Tutorial 4

1) a) $A\vec{v} = \lambda \vec{v}$ (as A is non singular, $\lambda \neq 0$)

$\operatorname{adj}A \cdot A\vec{v} = \operatorname{adj}A \cdot \lambda\vec{v}$

$\Rightarrow \lambda \operatorname{adj}A\, \vec{v} = |A| I \vec{v}$

$\Rightarrow (\operatorname{adj}A)\vec{v} = \dfrac{|A|}{\lambda}\vec{v}$

b) characteristic polynomial of $C = P_C(x) = \det(xI - C)$

$$P_{AB}(x) = \det(xI - AB)$$

$$= \frac{1}{\det(B^{-1})} \det(xI - AB)\det(B^{-1})$$

$$= \frac{1}{\det(B^{-1})} \det[(xI - AB)B^{-1}]$$

$$= \frac{1}{\det(B^{-1})} \det[xB^{-1} - A]$$

$$= \frac{1}{\det B \cdot \det B^{-1}} \det(B) \cdot \det[xB^{-1} - A]$$

$$= \det[B(xB^{-1} - A)]$$

$$= \det[xI - BA] = P_{BA}(x)$$

$\therefore AB$ and BA have same characteristic roots.

(c) $P_A(x) = \det(xI - A) = (x-\lambda)^r q(x)$

$B = A - \lambda I$

$$P_B(x) = \det(xI - B)$$

$$= \det(xI - A + \lambda I)$$

$$= \det((x+\lambda)I - A) = P_A(x+\lambda)$$

$$= x^r q(x+\lambda)$$

$\therefore 0$ is an eigenvalue of alg. mult. r of $A - \lambda I_n$

2) (c) $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$

$$P_A(x) = \det(xI - A) = \det\begin{pmatrix} x-3 & -10 & -5 \\ 2 & x+3 & 4 \\ -3 & -5 & x-7 \end{pmatrix}$$

$$= x^3 - 7x^2 + 16x - 12$$
$$= (x-2)(x^2 - 5x + 6)$$
$$= (x-2)(x-2)(x-3)$$

$2, 3$

$A\vec{v} = \lambda \vec{v}$

$$\begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 - 3R_1$

$5-30 \quad 5-15$

$-5+20$
$-4+10$
$5-30$

$$\begin{pmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & -25 & -10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$R_3 \to \frac{R_3}{5}, R_2 \to \frac{R_2}{3}$

$$\begin{pmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & -5 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$R_3 \to R_3 + R_2$

$$\begin{pmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 + 10x_2 + 5x_3 = 0$$
$$5x_2 + 2x_3 = 0$$
$$x_3 = 5k, \; x_2 = -2k$$
$$x_1 - 20k + 25k = 0 \Rightarrow x_1 = -5k$$

for $\lambda = 2$, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5k \\ -2k \\ 5k \end{pmatrix}$, $k \in \mathbb{R} \setminus \{0\}$ —— (1)

$+5 - \underbrace{20+25}_{5}$
$10 + 6 - 20$

$$\begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}\begin{pmatrix} -5 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -10 \\ -4 \\ 10 \end{pmatrix} = 2\begin{pmatrix} -5 \\ -2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$R_1 \to R_1/5$
$R_2 \to R_2/(-2)$

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 3 & 2 \\ 3 & 5 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$R_3 \to R_3 - 3R_1$

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 3 & 2 \\ 0 & -4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$2x_2 + x_3 = 0, \quad x_1 + 3x_2 + 2x_3 = 0$

$x_2 = k, \quad x_3 = -2k$

$x_1 + 3k - 4k = 0 \Rightarrow x_1 = k$

for $\lambda = 3$, $\left\{ \begin{pmatrix} k \\ k \\ -2k \end{pmatrix} : k \in \mathbb{R} \setminus \{0\} \right\}$

$$\begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

3) $\lambda_1 = \lambda_2 = 2$, $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $B\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & \cdot \\ 0 & 2 \end{pmatrix}$$

$a = 2 \ , \ c = 0$

$\sum \lambda_i = tr(A), \ \prod \lambda_i = \det(A)$

$a + d = 4 \Rightarrow d = 2$

$ad - bc = 4 \Rightarrow 4 - 0 = 4 \checkmark$

take $A = \begin{pmatrix} 2 & \pi \\ 0 & 2 \end{pmatrix}, \ B = \begin{pmatrix} 2 & e \\ 0 & 2 \end{pmatrix}$

$\begin{pmatrix} 2 & \pi \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

4) $\underline{a+b} = c + d$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (a+b) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

λ_1, λ_2

$\lambda_1 + \lambda_2 = \text{trace}(A) = a + d$

$\lambda_1 \lambda_2 = \det(A) = ad - bc$

$(a+b) + \lambda_2 = a + d \Rightarrow \lambda_2 = \underline{d - b}$

Eigenvalues are $a+b, \ d-b$

6) (a) $0 < \theta < \pi, \ A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\det(xI - A) = \det \begin{pmatrix} x - \cos\theta & \sin\theta \\ -\sin\theta & x - \cos\theta \end{pmatrix}$

$= (x - \cos\theta)^2 + \sin^2\theta$

$\sin\theta \neq 0$ for $\theta \in (0, \pi)$

$\therefore$ No real eigenvalues.

$\underline{\prod \lambda_i = \det(A)}$

(b) $AA^T = I = A^T A$

$A\vec{v} = \lambda \vec{v}$

$A^T A \vec{v} = A^T \lambda \vec{v} \Rightarrow I\vec{v} = \lambda A^T \vec{v}$

$\Rightarrow A^T \vec{v} = \frac{1}{\lambda} \vec{v}$

$\lambda \neq 0$ as A is non-singular

$$P_A(x) = \det(xI - A)$$

$$\det(B^T) = \det(B)$$

$$P_{A^T}(x) = \det(xI - A^T)$$

$$(xI - A)^T = (xI)^T - A^T = xI - A^T$$

$$\therefore P_A(x) = P_{A^T}(x)$$

* Characteristic polynomial of a square matrix and its transpose is same.

$\therefore 1/\lambda$ is also an eigenvalue of A.