- System of linear equations

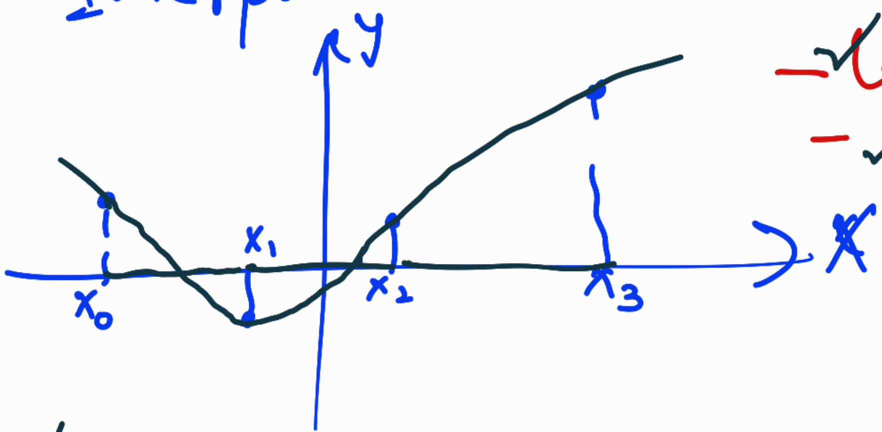
$$A_{100\times100}\, x_{100\times1} = b_{100\times1} \Rightarrow x = A^{-1}b$$

$$(x_1^0, x_2^0, x_3^0) = \left(\frac{D_1}{D}, \frac{D_2}{D}, D_3/D\right)$$

- Jacobi (✓) → $\vec{x}^{k+1} = D^{-1}b - D^{-1}R\,\vec{x}^{k}$
- Gauss Seidel (✓) $\vec{x}^{k+1} = (I + D^{-1}L)^{-1}(D^{-1}b - D^{-1}U\vec{x}^{k})$

- Algebraic & transcendental eqn.
  - Bisection
  - Fixed pt. iteration
  - Newton Raphson
  - ~~Regula falsi~~ ✗

- Interpolation



- ✓ Lagrange
- ✓ Newton forward
- ✓ Newton backward ✗

$$5x_1 - 2x_2 + 3x_3 = -1 \Rightarrow x_1^{(k)} = \frac{-1 + 2x_2^{(k-1)} - 3x_3^{(k-1)}}{5}$$

$$-3x_1 + 9x_2 + x_3 = 2 \Rightarrow x_2^{(k)} = \frac{2 + 3x_1^{(k-1)} - x_3^{(k-1)}}{9}$$

$$2x_1 - x_2 - 7x_3 = 3 \Rightarrow x_3^{(k)} = \frac{2x_1^{(k-1)} - x_2^{(k-1)} - 3}{7}$$

$$(x_1^0, x_2^0, x_3^0) \longrightarrow (x_1^1, x_2^1, x_3^1) \longrightarrow (x_1^2, x_2^2, x_3^2)$$

guess $(0, 0, 0)$

↓
⋮

3.14152
3.14153

take $x_1 \simeq \frac{D_1}{D}$, $x_2 \simeq \frac{D_2}{D}$, $x_3 \simeq \frac{D_3}{D}$

$$Ax = b$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$x_1^{k+1} = \frac{b_1 - a_{12} x_2^k - a_{13} x_3^k}{a_{11}}$$

$$= \frac{1}{a_{11}} \left(b_1 - 0 \cdot x_1^k - a_{12} x_2^k - a_{13} x_3^k \right)$$

$$A = D + R = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

$$R = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

$$\boxed{x^{(k+1)} = D^{-1} \left(b - R x^{(k)} \right)}$$

$$\boxed{x^{(k+1)} = D^{-1} b - \underbrace{D^{-1} R}_{\text{iteration matrix } (H)} x^{(k)}}$$

- If it has to converge for any initial guess, $\rho(H) < 1$

$\rho(H) = \max |\lambda_i|$, λ_i are eigenvalues of H.

shortcut: strictly diag. dominant.

$$|a_{11}| > |a_{12}| + \dots + |a_{1n}|$$

$$\vdots$$

$$|a_{nn}| > |a_{n1}| + \dots + |a_{nn-1}|$$

$$5x_1 - 2x_2 + 3x_3 = -1 \Rightarrow x_1^{(k)} = \frac{-1 + 2x_2^{(k-1)} - 3x_3^{(k-1)}}{5}$$

$$-3x_1 + 9x_2 + x_3 = 2 \Rightarrow x_2^{(k)} = \frac{2 + 3x_1^{(k)} - x_3^{(k-1)}}{9}$$

$$2x_1 - x_2 - 7x_3 = 3 \Rightarrow x_3^{(k)} = \frac{2x_1^{(k)} - x_2^{(k)} - 3}{7}$$

$$(x_1^0, x_2^0, x_3^0) \longrightarrow (x_1^1, x_2^0, x_3^0) \rightarrow (x_1^1, x_2^1, x_3^0)$$

guess

$(0,0,0)$

$$(x_1^2, x_2^1, x_3^1) \leftarrow (x_1^1, x_2^1, x_3^1)$$

$$Ax = b$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$x_1^k = \frac{b_1 - a_{12} x_2^{k-1} - a_{13} x_3^{k-1}}{a_{11}}$$

$$x_2^k = \frac{b_2 - a_{21} x_1^k - a_{23} x_3^{k-1}}{a_{22}}$$

$$x_3^k = \frac{b_3 - a_{31} x_1^k - a_{32} x_2^k}{a_{33}}$$

$$D = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

$$A = D + L + R$$

$$\vec{x}^{(k)} = D^{-1}(b - U\vec{x}^{(k-1)}) - D^{-1}L\,\vec{x}^{k}$$

$$\boxed{(I + D^{-1}L)\vec{x}^{(k)} = D^{-1}b - D^{-1}U\vec{x}^{(k-1)}}$$

$$\Rightarrow \boxed{\vec{x}^{k} = (I + D^{-1}L)^{-1}(D^{-1}b - D^{-1}U\vec{x}^{k-1})}$$

$$5x_1 - 2x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

	0	1	2	3
x_1	0	-0.2	0.146	
x_2	0	0.2222	0.2031	
x_3	0	-0.428	-0.517	

Jacobi

$$\vec{x}^{k+1} = D^{-1}b - D^{-1}R\vec{x}^{k}$$

$$= \begin{pmatrix} -1/5 \\ 2/9 \\ -3/7 \end{pmatrix} - \begin{pmatrix} 0 & -2/5 & 3/5 \\ -3/9 & 0 & 1/9 \\ -2/7 & 1/7 & 0 \end{pmatrix} \vec{x}$$

A B C

$$\begin{pmatrix} 1/5 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & -1/7 \end{pmatrix}\begin{pmatrix} 0 & -2 & 3 \\ -3 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2/5 & 3/5 \\ -3/9 & 0 & 1/9 \\ -2/7 & 1/7 & 0 \end{pmatrix}$$

Gauss Siedel

$$x^{k} = (I + D^{-1}L)^{-1}(D^{-1}b - D^{-1}Ux^{k-1})$$

$$D^{-1}L = \begin{pmatrix} 1/5 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & -1/7 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix}$$

$$I + D^{-1}L = \begin{pmatrix} 1 & 0 & 0 \\ -3/9 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{pmatrix} \rightarrow A$$

$$D^{-1}b = \begin{pmatrix} -1/5 \\ 2/9 \\ -3/7 \end{pmatrix} \rightarrow B$$

$$D^{-1}U = \begin{pmatrix} 1/5 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & -1/7 \end{pmatrix} \begin{pmatrix} 0 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2/5 & 3/5 \\ 0 & 0 & 1/9 \\ 0 & 0 & 0 \end{pmatrix} - C$$

$$x^k = \underbrace{A^{-1}B} - \underbrace{A^{-1}C}\, x^{k-1}$$

$$= \begin{pmatrix} -0.2 \\ 0.1555 \\ -0.507 \end{pmatrix} - \begin{pmatrix} 0 & -0.4 & 0.6 \\ 0 & -0.133 & 0.3111 \\ 0 & -0.095 & 0.1269 \end{pmatrix} x$$

	0	1	2	3	4	5
x_1	0	-0.2	0.1664	0.1903	0.1858	0.1855
x_2	0	0.1555	0.3339	0.333	0.331	0.331
x_3	0	-0.507	-0.427	-0.421	-0.422	-0.422

$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$
$$5x_1 - 3x_2 + 3x_3 = -1$$

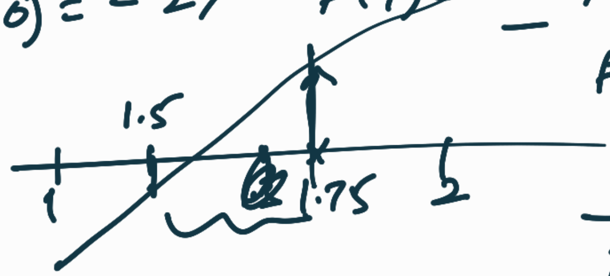
$$5x_1 - 3x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

Algebraic & transcendental

Bisection

$f(x) = x^3 - x - 2$

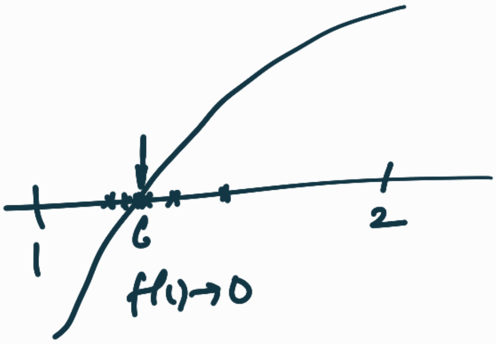
$f(0) = -2, \quad f(1) = -2, \quad f(2) = 4$



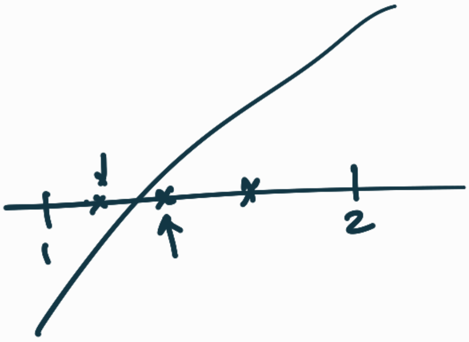
$f(a)<0 \quad f(b)>0$

a	b	$c=\frac{a+b}{2}$	$f(c)$
1	2	1.5	−0.125
1.5	2	1.75	1.609
1.5	1.75	1.625	0.666
1.5	1.625		

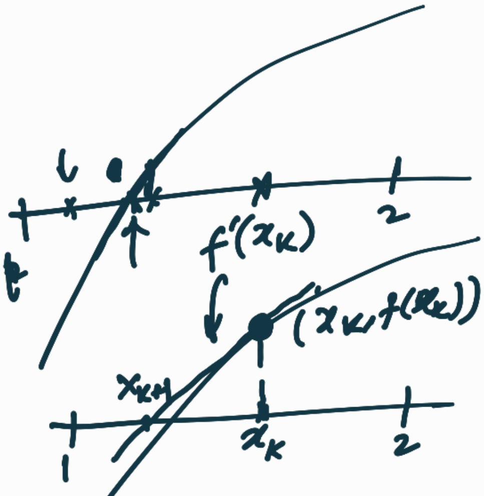
$\downarrow$ 0



11-12 iterations



Newton Raphson



$$f'(x_k) = \frac{f(x_k) - 0}{x_k - x_{k+1}}$$

$$\Rightarrow x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

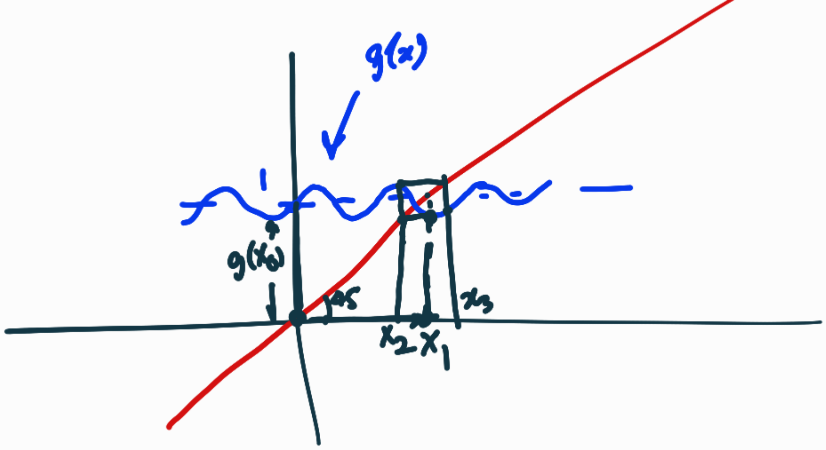
Fixed pt.

$\sin x = 10(x-1) \Rightarrow \sin x - 10(x-1) = 0 \rightarrow f(x) = 0$

$x = g(x)$

$$\underset{x}{x} = \underbrace{\frac{\sin x}{10} + 1}_{g(x)}$$

$$\boxed{x_{n+1} = g(x_n)}$$



$x_0, \quad x_1 = g(x_0),$

$g(x) = \frac{\sin x}{10} + 1$

x	0	1.084	1.0883	1.0885
n	0	1	2	3

Tutorial 6

1)
$$5x_1 - 3x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

Jacobi $\rightarrow x(k+1) = D^{-1}b - D^{-1}R\,x(k)$

Gauss Siedel $\rightarrow x(k+1) = (I + D^{-1}L)^{-1}(D^{-1}b - D^{-1}U\,x(k))$

$$D^{-1}b = \begin{pmatrix} -1/5 \\ 2/9 \\ -3/7 \end{pmatrix}, \quad D^{-1}R = \begin{pmatrix} 1/5 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & -1/7 \end{pmatrix} \begin{pmatrix} 0 & -3 & 3 \\ -3 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\Rightarrow D^{-1}R = \begin{pmatrix} 0 & -3/5 & 3/5 \\ -3/9 & 0 & 1/9 \\ -2/7 & 1/7 & 0 \end{pmatrix}$$

$$I + D^{-1}L = \begin{pmatrix} 1 & 0 & 0 \\ -3/9 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{pmatrix}$$

$$D^{-1}U = \begin{pmatrix} 0 & -3/5 & 3/5 \\ 0 & 0 & 1/9 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Jacobi} \rightarrow X(k+1) = \underbrace{\begin{pmatrix} -1/5 \\ 2/9 \\ -3/7 \end{pmatrix}}_{A} - \underbrace{\begin{pmatrix} 0 & -3/5 & 3/5 \\ -3/9 & 0 & 1/9 \\ -2/7 & 1/7 & 0 \end{pmatrix}}_{B} \underbrace{x(k)}_{C}$$

$$\text{Gauss Siedel} \rightarrow x(k+1) = \underbrace{\begin{pmatrix} -0.2 \\ 0.1555 \\ -0.5079 \end{pmatrix}}_{A} - \underbrace{\begin{pmatrix} 0 & -0.6 & 0.6 \\ -0.333 & -0.2 & 0.3111 \\ -0.238 & 0 & 0.1269 \end{pmatrix}}_{B} \underbrace{x(k)}_{C}$$

	n	0	1	2	3	4	5	6	7	8	9
jacobi	x_1	0	-0.2	0.1905	0.2324	0.2478	0.2534	0.2545	0.2549	0.2550	0.2550
	x_2	0	0.2222	0.2832	0.3432	0.3445	0.3505	0.3519	0.3522	0.3523	0.3523
	x_3	0	-0.4286	-0.5175	-0.4032	-0.4112	-0.4070	-0.4062	-0.4061	-0.4061	-0.4060
gauss seidel	x_1	0	-0.2	0.1980	0.2595	0.2564	0.2551	0.2550			
	x_2	0	0.1555	0.3447	0.3555	0.3527	0.3523	0.3523			
	x_3	0	-0.5079	-0.4212	-0.4052	-0.4057	-0.4060	-0.4060			

$$\boxed{(0.255,\ 0.352,\ -0.406)}\ \checkmark$$

2) $$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

Jacobi $x_{k+1} = D^{-1}b - D^{-1}R\,x_k$

gauss seidel $x_{k+1} = (I + D^{-1}L)^{-1}(D^{-1}b - D^{-1}U x_k)$

$$D^{-1}b = \begin{pmatrix} 1/12 \\ 28/5 \\ 76/13 \end{pmatrix} \to B, \quad D^{-1}R = \begin{pmatrix} 1/12 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/13 \end{pmatrix}\begin{pmatrix} 0 & 3 & -5 \\ 1 & 0 & 3 \\ 3 & 7 & 0 \end{pmatrix}$$

$$\Rightarrow D^{-1}R = \begin{pmatrix} 0 & 3/12 & -5/12 \\ 1/5 & 0 & 3/5 \\ 3/13 & 7/13 & 0 \end{pmatrix}$$

$$(I + D^{-1}L) = \begin{pmatrix} 1 & 0 & 0 \\ 1/5 & 1 & 0 \\ 3/13 & 7/13 & 1 \end{pmatrix} \to A$$

$$D^{-1}U = \begin{pmatrix} 0 & 3/12 & -5/12 \\ 0 & 0 & 3/5 \\ 0 & 0 & 0 \end{pmatrix} - C$$

$$\text{Jacobi} \to x_{k+1} = \underbrace{\begin{pmatrix} 1/12 \\ 28/5 \\ 76/13 \end{pmatrix}}_{A} - \underbrace{\begin{pmatrix} 0 & 3/12 & -5/12 \\ 1/5 & 0 & 3/5 \\ 3/13 & 7/13 & 0 \end{pmatrix}}_{B} \underbrace{x_k}_{C}$$

$$\text{gauss seidel } x_{k+1} = \underbrace{\begin{pmatrix} 0.0833 \\ 5.5833 \\ 2.8205 \end{pmatrix}}_{A} - \underbrace{\begin{pmatrix} 0 & 0.25 & -0.4167 \\ 0 & -0.05 & 0.6833 \\ 0 & -0.03 & -0.2718 \end{pmatrix}}_{B} \underbrace{x_k}_{C}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
Jacobi x_1	1	0.5	1.2231	0.8615	1.0566	0.9639	1.0143	0.9906	1.0036	0.9975	1.0009	0.9993	1.0002
Jacobi x_2	0	4.8	2.1308	3.4677	2.7777	3.1206	2.9433	3.0311	2.9855	3.0080	2.9963	3.0021	2.9991
Jacobi x_3	1	5.6154	3.1462	4.6166	3.7801	4.1066	3.9434	4.0273	3.9854	4.0070	3.9962	4.0018	3.9990
Gauss-Seidel x_1	1	0.5	0.1469	0.7413	0.9446	0.9897	0.9973						
Gauss-Seidel x_2	0	4.9	3.7153	3.1671	3.0310	3.0059	3.0024						
Gauss-Seidel x_3	1	3.0923	3.8080	3.9670	3.9937	3.9969	3.9970						

3)

$$5x_1 - 2x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

Jacobi → $x_k = \underbrace{\begin{pmatrix} -1/5 \\ 2/9 \\ -3/7 \end{pmatrix}}_{A} - \underbrace{\begin{pmatrix} 0 & -2/5 & 3/5 \\ -3/9 & 0 & 1/9 \\ -2/7 & 1/7 & 0 \end{pmatrix}}_{B} \underbrace{x_{k-1}}_{C}$

gauss siedel $x_k = \underbrace{\begin{pmatrix} -0.2 \\ 0.1555 \\ -0.507 \end{pmatrix}}_{A} - \underbrace{\begin{pmatrix} 0 & -0.4 & 0.6 \\ 0 & -0.133 & 0.3111 \\ 0 & -0.095 & 0.1269 \end{pmatrix}}_{B} \underbrace{x_{k-1}}_{C}$

n	0	1	2	3	4	5	6	7	8
Jacobi x_1	0	-0.2	0.1460	0.1917	0.1809	0.1854	0.1863	0.1861	0.1861
Jacobi x_2	0	0.222	0.2032	0.3284	0.3323	0.3293	0.3312	0.3313	0.3312
Jacobi x_3	0	-0.428	-0.5175	-0.4159	-0.4207	-0.4244	-0.4226	-0.4226	-0.4227
Gauss Seidel x_1	0	-0.2	0.1670	0.1909	0.1864	0.1861			
Gauss Seidel x_2	0	0.1555	0.3343	0.3335	0.3312	0.3312			
Gauss Seidel x_3	0	-0.507	-0.4286	-0.4247	-0.4226	-0.4227			

$\boxed{(0.186,\ 0.331,\ -0.423)}$

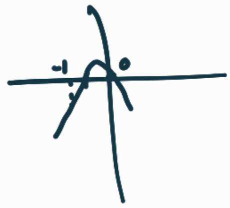
4)

$f(a)>0$ $f(b)<0$

$f(x) = \sin x$

a	b	c	$f(c)$
3	4	3.5	-0.357
3	3.5	3.25	-0.1082
3	3.25	3.125	0.0166
3.125	3.25	3.1875	-0.046
3.125	3.1875	...	...
⋮	⋮	⋮	⋮

9) $f(x) = 3x^3 + 4x^2 + 4x + 1 = 0$



$(-1, 0)$

$$x = \underbrace{\frac{1}{4}\left(-3x^3 - 4x^2 - 1\right)}_{g(x)}$$

$$g'(x) = -\frac{1}{4}\left(9x^2 + 8x\right) = -\frac{x}{4}(9x+8)$$

$$g'(-1) = -0.25$$

$$g'(-4/9) = 0.44$$

$\therefore$ $g'(x) \in [-0.25, 0.44]$ for $x \in [-1, 0]$

$\therefore$ $\underline{|g'(x)| < 1}$

$$\therefore \phi(x) = -\frac{1}{4}\left(3x^3 + 4x^2 + 1\right)$$

10) $x^3 + 2x^2 - 3x - 4 = 0$

$[-1.5, 0]$

$x = \frac{1}{3}\left(x^3 + 2x^2 - 4\right)$, $g'(x) = x^2 + \frac{4}{3}x$, $|g'(x)| < 1$

~~$x = 0.5$~~

~~$g(x)$~~

x	0.5	-1.125	-0.9642	-1.0123
n	0	1	2	3