INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR
Mid - Spring Semester Examination 2024
Department of Mathematics

Subject No.: **MA11004**; Subject: **Linear Algebra, Numerical and Complex Analysis**

Duration: 2hrs. **Total Marks:** 30

Instructions: (1) Any kind of calculator is not allowed. (2) Answer ALL the questions.

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1. (a) Consider the vector space $\mathbb{R}^3$ over $\mathbb{R}$ with respect to the usual addition and scalar multiplication. Let $W = \{(x, y, z) \in \mathbb{R}^3 : x^3 = -y^2x\}$ be a subset of $\mathbb{R}^3$. Verify whether W is a subspace of $\mathbb{R}^3$ or not. [3M]

   (b) Let $\mathbb{M}_{2\times 2}(\mathbb{R})$ be the vector space over $\mathbb{R}$ of all 2×2 real matrices with usual matrix addition and scalar multiplication. Let $B_1 = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$, and $B_3 = \begin{bmatrix} 1 & 3 \\ 9 & 13 \end{bmatrix}$. If $A = \begin{bmatrix} 3 & 2 \\ -4 & 1 \end{bmatrix}$, then find real numbers c_1, c_2 and c_3 such that
   $$A = c_1B_1 + c_2B_2 + c_3B_3.$$
   [3M]

2. (a) Let $\{v_1, v_2, v_3, v_4\}$ be a set of linearly independent vectors in a vector space over $\mathbb{R}$. Verify whether the set of vectors $\{v_1+v_2, v_2+v_3, v_3+v_4, v_4+v_1\}$ is linearly independent or dependent. [2M]

   (b) Verify whether the set of vectors $\{(1,1,0), (2,0,7), (3,3,3)\}$ is a basis of $\mathbb{R}^3$ or not. [1M]

   (c) Find all possible $\lambda \in \mathbb{R}$, for which the following matrix A has rank less than or equal to 2 :
   $$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{bmatrix}.$$
   [3M]

3. (a) Verify the existence of a nontrivial solution of the following system of linear equations:
   $$kx - 3y + z = 0$$
   $$x + ky - 3z = 0$$
   $$3x - y + 2z = 0,$$
   where $k^2 - 3k - 2 = 0$. Also find the dimension of the solution space of the above system. [3M]

P.T.O.

(b) Show that the following system of equations is consistent and hence find the solution(s):

$$\begin{aligned} x + 2y - z &= 3 \\ 3x - y + 2z &= 1 \\ 2x - 2y + 3z &= 2 \\ x - y + z &= -1. \end{aligned}$$

[3M]

4. (a) Let $\mathbb{M}_{2\times 2}(\mathbb{R})$ be the vector space of all 2×2 real matrices over $\mathbb{R}$ with respect to usual matrix addition and scalar multiplication. Let $T : \mathbb{R}^3 \to \mathbb{M}_{2\times 2}(\mathbb{R})$ be a linear transformation defined by

$$T(x, y, z) = \begin{bmatrix} x+y+z & y-z \\ y-z & x+2y \end{bmatrix}.$$

Then find a basis for the nullspace (kernel) of T. Also verify the Rank-Nullity Theorem for T by finding both rank and nullity of T. [3M]

(b) Let $\mathbb{P}_2(\mathbb{R})$ be the vector space over $\mathbb{R}$ of all polynomials in x of degree at most two with real coefficients, and with usual polynomial addition and scalar multiplication. Let $T : \mathbb{R}^3 \to \mathbb{P}_2(\mathbb{R})$ be a linear transformation defined by

$$T(a, b, c) = (3a - b + c)x^2 + (a + 2b)x + (b + 5c).$$

Find the matrix representation of T with respect to the ordered bases $\{(1, 1, 1), (2, 3, 0), (1, 0, 3)\}$ and $\{1 + x, x + 3x^2, 2 - x\}$ of $\mathbb{R}^3$ and $\mathbb{P}_2(\mathbb{R})$ respectively.

[3M]

5. (a) Find linearly independent eigenvectors corresponding to each eigenvalue of the matrix

$$A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}.$$

[3M]

(b) Verify whether $\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$ is diagonalizable or not. [1M]

(c) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & 1 & 1 \end{bmatrix}$. If $6A^{-1} = aA^2 + bA + cI$, for some $a, b, c \in \mathbb{R}$, then find the value of $a + b + c$ using Cayley–Hamilton theorem. [2M]

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