Dynamics of double layered ecosystem

Numerical Methods: Term Paper

Rukmani R

May 2024

1 Introduction

In realistic cases, we usually see multi trophic ecosystem, these trophic levels include producer(plants), primary consumer, secondary consumer, tertiary consumer and detritivores. Here, I take a very simple approach to the problem, with simplistic assumptions by considering only two levels. Let's consider classic Lotka Volterra model for two species, say Rabbits(Prey) and foxes(predator). Consider a situation where you grow 'R' number of rabbits and 'F' number of foxes together in a farm where you supply food to rabbits daily, but the only food to foxes are rabbits. We also ignore all other complications, seasonal effects, and other sources of food. Therefore, rules of this system are the prey grows exponentially in the absence of predator as we supply food to rabbit daily and ignore other complications. The predator eats the prey and in absence of prey, the predator decays exponentially as the only source of food for predator is prey and it eventually dies without the availability of food. The dynamical equations of the system are,

$$\frac{dR}{dt} = aR - bRF \tag{1}$$

$$\frac{dF}{dt} = -cF + dRF \tag{2}$$

where a,b,c,d are positive constants. a is the reproduction rate of prey, b is the predator feeding rate per prey creature, c is the predator mortality rate when prey is not available and d is the reproduction rate of predators per prey species.'R' is the population of rabbit and 'F' is the population of the foxes.

These equations are non linear and two dimensional, and has no exact solution. To describe the qualitative features of the system, we find the fixed points and analyse the nature of the fixed points. In this case, it is two equation with two unknowns and hence we get two fixed points. The fixed points are calculated by equating the time derivative of the variable(population of rabbits and foxes) to zero as at the fixed point the system is time invariant.

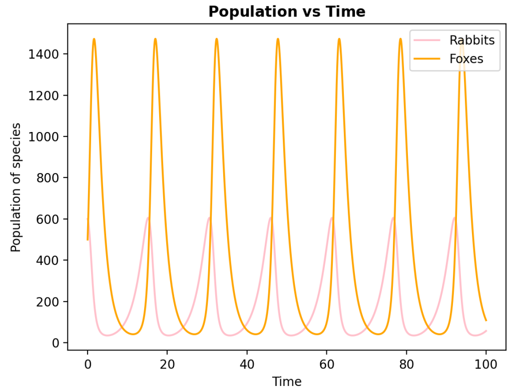
The fixed points are (0,0) and (c/d , a/b). To analyse the behaviour of this system, let's plot the phase portrait and the trajectories around the fixed points tells us about the nature of the fixed points. Analytically, we can do linear stability analysis and find if the fixed points are stable or unstable. Here, the point (0,0) is unstable fixed point, as a small perturbation from (0,0) grows exponentially. Applying small perturbation to the point (c/d , a/b), the perturbation neither grows nor decays, it exhibits oscillatory solution with frequency of oscillation $\sqrt{ca}$. This type of fixed point is called marginally stable fixed point.

2 Methodology

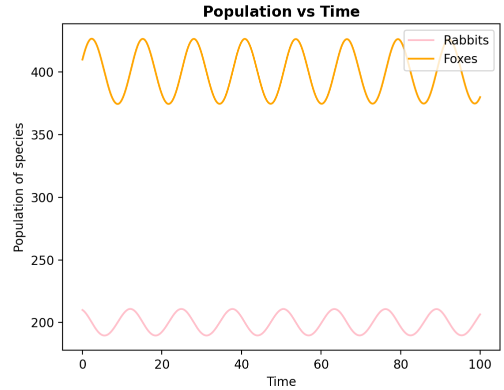
To solve this coupled differential equation, the default method RK45 is used. The population of rabbits and foxes with respect to time is computed, and the corresponding phase portrait is simulated. By using RK45, the phase portrait looks like a spiral, but clearly from the analytical calculation it should be a single lined loop. This might be because RK45 is not well suited for this problem. There are three kinds of equations suitable for Runge-Kutta methods which includes (1)the system is leading into equlibrium, (2)the system is unstable and (3)the system is time symmetric. Lotka Volterra model comes under first category and these kinds of problems are called as stiff problems. Stiffness refers to situations where the solution of an ODE changes rapidly on one scale but slowly on another, basically, the time scales are different for the two coupled differential equations. This makes other numerical methods like RK45 less efficient or accurate. "Radau" and "BDF" methods are well suited for stiff problems. Here, Radau algorithm is used to solve the equation. Radau method is based on Implicit Runge-kutta method. It computes the solution at the next time step by implicitly solving a system of equations. Radau method efficiently handles stiff problems and its preffered over other methods for its robustness and accuracy.

3 Results and implications

Using Radau algorithm, the coupled differential equation is solved for different initial conditions, and phase portrait is simulated for the values of parameters a = 0.4 , b = 0.001 , c = 0.6 , d = 0.003 . The fixed points under this condition are (0,0) and (200,400)

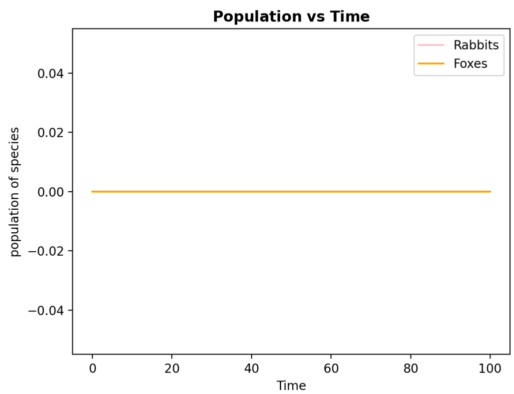


(a) for initial conditions: $R_0 = 600$, $F_0 = 500$

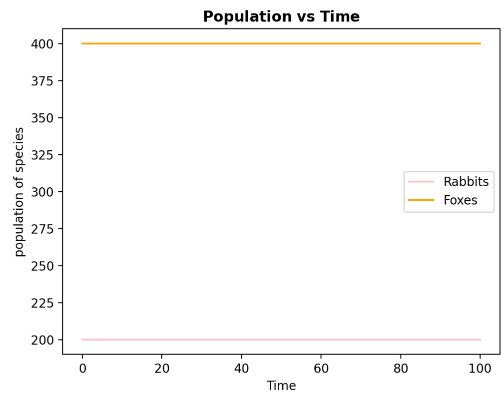


(b) for initial conditions: $R_0 = 210$, $F_0 = 410$

When we start from the initial condition (0,0) and (200,400), the fixed points, the system remains in the same state unless perturbed.



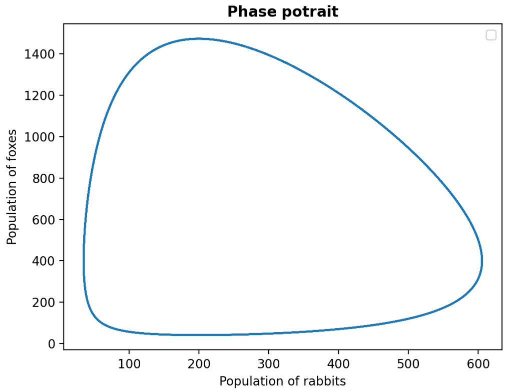
(a) for initial conditions: $R_0 = 0$, $F_0 = 0$



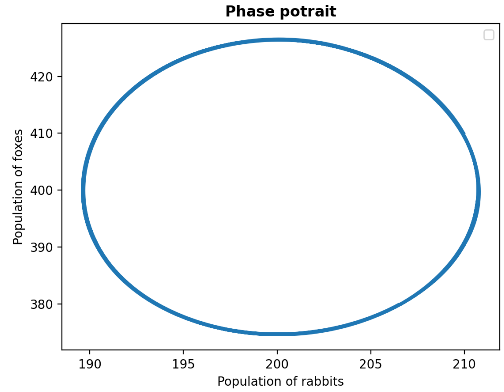
(b) for initial conditions: $R_0 = 200$, $F_0 = 400$

The phase portrait for the above conditions is plotted for the coupled differential equations solved using Radau method.

This trajectory shows that if we start from a particular initial condition, the system is forever stuck in a loop, and a small perturbation from the initial conditions takes the system to another loop. The system evloution,its trajectory is purely dependent on the initially conditions.

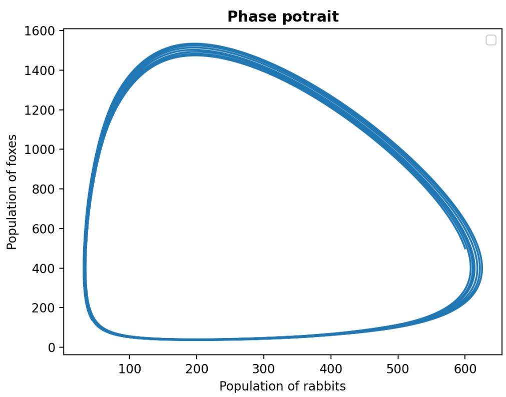


(a) for initial conditions: $R_0 = 600$, $F_0 = 500$

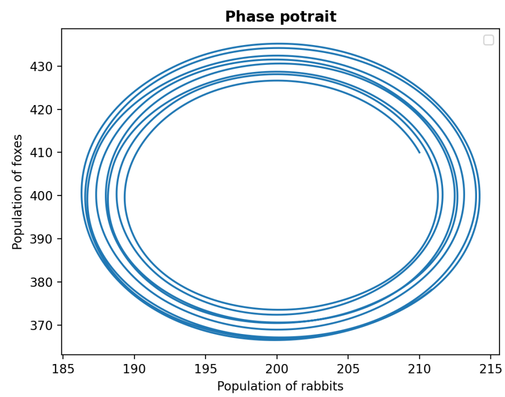


(b) for initial conditions: $R_0 = 210$, $F_0 = 410$

The phase portrait for the above conditions is plotted for the coupled differential equations solved using RK45 method.



(a) for initial conditions: $R_0 = 600$, $F_0 = 500$

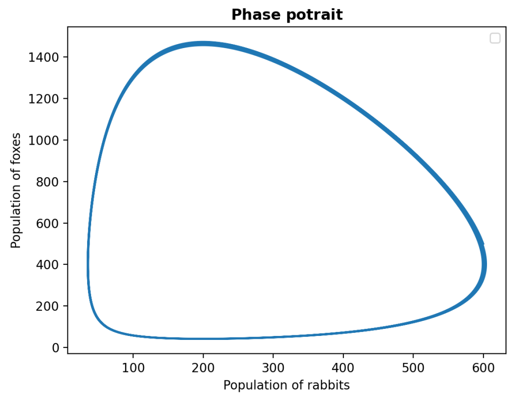


(b) for initial conditions: $R_0 = 210$, $F_0 = 410$

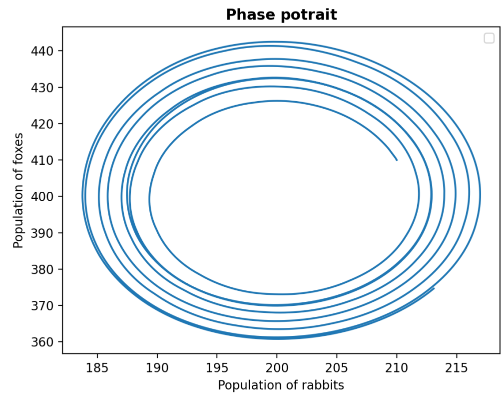
Here, by using RK45 algorithm to solve, we are getting spiral trajectory instead of single loop, this is clearly not correct in this case. This is because for stiff problems, Rk45 is not accurate and gives some error.

The phase portrait for the above conditions is plotted for the coupled differential equations solved using BDF method.

Using BDF methos also we get spiral trajectory, which is not correct. "Radau" method is more preferred as its more accurate for stiff problems.



(a) for initial conditions: $R_0 = 600$, $F_0 = 500$



(b) for initial conditions: $R_0 = 210$, $F_0 = 410$

The phase portrait for different initial conditions is plotted , and the nature of fixed points is analysed.

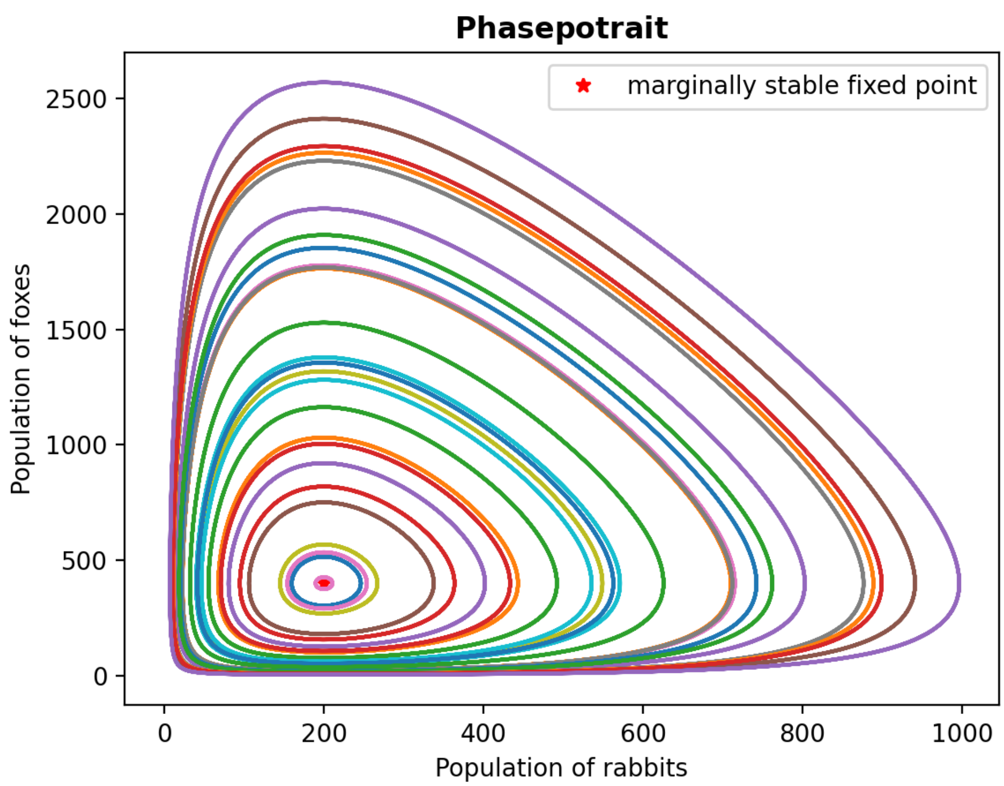


Figure 6: Phase portrait for different initial conditions

This shows that, (0,0) is unstable fixed point. The other fixed point (200,400) is a marginally stable fixed point because if slightly perturbed from this fixed point , it neither goes away completely nor comes back to it again. Instead enters a loop and stays forever in that loop unless perturbed. This type of fixed point, where it is neither stable nor unstable is called marginally stable fixed point.