$m \neq n \rightarrow$ Rectangular Mat ; $m=1 \rightarrow$ Row matrix

$[A]_{m \times n}$ $\quad m = n \longrightarrow$ Square matrix $\quad n=1 \rightarrow$ column matrix

Diagonal Matrix : $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

- Scalar matrix : A diagonal matrix whose all diagonals ~~are~~ elements are equal.

- Singular Matrix : $|A| = 0$

transpose of a matrix : $A' = [a_{ji}]$

Involuntary matrix : $A^2 = I$

Nilpotent Matrix : $A^2 = 0$

Transpose

$(A')' = A$

$(A+B)' = A' + B'$

$(kA)' = kA'$

$(AB)' = B'A'$

$(A^n)' = (A')^n$

Conjugate

$\bar{A} = [\bar{a}_{ij}] \rightarrow$ complex conjugate

$\overline{(\bar{A})} = A$

$\overline{(A+B)} = \bar{A} + \bar{B}$

$\overline{(kA)} = \bar{k}\bar{A}$

$\overline{(AB)} = \bar{B}\,\bar{A}$

$\overline{(A^n)} = (\bar{A})^n$

Transposed Conjugate

$A^\theta = (\bar{A})'$

$(A^\theta)^\theta = A$

$(kA)^\theta = \bar{k} A^\theta$

$(A^n)^\theta = (A^\theta)^n$

$(A+B)^\theta = A^\theta + B^\theta$

$(A \cdot B)^\theta = B^\theta \cdot A^\theta$

$\text{adj } A = [A_{ji}]'$ ... A_{ji} is cofactor of a_{ij}

- $\text{Trace}(AB) = \text{trace}(BA)$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

- $AA^{-1} = A^{-1}A = I$
- A is said to be invertible if inverse exists.
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^n)^{-1} = (A^{-1})^n$
- $\det(A^{-1}) = [\det(A)]^{-1}$

- Any System of L.E can be written (in any no. of unknowns) in the form of $\underline{\underline{AX = B}}$

  $X = A^{-1}B \rightarrow$ is unique solution where $|A| \neq 0$

- Symmetric Matrix : $A = [a_{ij}] = [a_{ji}]$; $A = A'$
- Skew Symmetric mat : $A = [a_{ij}] = [-a_{ji}]$; $A = -A'$
  - Diagonal elements of Skew symmetric matrix are 0.
- Hermitian mat $[a_{ij}] = [\overline{a_{ij}}]$; $A^{\theta} = A$
  - All diagonal matrix are real.
- Skew Hermitian $[a_{ij}] = [\overline{-a_{ij}}]$; $A = -A^{\theta}$
  - Diagonal elements are purely imaginary or 0.

- Row Echelon form :

- Rank of a matrix in echelon form is equal to no. of non-zero rows/columns of matrix.

- Normal form :

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} I_r \\ 0 \end{bmatrix} \quad \begin{bmatrix} I_r & 0 \end{bmatrix} \quad \begin{bmatrix} I_r \end{bmatrix}$$

<u>Thm:</u> If A be the matrix $m \times n$ of rank r, there exists non-singular matrix P & Q such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Linear Dependence and Independence

- Linear Dependence : $x_1, x_2, x_3 \dots x_n$ are said to be Linearly dependent if there exists 'n' no. $c_1, c_2 \dots c_n$ (not all zeros such that

$$c_1x_1 + c_2x_2 + \dots c_nx_n = 0$$

- Linear Independence : $x_1, x_2 \dots x_n$ are L.I if

$$c_1x_1 + c_2x_2 + \dots c_nx_n = 0 \quad \text{given} \quad c_1 = c_2 = \dots c_n = 0.$$

- A set containing only zero vector is L.D.
- A set containing only non-zero vector is L.I
- Every sub-set of L.I set is L.I.

<u>working Rules</u>:

take n scalars - $c_1, c_2, c_3 \dots c_n$ & put $c_1x_1 + c_2x_2 + \dots = 0$ → solve the system and find values of $c_1, c_2, \dots c_n$.

→ if $c_1 = c_2 = \dots = 0$ then only sol^n is L.I

→ ~~if $c_1 = c_2 =$~~ if atleast $c_i \neq 0$, then x_1, x_2 are L.D.

Alternate

1. Construct matrix A whose n columns are given vectors $\rightarrow x_1, x_2, x_3 \ldots x_n$.
2. Reduce A to R.E form by using elementry Row-op^n.
3. find $\rho(A)$
4. if $\rho(A) = n$ .... no of vector then L.I
   if $\rho(A) < n$ — " — L.D

if $|A| = 0$ then $\rho(A) < n$ — " — L.D

$A_{m \times n}$ matrix $\rho(A) = r \leq \min(m, n)$

dimension of solution space $= (n-r)$; i.e $n-r$ L.I solutions.

System of Linear Equations.

$AX = B \Rightarrow$ Non-Homogenous L.E

$[A:B] \Rightarrow$ augmented matrix

- Consistent
  - Unique Solution: $\rho(A) = \rho(A:B)$
  - Infinite Sol^n: $\rho(A) < \rho(A:B)$
- Inconsistent
  - (No-Solution): $\rho(A) \neq \rho(A:B)$

Characteristics Equations.

$$|A-\lambda I| = 0$$

- Roots of C.E are Characteristic Roots/Eigen Values/Latent roots.
- set of C.R of a matrix = spectrum of matrix.
- Any eqn $AX = \lambda X$ is c.vector/eigen vector.

- Caley Hamilton theorem:
  - Every square matrix satisfies its C.E
  - Also can be used to find inverse.
  - transpose of a square mat. has same C.E.

- **Orthogonal Matrix**

A: square matrix $\Rightarrow AA' = A'A = I$; $A' = A^{-1}$

- Rows & columns form orthonormal sets of vectors.

- Determinant of orthogonal matrix is ± 1

- **Unitary Matrix**

$A \rightarrow$ Sq. mat $\rightarrow$ unitary $\quad A \cdot A^{\theta} = A^{\theta} \cdot A = I$

$$A^{\theta} = A^{-1}$$

- Determinant of unitary mat. has absolute value 1.

- C.R of Unitary mat are of unit modulus.

- **Similar Matrix**

let A & B sq. matrix of order n then B is said to be similar to A if there exists a non-singular matrix P s.t

$$B = P^{-1}AP$$

Vector Spaces & subspaces

- V & $F \rightarrow$ two non-empty sets.

if in V $\left[\begin{array}{l} u * v \in V \;;\quad u, v \in V \\ u * v \text{ is unique} \end{array}\right]$ then $*$ is internal binary opⁿ on V.

- for each $u \in V$ and $a \in F$, $a * u \in V$ & unique, then $*$ is external binary opⁿ on V over F.

- A vector space over F is a non-empty set V (together with 2 operations : $+$, $\times$) such that $u+v$ is unique and for each $a \in F$ & $u \in V$, there is a unique $au \in V$ and it satisfies : ① Associativity ② Commutativity ③ existence of Identity ④ Existence of Inverse

Defⁿ 2:

Let $\langle V, + \rangle$ be an abelian group & $\langle F, +, \cdot \rangle$ be a field.
Define scalar multiplication such that for all $a \in F$, $u \in V$, $au \in V$.
Then we say that V is a vector space over F if ①, ②, ③, ④ are satisfied.

Vectors in R^n

- An ordered n-tuple of real numbers is called a real n-vector.

- A plane Vector is an ordered pair (a_1, a_2) of real nos.
- the set of plane Vectors (all ordered pairs of real nos.) is denoted by V_2 which is cartesian product $R \times R$: R^2.
- space vector : (a_1, a_2, a_3) : R^3.

Subspaces

- A non-empty subset W of a vector space V(F) is said to form a subspace of V if W is also a vector space over F with same addition and scalar multiplication as for V.

- Zero space (null space) consisting of single element zero vector and vector space V itself are subspaces of every vector space V(F) : Improper subspaces.
- All other subspaces, if any are called proper subspaces.

Theorems

① A necessary & sufficient condition for a non-empty subset W of a vector space V(F) to be a subspace of V is that W is closed under addition and scalar multiplication.

→ 1. Cond^n is necessary:

if W is a subspace of V, then by defination, W is a vector space over F ⇒

$u, v \in W \Rightarrow u + v \in W$ and

$u \in W, a \in F \Rightarrow au \in W$ ⇒ W is closed under + & ×.

Condition is sufficient: Let W be closed under add^n & scalar mult^n.

$\Rightarrow u, v \in W \Rightarrow u+v \in W$ and $u \in W, a \in F \Rightarrow au \in W$

Since 1 is multip. Identity of $F \Rightarrow$ ~~1 ∈ F~~ $-1 \in F$; $\therefore (-1)u \in W$

$\Rightarrow -u \in W$

— 0 is Additive inverse of F $\Rightarrow$ $u + -u \in W$ $\Rightarrow 0 \in W$

Thus zero element of V is also zero element of W.

Also $u, v \in W \Rightarrow u, v \in V$ and, so : $u+v = v+u$

Similarly cond^n of Asso. and other conditions of a vector space also hold in W since they hold in V.

$\therefore (W, +)$ is an abelian group. Also W is subset of V.

Hence W is subspace of V.

(2) The necessary & sufficient conditions for a non-empty subset W of a vector space $V(F)$ to be a subspace are :

1) $u - v \in W$ for $u, v \in W$ 2) $au \in W$ for $a \in F, u \in W$

Condition Necessary : → Let W be a subspace of $V(F)$

$\therefore W$ is also a vector space over F

$\Rightarrow$ 2) holds.

Now $v \in W \Rightarrow -v \in W$; $u, -v \in W \Rightarrow u + (-v) \in W$

$\Rightarrow u - v \in W$ .... 1) holds.

Conditions Sufficient : Let W be a non-empty subset of V s.t cond^n (i) & (ii) hold let $u \in W$

then, by condition (i) $u - u \in W \Rightarrow 0 \in W \Rightarrow$ Identity ele.

by condition (ii) $-1 \in F$ and $u \in W \Rightarrow -1.u \in W \Rightarrow -u \in W$

Additive Inverse.

Again: $u \in W$, $v \in W \Rightarrow u \in W, -v \in W \Rightarrow u - (-v) \in W$ (from (i))

$\Rightarrow u + v \in W \Rightarrow W$ is closed under vector Add^n.

Also $u, v \in W \Rightarrow u, v \in V$ and $u+v = v+u$ [$\because \langle V, + \rangle$ is abelian]

Similarly conditions of a vector space also holds in W since they hold in V and also W is subset of V.

Hence W is subspace of V.

~~② The necessary & sufficient conditions for a non-empty subset W of a vector space V(F) to be subspace:~~
~~① $u - v \in W$ for $u, v \in W$ ② au~~

③ A non-empty subset W of vector space V(F) is a subspace of V iff $au + bv \in W$ for $a, b \in F$ and $u, v \in W$.

<u>Condition Necessary</u>: Let W be a subspace of V(F). then by def^n: W is closed under vector addition and scaler multipl^n.

Since for any $a \in F, u \in W \Rightarrow au \in W$ & for any $b \in F, v \in W \Rightarrow bv \in W$

$\therefore$ for any $a, b \in F$ & $u, v \in W$ $au + bv \in W$.

Since $1 \in F$ → take $a = b = 1 \Rightarrow 1u + 1v \in W \Rightarrow u + v \in W$ for $u, v \in W$

$\Rightarrow$ W is closed under vector addition.

Again $a \in F, u \in W \Rightarrow au = au + 0v$ for any $v \in W$, $0 \in F$.

$\Rightarrow au = au + 0v \in W$

thus $a \in F, u \in W \Rightarrow au \in W$

$\Rightarrow$ W is closed under scaler multiplication

Hence W is subspace of V(F)

(Pg 25-27) → Point

Linear Sum of Subspaces.

- If W_1 & W_2 are two subspaces of a vector space $V(F)$, then we define Linear sum of W_1 and W_2 as:

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1 \ \& \ w_2 \in W_2\}$$

- Let $w_1 \in W_1$; ~~Since~~

Since $0 \in W_2 \Rightarrow w_1 + 0 \in W_1 + W_2$. thus $\Rightarrow$ $W_1 \subseteq W_1 + W_2$

similarly $W_2 \subseteq W_1 + W_2$

Hence $W_1 \cup W_2 \subseteq W_1 + W_2$

Theorems

① the Linear sum of two subspaces W_1 & W_2 is also a subspace of the vector space $V(F)$.

$\rightarrow$ ~~Et~~ Let W_1 & W_2 $\rightarrow$ 2 subspaces of $V(F)$; u, v be any two elements of $W_1 + W_2$.

then $\left.\begin{array}{l} u = w_1 + w_2 \\ v = w_1' + w_2' \end{array}\right\}$ for some: $w_1, w_1' \in W_1$ & $w_2, w_2' \in W_2$

Now $a, b \in F \Rightarrow au + bv = a(w_1 + w_2) + b(w_1' + w_2')$

$= (aw_1 + bw_1') + (aw_2 + bw_2')$

Since W_1 & W_2 are subspaces of V $\rightarrow$ $\therefore$ $aw_1 + bw_1' \in W_1$ & $aw_2 + bw_2' \in W_2$

thus: $au + bv = (aw_1 + bw_1') + (aw_2 + bw_2') \in W_1 + W_2$

Hence $W_1 + W_2$ is subspace of $V(F)$

Subspace Generated by a Set:

Let V be a vector space over F and $S \subseteq V$. then W is said to be subspace generated by S if W is the smallest subspace of V containing S and is denoted by $W = \langle S \rangle$

Thm: The Linear sum of two subspaces W_1 & W_2 of a vector space $V(F)$ is a subspace generated by union of W_1 & W_2 i.e $W_1 + W_2 = \langle W_1 + W_2 \rangle$

Direct Sum of Subspaces.

A vector space V is said to be direct sum of its two subspaces W_1 & W_2 if every $v \in V$ can be uniquely expressed as:

$v = \omega_1 + \omega_2$, where $\omega_1 \in W_1$, $\omega_2 \in W_2$

$V = W_1 \oplus W_2$

ex: If $W_1 = \{(a,b,0) : a,b \in R\}$, $W_2 = [(0,0,c) : c \in R]$ are two subspaces of $V_3(R)$ then any vector $(a,b,c) \in V_3(R)$ can be uniquely written as: $(a,b,c) = (a,b,0) + (0,0,c)$ $\Rightarrow$ $V_3 = V_1 \oplus V_2$

Thm: the necessary and sufficient condition for a vector space $V(F)$ to be a direct sum of its subspace W_1 & W_2 are: ① $V = W_1 + W_2$ (ii) $W_1 \cap W_2 = \{0\}$ $\Rightarrow$ (Disjoint subspaces)

Basis & Dim PDF $\Rightarrow$ Pg 1-5 $\rightarrow$ print

Dimensions of a Vector space

Defⁿ The no. of elements in any basis of a finite dimensional vector space $V(F)$ is called the dimension of vector space $V(F)$ & is denoted as $V(F)$

(i) Dimension of V is always less than or equal to no. of elements in any generating set.

(ii) If $\dim V = n$, then any linearly independent set can have at most n elements.

(iii) Dimensions of R, R^2, R^n .. are $1, 2, n$ — respectively.

Thm: Every Linearly Independent subset of a finitely generated vector space $V(F)$ can be extended to form a basis of V.

Thm: If W is subspace of a finitely dimensional vector space $V(R)$, then W is finite dimensional & $\dim W \leq \dim V$.

Thm: If W_1, W_2 are two sub-spaces of a finite dimension vector space $V(F)$, then $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

Rank and Nullity.

let $T: U \to V$ be a Linear transformation. the null space (or kernel) of T is subset of U consisting of all vectors u whose image under T is 0 and is denoted by $\ker T$ or $N(T)$ i.e $\ker T = N(T) = \{u \in U : T(u) = 0_V\}$

Thm:

let $T: U \to V$ be a L.T. Then T is one to one iff $N(T) = \{0\}$

Proof: let $T: U \to V$ be a L.T such that T is one to one.

let $u \in N(T)$ be arbitrary

$\Rightarrow T(u) = 0$ or $T(u) = T(0) \Rightarrow u = 0$ [$\because$ T is one-one]

~~let u ∈ N(T) be arbitrary~~

Thus $N(T) = \{0\}$ | conversely let $T: U \to V$ be L.T such that $N(T) = \{0\}$

let $u_1, u_2 \in U$ be arbitrary such that

$T(u_1) = T(u_2) \Rightarrow T(u_1) - T(u_2) = 0 \Rightarrow T(u_1 - u_2) = 0$

$\Rightarrow u_1 - u_2 = 0$

$u_1 = u_2$

which proves that T is one-one.

Theorem: If $U(F) \to V(F)$ is L.T, then show that $N(T)$, the null space of T, is a subspace of $U(F)$

(OR)

Let $T: U \to V$ be a homomorphism, then $\ker T$ is a subspace of U.

Proof: let $T: U \to V$ be a L.T, let $N(T) = \{u \in U : T(u) = 0\}$

obviously: $N(T) \neq \phi$, since $0 \in N(T)$ such that $T(0) = 0$.

Also, let $u_1, u_2 \in N(T)$ and $a, b \in F$ then,

$$T(au_1 + bu_2) = aT(u_1) + bT(u_2) = a.0 + b.0 = 0$$

$$\Rightarrow au_1 + bu_2 \in N(T)$$

Hence $N(T)$ or $\ker T$ is a subspace of U.

Range or Image of L.T

Def^n: let $T: U \to V$ be a LT. Then range of T i.e $R(T)$ ~~is a subspace of V.~~ is the set of all vectors in V that are images, under T of vectors in U. i.e $\text{Range}\, T = R(T) = \{T(u) : u \in U\}$

Thm: let $T: U \to V$ be a L.T then range of T i.e $R(T)$ is a subspace of V.

Proof Since $T(0) = 0$ so 0 is in $R(T)$ and thus $R(T) \neq \phi$

let $v_1, v_2 \in R(T)$. Then $v_1 = T(u_1)$ & $v_2 = T(u_2)$ for some u_1 & u_2 in U.

Now, for ~~Above~~ $a, b \in F$, $av_1 + bv_2 = aT(u_1) + bT(u_2)$

$$= T(au_1 + bu_2) \in R(T)$$

Since $au_1 + bu_2 \in U$ $\therefore$ $R(T)$ is subspace of V.

Linear transformations

- functions defined on vector spaces.
- Assumption → all vector spaces are over common field F.

Def^n let U & V be two vector spaces over the same Field F. then a map (function) $T: U \to V$ is called a linear transformation (or vector space homomorphism) if

(i) $T(u+v) = T(u) + T(v)$, for all $u,v \in U$

(ii) $T(au) = aT(u)$ for all $a \in F$, $u \in U$.

(i) L.T is also called Linear Mapping, Linear function or vector space homomorphism.

(ii) If $U=V$, then Linear transformation $T: U \to U$ is called Linear operator on U.

Ex.

1] A function $T: R^3 \to R^2$ defined by $T(x,y,z) = (x,y)$ is a Linear transformation and called projection.

2] The vector $(x_1, y_1) \in R^2$ is image of $(x_1, y_1, z_1) \in R^3$ under T