Contents

Chapter 1

Basic concepts

1.1 Why this course?

What we are interested in this course is finding either the maximum(or maximal) or the minimum(or minimal) value of a function subject to certain constraint. Let us formally write down what is a mathematical optimization problem. Having done that lets us write down what is convex optimization problem. Then, even before we think of solving the convex optimization problem, let us first be convinced why we are even studying convex optimization. As look ahead, I would say, the structure and properties of convex functions is such that it gives rise to computationally cheaper algorithms to solve the convex optimization problem. Then we might wonder, then what are these *super* properties?? So we begin with the properties of convex sets and convex functions and then move to convex optimization problems. When we look at the properties of convex sets and convex functions we should always question, which particular property of convex function is getting used.
You might wonder, after all, convexity only implies the following:
Definition(Convex set): A subset C of R^n is called convex if

$$\alpha x + (1-\alpha)y \in C \quad \forall x, y \in C, \quad \alpha \in [0,1] \tag{1.1}$$

that is, all the points on the line between any 2 points belonging to the set should also belong to the set. You might wonder to study this property do we really need a separate course?

But surprisingly, this turns out to be very strong property and leads to so many useful consequences, especially in showing the existence of solution and also in solving the problem. Thats exactly what we are doing in this course, that is, finding out what are all the useful consequence of this definition and how these consequence can be used to solve optimization problems.

When we do *convex analysis*, that is, where we study the properties of convex sets and convex functions we should always keep in mind and keep questioning ourselves as to why a particular property is important for convex optimization.
When we study the first 2 chapters, we essentially study the properties of convex sets and functions and also study how to characterize them, say for example, we characterize convex sets as intersection of hyperplanes or intersection of other convex sets. We need to prove such characterization is possible and prove its validity.
In the 3rd chapter we use the above properties and characterization to prove the existence of the solution to convex optimization under various conditions.
In the later chapters we make use of their properties and characterization to formulate and solve convex optimization problems.

1.2 Overview of today's lecture

In this lecture we look at the following topics:

1. What is an optimization problem? How do we put it in the standard form? What are the different kinds or class of optimization problems that we come across? When do we call problems equivalent?

2. What is convex optimization problem?

3. Why do we study convex optimization problems?

4. Basic definitions : convex sets, cones, generalized inequalities.

5. Properties of convex sets and functions and how these properties are important for solving convex optimization.

6. Operations on convex sets and convex functions. Why do we study the effect of these operations on convex sets? In most cases we will be studying what are all the operations that preserve convexity. Now the question that comes to our mind is, if the set or function is already convex then why do we need to transform it to some other function and see if the transformation preserves convexity. If the function is initially not convex, but the transformation makes it convex, then we might be interested in such transformations, but why do we study transformation of functions which are already convex.
   Such transformations are generally used to **check** whether the given problem/function is convex.

7. We also look at some of **topological properties** such as properties of relative interior and closures of convex sets. These properties become important in determining existence of solutions to convex optimization problems.

1.3 Introduction to optimization

A mathematical optimization problem can be written as

$$\text{minimize} \quad f_0(x) \tag{1.2}$$
$$\text{subject to} \quad f_i(x) \leq b_i, \quad i = 1, \cdots, m \tag{1.3}$$

where

- **Optimization function** : $f_0(x) : \mathbf{R}^n \to \mathbf{R}$
- **Optimization variable** : the vector $x = (x_1, x_2, \cdots, x_n)$
- **Constraint functions** : $f_i(x) \leq b_i, \quad i = 1, \cdots, m$

1.4 Introduction to convex optimization

1.4.1 Advantages of convex optimization

1.5 Convex sets

Definition: A subset C of R^n is called convex if

$$\alpha x + (1-\alpha)y \in C \quad \forall x, y \in C, \quad \alpha \in [0,1] \tag{1.4}$$

Note that we could look at more abstract convex sets. Because all we need is vector space in which the scalar multiplication and vector addition.
Note that empty set is convex by convention.

The following proposition gives some of the operations that preserve convexity.

Proposition 1.1.1:

1. The intersection $\cap_{i \in I} C_i$ of any collection $\{C_i | i \in I\}$ of convex sets is convex.
2. The vector sum $C_1 + C_2$ of two convex sets C_1 and C_2 is convex
3. The set λC is convex for any convex set C and scalar λ. Furthermore, if C is a convex set and λ_1 and λ_2 are positive scalars,
$$(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C \tag{1.5}$$
4. The closure and the interior of a convex set are convex.
5. The image and the inverse image of a convex set under an affine function are convex.

Proof:

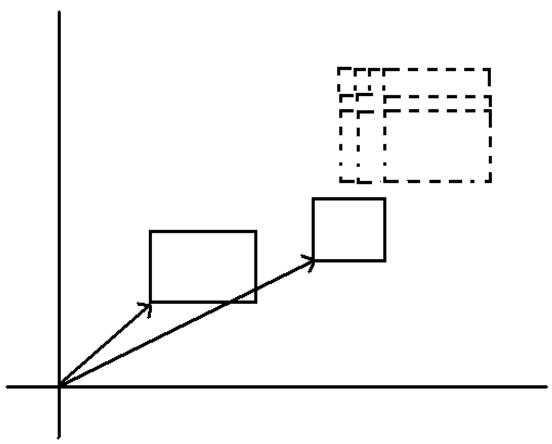
1. This follows directly from the fact that the points that belong to the intersection belong to each of the C_i and for any 2 points in the intersection, all the points along the line joining the two points lie in the intersection.

2. The vector sum $C_1 + C_2$ is defined as $\{x + y | x \in C_1, y \in C_2\}$
   We need to show that if $C_1 + C_2 = C$, $x_1, x_2 \in C_1$, $y_1, y_2 \in C_2$, $x_1 + y_1 \in C$ and $x_2 + y_2 \in C$, then $\alpha(x_1 + y_1) + (1 - \alpha)(x_2 + y_2) \in C$

$$\alpha(x_1 + y_1) + (1 - \alpha)(x_2 + y_2) \tag{1.6}$$

$$\alpha x_1 + (1 - \alpha)x_2 + \alpha y_1 + (1 - \alpha)y_2 \tag{1.7}$$

   $\alpha x_1 + (1 - \alpha)x_2 \in C_1$, $\alpha y_1 + (1 - \alpha)y_2 \in C_2$ and since if $x \in C_1$ and $y \in C_2$, $x + y \in C$, therefore, $\alpha x_1 + (1 - \alpha)x_2 + \alpha y_1 + (1 - \alpha)y_2 \in C$, therefore, $C_1 + C_2$ is convex.



   As can be seen in the above figure, the vector sum of convex sets is basically translation of one set with respect to the other. It basically leads to a smugged version, that is union of translated version of one set with respect to the vectors in the other set. Note that geometrically it may be difficult to visualize how exactly the resultant set turns out to be convex but it can proved to be convex.

3. The set λC is convex for any convex set C and scalar λ. Furthermore, if C is a convex and λ_1, λ_2 are positive scalars,

$$(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C \tag{1.8}$$

   Proof : In this problem what we want to show is if you scale the convex set, then the set, that is, the scaled set, remains convex.
   C is convex implies $\alpha x + (1 - \alpha)y \in C$ when $x, y \in C$ and $\alpha \in [0, 1]$. We need to show $\alpha(\lambda x) + (1 - \alpha)(\lambda y) \in C$ when $x, y \in C$, $\lambda x, \lambda y \in \lambda C$ and $\alpha \in [0, 1]$.

4. The closure and the interior of a convex set are convex.

   (a) Closure of a convex set is convex :

   (b) Interior of a convex set is convex : Basically what you need to prove is between any two interior points the line does not contain any boundary points, because of which, even though the boundary points have been exempted, the interior still remains convex. We can probably prove the above proposition using an equivalent statement:

A set is convex if and only if the line between any two interior points does not contain any boundary point.
or equivalently
Prove that if line between any two interior points contains a boundary point then the set is not convex.

5. The image and the inverse image of a convex set under an affine function are convex.

Theorem (Projection Theorem) : Let C be a non-empty closed convex set of R^n and let z be a vector in R^n. There exists a unique vector that minimizes $\|z-x\|$ over $x \in C$, called the projection of z on C. Furthermore, a vector x^* is the projection of z on C if and only if $(z-x^*)'(z-x)$.

1.6 Epigraph

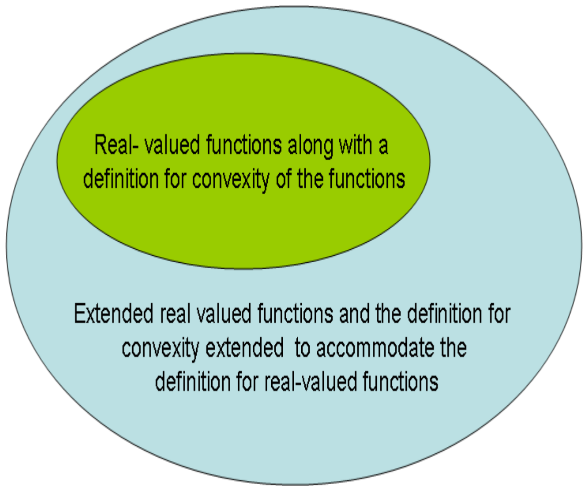
Before we formally define epigraph, we want to understand why are we even studying epigraphs in convex analysis. Basically, the main objects of convex analysis are convex sets and convex functions. Epigraph is a way to connect these two objects.

- Now the question is why do you want to connect the two?
- Under what situations does this connection of convex functions to convex sets become very useful?
- Now why suddenly epigraph is being discussed in the context of extended real valued functions? Why for dealing with extended real valued functions epigraph is necessary?
- Suppose we were dealing with only real valued functions then would we not require the concept of epigraph? Or is epigraph used generally to connect convex functions and convex sets so that the properties of convex functions can be proved or interpreted geometrically using epigraphs.

We need to see the following :

- The situations where even though we start off with real valued functions, it gives rise to extended real vlaued functions.
- The situation where the domain of the functions might be extended to give rise to extended real valued functions. The situations where you want to extend the function to the whole of R^n and avoid mentioning the domain of the functions. In this extension, it may not be possible to extend the function some functions to R^n without extending the range to extended real values.

In either case we have extended real valued functions. Now the question is how do we define convex functions for extended real valued functions because in case of extended real valued functions there is a possibility of having the forbidden sum of $+\infty$ and $-\infty$. Now the question is how do we define convexity for such functions? Also the definition should be such that when applied to real-valued functions, the previous definition should hold.



The concept of epigraph need not really be used for defining convexity of extended real valued function. Using the concept of epigraph to define convex functions gives a geometric approach to convex analysis by connecting the ideas of convex sets and convex functions. And at the same time, we see that using this way of defining convex functions, keeps the definition consistent to extended real valued functions as well therefore catering the definition to much bigger class of functions.

The *epigraph* of a function $f : X \rightarrow [-\infty, +\infty]$, where $X \subset R^n$ is defined to be the subset of R^{n+1} given by

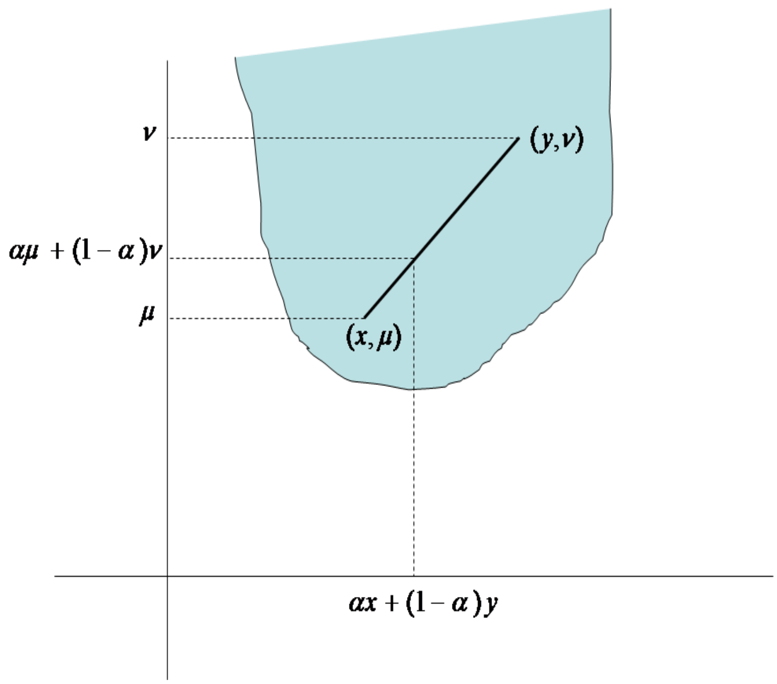
$$\text{epi}(f) = \{(x, w) | x \in X, w \in R, f(x) \leq w\} \tag{1.9}$$

The *effective domain* of f is defined to be the set

$$\text{dom}(f) = \{x \in X | f(x) < \infty\} \tag{1.10}$$

Definition: Let C be a convex subset of R^n. We say that an extended real-valued function $f : C \rightarrow [-\infty, +\infty]$ is convex if $\text{epi}(f)$ is a convex subset of R^{n+1}.

Show that the above definition leads to the interpolation property of convex functions (this property was initially used as the definition of convexity for real-valued functions).
Proof :
$\text{epi}(f)$ is convex implies $\alpha(x, \mu) + (1 - \alpha)(y, \nu) \in \text{epi}(f)$, for all $\alpha \in [0, 1]$
$\Rightarrow \alpha\mu + (1 - \alpha)\nu \geq f(\alpha x + (1 - \alpha)y)$
$\Rightarrow \alpha f(x) + (1 - \alpha)f(y) \geq f(\alpha x + (1 - \alpha)y)$
Hence it is consistent with the earlier definition of convexity for real valued functions.



Show that the above definition also implies:

1. The effective domain dom(f) is convex.
   Proof: The effective domain is the image of the epi(f) under affine transformation, that is, projection on to R^n. Since we already know that the convex set under affine transformation remains convex, the effective domain is convex. Note the definition of epigraph. It contains only those points (x, μ) such that $f(x) < \infty$. Therefore if the function itself is such that $f(x) = \infty$, then those points will not be included in the effective domain and neither will it be included in the epigraph. Therefore, the effective domain contains only those points for which $f(x) < \infty$.

2. The level sets $\{x \in C | f(x) \leq \gamma\} and \{x \in C | f(x) < \gamma\}$ are convex sets.

1.6.1 Closedness and Semicontinuity

Closed: If the epigraph of a function $f : X \to [-\infty, +\infty]$ is a closed set, we say that f is a closed function.
Lower semicontinuity: The function f is called lower semicontinous at a vector $x \in X$ if

$$f(x) \leq \liminf_{k\to\infty} f(x_k) \tag{1.11}$$

for every sequence $\{x_k\} \subset X$ with $x_k \to x$

Proposition 1.1.2: For a function $f : R^n \to [-\infty, +\infty]$, the following are equivalent:
1. The level set $V_\gamma = \{x | f(x) \leq \gamma\}$ is closed for every scalar γ.
2. f is lower semicontinous.
3. epi(f) is closed.

Proof:
To prove that if (1) is true then (2) is true, that is, we need to show that if the level set is closed for every scalar, then the function is semicontinous. That is we need to show that if the level set is closed for every scalar, then

$$f(x) \leq \liminf_{k \to \infty} f(x_k) \tag{1.12}$$

for every sequence $\{x_k\} \subset X$ with $x_k \to x$
Note that in this proposition the function need not be convex.

Chapter 2

Several variable differential calculus

Let us now review the concept of gradient and then discuss the first order condition of convex functions.

2.1 Basic Definitions

We define differentiation using limits in contrast to the geometric definition of derivatives. The advantage of working with analytically is

- we do not need to know the axioms of geometry and
- these definitions can be modified to handle functions with several variables of functions whose values can be modified to handle functions of several variables or functions whose values are vectors instead of scalar.
- One's geometric intuition becomes difficult to rely on once one has more than three dimensions.

Definition (ϵ-adherent point) Let X be a subset of R, let $\epsilon > 0$, and let $x \in R$. We say that x is ϵ-adherent to X if and only if there exists a $y \in X$ which is $\epsilon-$close to x (i.e. $|x - y| \leq \epsilon$).
Definition (Adherent point): Let X be a subset of R and let $x \in R$. We say that x is an adherent point of X if and only if it is $\epsilon-$adherent to X for every $\epsilon > 0$.
Definition (Limit point): Let X be a subset of the real line. We say that x is a *limit point* or a *cluster point* of X if and only if it is an adherent point of $X \backslash \{x\}$. We say that x is an *isolated point* of X of $x \in X$ and there exists some $\epsilon > 0$ such that $|x - y| > \epsilon$ for all $y \in X \backslash \{x\}$.
Lemma: Let X be a set of R and let $x \in R$. Then x is an adherent point of X if and only if there exists a sequence $(a_n)_{n=0}^{\infty}$, consisting entirely of elements in X, which converges to x.

From the above lemma we see that x is a limit point of X if and only if there exists a sequence $(a_n)_{n=0}^{\infty}$ consisting entirely of elements in X that are distinct from x and such that $(a_n)_{n=0}^{\infty}$ converges to x. It turns out that the set of adherent points splits into the set of limit points and the set of isolated points.

Definition(Convergence of functions at a point): Let X be a subset of R, let $f : X \to R$ be a function, let E be a subset of X, x_0 be an adherent point of E, and let L be a real number. We say that f *converges to* L *at* x_0 *in* E and write

$$\lim_{x \to x_0; x \in E} f(x) = L \tag{2.1}$$

iff f is ϵ-close to L near x_0 for every $\epsilon > 0$.

If f does not converge to any number L at x_0, we say that f diverges at x_0 and leave $\lim_{x \to x_0; x \in E} f(x)$ undefined.

Note the suttle difference between the convergence of sequences and convergence of functions at a point. In convergence of functions to a point we are not bothered about the sequence of x which converges to x_0. In other words, we have $\lim_{x \to x_0; x \in E} f(x) = L$ iff for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| \leq \epsilon$ for all $x \in E$ such that $|x - x_0| < \delta$.

Definition (Differentiability at a point): Let X be a subset of R and let $x_0 \in X$ be an element of X which is also a limit point of X. Let $f : X \to R$ be a function. If the limit

$$\lim_{x \to x_0; x \in X - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} \tag{2.2}$$

converges to some real number L, then we say that f *is differentiable at* x_0 *on* X *with derivative* L and write $f'(x_0) := L$.
If the limit does not exist or if x_0 is not an element of X or not a limit point of X, we leave $f'(x_0)$ undefined, and say that f is *not differentiable at* x_0 *on* X.

Remark: Note that we need x_0 to be a limit point in order for x_0 to be adherent to $X - \{x_0\}$, otherwise the limit

$$\lim_{x \to x_0; x \in X - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} \tag{2.3}$$

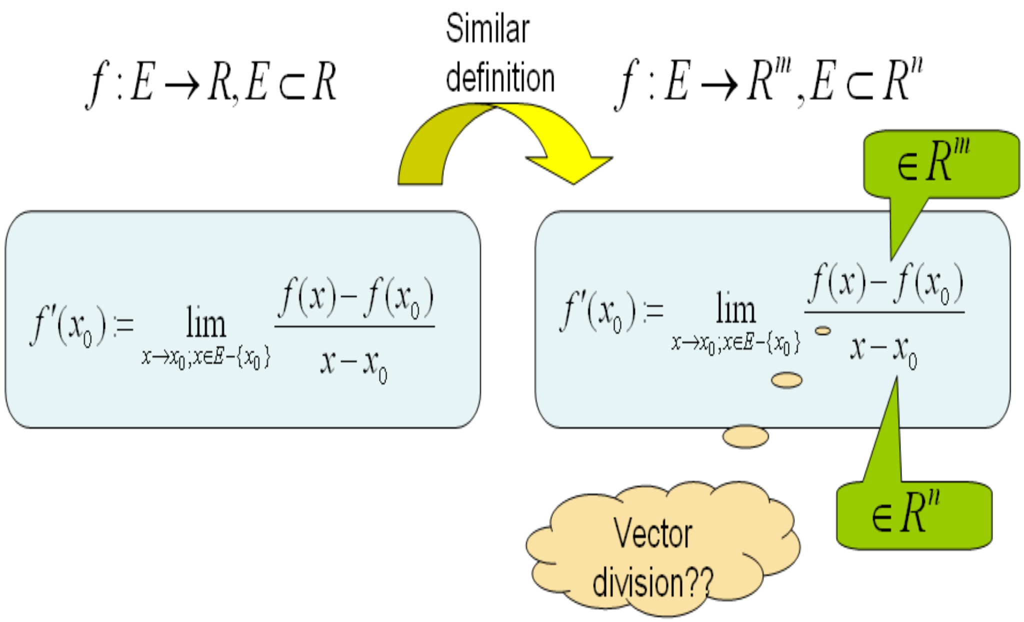
would autimatically be undefined. In particular we do not define the derivative of a function at an isolated point.

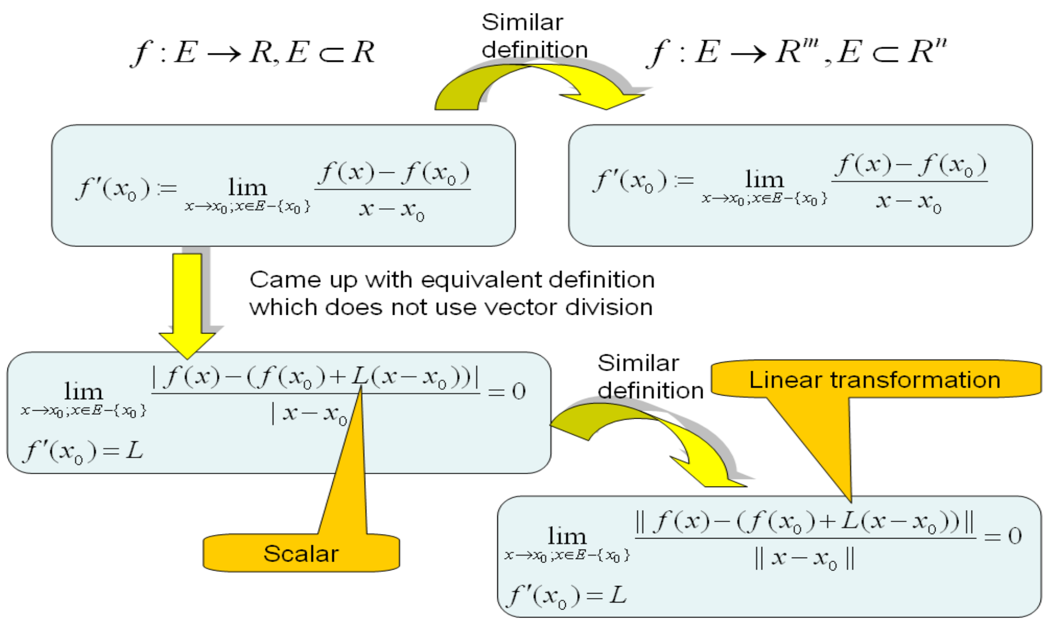
2.2 Derivatives in several variable calculus

Now we move on to understanding differentiation of functions of the form $f : R^n \to R^m$, that is, functions from one Euclidean space to another.
As we defined above, in single variable calculus, when one wants to differentiate a function $f : E \to R$ at a point x_0, where E is a subset of R that contains x_0, it is given as

$$\lim_{x \to x_0; x \in E - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} \tag{2.4}$$

One could try to mimic this definition in several variable case $f : E \to R^m$, where E is now a subset of R^m, however we encounter a difficulty in this case: the quantity $f(x) - f(x_0)$ will live in R^m and $x - x_0$ will live in R^n and we do not know how to divide an m-dimensional vector by an n-dimensional vector.
To get around this problem, we first rewrite the concept of derivative in a way which does not involve division of vectors. Instead we view differentiability at a point x_0 as an assertion that a function f is "approximately linear" near x_0.


$f : E \to R, E \subset R$
Similar definition
$f : E \to R^m, E \subset R^n$
$\in R^m$
$f'(x_0) := \lim_{x \to x_0; x \in E - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$
$f'(x_0) := \lim_{x \to x_0; x \in E - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$
$\in R^n$
Vector division??


$f : E \to R, E \subset R$
Similar definition
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$f'(x_0) := \lim_{x \to x_0; x \in E - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$
Came up with equivalent definition which does not use vector division
$\lim_{x \to x_0; x \in E - \{x_0\}} \frac{|f(x) - (f(x_0) + L(x - x_0))|}{|x - x_0|} = 0$
$f'(x_0) = L$
Similar definition
Linear transformation
Scalar
$\lim_{x \to x_0; x \in E - \{x_0\}} \frac{\| f(x) - (f(x_0) + L(x - x_0)) \|}{\| x - x_0 \|} = 0$
$f'(x_0) = L$

Chapter 3

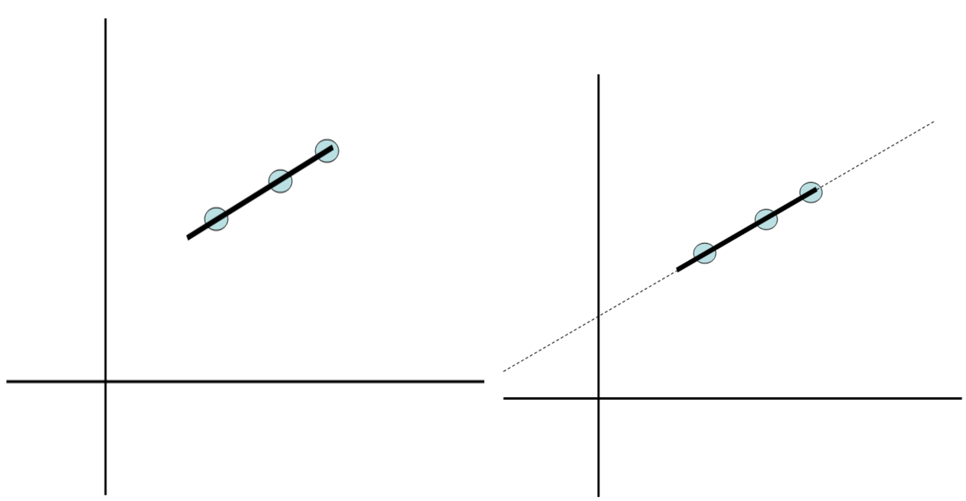
Relative Interior and Closures

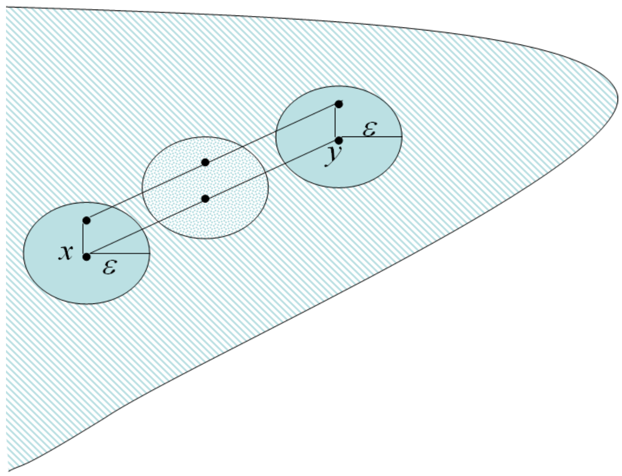
Let C be a nonempty convex subset of R^n. We know that closure of a convex set is convex. We also know that the interior of convex set is convex but possible empty. It turns out that convexity implies existence of interior points relative to the affine hull of C.

Discussion:

1. Note that for a convex set, the interior may be empty. But the interior relative to the affine hull is non empty. Note the difference between interior and relative interior. Consider the line segment shown below in R^2. It is a convex set. But its interior is empty. The entire set is a set of boundary points because if we consider a ball B^2 of radius ϵ around any point on the line there is always a point which is in the ball which does not belong to the line. That is, for any $\epsilon > 0$, there does not exist an open ball such that that ball is entirely contained in the line. But if we consider intersection of the sphere with the affine hull of the line segment, which is nothing but the line, then, there exists sphere such that it is entirely contained in the line segment.

2. If the set is convex, its closure is convex and its interior is convex. Interior is convex does not imply it is non-empty. Empty set by convention is convex As shown in the below figure, when we proved that the interior is convex, we started with two interior points and showed then showed that a point $z = \alpha x + (1 - \alpha)y$ is also interior point because allt the points in the ball will have a corresponding point belonging to C and therefore there exists a ball around z such that all the points in the ball belong to C. In this proof, we started with the assumption that interior is non-empty. We start by choosing two points from the interior.





3. Below we show that even if the interior is empty, convexity implies the relative interior w.r.t the affine hull is non-empty. That means w.r.t the affine hull the convex set cannot consist of just boundary points.

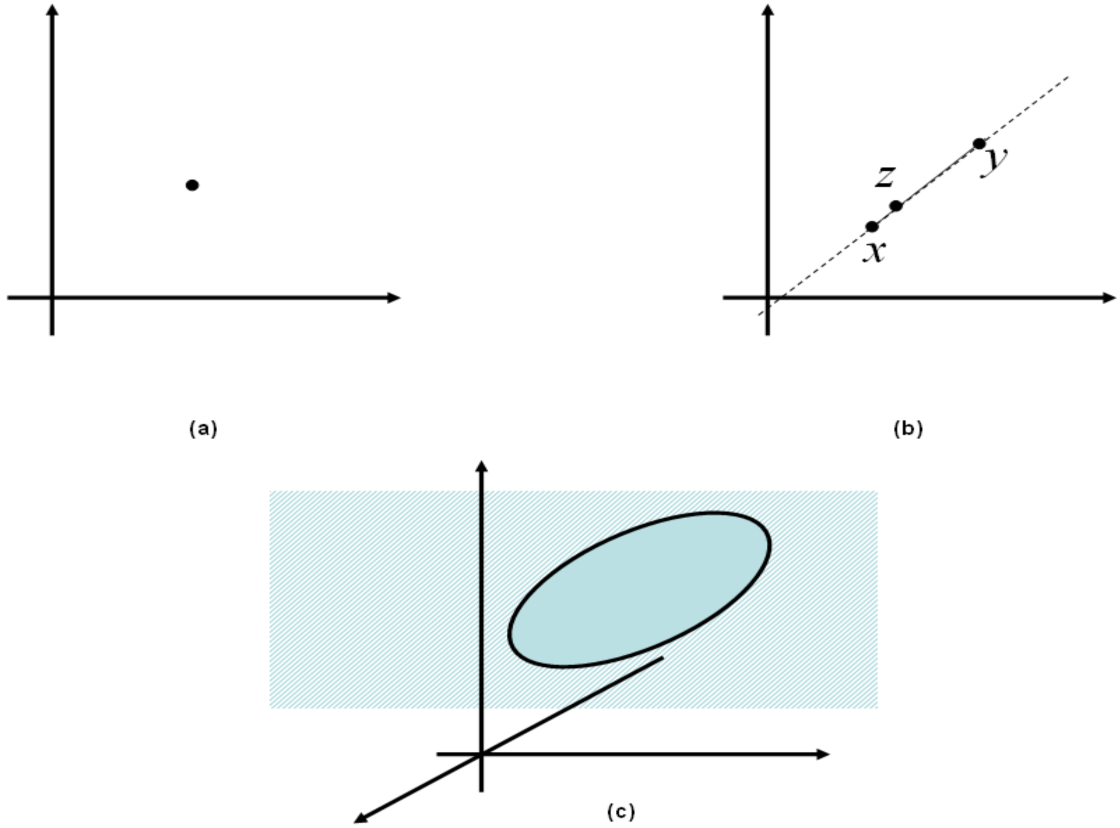
Definition (Relative interior point):
Definition (Relative interior):
Definition (Relatively open):
Definition (Relative boundary):
The most fundamental fact about relative interiors of convex sets is given in the following proposition:
Proposition: (Line Segment Principle) Let C be a nonempty convex set. If $x \in \mathrm{ri}(C)$ and $\bar{x} \in \mathrm{cl}(C)$, then all points on the line segment connecting x and $\bar{x}$, except possibly $\bar{x}$, belong to the $\mathrm{ri}(C)$.

Proposition: (Nonemptiness of Relative Interior) Let C be a nonempty convex set. Then:

1. $\mathrm{ri}(C)$ is a nonempty convex set, and has the same affine hull as C

2. if m is the dimension of $\mathrm{aff}(C)$ and $m > 0$, there exists vectors $x_0, x_1 \cdots x_m \in \mathrm{ri}(C)$ such that $x_1 - x_0, \cdots, x_m - x_0$ span the subspace parallel to $\mathrm{aff}(C)$.

Discussion: We know that ri(C) does not include the boundary points. Therefore, the question, is after excluding some points will affine of the ri(C) be the same as the aff(C). Also we need to show that C cannot consist of only relatively boundary points.

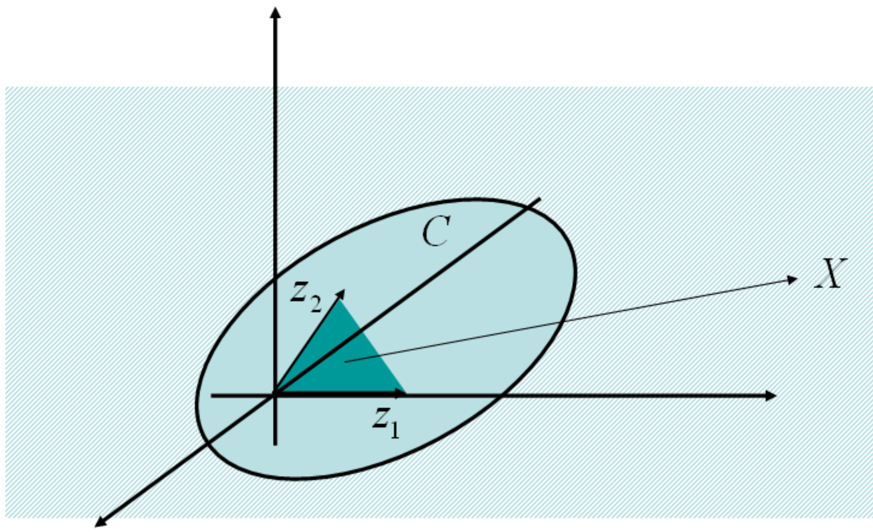


Proof:

Let us consider the case when $m = 0$. That is C is just one point.

Let us consider the case when $m = 1$. Let us consider 2 relatively boundary points. The aff(C) in this case is a line passing through the two relatively boundary points. Now since the set C is convex the set of points between these two points belongs to C. We need to show that these points are relative interior points. Rather, we need to show that these points are not relatively boundary points.

Consider linearly independent vectors, $z_1, z_2, \cdots, z_m$ of set C. Consider the set X which is convex linear combination of these vectors. Obviously, $X \subset C$. Ultimately we need to show that there exists a point $x \in C$, such that you can draw an open ball S around it such that $S \cap \text{aff}(C) \subset C$. If we are able to find such a point, then we have found a relatively interior point and hence the relative interior is non-empty.



Proposition: (Prolongation Lemma)

3.1 Calculus of relative interiors and closures

3.1.1 Relation between closures and relative interiors

Two convex sets have the same closure if and only if they have the same relative interior.

Proposition: Let C be a nonempty convex set.

1. cl(C) =cl(ri(C))

   Proof: We will show that cl$(C)\subseteq$ cl(ri(C)) and then cl(ri(C))$\subseteq$ cl(C) and hence prove the proposition.

   ri$(C)\subseteq C \Rightarrow$ cl(ri(C)) $\subseteq$ cl(C)

   To show that cl$(C)\subseteq$ cl(ri(C)), we show that every point $x \in$ cl(C) also belongs to cl(ri(C)).

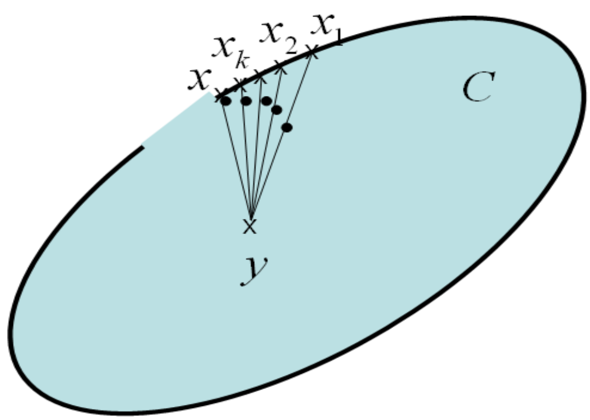
   We say that x is a *closure point of a subset of C of R^n* if there exists a sequence $\{x_k\} \subset C$ that converges to x. The *closure* of C, cl(C), is the set of all closure points of C.

   Consider a point $x \in$ cl(C). This means that there exists a sequence $\{x_k\} \subset C$ that converges to x. Since ri$(C) \subseteq C$, how are we sure that the sequence which belonged to C and which converged to x also belongs to the ri(C). The arguement is all the points in the sequence could have been relative boundary points which do not belong to the relative interior. So how do we show that there exists a sequence belonging to the relative interior which converges to x.

   Let us consider the worst case scenario where the entire sequence belonged to the relative bouundary of C, as shown in the figure. Since C is convex, we know that the relative interior is nonempty. Let us consider a relative interior point y. Now we know that all the points on the line segment from any relatively interior point and any point in the cl(C) belongs to ri(C). Consider the sequence

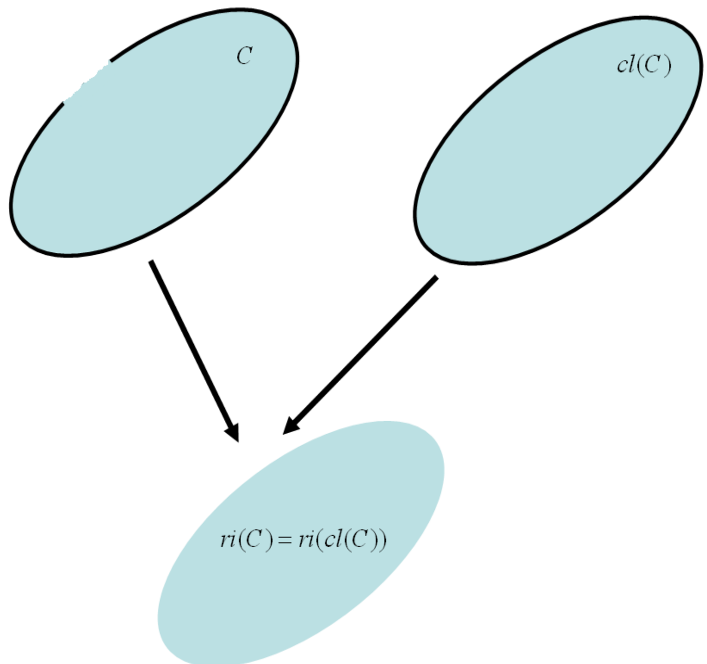
$$z_k = \frac{1}{k}y + \frac{1}{1-k}x_k, \quad \text{for all } k > 0 \tag{3.1}$$

$\{z_k\}$ consists of all relatively interior points and $z_k \to x$. Therefore, $x \in \mathrm{cl}(\mathrm{ri}(C))$

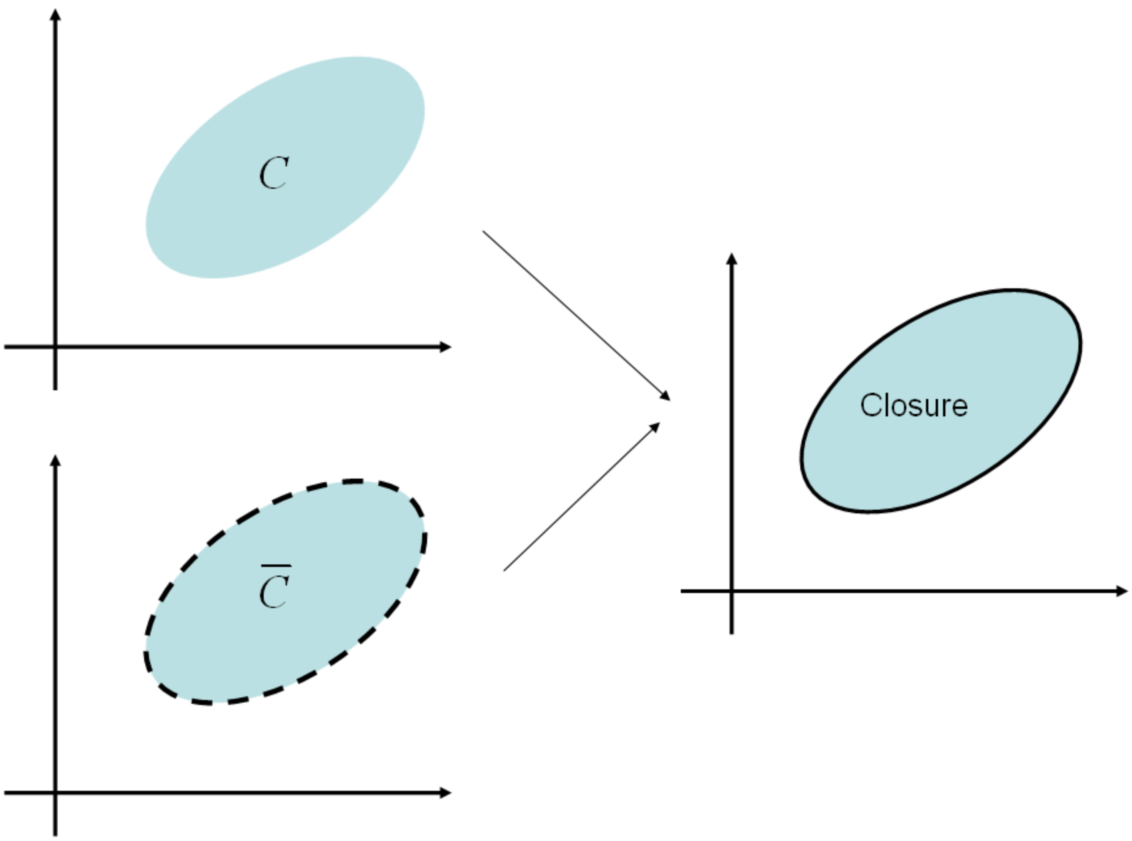


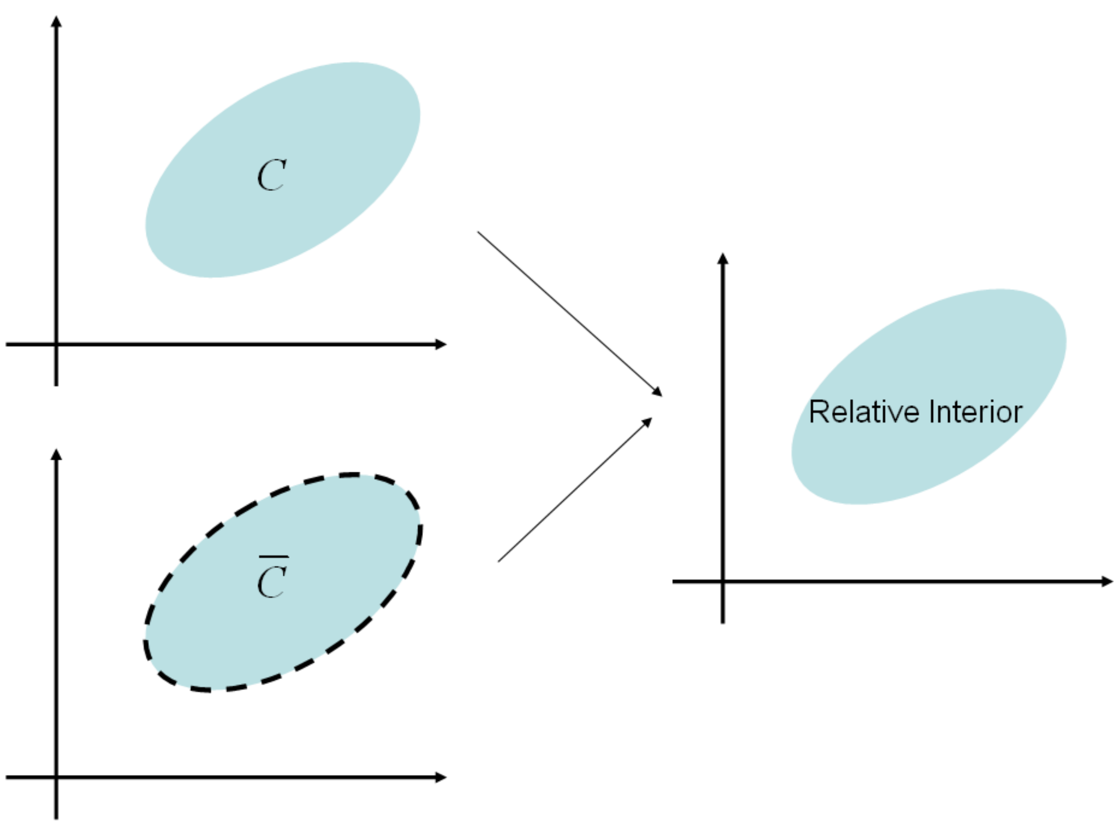
2. $\mathrm{ri}(C) = \mathrm{ri}(\mathrm{cl}(C))$
   First we shall show that $\mathrm{ri}(C) \subseteq \mathrm{ri}(\mathrm{cl}(C))$
   A point x is said to be relative interior point of C is there exists a open sphere S centred at x, of radius $\epsilon > 0$, such that $S \cap \mathrm{aff}(C) \subset C$.
   Let x be a relative interior point of C. This means there exists an open sphere S centred at x and radius $\epsilon > 0$ such that $S \cap \mathrm{aff}(C) \subset C$.
   We know that, $C \subseteq \mathrm{cl}(C) \Rightarrow S \cap \mathrm{aff}(C) \subset C \subseteq \mathrm{cl}(C)$.
   Also, $\mathrm{aff}(C) = \mathrm{aff}(\mathrm{cl}(C)) \Rightarrow S \cap \mathrm{aff}(\mathrm{cl}(C)) \subset C \subseteq \mathrm{cl}(C)$.
   Therefore, $x \in \mathrm{ri}(\mathrm{cl}(C))$.

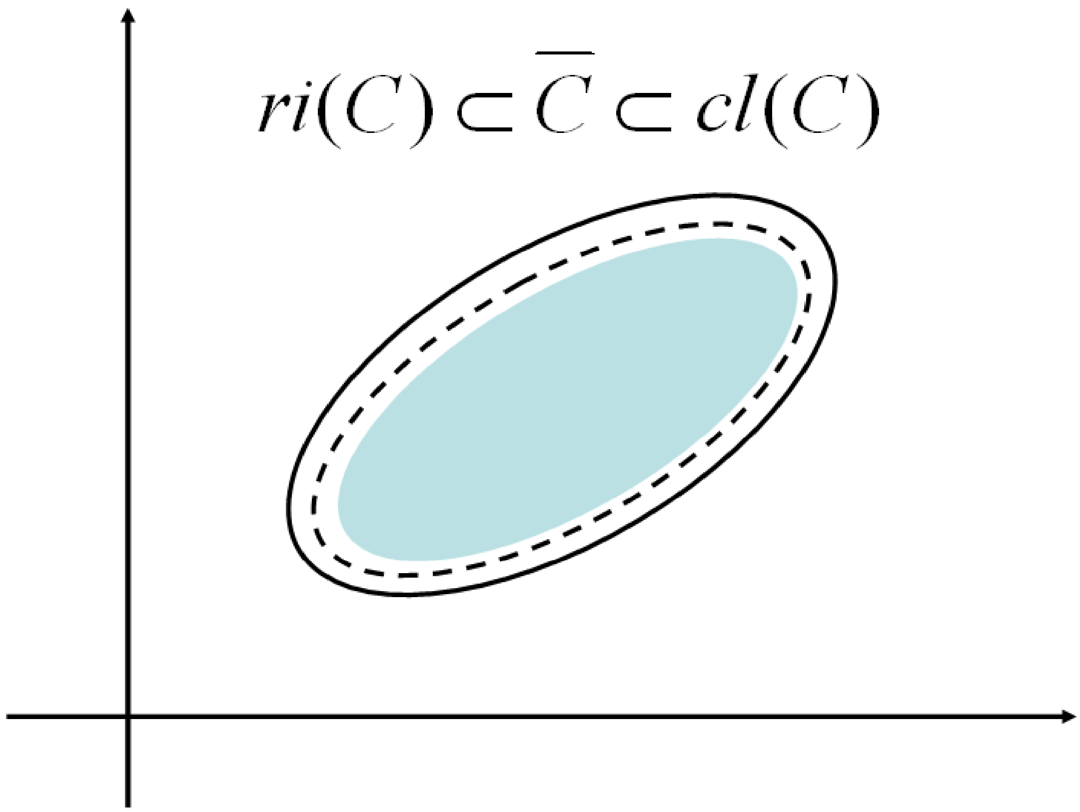
   Now we need to prove the converse. We need to show that $\mathrm{ri}(\mathrm{cl}(C)) \subseteq \mathrm{ri}(C)$.
   Let us think about the worst case scenarios, that is, think about those points belonging to $\mathrm{ri}(\mathrm{cl}(C))$ about whose membership to $\mathrm{ri}(C)$ we have doubt.
   The question is : Is there a point x which is a relative interior point of $\mathrm{cl}(C)$ and will seize being a relative interior point of C.
   Consider a point x which is a relative interior point of $\mathrm{cl}(C)$ such that there exists an open sphere S of radius ϵ such that $S \cap \mathrm{aff}(\mathrm{cl}(C)) \subseteq \mathrm{cl}(C)$. Let this sphere contain a point z such that $z \in \mathrm{cl}(C)$. But $z \notin C$. We need to show that even under such a scenario, x will remain relative interior point of C. We know that from the line segment theorem, that the line segment from x to z will contain relative interior points. Consider the line segment between x to z. Let the distance be δ. Now reduce the radius to $\delta/2$. So that the sphere will contain all points belonging to C.



3.


C
$\overline{C}$
Closure


C
$\overline{C}$
Relative Interior



3.1.2 Under linear transformation

We now consider the image of a convex C under linear transformation A. Here are a few observations

- Relative interiors are preserved, that is, $A \cdot \mathrm{ri}(C) = \mathrm{ri}(A \cdot C)$.
- Image of closed convex set under linear transformation need not be closed.
- If C is closed and bounded, then the image is closed.

The above observations are summarized in the following proposition:

Proposition: Let C be a nonempty convex subset of R^n and let A be an $m \times n$ matrix.

1. We have $A \cdot \mathrm{ri}(C) = \mathrm{ri}(A \cdot C)$
2. We have $A \cdot \mathrm{cl}(C) \subset \mathrm{cl}(A \cdot C)$. Furthermore, if C is bounded, then $A \cdot \mathrm{cl}(C) = \mathrm{cl}(A \cdot C)$

Proof:

1. $A \cdot \mathrm{ri}(C) = \mathrm{ri}(A \cdot C)$

   To prove this we have to show the following:

   (a) $A \cdot \mathrm{ri}(C) \subseteq \mathrm{ri}(A \cdot C)$
   (b) $\mathrm{ri}(A \cdot C) \subseteq A \cdot \mathrm{ri}(C)$

(a) To prove $A \cdot \mathrm{ri}(C) \subseteq \mathrm{ri}(A \cdot C)$
Discussion: We know that if C is convex, $\mathrm{ri}(C)$ is convex, and image of a convex set under affine transformation is convex, therefore $A \cdot \mathrm{ri}(C)$ is convex.
Consider a point $x \in A \cdot \mathrm{ri}(C)$. Let us think of a scenario where it is possible that this point does not belong to $\mathrm{ri}(A \cdot C)$. Then we shall show that such a situation does not arise and hence prove (a). If x falls on the relative boundary of $A \cdot C$ then, $x \notin \mathrm{ri}(A \cdot C)$. We need to show that every $x \in A \cdot \mathrm{ri}(C)$ also belongs to $\mathrm{ri}(A \cdot C)$.

2.