Tutorial-9

1) (a) $(x^2D^2 - 3xD + 4)y = 2x^2$

$x = e^z$

$x\frac{dy}{dx} = \frac{dy}{dz}, \quad x^2\frac{d^2y}{dx^2} = \left(\frac{d^2}{dz^2} - \frac{d}{dz}\right)y$

$x^3\frac{d^3y}{dx^3} = \frac{d}{dz}\left(\frac{d}{dz} - 1\right)\left(\frac{d}{dz} - 2\right)y$

$D = \frac{d}{dz}$

D^2-3D+2

$[D(D-1) - 3D + 4]y = 2e^{2z}$

$\Rightarrow (D^2 - 4D + 4)y = 2e^{2z}$

$C.F. = y = e^{2z}(c_1 + c_2 z)$
$= x^2(c_1 + c_2 \ln x)$

$P.I. = \frac{1}{(D-2)^2} 2e^{2z} = \cancel{2} \cdot e^{2z} \cdot \frac{z^2}{\cancel{2}}$
$= x^2(\ln x)^2$

$\therefore y = x^2(c_1 + c_2 \ln x) + x^2(\ln x)^2$

(b) $(x^2D^2 + 7xD + 13)y = \ln x$

$\ln x = z \Rightarrow x = e^z$

$(D(D-1) + 7D + 13)y = z$

$\Rightarrow (D^2 + 6D + 13)y = z$

$\frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$

$CF = e^{-3z}(c_1 \cos 2z + c_2 \sin 2z)$
$= \frac{1}{x^3}(c_1 \cos(2\ln x) + c_2 \sin(2\ln x))$

$$PI = \frac{1}{D^2+6D+13} z$$

$$= \frac{1}{13\left(1+\frac{D^2+6D}{13}\right)} z$$

$$= \frac{1}{13}\left(1+\frac{D^2+6D}{13}\right)^{-1} z$$

$$= \frac{1}{13}\left(1-\frac{D^2+6D}{13}\right) z$$

$$= \frac{1}{13}\left(z-\frac{6}{13}\right) = \frac{\ln x}{13} - \frac{6}{169}$$

$$\therefore y = \frac{1}{x^3}\left(c_1\cos(2\ln x) + c_2\sin(2\ln x)\right)$$

$$+ \frac{\ln x}{13} - \frac{6}{169}$$

(c) $(x^2D^2 - 4xD + 6)y = x^4$

$x = e^z \Rightarrow z = \ln x$

$(D^2 - D - 4D + 6)y = e^{4z}$

$\Rightarrow (D^2 - 5D + 6)y = e^{4z}$

$CF = c_1e^{2z} + c_2e^{3z} = c_1x^2 + c_2x^3$

$PI = \frac{e^{4z}}{16-20+6} = \frac{1}{2}x^4$

$y = \frac{1}{2}x^4 + c_1x^2 + c_2x^3$

d) $(D^2-1)y = \cancel{e^{mz}}\ e^{mz}$

$CF = c_1e^{z} + c_2e^{-z} = c_1x + \frac{c_2}{x}$

$PI = \frac{1}{(D+1)(D-1)}e^{mz} = \frac{1}{m^2-1}e^{mz} \quad (m \neq \pm 1)$

$y = \frac{x^m}{m^2-1} + c_1x + \frac{c_2}{x}$

e) $(D^2 - D - 3D + 5)y = \sin z$

$\Rightarrow (D^2 - 4D + 5)y = \sin z$

$CF = e^{2z}(c_1 \cos z + c_2 \sin z)$

$= x^2(c_1 \cos(\ln x) + c_2 \sin(\ln x))$

$\frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$

$P.I = \frac{1}{D^2 - 4D + 5} \sin z = \frac{1}{-1 - 4D + 5} \sin z$

$= \frac{(4+4D)}{(4-4D)(4+4D)} \sin z$

$= \frac{4+4D}{16+16} \sin z$

$= \frac{1+D}{8} \sin z = \frac{1}{8}(\sin z + \cos z)$

$\therefore y = x^2(c_1 \cos(\ln x) + c_2 \sin(\ln x))$
$+ \frac{1}{8}(\cos(\ln x) + \sin(\ln x))$

f) $(D^2 - 2D + 1)y = 2z$

$CF = e^{z}(c_1 + c_2 z) = x(c_1 + c_2 \ln x)$

$PI = \frac{1}{D^2 - 2D + 1} 2z = (1 + D^2 - 2D)^{-1} 2z$

$= (1 - D^2 + 2D) 2z$

$= 2z + 4$

$= 2\ln x + 4$

$y = x(c_1 + c_2 \ln x) + 2\ln x + 4$

g) $(D^2 - D - (2m-1)D + m^2 + n^2)y = n^2 z e^{mz}$

$\Rightarrow (D^2 - 2mD + m^2 + n^2)y = n^2 z e^{mz}$

$(D-m)^2 - (ni)^2$

$CF = e^{mz}(c_1 \cos nz + c_2 \sin nz)$

$= x^m(c_1 \cos(n \ln x) + c_2 \sin(n \ln x))$

$PI = \frac{1}{(D-m)^2 + n^2} n^2 z e^{mz} = n^2 e^{mz} \frac{1}{D^2 + n^2} z$

$$\frac{1}{D^2+n^2}z = \frac{1}{n^2\left(1+\frac{D^2}{n^2}\right)}z$$

$$= \frac{1}{n^2}\left(1-\frac{D^2}{n^2}\right)z = \frac{z}{n^2}$$

$$y = x^m\left(c_1\cos(\ln x^n) + c_2\sin(\ln x^n)\right)$$

$$+ x^m \ln x$$

h) $(D^2-2D+2)y = e^z z$

$CF = e^z(c_1\cos z + c_2\sin z)$ $\quad (D-1)^2+1$

$= x(c_1\cos(\ln x) + c_2\sin(\ln x))$

$$PI = \frac{1}{(D-1)^2+1} z e^z = e^z \frac{1}{D^2+1} z = z e^z$$

$$= x \ln x$$

$$y = x(c_1\cos \ln x + c_2 \sin \ln x) + x\ln x$$

i) $(D^2-4D+5)y = e^{2z}\sin z$

$\hookrightarrow (D-2)^2+1$

$CF = e^{2z}(c_1\cos z + c_2\sin z)$

$= x^2(c_1\cos\ln x + c_2\sin\ln x)$

$$PI = e^{2z}\frac{1}{D^2+1}\sin z \cancel{= -\frac{1}{2}e^{2z}\sin z}$$

$$\cancel{= -\frac{1}{2}x^2 \sin\ln x}$$

~~$y = x^2(c_1\cos\ln x + c_2\sin\ln x)$~~

$$\frac{1}{D^2+1}\sin z = \mathrm{Im}\left(\frac{1}{D^2+1}e^{iz}\right)$$

$$= \mathrm{Im}\left(\frac{1}{(D+i)(D-i)}e^{iz}\right)$$

$$= \mathrm{Im}\left(\frac{1}{2i}e^{iz}z\right)$$

$$= -\frac{1}{2}z\cos z$$

$$\therefore y = x^2(c_1 \cos\ln x + c_2 \sin\ln x) - \frac{1}{2}x^2 \ln x \cos\ln x$$

j) $[D(D-1)(D-2)(D-3) + 6D(D-1)(D-2)$

$+ 9D(D-1) + 3D + 1]\, y = (1+z)^2$

$\cancel{D(D^3 - 6D^2 + 11D - 6)}$

$D^4 - 6D^3 + 11D^2 - 6D + 6D^3 - 18D^2 + 12D + 9D^2 - 9D + 3D + 1$

$= D^4 + 2D^2 + 1$

$D^2 - 3D + 2$

$(D^4 + 2D^2 + 1)\, y = (1+z)^2$

$\frac{20}{-18}\atop 2$

CF : $(D^2+1)^2 y = 0 \Rightarrow (D+i)^2(D-i)^2 y = 0$

$y = (c_1 + c_2 z)\cos z + (c_3 + c_4 z)\sin z$

PI : $\frac{1}{D^4 + 2D^2 + 1}(z^2 + 2z + 1)$

$= (1 - D^4 - 2D^2)(z^2 + 2z + 1)$

$= (z^2 + 2z + 1 - 2.2.1)$

$= z^2 + 2z - 3$

$$y = (c_1 + c_2 \ln x)\cos\ln x + (c_3 + c_4 \ln x)\sin\ln x + (\ln x)^2 + 2(\ln x) - 3.$$

27 a) $(1+x)^2 y'' - 4(1+x)y' + 6y = 6(1+x)$

$1 + x = e^z \Rightarrow dx = e^z dz$

$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{(1+x)} dy/dz$

$\Rightarrow (1+x)y' = dy/dz$

$D \equiv d/dz$

$(1+x)^2 y'' = \frac{d^2y}{dz^2} - dy/dz$

$(D^2 - 5D + 6)y = 6e^z$

CF $= c_1 e^{2z} + c_2 e^{3z}$, PI $= \frac{\cancel{6}^{3} e^z}{\cancel{2}} \Rightarrow y = 3(1+x) + c_1(1+x)^2 + c_2(1+x)^3$

(b) $(x+1)^2 y'' + (x+1)y' = (2x+3)(2x+4)$

$x+1 = e^z \Rightarrow z = \ln(1+x)$

$D = \frac{d}{dz}$, $(2x+3)(2x+4)$
$= 2(x+1+1)(2x+2+1)$

$D^2 y = (2e^z+1)\cdot 2(e^z+1)$
$= 2(2e^{2z} + e^z + 2e^z + 1)$
$= 2(2e^{2z} + 3e^z + 1)$
$= 4e^{2z} + 6e^z + 2$

$y = \int (2e^{2z} + 6e^z + 2z + c_1)\,dz$

$\Rightarrow y = e^{2z} + 6e^z + z^2 + c_1 z + c_2$
$= (1+x)^2 + 6(1+x) + \ln^2(1+x)$
$+ c_1 \ln(1+x) + c_2$

(c) $(1+2x)^2 y'' - 6(1+2x)y' + 16y = 8(1+2x)^2$

$1+2x = e^z \Rightarrow z = \ln(1+2x)$

$\Rightarrow 2dx = e^z dz$

$\Rightarrow \frac{dz}{dx} = \frac{2}{1+2x}$ $D = d/dz$

$(1+2x)\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}(1+2x) = 2Dy$

$2D^2 y = \frac{d}{dz}\left((1+2x)\frac{dy}{dx}\right)$
$= 2\,\frac{\cancel{2}}{2}(1+2x)\frac{dy}{dx} + (1+2x)^2 \frac{d^2y}{dx^2}\frac{1}{2}$

$\Rightarrow 2D^2 y = 2Dy + \frac{1}{2}(1+2x)^2\frac{d^2y}{dx^2}$

$\Rightarrow (1+2x)^2 \frac{d^2y}{dx^2} = 4(D^2 - D)y$

$(4D^2 - 4D - 12D + 16)y = 8e^{2z}$

$\Rightarrow (D^2 - 4D + 4)y = 2e^{2z}$

$$CF = e^{2z}(c_1 + c_2 z)$$
$$= (1+2x)^2 (c_1 + c_2 \ln(1+2x))$$

$$PI = \frac{1}{(D-2)^2} 2e^{2z} = \cancel{2} e^{2z} \frac{z^2}{\cancel{2}!}$$
$$= (1+2x)^2 \cdot \ln^2(1+2x)$$

$$y = (1+2x)^2 \left[c_1 + c_2 \ln(1+2x) + \ln^2(1+2x) \right]$$

3Y a) $y'' - 2y' = e^x \sin x$

$\cancel{(D^2 - 2D)y = 0}$
$$(y')' = 2(y')$$
$$y' = c_1 e^{2x}$$
$$\cancel{y} y = c_1' e^{2x} + c_2$$
Let $y_p = L_1(x)(1) + L_2(x)(e^{2x})$
y_p
$$y_1 = 1, \; y_2 = e^{2x}$$
$$y' = L_1' + L_2' e^{2x} + 2L_2 e^{2x}$$
$$L_1' + L_2' e^{2x} = 0$$
$$\Rightarrow y' = 2L_2 e^{2x}$$
$$y'' = 2(L_2' e^{2x} + L_2 \cdot 2e^{2x})$$
$$= 2L_2' e^{2x} + 4L_2 e^{2x}$$
$$2L_2' e^{2x} + \cancel{4L_2 e^{2x}} - \cancel{4L_2 e^{2x}} = e^x \sin x$$
$$L_2 = \int \frac{1}{2} e^{-x} \sin x \, dx = -\frac{1}{4} e^{-x}(\sin x + \cos x)$$
$$L_1 = -\frac{1}{2} \int e^x \sin x \, dx = -\frac{1}{4} e^x (\sin x - \cos x)$$
$$y_p = -\frac{1}{4} e^x (\sin x - \cos x) - \frac{1}{4} e^x (\sin x + \cos x)$$
$$= -\frac{1}{2} e^x \sin x$$
$$\therefore y = Ae^{2x} + B - \frac{1}{2} e^x \sin x$$

b) $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

$m^2 - 6m + 9 = (m-3)^2$

$y_1(x) = e^{3x}$

$y_2(x) = xe^{3x}$

$y_p : y = L_1 y_1 + L_2 y_2$

$L_1' y_1 + L_2' y_2 = 0$

$L_1' y_1' + L_2' y_2' = \frac{e^{3x}}{x^2}$

$L_1' e^{3x} + L_2' x e^{3x} = 0$

$\Rightarrow L_1' = -x L_2'$

$L_1' 3e^{3x} + L_2' (e^{3x} + 3xe^{3x}) = \frac{e^{3x}}{x^2}$

$\Rightarrow 3L_1' + L_2'(1+3x) = \frac{1}{x^2}$

$\Rightarrow -\cancel{3x L_2'} + L_2' + \cancel{3x L_2'} = \frac{1}{x^2}$

$\Rightarrow L_2' = \frac{1}{x^2} \Rightarrow L_2 = \int \frac{1}{x^2} dx$

$= -\frac{1}{x}$

$L_1 = \int -x \cdot \frac{1}{x^2} dx$

$= -\ln x$

$\therefore y = e^{3x}(A + Bx) + (-\ln x) e^{3x}$

$+ (-1/x)(xe^{3x})$

$= e^{3x}(A' + Bx - \ln x)$

c) $y'' - 2y' + y = e^x \ln x$

$y_1(x) = e^x \, ; \, y_2(x) = xe^x$

$y_p : y = L_1 y_1 + L_2 y_2$

$L_1' y_1 + L_2' y_2 = 0$

$L_1' y_1' + L_2' y_2' = e^x \ln x$

$$L_1' + x L_2' = 0 \Rightarrow L_1' = -x L_2'$$

$$L_1' e^x + L_2'(xe^x + e^x) = e^x \ln x$$

$$\Rightarrow L_1' + L_2'(x+1) = \ln x$$

$$\Rightarrow L_2' = \ln x$$

$$\Rightarrow L_2 = \int \ln x \, dx = x\ln x - x$$

$$L_1 = \int -x \ln x = -\left[\frac{x^2}{2}\ln x - \int \frac{1}{x}\cdot\frac{x^2}{2}dx\right]$$

$$= -\frac{x^2}{2}\ln x + \frac{x^2}{4}$$

$$y_p = \left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right)e^x + (x\ln x - x)xe^x$$

$$= \left(\frac{x^2}{2}\ln x - \frac{3x^2}{4}\right)e^x$$

$$= \frac{x^2e^x}{4}(2\ln x - 3)$$

$$\therefore y = c_1 e^x + c_2 x e^x + \frac{x^2e^x}{4}(2\ln x - 3)$$

d) $y''' + y' = \tan x$

$$m^3 + m = 0 \Rightarrow m(m^2+1) = 0, \quad m = 0, \pm i$$

$y_1(x) = 1$, $y_2(x) = \cos x$, $y_3(x) = \sin x$

Let $y_p: y = L_1 y_1 + L_2 y_2 + L_3 y_3$

$$L_1' y_1 + L_2' y_2 + L_3' y_3 = 0$$

$$L_1' y_1' + L_2' y_2' + L_3' y_3' = 0$$

$$L_1' y_1'' + L_2' y_2'' + L_3' y_3'' = \tan x$$

$$L_1' + L_2' \cos x + L_3' \sin x = 0$$

$$-L_2' \sin x + L_3' \cos x = 0$$

$$-L_2' \cos x - L_3' \sin x = \tan x$$

$\cancel{L_1' = 0 \Rightarrow L_1}$

$\cancel{L_2' \sin x \cos x + L_3' \sin^2 x = 0}$

$\cancel{L_2' \sin x \cos x + L_3' \cos^2 x = 0}$

$2L_3' = 0$

$$L_2' = L_3' \cot x$$

$$L_3' \frac{\cos^2 x}{\sin x} + L_3' \sin x = -\frac{\sin x}{\cos x}$$

$$\Rightarrow L_3' = -\frac{\sin^2 x}{\cos x}$$

$$L_3 = -\int \frac{1-c^2}{c}dx$$

$$= -\ln|\sec x + \tan x| + \sin x$$

$$\therefore L_1' = \tan x \Rightarrow L_1 = \int \tan x \, dx = \ln(\sec x)$$

$$L_2 = \int -\frac{\sin^2 x}{\cos x}\cdot\frac{\cos x}{\sin x}dx = \cos x$$

$$y_p = \ln(\sec x) + \cos^2 x + \sin x(\sin x + \ln(\sec x - \tan x)$$
$$= \ln(\sec x) + \sin x \ln(\sec x - \tan x) + 1$$

$$\therefore y = c_1 + c_2 \cos x + c_3 \sin x + \ln(\sec x) + \sin x \ln(\sec x - \tan x)$$

e) $y''' - 2y'' - 21y' - 18y = 3 + 4e^{-x}$

$m^3 - 2m^2 - 21m - 18 = 0$, 6, -3

$\beta + \gamma = 3$

$\alpha\beta\gamma = 18$

~~1 - 2 + 21 - 18~~

$\alpha = -1$

$\beta = 6$

$\gamma = -3$

$y_1(x) = e^{-x}$, $y_2(x) = e^{6x}$, $y_3(x) = e^{-3x}$

$y_p : y = c_1 y_1 + c_2 y_2 + c_3 y_3$

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3+4e^{-x} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} e^{-x} & e^{6x} & e^{-3x} \\ -e^{-x} & 6e^{6x} & -3e^{-3x} \\ e^{-x} & 36e^{6x} & 9e^{-3x} \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3+4e^{-x} \end{bmatrix}$$

$R_1 \to R_1 + R_2$, $R_3 \to R_3 + R_2$, $R_2 \to R_2(-e^{x})$

$6 \times (-2)$

12

18

$$\begin{bmatrix} 0 & 7e^{6x} & -2e^{-3x} \\ 1 & -6e^{7x} & 3e^{-2x} \\ 0 & 42e^{6x} & 6e^{-3x} \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3+4e^{-x} \end{bmatrix}$$

$R_3 \to R_3 - 6R_1$

$$\begin{bmatrix} 0 & 7e^{6x} & -2e^{-3x} \\ 1 & -6e^{7x} & 3e^{-2x} \\ 0 & 0 & 18e^{-3x} \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3+4e^{-x} \end{bmatrix}$$

$$18e^{-3x} L_3' = 3 + 4e^{-x}$$

$$\Rightarrow L_3' = \frac{1}{18}(3e^{3x} + 4e^{2x})$$

$$L_3 = \frac{1}{18}(e^{3x} + 2e^{2x})$$

$$7e^{6x} L_2' - 2e^{-3x} \frac{1}{18}(3e^{3x} + 4e^{2x}) = 0$$

$$\Rightarrow 7e^{6x} L_2' - \frac{1}{9}(3 + 4e^{-x}) = 0$$

$$L_2' = \frac{1}{63}(3e^{-6x} + 4e^{-7x})$$

$$L_2 = \frac{1}{63}\left(-\frac{3}{6}e^{-6x} - \frac{4}{7}e^{-7x}\right)$$

$$= -\frac{1}{63}\left(\frac{1}{2}e^{-6x} + \frac{4}{7}e^{-7x}\right)$$

$$L_1' - 6e^{7x} L_2' + 3e^{-2x} L_3' = 0$$

$$L_1' - 6e^{7x} \cdot \frac{1}{63}(3e^{-6x} + 4e^{-7x}) + \frac{3e^{-2x}}{18}(3e^{3x} + 4e^{2x}) = 0$$

$$\Rightarrow L_1' - \frac{2}{21}(3e^{x} + 4) + \frac{1}{6}(3e^{x} + 4) = 0$$

$$\Rightarrow L_1' = \left(\frac{2}{21} - \frac{1}{6}\right)(3e^{x} + 4)$$

$$= \left(\frac{4 - 7}{42}\right)(3e^{x} + 4) = -\frac{3}{42}(3e^{x} + 4)$$

$$= -\frac{1}{14}(3e^{x} + 4)$$

$$L_1 = -\frac{1}{14}(3e^{x} + 4x)$$

or

$$L_1' = \frac{W_1}{W}$$

$$= \frac{\begin{vmatrix} 0 & e^{6x} & e^{-3x} \\ 0 & 6e^{6x} & -3e^{-3x} \\ 3+4e^{-x} & 36e^{6x} & 9e^{-3x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & e^{6x} & e^{-3x} \\ -e^{-x} & 6e^{6x} & -3e^{-3x} \\ e^{-x} & 36e^{6x} & 9e^{-3x} \end{vmatrix}}$$

$$= \frac{(3+4e^{-x})(-9e^{3x})}{e^{2x}\begin{vmatrix} 1 & 1 & 1 \\ -1 & 6 & -3 \\ 1 & 36 & 9 \end{vmatrix}}$$

$$= \frac{-9e^{x}(3+4e^{-x})}{\begin{vmatrix} 0 & 7 & -2 \\ -1 & 6 & -3 \\ 0 & 42 & 6 \end{vmatrix}}$$

$$= \frac{-9e^{x}(3+4e^{-x})}{42 \times 9}$$

$$L_1' = -\frac{1}{14}(3e^{x} + 4)$$

$$\therefore y_p = -\frac{1}{14}(3e^{x} + 4x)e^{-x} - \frac{1}{63}\left(\frac{1}{2}e^{-6x} + \frac{4}{7}e^{-7x}\right)e^{6x} + \frac{1}{18}(e^{3x} + 2e^{2x})e^{-3x}$$

$$= -\frac{1}{14}(3 + 4xe^{-x}) - \frac{1}{63}\left(\frac{1}{2} + \frac{4}{7}e^{-x}\right) + \frac{1}{18}(1 + 2e^{-x})$$

$$= -\frac{1}{6} - \frac{2}{7}xe^{-x} + \frac{5}{49}e^{-x}$$

$$\therefore y = c_1 e^{-x} + c_2 e^{6x} + c_3 e^{-3x} - \frac{1}{6} - \frac{2}{7}xe^{-x}$$

f) $y'' - 2y' + 2y = e^{x}\tan x$

$$m^2 - 2m + 1 + 1 = (m-1)^2 - i^2$$
$$1+i,\ 1-i$$

$y_1(x) = e^{x}\cos x$

$y_2(x) = e^{x}\sin x$

$$y_p : y = L_1 y_1 + L_2 y_2$$

$$L_1' y_1 + L_2' y_2 = 0$$
$$L_1' y_1' + L_2' y_2' = e^{x}\tan x$$

$$\Rightarrow L_1' \cos x + L_2' \sin x = 0$$
$$\Rightarrow L_1' = -L_2' \tan x$$

$$\Rightarrow L_1' \cancel{e^{x}}(\cos x - \sin x) + L_2' \cancel{e^{x}}(\sin x + \cos x) = \cancel{e^{x}}\tan x$$

$$\Rightarrow -L_2' \tan x(\cos x - \sin x) + L_2'(\sin x + \cos x) = \tan x$$

$$\Rightarrow L_2'\left(\cancel{\sin x} + \cos x - \cancel{\sin x} + \frac{\sin^2 x}{\cos x}\right) = \tan x$$

$$\Rightarrow \frac{L_2'}{\cos x} = \frac{\sin x}{\cos x} \Rightarrow L_2 = -\cos x$$

$$L_1' = -\sin x \cdot \frac{\sin x}{\cos x}$$
$$= \cos x - \sec x$$
$$L_1 = \sin x - \ln|\sec x + \tan x|$$

$$y_p = -\cos x\, e^{x}\sin x + (\sin x - \ln|\sec x + \tan x|)\, e^{x}\cos x$$
$$= -e^{x}\cos x \ln|\sec x + \tan x|$$

$$\therefore y = e^{x}(A\cos x + B\sin x) - e^{x}\cos x \ln|\sec x + \tan x|$$

47) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$

$y_1(x) = 1$ $\qquad m^2 - 2m = 0$

$y_2(x) = e^{2x}$ $\qquad m(m-2) = 0$

$y_p = L_1 y_1 + L_2 y_2$

$L_1' y_1 + L_2' y_2 = 0$

$L_1' y_1' + L_2' y_2' = e^x \sin x$

$\Rightarrow L_1' = -e^{2x} L_2'$

$L_2'(2e^{2x}) = e^x \sin x$

$\Rightarrow L_2 = \int \frac{1}{2} e^{-x} \sin x \, dx$

$\int e^{-x} \sin x \, dx = \mathrm{Im}\left[\int e^{-x} e^{ix} dx\right] = \mathrm{Im}\left(\frac{e^{-x} e^{ix}}{(i-1)(i+1)}(i+1)\right)$

$= -\frac{e^{-x}}{2} \mathrm{Im}\left[(i+1)(\cos x + i \sin x) \right]$

$= -\frac{e^{-x}}{2}(\cos x + \sin x)$

$L_2 = -\frac{1}{4} e^{-x}(\cos x + \sin x)$

$L_1 = \int -\frac{1}{2} e^{x} \sin x \, dx$

$\int e^x \sin x \, dx = \mathrm{Im}\left[\int e^x e^{ix} dx\right]$

$= \mathrm{Im}\left[\frac{e^x e^{ix}(1-i)}{(1+i)(1-i)}\right]$

$= \frac{e^x}{2} \mathrm{Im}\left((1-i)(\cos x + i \sin x)\right)$

$= \frac{e^x}{2}(\sin x - \cos x)$

$L_1 = -\frac{1}{4} e^x (\sin x - \cos x)$

$y_p = -\frac{1}{4} e^x(\sin x - \cos x) - \frac{1}{4} e^x (\cos x + \sin x)$

$= -\frac{1}{2} e^x \sin x$ $\qquad \Rightarrow y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x$

$y(0) = 0$

$\Rightarrow c_1 + c_2 = 0$

$y'(0) = 0$

$2c_2 - \frac{1}{2} = 0$

$\Rightarrow c_2 = 1/4, \; c_1 = -1/4$

$\therefore y = -\frac{1}{4} + \frac{1}{4} e^{2x} - \frac{1}{2} e^x \sin x$