Spring 2023 : Linear Algebra

1) a) (i) $W_1 = \{a_0 + a_1 x + a_2 x^2 : a_0 \geq 0, a_1 \geq 0, a_2 \geq 0\}$

Let us consider $\vec{u}, \vec{v} \in W_1$

$\vec{u} = a_0 + a_1 x + a_2 x^2$,

$\vec{v} = b_0 + b_1 x + b_2 x^2$, $a_i, b_i \geq 0$

Consider scalars $\alpha = b_0$, $\beta = -2a_0$

$\alpha\vec{u} + \beta\vec{v}$

$= \alpha(a_0 + a_1 x + a_2 x^2) + \beta(b_0 + b_1 x + b_2 x^2)$

$= \underbrace{-a_0 b_0} + (a_1 b_0 - 2a_0 b_1)x + (a_2 b_0 - 2a_0 b_2)x^2$

for $a_0 > 0$, $b_0 > 0$, $-a_0 b_0 < 0$

$\therefore \alpha\vec{u} + \beta\vec{v} \notin W_1$ for some $\vec{u}, \vec{v} \in W_1$ and $\alpha, \beta \in \mathbb{R}$

$\therefore$ $\boxed{W_1 \text{ is not a subspace}}$.

(ii) $W_2 = \{a_0 + 2a_1 x + 3a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}, a_0 + a_1 + a_2 = 0\}$

Let us consider $\vec{u}, \vec{v} \in W_2$

$\vec{u} = a_0 + 2a_1 x + 3a_2 x^2$

$\vec{v} = b_0 + 2b_1 x + 3b_2 x^2$, $a_i, b_i \in \mathbb{R}$, $\sum a_i = 0 = \sum b_i$

Consider scalars $\alpha, \beta \in \mathbb{R}$

$\alpha\vec{u} + \beta\vec{v} = (\alpha a_0 + \beta b_0) + 2(\alpha a_1 + \beta b_1)x + 3(\alpha a_2 + \beta b_2)x^2$

$(\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1) + (\alpha a_2 + \beta b_2)$

$= \alpha(a_0 + a_1 + a_2) + \beta(b_0 + b_1 + b_2) = \alpha(0) + \beta(0) = 0$

$\therefore \alpha\vec{u} + \beta\vec{v} \in W_2 \; \forall \; \vec{u}, \vec{v} \in W_2$ and $\alpha, \beta \in \mathbb{R} \Rightarrow$ $\boxed{W_2 \text{ is a subspace.}}$

(b) $V_1 = V_2 = \mathbb{P}_2(\mathbb{R})$

$T : V_1 \to V_2$

$T(a+bx+cx^2) = b + 2cx + ax^2$

$B_1 = \{1+x, 1+x^2, 2\}$

$B_2 = \{1+x+x^2, 1-x+x^2, 2\}$

$T(1+x) = 1+x^2$

$T(1+x^2) = 2x + x^2$

$T(2) = 2x^2$

$1+x^2 = c_1(1+x+x^2) + c_2(1-x+x^2) + c_3(2)$

Comparing coefficients of x^2 we get

$1 = c_1 + c_2$

Comparing coefficient of x we get.

$c_1 = \frac{1}{2} = c_2$

$0 = c_1 - c_2$

Comparing coefficient of constant term we get

$1 = c_1 + c_2 + 2c_3$

$\Rightarrow 2c_3 + \cancel{1} = \cancel{1} \Rightarrow c_3 = 0$.

$\therefore T(1+x) = \frac{1}{2}(1+x+x^2) + \frac{1}{2}(1-x+x^2) + 0(2)$. —(i)

$2x + x^2 = c_4(1+x+x^2) + c_5(1-x+x^2) + c_6(2)$

Comparing coefficient of x^2 we get

$1 = c_4 + c_5$

Comparing coefficient of x we get

$c_4 = 3/2$

$c_5 = -1/2$

$2 = c_4 - c_5$

Comparing coefficient of constant term we get

$0 = c_4 + c_5 + 2c_6 \Rightarrow 2c_6 = -1 \Rightarrow c_6 = -\frac{1}{2}$

$\Rightarrow T(1+x^2) = \frac{3}{2}(1+x+x^2) - \frac{1}{2}(1-x+x^2) - \frac{1}{2}(2)$ —(ii)

$T(2) = 2x^2 = 1(1+x+x^2) + 1(1-x+x^2) + (-1)2$ — (iii)

$$\therefore \boxed{M = \begin{pmatrix} 1/2 & 3/2 & 1 \\ 1/2 & -1/2 & 1 \\ 0 & -1/2 & -1 \end{pmatrix}}$$

2)(a) $p(t) = t^2 + 4t - 3$

$t^2 + 4t - 3 = c_1(t^2 - 2t + 5) + c_2(2t^2 - 3t) + c_3(t + 3)$

$$\begin{aligned} c_1 + 2c_2 \quad &= 1 \\ -2c_1 - 3c_2 + c_3 &= 4 \\ 5c_1 \quad + 3c_3 &= -3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ -2 & -3 & 1 & 4 \\ 5 & 0 & 3 & -3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & -10 & 3 & -8 \end{array}\right] \quad R_3 \to R_3 + 10R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 13 & 52 \end{array}\right]$$

$-3+4$

$4+2$

$6+10$

$-3-5$

$\Rightarrow 13c_3 = 52 \Rightarrow c_3 = 4$

$c_2 + c_3 = 6 \Rightarrow c_2 = 6 - 4 = 2$

$c_1 + 2c_2 = 1 \Rightarrow c_1 + 4 = 1 \Rightarrow c_1 = -3$

-10 $3+10$

$-8+60$

$$\therefore \boxed{p(t) = -3p_1(t) + 2p_2(t) + 4p_3(t)}$$

(b)

$\cancel{A\begin{pmatrix}1\\1\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix}}$

$D = P^{-1}AP$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$D^2 = P^{-1}APP^{-1}AP$
$= P^{-1}A^2P$
$\Rightarrow A^2 = PD^2P^{-1}$

$D^2 = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$

$P^{-1} = \frac{\text{adj } P}{\det P} = \frac{\text{adj } P}{-1} = -1\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$
$= \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

$A^2 = PD^2P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 9 & 8 \\ 9 & 4 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

$A^2 = \begin{pmatrix} -1 & 10 \\ -5 & 14 \end{pmatrix}$

$A^2\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix} -1 & 10 \\ -5 & 14 \end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} = \boxed{\begin{pmatrix}26\\22\end{pmatrix}}$

~~9+8~~
18-8
~~-9+9~~
18-4
$-4+30$
$-20+42$

3)(a)

$\{(2,3,-1),(9,8,0),(4,-5,a+b)\}$

$\{(2,3,-1),(9,8,0),(1,-15,a+3b)\}$

$$\begin{pmatrix} 2 & 3 & -1 \\ 9 & 8 & 0 \\ 4 & -5 & a+b \end{pmatrix} \quad R_3 \to R_3 - 2R_1$$

$$\begin{pmatrix} 2 & 3 & -1 \\ 9 & 8 & 0 \\ 0 & -11 & a+b+2 \end{pmatrix} \quad R_2 \to R_2 - \frac{9R_1}{2}$$

$$\begin{pmatrix} 2 & 3 & -1 \\ 0 & -5.5 & 4.5 \\ 0 & -11 & a+b+2 \end{pmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\begin{pmatrix} \boxed{2} & 3 & -1 \\ 0 & \boxed{-5.5} & 4.5 \\ 0 & 0 & \boxed{a+b-7} \end{pmatrix}$$

$a+b-7=0 \Rightarrow a+b=7$ —(i)

$\cancel{4-4}$

40

$a+b+2$

$9 - \frac{9}{2} \times 2$

$8 - \frac{9}{2} \times 3$

$\frac{55}{55}$

$-11 + 11$

$-11 + 11$

$a+b+2 = 9$

$$\begin{pmatrix} 2 & 3 & -1 \\ 9 & 8 & 0 \\ 1 & -15 & a+3b \end{pmatrix}$$

$R_1 \to R_1 - 2R_3$

$R_2 \to R_2 - 9R_3$

$$\begin{pmatrix} 0 & 33 & -2a-6b-1 \\ 0 & 143 & -9a-27b \\ 1 & -15 & a+3b \end{pmatrix} \quad R_3 \to R_3 - \frac{3}{13}R_2$$

$$\begin{pmatrix} 0 & 0 & \boxed{-2a-6b-1+\frac{3}{13}(9a+27b)} \\ 0 & \boxed{143} & -9a-27b \\ \boxed{1} & -15 & a+3b \end{pmatrix}$$

$3+2(-15)$

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$-1-2(a+3b)$

$-2(a+3b)-1+\frac{27}{13}(a+3b)=0$

$\Rightarrow \left(\frac{27}{13}-2\right)(a+3b)=1 \Rightarrow a+3b=13$ —(ii)

$$a+b=7$$
$$a+3b=13$$
$$2b=6 \Rightarrow \boxed{\begin{matrix} b=3 \\ a=4 \end{matrix}}$$

(b) $2x+3y+5z=9$

$7x+3y-5z=8$

$2x+3y+\lambda z=\mu$

$$\left[\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{2 \quad 3 \quad 5}^{A}} & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$
$$\tilde{A}$$

$R_3 \to R_3 - R_1$

$R_2 \to R_2 - \frac{7R_1}{2}$

$$\left[\begin{array}{ccc|c} \boxed{2} & 3 & 5 & 9 \\ 0 & \boxed{-7.5} & -22.5 & -23.5 \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right]$$

(i) For no sol^n, $\text{rank}(A) \neq \text{rank}(\tilde{A})$

$\therefore \lambda-5=0$ but $\mu-9 \neq 0$

$\Rightarrow \boxed{\lambda=5, \mu \neq 9}$

(ii) For unique sol^n, $\text{rank}(A) = \text{rank}(\tilde{A}) = 3$

$\boxed{\lambda \neq 5}$

(iii) For infinite no. of sol^n, $\text{rank}(A) = \text{rank}(\tilde{A}) \neq 3$

$\Rightarrow \boxed{\lambda=5, \mu=9}$

4)(a) $U = \{(x,y,z) \in \mathbb{R}^3 : x+2y+3z=0\}$

$V = \{(x,y,z) \in \mathbb{R}^3 : 2x+3y+4z=0\}$

$U \cap V = \{(x,y,z) \in \mathbb{R}^3 : x+2y+3z=0$ and $2x+3y+4z=0\}$

$x+2y+3z=0$

$2x+3y+4z=0$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$$

3-4

4-6

$x+2y+3z=0$

$-y-2z=0$

$z=k,\ y=-2k$

$x+2(-2k)+3(k)=0$

$\Rightarrow x-4k+3k=0 \Rightarrow x=k$

$\therefore U \cap V = \{(k,-2k,k) : k \in \mathbb{R}\}$

a basis of $U \cap V$ is $(1,-2,1)$

$\dim(U \cap V) = 1$

(b) $2x+3y+z=0$

$5x+7y+2z=0$

$6x+2y+\alpha z=0$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 5 & 7 & 2 & 0 \\ 6 & 2 & \alpha & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - \frac{5R_1}{2} \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -7 & \alpha-3 & 0 \end{array}\right] \quad R_2 \to -2R_2$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -7 & \alpha-3 & 0 \end{array}\right]$$

$R_3 \to R_3 + 7R_2$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \alpha+4 & 0 \end{array}\right]$$

$\cancel{-7+7}$

$\cancel{\alpha-3+7}$

$\cancel{-7+7}$

$\cancel{\alpha-3+7}$

for infinite solⁿ, $\alpha+4=0$

$\Rightarrow \boxed{\alpha=-4}$

$y+z=0,\ z=k,\ y=-k$

$2x+3y+z=0$

$\Rightarrow 2x+3(-k)+k=0$

$\Rightarrow 2x=2k \Rightarrow x=k$

$\therefore$ Solution space $=\left\{\begin{pmatrix} k \\ -k \\ k \end{pmatrix} : k\in\mathbb{R}\right\}$

a basis for this is $\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$

$\therefore \boxed{\text{Dim (Sol. space)} = 1}$

5) (a) $$\left[\begin{array}{ccc|c} 0 & 3 & 1 & -9 \\ 4 & 2 & 1 & 0 \\ 16 & 3 & 1 & -1 \end{array}\right]$$

$R_3 \to R_3 - 4R_2$

$$\left[\begin{array}{ccc|c} 0 & 3 & 1 & -9 \\ 4 & 2 & 1 & 0 \\ 0 & -5 & -3 & -1 \end{array}\right]$$

$R_2 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} 4 & 2 & 1 & 0 \\ 0 & 3 & 1 & -9 \\ 0 & -5 & -3 & -1 \end{array}\right]$$

$R_3 \to R_3 + \frac{5R_2}{3}$

$$\left[\begin{array}{ccc|c} 4 & 2 & 1 & 0 \\ 0 & 3 & 1 & -9 \\ 0 & 0 & -4/3 & -16 \end{array}\right]$$

$$-\frac{4}{3}z = -16 \Rightarrow z = 12$$

$$3y + z = -9 \Rightarrow 3y + 12 = -9$$

$$\Rightarrow 3y = -21$$

$$\Rightarrow y = -7$$

$$4x + 2y + z = 0$$

$$\Rightarrow 4x - 14 + 12 = 0 \Rightarrow 4x = 2 \Rightarrow x = 1/2$$

$$\boxed{(x, y, z) = (1/2, -7, 12)}$$

(b) $T : M_{2\times2}(\mathbb{R}) \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b, c+d).$$

$$\ker(T) = \{ A : A \in M_{2\times2}(\mathbb{R}), T(A) = (0,0) \}$$

$$a+b = 0, \quad c+d = 0$$

$$\therefore A = \begin{pmatrix} a & -a \\ c & -c \end{pmatrix} = a\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

$$\therefore \ker(T) = \text{span}\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$$

$$\therefore \boxed{\text{nullity} = 2}$$

$$\text{range}(T) = \left\{ P : P \in \mathbb{R}^2,\ T(A) = P \text{ for } A \in M_{2\times2} \right\}$$

$$(a+b)(1,0) + (c+d)(0,1)$$

$\therefore$ range $(T) = \text{span}\{(1,0), (0,1)\}$

$\therefore$ rank $= 2$

$$\boxed{\text{nullity} + \text{rank} = 4 = \dim(M_{2\times2}(\mathbb{R}))}$$