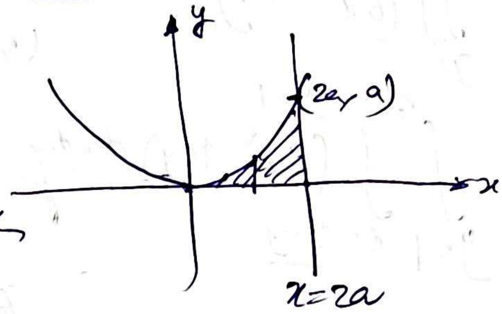
Problem Sheet-12

1) (i) $\iint_D xy\, dA$



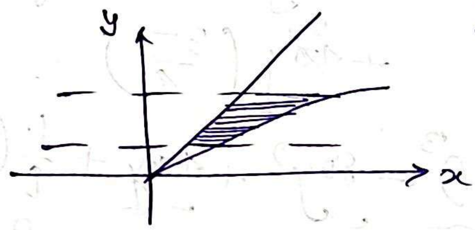
$$= \int_{x=0}^{2a} \int_{y=0}^{x^2/4a} xy\, dy\, dx$$

$$= \int_{x=0}^{2a} \frac{x y^2}{2}\Big|_0^{x^2/4a} dx$$

$$= \int_{x=0}^{2a} \frac{x}{2} \cdot \frac{x^4}{16a^2}\, dx = \int_{x=0}^{2a} \frac{x^5}{32a^2}\, dx = \frac{x^6}{32 \times 6a^2}\Big|_0^{2a}$$

$$= \frac{2\cancel{64}\, a^6}{\cancel{32} \times 6a^2} = \frac{1}{3} a^4$$

(ii) $\iint_D e^{x/y}\, dA$



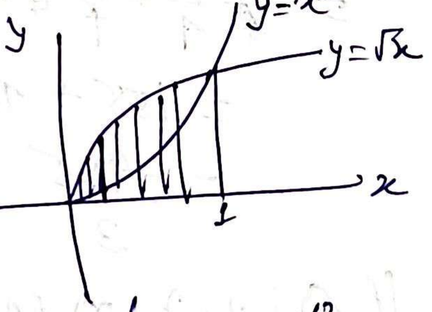
$$\int_{y=1}^{2} \int_{x=y}^{y^3} e^{x/y}\, dx\, dy$$

$$= \int_{y=1}^{2} y e^{x/y}\Big|_y^{y^3} dy = \int_1^2 y(e^{y^2} - e)\, dy = \frac{1}{2} e^{y^2}\Big|_1^2 - \frac{e y^2}{2}\Big|_1^2$$

$$= \frac{1}{2}(e^4 - e) - \frac{e}{2}(4-1)$$

$$= \frac{1}{2}(e^4 - e) - \frac{3e}{2} = \frac{e^4}{2} - 2e$$

(iii) $\iint_D (4xy - y^3)\, dA$



$$\int_{x=0}^{1} \int_{y=\sqrt{x}}^{x^3} (4xy - y^3)\, dy\, dx$$

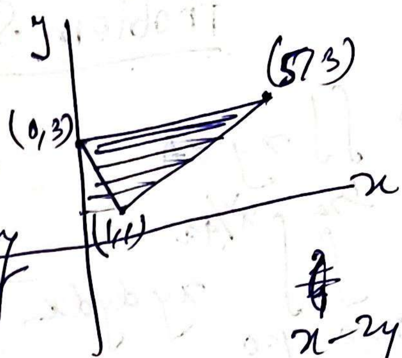
$$= \int_0^1 \left(2xy^2 - \frac{y^4}{4}\right)\Big|_{\sqrt{x}}^{x^3} dx = \int_0^1 \left(2x^7 - \frac{x^{12}}{4} - 2x^2 + \frac{x^2}{4}\right) dx$$

$$= \frac{2}{8} - \frac{1}{4 \times 13} - \frac{2}{3} + \frac{1}{12}$$

$$= -\frac{55}{156}$$

$\frac{xy^2}{x}$ $\quad x \quad x^{1/2}$

(iv) $\iint_D (6x^2-40y)\,dA$



$$= \int_{y=1}^{3}\int_{x=\frac{3-y}{2}}^{2y-1} (6x^2-40y)\,dx\,dy$$

$x-2y=-1$

$1-2=-1$

$2x+y=3$

$2\frac{6x^3}{3}$

$$= \int_{y=1}^{3} 2x^3-40xy\Big|_{\frac{3-y}{2}}^{2y-1}\,dy$$

$$= 2(2y-1)^3 - 40y(2y-1) + \frac{2}{8}(\cancel{3-y}\;y-3)^3 + \cancel{40}\,20y\left(\frac{3-y}{\cancel{2}}\right)$$

$$= \int_1^3 2(2y-1)^3 - 80y^2 + 40y + \frac{1}{4}(y-3)^3 + 60y - 20y^2\,dy$$

$$= \int_1^3 \left\{2(2y-1)^3 + \frac{1}{4}(y-3)^3 - 100y^2 + 100y\right\}dy$$

$$= \frac{(2y-1)^4}{4} + \frac{(y-3)^4}{16} - \frac{100y^3}{3} + 50y^2\Big|_1^3$$

$$= \frac{5^4}{4} - \frac{1}{4} - 1 - \frac{100}{3}\times 26 + 50\times 8$$

16 $\quad 8$

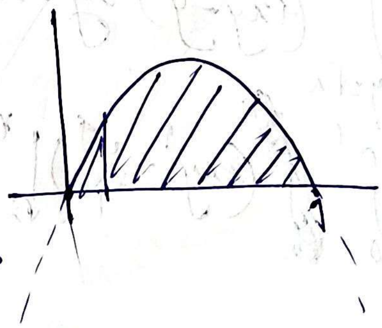
$2^4=16$

$3^3=27$

$$= 155 + 400 - \frac{2600}{3}$$

$$= \frac{-935}{3}$$

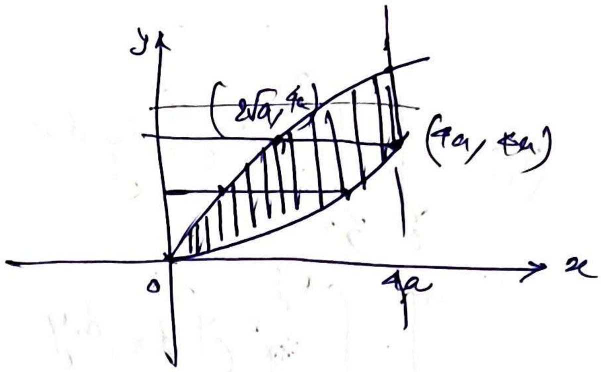
(v) $\iint_D (x^2+2xy^2+2)\,dA$



$$\int_{x=0}^{1}\int_{y=0}^{x-x^2} (x^2+2xy^2+2)\,dy\,dx$$

$$= \int_0^1 (x^2+2)(x-x^2) + \frac{2x}{3}(x-x^2)^3\,dx = \frac{27}{70}$$

27) (i) $\displaystyle\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{a}\,x} dy\, dx$



$2\sqrt{a}\cdot 4a$

$2\sqrt{a}x = 4a$

$x = 2\sqrt{a}$

$= \displaystyle\int_{y=0}^{4a}\int_{x=\frac{y}{2\sqrt{a}}}^{\sqrt{4ay}} dx\,dy$

$+ \displaystyle\int_{y=4a}^{8a\sqrt{a}}\int_{x=\frac{y}{2\sqrt{a}}}^{4a} dx\,dy$

$= \displaystyle\int_0^{4a}\left(2\sqrt{a}\sqrt{y} - \frac{y}{2\sqrt{a}}\right)dy + \int_{4a}^{8a\sqrt{a}}\left(4a - \frac{y}{2\sqrt{a}}\right)dy$

$= \frac{2}{3}2\sqrt{a}\,y^{3/2}\Big|_0^{4a} + 4a(8a\sqrt{a} - 4a) - \frac{1}{4\sqrt{a}}(8a\sqrt{a})^2$

$= \frac{4}{3}\sqrt{a}\,8\,a^{3/2} + 4a\cdot 4a(2\sqrt{a}-1) - \frac{1}{4\sqrt{a}}64a^3$

$2^{2\times 3/2}$

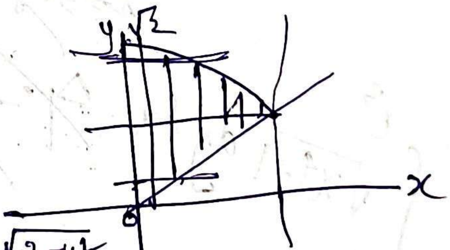
$= \frac{32}{3}a^2 + 16a^2(2\sqrt{a}-1) - 16a^2\sqrt{a}$

$\frac{3}{2}+\frac{1}{2}=\frac{4}{2}$

$= \frac{32a^2}{3} - 16a^2 + 16a^2\sqrt{a}$

$= 16a^2\sqrt{a} - \frac{16a^2}{3} = \frac{16}{3}a^2(3\sqrt{a}-1)$

(ii) $\displaystyle\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$



$y^2+x^2=2$

$= \displaystyle\int_{y=0}^{1}\int_{x=0}^{y}\frac{x}{\sqrt{x^2+y^2}}\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{\sqrt{2-y^2}}\frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$

$= \displaystyle\int_{y=0}^{1}\sqrt{x^2+y^2}\Big|_0^{y}dy + \int_1^2 \sqrt{x^2+y^2}\Big|_0^{\sqrt{2-y^2}}dy$

$\frac{1}{2}/\cancel{\ }\,2\sqrt{2}$

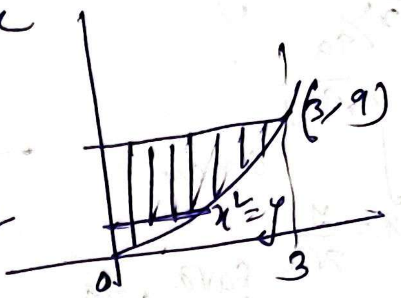
$= \displaystyle\int_0^1 \sqrt{2y^2} - \sqrt{y^2}\,dy + \int_1^2 (\sqrt{2} - \sqrt{y^2})\,dy$

$\sqrt{2y^2+y}$

$= (\sqrt{2}-1)\frac{y^2}{2}\Big|_0^1 + \sqrt{2}(2-1) - \frac{y^2}{2}\Big|_1^2$

$= \frac{\sqrt{2}-1}{2} + \sqrt{2} - \frac{3}{2} = \frac{3\sqrt{2}}{2} - 2.$

(iii) $\int_0^3\int_{x^2}^9 x^3 e^{y^3}\,dy\,dx$



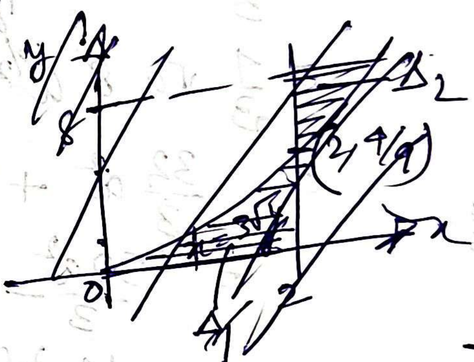
$$\int_{y=0}^{9}\int_{x=0}^{\sqrt{y}} x^3 e^{y^3}\,dx\,dy$$

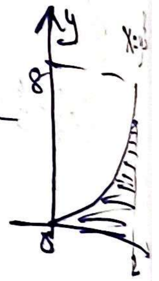
$$= \int_0^9 \frac{x^4}{4} e^{y^3}\Big|_0^{\sqrt{y}}\,dy$$

$$= \int_0^9 \frac{e^{y^3}(3y^2)}{4\times 3}\,dy = \frac{1}{12}\int_0^9 e^{y^3}(3y^2)\,dy$$

$$= \frac{1}{12} e^{y^3}\Big|_0^9 = \frac{1}{12}\left(e^{729}-1\right)$$

(iv) $\int_0^8\int_{y^{1/3}}^2 \sqrt{x^4+1}\,dx\,dy$





$\cancel{x=3\sqrt{y}}\ \ y = x^2/9$

$\cancel{x=3\sqrt{8}}$

$$\int_{x=0}^{2}\int_{y=0}^{x^3}\sqrt{x^4+1}\,dy\,dx \quad \cancel{= \int_{x=0}^{2}\int_{y=0}^{x^2/9}\sqrt{x^4+1}\,dy\,dx}$$

$$= \int_0^2 \frac{1}{4}\sqrt{x^4+1}\,4x^3\,dx \quad \cancel{= \int_0^2 (x^4+1)^{1/2}\frac{x^2}{9}\,dx}$$

$$= \frac{1}{4}(x^4+1)^{3/2}\times\frac{2}{3}\Big|_0^2$$

$\cancel{x^2 = \tan\theta}$

$\cancel{2x\,dx = \sec^2\theta\,d\theta}$

$$\cancel{\int_0^{\tan^{-1}(4)} \frac{\sec\theta\tan\theta}{9}\,\frac{\sec^2\theta\,d\theta}{2\sqrt{\tan\theta}}}$$

$$= \frac{1}{4}\cdot\frac{2}{3}\left[17^{1.5}-1\right]$$

$$= \frac{1}{6}\left[17^{1.5}-1\right]$$

$$\cancel{= \frac{1}{18}\int_0^{\tan^{-1}4}\sqrt{\tan\theta}\,\sec^3\theta\,d\theta}$$

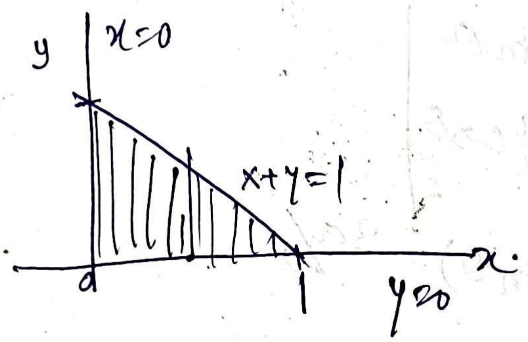
$\cancel{\frac{\sqrt{t}\,dt}{t}}$

3) $\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}}\, dy\, dx$

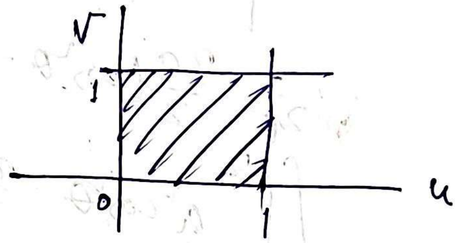
$x+y=u$
$y=uv$
$x+uv=u \Rightarrow x=u(1-v)$
$y=uv$

$$J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$=\begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = (1-v)u+uv$$
$$=u$$



$u(1-v)=0 \Rightarrow u=0,\ v=1$
$y=0 \Rightarrow v=0,$
$x+y=1 \Rightarrow u=1$



$$\int_0^1\int_0^1 u e^{\frac{uv}{u}}\, du\, dv$$

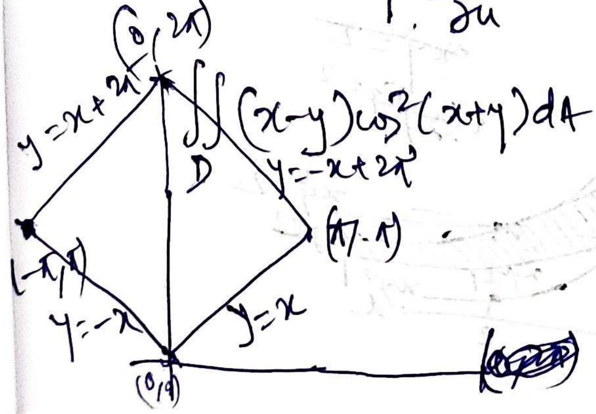
$$=\int_0^1\int_0^1 u e^{v}\, du\, dv = \frac{u^2}{2}\Big|_0^1 e^v\Big|_0^1$$
$$=\frac{1}{2}(e-1).$$

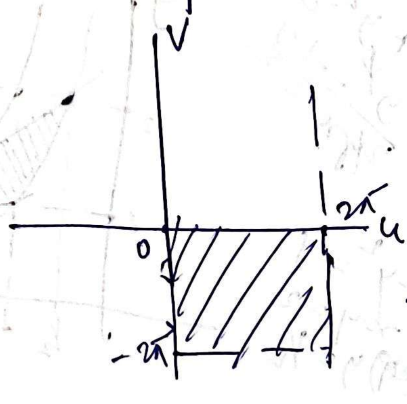
4) $T: x=\frac{1}{2}(u+v),\ y=\frac{1}{2}(u-v)$

$u=x+y$
$v=x-y$

$$J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{4}-\frac{1}{4}=-\frac{1}{2}$$

$\iint_D (x-y)\cos^2(x+y)\, dA$

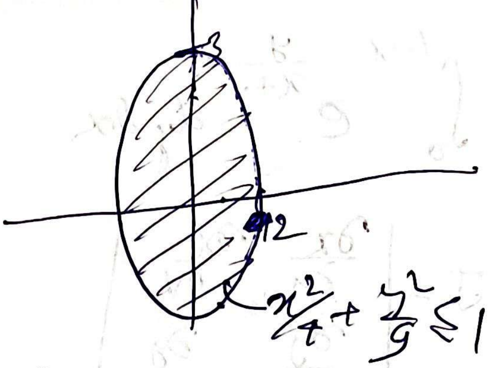




$y=x \Rightarrow v=0$
$y=x+2\pi \Rightarrow v=-2\pi$
$y=-x \Rightarrow u=0$
$y=-x+2\pi \Rightarrow u=2\pi$

$$\int_{-2\pi}^{0}\int_0^{2\pi} \frac{1}{2} v\cos^2 u\, du\, dv$$
$$=\frac{\pi}{4}(-4\pi^2)=-\pi^3$$

5) (i) $\iint_D x^2\,dx\,dy$



~~$x=2\cos$~~

$x = a\cos\theta,$
$y = \frac{3a}{2}\sin\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -a\sin\theta \\ \frac{3}{2}\sin\theta & \frac{3a}{2}\cos\theta \end{vmatrix}$$

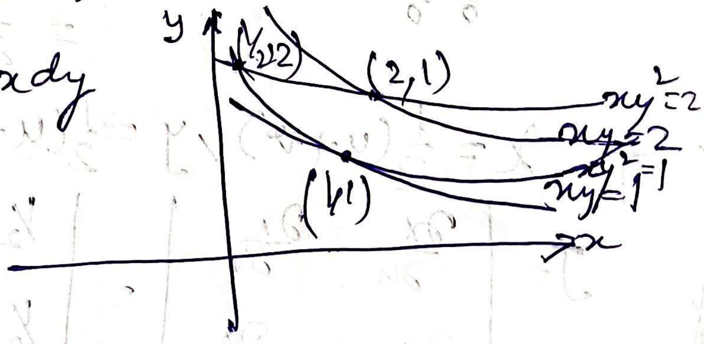
$$= \frac{3a}{2}(\cos^2\theta + \sin^2\theta) = 3a/2$$

$$\int_{\theta=0}^{2\pi}\int_{a=0}^{2} a^2\cos^2\theta \, \frac{3a}{2}\, da\, d\theta$$

$$= \pi \cdot \frac{3}{2}\, \frac{a^4}{4}\Big|_0^2$$

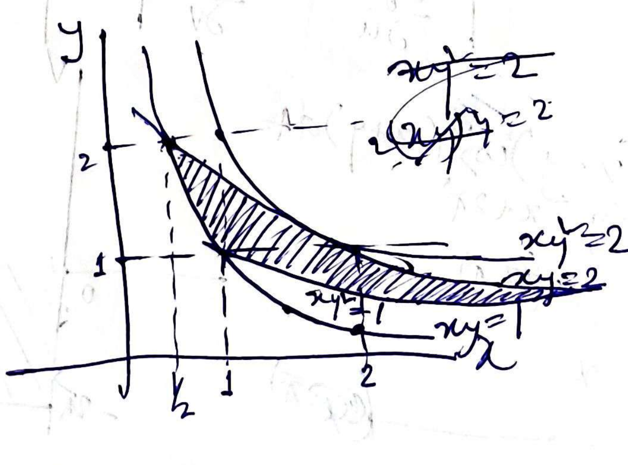
$$= \frac{3\pi}{2}\cdot\frac{16}{4}\cdot 2 = 6\pi$$

(ii) $\iint_D y^2\,dx\,dy$



$xy = 1$
$\Rightarrow xy' + y = 0$
$y' = -y/x$

$xy^2 = 1$
$\Rightarrow x(2yy') + y^2 = 0$
$\Rightarrow y' = \frac{-y^2}{2xy} = -\frac{1}{2}\frac{y}{x}$



$xy^2 = 1$
$xy = 2$
$y = 1/2$

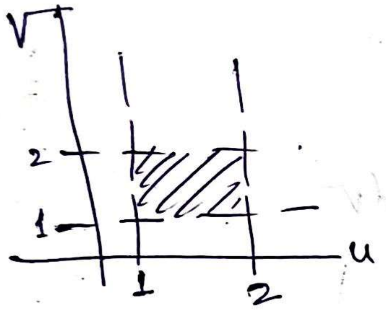
$u = xy,\ v = xy^2$

$$\Rightarrow xy \cdot y = v \Rightarrow y = v/u$$

$$x = \frac{u}{v/u} = u^2/v.$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix}$$

$$= \frac{2}{v} - \frac{1}{v} = 1/v$$



$$\int_1^2\int_1^2 \frac{v^{\cancel{2}}}{u^2} \frac{1}{\cancel{v}}\, du\, dv$$

$$= \frac{v^2}{2}\Big|_1^2 \ \frac{1}{u}\Big|_2^1$$

$$= \frac{3}{2}\left(1 - \frac{1}{2}\right) = \frac{3}{4}$$

(iii) $\iint_D (x+y)^2\, dx\, dy$

$u = x + y$

$v = 2x - y$

$\Rightarrow x = \frac{u+v}{3}$

$y = u - \frac{u+v}{3} = \frac{2u - v}{3}$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{vmatrix}.$$

$$= -\frac{1}{9} - \frac{2}{9} = -\frac{\cancel{3}1}{\cancel{9}3} = -1/3.$$

$$\int_0^3\int_0^1 u^2 \frac{1}{3}\, du\, dv = \frac{1}{3}\,\frac{1}{\cancel{3}}\cdot\cancel{3} = 1/3$$

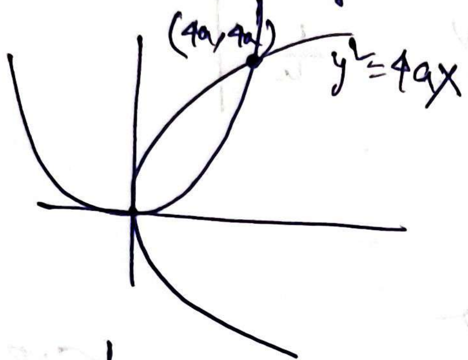
6> $u = \frac{y^2}{x},\ v = \frac{x^2}{y}$

$$uv^2 = \frac{y^2}{x}\cdot\frac{x^4}{y^2} = x^3 \Rightarrow x = (uv^2)^{1/3}$$

$$y = (u^2v)^{1/3}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v^{2/3}\frac{1}{3}u^{-2/3} & u^{1/3}\frac{2}{3}v^{-1/3} \\ v^{1/3}\frac{2}{3}u^{-1/3} & u^{2/3}\frac{1}{3}v^{-2/3} \end{vmatrix}$$

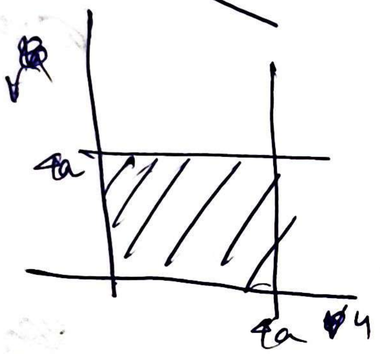
$$= \frac{1}{9} - \frac{4}{9} = -3/9 = -1/3$$



$$\iint_D dx\,dy$$

$$= \int_0^{4a}\int_0^{4a} \left(\frac{1}{3}\right) du\,dv$$

$$= \frac{16a^2}{3}$$



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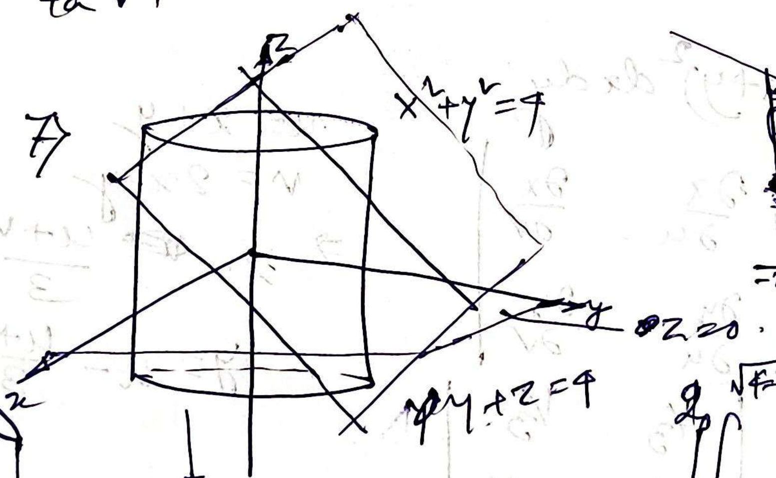
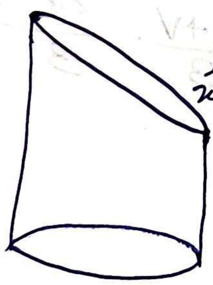
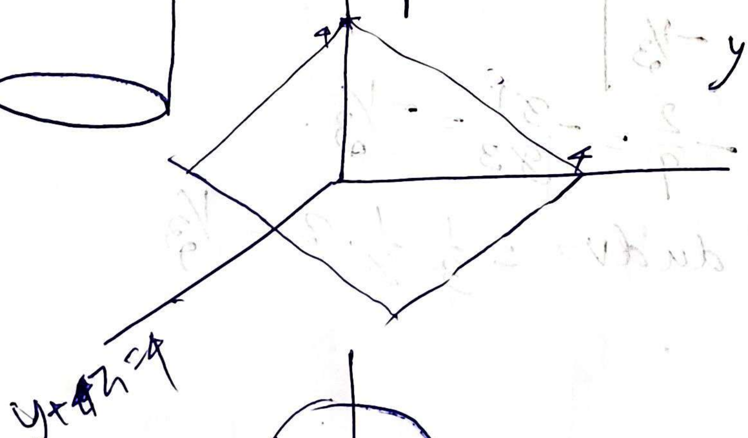
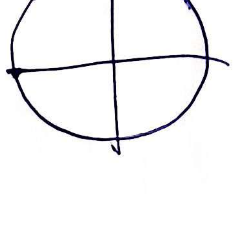
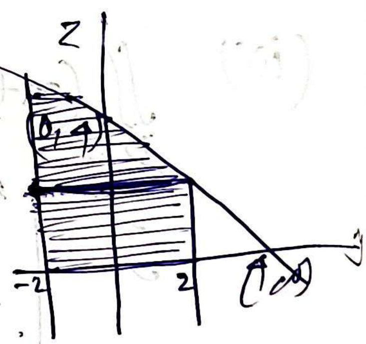










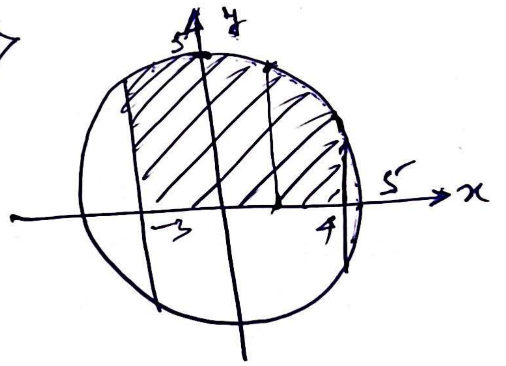
$$\int_{y=-2}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4-y)\,dx\,dy$$

$$\int_{-2}^{2} 2(4-y)\sqrt{4-y^2}\,dy$$

$$= 16\pi$$

8) $S = \iint \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$

9)



$$\int_{-3}^{4}\int_{0}^{\sqrt{25-x^2}} \cancel{dx\,dy}\; dy\, dx$$

$$= \int_{-3}^{4} \sqrt{25-x^2}\, dx$$

$$= \frac{x}{2}\sqrt{25-x^2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\Big|_{-3}^{4}$$

$$= \frac{4}{2}\sqrt{9} + \frac{3}{2}\sqrt{16} + \frac{25}{2}\sin^{-1}(4/5) + \frac{25}{2}\sin^{-1}(3/5)$$

$$= 6 + 6 + \frac{25}{2}\frac{\pi}{2}$$

$$= 12 + \frac{25\pi}{4}$$