Tutorial - 10

1) (i) $\int_1^\infty \frac{1}{x^4}\,dx = \lim_{r\to\infty}\int_1^r x^{-4}dx = \frac{1}{x^3}\left(\frac{-1}{3}\right)\Big|_1^r$

$= \frac{1}{3x^3}\Big|_r^1$

$= \frac{1}{3}\left(1-\frac{1}{r^3}\right)$

$\lim_{r\to\infty}\frac{1}{3}\left(1-\frac{1}{r^3}\right) = 1/3$

(ii) $\int_{10}^\infty \frac{1}{x\ln x}\,dx = \lim_{r\to\infty}\int_{10}^r \frac{1}{x\ln x}\,dx$ $\quad \frac{1}{t}dt \quad \ln(t)$

$= \lim_{r\to\infty}\ln(\ln x)\Big|_{10}^r$

divergent

(iii) $\int_0^1 \frac{\ln x}{x^2}\,dx = \lim_{r\to 0^+}\int_r^1 \frac{\ln x}{x^2}\,dx$

$\int_r^1 \frac{\ln x}{x^2}\,dx = \ln x\left(\frac{-1}{x}\right)\Big|_r^1 + \int_r^1 \frac{1}{x}\frac{1}{x}\,dx$

$= \frac{\ln r}{r} + \frac{1}{x}\Big|_1^r$ $\quad \frac{x^{-2+1}}{-2+1} \quad x^{-2} \quad \frac{x^{-2+1}}{-2+1}$

$= \frac{\ln r}{r} + \frac{1}{r} - 1$

$= \frac{1+\ln r}{r} - 1$

$\lim_{r\to 0^+}\left(\frac{1+\ln r}{r} - 1\right)$ dne $\Rightarrow$ divergent

(iv) $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

$= \lim_{r_1\to\infty}\int_{-r_1}^c \frac{1}{2}\frac{2x}{x^2+1}\,dx + \lim_{r_2\to\infty}\int_c^{r_2}\frac{1}{2}\frac{2x}{x^2+1}\,dx$

$= \lim_{r_1\to\infty}\frac{1}{2}\ln(x^2+1)\Big|_{-r_1}^c + \lim_{r_2\to\infty}\frac{1}{2}\ln(x^2+1)\Big|_c^{r_2}$

$= \frac{1}{2}\lim_{r_1\to\infty}\ln\left(\frac{c^2+1}{r_1^2+1}\right) + \frac{1}{2}\lim_{r_2\to\infty}\ln\left(\frac{r_2^2+1}{c^2+1}\right) \to$ dne

$\therefore \int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx \to$ divergent.

$\int t^{-1/3}\,dt$

$\frac{t^{-1/3+1}}{-1/3+1} = \frac{t^{2/3}}{2/3}$

(v) $\int_1^2 \frac{4x}{(x^2-4)^{1/3}}\,dx$

$= \lim_{r\to 2^-} \int_1^r 2\frac{2x}{(x^2-4)^{1/3}}\,dx = \lim_{r\to 2^-} 2\times\frac{3}{2}(x^2-4)^{2/3}\Big|_1^r$

$= \lim_{r\to 2^-} 3\left[(r^2-4)^{2/3} - 9^{1/3}\right] = -3^{\frac{5}{3}}$

convergent

$\frac{(t^3)^2}{9}$

$t^{2/3}$

(vi) $\int_1^{\infty} \frac{x+1}{x^{3/2}}\,dx = \lim_{r\to\infty}\int_1^r (x^{-1/2}+x^{-3/2})\,dx$

$1-3/2$

$= (2x^{1/2} - 2x^{-1/2})\Big|_1^r$

$x^{-1/2+1}$

$= 2\left(\sqrt{r}-\frac{1}{\sqrt{r}}\right)$

$-3/2+1$

$\lim_{r\to\infty} 2\left(\sqrt{r}-\frac{1}{\sqrt{r}}\right) \to$ dne

$\therefore \int_1^{\infty} \frac{x+1}{x^{3/2}}\,dx \to$ divergent

$-1/2+1$

(vii) $\int_{-\infty}^{0} \frac{1}{(3-x)^{1/2}}\,dx = \lim_{r\to\infty}\int_{-r}^{0} \frac{1}{(3-x)^{1/2}}\,dx$

$= \lim_{r\to\infty} 2(3-x)^{1/2}\Big|_0^{-r}$

$= \lim_{r\to\infty} 2(\sqrt{3+r}-\sqrt{3}) \to$ dne

$\therefore \int_{-\infty}^{0} \frac{1}{(3-x)^{1/2}}\,dx \to$ divergent.

(viii) $\int_{-2}^{\infty} \sin x\,dx = \lim_{r\to\infty} \cos x\Big|_r^{-2} = \lim_{r\to\infty}(\cos 2 - \cos r)$
$\to$ dne

$\int_{-2}^{\infty} \sin x\,dx \to$ divergent

(ix) $\int_1^{\infty} \frac{1}{(1+x)\sqrt{x}}\,dx = \lim_{r\to\infty}\int_1^r \frac{2}{(1+(\sqrt{x})^2)(2\sqrt{x})}\,dx$

$= 2\lim_{r\to\infty}\int_1^{\sqrt{r}} \frac{dt}{1+t^2}$

$= 2\lim_{r\to\infty}\left(\tan^{-1}\sqrt{r}-\frac{\pi}{4}\right)$

$= \pi/2$

Convergent

$\sqrt{x}=t$

$\Rightarrow \frac{1}{2\sqrt{x}}\,dx = dt$

(x) $\int_1^2 \frac{1}{x\ln^2 x}\,dx$

$\ln x = t$

$t^{-2}dt$

$\frac{t^{-2+1}}{-2+1} = -\frac{1}{t}$

$= \lim_{r\to 1^+} \int_r^2 \frac{1}{x\ln^2 x}\,dx$

$= \lim_{r\to 1^+} \frac{1}{\ln x}\Big|_2^r$

$= \lim_{r\to 1^+} \left(\frac{1}{\ln r} - \frac{1}{\ln 2}\right) \to$ dne

$\therefore \int_1^2 \frac{1}{x\ln^2 x}\,dx \to$ divergent

2>(i) $\int_0^1 \frac{1}{x^2+\sqrt{x}}\,dx$

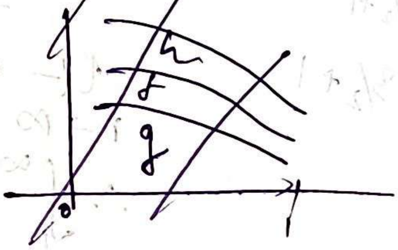
$0 < x < 1$

$x^2 < \sqrt{x}$

Let $f(x) = \frac{1}{x^2+\sqrt{x}}$

$g(x) = \frac{1}{2\sqrt{x}}$

$h(x) = \frac{1}{2x^2}$



for $0 < x \le 1$, $0 < g \le f \le h$

$\cancel{\lim_{r\to 0^+} \int_r^1 \frac{1}{2\sqrt{x}}\,dx =}$

~~We know~~ that $\int_a^b \frac{dx}{(x-a)^p}$ converges for $p < 1$

$\therefore$ g converges, h diverges.

$\int \frac{1}{\sqrt{x}\left((\sqrt{x})^3+1\right)}\,dx$

$\sqrt{x} = t$

$\frac{1}{2\sqrt{x}}dx = dt$

$= \int \frac{2\,dt}{t^3+1}$

$= 2\int \frac{t^2-(t^2-1)}{t^3+1}\,dt = \frac{2}{3}\int \frac{3t^2}{t^3+1}\,dt - 2\int \frac{(t+1)(t-1)}{(t+1)(t^2-t+1)}\,dt$

$= \frac{2}{3}\ln(x^{3/2}+1) - 2\int \frac{t-1}{t^2-t+1}\,dt$

$$\frac{1}{2}\int \frac{2t-2}{t^2-t+1}\,dt = \frac{1}{2}\int \frac{\cancel{(t-1)}\,2t-1-1}{t^2-t+1}\,dt$$

$\cancel{t^2-t+1}$

$$= \frac{1}{2}\int \frac{2t-1}{t^2-t+1}\,dt - \frac{1}{2}\int \frac{dt}{t^2-t+1}$$

$$= \frac{1}{2}\ln(t^2-t+1) - \frac{1}{2}\int \frac{dt}{(t-1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{2}\ln(t^2-t+1) - \frac{1}{2}\,\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-1/2}{\sqrt{3}/2}\right)$$

$$= \frac{1}{2}\ln(x-\sqrt{x}+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\sqrt{x}-1}{\sqrt{3}}\right)$$

$$\int \frac{dx}{x^2+\sqrt{x}} = \frac{2}{3}\ln(x^{3/2}+1) - \ln(x-\sqrt{x}+1) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\sqrt{x}-1}{\sqrt{3}}\right)$$

$$\int_0^1 \frac{dx}{x^2+\sqrt{x}} = \lim_{r\to 0^+}\int_r^1 \frac{dx}{x^2+\sqrt{x}}$$

$$-\ln(1)+\ln(r-\sqrt{r}+1)$$

$$= \lim_{r\to 0^+}\left[\frac{2}{3}\left(\ln 2 - \ln(r^{3/2}+1)\right) + \frac{2}{\sqrt{3}}\left(\frac{\pi}{6} - \tan^{-1}\left(\frac{2\sqrt{r}-1}{\sqrt{3}}\right)\right)\right]$$

$$= \frac{2}{3}\ln 2 + \frac{2}{\sqrt{3}}\cancel{\left(\frac{\pi}{6}+\frac{\pi}{6}\right)}\ \frac{\pi}{3}$$

$$= \frac{2}{3}\ln 2 + \frac{2\pi}{3\sqrt{3}}$$

$\therefore \int_0^1 \frac{1}{x^2+\sqrt{x}}\,dx$ is convergent.

OR

$$\lim_{x\to 0^+}\frac{\sqrt{x}}{x^2+\sqrt{x}} = \lim_{x\to 0^+}\frac{1}{x^{3/2}+1} = 1$$

for $\mu = 1/2$, $\lim_{x\to 0^+} x^\mu f(x)$ exists (from μ test).

$\therefore \int_0^1 f(x)\,dx$ exists

OR

Let $g(x) = \frac{1}{\sqrt{x}}$

$$\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}\frac{\sqrt{x}}{x^2+\sqrt{x}} = 1$$

$\int_{0^+}^1 \frac{1}{\sqrt{x}}\,dx$ converges, so $\int_{0^+}^1 \frac{1}{x^2+\sqrt{x}}$ also converges.

(ii) $\displaystyle\int_0^\infty \frac{1}{x+e^x}\,dx$

Let $g(x) = \frac{1}{e^x}$, $f(x) = \frac{1}{x+e^x}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{e^x}{x+e^x} = 1$$

$$\int_0^\infty e^{-x}\,dx = \lim_{r\to\infty} \left. e^{-x} \right|_r^0$$

$$= \lim_{r\to\infty} (1 - e^{-r}) = 1$$

$\therefore \int_0^\infty f(x)\,dx$ also converges.

(iii) $\displaystyle\int_0^\infty \frac{1}{x^2+xe^x}\,dx$

$$= \underbrace{\int_0^1 \frac{1}{x^2+xe^x}\,dx}_{2^{nd}\text{ kind}} + \int_1^\infty \underbrace{\frac{dx}{x^2+xe^x}}_{1^{st}\text{ kind}} = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$$

Let $g(x) = \frac{1}{x}$, $h(x) = \frac{1}{xe^x}$

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{x}{x^2+xe^x} = \lim_{x\to 0^+} \frac{1}{x+e^x} = 1.$$

$$\lim_{x\to\infty} \frac{f(x)}{h(x)} = \lim_{x\to\infty} \frac{xe^x}{x^2+xe^x} = \lim_{x\to\infty} \frac{e^x}{e^x+x} = 1.$$

as $\int_0^1 g(x)\,dx$ diverges, so $\int_0^1 f(x)\,dx$ diverges.

hence $\int_0^\infty f(x)\,dx$ diverges.

(iv) $\displaystyle\int_0^\infty \frac{1-\cos x}{x^2}\,dx$. ~~proper integral~~ ~~hence convergent~~.

$$\cancel{\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}}$$

$$\cancel{\lim_{x\to\infty} \frac{1-\cos x}{x^2} = 0}$$

$\cancel{f(x) = \frac{1-\cos x}{x^2}}$

$\cancel{g(x) =}$

$$\int_0^\infty \frac{2\sin^2 x/2}{(x^2/4)\,4}\,dx = \frac{1}{2}\int_0^\infty \frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2}\,dx$$

$x = 2u$

$dx = 2\,du$

$$= \int_0^\infty \frac{\sin^2 u}{u^2}\,du = \int_0^1 \frac{\sin^2 u}{u^2}\,du + \int_1^\infty \frac{\sin^2 u}{u^2}\,du$$

proper integral

$\frac{\sin^2 u}{u^2} \leq \frac{1}{u^2}$

$\int_1^\infty \frac{1}{u^2} du$ is convergent. $\therefore \int_1^\infty \frac{\sin^2 u}{u^2} du$ is convergent

$\therefore \int_0^\infty \frac{1-\cos x}{x^2} dx$ is convergent.

(v) $\int_1^\infty \frac{x}{(1+x)^3} dx \xrightarrow{\text{OR}} \int_1^\infty \frac{x+1-1}{(x+1)^3} dx$.

$f(x) = \frac{x}{(1+x)^3}$, $g(x) = \frac{1}{x^2}$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^3}{(1+x)^3} = 1$

$\int_1^\infty \frac{1}{x^2} dx$ converges $\therefore \int_1^\infty \frac{x}{(1+x)^3} dx$ converges.

(vi) $\int_1^\infty \frac{x\,dx}{3x^4+5x^2+1}$

$f(x) = \frac{x}{3x^4+5x^2+1}$, $g(x) = \frac{1}{3x^3}$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{3x^4}{3x^4+5x^2+1} = 1$

$\int_1^\infty \frac{1}{3x^3} dx$ converges $\therefore \int_1^\infty \frac{x\,dx}{3x^4+5x^2+1}$ converges.

(vii) $\int_{-\infty}^\infty e^{-|x|} dx = \int_{-\infty}^0 e^x dx + \int_0^\infty e^{-x} dx$

$= \lim_{r_1\to\infty} \int_{-r_1}^0 e^x dx + \lim_{r_2\to\infty} \int_0^{r_2} e^{-x} dx$

$= \lim_{\substack{r_1\to\infty \\ r_2\to\infty}} e^x \Big|_{-r_1}^0 + e^{-x} \Big|_{r_2}^0$

$= \lim_{\substack{r_1\to\infty \\ r_2\to\infty}} (2 - e^{-r_1} - e^{-r_2}) = 2.$

(viii) $\int_0^\infty \frac{\cos x}{e^x}\,dx$

Let $g(x) = \frac{1}{e^x}$, $f(x) = \cos x$

for $x \in [0, \infty)$, range of g is $(0, 1]$.

So, g is bounded. Also g is monotone

$\lim_{x \to \infty} g(x) = 0$

$\left|\int_0^b \cos x\,dx\right| = |\sin b| \le 1$

$\therefore \int_0^\infty \cos x \cdot \frac{1}{e^x}\,dx$ converges

by Dirichlet's test.

(ix) $\int_1^\infty e^{x + x^{-1}}\,dx$

$f(x) = e^{\frac{x^2+1}{x}}$

$g(x) = e^x$

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} e^{1/x}$

$= 1$

$\int_1^\infty e^x\,dx$ diverges, so $\int_1^\infty e^{x+x^{-1}}\,dx$ diverges.

(x) $\int_0^1 \frac{e^x}{x^2}\,dx$

$f(x) = e^x/x^2$, $g(x) = \frac{1}{2} + \frac{1}{x} + \frac{1}{x^2}$

$f, g > 0$ for $x \in (0, 1]$

$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{e^x}{x^2\left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{2}\right)}$

$= \lim_{x \to 0^+} \frac{e^x}{1 + x + x^2/2} = 1$

$\int_0^1 g(x)\,dx$ diverges

$\therefore \int_0^1 f(x)\,dx$ diverges.

37 (i) $\displaystyle\int_0^1 \frac{dx}{(x+2)\sqrt{x(1-x)}}$

$$= \int_0^{1/2} \frac{dx}{(x+2)\sqrt{x(1-x)}} + \int_{1/2}^1 \frac{dx}{(x+2)\sqrt{x(1-x)}}$$

$1-x = u$

$\Rightarrow -dx = du$ $\qquad \cancel{t=u+2}$

$$\int_{1/2}^{0} \frac{-du}{\sqrt{u(1-u)}\cdot(3-u)}$$

$$= \int_{0+}^{1/2} \frac{dx}{(x+2)\sqrt{x(1-x)}} + \int_{0+}^{1/2} \frac{dx}{(3-x)\sqrt{x(1-x)}}$$

$$f(x) = \frac{1}{(x+2)\sqrt{x(1-x)}}, \quad g(x) = \frac{1}{2\sqrt{x}}$$

$f, g > 0$ for $x \in (0, \frac{1}{2}]$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{2\sqrt{x}}{(x+2)\sqrt{x(1-x)}}$$

$= 1$ $\qquad \displaystyle\int_{0+}^{1/2} \frac{1}{2\sqrt{x}}\,dx$ converges

so, $\displaystyle\int_{0+}^{1/2} \frac{dx}{(x+2)\sqrt{x(1-x)}}$ converges

$p, q > 0$ for $x \in (0, \frac{1}{2}]$

$$p(x) = \frac{1}{(3-x)\sqrt{x(1-x)}}, \quad q(x) = \frac{1}{3\sqrt{x}}$$

$$\lim_{x\to 0+} \frac{p(x)}{q(x)} = \lim_{x\to 0+} \frac{3\sqrt{x}}{(3-x)\sqrt{x(1-x)}} = 1$$

$\displaystyle\int_{0+}^{1/2} \frac{1}{3\sqrt{x}}\,dx$ converges,

so, $\displaystyle\int_{0+}^{1/2} \frac{1}{(3-x)\sqrt{x(1-x)}}\,dx$ converges.

$\therefore \displaystyle\int_0^1 \frac{dx}{(x+2)\sqrt{x(1-x)}}$ converges.

(ii) $\int_0^\infty x^{-1/2} e^{-x}\,dx = \int_0^1 \frac{1}{\sqrt{x}} e^{-x}\,dx + \int_1^\infty \frac{1}{\sqrt{x}} e^{-x}\,dx$

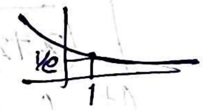
$\Downarrow$ type 2 $\qquad\qquad$ $\Downarrow$ type 1

$f(x) = \frac{1}{\sqrt{x}} e^{-x}$, $g(x) = \frac{1}{\sqrt{x}}$, $f, g > 0$ for $x \in (0,1]$

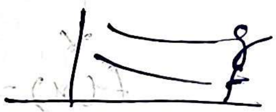
$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} e^{-x} = 1$

$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges $\therefore \int_0^1 \frac{1}{\sqrt{x}} e^{-x}\,dx$ converges.

$\int_1^\infty \frac{1}{\sqrt{x}} e^{-x}\,dx$



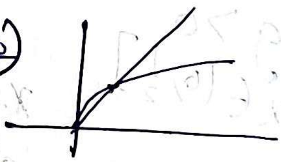
$f(x) = \frac{1}{\sqrt{x}} e^{-x}$



$\cancel{g(x) = \frac{1}{e\sqrt{x}}}$

$\cancel{0 < f \le g \text{ for } x \in [1,\infty)}$

$\cancel{\int_1^\infty \frac{1}{e\sqrt{x}}\,dx \text{ diverges}}$



$x > 1$

$x > \sqrt{x}$

$\Rightarrow -x < -\sqrt{x}$

$e^{-x} < e^{-\sqrt{x}}$

$\frac{1}{\sqrt{x}} e^{-x} < \frac{1}{\sqrt{x}} e^{-\sqrt{x}}$

Let $g(x) = \frac{1}{\sqrt{x}} e^{-\sqrt{x}}$

$0 < f < g$ for $x \in (1,\infty)$

$\int_1^\infty \frac{2}{2\sqrt{x}} e^{-\sqrt{x}}\,dx = 2\int_1^\infty e^{-t}\,dt = \frac{2}{e}$

$\cancel{\sqrt{x} = t}$

$\cancel{\frac{1}{2\sqrt{x}}dx = dt}$

$\therefore$ as g converges so f also converges.

$\therefore \int_0^\infty x^{-1/2} e^{-x}\,dx$ converges.

(iii) $\displaystyle\int_1^{\infty} \frac{1}{x^{1/2}(1+x)^{1/4}}\,dx$

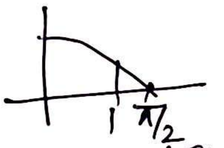
$f(x) = \dfrac{1}{x^{1/2}(1+x)^{1/4}}$, $g(x) = \dfrac{1}{x^{1/2}x^{1/4}} = \dfrac{1}{x^{3/4}}$

$f, g > 0$ for $x \in [1, \infty)$

$\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{1/4}}{(1+x)^{1/4}} = 1.$

$\displaystyle\int_1^{\infty} g(x)\,dx$ diverges, so $\displaystyle\int_1^{\infty} f(x)\,dx$ diverges.

(iv) $\displaystyle\int_0^{\infty} \frac{\cos x}{\sqrt{x^3+x}}\,dx = \underbrace{\int_0^1 \frac{\cos x}{\sqrt{x^3+x}}\,dx}_{\text{type 2}} + \underbrace{\int_1^{\infty} \frac{\cos x}{\sqrt{x^3+x}}\,dx}_{\text{type 1}}$



$f(x) = \dfrac{\cos x}{\sqrt{x^3+x}}$, $g(x) = \dfrac{1}{\sqrt{x}}$

$f, g > 0$ for $x \in (0, 1]$, $1 < \pi/2$

$\displaystyle\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\cos x \sqrt{x}}{\sqrt{x(1+x^2)}} = 1.$

$\displaystyle\int_0^1 g(x)\,dx$ converges

$\therefore \displaystyle\int_0^1 f(x)\,dx$ converges. (Limit Comparison test)

Let $p(x) = \cos x$, $q(x) = \dfrac{1}{\sqrt{x^3+x}}$

range of q is $(0, \frac{1}{\sqrt{2}}]$. So q is bounded.

Also, q is monotone.

$\displaystyle\lim_{x\to\infty} q(x) = 0.$

$\displaystyle\left|\int_1^b p(x)\,dx\right| = \left|\int_1^b \cos x\,dx\right| = |\sin b - \sin 1| \le 1 + \sin 1$

also p is integrable.

$\therefore$ By Dirichlet's test $\displaystyle\int_1^{\infty} p(x)q(x)\,dx$ converges.

$\therefore \displaystyle\int_0^{\infty} \frac{\cos x}{\sqrt{x^3+x}}\,dx$ converges.

(v) $\displaystyle\int_0^1 \frac{x^{p-1}}{1-x}\,dx$

if $p=1$

$\displaystyle\int_{0^+}^{1^-} \frac{1}{1-x}\,dx$

$= \ln(1-x)\Big|_{1^-}^{0}$

$\to$ dne.

$1-x=u$

$\Rightarrow -dx = du$

$\cancel{\displaystyle\int_1^0 \frac{(1-u)^{p-1}}{u}(-du)}$

$\cancel{= \displaystyle\int_0^1 \frac{(1-u)^{p-1}}{u}\,du}$

if $p>1$, $\dfrac{x^{p-1}}{1-x}$ is discont. at $x=1$

if $p<1$, $\dfrac{x^{p-1}}{1-x}$ is discont at $x=1,0$

if $p<1$, $\displaystyle\int_{0^+}^{1/2} \frac{1}{x^{1-p}(1-x)}\,dx + \int_{1/2}^{1^-} \frac{1}{x^{1-p}(1-x)}\,dx$

$1-x=u$

$\Rightarrow -dx = du$

$\displaystyle\int_{1/2}^{0} \frac{-du}{(1-u)^{1-p}u}$

$\displaystyle\int_{0^+}^{1/2} \frac{dx}{x^{1-p}(1-x)} + \int_{0^+}^{1/2} \frac{dx}{x(1-x)^{1-p}}$

$f(x) = \dfrac{1}{x^{1-p}(1-x)}$, $g(x) = \dfrac{1}{x^{1-p}}$

$x^{1-p} - x^{2-p}$

$\min(1-p, 2-p)$

$= \underline{1-p}$

$\displaystyle\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{x^{1-p}}{x^{1-p}(1-x)} = 1$

$\displaystyle\int_{0^+}^{1/2} g(x)\,dx$ converges if $1-p<1$

$\Rightarrow p>0$

Now let $f(x) = \dfrac{1}{x(1-x)^{1-p}}$, $g(x) = \dfrac{1}{x}$

$\displaystyle\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{x}{x(1-x)^{1-p}} = 1$

but $\displaystyle\int_{0^+}^{1/2} \frac{1}{x}\,dx$ diverges $\therefore \displaystyle\int_{0^+}^{1/2} \frac{dx}{x(1-x)^{1-p}}$ diverges

$\therefore \displaystyle\int_0^1 \frac{x^{p-1}}{1-x}\,dx$ $\boxed{\text{diverges for } p \le 1}$.

if $p>1$

$$\int_0^{1^-} \frac{x^{p-1}}{1-x}\,dx \qquad 1-x=u \Rightarrow -dx=du$$

$$\int_1^0 \frac{(1-u)^{p-1}}{u}(-du) = \int_{0+}^1 \frac{(1-x)^{p-1}}{x}\,dx$$

$$f(x)=\frac{(1-x)^{p-1}}{x},\ g(x)=\frac{1}{x}$$

$$\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}(1-x)^{p-1}=1$$

but $\int_0^1 g(x)\,dx$ diverges

so $\int_0^1 \frac{(1-x)^{p-1}}{x}\,dx$ diverges.

$\therefore \int_0^1 \frac{x^{p-1}}{1-x}\,dx$ $\boxed{\text{diverges } \forall p}$.

4) $\int_0^{\pi/2} \frac{x^m}{\sin^n x}\,dx$ $\qquad f(x)=\frac{x^m}{\sin^n x},\ g(x)=\frac{1}{x^{n-m}}$

$= \cancel{\int_0^{\pi/2} \frac{x^{m-n}}{\left(\frac{\sin^n x}{x^n}\right)}\,dx}$ $\qquad \lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}\frac{x^m}{\sin^n x}\cdot\frac{x^n}{x^m}$

$= 1$

$\int_0^{\pi/2} g(x)\,dx$ converges if $n-m<1$

$\therefore \int_0^{\pi/2}\frac{x^m}{\sin^n x}\,dx$ converges if $n<m+1$

5) $\int_0^1 \frac{\sin\left(\frac{1}{x}\right)}{\sqrt{x}}\,dx$

$$f(x)=\frac{\sin\frac{1}{x}}{\sqrt{x}},\ g(x)=\frac{1}{\sqrt{x}}$$

$0<f\le g$ for $x\in(0,1]$.

$\int_0^1 g(x)\,dx$ converges

$\therefore \int_0^1 f(x)\,dx$ converges.

6) $\int_0^{\infty}\left(\frac{1}{x+1}-\frac{1}{e^x}\right)\frac{1}{x}\,dx$

$= \int_0^{\infty} \frac{e^x - x - 1}{x(x+1)e^x}\,dx$

$\lim_{x\to 0} \frac{e^x - x - 1}{x(x+1)e^x} = 1/2$

$\lim_{x\to \infty} \frac{e^x - x - 1}{x(x+1)e^x} = 0$

$\therefore \frac{e^x - x - 1}{x(x+1)e^x}$ is bounded

$\therefore$ integral is of 1st kind.

$f(x) = \frac{e^x - x - 1}{x(x+1)e^x}, \quad g(x) = \frac{1}{x^2}$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{(e^x - x - 1)x^2}{x(x+1)e^x} = 1$

$\int_0^{\infty} f(x)\,dx = \underbrace{\int_0^1 f(x)\,dx}_{\substack{\text{proper integral} \\ \to \text{convergent}}} + \int_1^{\infty} f(x)\,dx$

$\int_1^{\infty} g(x)\,dx$ converges $\Rightarrow \int_1^{\infty} f(x)\,dx$ converges

$\therefore \int_0^{\infty} f(x)\,dx$ converges.

7) $\int_0^{\infty} e^{-x^2}\,dx = \underbrace{\int_0^1 e^{-x^2}\,dx}_{\text{proper integral}} + \underbrace{\int_1^{\infty} e^{-x^2}\,dx}_{\text{improper integral of first kind}}$

for $x > 1$
$x^2 > x$
$\Rightarrow -x^2 < -x$
$e^{-x^2} < e^{-x}$

$\int_1^{\infty} e^{-x}\,dx$ converges $\therefore \int_1^{\infty} e^{-x^2}\,dx$ converges

$\therefore \int_0^{\infty} e^{-x^2}\,dx$ converges

$x^{-1/2+1} = \frac{x^{1/2}}{1/2}$

8) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

$= \lim_{r\to 0^+} \int_r^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{r\to 0^+}\left[\ln x \; 2\sqrt{x}\Big|_r^1 - \int_r^1 \frac{1}{x}\, 2\sqrt{x}\,dx\right]$

$= \lim_{r\to 0^+}\left[-2\sqrt{r}\ln r - 4\sqrt{x}\Big|_r^1\right]$

$= \lim_{r\to 0^+}\left[-2\sqrt{r}\ln r - 4 + 4\sqrt{r}\right]$

$r^{-1/2}$

$\lim_{r\to 0} \sqrt{r}\ln r = \lim_{r\to 0} \frac{\ln r}{(1/\sqrt{r})}$ L'Hospital rule.

$= \lim_{r\to 0} \frac{1/r}{-\frac{1}{2} r^{-3/2}} = 0.$

$\therefore \int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = -4.$

9) $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

if $m=1$, $\int_0^1 (1-x)^{n-1}\,dx = \int_0^1 x^{n-1}\,dx$

$= \int_0^1 \frac{1}{x^{1-n}}\,dx$

convergent if

$1-n<1 \Rightarrow n>0.$

Similarly if $n=1$, $B(m,n)$ is convergent if $m>0$.

if $m>1$, $\int_0^1 \frac{x^{m-1}}{(1-x)^{1-n}}\,dx$

$= \int_0^1 \frac{(1-x)^{m-1}}{x^{1-n}}\,dx$

Let $g(x) = \frac{1}{x^{1-n}}$, $f(x) = \frac{(1-x)^{m-1}}{x^{1-n}}$

$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = 1$ $\int_0^1 g(x)\,dx$ converges for $1-n<1 \Rightarrow n>0$

If $m < 1$

$$B(m,n) = \int_0^1 \frac{1}{x^{1-m}} \frac{1}{(1-x)^{1-n}} dx$$

$$= \int_0^{1/2} \frac{1}{x^{1-m}(1-x)^{1-n}} dx + \int_{1/2}^1 \frac{dx}{x^{1-m}(1-x)^{1-n}}$$

Consider $f(x) = \dfrac{1}{x^{1-m}(1-x)^{1-n}}$

and $g(x) = \dfrac{1}{x^{1-m}}$

$\lim\limits_{x \to 0^+} \dfrac{f(x)}{g(x)} = 1$ $\int_0^{1/2} g(x)\,dx$ converges for $1-m < 1 \Rightarrow m > 0$.

$1-x = u$
$\Rightarrow -dx = du$

$$\int_{1/2}^0 \frac{-du}{(1-u)^{1-m} u^{1-n}} = \int_0^{1/2} \frac{dx}{x^{1-n}(1-x)^{1-m}}$$

converges for $1-n < 1 \Rightarrow n > 0$.

$\therefore B(m,n)$ converges for $m > 0, n > 0$

10) $\displaystyle\int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x} dx$

Let $\phi(a) = \displaystyle\int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x}$

$$\phi'(a) = \int_0^\infty \frac{1}{x} \frac{1}{1+a^2x^2} x\, dx$$

$$= \frac{1}{a^2} \int_0^\infty \frac{dx}{x^2 + 1/a^2} = \frac{1}{a^2} \frac{1}{(1/a)} \tan^{-1}(ax)\Big|_0^\infty$$

$$= \frac{1}{a} \frac{\pi}{2} \Rightarrow \phi(a) = \frac{\pi}{2} \ln a + c$$

$\phi(b) = 0 \Rightarrow \frac{\pi}{2} \ln b + c = 0 \Rightarrow c = -\frac{\pi}{2} \ln b$

$$\therefore \phi(a) = \frac{\pi}{2} \ln(a/b)$$

11) $\psi(a) = \int_0^\infty \frac{\phi(ax) - \phi(bx)}{x} dx$

$\Rightarrow \psi'(a) = \int_0^\infty \phi'(ax) dx = \frac{1}{a}\int_0^\infty \phi'(x) dx$

$= \frac{1}{a}\left(\lim_{x\to\infty}\phi(x) - \lim_{x\to 0}\phi(x)\right)$

$\Rightarrow \psi(a) = \left(\lim_{x\to\infty}\phi(x) - \lim_{x\to 0}\phi(x)\right) \ln a + c$

$\psi(b) = 0 \Rightarrow c = -\left(\lim_{x\to\infty}\phi(x) - \lim_{x\to 0}\phi(x)\right)\ln b$

$\therefore \psi(a) = \left(\lim_{x\to\infty}\phi(x) - \lim_{x\to 0}\phi(x)\right)\ln\left(\frac{a}{b}\right)$

$\int_0^\infty \frac{\frac{\sin x}{x} - \frac{\sin 2x}{2x}}{x} dx = \left(\lim_{x\to\infty}\frac{\sin x}{x} - \lim_{x\to 0}\frac{\sin x}{x}\right)\ln\left(\frac{1}{2}\right)$

$= \ln 2$