Fourier Series : An Introduction

Before we get into Fourier Series, let us review the what series and convergence of series means. Also we will look at convergence of functions.

1 Continuous functions

Let us consider the following definition from [3].
Definition 13.1.1: (Continous functions) Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space and let $f : X \to Y$ be a function. If $x_0 \in X$, we say that f is *continuous at* x_0 iff for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ whenever $d_X(x, x_0) < \delta$. We say that f is *continuous* iff it is continuous at every point $x \in X$.

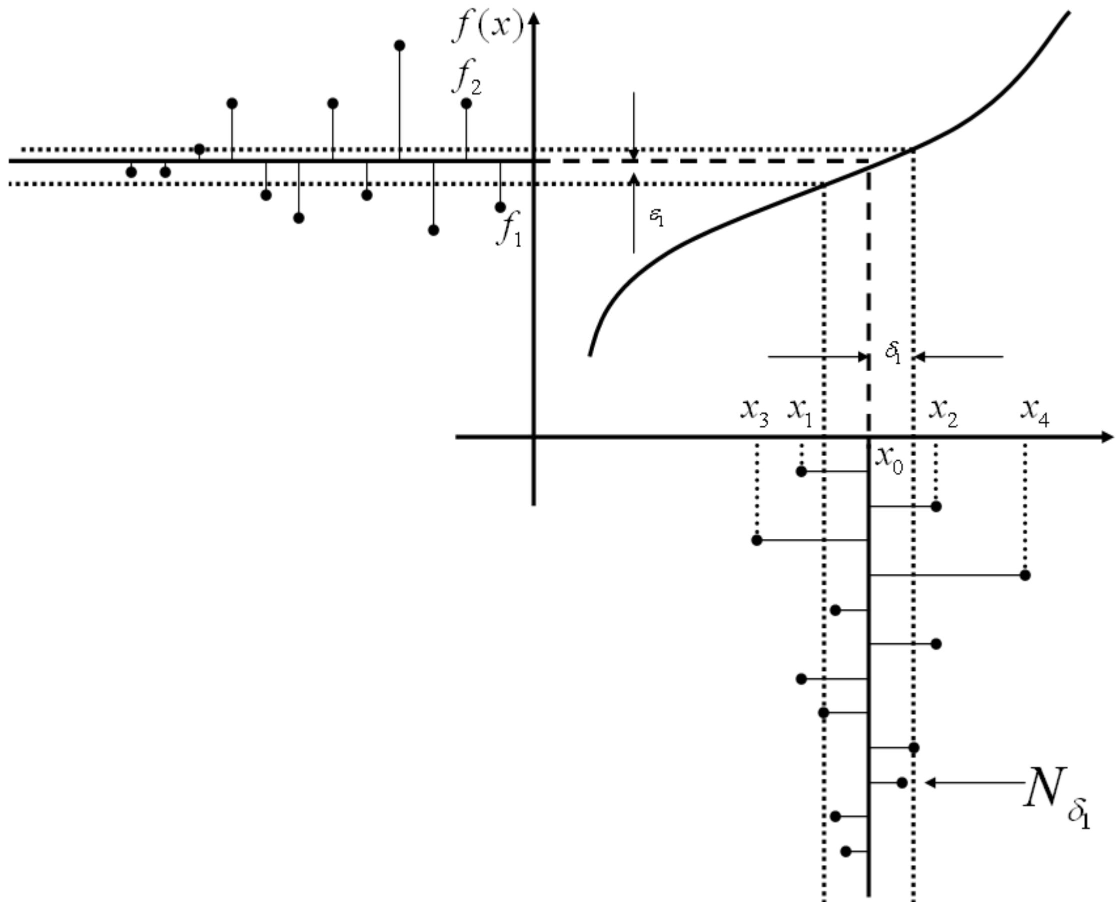
Note that in the above definition, it says that for every $\epsilon > 0$ there exists a δ and not the other way round. It is important to realize the significance. Infact, this point is stressed in the below theorems

Theorem 13.1.4: (Continuity preserves convergence) [3] Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Let $f : X \to Y$ be a function, and let $x_0 \in X$ be a point in X. Then the following three statememts are logically equivalent:
(a) f is continuous at x_0.
(b) Whenever $(x_n)_{n=1}^{\infty}$ is a sequence in X which converges to x_0 with respect to the metric d_X, the sequence $(f(x_n))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.
(c) For every open set $V \subset Y$ that contains $f(x_0)$, there exists an open set $U \subset X$ containing x_0 such that $f(U) \subseteq V$.
Proof:

1. We need to show that $(a) \Leftrightarrow (b)$.

   - **If part:** If f is continuous at x_0, then show that whenever $(x_n)_{n=1}^{\infty}$ is a sequence in X which converges to x_0 with respect to the metric d_X, the sequence $(f(x_n))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.
     Proof:
     - Given that f is continuous, that means, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \varepsilon$.
     - Now consider a sequence $\{x_n\}_{n=1}^{\infty}$ which converges to x_0. We need to show that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$, that is, for every $\varepsilon_1 > 0$ there exists an N_{ε_1} such that for all $n \geq N_{\varepsilon_1}$, $d_Y(f(x), f(x_0)) < \varepsilon_1$.

- Choose ε_1.
- It is given that, for every $\varepsilon_1 > 0$, $\exists \delta_1 > 0$ such that if $d_X(x, x_0) < \delta_1$, then $d_Y(f(x), f(x_0)) < \varepsilon_1$
- Since the sequence $(x_n)_{n=1}^{\infty}$ converges, for every $\delta_1 > 0$, there exists N_{δ_1}, such that for all $n > N_{\delta_1}$, $d_X(x, x_0) < \delta_1$.
- $d_X(x_n, x_0) < \delta_1$, $\forall n > N_{\delta_1} \Rightarrow d_Y(f(x_n), f(x_0)) < \varepsilon_1$, $\forall n > N_{\delta_1}$.
- Hence for every ε_1 $\exists N$ such that $\forall n > N$, $d_Y(f(x_n), f(x_0)) < \varepsilon_1$. Hence, $(f(x_n))_{n=1}^{\infty}$ converges to $f(x_0)$. Hence proved.



- **Only if part:** Whenever $(x_n)_{n=1}^{\infty}$ is a sequence in X which converges to x_0 with respect to the metric d_X, if the sequence $(f(x_n))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y, then show that f is continuous.
  - Given that: if $(x_n)_{n=1}^{\infty}$ converges to x_0, then $(f(x_n))_{n=1}^{\infty}$ converges to $f(x_0)$, that is, if for every $\delta_1 > 0$, there exists N_{δ_1} such that for all $n > N_{\delta_1}$, $d_X(x_n, x_0) < \delta_1$, then for every $\varepsilon_1 > 0$, there exists N_{ε_1} such that $d_Y(f(x_n), f(x_0)) < \varepsilon_1$.

- Consider an open set $V \subset Y$ containing $f(x_0)$. Now since this is an open set, every point is an interior point. Therefore, there exists $\varepsilon_2 > 0$ such that an open ball $\mathcal{B}(f(x_0), \varepsilon_2) \cap V \subseteq V$.
- Consider a sequence $(x_n)_{n=1}^{\infty}$ converges to x_0, by hypothesis, the corresponding sequence $(f(x_n))_{n=1}^{\infty}$ converges to $f(x_0)$.
- The sequence $(f(x_n))_{n=1}^{\infty}$ converges to $f(x_0)$, implies for every ε_1 there exists N_{ε_1} such that for all $n > N_{\varepsilon_1}$, $d_Y(f(x_n), f(x_0)) < \varepsilon_1$.
- Set $\varepsilon_1 = \varepsilon_2$.
- Consider the image $U = f^{-1}(\mathcal{B}(f(x_0), \varepsilon_1))$. $\{x_0, (x_n)_{n=N_{\varepsilon_1}}^{\infty}\} \subset U$.
-

2.

Theorem 13.1.5: [**3**] Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a function, and let $x_0 \in X$ be a point in X. Then the following three statememts are logically equivalent:
(a) f is continuous at x_0.
(b) Whenever $(x_n)_{n=1}^{\infty}$ is a sequence in X which converges to x_0 with respect to the metric d_X, the sequence $(f(x_n))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.
(c) Whenever V is an open set in Y, the set $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is an open set in X.
(d) Whenever F is an closed set in Y, the set $f^{-1}(F) := \{x \in X : f(x) \in F\}$ is an closed set in X.
Remark : It may seem strange that continuity ensures that the inverse image of an open set is open. One may guess instead that the reverse should be true, that the forward image of an open set is open; but this is not true as discussed in the below exerise.

Proof:

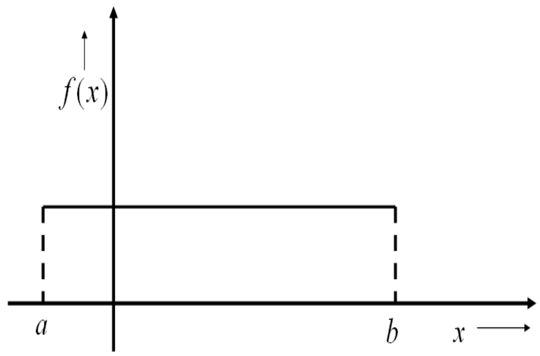
1. $(b) \Leftrightarrow (c)$

   - **If part:** We need to show if whenever $(x_n)_{n=1}^{\infty}$ converges to x_0, the $(f(x_n))_{n=1}^{\infty}$ converges to $f(x_0)$, then inverse image of open set in Y maps to open set in X. Consider an open set V in Y. Let $U = f^{-1}(V) := \{x \in X | f(x) \in V\}$ is the inverse image in X. To show that U is open set, we need to show that for every $x \in U$, there exists a ball $B(x, \epsilon)$ for some $\epsilon > 0$.
   - **Only If part:**

2.

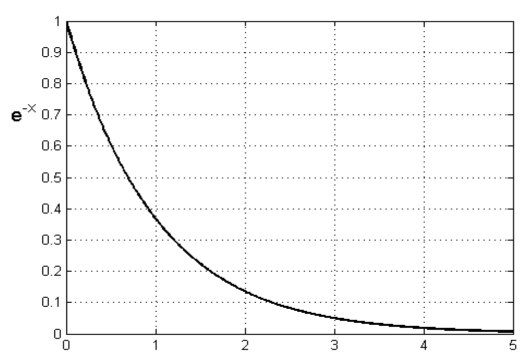
Examples :

1. **Exercise 12.5.4** [**3**] Let (R, d) be the real line with the standard metric. Give an example of a continuous function $f : R \to R$, and an open set $V \subset R$, such that the image $f(V) := \{f(x) : x \in V\}$ of V is not open.



Consider the above figure. Let $V = (a, b)$. $f(V) = \{f(x) : x \in V\} = \{c\}$. As can be seen in the above figure, $f(x) = c$ is singleton set which is closed set.

2. **Exercise 12.5.5** [**3**]Let (R, d) be the real line with the standard metric. Give an example of a continuous function $f : R \to R$, and a closed set $F \subseteq R$, such that $f(F)$ is not closed.
   Note that a continuous function on a closed and bounded interval are always bounded and we are looking for an example of a forward image of closed set which is not closed. We have the following theorem:
   Theorem 13.3.1 (Continuous maps preserve compactness) Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let $K \subseteq X$ be any compact subset of X. Then the image $f(K) := \{f(x) : x \in K\}$ of K is also compact.
   From the above theorem it is clear that if we are looking for an example of a closed set whose image is not closed under the continuous function, then we have to take the image of closed and unbounded set. Consider the set $X = \{x \in R : x \geq 0\}$. This set is a closed set in R because it contains all its closure points. In other words, all the limit points of all the sequences belonging to X also belongs to the set. Let $f(x) = e^{-x}$. $f(X) = \{f(x) : x \in X\} = (0, 1]$. Therefore it is not closed set.



2 Series and Convergence

The concepts in this section are from [3].

2.1 Series

Definition 7.1.1 (Finite series). Let m, n be integers, and let $(a_i)_{i=m}^{n}$ be a finite sequence of real numbers, assigning a real number a_i to each integer i between m and n inclusive (i.e,$m \leq i \leq n$). Then we define the finite sum (or finite series) $\sum_{i=m}^{n} a_i$ by the recursive formula

$$\sum_{i=m}^{n} a_i := 0 \quad \text{whenever } n < m \tag{1}$$

$$\sum_{i=m}^{n+1} a_i := \left(\sum_{i=m}^{n} a_i\right) + a_{n+1} \quad \text{whenever } n \geq m-1 \tag{2}$$

Remark 7.1.2. The difference between "sum" and "series" is a subtle linguistic one. Strictly speaking, a series is an expression of the form $\sum_{i=m}^{n} a_i$; this series is mathematically (but not semantically) equal to a real number, which is then the sum of that series. For instance, $1 + 2 + 3 + 4 + 5$ is a series, whose sum is 15.
This difference would probably be important in the case of infinite series because, in that case, a series might not converge and might not be equal to any real number.

Definition 7.2.1 (Formal infinite series). A (formal) infinite series is any expression of the form

$$\sum_{n=m}^{\infty} a_n$$

where m is an integer, and a_n is a real number for any integer $n > m$. We sometimes write this series as

$$a_m + a_{m+1} + a_{m+2} + \cdots$$

.At present, this series is only defined *formally*; we have not set this sum equal to any real number; the notation $a_m + a_{m+1} + a_{m+2} + \cdots$ is of course designed to look very suggestively like a sum, but is not actually a finite sum because of the $\cdots$ symbol. To rigourously define what the series actually sums to, we need another definition.
Note that when we say *formal* infinite series, it means it an expression of the form $\sum_{n=m}^{\infty} a_n$.

Definition 7.2.2 (Convergence of series). Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer$N > m$, we define the Nth partial sum S_N of this series to be $S_N := \sum_{n=m}^{N} a_n$; of course, S_N is a real number. If the sequence $(S_N)_{n=m}^{\infty}$ converges to some limit L as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is *convergent*, and converges to L; we also write $L = \sum_{n=m}^{\infty} a_n$, and say that L is the sum of the infinite series $\sum_{n=m}^{\infty} a_n$. If the partial sums S_N diverge, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is *divergent*, and we do not assign any real number value to that series.

Now we ask the question, when does the series converge?

- **"Tail" of the sequence is eventually small**
  Proposition 7.2.5 Let $\sum_{n=m}^{\infty} a_n$ an be a formal series of real numbers. Then $\sum_{n=m}^{\infty} a_n$

an converges if and only if, for every real number $\varepsilon > 0$, there exists an integer $N > m$ such that

$$\sum_{n=p}^{q} a_n \leq \varepsilon \quad \text{for all } p, q \geq N$$

.

- **Zero test** : Let $\sum_{n=m}^{\infty} a_n$ an be a convergent series of real numbers. Then we must have $\lim_{n\to\infty} a_n = 0$. To put this another way, if $\lim_{n\to\infty} a_n$ is non-zero or divergent, then the series $\sum_{n=m}^{\infty} a_n$ is divergent.

- If a sequence $(a_n)_{n=m}^{\infty}$ does converge to zero, then the series $\sum_{n=m}^{\infty} a_n$ may or may not be convergent; it depends on the series.

Definition 7.2.8 (Absolute convergence). Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. We say that this series is absolutely convergent iff the series $\sum_{n=m}^{\infty} |a_n|$ is convergent.

2.2 Convergence of functions

Definition (Pointwise Convergence)
Definition (Uniform Convergence)
Definition 14.5.2 (Infinite series). Let (X, d_X) be a metric space. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from X to R, and let f be another function from X to R. If the partial sums $\sum_{n=1}^{N} f^{(n)}$ converge pointwise to f on X as $N \to \infty$, we say that the infinite series $\sum_{n=1}^{\infty} f^{(n)}$ **converges pointwise** to f, and write $f = \sum_{n=1}^{\infty} f^{(n)}$.
If the partial sums $\sum_{n=1}^{N} f^{(n)}$ converge uniformly to f on X as $N \to \infty$, we say that the infinite series $\sum_{n=1}^{\infty} f^{(n)}$ **converges uniformly** to f, and again write $f = \sum_{n=1}^{\infty} f^{(n)}$. (Thus when one sees an expression such as $\sum_{n=1}^{\infty} f^{(n)}$, one should look at the context to see in what sense this infinite series converges.)

The following theorem can be found in the work of Mr.Cauchy: If the various terms of the series $u0 + u1 + u2 + \cdots$ are continuous functions, then the sum s of the series is also a continuous function of x. But it seems to me that this theorem admits exceptions. For example the series $sinx\frac{1}{2}sin2x + \frac{1}{3}sin3x \cdots$ is discontinuous at each value $(2m + 1)\pi$ of x. - (Abel 1826, Oeuvres, vol. 1, p. 224-225)

The Cauchy-Bolzano era (first half of 19th century) left analysis with two important gaps: first the concept of uniform convergence, which clarifies the limit of continuous functions and the integral of limits; second the concept of uniformcontinuity, which ensures the integrability of continuous functions. Both gaps were filled byWeierstrass and his school (second half of 19th century).

The Limit of a Sequence of Functions
We consider a sequence of functions $f_1, f_2, f_3, \cdots : A \to R$. For a chosen $x \in A$ the values $f_1(x), f_2(x), f_3(x), \cdots$ are a sequence of numbers. If the limit

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{3}$$

exists for all $x \in A$, we say that $\{f_n(x)\}$ converges pointwise on A to $f(x)$.
Cauchy announced in his Cours (1821, p. 131; Oeuvres II.3, p. 120) that if (3) converges

for all x in A and if all $f_n(x)$ are continuous, then $f(x)$i s also continuous. Here are four counterexamples to this assertion; the first one is due to Abel.

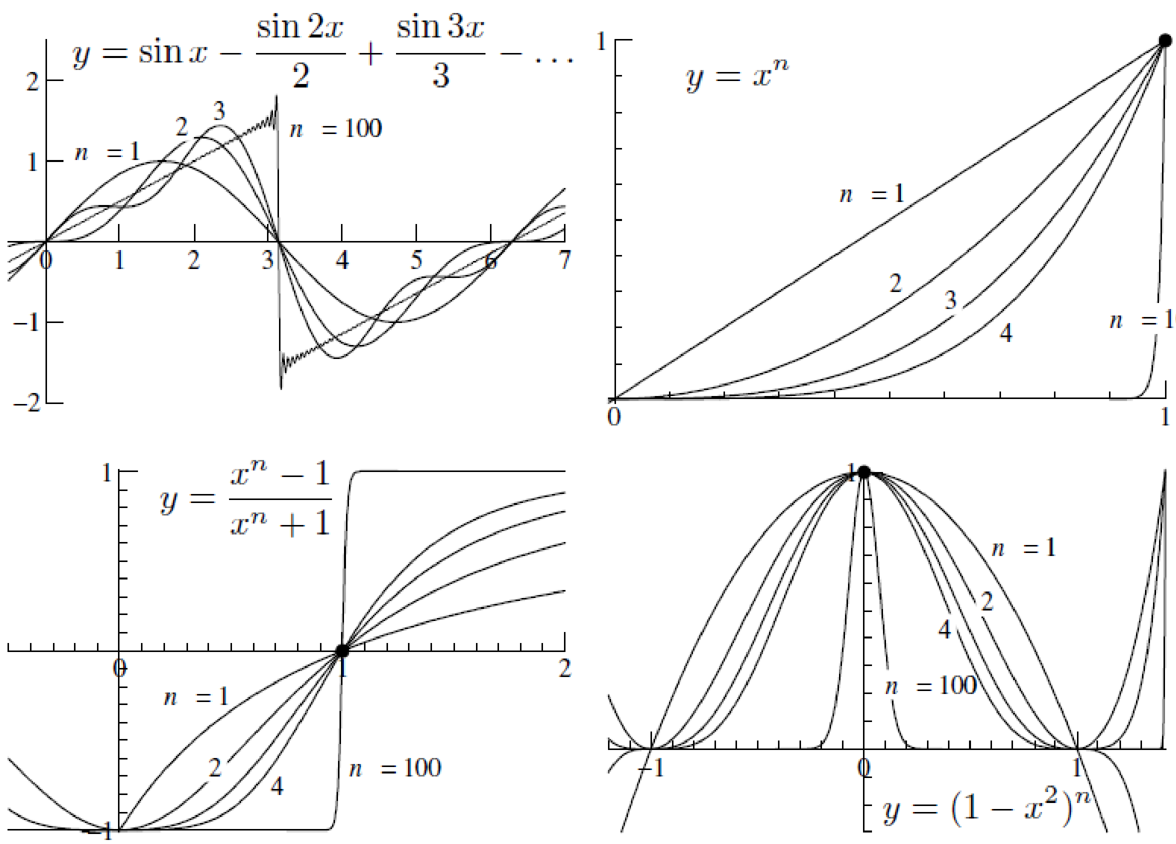


Figure 1: Sequences of continuous functions with a discontinuous limit

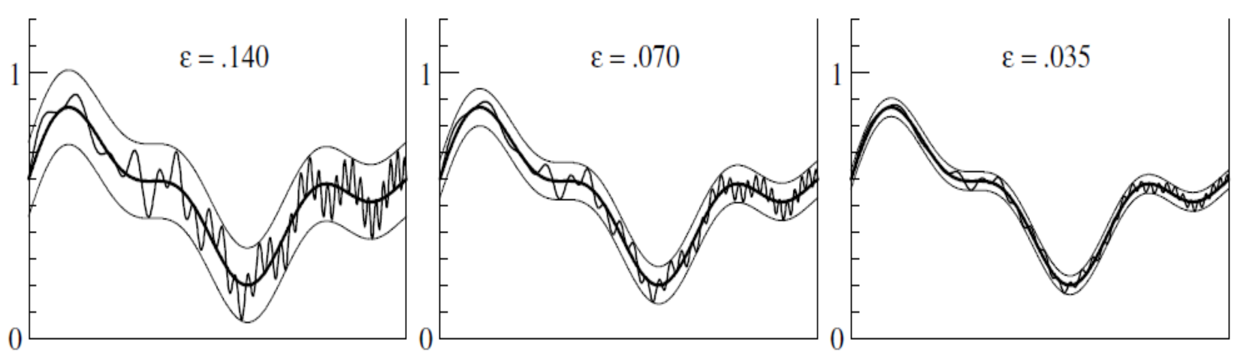


Figure 2: Sequence of uniformly convergent functions

We look at the upper right picture of Fig.1. The closer x is chosen to the point $x = 1$, the slower is the convergence and the larger we must take n in order to obtain the prescribed precision ε. This allows the discontinuity to be created. We must therefore require that, for a given $\varepsilon > 0$, the difference $f_n(x) f(x)$ be smaller than ε for all $x \in A$, if, of course, $n \geq N$.

Theorem (Weierstrasss lectures of 1861): If $f_n : A \to R$ are continuous functions and if $f_n(x)$ converges uniformly on A to $f(x)$, then $f : A \to R$ is continuous.
Proof:

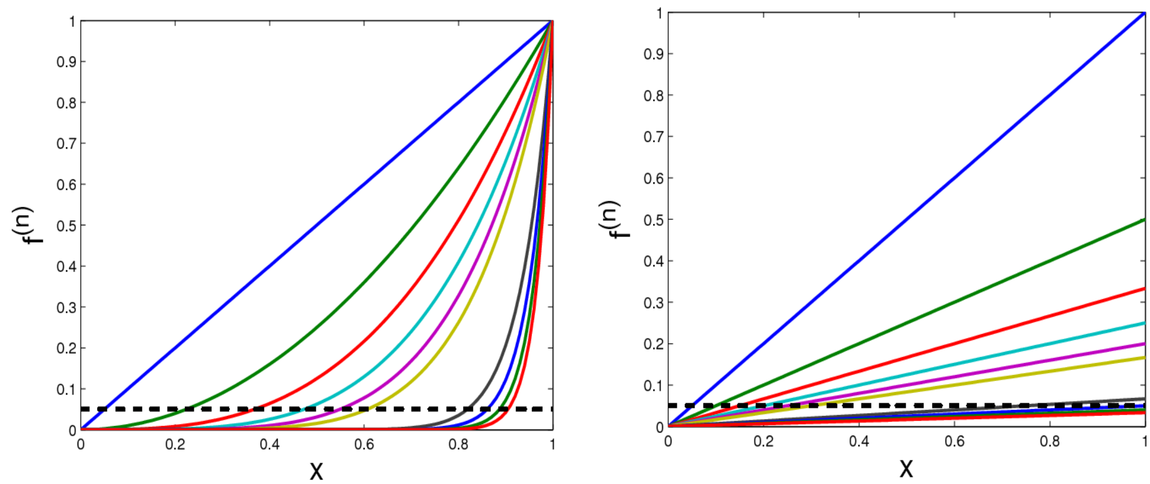


Figure 3: Uniform convergence of continuous functions to continuous function.

Let us compare the two figures to understand what is the behaviour of continous function in uniform convergence that the limit function is also continuous.

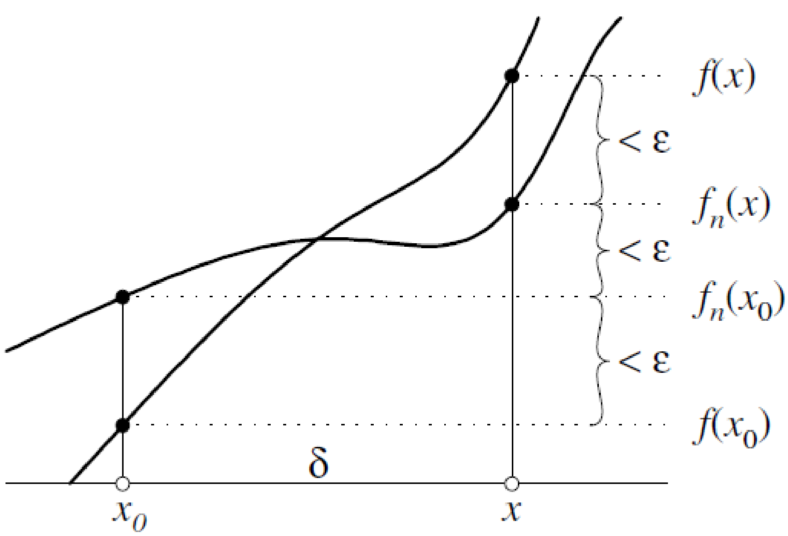


Figure 4: Uniform convergence of continuous functions to continuous function.

3 Uniqueness of Fourier Series

If we were to assume that the Fourier series of functions f converge to f in an appropriate sense, then we could infer that a function is uniquely determined by its Fourier coefficients. This would lead to the following statement: if f and g have the same Fourier coefficients, then f and g are necessarily equal. By taking the difference $f - g$, this proposition can be reformulated as: if $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$.
As stated, this assertion cannot be correct without reservation, since calculating Fourier coefficients requires integration, and we see that, for example, any two functions which differ at finitely many points have the same Fourier series.
However, we do have the following positive result.
Theorem 2.1 Suppose that f is an integrable function on the circle with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(\theta_0) = 0$ whenever f is continuous at the point θ_0.
Proof: It is given that $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Without loss of generality let us consider a point at which the function is continuous, say, $\theta_0 = 0$. We need to show that $f = 0$ at this point. We prove this by contradiction.
Let us consider that the function is such that $f(0) \neq 0$. It is continuous at 0.

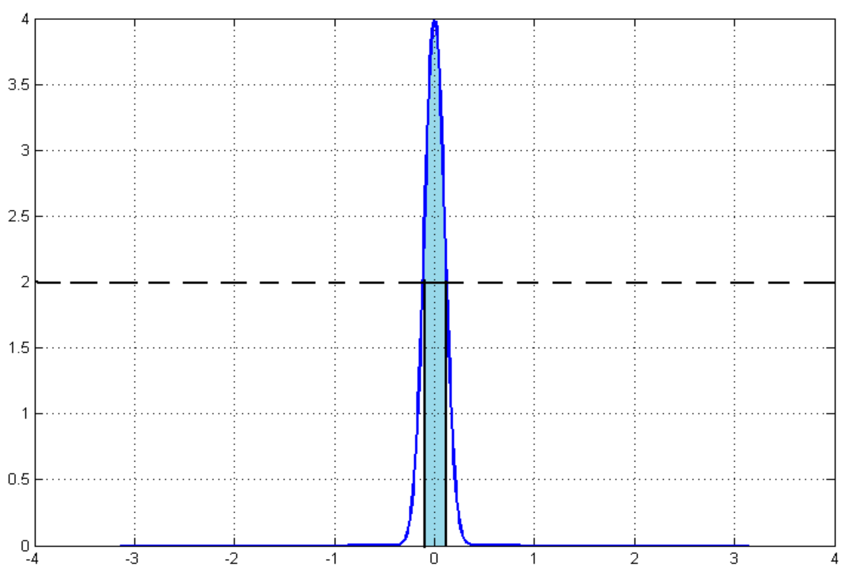


Figure 5: Function continuous at 0 and $f(0) \neq 0$

The intuition behind the contradiction is that, if the function is continous at a point, say 0, then, we will always be able to find a finite area under the curve $f(x)\cos(x)$ because of the continuity condition. Therefore, the coefficients will not be zero. Hence resulting in a contradiction. In the proof below we make this intuition rigourous. Thus if the coeefficients are all zero then the function should be equal to 0 at all points which are continuous.

Corollary: If f is continuous on the circle and $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$.

Corollary: Suppose that f is a continuous function on the circle and that the Fourier series of f is absolutely convergent, $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Then, the Fourier series converges uniformly to f, that is,

$$\lim_{N\to\infty} S_N(f)(\theta) = f(\theta) \qquad \text{uniformly in } \theta \tag{4}$$

Before we proceed to the proof, let us first review some of the definition and the properties of series and convergence.

4 Term-by-term integration of nonuniformly converging series

References

[1] Real analysis and Probability, Dudley - Chapter 2.

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[3] Analysis II, Terrance Tao - Chapter 13

[4] Fourier Analysis : An Introduction, Princeton Lectures in Analysis, Elias.M.Stein and Rami Shakarchi