FULL TITLE

1. Rank Nullity Theorem

It is also called then fundamental theorem of Linear Maps because of its importance in linear transformation.

1.1. Stating and Proof.

Theorem 1. ***Rank-Nullity Theorem:*** *Suppose V is finite-dimensional and $T \in \mathcal{L} : (V, W)$. Then range T is finite-dimensional and*

$$Rank(T) \; + \; Nullity(T) \; = \dim(V), \; or$$
$$\dim(Range(T)) \; + \; \dim(Ker(T)) \; = \dim(V)$$

Proof. Let $\phi_1, \phi_2, \ldots, \phi_l$ be the minimum vectors will that span $Ker(T)$, where l is the $\dim(Ker(T))$.
And $v_1, v_2, \ldots, v_{n-l}$ be the vectors that will span the remaining vector space $V$where, n is the $\dim(V)$. Their linear transformation must be independent and should span the entire range of T in W, let $\dim(Range(T))$ be k. *We will prove this claim later.*
Then it becomes quite easy to see why the rank nullity theorem holds. □

Claim 1. *$w_1, w_2, \ldots, w_{n-l}$ are linearly independent and span the entire $Range(T)$.*

1.2. Applications.

We can now prove that some of the mapping can not be surjective or injective.

Theorem 2 (A map to a smaller dimensional space is not injective)**.** *Suppose V and W are finite-dimensional vector spaces such that $\dim(V)$ and $\dim(W)$, then there is no injective linear mapping from V to W.*

Proof. Let $T \in \mathcal{L}$:$(V,\, W)$. Then

$$\begin{aligned} \dim(null\, T) &= \dim(V) - \dim(Range(T)) \\ &\geq \dim(V) - \dim(W) \\ &> 0 \end{aligned}$$

□

Proof. □

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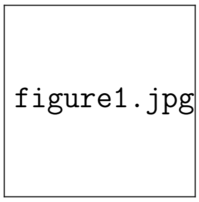
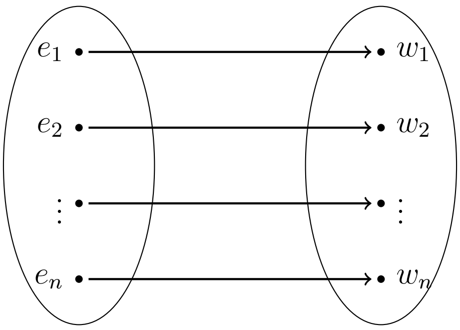
2. Matrix Representation of Linear Transformation

A vector is represented in different ways on the coordinate system depending on what its basis is. Suppose the vector (2,1) in R^2 is represented with basis as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



But when the basis is changed the vector is represented differently and on a different basis.
The vector V can be represented as $a_1b_1 + a_2b_2$ and in matrix form can be represented as $V\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.
Suppose there are is a linear transformation T that maps a vector V from R^n to W in R^m. Let the basis be $B = \{e_1, \ldots, e_n\}$ and $B' = \{e'_1, \ldots, e'_m\}$ The map can be represented as



Suppose the basis for B is $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ and the upon applying the transformation the basis for B' is $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$
Now the question is how should we connect α_i to the β_j such that we

reach to our answer.

$$\begin{gathered}\text{We know that } Tv = w \\ v = \sum_{i=1}^{n} \alpha_i e_i \\ \text{Applying linear transformation T on both sides we get} \\ Tv = \sum_{i=1}^{n} \alpha_i T\left(e_i\right)\end{gathered}$$

$$\begin{gathered}\text{Now what is } Te_i \text{ ?} \\ \text{In this case } e_i \to w_i\end{gathered}$$

$$\therefore Tv = \sum_{i=1}^{n} \alpha_i w_i$$

But we want w_i to be represented in terms of Standard Basis.

$$w_i = \begin{pmatrix} w_{1i} \\ w_{2i} \\ \vdots \\ w_{mi} \end{pmatrix} \text{(This is the representation of } w_i \text{ in standard basis)}$$

$$\therefore w_i = \sum_{i=1}^{m} w_{ij} e'_j$$

Hence now the Tv can be transformed as follows

$$\begin{aligned} Tv = & \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} e_j \\ Tv = & \sum_{j=1}^{m} \left(\sum_{i=1}^{n} w_{ij} \alpha_i\right) e'_j \\ & \text{Call the quantity } \left(\sum_{i=1}^{n} w_{ij} \alpha_i\right) = \beta_j \\ \therefore Tv = & \sum_{j=1}^{m} \beta_j e'_j \end{aligned}$$

This can now be represented in the matrix form as follows

$$Tv = \begin{pmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ w_{m,1} & w_{m,2} & \cdots & w_{m,n} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Here we can clearly see that $Te_i = \begin{pmatrix} w_{1,i} \\ \vdots \\ w_{m,i} \end{pmatrix}$ That is the i^{th} column in the matrix.

Hence to get the first column and eventually Te_1 we must place the $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ equal to $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Similarly to get the second column we must place the $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ equal to $\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ and so on.

Example 1: $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$. Given

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} -3 \\ 1 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Find the linear transformation $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$.

Solution: The first question that arises is where the standard basis goes. The vector on the LHS be in the standard basis $(1, 0)$ and $(0, 1)$.

$$\begin{aligned} \begin{pmatrix} 1 \\ 2 \end{pmatrix} & \xrightarrow{T} \begin{pmatrix} -3 \\ 1 \end{pmatrix} \\ -2 \cdot \left(\begin{pmatrix} -3 \\ 1 \end{pmatrix} \right. & \xrightarrow{T} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ = \begin{pmatrix} 7 \\ 0 \end{pmatrix} & \xrightarrow{T} \begin{pmatrix} -5 \\ -3 \end{pmatrix} \\ \Longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \xrightarrow{T} \frac{1}{7} \begin{pmatrix} -5 \\ -3 \end{pmatrix} \end{aligned}$$

Similarly we find for the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and we get it to be $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{T} \frac{1}{7} \begin{pmatrix} -8 \\ 5 \end{pmatrix}$.

Now we know the linear transformation for both the standard basis vectors. Now we can push these values in the matrix and get the value for the corresponding β depending on the value of α.

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \frac{-5}{7} & \frac{-8}{7} \\ \frac{-3}{7} & \frac{5}{7} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$