Tutorial - 8

1) a) $(D^3 - 3D^2 + 4)y = 0$

$\Rightarrow (D+1)(D^2 - 4D + 4)y = 0$

$\Rightarrow y = c_1 e^{-x} + e^{2x}(c_2 + c_3 x)$

b) $(D^2 - 3D + 2)y = 0$

$\Rightarrow (D-1)(D-2)y = 0$

$\Rightarrow y = c_1 e^{x} + c_2 e^{2x}$

c) $(D^2 + a^2)y = 0$

$y = c_1 \cos ax + c_2 \sin ax$

d) $(2D^2 - 4D + 8)y = 0$

$\Rightarrow (D^2 - 2D + 4)y = 0$

$\frac{2 \pm \sqrt{4-16}}{2}$

$= 1 \pm \sqrt{1-4}$

$= 1 \pm \sqrt{3}i$

$y = e^{x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$

e) $(D^4 + 2D^2 + 1)y = 0$

$(D^2+1)^2 y = 0$

$\Rightarrow (D+i)^2 (D-i)^2 y = 0$

$y = (p_1 + p_2 x)\cos x + (q_1 + q_2 x)\sin x$

f) $(D^2+1)^2 \left(\frac{d^2y}{dx^2} + y\right)^3 \left(\frac{d^2y}{dx^2} + \frac{dy}{dx} + y\right)^2 = 0$

$\cancel{(D^2+1)y = 0}$, $\cancel{(D^2+D+1)y = 0}$

$\cancel{y = c_1\cos x + c_2 \sin x + e^{-x/2}\left(c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right)}$

g) $(D^5 - 3D^4 + 3D^3 - D^2)y = 0$

$\Rightarrow D^2(D^3 - 3D^2 + 3D - 1)y = 0$

$\Rightarrow D^2 (D-1)^3 y = 0$

$y = c_1 + c_2 x + e^{x}(c_3 + c_4 x + c_5 x^2)$

h) $(D^4 - m^4)y = 0$

$\Rightarrow (D-m)(D+m)(D+mi)(D-mi)y = 0$

$\Rightarrow y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$

i) $(D^4 - 5D^3 + 6D^2 + 4D - 8)y = 0$

$\cancel{1-5+6+4-8}$

$\cancel{16-40+24+8-8}$

$(D-2)^3(D+1)y = 0$

$y = c_1 e^{-x} + e^{2x}(c_2 + c_3 x + c_4 x^2)$

2) a) $(D^2-3D+2)x=0$

$\Rightarrow (D-1)(D-2)x=0$

$x=c_1e^{t}+c_2e^{2t}$

$x(0)=c_1+c_2=2$

$x'(t)=c_1e^{t}+c_2\cdot 2e^{2t}$

$x'(0)=c_1+2c_2=0$

$c_2=-2$

$c_1=4$

$\therefore x=4e^{t}-2e^{2t}$

b) $(D^2+4D+5)y=0$

$\frac{-4\pm\sqrt{16-20}}{2}$

$=-2\pm\sqrt{4-5}$

$=-2\pm i$

$y=e^{-2x}(c_1\cos x+c_2\sin x)$

$y(0)=c_1=1$

$y'(x)=e^{-2x}(-c_1\sin x+c_2\cos x)-2e^{-2x}(c_1\cos x+c_2\sin x)$

$y'(0)=c_2-2c_1=0$

$\Rightarrow c_2=2$

$y=e^{-2x}(\cos x+2\sin x)$

c) $(D^2-2D+10)y=0$

$\Rightarrow y=e^{x}(c_1\cos 3x+c_2\sin 3x)$

$y'(x)=e^{x}(-3c_1\sin 3x+3c_2\cos 3x)+e^{x}(c_1\cos 3x+c_2\sin 3x)$

$y(0)=c_1=4$

$y'(0)=3c_2+c_1=1$

$c_1=4,\ c_2=-1$

$y=e^{x}(4\cos 3x-\sin 3x)$

d) $(D^2-2D-8)y=0$

$1\pm\sqrt{9}=1\pm 3$

$4,\ -2$

$y=c_1e^{4t}+c_2e^{-2t}$

$y'=4c_1e^{4t}-2c_2e^{-2t}$

$2y(0)=2c_1+2c_2=2\alpha$

$y'(0)=4c_1-2c_2=2\pi$

$6c_1=2\alpha+2\pi \Rightarrow c_1=\frac{\pi+\alpha}{3}$

$c_2=\alpha-\frac{\pi+\alpha}{3}$

$=\frac{2\alpha-\pi}{3}$

e) $(D^3-5D^2-22D+56)y=0$

$\cancel{8-20-44+56}$ $(D-2)(D-7)(D+4)y=0$

$y=c_1e^{2x}+c_2e^{-4x}+c_3e^{7x}$

$y'=2c_1e^{2x}-4c_2e^{-4x}+7c_3e^{7x}$

$y''=4c_1e^{2x}+16c_2e^{-4x}+49c_3e^{7x}$

$c_1+c_2+c_3=1$

$2c_1-4c_2+7c_3=-2$

$4c_1+16c_2+49c_3=-4$

$c_1=13/15,\ c_2=14/33,\ c_3=-16/55$

$y=\frac{13}{15}e^{2x}+\frac{14}{33}e^{-4x}-\frac{16}{55}e^{7x}$

f) $(D^2+D+1)y=0$

$\frac{-1\pm\sqrt{3}}{2}$

$y=e^{-x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)$

$c_1=2,\ c_2=2\sqrt{3},$

3) a) $(D^2-5D+6)y = e^{3x}$

$(D-2)(D-3)y = e^{3x}$

$C.F. = c_1 e^{2x} + c_2 e^{3x}$

$P.I = \frac{1}{(D-2)(D-3)} e^{3x}$

$= \frac{1}{(3-2)} x e^{3x} = x e^{3x}$

$\therefore y = c_1 e^{2x} + (c_2 + x) e^{3x}$

b) $(D^2-3D+2)y = \cosh x$

$\Rightarrow (D-1)(D-2)y = \cosh x$ $\quad \cosh x = \frac{e^x + e^{-x}}{2}$

$C.F. = c_1 e^x + c_2 e^{2x}$

$P.I. = \frac{1}{(D-1)(D-2)} \left(\frac{e^x + e^{-x}}{2}\right)$

$= \frac{1}{2} \frac{1}{(D-1)(D-2)} e^x + \frac{1}{2} \frac{1}{(D-1)(D-2)} e^{-x}$

$= \frac{1}{2} \frac{1}{(1-2)} x e^x + \frac{1}{2} \frac{1}{(-1-1)(-1-2)} e^{-x}$

$P.I. = -\frac{1}{2} x e^x + \frac{1}{12} e^{-x}$

$y = \left(c_1 - \frac{x}{2}\right) e^x + c_2 e^{2x} + \frac{1}{12} e^{-x}$

c) $(D^2-4)y = e^x + \sin 3x$

$C.F. = c_1 e^{2x} + c_2 e^{-2x}$

$P.I. = \frac{1}{(D-2)(D+2)} (e^x + \sin 3x)$

$= \frac{1}{D^2-4} e^x + \frac{1}{D^2-4} \sin 3x$

$= \frac{1}{1-4} e^x + \frac{1}{-9-4} \sin 3x$

$= -\frac{1}{3} e^x - \frac{1}{13} \sin 3x$

$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} e^x - \frac{1}{13} \sin 3x$

d) $(D^3+D^2-D-1)y = \cos 2x$

C.F.: $(D^3+D^2-D-1)y=0$

$\Rightarrow (D-1)(D^2+2D+1)y=0$

$\Rightarrow (D-1)(D+1)^2 y=0$

CF $= c_1 e^x + (c_2+c_3x)e^{-x}$

$$PI = \cancel{\frac{1}{(D-1)(D+1)^2}} \frac{1}{(D^3+D^2-D-1)}\cos 2x$$

$$= \frac{1}{(-4)D+(-4)-D-1}\cos 2x$$

$$= \frac{1}{-5D-5}\cos 2x$$

$$= -\frac{1}{5}\frac{(D-1)}{(D+1)(D-1)}\cos 2x$$

$$= -\frac{1}{5}\frac{D-1}{(-4-1)}\cos 2x$$

$$= \frac{1}{25}(-2\sin 2x - \cos 2x).$$

$$y = c_1e^x + (c_2+c_3x)\bar{e}^x - \frac{1}{25}(2\sin 2x + \cos 2x)$$

e) $(D^2-4)y = x\sinh x$

$$= x\left(\frac{e^x - e^{-x}}{2}\right)$$

C.F. $= c_1e^{2x} + c_2e^{-2x}$

$$P.I. = \frac{1}{(D+2)(D-2)}\frac{xe^x}{2} - \frac{1}{(D+2)(D-2)}\frac{xe^{-x}}{2}$$

$$= \frac{e^x}{2}\frac{1}{(D+3)(D-1)}x - \frac{\bar{e}^x}{2}\frac{1}{(D+1)(D-3)}x$$

$$\frac{1}{D^2+2D-3} = -\frac{1}{03}\left[1+\frac{D^2+2D}{-3}\right]^{-1}$$

$$= -\frac{1}{3}\left[1+\frac{D^2+2D}{3}\right]$$

~~D+2~~ D^2+2D-3

$-3\left(1+\frac{D^2+2D}{-3}\right)$

$$\frac{1}{D^2+2D-3}x = -\frac{1}{3}\left(1+\frac{2D}{3}\right)x$$

$$= -\frac{1}{3}\left(x+\frac{2}{3}\right) = -\frac{x}{3}-\frac{2}{9}$$

$$\frac{1}{D^2-2D-3}x = -\frac{1}{3}\left(x-\frac{2}{3}\right) = -\frac{x}{3}+\frac{2}{9}$$

$$P.I = \frac{e^x}{2}\left[-\frac{x}{3}-\frac{2}{9}\right] - \frac{e^{-x}}{2}\left(-\frac{x}{3}+\frac{2}{9}\right)$$

$$y = c_1e^{2x}+c_2e^{-2x}-\frac{xe^x}{6}+\frac{xe^{-x}}{6}-\frac{e^x}{9}-\frac{e^{-x}}{9}$$

8) $(D^2-4)y = x^2$

$CF = c_1e^{2x}+c_2e^{-2x}$ $\qquad D^2-4 = -4\left(1+\frac{D^2}{-4}\right)$

$$P.I = \frac{1}{D^2-4}x^2$$

$$= -\frac{1}{4}\left(1-\frac{D^2}{4}\right)^{-1}x^2 \qquad \frac{d^2}{dx^2}(x^2) = 2x$$

$$= -\frac{1}{4}\left[1+\frac{D^2}{4}\right]x^2$$

$$= -\frac{1}{4}\left(x^2+\frac{1}{4}\times 2\right)$$

$$= -\frac{x^2}{4}-\frac{1}{8}$$

$$\therefore y = c_1e^{2x}+c_2e^{-2x}-\frac{x^2}{4}-\frac{1}{8}$$

9) $\frac{d^2y}{dx^2}+4y = e^x+\sin 3x+x^2$

$(D^2+4)y = e^x+\sin 3x+x^2$

$CF = c_1\cos 2x+c_2\sin 2x$

$$P.I. = \frac{1}{D^2+4}(e^x+\sin 3x+x^2)$$

$$= \frac{1}{5}e^x-\frac{1}{5}\sin 3x+\frac{x^2}{4}-\frac{1}{8}$$

$$\frac{1}{D^2+4}x^2 = \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^2 = \frac{1}{4}\left(1-\frac{D^2}{4}\right)x^2 = \frac{1}{4}\left(x^2-\frac{1}{2}\right)$$

$$y = c_1\cos 2x+c_2\sin 2x+\frac{1}{5}e^x-\frac{1}{5}\sin 3x+\frac{x^2}{4}-\frac{1}{8}$$

h) $(D^2-2D+1)y = xe^x \sin x$

$C.F. = e^x(c_1+c_2x)$

$P.I = \frac{1}{(D-1)^2} xe^x \sin x$

$= e^x \frac{1}{D^2} x\sin x$

$= e^x \int\left(\int x\sin x\,dx\right)dx$

$\int x\sin x\,dx = x(-\cos x) + \int \cos x\,dx$

$= -x\cos x + \sin x$

$\int(-x\cos x + \sin x)\,dx$

$= -\cos x - \int x\cos x\,dx$

$= -\cos x - \left[x\sin x - \int \sin x\,dx\right]$

$= -\cos x - [x\sin x + \cos x]$

$= -x\sin x - 2\cos x$

$\therefore y = e^x(c_1+c_2x) - e^x(x\sin x + 2\cos x)$

i) $(D^2+5D+6)y = e^{-2x}\sin 2x$

$(D+2)(D+3)$

$CF = c_1e^{-2x} + c_2e^{-3x}$

$P.I. = \frac{1}{(D+3)(D+2)} e^{-2x}\sin 2x$

j) $(D^2+1)y = \operatorname{cosec} x$

$CF = c_1\cos x + c_2\sin x$

$P.I. = \frac{1}{D^2+1}\operatorname{cosec} x = \frac{(D+i)-(D-i)}{2i(D+i)(D-i)}\operatorname{cosec} x$

$= \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\operatorname{cosec} x$

$$\frac{1}{D-i}\cos x = e^{ix}\int \cos x \, e^{-ix} \, dx$$

$$\cancel{\frac{d}{dx}(\cos x) = \sin x \, dx}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\frac{1}{D-i} \cdot \frac{2i}{e^{ix} - e^{-ix}} = 2i\, e^{ix} \int \frac{e^{-ix}}{e^{ix} - e^{-ix}} dx$$

$$= 2i e^{ix} \int \frac{1}{e^{2ix} - 1} dx$$

$$= i e^{ix} \int \frac{(e^{ix}+1) - (e^{ix}-1)}{(e^{ix}-1)(e^{ix}+1)} dx$$

$$= i e^{ix} \int \left(\frac{e^{-ix}}{1 - e^{-ix}} - \frac{e^{-ix}}{1 + e^{-ix}} \right) dx$$

$$= e^{ix} \int \frac{d(e^{-ix}-1)}{(e^{-ix}-1)} + \frac{d(e^{-ix}+1)}{(e^{-ix}+1)}$$

$$= e^{ix} \ln(e^{-2ix} - 1)$$

$$\frac{1}{D+i}\cos x = e^{-ix} \ln(e^{2ix} - 1)$$

$$P.I. = \frac{1}{2i}\left(e^{ix} \ln(e^{-2ix} - 1) - e^{-ix} \ln(e^{2ix} - 1) \right)$$

$$e^{2ix} - 1 = \cos 2x + i \sin 2x - 1$$

$$= -2\sin^2 x + 2i \sin x \cos x$$

$$= \cancel{2\sin x}\; 2\sin x(-\sin x + i\cos x)$$

$$\cos(x + \pi/2) = -\sin x$$

$$\sin(x + \pi/2) = \cos x$$

$$= 2\sin x \, e^{i(x+\pi/2)}$$

$$e^{-2ix} - 1 = 2\sin x \, e^{-i(x+\pi/2)}$$

$$\frac{1}{2i}\left[e^{ix}\left[\ln(2\sin x) - (x+\tfrac{\pi}{2})i\right] - e^{-ix}\left[\ln(2\sin x) + (x+\tfrac{\pi}{2})i\right]\right]$$

$$= \sin x \ln(2\sin x) - \left(x + \frac{\pi}{2}\right)\cos x$$

$$\therefore y = c_1' \cos x + c_2' \sin x - x\cos x + \sin x \ln(\sin x)$$

k) $(D^3+1)y = \sin 3x - \cos^2 \frac{x}{2}$ $\quad (D+1)(D^2-D+1)$ $\quad \frac{1 \pm \sqrt{3}\,i}{2}$

$$CF = c_1 e^{-x} + e^{x/2}\left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right)$$

$$P.I. = \frac{1}{D^3+1}\left(\sin 3x - \cos^2 \frac{x}{2}\right)$$

$$= \frac{1}{D^3+1}\left(\sin 3x - \left(\frac{1+\cos x}{2}\right)\right)$$

$$= \frac{1}{D^3+1}\sin 3x + \frac{1}{D^3+1}\left(-\tfrac{1}{2}\right) + \frac{1}{D^3+1}\left(-\tfrac{1}{2}\cos x\right)$$

$$\frac{1}{D(-9)+1}\sin 3x$$

$$= \frac{(1+9D)}{(1-9D)(1+9D)}\sin 3x = \frac{1+9D}{1-81D^2}\sin 3x$$

$$\frac{1}{f(D)}c = \frac{1}{f(0)}c$$

c is const.

$$= \frac{(1+9D)}{1-81(-9)}\sin 3x = \frac{1}{730}(1+9D)\sin 3x$$

$$\frac{1}{D^3+1}\left(-\tfrac{1}{2}\right) = -\tfrac{1}{2}$$

$$= \frac{1}{730}(\sin 3x + 9 \times 3 \cos 3x)$$

$$= \frac{1}{730}(\sin 3x + 27 \cos 3x)$$

$$\frac{1}{D^3+1}\cos x = \frac{1}{D(-1)+1}\cos x = \frac{(1+D)}{(1-D)(1+D)}\cos x$$

$$= \frac{1+D}{2}\cos x$$

$$= \frac{1}{2}(\cos x - \sin x)$$

$$P.I. = \frac{1}{730}(\sin 3x + 27\cos 3x) - \frac{1}{2} - \frac{1}{4}(\cos x - \sin x)$$

$$y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right) + \frac{1}{730}(\sin 3x + 27\cos 3x) - \frac{1}{2} - \frac{1}{4}(\cos x - \sin x)$$

c) $H = \begin{bmatrix} y & y' & y'' \\ u & u' & u'' \\ v & v' & v'' \end{bmatrix}$

$u = x, \; v = e^x$

$\det H = 0$

(i) $\begin{vmatrix} y & y' & y'' \\ x & 1 & 0 \\ e^x & e^x & e^x \end{vmatrix} = 0$

$\Rightarrow ye^x + y'(-xe^x) + y'' e^x(x-1) = 0$

$\Rightarrow e^x(x-1)y'' - xe^x y' + e^x y = 0$

$\Rightarrow (x-1)y'' - xy' + y = 0$

(ii) $u = 1/x, \; v = e^{-x}$

$\begin{vmatrix} y & y' & y'' \\ 1/x & -1/x^2 & 2/x^3 \\ e^{-x} & -e^{-x} & e^{-x} \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} y & y' & y'' \\ x^2 & -x & 2 \\ 1 & -1 & 1 \end{vmatrix} = 0$

$\Rightarrow (x-x^2)y'' + (2-x^2)y' + (2-x)y = 0$