Lebesgue's Integration Theory: Its Origin and Development

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Chapter 1

Introduction

We are essentially interested in coming up with a definition of integration such that we will be able to include as big a class of functions as possible under this definition as integrable functions. This definition of integration should also be consistent with our intuition of area under the curve.
If we look at the development of the concept of integration, we see that the criteria of integrability was the one which underwent changes to include more and more functions in the class of integrable functions.
Each time a mathematician came up with a criteria for integrability, there were a class of functions which were the troublemakers. These trouble-makers were identified and then the criteria for integrability was modified.
During 18th century, function itself was not defined properly. Function was defined as an analytical expression. As the definition of function underwent refinement and as and when examples of all sorts of weird functions were discovered, the definition of integration started falling apart. Rather, it was realized, that a very small class of functions are integrable.
Let us illustrate this process of development of the definition of integration below.
Example 1: Let us consider the function to be defined as a expression giving a single value for all values of x. Functions are those correspndances between x and $f(x)$ of the form f(x)= p, for some value $p \in R$. When the definition of function is as above the integration of a function bounded function between $(a, b), b > a$ is defined as

$$\int_b^a f(x) = f(x)(b-a) \tag{1.1}$$

Under such a definition, integration is defined only for functions which take a single value for all values of x. Now suppose functions of the form shown in figure 1.1(b) was discovered at a later stage or say the functions of the above form we admitted to be 'functions' , then for such functions, integration was not defined and hence the definition of integration had to be modified.
Suppose functions are of the form they take finite set of values, say y_i, over the regions δ_i, then, integration is defined as

$$\int_b^a f(x) = \sum_{i=1}^{L} y_i \delta_i \tag{1.2}$$

But again, when functions were discovered to be of the from shown in 1.1(c) or those where in L in the above definition was infinite, then again the above definition of integration would not hold.

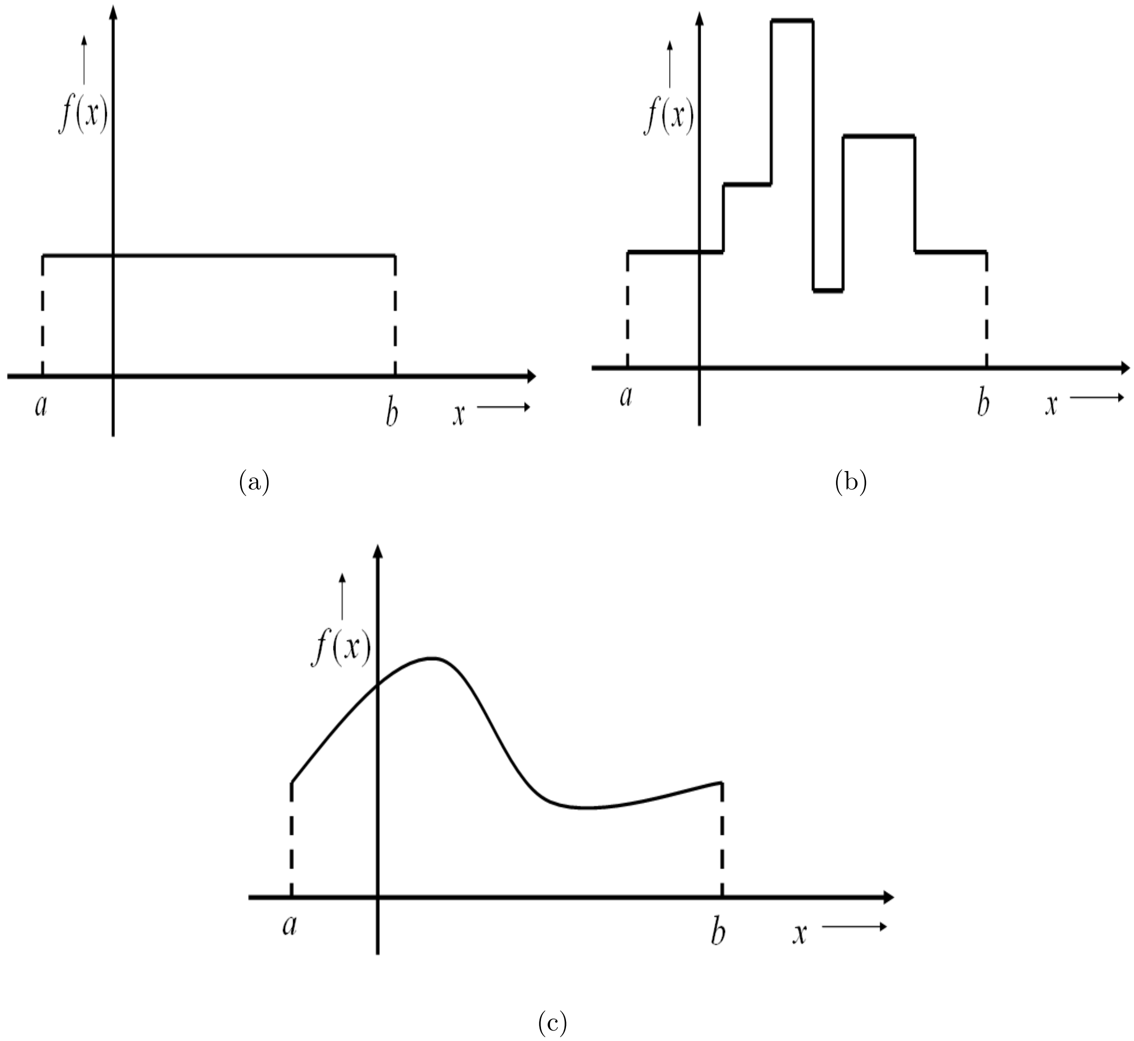

Figure 1.1: Illustration of development of definition of function and integration

Below sections we look at development of defintion of functions and definition of integration. One other concept whose developemnt is closely related to the defintion of area. As we will see later in this lecture, integration is closely dependent on the way area, or more specifically, how measure is defined.

1.1 Euler : Arbitrary functions

Although the notion of a function did not originate with Euler, it was he who first gave it prominence by treating the calculus as a formal theory of functions. In 1748 he defined a

function of a variable quantity as "an analytical expression" composed in any way of that variable and constants. The key to this definition is the notion of an analytical expression, which Euler evidently understood to be the common characteristic of all known functions.
Arbitrary functions: It was also Euler, however, who initiated a viewpoint that eventually led to the introduction of the modern concept of a function. In his pioneering study of partial differential equations of 1734, Euler *admitted "arbitrary functions" into the integral solutions.* And, in answer to Jean d'Alembert-who maintained that these arbitrary functions must be given by a single algebraic or transcendental equation in order to be the proper object of mathematical analysis-Euler clarified his earlier pronouncement by contending that the **curves which the arbitrary functions represent need not be subject to any law but may be "irregular" and "discontinuous," i.e., formed from the parts of many curves or traced freehand in the plane.**
But note that, at that point of time, attention was focused upon the fact that arbitrary functions are not determined by a single equation rather than upon their properties as correspondences $x \to f(x)$ between real numbers.

1.2 Fourier: Trigonometric representation of 'arbitrary functions'

Fourier's proposition can be stated as:
Any (bounded) function f defined on $(-a, a)$ can be expressed in the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n\text{cos}(n\pi x/a) + b_n\text{sin}(n\pi x/a)\} \tag{1.3}$$

where the coefficients are given by

$$a_n = \frac{1}{a}\int_a^b f(x)\text{cos}(n\pi x/a)dx \tag{1.4}$$

$$b_n = \frac{1}{a}\int_a^b f(x)\text{sin}(n\pi x/a)dx \tag{1.5}$$

Let us consider

1.3 Integral Calculus

The integral calculus is, in fact, much older than the differential calculus, because the computation of areas, surfaces, and volumes occupied the greatest mathematicians since antiquity: Archimedes, Kepler, Cavalieri, Viviani, Fermat, Gregory St.Vincent, Guldin, Gregory, Barrow. The decisive breakthrough came when Newton, Leibniz, and Joh.Bernoulli discovered independently that integration is the inverse operation of differentiation, thus reducing all efforts of the above researchers to a couple of differentiation rules. The integral sign is due to Leibniz (1686), the term integral is due to Joh.Bernoulli and was published by his brother Jac.Bernoulli (1690).

The integral is given by the formula

$$\int_a^b f(x)dx = F(b)F(a) \tag{1.6}$$

where $F(x)$ is a primitive of $f(x)$. We have implicitly assumed that such a primitive always exists and is unique (up to an additive constant). Here, we will give a precise definition of $\int_a^b f(x)dx$ independent of differential calculus. This allows us to interpret $\int_a^b f(x)dx$ for a larger class of functions, including discontinuous functions or functions for which a primitive is not known.

- **Cauchy**(1823) described, as rigorously as was then possible, the integral of a continuous function as the limit of a sum.
- **Riemann** (1854), merely as a sideremark in his habilitation thesis on trigonometric series, defined the integral for more general functions.
- Riemann's work was further extended by **Du Bois-Reymond** and **Darboux**.
- Still more general theories, are due to **Lebesgue** (in 1902) and **Kurzweil** in 1957.

1.3.1 Uniform convergence and term-by-term integration

Theorem: Consider a sequence $f_n(x)$ of integrable functions and suppose that it converges uniformly on $[a, b]$ to a function $f(x)$. Then $f : [a, b] \to R$ is integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx \tag{1.7}$$

Corollary: Consider a sequence $f_n(x)$ of integrable functions and suppose that the series

$$\sum_{n=0}^{\infty} \int_a^b f_n(x)dx = \int_a^b \sum_{n=0}^{\infty} f_n(x)dx \tag{1.8}$$

Darboux was familiar with the papers of Heinee and Cantor and supplied an elegant proof which established both the integrability of a uniformly convergent series of integrable functions and the validity of term-by-term integration. To indicate what can happen when a series converges nonuniformly, Darboux introduced the series $\sum_{n=1}^{\infty} u_n(x)$, where

$$u_n(x) = -2n^2 x e^{-n^2x^2} + 2(n+1)^2 x e^{-(n+1)^2x^2}$$

This series converges to the continuous function $-2xe^{-x^2}$, which, when integrated over $[0, x]$, yields

$$\int_0^x (-2te^{-t^2})dt = e^{-x^2} - 1$$

But term-by-term integration of the series over $[0, x]$ yields

$$\sum_{n=1}^{\infty} \int_0^x u_n(t)dt = [e^{-n^2x^2} - e^{-(n+1)^2x^2}] = e^{-x^2}$$

The series fails to converge uniformly because, as Darboux realized, $|R_n(1/n)| = 2n/e$, where $R_n(x) = \sum_{n=0}^{\infty} u_k(x)$. In other words, the series is **not uniformly bounded** on $[0, x]$. (A series $u(x) = sum_{n=0}^{\infty} u_k(x)$ or, equivalently, the sequence $S_n(x) = \sum_{n=1}^{n} u_k(x)$, is said to be uniformly bounded on $[a, b]$ if there exists a number B such that $|S_n(x)| < B$ for all x in $[a, b]$ and all n.) But Darboux did not introduce the **notion of uniform boundedness** and apparently had no idea of **its importance for term-by- term integration.**

1.3.2 Negligible Sets

1.4 Cantor Set

e Cantor set is a set of points lying on a single line segment that has a number of remarkable and deep properties. It was discovered in 1874 by Henry John Stephen Smith and introduced by German mathematician Georg Cantor in 1883.

Through consideration of it, Cantor and others helped lay the foundations of modern general topology. Although Cantor himself defined the set in a general, abstract way, the most common modern construction is the Cantor ternary set, built by removing the middle thirds of a line segment. Cantor himself only mentioned the ternary construction in passing, as an example of a more general idea, that of a perfect set that is nowhere dense.

1.4.1 Construction and Formula of the ternary set

The Cantor ternary set is created by repeatedly deleting the open middle thirds of a set of line segments. One starts by deleting the open middle third $(13, 23)$ from the interval $[0, 1]$, leaving two line segments: $[0, 13] \cup [23, 1]$. Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $[0, 19] \cup [29, 13] \cup [23, 79] \cup [89, 1]$. This process is continued ad infinitum. The Cantor ternary set contains all points in the interval [0, 1] that are not deleted at any step in this infinite process.

1.4.2 Properties

Composition

Proposition: Prove that Cantor set is non-empty.
Proof: In each step, the middle third portion is removed. It is an open set and leaves behind the end points. Again within each interval the middle third portion is removed, thus the previous end points remain. Therefore the set is not empty.

Proposition: Prove that the Cantor set does not contain only endpoints of intervals.

Cardinality

Proposition: Cantor set is uncountable.
Proposition: Cantor ternary set has zero measure.

Example of nowhere dense set with positive measure

Perfect Nowhere dense set

Proposition: Cantor Set is perfect set which is nowhere dense.
Proof: Nowhere dense set is defined as the set whose closure has empty interior. That means that we need to show that there the closure of the set does not contain any interior points. Since each each time we remove middle third of open interval, what is left behind is union of closed intervals. Now we know that finite union of closed set is true. It is just that we cannot say arbitrary unions are closed in every case. But in this case we can show that the set closed because the set contains all its boundary points. The set is closed. That means the closure of the set is the set itself.
We now show that the interior of the set is empty because, for no $\epsilon > 0$, can you find a ball $B(x_0, \epsilon)$ such that $B \cap C \in C$, where C is the Cantor set. Because whichever ball you consider around a point x_0, the middle third portion of the interval will not belong to the set. Therefore the Cantor set does not contain any interior point.

1.5 Lipschitz

Lipschitz began by listing the possible ways in which a function could fail to satisfy the conditions under which Dirichlet had proved his theorem. Essentially, Dirichlet required the function to be monotonic in a small interval around x and also it had to have finite number of discontinuities in the interval. Therefore, one way where the **function would fail** was if a bounded and piecewise monotonic function f was **discontinuous at an infinite number of points** in the interval $[-\pi, \pi]$.

Lipschitz realized that Dirichlet's proof would apply if the **integral concept could be extended to** f, and he concluded that this could easily be accomplished for functions **satisfying Dirichlet's condition** on the set D of **discontinuities** (D is **nowhere dense**).

The reason the extension seemed simple to him is that he believed that by "an appropriate argument" it could be shown that if D is nowhere dense, then D', the set of limit points of D, must be finite.

1.6 Differentiation

The fundamental theorem of calculus is a theorem that links the concept of the derivative of a function with the concept of the integral.

The first part of the theorem, sometimes called the **first fundamental theorem of calculus**, shows that an indefinite integration can be reversed by a differentiation. This part of the theorem is also important because it guarantees the existence of antiderivatives for continuous functions.
The process of solving for antiderivatives is called antidifferentiation (or indefinite integration) and its opposite operation is called differentiation, which is the process of finding a derivative.

Let us now understand the concepts with respect to metric space. Let (X, d_X) and (Y, d_Y) be the metric spaces. Let us consider a set $\mathcal{F}$ of all functions from X to Y, that is, $\mathcal{F} = \{f | f : X \to Y\}$.
Let us consider a map (we can call it function because every function $f \in \mathcal{F}$ has only one derivative) called derivative $\mathcal{D} : \mathcal{F} \to \mathcal{F}$. Suppose $f'(x) = \mathcal{D}(f(x))$, the derivative at a point x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1.9}$$

If the limit exists for all values of x, then the function is said to be differentiable function. An **antiderivative, primitive integral or indefinite integral** of a function f is a differentiable function F whose derivative is equal to f, i.e., $F' = f$.

Does integrability guarantee the existence of antiderivative?
Suppose a function f is said to be integrable on $[a, b]$, that means, it satisfies the integrability criterion (that may be Cauchy criterion or Riemann criterion). Now we want to know if there exists a function F such that $D(F) = f$?

We could also put it the other way round.

What conditions on a function F on $[a, b]$ guarantee that $D(F)$ exists, that this function is integrable and that moreover

$$F(b) - F(a) = \int_a^b D(F)(x)dx \tag{1.10}$$

Antiderivatives are related to definite integrals through the fundamental theorem of calculus: the definite integral of a function over an interval is equal to the difference between the values of an antiderivative evaluated at the endpoints of the interval.
The second part, sometimes called the **second fundamental theorem of calculus**, allows one to compute the definite integral of a function by using any one of its infinitely many antiderivatives. This part of the theorem has invaluable practical applications, because it markedly simplifies the computation of definite integrals.

Theorem: First fundamental theorem of Calculus Let $a < b$ be real numbers, and let $f : [a, b] \to R$ be (Riemann) integrable function. Let $F : [a, b] \to R$ be the function

$$F(x) = \int_{[a,x]} f \tag{1.11}$$

Then F is continuous. Furthermore, if $x_0 \in [a, b]$ and f is continuous at x_0, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.
Informally, the first fundamental theorem of calculus asserts that

$$\left(\int_{[a,x]} f \right)'(x) = f(x) \tag{1.12}$$

given a certain number of assumption on f. Roughly, this means that the derivative of an integral recovers the original function.
Now we show the reverse, that the integral of a derivative recovers the original function.
Definition (Antiderivatives) . Let I be a bounded interval, and let $f : I \to R$ be a function. We say that a function $F : I \to R$ is an antiderivative of f if F is differentiable on I and $F'(x) = f(x)$ for all $x \in I$.
Theorem: Second Fundamental Theorem of Calculus Let $a < b$ be real numbers, and let $f : [a, b] \to R$ be a (Riemann) integrable function. If $F : [a, b] \to R$ is an antiderivative of f,then

$$\int_{[a,b]} f = F(b) - F(a) \tag{1.13}$$

Informally, second theorem asserts that

$$\int_{[a,b]} F' = F(b) - F(a) \tag{1.14}$$

that is, integral of a derivative recovers the original function. We essentially want to understand the conditions under which the two theorems exists. Also more importantly, the question was **when is a function differentiable?** Just like how we had the question about integration.

Initially, until a counter example was discovered, it was believed that all continuous functions are differentiable.

Chapter 2

Development of Riemann's Ideas

Chapter 3

Set Theory and the Theory of Integration

3.1 Nowhere Dense sets and their measure theoretic properties

3.1.1 Nowhere dense sets with positive content

There were two instances when the idea of nowhere dense sets with positive outer content came to light

- Dini had rejected Hankel's proof that a**pointwise discontinuous function is necessarily integrable**, but was unable to construct an example showing that not only the proof but also the proposition itself is erroneous. (Such a counterexample amounts to constructing a nowhere dense set of positive outer content.)
- In connection with the **existence of bounded, nonintegrable derivatives**, Dini had posited the existence of continuous, nonconstant functions with densely distributed intervals of invariability and, thereby, the existence of nowhere dense sets of positive outer content, although he was again unable to produce an example.

3.1.2 Volterra's example

Volterra's discovery of nowhere dense sets of positive outer content confirmed Dini's opinions. **Volterra's example of nowhere dense set:** The interval $[0, 1]$ is divided into an infinite number of intervals by the sequence $1 > c_1 > c_2 > \cdots$, where:

1. $c_n \to 0$
2. $1 - c_1 = \frac{1}{2^{2.1}}(1 - 0)$
3. $c_n - c_{n+1} < 1 - c_1$

The interval $(c_1, 1)$ is excluded from further subdivision and the same procedure is applied to each of the (c_{n+1}, c_n). That is, let $c_n > c_{n,1} > c_{n,2} > \cdots > c_{n,k} > \cdots$ where:

1. $c_{n,k} \rightarrow c_{n+1}$
2. $c_n - c_{n,1} = \frac{1}{2^{2.2}}(c_n - c_{n+1})$
3. $c_{n,k} - c_{n,k+1} < c_n - c_{n,1}$.

In this manner each of the intervals (c_{n+1}, c_n) is divided into an infinity of intervals $(c_{n,k}, c_{n,k+1})$, for $k = 1, 2, 3, \cdots$. The same procedure is then applied to each of these intervals the interval $(c_{n,1}, c_n)$ being excluded from further subdivision-and so on ad infinitum.
Let G denote the set of points of subdivision, that is, the c's together with 0 and 1, and let $\overline{G}$ denote the set of points of G together with all its limit points. Show that G and $\overline{G}$ are then examples of nowhere dense sets.
Proof:
Definition (Limit point of a set): Let X be subset of real line. We say that x is a limit point (or a cluster point) of X iff it is an adherent point of $X \backslash \{x\}$. We say that x is an isolated point of X if $x \in X$ and there exists some $\epsilon > 0$ such that $|x - y| > \epsilon$ for all $y \in X \backslash \{x\}$.
Definition (Adherent Point): Let X be a subset of R, and let $x \in R$. Then x is an adherent point of X if and only if there exists a sequence $(a_n)_{n=0}^{\infty}$ consisting entirely of elements in X, which converges to x.
Definition (Nowhere dense set): Nowhere dense set is defined as set whose closure has empty interior.

To show that the closure of G, that is, $\overline{G}$ is nowhere dense, we need to show that the interior is of $\overline{G}$ is empty. That is, there does not exists any $x \in \overline{G}$ which is an interior point.
Definition (Interior Point): Let X be a subset of real line. A point $x \in X$ is said to be an interior point if there exists a ball $\mathcal{B}(x, \epsilon)$ for $\epsilon > 0$ such that $\mathcal{B}(x, \epsilon) \cap X \neq \emptyset$

3.1.3 Volterra's example of nonintegrable bounded derivative

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