Functions of Single Variable

Function - if one qty depends on other, so that it assumes a definate value when a system of definate values is given to other.

i.e if a symbol y has one definate value for every value of variable x, then it is called a function of $x \Rightarrow y = f(x)$

explicit fn: when it is expressed directly in terms of Indep variables $y = f(x)$

- Implicit fn: $f(x,y) = 0$

- Even fn $\rightarrow$ $f(-x) = x$
- odd fn $\rightarrow$ $f(-x) = -f(x)$

Limit: A fn $f(x)$ is said to tend to a limit A as x tend to a, if corresponding to any arbitrary chosen positive no ε, however small (but not zero) there exists a positive no δ such that

$$|f(x) - A| < \varepsilon$$

for all values of x for which $0 < |x-a| < \delta$

$\rightarrow$ We write it as $\lim\limits_{x \to a} f(x) = A$

- $\lim_{x \to a} f(x)$ exists only if LHL = RHL

- if $f_1(x) \to A$ & $f_2(x) \to B$ as $x \to a$ where A & B are both finite, we have

① $\lim_{x \to a} \{f_1(x) \pm f_2(x)\} = A \pm B$ ② $\lim_{x \to a} \{f_1(x) \cdot f_2(x)\} = A \cdot B$

③ $\lim_{x \to a} \{f_1(x)/f_2(x)\} = \frac{A}{B}$ → (Provided $B \neq 0$)

lt of $f(x)$ as $x \to a$ is not necessary the same as value of function at $x = a$, In fact the limit of $f(x)$ as $x \to a$ may exist if function $f(x)$ is not defined at $x = a$.

Continuity

A function $f(x)$ defined for $x=a$ is said to be continuous at $x=a$ if

(i) $f(a)$ i.e the value of $f(x)$ at $x=a$ is a definate no.

(ii) the limit of function $f(x)$ as $x \to a$ exists and is equal to $f(x)$ at $x=a$

- continuous if $\lim_{x \to a} f(x) = \underbrace{f(a+0)}_{RHL} = \underbrace{f(a-0)}_{LHL} = f(a)$

- A function is said to be continuous in (a,b) if it is continuous at every point of that Interval.

- A function $f(x)$ is said to be continuous at $x=a$, if for any arbitrary chosen positive no ϵ (however small) but not zero, we can find a corresponding no. δ s.t

$$|f(x)-f(a)| < \epsilon$$

for all values of x for which $|x-a| < \delta$

Differentiability

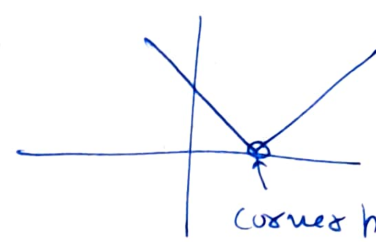
A function is said to be differentiable at $x=a$, if both

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}, \quad h>0 \quad \& \quad \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}, \quad h>0$$

RHL → Progressive Derivative　　　　LHL → Regressive Derivative

exists & have a common value (finite or infinite). This common value is called derivative of $f(x)$ at point $x=a$

Thm: Continuity is a necessary condition but not sufficient condition for existence of finite Derivative



corner point

↓

Derivative D.N.E.

Indeterminate Forms

A fraction whose numerator & denominator both tend to zero as $x \to a$ is called the indeterminate form $\to \frac{0}{0}$

it does not mean limit will not exist

other forms $\to$ $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 1^{∞}, 0^0, ∞^0

- $\frac{0}{0}$ form $\to$ L'Hospital's Rule

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}$$ ... Derived from Taylor's thm

... Do not go on applying this rule if the form is not $\frac{0}{0}$

- Methods of expansion

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \quad |x| < 1$$

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots ; \ (|x| < 1)$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) ; \ (|x| < 1)$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots ;$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots ;$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots ; \ \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Also remember →

$\log 1 = 0$; $\log e = 1$; $\log \infty = \infty$; $\log 0 = -\infty$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \cos x = 1$$

$$\lim_{x \to 0} (1+x)^{1/x} = e$$

$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \to 0} (1+mx)^{1/x} = e^m \qquad \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

Maxima, Minima (Single variable)

- A function is said to be maximum at $x=a$, if there exists a positive no. δ such that

$$f(a+h) < f(a)$$

for all values of h, other than 0 in $(-\delta, \delta)$

- A f^n is said to be minimum at $x=a$, if there exists a positive no. δ such that,

$$f(a+h) > f(a)$$

for all values of h, other than 0 in $(-\delta, \delta)$

- Maximum & Minimum values are also called its extreme values or turning values and the points at which they attain are called points of maxima & minima.

- Properties

(i) Atleast one maximum or one minimum must lie between two equal values of a function.

(ii) there may be several maximum or minimum values of same function.

(iii) Maximum & minimum values must occour alternately.

(iv) A f^n, $y=f(x)$ is maximum at $x=a$, if dy/dx changes sign from +ve to -ve as x passes through a.

(v) A function $y = f(x)$ is minimum at $x = a$ if dy/dx changes sign -ve to +ve as x passes through a.

(vi) If the sign of dy/dx does not change then y is neither maximum, nor minimum.

- <u>Necessary condition</u>: for $f(x)$ to be a maximum or a minimum at $x = a$ is that $f'(a) = 0$

- A function is said to be stationary if $f'(a) = 0$ at $f(x) = a$.

<u>Working Rule for Maxima & Minima</u>

(i) find $f'(x)$ & equate it to 0

(ii) Solve the resulting eq^n for x. let its roots be $a_1, a_2, \ldots$ then $f(x)$ is stationary at $x = a_1, a_2, \ldots$; thus $x = a_1, a_2 \ldots$ are only points at which $f(x)$ can be maximum or a minimum.

(iii) find $f''(x)$ & substitute in it by turns $x = a_1, a_2, \ldots$

(iv) If $f''(a)$ is -ve, we have a maximum

If $f''(a)$ is +ve, we have a minimum

(5) If $f''(x) = 0$, find $f'''(x)$ & put $x = a_1$

If $f'''(x) = 0$, find $f^n(x)$ ...

(6) Repeat the above process for each root of eq^n $f'(x) = 0$

Mean Value theorems

(1) Rolle's theorem

If a function $f(x)$ is such that

(i) $f(x)$ is continuous in closed interval $a \le x \le b$

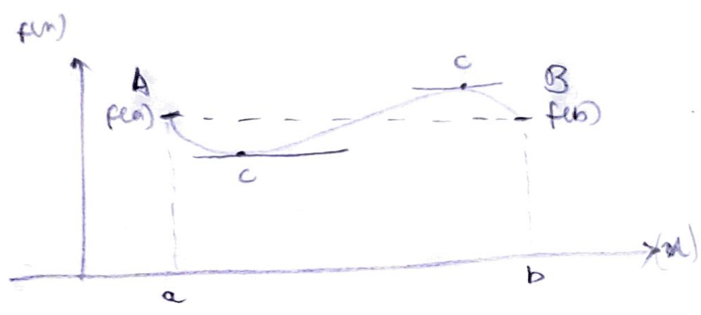
(ii) $f'(x)$ exists for every point in open int. $\rightarrow$ $a < x < b$

(iii) $f(a) = f(b)$,

then there exists atleast one value of x, say c,

where $a < c < b$, s.t $f'(c) = 0$

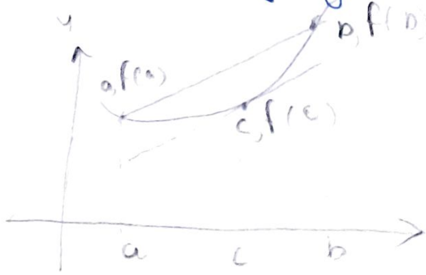
Note: there may be more than of one point like c at which $f'(x)$ vanishes.



(2) Lagranges mean value theorem (1st M.V.T)

If a function $f(x)$ is (i) continuous in closed interval $[a,b]$
(ii) differentiable in (a,b)

then there exists at least one value 'c' of x lying in open Interval (a,b) s.t →



$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(c)}$$

Another form: If (i), (ii), then $f(a+h) - f(a) = h f'(a+\theta h)$

Some Imp Deductions

(i) If a function $f(x)$ be such that $f'(x)$ is 0 throughout the Interval (a,b) then $f(x)$ must be constant throughout interval

(ii) If $f(x)$ & $\phi(x)$ be two functions such that $f'(x) = \phi'(x)$ throughout the interval (a,b), then $f(x)$ & $\phi(x)$ differ only by a constant

(iii) If $f(x)$ is

(i) continuous in closed interval $[a,b]$,

(ii) differentiable in (a,b)

(iii) $f'(x)$ is $-ve$ in (a,b), then

f(x) is monotonically decreasing function in the closed interval [a,b].

③ Cauchys MVT (2nd MVT)

If two functions f(x) & g(x) are

(i) continuous in closed interval [a,b]

(ii) differentiable in open interval (a,b)

(iii) $g'(x) \neq 0$ for any point in (a,b), then there is exists at least one value c of x in (a,b) s.t

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \quad ; \quad a<c<b}$$

④ Taylors theorem with Lagranges form of remainder after n terms.

If f(x) is a single valued function of x s.t

(i) all the derivatives of f(x) upto $(n-1)^{th}$ are continuous $a \le x \le a+h$ and

(ii) $f^n(x)$ exists in $a<x<a+h$, then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

$$+ \frac{h^{m-1}}{(m-1)!} f^{(m-1)}(a) + \frac{h^m}{m!} f^m(a+\theta h), \text{ where } 0<\theta<1$$

If we take $m=1$; we observe LMVT is a particular case of Taylors theorem.

Corollary: (Maclaurin's development); Instead of considering the interval $[a, a+h]$, let us take the interval $[0, x]$, then changing a to 0 & h to x in taylors thm, we get,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots \frac{x^m}{m!} f^m(\theta x)$$

which is known as Maclaurins theorem or

Functions of Two Variables.

Neighbourhood of a point

- Set of values (x_1, y_1) other than $\{a, b\}$, that satisfy

$$|x_1 - a| < \delta \quad ; \quad |y_1 - b| < \delta$$

is said to form a nbd of point (a, b), thus a nbd is a square $\longrightarrow$ $(a-\delta, a+\delta, b-\delta, b+\delta)$

- there may also be a circular nbd.

Limit of a function

A function tends to a limit l, when (x, y) tends to (a, b) if for every arbitrary small positive no ε, there corresponds a positive no. δ, such that

$$|f(x,y) - l| < \varepsilon \quad ; \quad \text{whenever}$$
$$|(x,y) - (a,b)| < \delta$$

for every point (x, y) other than (a, b) of nbd N.

$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

★ There must be no assumption of any relation b/w the Independent variables as they tend to their respective limits.

• The Simultaneous Limit postulates that by whatever path the points is approached, the function f attains the same Limits

- If we approach point $(0,0)$ along different paths, $(m_1, m_2$, line parallel to co-ordinate axes) we get different Limits $\rightarrow$ means Limit Does not exist.

Continuity

A function is said to be continuous at point (a,b) of its domain of defination if for $\varepsilon > 0$, there exists a nbd N of (a,b) such that,

$$|f(x,y) - f(a,b)| < \varepsilon, \text{ for all } (x,y) \in N$$

Note: Def^n requires $\rightarrow$ f is defined in a certain nbd of (a,b) and moreover the limit of f when $(x,y) \rightarrow (a,b)$ exists and equal to the value $f(a,b)$

• If $f(x,y)$ is continuous at (a,b) then $f(x,b)$ is continuous at $x=a$ and $f(a,y)$ is continuous $y=b$.
but converse is not true

example

continuity of $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$

$$|f(x,y) - 0| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \left|\frac{xy}{(x^2+y^2)}\right| \left|\sqrt{x^2+y^2}\right|$$

$\le \frac{1}{2}$ [AM $\ge$ GM]

$\le \frac{1}{2}\sqrt{x^2+y^2}$

$\le \frac{\sqrt{x^2+y^2}}{2} < \varepsilon$ [take $\delta = \varepsilon$]

or directly use →

$x = r\cos\theta$

$y = r\sin\theta$

$$|f(x,y) - f(0,0)| = \frac{xy}{\sqrt{x^2+y^2}} = \left|\frac{r^2\sin\theta\cos\theta}{r}\right| = r|\sin\theta||\cos\theta|$$

$\le r$

$\le \sqrt{x^2+y^2}$

$\le \varepsilon$

$\because \sin\theta \le 1$, $\cos\theta \le 1$

whenever $x^2+y^2 < \varepsilon^2 = \delta$

$\therefore |f(x,y) - f(0,0)| < \varepsilon$ whenever $x^2+y^2 < \delta$

$\therefore f(x,y)$ is continuous at $(0,0)$.

- A function is said to be continuous in a region if it is continuous at every point is same region.

Partial Derivatives.

- ordinary derivative of a function of several variables w.r.t one of the independent variables, keeping all other independent variables constant.

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} \qquad \frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

P.D at a particular point (a,b) are often denoted by →

$$\left[\frac{\partial f}{\partial x}\right]_{(a,b)}, \quad \frac{\partial f(a,b)}{\partial x} \quad \text{or} \quad f_x(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a,b)}{k}$$

Note: Unlike the situation for functions of one variable, the existence of the first partial derivatives at a point does not imply continuity at that point.

- The value of f_x & f_y at a point (a,b) depend only on the value of f along two lines through (a,b) respectively parallel to co-ordinate axes.

- This information is incomplete and tells us nothing at all about the behaviour of function f as the point (a,b) is approached along lines not parallel to axes.

A Sufficient Condition for Continuity.

- A sufficient condition that a function f be continuous at (a,b) is that one of the partial Derivatives exists & is bounded in a nbd of (a,b) & other exists at (a,b).

- A sufficient condition that a function be continuous in a closed region is that both partial derivatives exists & are bounded throughout region.

• If f^n is differentiable at (a, b), then it is continuous and both P.D exists. But converse is not true.

7/10/23

Differentiability.

Let (x,y), $(x+\delta x, y+\delta y)$ be two neighbouring points in domain of function f. The change δf in function as the point changes from $(x,y) \to (x+\delta x, y+\delta y)$ is given by:

$$\delta f = f(x+\delta x, y+\delta y) - f(x,y)$$

- Differential of f at (x,y) $\Rightarrow$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = f_x dx + f_y dy$$

- A function differentiable at a point is necessarily continuous & processes partial derivatives there at.

Sufficient Condition for Differentiability.

If (a,b) be a point of domain of defination of a function f such that:

(i) f_x is continuous at (a,b)

(ii) f_y exists at (a,b)

— one of the P.D is to be continuous & other merely to exist at the point.

then f is differentiable at (a,b)

[Cond^n of existence of one P.D & continuity of other is sufficient to ensure the function is differentiable But cond^n of continuity is not necessary.]

• Partial Derivatives (Higher Order)

- 2^{nd} order P.D at (a,b) are denoted by -

$$f_{xx}(a,b) = \lim_{h\to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h}$$

$$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$

$$f_{yx}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}$$

$$f_{yy}(a,b) = \lim_{k\to 0} \frac{f_y(a,b+k) - f_y(a,b)}{k}$$

Sufficient Condn for equality of f_{xy} & f_{yx}

→ Youngs theorem:

- If f_x & f_y are both differentiable at a point (a,b) of the domain f, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

→ Schwartz Theorem.

- If f_y exists in a certain nbd of point (a,b) of domain of defination of a function f, & f_{yx} is continuous at (a,b) then $f_{xy}(a,b) = f_{yx}(a,b)$

- The conditions are sufficient but not Necessary.

Homogenous functions.

- An exp^n in which every term is of the same degree

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots a^n y^n$$

- If we get $f(tx, ty) = t f(x,y)$ then it is homogenous.

Eulers theorem.

If u is a function homogenous of x & y of degree n,
then ~~xd~~ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

→ can be extended to a function of any no. of Variable

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \dots x_n\frac{\partial f}{\partial x_n} = nf$$

Eulers 2nd theorem.

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

Conditions at (a, b)	Result at (a,b)
① f_x & f_y exists + one P.D is bounded	f is continuous.
② f_x & f_y exists + one P.D is cont.	f is Differentiable
③ f_{xy} & f_{yx} are continuous ④ f_x exists in nbd (a,b) + f_{xy} cont. ⑤ both f_x & f_y differentiable	$f_{xy} = f_{yx}$

Maxima & Minima

Defⁿ let $f(x,y)$ be a function → continuous for all values in nbd of their values a & b respectively.
then $f(a,b)$ is said to be maximum or minimum value of $f(x,y)$ according as $f(a+h, b+k)$ is less or greater than $f(a,b)$ for all sufficiently small independent values of h & k, positive or negative (provided both are not equal to 0).

Necessary conditions - that $f(x,y)$ should have a maximum or a minimum at $x=a, y=b$ is that,

$$\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\y=b}} = 0 \quad \& \quad \left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\y=b}} = 0$$

Stationary & extreme points.

A point (a,b) is called stationary point, if both 1st order partial derivative of the function $f(x,y)$ vanish at that point.

A stationary point which is either a maximum or a minimum is called an extreme point & value of function at that point is called extreme value.

Sufficient conditions for Maxima & Minima.

let $r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a \\ y=b}}$ $\quad s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=a \\ y=b}}$ $\quad t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a \\ y=b}}$

If $\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a \\ y=b}} = 0$ & $\left(\frac{\partial f}{\partial y}\right)_{\substack{x=a \\ y=b}} = 0$ i.e if necessary conditions

for existence of maxima ~~&~~ or minima are satisfied, we have

(i) $rt - s^2 > 0$ .... $f(x,y)$ will have maximum or minimum at (a,b)

(ii) $rt - s^2 < 0$ ... neither min, nor max

(iii) $rt - s^2 = 0$ .... case is doubtful.

Working Rule for Maxima & Minima

Suppose $f(x,y)$ is a given function of x & y

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ & solve simultaneous eq^n s $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

- One of the variable may be eliminated. (to factorize eq^n s)

- In latter case, each factor of 1^{st} eq^n must be solved in conjunction with each factor of 2^{nd} eq^n.

- Suppose, solving these equations, we get the pair of values of x & y as (a_1, b_1) (a_2, b_2) etc. then all these pairs of roots will give stationary values of $f(x,y)$.

To discuss maximum or minimum at $x = a_1$, $y = b_1$,

we should find

$$r = \left(\frac{\partial^2 u}{\partial x^2}\right)_{\substack{x=a_1 \\ y=b_1}} \qquad s = \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{\substack{x=a_1 \\ y=b_1}} \qquad t = \left(\frac{\partial^2 u}{\partial y^2}\right)_{\substack{x=a_1 \\ y=b_1}}$$

then calculate $rt - s^2$

If $rt - s^2 > 0$ & r is negative $f(x,y)$ is max at $x = a_1$; $y = b_1$

If $rt - s^2 > 0$ & r is positive, $f(x,y)$ is min at $x = a_1$, $y = b_1$,

If $rt - s^2 < 0$, $f(x,y)$ is neither max, nor min.

If $rt - s^2 = 0$, case - doubtful, further Investig^n required.

§ 1. Definition.

If $u_1, u_2, \ldots, u_n$ are functions of n independent variables $x_1, x_2, \ldots, x_n$ then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \ldots, u_n$ with respect to $x_1, x_2, \ldots, x_n$ and is denoted either by $\frac{\partial (u_1, u_2, \ldots, u_n)}{\partial (x_1, x_2, \ldots, x_n)}$ or by $J(u_1, u_2, \ldots, u_n)$. The second notation is used when there is no doubt as regards the independent variables.

Thus if u and v are functions of two independent variables x and y, we have

$$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = J(u, v).$$

Similarly if u, v and w are functions of three independent variables x, y and z, we have

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = J(u, v, w).$$

If det = 0; then functions are Dependent

Note. If the functions $u_1, u_2, \ldots, u_n$ of n independent variables $x_1, x_2, \ldots, x_n$ are of the following forms,

$u_1 = f_1(x_1), u_2 = f_2(x_1, x_2), \ldots, u_n = f_n(x_1, x_2, \ldots, x_n)$, then

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \ldots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \ldots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} \ldots \ldots \frac{\partial u_n}{\partial x_n},$$

i.e., in such cases the Jacobian reduces to the principal diagonal term of the determinant.

Solved Examples

Ex. 1. *If $x = c \cos u \cosh v, y = c \sin u \sinh v$, prove that*

$$\frac{\partial(x, y)}{\partial(u, v)} = \tfrac{1}{2} c^2 (\cos 2u - \cosh 2v).$$

(Agra 1982)

Sol. We have

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -c \sin u \cosh v & c \cos u \sinh v \\ c \cos u \sinh v & c \sin u \cosh v \end{vmatrix}$$

$$= -c^2 \sin^2 u \cosh^2 v - c^2 \cos^2 u \sinh^2 v$$

$$= -\tfrac{1}{2} c^2 [(1 - \cos 2u) \cosh^2 v + (1 + \cos 2u) \sinh^2 v]$$

$$= -\tfrac{1}{2} c^2 [\cosh^2 v + \sinh^2 v - \cos 2u (\cosh^2 v - \sinh^2 v)]$$

$$= -\tfrac{1}{2} c^2 (\cosh 2v - \cos 2u) = \tfrac{1}{2} c^2 (\cos 2u - \cosh 2v).$$

Ex. 2. *If $x = u(1 + v), y = v(1 + u)$, find the Jacobian of x, y with respect to u, v.*

Sol. We have

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 + v & u \\ v & 1 + u \end{vmatrix}$$

$$= (1 + v)(1 + u) - uv = 1 + u + v + uv - uv = 1 + u + v.$$

$$= (-1)^{n-1} \cdot \frac{r}{y_n} \begin{vmatrix} \frac{\partial y_1}{\partial \theta_1} & \frac{\partial y_1}{\partial \theta_2} & \cdots & \frac{\partial y_1}{\partial \theta_{n-1}} \\ \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \cdots & \frac{\partial y_2}{\partial \theta_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_{n-1}}{\partial \theta_1} & \frac{\partial y_{n-1}}{\partial \theta_2} & \cdots & \frac{\partial y_{n-1}}{\partial \theta_{n-1}} \end{vmatrix},$$

expanding the determinant along the nth row

$$= (-1)^{n-1} \cdot \frac{r}{y_n} \begin{vmatrix} \frac{\partial y_1}{\partial \theta_1} & 0 & 0 & \cdots & 0 \\ \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_{n-1}}{\partial \theta_1} & \frac{\partial y_{n-1}}{\partial \theta_2} & \frac{\partial y_{n-1}}{\partial \theta_3} & \cdots & \frac{\partial y_{n-1}}{\partial \theta_{n-1}} \end{vmatrix}$$

$$= (-1)^{n-1} \cdot \frac{r}{y_n} \cdot \frac{\partial y_1}{\partial \theta_1} \cdot \frac{\partial y_2}{\partial \theta_2} \cdot \frac{\partial y_3}{\partial \theta_3} \cdot \cdots \frac{\partial y_{n-1}}{\partial \theta_{n-1}}$$

$$= (-1)^{n-1} \cdot \frac{r}{y_n} \cdot (-r \sin \theta_1)(-r \sin \theta_1 \sin \theta_2) \ldots .$$
$$. (-r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-1})$$

$$= (-1)^{n-1} \cdot \frac{r}{y_n} (-1)^{n-1} r^{n-1} \sin^{n-1} \theta_1 \sin^{n-2} \theta_2 \ldots$$
$$. \sin^2 \theta_{n-2} \sin \theta_{n-1}$$

$$= \frac{r}{r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \sin \theta_{n-1}} \cdot r^{n-1} \sin^{n-1} \theta_1 \sin^{n-2} \theta_2$$
$$\ldots \sin^2 \theta_{n-2} \sin \theta_{n-1}$$

$$= r^{n-1} \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \ldots \sin \theta_{n-2}.$$

§ 2. Case of Functions of Functions.

We shall establish the formula for two variables and the result can be easily extended to any number of variables.

Theorem. *If u_1, u_2 are functions of y_1, y_2 and y_1, y_2 are functions of x_1, x_2, then*

$$\frac{\partial (u_1, u_2)}{\partial (x_1, x_2)} = \frac{\partial (u_1, u_2)}{\partial (y_1, y_2)} \cdot \frac{\partial (y_1, y_2)}{\partial (x_1, x_2)}.$$

(Meerut 1991)

Integration.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \operatorname{cosec}^2 dx = -\cot x + c$$

$$\int \sec x \tan x = \sec x + c$$

$$\int \operatorname{cosec} x \cot x = -\operatorname{cosec} x + c$$

$$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + c$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + c$$

$$\int x \, dx = \frac{x^2}{2} + c$$

$$\int \tan x = \log|\sec x| + c$$

$$\int \cot x = \log|\sin x| + c$$

$$\int \sec x = \log|\sec x + \tan x| + c$$

$$\int \operatorname{cosec} x = \log|\operatorname{cosec} x - \cot x| + c$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x + \sqrt{x^2+a^2}\right| + c$$

Partial fractions

$$\frac{px+q}{(x-a)(x-b)}, \quad a \neq b \implies \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

$$\frac{(px+q)}{(x-a)^2} \longrightarrow \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$$

$$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} \longrightarrow \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$$

$$\frac{px^2+qx+r}{(x-a)^2(x-b)} \longrightarrow \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$$

$$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} \longrightarrow \frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$$

$$\int f.g\,dx = f\int g - \int\left[df \int g\,dx\right]dx$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2-a^2}\right| + c$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2+a^2}\right| + c$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

Some properties of Definate Integrals.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ;\text{f f is even} \\ 0 & ;\text{if f is odd.} \end{cases}$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad , \text{if } f(2a-x) = f(x)$$
$$\quad 0, \text{ if } f(2a-x) = -f(x)$$

$$\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$$

$$\int \operatorname{cosec} x\,dx = \log \tan\left(\frac{x}{2}\right)$$

$$\int \sec x\,dx = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$\int e^x [f(x) + f'(x)]\,dx = e^x f(x)$$

Beta Functions

- The first Eulerian Integral $\to \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ $\quad m>0$, $n>0$

is called Beta function $\Rightarrow B(m,n)$

- $B(n,m) = B(m,n)$
- $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$; $m,n>0$
- $B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$ if $m,n \in I^+$
- P.T $\to \int_0 \frac{x^{m+1}+x^{n+1}}{(1+x)^{m+n}}\,dx = B(m,n)$

Dirichlets theorem

If V is a region bounded by $x\geq 0$, $y\geq 0$ & $x+y+z\leq 1$ then

$$\iiint_V x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)}$$

- Liouville's extension of Dirichlets Thm:

If x,y,z are all positive such that $h_1 \leq (x+y+z) \leq h_2$

then $\iiint F(x+y+z)\,x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(h)\,h^{l+m+n-1}\,dh$$