LECTURE 5 AND 6

SHUBH AGARWAL, SAKSHAM CHHIMWAL, ABHIRAM K,
AND RISHABH SHARAD

1. Rank Nullity Theorem

It is also called then fundamental theorem of Linear Maps because of its importance in linear transformation.

1.1. Statement and Proof.

Theorem 1. ***Rank-Nullity Theorem:*** *Suppose* V *is finite-dimensional and* $T \in \mathcal{L} : (V, W)$*. Then range* T *is finite-dimensional and*

$$Rank(T) \, + \, Nullity(T) \, = \dim(V), \text{ or}$$

$$\dim(Range(T)) \, + \, \dim(Ker(T)) \, = \dim(V)$$

Proof. Let $\phi_1, \phi_2, \ldots, \phi_l$ be the minimum vectors will that span $Ker(T)$, where l is the $\dim(Ker(T))$.
Andv$_1, v_2, \ldots, v_{n-l}$ be the vectors that will span the remaining vector space V where, n is the $\dim(V)$. Their linear transformation
$(w_1, w_2, \ldots, w_{n-l})$ must be independent and should span the entire range of T in W, let $\dim(Range(T))$ be k.

We will prove this claim later.
Then it becomes quite easy to see why the rank nullity theorem holds. Because,

$$n - l = k \;\; or \;\; n = l + k$$

Hence, Proved. □

Claim 1. $w_1, w_2, \ldots, w_{n-l}$ *are linearly independent and span the entire* $Range(T)$*.*

Date: January 24, 2023.
2020 *Mathematics Subject Classification.* 2023 Mathematics for Data Science, CS-427.
This paper is in final form.

Proving Claim. $w_1, w_2, \ldots, w_{n-l}$ are linear transformation of basis $v_1, v_2, \ldots, v_{n-l}$. So,

$$\begin{aligned} \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_{n-l} w_{n-l} &= 0 \\ or, \sum_{i=1}^{n-l} \alpha_i w_i &= 0 \\ or, \sum_{i=1}^{n-l} \alpha_i T v_i &= 0 \\ or, T(\sum_{i=1}^{n-l} \alpha_i v_i) &= 0 \end{aligned}$$

So $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{n-l} v_{n-l} \in \dim(nullT)$.
$Since$,$\phi_1, \phi_2, \ldots, \phi_l$ spans *null T*, we can write:

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{n-l} v_{n-l} = \beta_1 \phi_1 + \beta_2 \phi 2 + \ldots + \beta_l \phi_l$$

But we also know that $\phi_1 \, to \, \phi_l$ and $v_1 \, to \, v_{n-l}$ are linearly independent, which means the onlu way to satisfy above equation is to have $\alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = \beta_1 = \ldots = \beta_l = 0$.
$Thus$Tv$_1, Tv_2, \ldots, Tv_{n-l}$ are linearly independent and span the entire *Range*(T). Hence $w_1, w_2, \ldots, w_{n-l}$ will span the entire *Range*(T). □

1.2. **Applications.** We can now prove that some of the mapping can not be surjective or injective.

Theorem 2 (A map to a smaller dimensional space is not injective)**.** *Suppose* V *and* W *are finite-dimensional vector spaces such that* $\dim(V)$ *and* $\dim(W)$*, then there is no injective linear mapping from* V *to* W*.*

Proof. Let $T \in \mathcal{L}$:$(V,\, W)$. Then

$$\begin{aligned} \dim(null\, T) &= \dim(V) - \dim(Range(T)) \\ &\geq \dim(V) - \dim(W) \\ &> 0 \end{aligned}$$

The equality above comes from the fundamental theorem of Linear Maps and the strict inequality at the end shows the dimension of T is greater than 0, means vector other than 0 are present in T. Hence the mapping is not injective. □

Theorem 3 (A map to a larger dimensional space is not surjective)**.**
Suppose V *and* W *are finite-dimensional vector spaces such that* $\dim(V) and \dim(W)$*,*
then there is no surjective linear mapping from V *to* W*.*

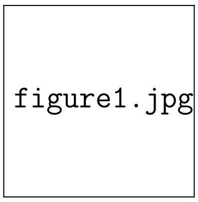
Proof. Let $T \in \mathcal{L}$:(V, W). Then

$$\begin{aligned} \dim(range\, T) &= \dim(V) - \dim(null\, T) \\ &\leq \dim(V) \\ &< \dim(W) \end{aligned}$$

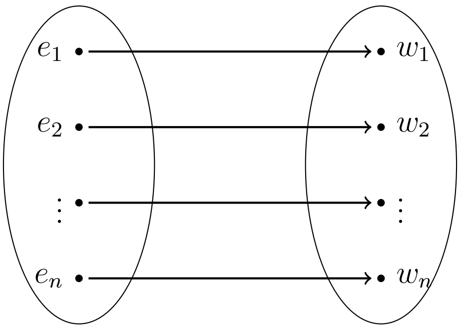
Here again the equality comes from the Rank-Nullity Theorem, and the rest is just normal inference.
At the end we have range of T ¡ the dimension of W. Hence the mapping is not surjective. ☐

2. Matrix Representation of Linear Transformation

A vector is represented in different ways on the coordinate system depending on what its basis is. Suppose the vector (2,1) in R^2 is represented with basis as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



But when the basis is changed the vector is represented differently and on a different basis.
The vector V can be represented as $a_1b_1 + a_2b_2$ and in matrix form can be represented as $V \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.
Suppose there are is a linear transformation T that maps a vector V from R^n to W in R^m. Let the basis be $B = \{e_1, \ldots, e_n\}$ and $B' = \{e'_1, \ldots, e'_m\}$ The map can be represented as



Suppose the basis for B is $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ and the upon applying the transformation the basis for B' is $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$

Now the question is how should we connect α_i to the β_j such that we reach to our answer.

We know that $Tv = w$

$v = \sum_{i=1}^{n} \alpha_i e_i$

Applying linear transformation T on both sides we get

$Tv = \sum_{i=1}^{n} \alpha_i T(e_i)$

Now what is Te_i ?

In this case $e_i \to w_i$

$\therefore Tv = \sum_{i=1}^{n} \alpha_i w_i$

But we want w_i to be represented in terms of Standard Basis.

$$w_i = \begin{pmatrix} w_{1i} \\ w_{2i} \\ \vdots \\ w_{mi} \end{pmatrix} \text{(This is the representation of } w_i \text{ in standard basis)}$$

$$\therefore w_i = \sum_{i=1}^{m} w_{ij} e'_j$$

Hence now the Tv can be transformed as follows

$$Tv = \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} e_j$$

$$Tv = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} w_{ij}\alpha_i\right) e'_j$$

Call the quantity $\left(\sum_{i=1}^{n} w_{ij}\alpha_i\right) = \beta_j$

$$\therefore Tv = \sum_{j=1}^{m} \beta_j e'_j$$

This can now be represented in the matrix form as follows

$$Tv = \begin{pmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ w_{m,1} & w_{m,2} & \cdots & w_{m,n} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Here we can clearly see that $Te_i = \begin{pmatrix} w_{1,i} \\ \vdots \\ w_{m,i} \end{pmatrix}$ That is the i^{th} column in the matrix.

Hence to get the first column and eventually Te_1 we must place the $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ equal to $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Similarly to get the second column we must place the $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ equal to $\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ and so on.

Example 1: $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$. Given

$$\begin{pmatrix}1\\2\end{pmatrix} \xrightarrow{T} \begin{pmatrix}-3\\1\end{pmatrix}$$
$$\begin{pmatrix}-3\\1\end{pmatrix} \xrightarrow{T} \begin{pmatrix}1\\2\end{pmatrix}$$

Find the linear transformation $\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} \xrightarrow{T} \begin{pmatrix}\beta_1\\\beta_2\end{pmatrix}$.

Solution: The first question that arises is where the standard basis goes. The vector on the LHS be in the standard basis $(1,0)$ and $(0,1)$.

$$\begin{pmatrix}1\\2\end{pmatrix} \xrightarrow{T} \begin{pmatrix}-3\\1\end{pmatrix}$$
$$-2\cdot\left(\begin{pmatrix}-3\\1\end{pmatrix} \xrightarrow{T} \begin{pmatrix}1\\2\end{pmatrix}\right)$$
$$= \begin{pmatrix}7\\0\end{pmatrix} \xrightarrow{T} \begin{pmatrix}-5\\-3\end{pmatrix}$$
$$\implies \begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{T} \frac{1}{7}\begin{pmatrix}-5\\-3\end{pmatrix}$$

Similarly we find for the vector $\begin{pmatrix}0\\1\end{pmatrix}$ and we get it to be $\begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{T} \frac{1}{7}\begin{pmatrix}-8\\5\end{pmatrix}$.

Now we know the linear transformation for both the standard basis vectors. Now we can push these values in the matrix and get the value for the corresponding β depending on the value of α.

$$\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} = \begin{pmatrix}\frac{-5}{7} & \frac{-8}{7}\\ \frac{-3}{7} & \frac{5}{7}\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}$$

Email address, Shubh: `210020047@iitdh.ac.in`
URL: `https://shubhagarwal-dev.github.io/`

Email address, Saksham: `210010046@iitdh.ac.in`

Email address, Abhiram: `210150001@iitdh.ac.in`

Email address, Rishabh: `210020036@iitdh.ac.in`