

LECTURE 5 AND 6

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1. RANK NULLITY THEOREM

It is also called then fundamental theorem of Linear Maps because of its importance in linear transformation.

1.1. Statement and Proof.

Theorem 1. *Rank-Nullity Theorem:* Suppose V is finite-dimensional and $T \in \mathcal{L} : (V, W)$. Then range T is finite-dimensional and

$$\begin{aligned} \text{Rank}(T) + \text{Nullity}(T) &= \dim(V), \text{ or} \\ \dim(\text{Range}(T)) + \dim(\text{Ker}(T)) &= \dim(V) \end{aligned}$$

Proof. Let $\phi_1, \phi_2, \dots, \phi_l$ be the minimum vectors will that span $\text{Ker}(T)$, where l is the $\dim(\text{Ker}(T))$.

And v_1, v_2, \dots, v_{n-l} be the vectors that will span the remaining vector space V where, n is the $\dim(V)$. Their linear transformation $(w_1, w_2, \dots, w_{n-l})$ must be independent and should span the entire range of T in W , let $\dim(\text{Range}(T))$ be k .

We will prove this claim later.

Then it becomes quite easy to see why the rank nullity theorem holds. Because,

$$n - l = k \text{ or } n = l + k$$

Hence, Proved. □

Claim 1. w_1, w_2, \dots, w_{n-l} are linearly independent and span the entire $\text{Range}(T)$.

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Proving Claim. w_1, w_2, \dots, w_{n-l} are linear transformation of basis v_1, v_2, \dots, v_{n-l} . So,

$$\alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_{n-l} w_{n-l} = 0$$

$$\text{or, } \sum_{i=1}^{n-l} \alpha_i w_i = 0$$

$$\text{or, } \sum_{i=1}^{n-l} \alpha_i T v_i = 0$$

$$\text{or, } T\left(\sum_{i=1}^{n-l} \alpha_i v_i\right) = 0$$

So $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{n-l} v_{n-l} \in \dim(\text{null } T)$.

Since $\phi_1, \phi_2, \dots, \phi_l$ spans $\text{null } T$, we can write:

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{n-l} v_{n-l} = \beta_1 \phi_1 + \beta_2 \phi_2 + \dots + \beta_l \phi_l$$

But we also know that ϕ_1 to ϕ_l and v_1 to v_{n-l} are linearly independent, which means the only way to satisfy above equation is to have $\alpha_1 = \alpha_2 = \dots = \alpha_{n-l} = \beta_1 = \beta_2 = \dots = \beta_l = 0$.

Thus $Tv_1, Tv_2, \dots, Tv_{n-l}$ are linearly independent and span the entire $\text{Range}(T)$. Hence w_1, w_2, \dots, w_{n-l} will span the entire $\text{Range}(T)$. \square

1.2. Applications. We can now prove that some of the mapping can not be surjective or injective.

Theorem 2 (A map to a smaller dimensional space is not injective). *Suppose V and W are finite-dimensional vector spaces such that $\dim(V) > \dim(W)$, then there is no injective linear mapping from V to W .*

Proof. Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned} \dim(\text{null } T) &= \dim(V) - \dim(\text{Range}(T)) \\ &\geq \dim(V) - \dim(W) \\ &> 0 \end{aligned}$$

The equality above comes from the fundamental theorem of Linear Maps and the strict inequality at the end shows the dimension of $\text{null } T$ is greater than 0, means vector other than 0 are present in $\text{null } T$. Hence the mapping is not injective. \square

Theorem 3 (A map to a larger dimensional space is not surjective). *Suppose V and W are finite-dimensional vector spaces such that $\dim(V) < \dim(W)$, then there is no surjective linear mapping from V to W .*

Proof. Let $T \in \mathcal{L}(V, W)$. Then

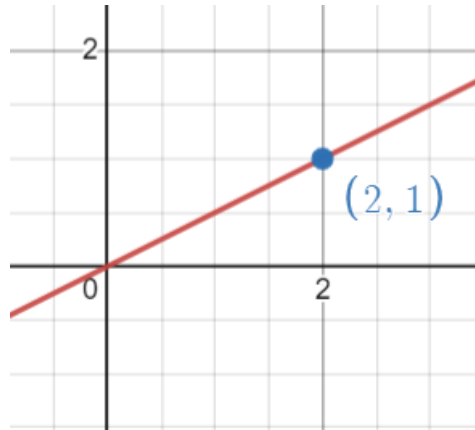
$$\begin{aligned} \dim(\text{range } T) &= \dim(V) - \dim(\text{null } T) \\ &\leq \dim(V) \\ &< \dim(W) \end{aligned}$$

Here again the equality comes from the Rank-Nullity Theorem, and the rest is just normal inference.

At the end we have range of T is less than the dimension of W . Hence the mapping is not surjective. \square

2. MATRIX REPRESENTATION OF LINEAR TRANSFORMATION

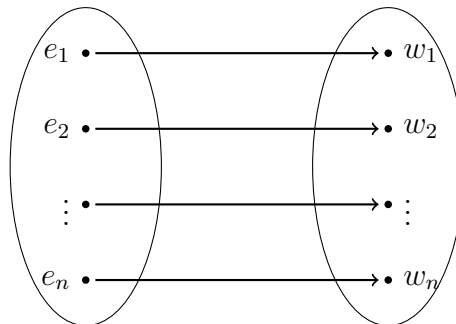
A vector is represented in different ways on the coordinate system depending on what its basis is. Suppose the vector $(2,1)$ in R^2 is represented with basis as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



But when the basis is changed the vector is represented differently and on a different basis.

The vector V can be represented as $a_1b_1 + a_2b_2$ and in matrix form can be represented as $V \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

Suppose there is a linear transformation T that maps a vector V from R^n to W in R^m . Let the basis be $B = \{e_1, \dots, e_n\}$ and $B' = \{e'_1, \dots, e'_m\}$. The map can be represented as



Suppose the basis for B is $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ and the upon applying the transformation the basis for B' is $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$

Now the question is how should we connect α_i to the β_j such that we reach to our answer.

We know that $Tv = w$

$$v = \sum_{i=1}^n \alpha_i e_i$$

Applying linear transformation T on both sides we get

$$Tv = \sum_{i=1}^n \alpha_i T(e_i)$$

Now what is Te_i ?

In this case $e_i \rightarrow w_i$

$$\therefore Tv = \sum_{i=1}^n \alpha_i w_i$$

But we want w_i to be represented in terms of Standard Basis.

$$w_i = \begin{pmatrix} w_{1i} \\ w_{2i} \\ \vdots \\ w_{mi} \end{pmatrix} \text{ (This is the representation of } w_i \text{ in standard basis)}$$

$$\therefore w_i = \sum_{j=1}^m w_{ij} e'_j$$

Hence now the Tv can be transformed as follows

$$\begin{aligned} Tv &= \sum_{i=1}^n \sum_{j=1}^m w_{ij} e_j \\ Tv &= \sum_{j=1}^m \left(\sum_{i=1}^n w_{ij} \alpha_i \right) e'_j \\ \therefore Tv &= \sum_{j=1}^m \beta_j e'_j \end{aligned}$$

Call the quantity $\left(\sum_{i=1}^n w_{ij} \alpha_i \right) = \beta_j$

This can now be represented in the matrix form as follows

$$Tv = \begin{pmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ w_{m,1} & w_{m,2} & \cdots & w_{m,n} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Here we can clearly see that $Te_i = \begin{pmatrix} w_{1,i} \\ \vdots \\ w_{m,i} \end{pmatrix}$ That is the i^{th} column in the matrix.

Hence to get the first column and eventually Te_1 we must place the

$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ equal to $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Similarly to get the second column we must

place the $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ equal to $\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ and so on.

Example 1: $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$. Given

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} -3 \\ 1 \end{pmatrix} \\ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Find the linear transformation $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$.

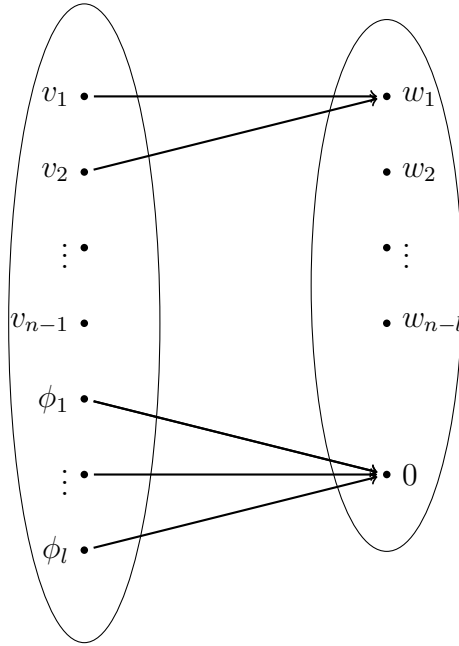
Solution: The first question that arises is where the standard basis goes. The vector on the LHS be in the standard basis $(1, 0)$ and $(0, 1)$.

$$\begin{aligned} & \begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} -3 \\ 1 \end{pmatrix} \\ -2 \cdot & \left(\begin{pmatrix} -3 \\ 1 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) \\ &= \begin{pmatrix} 7 \\ 0 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} -5 \\ -3 \end{pmatrix} \\ \implies & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{T} \frac{1}{7} \begin{pmatrix} -5 \\ -3 \end{pmatrix} \end{aligned}$$

Similarly we find for the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and we get it to be $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{T} \frac{1}{7} \begin{pmatrix} -8 \\ 5 \end{pmatrix}$.

Now we know the linear transformation for both the standard basis vectors. Now we can push these values in the matrix and get the value for the corresponding β depending on the value of α .

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \frac{-5}{7} & \frac{-8}{7} \\ \frac{-3}{7} & \frac{5}{7} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$



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