Tutorial-7

67 (i) $\dfrac{dy}{dx} + \dfrac{1-2x}{x^2} y = 1$ $\quad -2+1$

$$IF = e^{\int \frac{1-2x}{x^2} dx} = e^{-\frac{1}{x} - \ln(x^2)} = \frac{1}{x^2} e^{-1/x} \quad \frac{-x}{x^2}$$

$-\frac{1}{t}$

$$y \frac{1}{x^2} e^{-1/x} = \int e^{-1/x} \left(\frac{1}{x^2}\right) dx$$

$$= e^{-1/x} + c$$

$$\Rightarrow \frac{y}{x^2} = 1 + c e^{1/x}$$

$$\Rightarrow y = x^2 (1 + c e^{1/x}).$$

(ii) $\sqrt{a^2+x^2} \dfrac{dy}{dx} + y = \sqrt{a^2+x^2} - x$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{\sqrt{a^2+x^2}} y = 1 - \frac{x}{\sqrt{a^2+x^2}}$$

~~IF = exp~~ $IF = e^{\int \frac{1}{\sqrt{a^2+x^2}} dx} = e^{\ln(x+\sqrt{a^2+x^2})}$

$$= x + \sqrt{a^2+x^2}$$

$$y(x + \sqrt{a^2+x^2}) = \int \left(\frac{\sqrt{a^2+x^2} - x}{\sqrt{a^2+x^2}}\right)(\sqrt{a^2+x^2}+x)\, dx$$

$$= \int \frac{a^2 + \cancel{x^2} - \cancel{x^2}}{\sqrt{a^2+x^2}} dx$$

$$\Rightarrow y(x + \sqrt{x^2+a^2}) = a^2 \ln(x + \sqrt{x^2+a^2}) + c.$$

(iii) $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x) e^x \sec y$

$$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x) e^x$$

$$-\sin y = u \Rightarrow \cos y \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{du}{dx} + \frac{1}{1+x} u = (1+x) e^x$$

$$IF = e^{\int \frac{1}{1+x} dx} = (1+x)$$

$$(-\sin y)(1+x) = \int e^x (1+x)^2 dx = e^x (x^2+1) + c$$

(iv) $y(2xy+e^{x})\,dx - e^{x}\,dy = 0$

$\Rightarrow \frac{dy}{dx} = y(1+2xe^{-x}y)$

$\Rightarrow \frac{dy}{dx} - y = 2xe^{-x}y^2$

$n=2$

$P=-1,\ Q = 2xe^{-x}$

$v = \frac{1}{y^{2-1}} = 1/y$, $\frac{dv}{dx} + (-1)(-1)v = 2xe^{-x}(-1)$

$\Rightarrow \frac{dv}{dx} + v = -2xe^{-x}$

$IF = e^{x}$

$ve^{x} = \int -2x\,dx = -x^2 + c$

$\Rightarrow \frac{1}{y}e^{x} = c - x^2 \Rightarrow y = \frac{e^{x}}{c-x^2}$

(v) $y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0$

$\Rightarrow y^2\frac{dx}{dy} + (x - 1/y) = 0$

$\Rightarrow \frac{dx}{dy} + \frac{1}{y^2}x = \frac{1}{y^3}$

$IF = e^{\int \frac{1}{y^2}dy} = e^{-1/y}$

$xe^{-1/y} = \int e^{-1/y}\frac{1}{y^3}dy = \frac{e^{-1/y}(y+1)}{y} + c$

$\Rightarrow x = ce^{1/y} + \frac{y+1}{y}$

(vi) $\frac{dy}{dx} + \frac{y}{x}\ln y = \frac{y}{x^2}(\ln y)^2$

$\Rightarrow \frac{1}{y(\ln y)^2}\frac{dy}{dx} + \left(\frac{1}{\ln y}\right)\frac{1}{x} = \frac{1}{x^2}$

$-\frac{1}{\ln y} = u \Rightarrow \frac{du}{dx} = \frac{1}{(\ln y)^2}\frac{1}{y}\frac{dy}{dx}$

$\frac{du}{dx} - \frac{1}{x}u = \frac{1}{x^2}$

$IF = e^{-\int 1/x\,dx} = e^{-\ln(x)} = \frac{1}{x}$

$u\frac{1}{x} = \int \frac{1}{x^3}dx \Rightarrow \left(-\frac{1}{\ln y}\right)\frac{1}{x} = -\frac{1}{2x^2} + c$

$\boxed{\frac{1}{x\ln y} = c + \frac{1}{2x^2}}$

(vii) $\cos x \frac{dy}{dx} - y\sin x + y^2 = 0$

$\Rightarrow \frac{dy}{dx} - \tan x \, y = -y^2 \sec x$

$P = -\tan x, \; Q = -\sec x, \; n = 2$

$v = \frac{1}{y^{2-1}} = \frac{1}{y}$

$\frac{dv}{dx} + (-\tan x)(-1)\, v = (-\sec x)(-1)$

$\Rightarrow \frac{dv}{dx} + \tan x \, v = \sec x$

$\frac{1}{y} \sec x = \tan x + c.$

(viii) $y\,dy + by^2 dx = a\cos x \, dx$

$\cancel{\Rightarrow y + by^2 \frac{dx}{dy}}$

$y\frac{dy}{dx} + by^2 = a\cos x$

$y^2 = u$

$\Rightarrow 2y\frac{dy}{dx} = \frac{du}{dx}$

$\Rightarrow \frac{1}{2}\frac{du}{dx} + bu = a\cos x$

$\Rightarrow \frac{du}{dx} + 2bu = 2a\cos x$

$IF = e^{\int 2b\,dx} = e^{2bx}$

$y^2 e^{2bx} = \int e^{2bx} \cdot 2a\cos x \, dx$

$= \frac{2a e^{2bx}(2b\cos x + \sin x)}{4b^2+1} + c.$

5) (i) $x^2 y\,dx - (x^3 + y^3)\,dy = 0$

$\frac{\partial M}{\partial y} = x^2, \; \frac{\partial N}{\partial x} = -3x^2$

$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-4x^2}{x^2 y} = -4/y$

$IF = e^{\int -4/y \, dy} = e^{-4\ln y} = e^{\ln \frac{1}{y^4}} = 1/y^4$

$$\frac{1}{y^4}\left(x^2y\,dx - (x^3+y^3)\,dy\right) = 0$$

$$\Rightarrow \underbrace{\frac{x^2}{y^3}}_{M}\,dx - \underbrace{\frac{x^3+y^3}{y^4}}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{-3x^2}{y^4}, \quad \frac{\partial N}{\partial x} = -\frac{3x^2}{y^4}$$

$$\int M\,dx + \int (\text{terms of } N \text{ free from } x)\,dy = c$$

$$\Rightarrow \frac{x^3}{3y^3} - \ln y = c.$$

(ii) $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0$

$$\frac{\partial M}{\partial y} = 6xy, \quad \frac{\partial N}{\partial x} = 6xy$$

$$\int M\,dx + \int (\text{terms of } N \text{ free from } x)\,dy = c.$$

$$\Rightarrow \frac{x^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} = c$$

(iii) $(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$

$$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y, \quad \frac{\partial N}{\partial x} = 2x - \tan^2 y$$

$$\int M\,dx + \int (\text{terms of } N \text{ free from } x)\,dy = c$$

$$\Rightarrow x^2y + xy - x\tan y + \tan y = c$$

(iv) $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

$$\Rightarrow y^2\,dx + (x^2 - xy)\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2x - y$$

$$\text{IF} = \frac{1}{xy^2 + x^2y - xy^2} = \frac{1}{x^2y}$$

$$\frac{y}{x^2}\,dx + \left(\frac{1}{y} - \frac{1}{x}\right)dy = 0 \Rightarrow \int M\,dx + \int (\text{terms of } N \text{ free from } x)\,dy = c$$

$$\partial M/\partial y = 1/x^2, \quad \partial N/\partial x = 1/x^2 \Rightarrow -\frac{y}{x} + \ln y = c$$

(v) $x\frac{dy}{dx} + (3x+1)y = xe^{-2x}$

$\Rightarrow \underbrace{x}_{N}dy + [\underbrace{(3x+1)y - xe^{-2x}}_{M}]dx = 0$

$$\frac{\partial M}{\partial y} = 3x+1, \quad \frac{\partial N}{\partial x} = 1$$

$$IF = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx} = e^{\int 3dx} = e^{3x}$$

$\underbrace{[(3x+1)e^{3x}y - xe^{x}]}_{M} dx + \underbrace{xe^{3x}}_{N} dy = 0$

$\int M dx + \int (\text{terms of } N \text{ free from } x)\, dy = c$

$\Rightarrow \int (3x+1)e^{3x}y - xe^{x}\, dx = c$

$\Rightarrow xye^{3x} - e^{x}(x-1) = c$.

4/(i) $x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\left(\frac{y}{x}\right) - x\right)dx$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x}y - \operatorname{cosec}(y/x)$

$y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\cancel{v} + x\frac{dv}{dx} = \cancel{v} - \operatorname{cosec}(v)$

$\Rightarrow \int \sin v\, dv = \int -\frac{dx}{x}$

$\Rightarrow \cos v = \ln(x) + c$

$\Rightarrow \cos\left(\frac{y}{x}\right) = \ln(x) + c$

(ii) $2xy\,dy = (y^2 - x^2)dx$

$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{v^2-1}{2v} \Rightarrow x\frac{dv}{dx} = \frac{v^2-1-2v^2}{2v} = \frac{-(1+v^2)}{2v}$

$\Rightarrow \int \frac{2v}{1+v^2}dv = -\int\frac{dx}{x} \Rightarrow \ln(1+v^2) = -\ln(x) + \ln c \Rightarrow x\left(1+\frac{y^2}{x^2}\right) = c$

(iii) $(x+2y-3)dy = (2x-y+1)dx$

$\Rightarrow \int d(xy) + \int (2y-3)dy - \int (2x+1)dx = \int 0$

$\Rightarrow xy + y^2 - 3y - x^2 - x = c$

3) (i) $x\cos\left(\frac{y}{x}\right)(ydx+xdy) = y\sin\left(\frac{y}{x}\right)\frac{(xdy-ydx)}{x^2}x^2$

$\Rightarrow \cancel{x}\cos(y/x)\, d(xy) = \cancel{x}y\sin(y/x)\, d\left(\frac{y}{x}\right)$

$\Rightarrow \frac{d(xy)}{xy} = \tan(y/x)\, d(y/x)$

$\Rightarrow \ln(xy) = \ln\left(\sec\frac{y}{x}\right) + \ln c.$

$\Rightarrow xy = c\sec\frac{y}{x}$

$x=1, y=\pi$

$\Rightarrow \pi = c\sec\pi \Rightarrow c = -\pi$

$\Rightarrow xy + \pi\sec\frac{y}{x} = 0$

(ii) $(xy^2 - e^{\frac{1}{x^3}})dx - x^2y\,dy = 0$, $y(1)=0$

$\Rightarrow x^3 \frac{y}{x}\frac{(ydx - xdy)}{x^2} - e^{1/x^3}dx = 0$

$\Rightarrow -x^4\left(\frac{y}{x}\right)d(y/x) - e^{1/x^3}dx = 0$

$\Rightarrow \left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) + \frac{1}{x^4}e^{\frac{1}{x^3}}dx = 0$

$\left(-\frac{1}{3}\right)\int -\frac{3}{x^4}e^{1/x^3}dx$

$= \left(-\frac{1}{3}\right)e^{1/x^3}$

$\frac{1}{2}\left(\frac{y}{x}\right)^2 - \frac{1}{3}e^{1/x^3} = c$

$x=1, y=0$

$\Rightarrow c = -\frac{1}{3}e$

$\frac{1}{2}\frac{y^2}{x^2} - \frac{1}{3}e^{1/x^3} = -\frac{1}{3}e$

(iii) $x\cos x\frac{dy}{dx} + y(x\sin x+\cos x) = 1$, $y(0)=1$

$$\Rightarrow \frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right)y = \frac{1}{x}\sec x$$

$$IF = e^{\int (\tan x + \frac{1}{x})\,dx} = e^{\ln(\sec x)+\ln(x)}$$

$$= x\sec x$$

$$y\,x\sec x = \tan x + c$$

$$x=0,\ y=1 \Rightarrow c=0$$

$$\Rightarrow xy\frac{1}{\cos x} = \frac{\sin x}{\cos x}$$

$$\Rightarrow xy = \sin x$$

~~1) (a)(i)~~ ~~order~~ ~~degree~~

1)

DE	order	degree
i) $\left(\frac{d^2x}{dt^2}\right)^2 + \frac{d^2x}{dt^2} + t\frac{dx}{dt} = 0$	2	2
ii) $t\frac{d^2y}{dt^2} + t^2\frac{dy}{dt} - \sqrt{y}\cos t = 2t^2-3t+4$	2	1
iii) $y + \left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{dy}{dx}\right)^{10}$	2	2
iv) $\left(\frac{d^2y}{dx^2}\right)^2 = \frac{1}{16}\left[2+\left(\frac{dy}{dx}\right)^2\right]^7$	2	2
v) $\left(1+\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx}\right)^2$	2	3