FULL TITLE

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1. Rank Nullity Theorem

It is also called then fundamental theorem of Linear Maps because of its importance in linear transformation.

1.1. **Stating and Proof.**

Theorem 1. ***Rank-Nullity Theorem:*** *Suppose V is finite-dimensional and $T \in \mathcal{L} : (V, W)$. Then range T is finite-dimensional and*

$$Rank(T)\ +\ Nullity(T)\ = \dim(V),\ or$$

$$\dim(Range(T))\ +\ \dim(Ker(T))\ = \dim(V)$$

Proof. Let $\phi_1, \phi_2, \ldots, \phi_l$ be the minimum vectors will that span $Ker(T)$, where l is the $\dim(Ker(T))$.
Andv$_1, v_2, \ldots, v_{n-l}$ be the vectors that will span the remaining vector space V where, n is the $\dim(V).Theirlineartransformation$
(w$_1, w_2, \ldots, w_{n-l}$) must be independent and should span the entire range of T in W, let $\dim(Range(T))$ be k.

We will prove this claim later.
Then it becomes quite easy to see why the rank nullity theorem holds. Because,

$$n - l = k\, or\, n = l + k$$

Hence, Proved. □

Claim 1. *$w_1, w_2, \ldots, w_{n-l}$ are linearly independent and span the entire $Range(T)$.*

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Proving Claim. $w_1, w_2, \ldots, w_{n-l}$ are linear transformation of basis $v_1, v_2, \ldots, v_{n-l}$. So,

$$\begin{aligned}
\alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_{n-l} w_{n-l} &= 0 \\
or, \sum_{i=1}^{n-l} \alpha_i w_i &= 0 \\
or, \sum_{i=1}^{n-l} \alpha_i T v_i &= 0 \\
or, T(\sum_{i=1}^{n-l} \alpha_i v_i) &= 0
\end{aligned}$$

So $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{n-l} v_{n-l} \in \dim(nullT)$.
Since,$\phi_1, \phi_2, \ldots, \phi_l$ spans *null T*, we can write:

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{n-l} v_{n-l} = \beta_1 \phi_1 + \beta_2 \phi 2 + \ldots + \beta_l \phi_l$$

But we also know that $\phi_1 \, to \, \phi_l$ and $v_1 \, to \, v_{n-l}$ are linearly independent, which means the onlu way to satisfy above equation is to have $\alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = \beta_1 = \ldots = \beta_l = 0$.
$Thus$Tv$_1, Tv_2, \ldots, Tv_{n-l}$ are linearly independent and span the entire *Range*(T). Hence $w_1, w_2, \ldots, w_{n-l}$ will span the entire *Range*(T). □

1.2. **Applications.** We can now prove that some of the mapping can not be surjective or injective.

Theorem 2 (A map to a smaller dimensional space is not injective)**.** *Suppose V and W are finite-dimensional vector spaces such that* $\dim(V)$ *and* $\dim(W)$*, then there is no injective linear mapping from V to W.*

Proof. Let $T \in \mathcal{L}$:(V, W). Then

$$\begin{aligned}
\dim(null\, T) &= \dim(V) - \dim(Range(T)) \\
&\geq \dim(V) - \dim(W) \\
&> 0
\end{aligned}$$

The equality above comes from the fundamental theorem of Linear Maps and the strict inequality at the end shows the dimension of T is greater than 0, means vector other than 0 are present in T. Hence the mapping is not injective. □

Theorem 3 (A map to a larger dimensional space is not surjective)**.** *Suppose V and W are finite-dimensional vector spaces such that* $\dim(V) and \dim(W)$*, then there is no surjective linear mapping from V to W.*

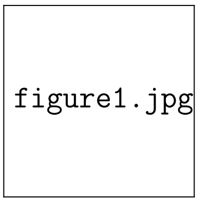
Proof. Let $T \in \mathcal{L}$:(V, W). Then

$$\begin{aligned} \dim(range\, T) &= \dim(V) - \dim(null\, T) \\ &\leq \dim(V) \\ &< \dim(W) \end{aligned}$$

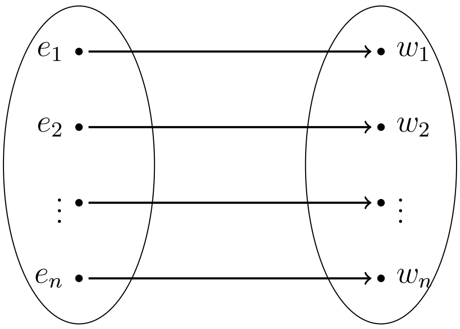
Here again the equality comes from the Rank-Nullity Theorem, and the rest is just normal inference.
At the end we have range of T ¡ the dimension of W. Hence the mapping is not surjective. ☐

2. Matrix Representation of Linear Transformation

A vector is represented in different ways on the coordinate system depending on what its basis is. Suppose the vector (2,1) in R^2 is represented with basis as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



But when the basis is changed the vector is represented differently and on a different basis.
The vector V can be represented as $a_1b_1 + a_2b_2$ and in matrix form can be represented as $V \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.
Suppose there are is a linear transformation T that maps a vector V from R^n to W in R^m. Let the basis be $B = \{e_1, \ldots, e_n\}$ and $B' = \{e'_1, \ldots, e'_m\}$ The map can be represented as



Suppose the basis for B is $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ and the upon applying the transformation the basis for B' is $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$

Now the question is how should we connect α_i to the β_j such that we reach to our answer.

We know that $Tv = w$

$v = \sum_{i=1}^{n} \alpha_i e_i$

Applying linear transformation T on both sides we get

$Tv = \sum_{i=1}^{n} \alpha_i T(e_i)$

Now what is Te_i ?

In this case $e_i \to w_i$

$\therefore Tv = \sum_{i=1}^{n} \alpha_i w_i$

But we want w_i to be represented in terms of Standard Basis.

$$w_i = \begin{pmatrix} w_{1i} \\ w_{2i} \\ \vdots \\ w_{mi} \end{pmatrix} \text{(This is the representation of } w_i \text{ in standard basis)}$$

$$\therefore w_i = \sum_{i=1}^{m} w_{ij} e'_j$$

Hence now the Tv can be transformed as follows

$$Tv = \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} e_j$$

$$Tv = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} w_{ij} \alpha_i\right) e'_j$$

Call the quantity $\left(\sum_{i=1}^{n} w_{ij} \alpha_i\right) = \beta_j$

$$\therefore Tv = \sum_{j=1}^{m} \beta_j e'_j$$

This can now be represented in the matrix form as follows

$$Tv = \begin{pmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ w_{m,1} & w_{m,2} & \cdots & w_{m,n} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Here we can clearly see that $Te_i = \begin{pmatrix} w_{1,i} \\ \vdots \\ w_{m,i} \end{pmatrix}$ That is the i^{th} column in the matrix.

Hence to get the first column and eventually Te_1 we must place the $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ equal to $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Similarly to get the second column we must place the $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ equal to $\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ and so on.

Example 1: $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$. Given

$$\begin{pmatrix}1\\2\end{pmatrix} \xrightarrow{T} \begin{pmatrix}-3\\1\end{pmatrix}$$
$$\begin{pmatrix}-3\\1\end{pmatrix} \xrightarrow{T} \begin{pmatrix}1\\2\end{pmatrix}$$

Find the linear transformation $\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} \xrightarrow{T} \begin{pmatrix}\beta_1\\\beta_2\end{pmatrix}$.

Solution: The first question that arises is where the standard basis goes. The vector on the LHS be in the standard basis $(1,0)$ and $(0,1)$.

$$\begin{pmatrix}1\\2\end{pmatrix} \xrightarrow{T} \begin{pmatrix}-3\\1\end{pmatrix}$$
$$-2\cdot\left(\begin{pmatrix}-3\\1\end{pmatrix} \xrightarrow{T} \begin{pmatrix}1\\2\end{pmatrix}\right)$$
$$= \begin{pmatrix}7\\0\end{pmatrix} \xrightarrow{T} \begin{pmatrix}-5\\-3\end{pmatrix}$$
$$\implies \begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{T} \frac{1}{7}\begin{pmatrix}-5\\-3\end{pmatrix}$$

Similarly we find for the vector $\begin{pmatrix}0\\1\end{pmatrix}$ and we get it to be $\begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{T} \frac{1}{7}\begin{pmatrix}-8\\5\end{pmatrix}$.

Now we know the linear transformation for both the standard basis vectors. Now we can push these values in the matrix and get the value for the corresponding β depending on the value of α.

$$\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} = \begin{pmatrix}\frac{-5}{7} & \frac{-8}{7}\\ \frac{-3}{7} & \frac{5}{7}\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}$$

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