,315	-0,097

Option 1: H(X) = 0,529+0,521+0,454+0,141 = **1,64 [bit]**

Option 2: H(X) = 0,367+0,361+0,315+0,097= **1,14 [nat]**

(b) CE(B(n=3, p=0,3) || B(n=3, q=0,5)) - ? (1 point)

$$CE(P||Q) = -\sum_i P(X = x_i) \ln Q(X = x_i)$$

x_i	0	1	2	3
$P(X = x_i)$	$0,7^3 = 0,343$	$3 * 0,3 * 0,7^2 = 0,441$	$3 * 0,7 * 0,3^2 = 0,189$	$0,3^3 = 0,027$
$Q(X = x_i)$	$0,5^3 = 0,125$	$3 * 0,5^3 = 0,375$	$3 * 0,5^3 = 0,375$	$0,5^3 = 0,125$
$\ln(Q(X = x_i))$	-2,079	-0,981	-0,981	-2,079
$P(X = x_i) \ln(Q(X = x_i))$	-0,713	-0,432	-0,185	-0,056

CE(P||Q) = 0,713+0,432+0,185+0,056 = **1,387**

Problem 2 (2 points)

x	0	5	10
$P(K_n = x)$	1/2	1/4	1/4
Net gain	0 - 1 = -1	5 - 1 = 4	10-1 = 9

Maximize long-term growth rate ⇒ consider Kelly criterion

$$0,5\ln(1-f) + 0,25\ln(1+4f) + 0,25\ln(1+9f) \to \max_f$$

f - fraction of wealth betted each round

$f^* = 0,42$

Computation in WA

📝 Comment on mistakes

- Note in this task the gain is equal to the value of K itself, so there is no need to calculate multipliers as inverse of probabilities.
- In logs have net return, not gross return

📚 Useful links

- Beautiful visualization ov Kelly Criterion: **https://youtu.be/_FuuYSM7yOo**
- Kelly 1956 paper: https://www.princeton.edu/~wbialek/rome/refs/kelly_56.pdf

Problem 3 (6 points)

	H=A	H=B	H=C
$P(H = h, S = 1)$	0,1	0,2	0,3
$P(H = h, S = 2)$	0,2	0,1	0,1
$P(H = h)$	0,3	0,3	0,4
Gain	$\frac{10}{3}$	$\frac{10}{3}$	$\frac{5}{2}$

(a) Optimal bet for each signal - ? (2 points)

At the lecture showed, that expected return is maximized for the bet proportional to the marginal probabilities:

$$\mathrm{E}(\ln \frac{V_1}{V_0}|S=s) = \sum_h P(h|S=s)\ln \frac{V_0 b(h|S=s)\frac{1}{p(h)}}{V_0}$$

⇒ optimal bet : $b^*(h|S=s) = P(h|S=s)$

	H=A	H=B	H=C
$P(H = h, S = 1)$	0,1	0,2	0,3
$P(H = h, S = 2)$	0,2	0,1	0,1
$P(H = h \vert S = 1) = b^*(h \vert S = 1)$	$\frac{0,1}{0,6} = \frac{1}{6}$	$\frac{0,2}{0,6} = \frac{1}{3}$	$\frac{0,3}{0,6} = \frac{1}{2}$
$P(H = h \vert S = 2) = b^*(h \vert S = 2)$	$\frac{0,2}{0,4} = \frac{1}{2}$	$\frac{0,1}{0,4} = \frac{1}{4}$	$\frac{0,1}{0,4} = \frac{1}{4}$

📝 **Comment on mistakes**

- Need to calculate optimal portion to bet on each horse (3 fractions, not 1 unique fraction tha to be put in the bet in general)
- Explicit maximization of Kelly criterion was acceptable for this task, but in general, usage of the trick from lecture is more elegant way to solve the problem

(b) $H(W|S)-?$ (2 points)

$$H(W|S) = -\sum_{h,s} P(H = h, S = s) \ln P(H = h|S = s)$$

$$H(W|S) = -(0,1\ln(1/6) + 0,2\ln(1/3) + 0,3\ln(1/2) + 0,2\ln(1/2) + 0,1\ln(1/4) + 0,1\ln(1/4)) = 1,023$$

(c) $r^{LT}-?$ (2 points)

$$r^{LT} = \mathrm{E}(\ln \frac{V_1}{V_0}) = \sum_{h,s} P(h,s) \ln b(h|s) - \sum_{h,s} P(h,s) \ln(P(h)) = -\sum_{h} \ln(P(h)) \sum_{s} P(h,s) - (-\sum_{h,s} P(h,s) \ln b(h$$

$$H(W) = -\sum_{h} P(h) \ln P(h) = -(0,3\ln(0,3) + 0,3\ln(0,3) + 0,4\ln(0,4)) = 1,089$$

$\Rightarrow r^{LT} = 1,089 - 1,023 = 0,066$

📝 **Comment on mistakes**

- In calculating the “entropy of horse” have log of the <u>marginal probability,</u> not the probability of the intersection