IC112 Calculus
Assignment 5

1. Check for the existence of the limit.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$,

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$.

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$,

   (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2-y^2}$.

2. (a) Consider $f(x,y) = \dfrac{x+y}{x-y}$ for $x \neq y$. Show that

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 1, \quad \lim_{y\to 0}\lim_{x\to 0} f(x,y) = -1.$$

   What can you say about $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$?

   (b) Let

$$f(x,y) = \begin{cases} x\sin\left(\dfrac{1}{y}\right) & y \neq 0, \\ 0 & y = 0. \end{cases}$$

   Compute $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$, $\displaystyle\lim_{x\to 0}\lim_{y\to 0} f(x,y)$, and $\displaystyle\lim_{y\to 0}\lim_{x\to 0} f(x,y)$, if they exist.

3. Show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ and $\displaystyle\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ exist, but $\displaystyle\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ **does not** exist, where

$$f(x,y) = \begin{cases} y + x\sin\left(\dfrac{1}{y}\right) & y \neq 0, \\ 0 & y = 0. \end{cases}$$

4. Show that $\displaystyle\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ exists, but $\displaystyle\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ and $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ **do not** exist, where

$$f(x,y) = \begin{cases} y\sin\left(\dfrac{1}{x}\right) + \dfrac{xy}{x^2+y^2} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

5. Show that the following functions are continuous at $(0,0)$:

   (a) $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$

   (b) $f(x,y) = \begin{cases} \dfrac{x^3y^3}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$

6. Show that the following functions are **discontinuous** at $(0,0)$:

   (a) $f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$

   (b) $f(x,y) = \begin{cases} \dfrac{x^2y}{x^3+y^3} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$

7. Show that the partial derivatives f_x and f_y exist everywhere in $[-1,1]\times[-1,1]$, but $f(x,y)$ is discontinuous at the origin, where

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & x^2+y^2 \neq 0, \\ 0 & x^2+y^2 = 0. \end{cases}$$

8. Evaluate f_x, f_y, $f_x(0,0)$, and $f_y(0,0)$ using the definition for:

(a) $f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2} & (x,y)\neq(0,0),\\ 0 & (x,y)=(0,0).\end{cases}$ (b) $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & x^2+y^2\neq 0,\\ 0 & x^2+y^2=0.\end{cases}$

9. Show that $f(x,y)$ possesses first partial derivatives everywhere (including the origin) but is discontinuous at the origin, where

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2} & (x,y)\neq(0,0),\\ 0 & (x,y)=(0,0).\end{cases}$$

10. Show that the following functions are continuous at the origin and possess partial derivatives there, but are **not differentiable** at the origin:

(a) $f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2} & x^2+y^2\neq 0,\\ 0 & x^2+y^2=0.\end{cases}$ (b) $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & x^2+y^2\neq 0,\\ 0 & x^2+y^2=0.\end{cases}$