2024_ Mid Spring

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$x^3 + y^2 x = 0$

$\Rightarrow x(x^2+y^2) = 0$

$\boxed{x=0}$

1) (a) $V = (\mathbb{R}^3(\mathbb{R}), +, \cdot)$

$W = \{(x,y,z) \in \mathbb{R}^3 : x^3 = -y^2 x\}$

$\vec{u_1}, \vec{u_2} \in W$

$\vec{u_1} = (x_1, y_1, z_1) : x_1^3 = -y_1^2 x_1$

$\vec{u_2} = (x_2, y_2, z_2) : x_2^3 = -y_2^2 x_2$

$\alpha \vec{u_1} + \beta \vec{u_2} = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$\cancel{(\alpha x_1 + \beta x_2)^3}$ | $\cancel{-(\alpha y_1 + \beta y_2)^2 (\alpha x_1 + \beta x_2)}$

$\therefore x_1 = 0 = x_2$

$\alpha x_1 + \beta x_2 = 0$

$\therefore \alpha \vec{u_1} + \beta \vec{u_2} \in W$

$\therefore W$ is a subspace.

$x^3 = -y^2 x$

$x^2 + y^2 = 0$

$\cancel{(0, 0, c)}$

(c_1, c_2)

(b)

$$A = \begin{bmatrix} 3 & 2 \\ -4 & 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 3 \\ 9 & 13 \end{bmatrix}$$

$c_1 + c_2 + c_3 = 3$

$c_1 + 2c_2 + 3c_3 = 2$

$c_1 + 4c_2 + 9c_3 = -4$

$3c_1 + 7c_2 + 13c_3 = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 2 \\ 1 & 4 & 9 & -4 \\ 3 & 7 & 13 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - 3R_1$

A

$\bar{A}$

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$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 8 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - 3R_2$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & \boxed{2} & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$8-6$

$-7+3$

$2c_3 = -4 \Rightarrow \boxed{c_3 = -2}$

$c_2 + 2c_3 = -1 \Rightarrow c_2 - 4 = -1 \Rightarrow \boxed{c_2 = 3}$

$c_1 + c_2 + c_3 = 3 \Rightarrow c_1 + \cancel{3} - 2 = \cancel{3} \Rightarrow \boxed{c_1 = 2}$

2) (a) $\{v_1, v_2, v_3, v_4\} \to L.I$

$\alpha(v_1+v_2) + \beta(v_2+v_3) + \gamma(v_3+v_4) + \delta(v_4+v_1) = 0$

$\Rightarrow (\alpha+\delta)v_1 + (\alpha+\beta)v_2 + (\gamma+\beta)v_3 + (\gamma+\delta)v_4 = 0$

$\alpha + \delta = 0$

$\alpha + \beta = 0 \Rightarrow \alpha = t, \beta = -t, \gamma = t, \delta = -t$

$\beta + \gamma = 0$

$\gamma + \delta = 0$

$\therefore$ ~~Not~~ Linearly dependent.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & 0 \end{array}\right]$$

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(b) Basis $\rightarrow$ $\boxed{LS(B) = V}$

B is linearly independents

$$\mathbb{R}^3 \quad \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 7 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{-2} & 7 \\ 0 & 0 & \boxed{3} \end{bmatrix}$$

$$\text{Span}\{(1,1,0), (2,0,7), (3,3,3)\} = \mathbb{R}^3 \qquad (x,y,z)$$

$$c_1(1,1,0) + c_2(2,0,7), c_3(3,3,3)$$

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

$$(1,0,0) = c_1(1,1,0) + c_2(2,0,7) + c_3(3,3,3)$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 0 & 7 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

rank A = rank $\bar{A}$ = 3

$c_1 + 2c_2 + 3c_3 = 1$

$c_1 + 3c_3 = 0$

$7c_2 + 3c_3 = 0$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 7 & 3 & 0 \end{array}\right)$$

(c) $\text{rank}(A) \le 2$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{bmatrix} \qquad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & \lambda - 1 \\ 0 & 2 & -4 & \lambda^2 - 5 \end{bmatrix} \qquad R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & -2 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 2\lambda - 3 \end{bmatrix}$$

$\cancel{7-5}$

$\cancel{1-5}$

$\lambda^2 - 5 - 2(\lambda - 1)$

$\lambda^2 - 5 - 2\lambda + 2$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\Rightarrow (\lambda + 1)(\lambda - 3) = 0$$

$$\boxed{\lambda = 3, -1}$$

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$R = R_i + \frac{r_2 R_L}{r_2 + R_L}$

$\therefore r_2 << R_L$

$\approx R_i + r_2$

3) (a)

$kx - 3y + z = 0$

$x + ky - 3z = 0$

$3x - y + 2z = 0$

$$\left[\begin{array}{ccc|c} k & -3 & 1 & 0 \\ 1 & k & -3 & 0 \\ 3 & -1 & 2 & 0 \end{array}\right]$$

~~$R_1 \to R_1 - kR_2$~~

$2k^2 + 27 - 1 - 3k + 6 - 3k = 0$

$\Rightarrow 2k^2 - 6k + 32 = 0$

$\Rightarrow k^2 - 3k + 16 = 0$

~~$k = \frac{3 \pm \sqrt{9 - 64}}{2}$~~

$k(2k-3) + 3(2+9) + 1(-1-3k)$

$2k^2 - 3k + 33 - 1 - 3k$

$2k^2 - 6k + 32$

~~$k^2 - 3k + 16 = 0$~~

$\det = 2k^2 + 27 - 1 - 3k + 6 - 3k$

$\Rightarrow 2k^2 - 6k + 32$

$= 2(k^2 - 3k + 16)$

$k^2 - 3k - 2 = 0$

$(k \ \ )(k$

dimension $= 0$

$\{(0,0,0)\}$

(b) $z = 3$

$x + 2y - z = 3$

$3x - y + 3z = 1$

$2x - 2y + 3z = 2$

$x - y + z = -1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 2 & -4 \\ 0 & 0 & 10 & 4 \\ 0 & -7 & 5 & -8 \end{array}\right]$$

$\boxed{z = 4 = y, \ x = -1}$

$x + 2y - z = 3$

$x + 8 - 4 = 3$

$x = -4$

$-3y + 27 + 8 = -4$

$-3y + 8 = -4$

$3y = 12$

$2 + 3$

$1 - 9$

$1 - 6$

$5 - 4$

$-4 + 8$

$3 + 2$

-6

$-1 + 8 - 4$

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4) (a) $T: \mathbb{R}^3 \to M_{2\times 2}(\mathbb{R})$

$$T(x,y,z) = \begin{bmatrix} x+y+z & y-z \\ y-z & x+2y \end{bmatrix}$$

$x+y+z = 0$

$y - z = 0$

$x+2y = 0$

$$x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + z\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \quad R_2 \to R_2 - R_1$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x+y+z=0$

$y-z=0$

$y=t=z$

$x=-2t$

nullity = 1

rank = 2

rank + nullity = 3

$$\left\{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\}$$

(b) $$M = \begin{pmatrix} 10/3 & 17/3 & 13/3 \\ 1 & 1 & 2 \\ 4/3 & -4/3 & 16/3 \end{pmatrix}$$

5) (a) $$A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$$

$$P_A(x) = \det(xI - A) = \det\begin{bmatrix} x-7 & 0 & 3 \\ 9 & x+2 & -3 \\ -18 & 0 & x+8 \end{bmatrix}$$

$54 - 56$

$-56 + 54$

$= (x+2)\left[(x-7)(x+8) + 54\right]$

$= (x+2)\left[x^2 + x - 2\right]$

$= (x+2)(x-1)(x+2)$

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$$\lambda = 1, -2$$

$$A\vec{v} = \lambda\vec{v}$$

$$\begin{pmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = 1\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 6 & 0 & -3 \\ -9 & -3 & 3 \\ 18 & 0 & -9 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & -3 & 0 \\ -9 & -3 & 3 & 0 \\ 18 & 0 & -9 & 0 \end{array}\right] \quad R_1 \to R_1/3,\ R_2 \to R_2/3$$

$$R_3 \to R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ -3 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \to R_2 + \frac{3}{2}R_1$$

$-3+\frac{3}{2}\times$

$\frac{3}{2}(-1)$

$-3/2$

$$\left[\begin{array}{ccc|c} \boxed{2} & 0 & -1 & 0 \\ 0 & \boxed{-1} & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$2V_1 - V_3 = 0 \Rightarrow V_1 = k,\ V_3 = 2k$$

$$-V_2 - \frac{V_3}{2} = 0 \Rightarrow V_2 = -k$$

$$\therefore E_{\lambda=1} = \left\{\begin{pmatrix} k \\ -k \\ 2k \end{pmatrix} : k \in \mathbb{R}\setminus\{0\}\right\}$$

$$= \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right\} \setminus \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$\lambda = -2$

$A\vec{v} = \lambda \vec{v}$

$$\begin{pmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = -2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 9 & 0 & -3 \\ -9 & 0 & 3 \\ 18 & 0 & -6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$R_2 \quad \left[\begin{array}{ccc|c} 9 & 0 & -3 & 0 \\ -9 & 0 & 3 & 0 \\ 18 & 0 & -6 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} \boxed{9} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$3\frac{9\alpha}{\cancel{3}}$

$9v_1 - 3v_3 = 0 \Rightarrow v_1 = \alpha,\ v_3 = 3\alpha$

$v_2 = \beta$

$$\begin{pmatrix} \alpha \\ \beta \\ 3\alpha \end{pmatrix}$$

$$\therefore E_{\lambda=2} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \setminus \left\{ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\}$$

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5)(b)

$$A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

$$P_A(x) = \det(xI - A)$$

$$= \begin{vmatrix} x-2 & -2 \\ 0 & x-2 \end{vmatrix}$$

$$= (x-2)^2$$

$\lambda = 2$ $\quad alg(2) = 2$

λI

$$A\vec{v} = \lambda \vec{v}$$

$$\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x_2 = 0 \Rightarrow x_2 = 0$$

$$\begin{pmatrix} t \\ 0 \end{pmatrix} \quad t\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \dim = 1$$

$geo(2) = 1$

$geo(2) \neq alg(2)$

$\therefore$ not diagonalizable.

$T: \mathbb{R}^3 \to P_2(\mathbb{R})$

$T(a,b,c) = (3a-b+c)x^2 + (a+2b)x + (b+5c)$

$\{(1,1,1), (2,3,0), (1,0,3)\}$

$\{1+x, x+3x^2, 2-x\}$

$x^2 = c_1(1+x) + c_2(x+3x^2) + c_3(2-x)$

comparing coefficients we have

$3c_2 = 1 \Rightarrow c_2 = 1/3$

$c_1 + c_2 - c_3 = 0 \Rightarrow c_3 - c_1 = 1/3$

$c_1 + 2c_3 = 0 \Rightarrow c_1 + 2c_3 = 0$

$3c_3 = 1/3 \Rightarrow c_3 = 1/9$

$c_1 = \frac{1}{9} - \frac{3}{9} = -2/9$

$x^2 = -\frac{2}{9}(1+x) + \frac{1}{3}(x+3x^2) + \frac{1}{9}(2-x)$

$x = c_4(1+x) + c_5(x+3x^2) + c_6(2-x)$

comparing coefficients we have

$c_5 = 0$

$1 = c_4 + c_5 - c_6 \Rightarrow c_4 - c_6 = 1$

$c_4 + 2c_6 = 0 \Rightarrow c_4 + 2c_6 = 0$

$3c_6 = -1 \Rightarrow c_6 = -1/3$

$c_4 = 1 - 1/3 = 2/3$

$x = \frac{2}{3}(1+x) + 0(x+3x^2) + \left(-\frac{1}{3}\right)(2-x)$

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$$1 = c_7(1+x) + c_8(x+3x^2) + c_9(2-x)$$

Comparing coefficients we have

$$c_8 = 0$$

$$c_7 + c_8 - c_9 = 0 \Rightarrow c_7 - c_9 = 0$$

$$1 = c_7 + 2c_9 \Rightarrow c_7 + 2c_9 = 1$$

$$3c_9 = 1 \Rightarrow c_9 = 1/3$$

$$c_7 = 1/3$$

$$1 = \frac{1}{3}(1+x) + 0(x+3x^2) + \frac{1}{3}(2-x)$$

$$T(1,1,1) = 3x^2 + 3x + 6$$

$$= 3\left[-\frac{2}{9}\bar{v}_1 + \frac{1}{3}\bar{v}_2 + \frac{1}{9}\bar{v}_3\right]$$

$$+3\left[\frac{2}{3}\bar{v}_1 - \frac{1}{3}\bar{v}_3\right]$$

$$+6\left[\frac{1}{3}\bar{v}_1 + \frac{1}{3}\bar{v}_3\right]$$

$$-\frac{2}{3}+2+2 \qquad 4-\frac{2}{3} \qquad \frac{1}{3}-1+2 \qquad \frac{1}{3}+1$$

$$T(1,1,1) = \frac{10}{3}\bar{v}_1 + \bar{v}_2 + \frac{4}{3}\bar{v}_3$$

$$T(2,3,0) = 3x^2 + 8x + 3$$

$$-\frac{2}{3}+\frac{16}{3}+1 \qquad \frac{14}{3}+1$$

$$= 3\left[-\frac{2}{9}\bar{v}_1 + \frac{1}{3}\bar{v}_2 + \frac{1}{9}\bar{v}_3\right]$$

$$+8\left[\frac{2}{3}\bar{v}_1 - \frac{1}{3}\bar{v}_3\right] + 3\left[\frac{1}{3}\bar{v}_1 + \frac{1}{3}\bar{v}_3\right]$$

$$T(2,3,0) = \frac{17}{3}\bar{v}_1 + \bar{v}_2 - \frac{4}{3}\bar{v}_3$$

$$\frac{2}{3}-\frac{1}{3}+5 \qquad \frac{1}{3}+5$$

$$T(1,0,3) = 6x^2 + x + 15$$

$$= 6\left[-\frac{2}{9}\bar{v}_1 + \frac{1}{3}\bar{v}_2 + \frac{1}{9}\bar{v}_3\right] + \frac{2}{3}\bar{v}_1 - \frac{1}{3}\bar{v}_3$$

$$+15\left[\frac{1}{3}\bar{v}_1 + \frac{1}{3}\bar{v}_3\right]$$

$$-\frac{4}{3}+\frac{2}{3}+5 \qquad \frac{2}{3}+5$$

$$= \frac{13}{3}\bar{v}_1 + 2\bar{v}_2 + \frac{16}{3}\bar{v}_3$$

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$$M = \begin{pmatrix} 10/3 & 17/3 & 13/3 \\ 1 & 1 & 2 \\ 4/3 & -4/3 & 16/3 \end{pmatrix}$$

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(C) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

$P_A(x) = \det(xI - A)$

$= \begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-4 & 2 \\ 0 & -1 & x-1 \end{vmatrix}$

$= (x-1)^2 (x-4)$

$= (x^2 - 2x + 1)(x-4)$

$= x^3 - 2x^2 + x - 4x^2 + 8x - 4$

$= x^3 - 6x^2 + 9x - 4$

$A^3 - 6A^2 + 9A - 4I = 0$

$A^2 - 6A + 9I - 4A^{-1} = 0$

$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$

$\Rightarrow 6A^{-1} = \frac{6}{4}(A^2 - 6A + 9I)$

$\frac{6}{4}(1 - 6 + 9) = 6$ (Ans)