BITS Pilani
Pilani Campus

Mechatronics AEZG511

Lecture

Transfer functions

Transfer function

Gain = Output / Input = x(t)/y(t)

Gain of inverting amplifier is $V_{out} / V_{in} = -R_2 / R_1$

Whereas differential equation cannot be expressed as a gain equation as simple ratio

Therefore Laplace transforms are used

$$\text{Transfer Function} = \frac{\text{Laplace transformation of output}}{\text{Laplace transformation of input}}$$

Laplace transform of derivatives/Integrals

$$\mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right\} = sF(s) - f(0)$$

$$\mathcal{L}\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\right\} = s^2F(s) - sf(0) - \frac{\mathrm{d}}{\mathrm{d}t}f(0)$$

The Laplace transform of the integral of a function $f(t)$ which has a Laplace transform $F(s)$ is given by

$$\mathcal{L}\left\{\int_0^t f(t)\,\mathrm{d}t\right\} = \frac{1}{s}F(s)$$

For example, the Laplace transform of the integral of the function e^{-t} between the limits 0 and t is given by

$$\mathcal{L}\left\{\int_0^t \mathrm{e}^{-t}\,\mathrm{d}t\right\} = \frac{1}{s}\mathcal{L}\{\mathrm{e}^{-t}\} = \frac{1}{s(s+1)}$$

Example 1 – Transfer function

Consider a system where the relationship between the input and the output is in the form of a first-order differential equation. The differential equation of a first-order system is of the form

$$a_1 \frac{\mathrm{d}x}{\mathrm{d}t} + a_0 x = b_0 y$$

where a_1, a_0, b_0 are constants, y is the input and x the output, both being functions of time. The Laplace transform of this, with all initial conditions zero, is

$$a_1 s X(s) + a_0 X(s) = b_0 Y(s)$$

and so we can write the transfer function $G(s)$ as

$$G(s) = \frac{X(s)}{Y(s)} = \frac{b_0}{a_1 s + a_0}$$

This can be rearranged to give

$$\boxed{G(s) = \frac{b_0/a_0}{(a_1/a_0)s + 1} = \frac{G}{\tau s + 1}}$$

Example 2 – Transfer Function

■ Problem

Obtain the transfer functions $X(s)/F(s)$ and $X(s)/G(s)$ for the following equation.

$$5\ddot{x} + 30\dot{x} + 40x = 6f(t) - 20g(t)$$

■ Solution

Using the derivative property with zero initial conditions, we can immediately write the equation as

$$5s^2X(s) + 30sX(s) + 40X(s) = 6F(s) - 20G(s)$$

Solve for $X(s)$.

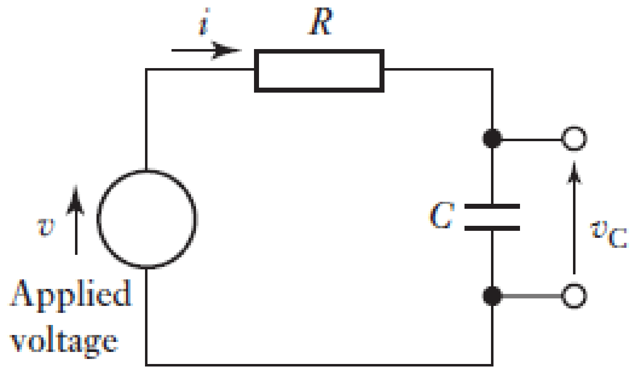
$$X(s) = \frac{6}{5s^2 + 30s + 40}F(s) - \frac{20}{5s^2 + 30s + 40}G(s)$$

When there is more than one input, the transfer function for a specific input can be obtained by temporarily setting the other inputs equal to zero (this is another aspect of the superposition property of linear equations). Thus, we obtain

$$\frac{X(s)}{F(s)} = \frac{6}{5s^2 + 30s + 40} \qquad \frac{X(s)}{G(s)} = -\frac{20}{5s^2 + 30s + 40}$$

First order system example

$$v = RC\frac{dv_C}{dt} + v_C$$



The Laplace transform is:

$$V(s) = RCsV_C(s) + V_C(s)$$

Thus $V(s)$ is the Laplace transform of the input voltage v and $V_C(s)$ is the Laplace transform of the output voltage v_C. Rearranging gives:

$$\boxed{\frac{V_C(s)}{V(s)} = \frac{1}{RCs+1}}$$

The above equation thus describes the relationship between the input and output of the system when described as s functions.

First order system problem

Example

A thermocouple which has a transfer function linking its voltage output V and temperature input of:

$$G(s) = \frac{30 \times 10^{-6}}{10s + 1} \text{ V/°C}$$

Determine the response of the system when it is suddenly immersed in a water bath at 100°C.

The output as an s function is:

$$V(s) = G(s) \times \text{input } (s)$$

The sudden immersion of the thermometer gives a step input of size 100°C and so the input as an s function is $100/s$. Thus:

$$V(s) = \frac{30 \times 10^{-6}}{10s + 1} \times \frac{100}{s} = \frac{30 \times 10^{-4}}{10s(s + 0.1)} = 30 \times 10^{-4} \frac{0.1}{s(s + 0.1)}$$

The fraction element is of the form $a/s(s + a)$ and so the output as a function of time is:

$$V = 30 \times 10^{-4} (1 - e^{-0.1t}) \text{ V}$$

Second Order system

For a second-order system, the relationship between the input y and the output x is described by a differential equation of the form

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_0 y$$

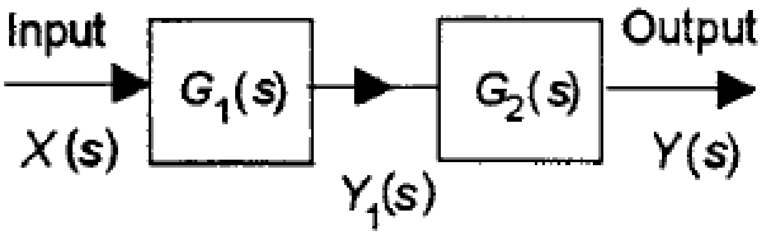
where a_2, a_1, a_0 and b_0 are constants. The Laplace transform of this equation, with all initial conditions zero, is

$$a_2 s^2 X(s) + a_1 s X(s) + a_0 X(s) = b_0 Y(s)$$

Hence

$$G(s) = \frac{X(s)}{Y(s)} = \frac{b_0}{a_2 s^2 + a_1 s + a_0}$$

Systems in series

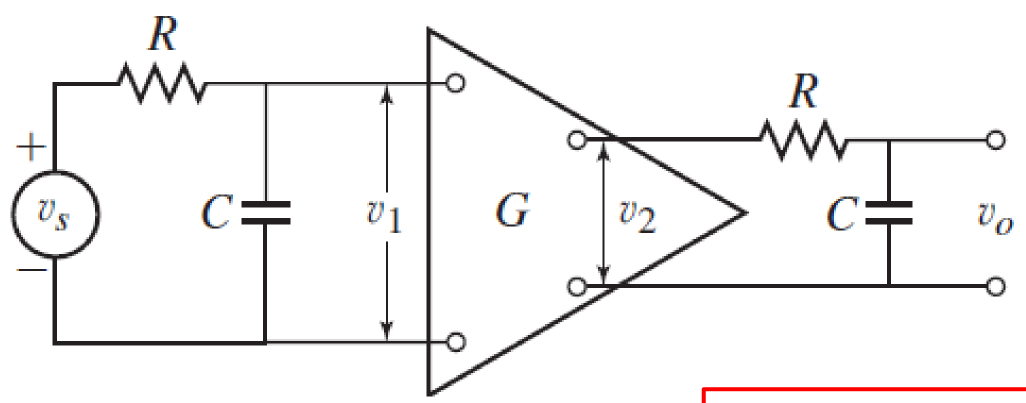


$$Y(s) = G_2(s)Y_1(s) = G_2(s)G_1(s)X(s)$$

The overall transfer function $G(s)$ of the system is $Y(s)/X(s)$ and so:

$$G_{overall}(s) = G_1(s)G_2(s)$$

First order system example



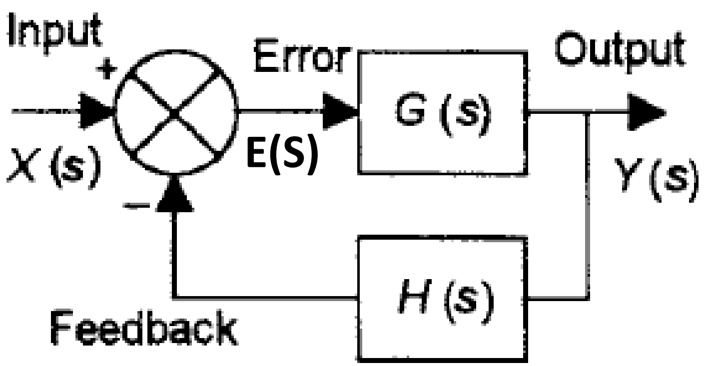
$$\boxed{\frac{V_1(s)}{V_s(s)} = \frac{1}{RCs+1}}$$

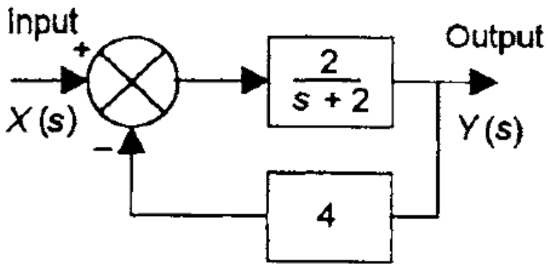
$$\boxed{V_2(s) = GV_1(s)}$$

$$\boxed{\frac{V_o(s)}{V_2(s)} = \frac{1}{RCs+1}}$$

$$\frac{V_o(s)}{V_s(s)} = \frac{V_o(s)}{V_2(s)}\frac{V_2(s)}{V_1(s)}\frac{V_1(s)}{V_s(s)} = \frac{1}{RCs+1}G\frac{1}{RCs+1} = \frac{G}{R^2C^2s^2+2RCs+1}$$

Systems with feedback loops





$$\text{Error}(s) = X(s) - H(s)Y(s)$$

Since G(S) = Y(S)/(E(S)

$$G(s) = \frac{Y(s)}{X(s) - H(s)Y(s)}$$

$$G_{\text{overall}}(s) = \frac{\frac{2}{s+2}}{1 + 4 \times \frac{2}{s+2}} = \frac{2}{s+10}$$

$$\text{overall transfer function} = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Second Order system

What will be the state of damping of a system having the following transfer function and subject to a unit step input?

$$G(s) = \frac{1}{s^2 + 8s + 16}$$

The output $Y(s)$ from such a system is given by:

$$Y(s) = G(s)X(s)$$

For a unit step input $X(s) = 1/s$ and so the output is given by:

$$Y(s) = \frac{1}{s(s^2 + 8s + 16)} = \frac{1}{s(s+4)(s+4)}$$

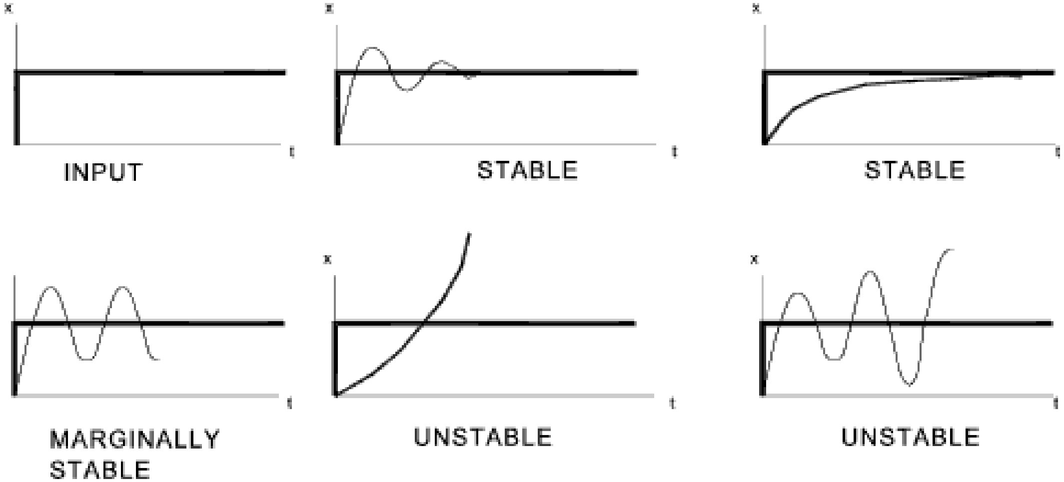
The roots of $s^2 + 8s + 16$ are $p_1 = p_2 = -4$. Both the roots are real and the same and so the system is critically damped.

Stability of system

A system is **stable** if, when it is given an input, it has transients which die away with time.

Final state is steady state condition

It is **unstable**, if transients do not die away with time but increase in size and steady state is not attained.



Stability of system

For a transfer function, value of **S** which make the transfer function infinite is termed as **Poles**

$$Y(s) = \frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$$

$$y = \tfrac{1}{2} - e^{-t} + \tfrac{1}{2}e^{-2t}$$

Only If all poles are negative, then system is stable

$$Y(s) = \frac{1}{s(s-1)(s-2)} = \frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}$$

$$y = \tfrac{1}{2} - e^{+t} + \tfrac{1}{2}e^{+2t}$$

Even If one of poles is positive, then system is unstable

Stability of system

$$G(s) = \frac{1}{s^2 + 2s + 4}$$

we have the roots of the quadratic given by:

$$s = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm j1.73$$

$$G(s) = \frac{1}{s^2 - 2s + 4}$$

we have the roots of the quadratic given by:

$$s = \frac{2 \pm \sqrt{4-16}}{2} = +1 \pm j1.73$$

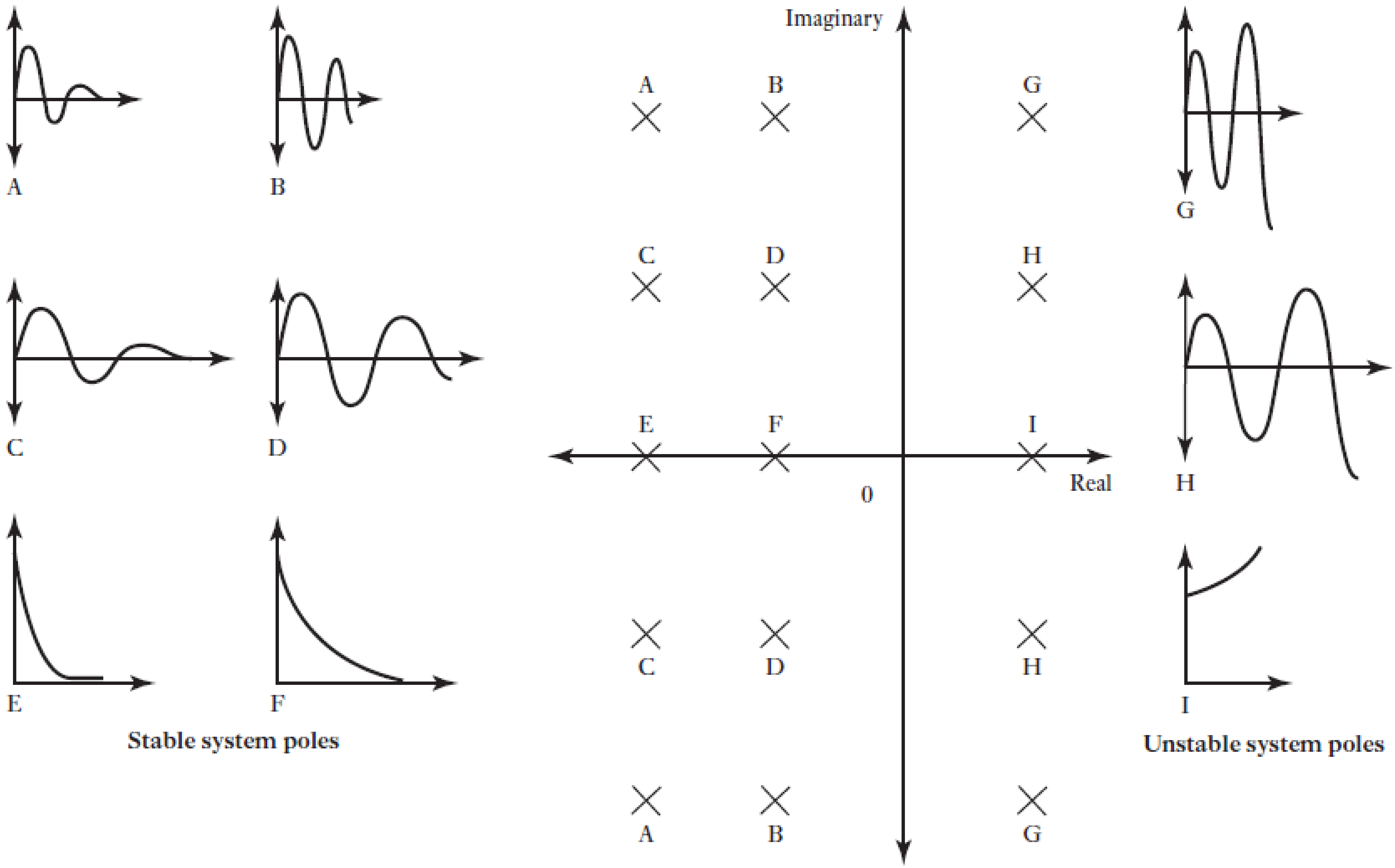
A system is stable , if the real part of all its poles is negative

A system is unstable, if the real part of any of its poles is positive

Matlab code to view the graph

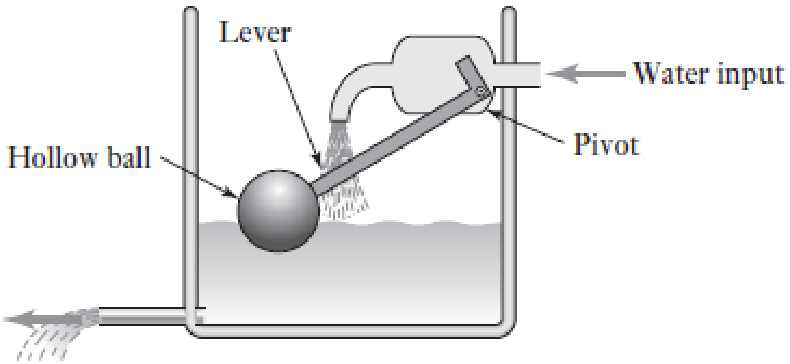
```
syms s, t
num = [ 1];
den  = [1  2 4];
step (num, den)
```

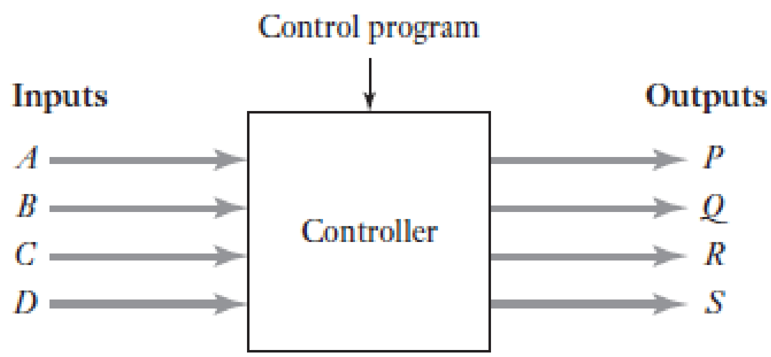
Stability of system



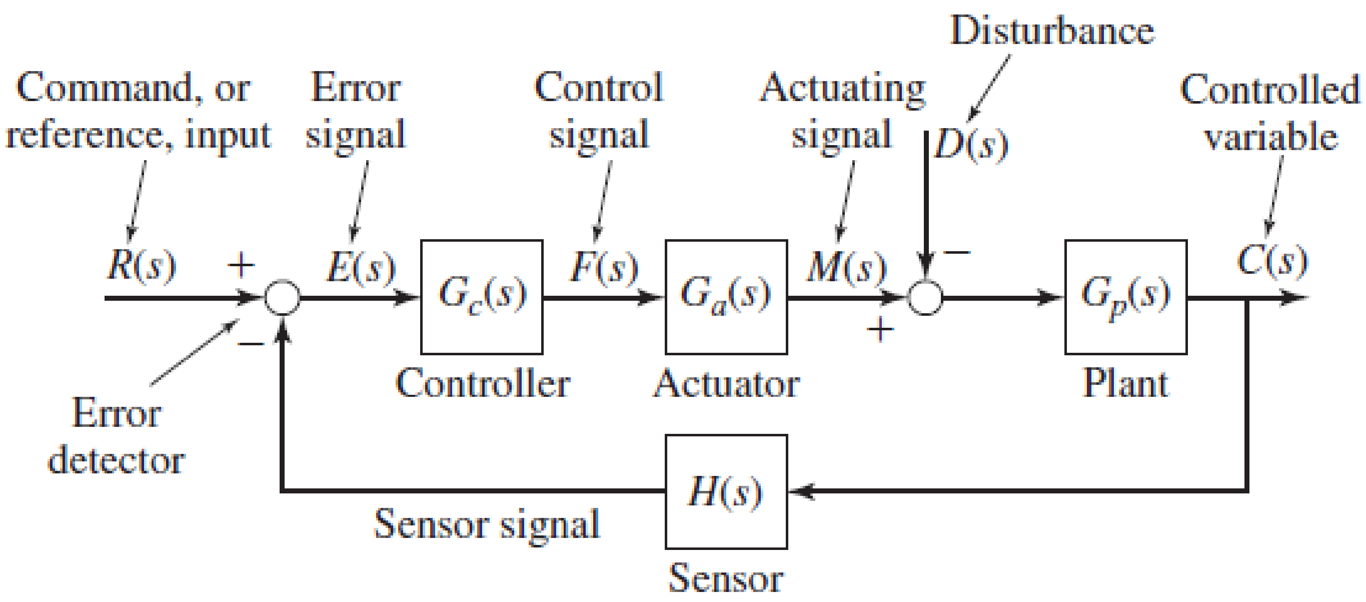
Closed Loop Controllers

Continuous and discrete control process

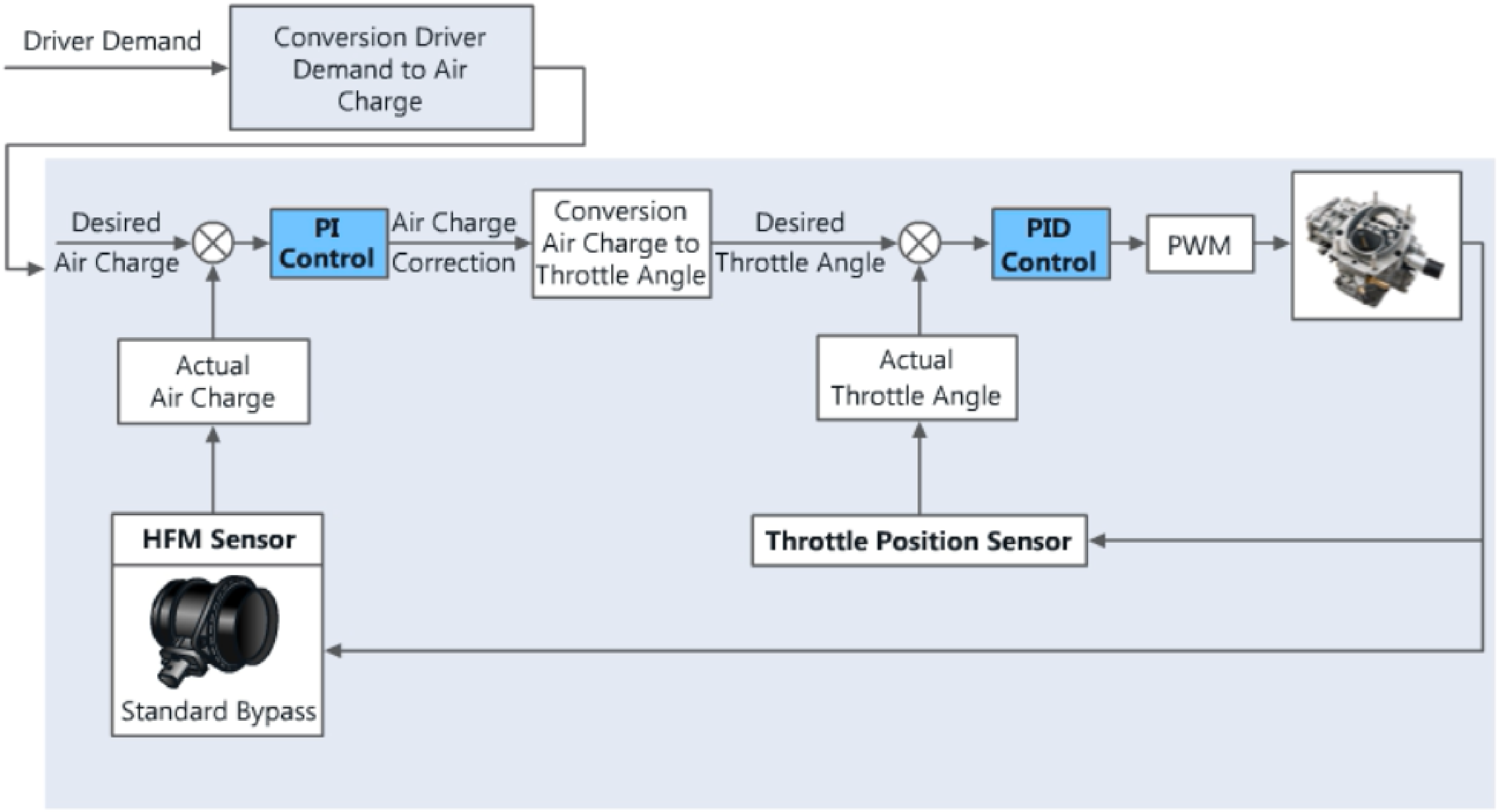




Control system terminology



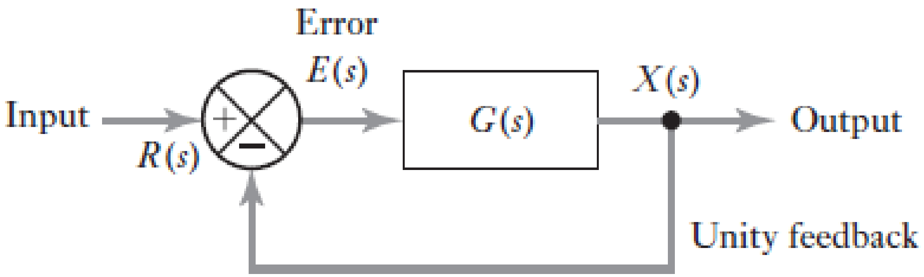
Control systems



Terminology

- **Lag**
  - Time required for the system to make necessary response

- **Steady State error**

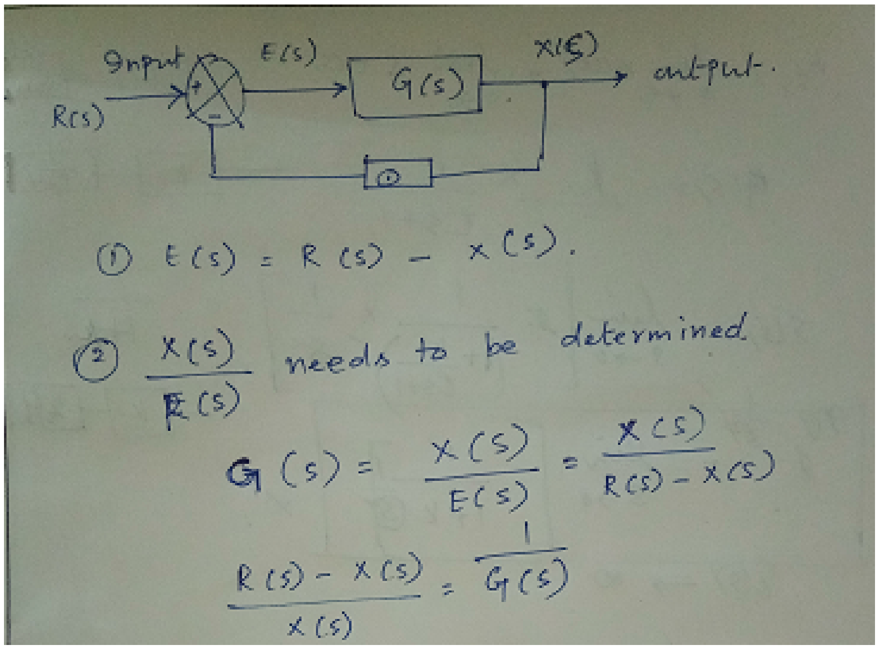


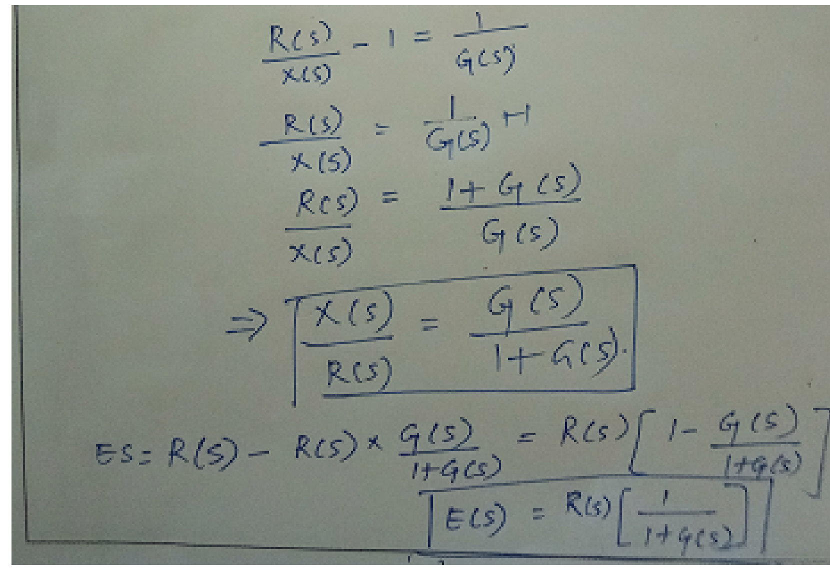
$$E(s) = R(s) - X(s)$$

$$\frac{X(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{G(s)}{1 + G(s)}$$

$$E(s) = R(s) - X(s) = R(s) - \frac{G(s)R(s)}{1 + G(s)} = \frac{1}{1 + G(s)}R(s)$$

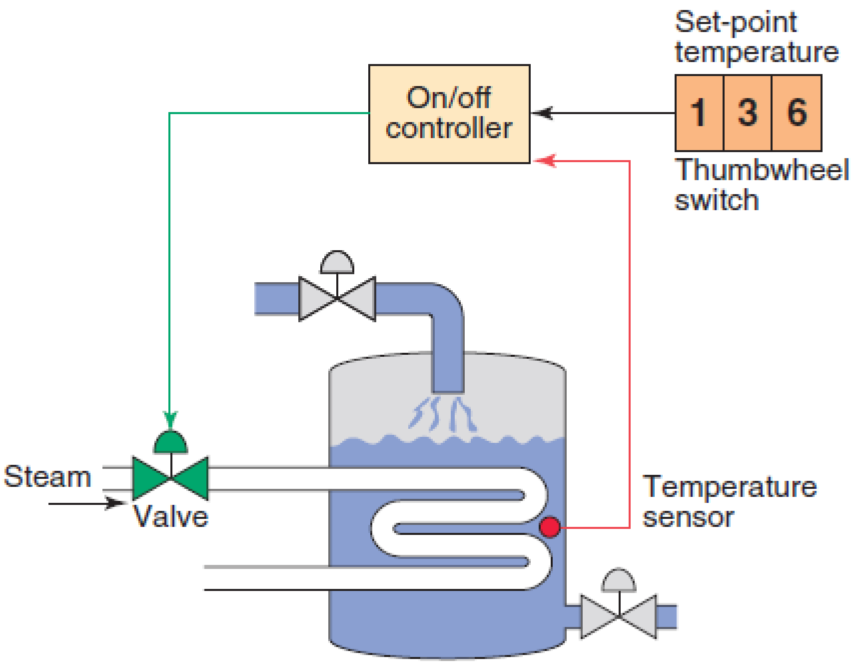
Steady state error



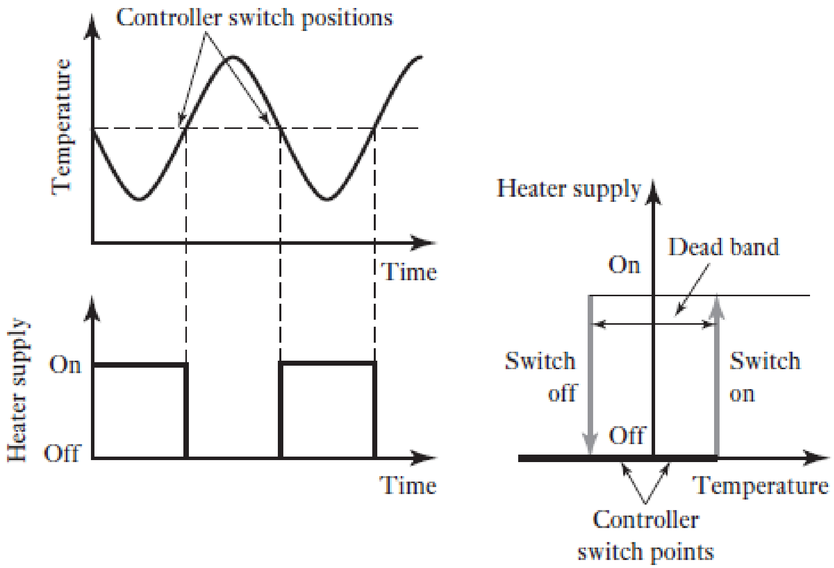


$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

Control modes – On/Off



On/off controlled liquid heating system.



Control modes – Proportional (P)

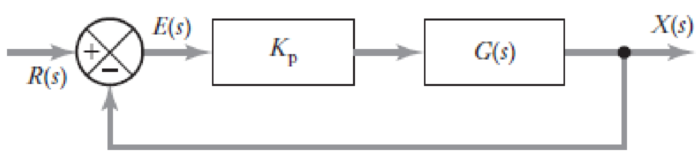
$$\text{controller output} = K_{\mathrm{P}}e$$

where e is the error and K_{P} a constant. Thus taking Laplace transforms,

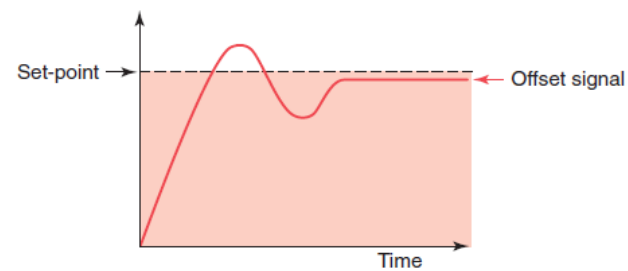
$$\text{controller output}\,(s) = K_{\mathrm{P}}E(s)$$

and so K_{P} is the transfer function of the controller.

$$E(s) = \frac{1}{1 + K_{\mathrm{P}}G(s)}R(s)$$



Unity feedback

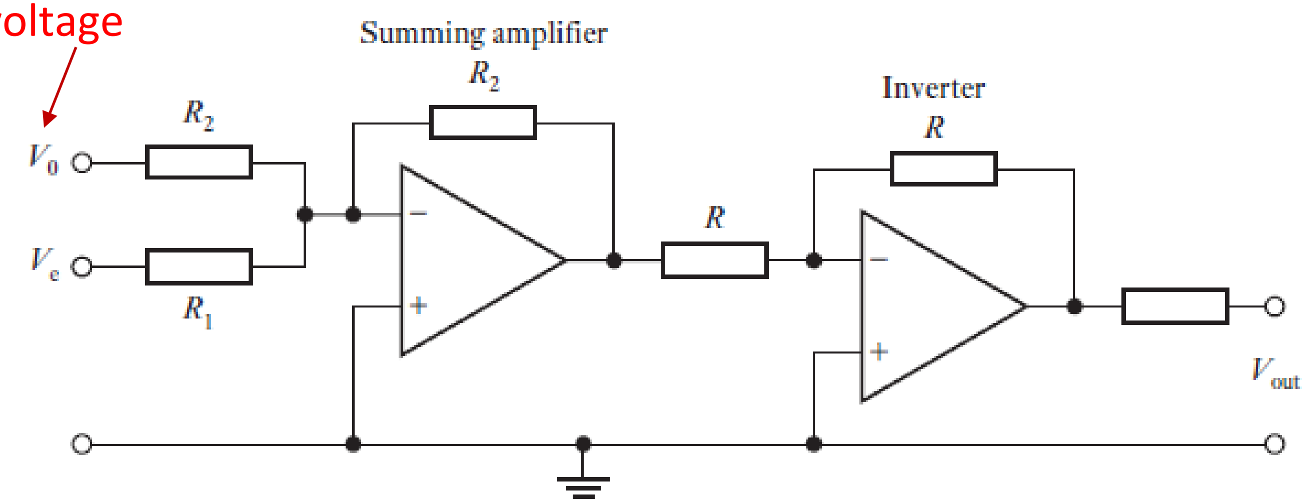


and so, for a step input, the steady-state error is

$$e_{\mathrm{SS}} = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\left[s\frac{1}{1 + K_{\mathrm{P}}G(s)}\frac{1}{s}\right]$$

Proportional (P) Hardware

Zero error voltage

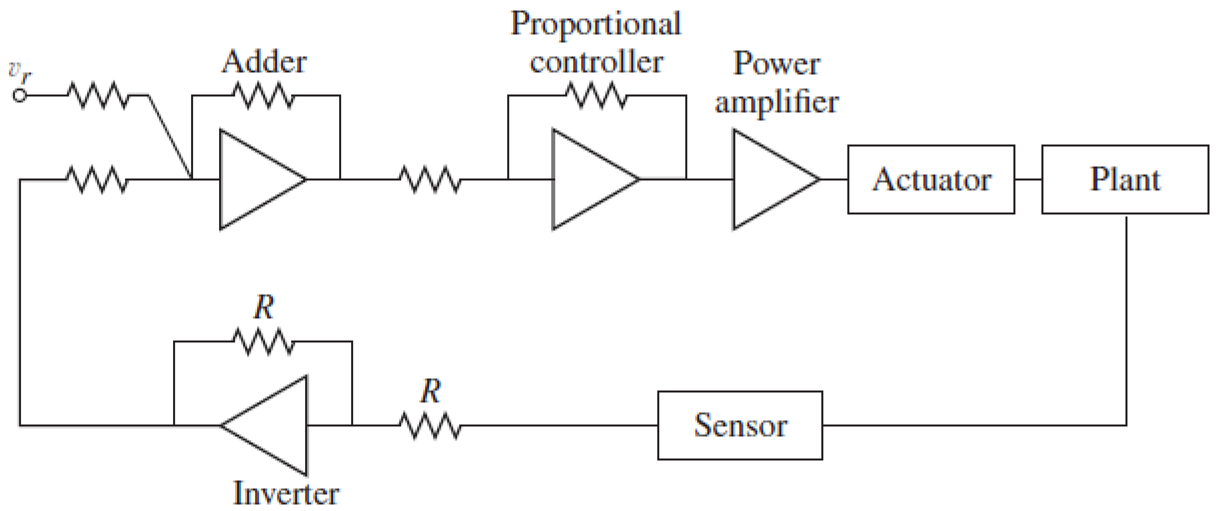


$$V_{out} = -\frac{R_2}{R_1} V_e - V_0$$

$$V_{out} = \frac{R_2}{R_1} V_e + V_0$$

$$V_{out} = K_P V_e + V_0$$

Proportional (P) Hardware

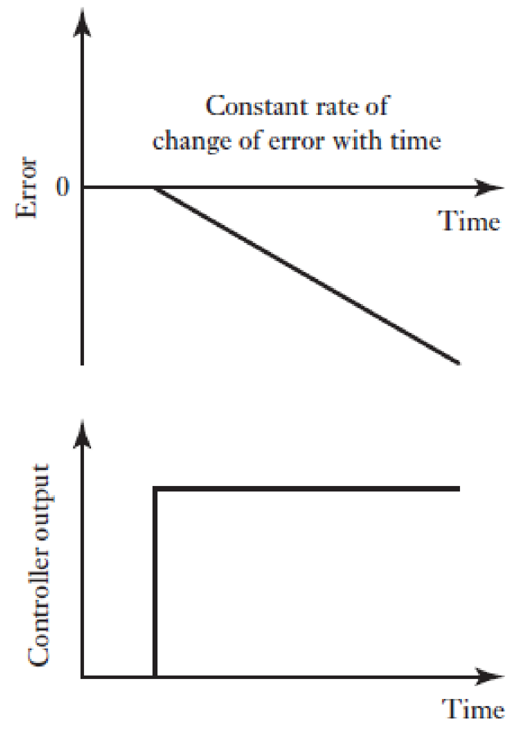


Derivative (D)

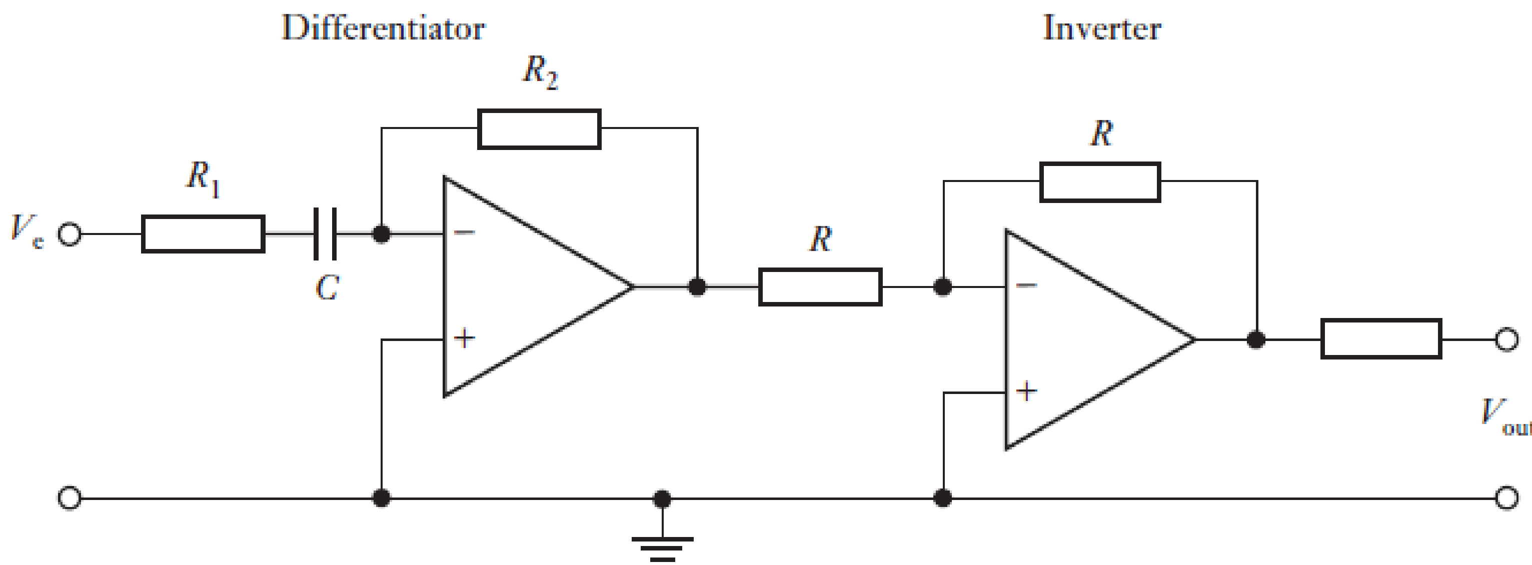
$$\text{controller output} = K_D \frac{de}{dt}$$

K_D is the constant of proportionality. The transfer function is obtained by taking Laplace transforms, thus

$$\text{controller output}(s) = K_D s E(s)$$



Derivative (D) – Hardware



Does not respond to steady state errors!

PD control

With proportional plus derivative control the controller output is given by

$$\text{controller output} = K_P e + K_D \frac{de}{dt}$$

K_P is the proportionality constant and K_D the derivative constant, de/dt is the rate of change of error. The system has a transfer function given by

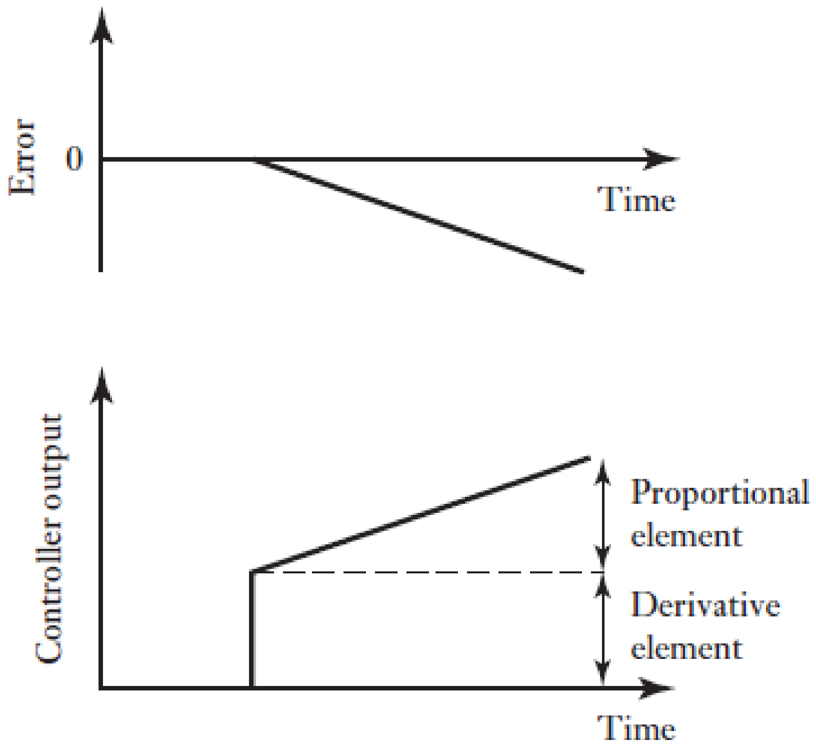
$$\text{controller output}(s) = K_P E(s) + K_D s E(s)$$

Hence the transfer function is $K_P + K_D s$. This is often written as

$$\text{transfer function} = K_D\left(s + \frac{1}{T_D}\right)$$

where $T_D = K_D/K_P$ and is called the **derivative time constant.**

PD control



Ideal for fast changing processes

Integral Control

The **integral mode** of control is one where the rate of change of the control output I is proportional to the input error signal e:

$$\frac{dI}{dt} = K_I e$$

K_I is the constant of proportionality and has units of $1/s$. Integrating the above equation gives

$$\int_{I_0}^{I_{out}} dI = \int_0^t K_I e \, dt$$

$$I_{out} - I_0 = \int_0^t K_I e \, dt$$

I_0 is the controller output at zero time, I_{out} is the output at time t.

The transfer function is obtained by taking the Laplace transform. Thus

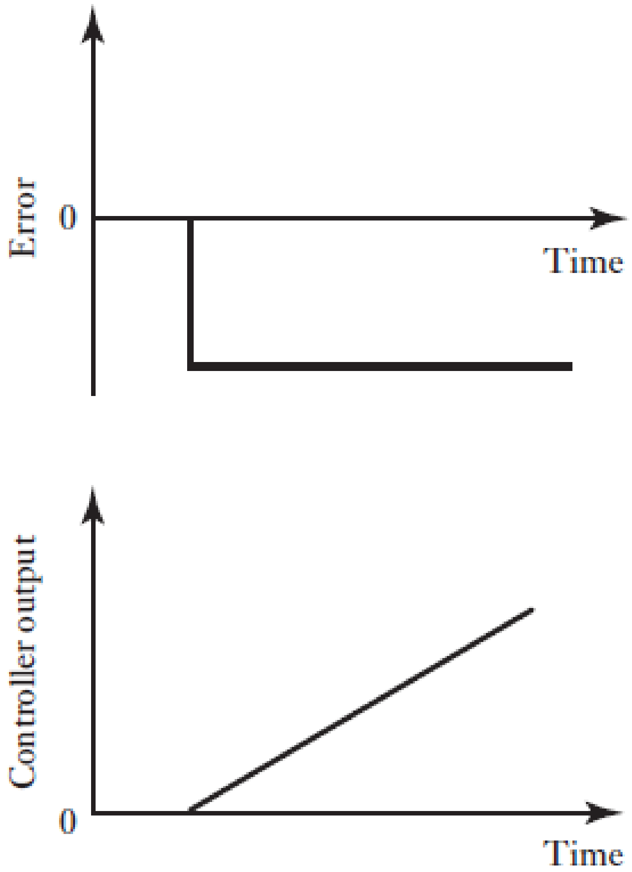
$$(I_{out} - I_0)(s) = \frac{1}{s} K_I E(s)$$

Laplace transform of an integral (Refer Appendix-Bolton)

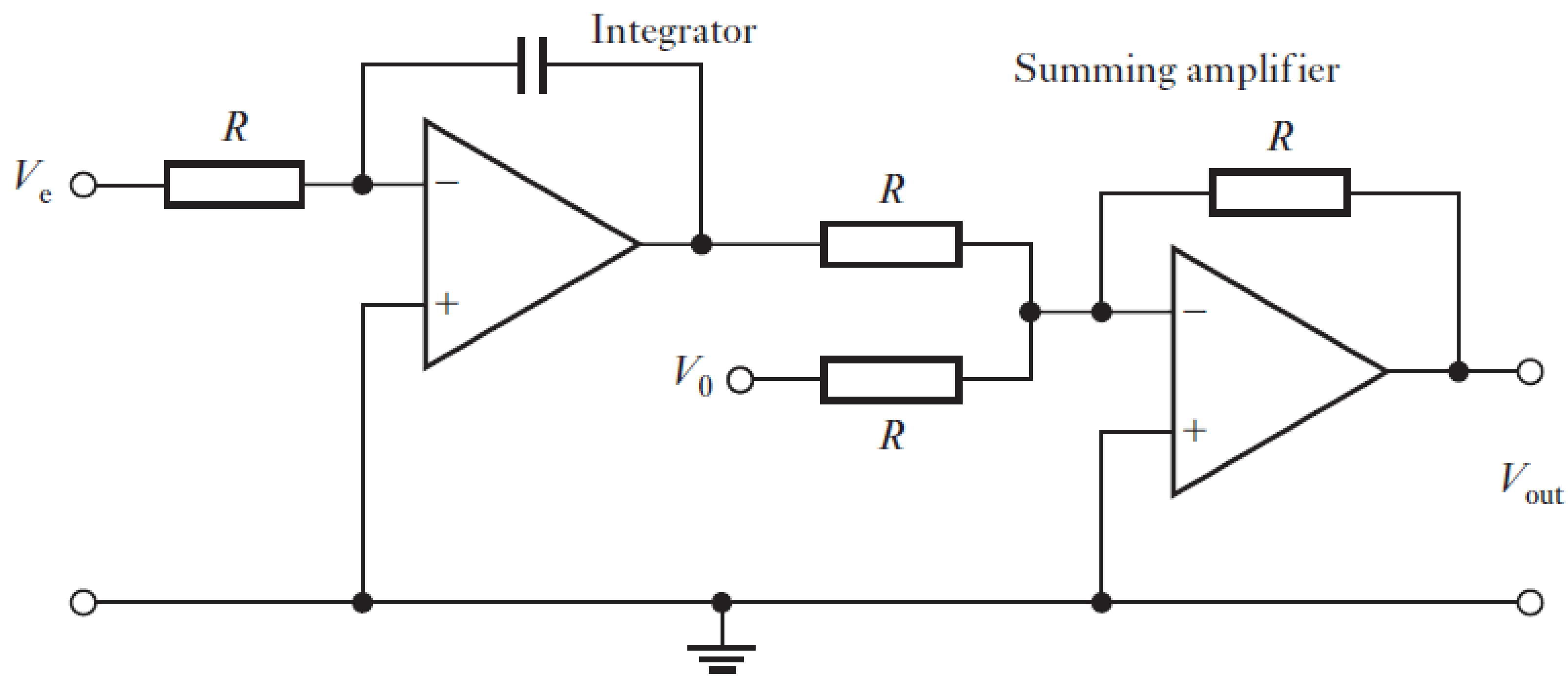
and so

$$\text{transfer function} = \frac{1}{s} K_I$$

Integral (I) control



Integral (I) control - Hardware



P I control

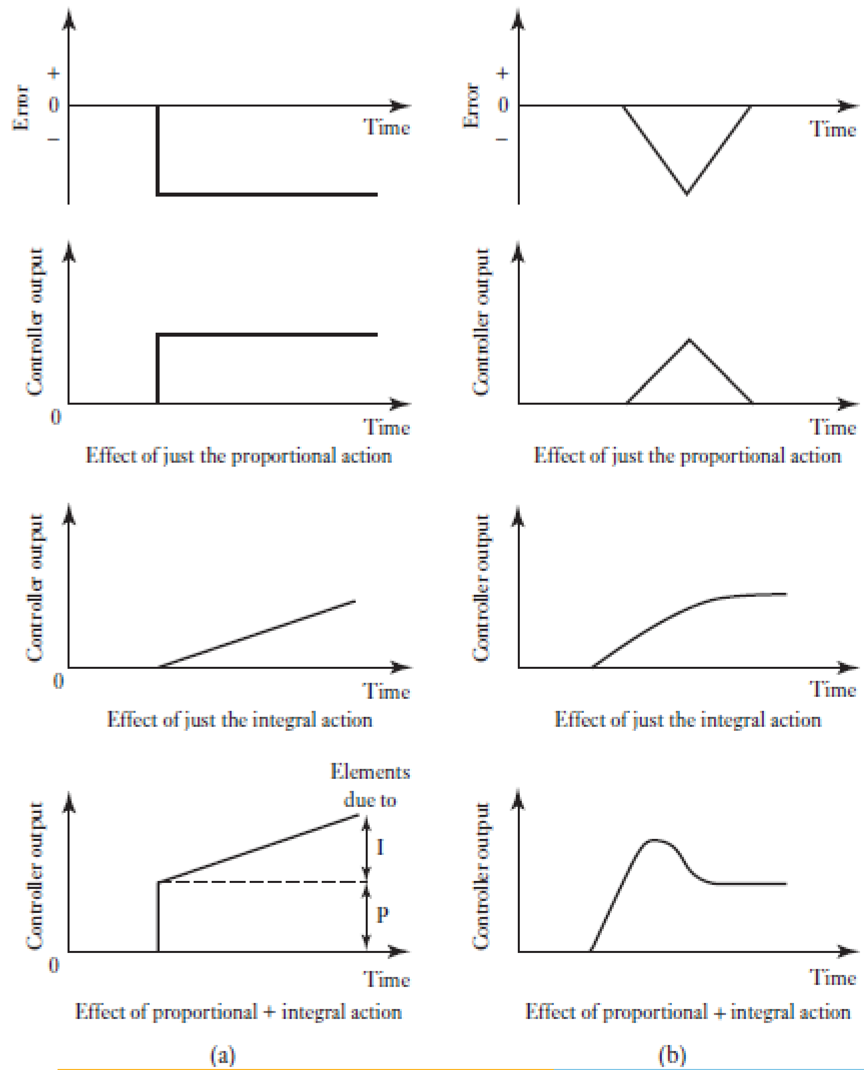
$$\text{controller output} = K_P e + K_I \int e \, dt$$

where K_P is the proportional control constant, K_I the integral control constant and e the error e. The transfer function is thus

$$\text{transfer function} = K_P + \frac{K_I}{s} = \frac{K_P}{s}\left(s + \frac{1}{T_I}\right)$$

where $T_I = K_P / K_I$ and is the **integral time constant.**

P I control



Error
+
0
−
Time
Controller output
0
Time
Effect of just the proportional action
Controller output
0
Time
Effect of just the integral action
Elements
due to
I
P
Controller output
0
Time
Effect of proportional + integral action
(a)
Error
+
0
−
Time
Controller output
Time
Effect of just the proportional action
Controller output
Time
Effect of just the integral action
Controller output
Time
Effect of proportional + integral action
(b)

P I D control

$$\text{controller output} = K_P e + K_I \int e \, dt + K_D \frac{de}{dt}$$

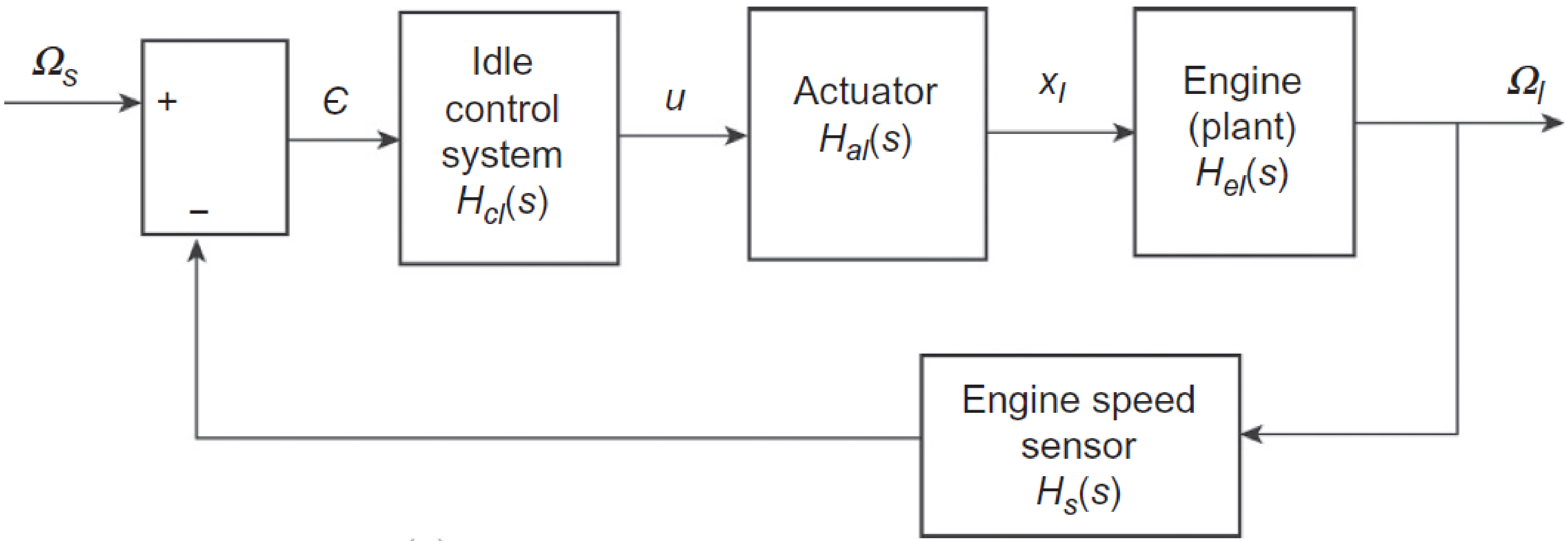
where K_P is the proportionality constant, K_I the integral constant and K_D the derivative constant. Taking the Laplace transform gives

$$\text{controller output}(s) = K_P E(s) + \frac{1}{s} K_I E(s) + s K_D(s)$$

and so

$$\text{transfer function} = K_P e + \frac{1}{s} K_I + s K_D = K_P \left(1 + \frac{1}{T_I s} + T_D s \right)$$

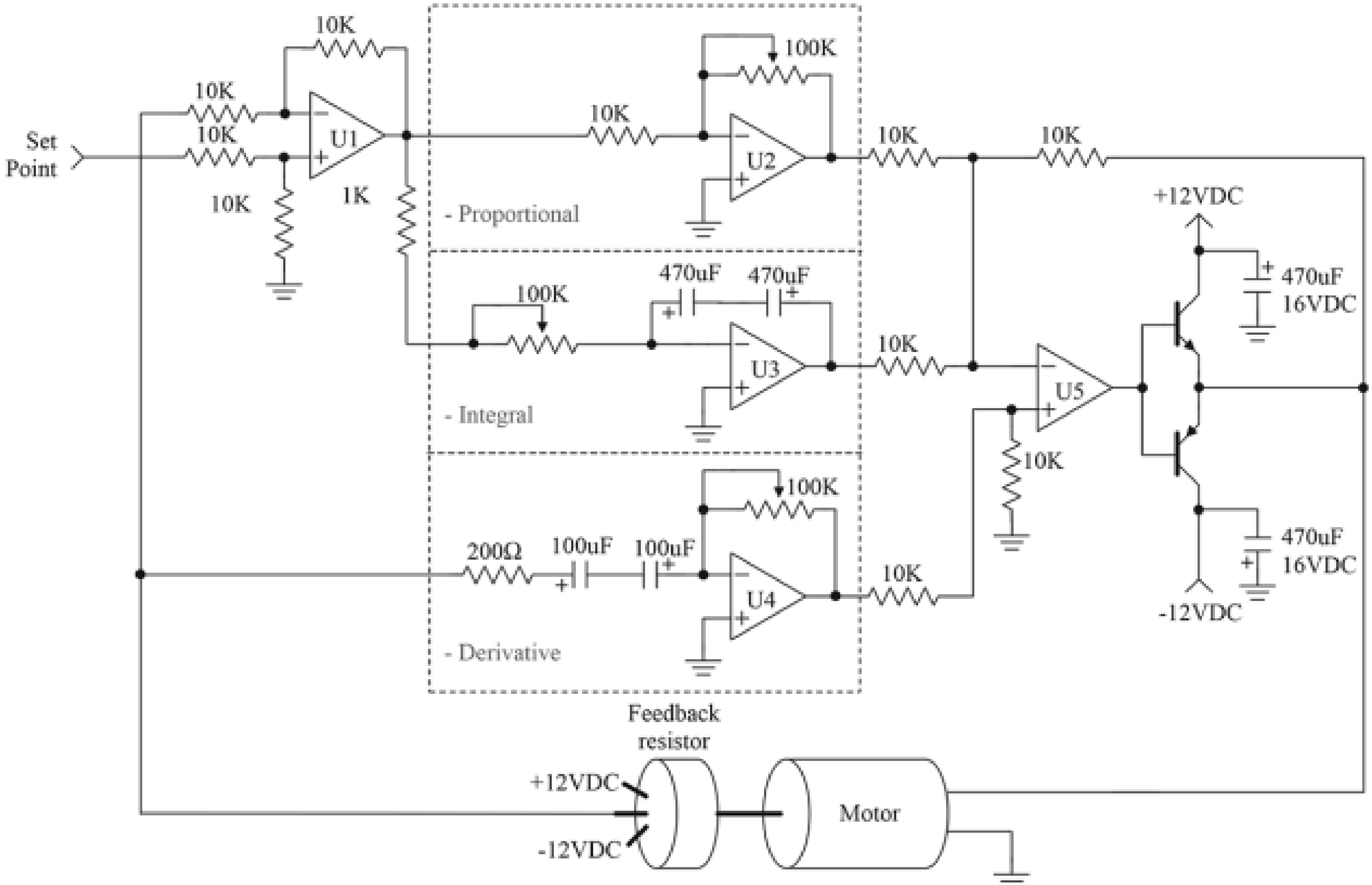
Idle speed control



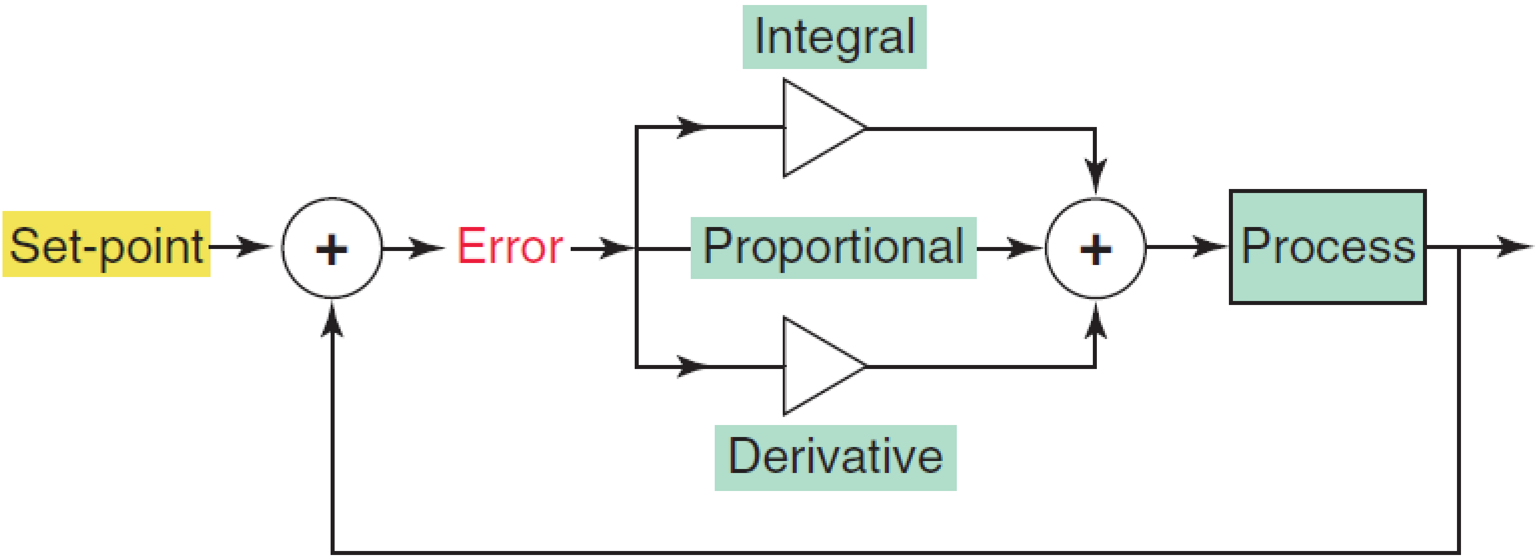
$$H_{CLI}(s) = \frac{\Omega_I(s)}{\Omega_S(s)}$$

$$= \frac{H_{CI}(s)H_{pI}}{I + H_s(s)H_{CI}(s)H_{pI}(s)}$$

P I D control - Hardware



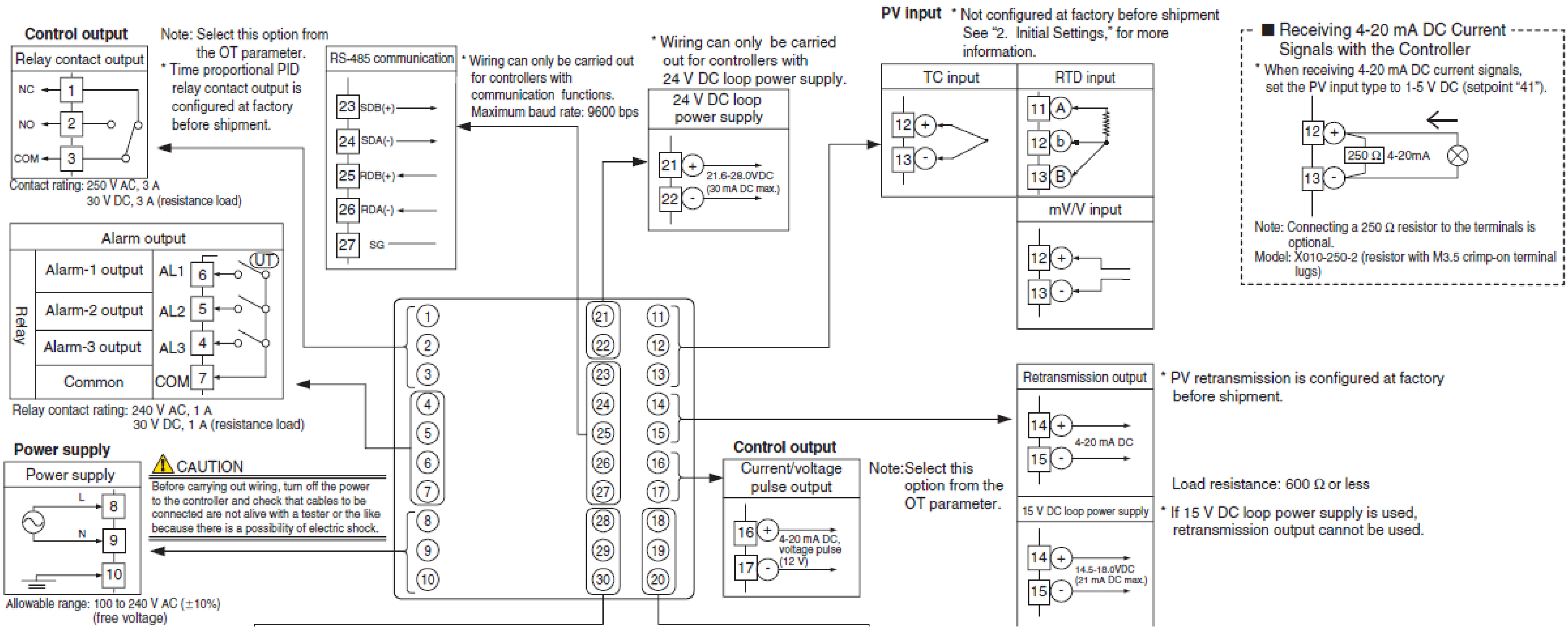
Typical PID combination



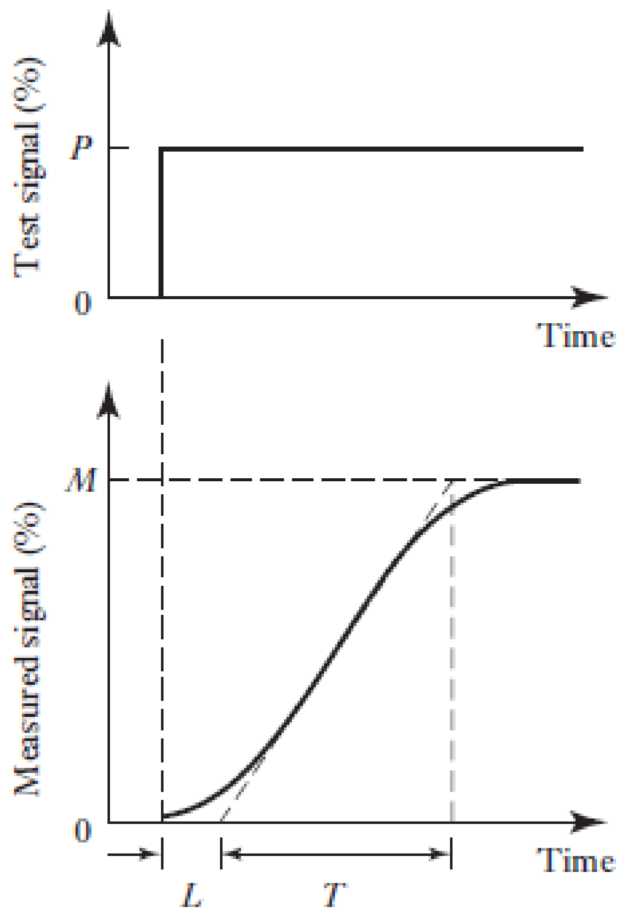
PID controller

Commercial PID

Commercial PID



Ziegler – Nichols – Process Reaction



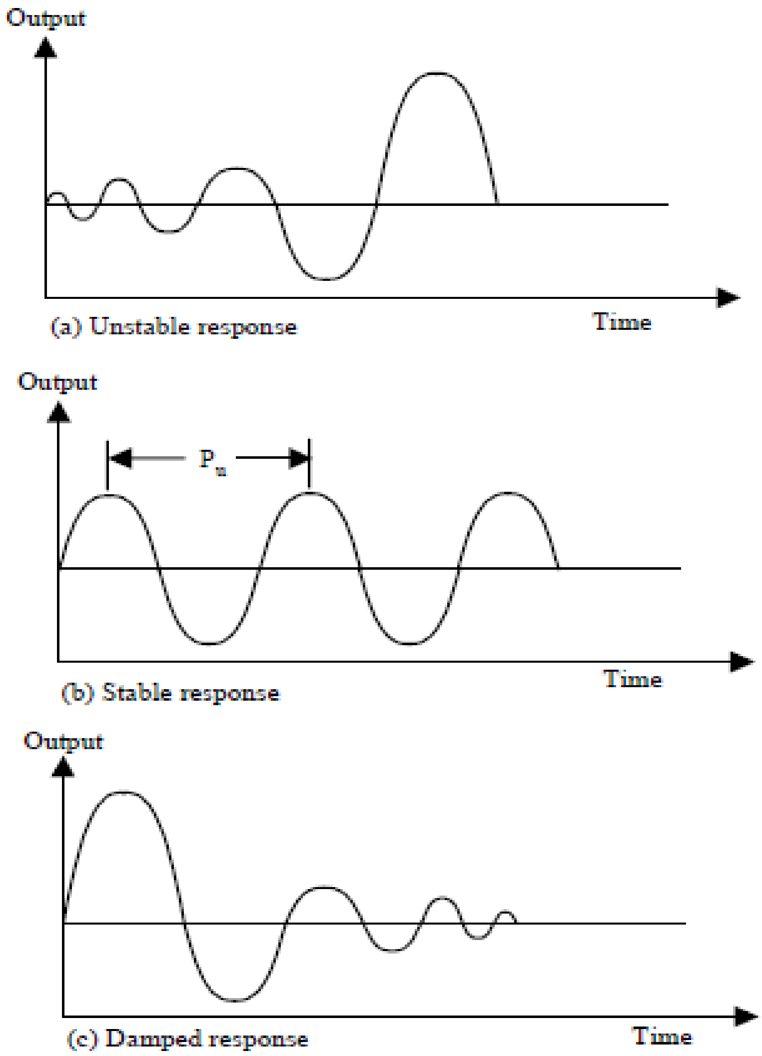
Open the control loop, no control action is allowed

Control mode	K_P	T_I	T_D
P	P/RL		
PI	$0.9P/RL$	$3.33L$	
PID	$1.2P/RL$	$2L$	$0.5L$

R is maximum gradient(Slope) = M/T

Study the example problem given in Bolton

Ziegler-Nichols – Ultimate cycle



The method describes the procedure to find out constants like gain, Integral time, derivative time

Procedure

Step 1:

Remove the integral and derivative action from the controller by setting

a) Derivative time to zero,
b) Integral time to zero
c) Proportional gain to one.

Step 2:

Run the system in automatic mode and control loop.

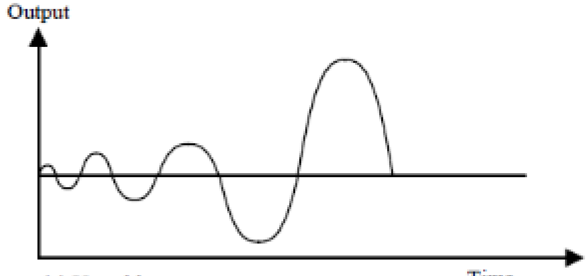
Procedure

Step 3:

Upset the process (Say change the set point)

Step 4:

If the response curve does not damp but is unstable (Like below)



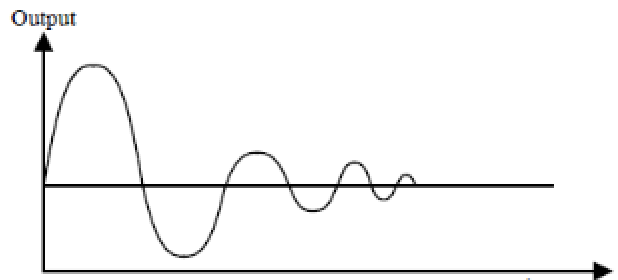
(a) Unstable response

then gain is too high. Reduce the gain

Procedure

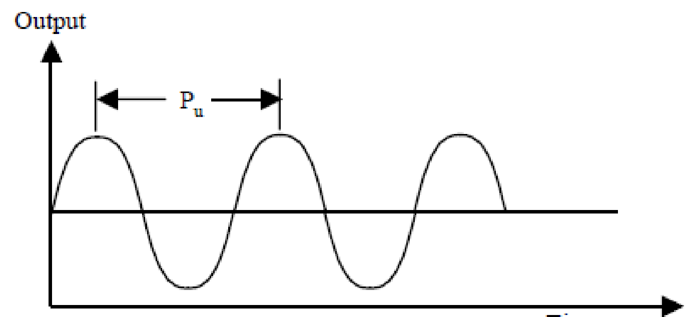
Step 5:

If the response curve damps out



(c) Damped response

then gain is too low. Increase the gain till you get the stable response.



(b) Stable response

Procedure

Record the value of Time (P_u) and ultimate gain (S_u) which generated stable response.

PI control:

$$K_c = 0.45 S_u$$

$$T_i = \frac{P_u}{1.2}$$

PD control:

$$K_c = 0.6 S_u$$

$$T_d = \frac{P_u}{8}$$

PID control:

$$K_c = 0.6 S_u$$

$$T_i = 0.5 P_u$$

$$T_d = \frac{P_u}{8}$$

Set the calculated parameters in the controller

Other tuning procedures

Manual:

- ✓ Operator estimates the tuning parameters required to give the desired controller response
- ✓ Proportional , integral, and derivative terms must be adjusted and tuned individually to a particular system using trial and error method.

Auto tune:

- The controller takes care of calculating and setting PID parameters
  - ✓ Measures sensor
  - ✓ Calculates error, sum of error, rate of change of error
  - ✓ Calculates desired parameter with PID equations
  - ✓ Updates control output

Thank you