Tutorial - 11

1) a) $\int_0^\infty e^{-x^3}\sqrt{x}\,dx$

$x^3 = u$

$\Rightarrow 3x^2 dx = du$

$\int_0^\infty e^{-u} u^{1/6} \frac{du}{3u^{2/3}} = \frac{1}{3}\int_0^\infty e^{-u} u^{-1/2} du = \frac{1}{3}\int_0^\infty e^{-u} u^{1/2-1} du$

$= \frac{1}{3}\Gamma(1/2) = \frac{\sqrt{\pi}}{3}$

(b) $\int_0^\infty e^{-a^2x^2} dx$

$a^2x^2 = u \Rightarrow a^2(2x\,dx) = du$

$\int_0^\infty e^{-u} \frac{du}{2a^{\cancel{2}}\frac{\sqrt{u}}{\cancel{a}}} = \frac{1}{2a}\int_0^\infty e^{-u} u^{1/2-1} du = \frac{1}{2a}\Gamma(1/2) = \frac{\sqrt{\pi}}{2a}$

(c) $\int_0^1 x^3(1-x^2)^{5/2} dx$

$x^2 = u$

$\Rightarrow 2x\,dx = du$

$\frac{1}{2}\int_0^1 u(1-u)^{5/2} du = \frac{1}{2}\int_0^1 u^{2-1}(1-u)^{7/2-1} du$

$= \frac{1}{2}B(2, 7/2) = \frac{1}{\cancel{2}} \times \frac{1!}{\frac{7}{\cancel{2}} \times \frac{9}{2}}$

$= \frac{2}{63}$

(d) $\int_0^{\pi/2} \sin^m x\, dx$

$\sin^2 x = u$

~~$\Rightarrow \cos x\, dx$~~

$\Rightarrow 2\sin x \cos x\, dx = du$

$\Rightarrow dx = \frac{du}{2\sqrt{u}\sqrt{1-u}}$

$\int_0^1 (\sqrt{u})^m \frac{du}{2\sqrt{u}\sqrt{1-u}}$

$= \frac{1}{2}\int_0^1 u^{\frac{m-1}{2}}(1-u)^{-1/2}$

$= \frac{1}{2}\int_0^1 u^{\frac{m+1}{2}-1}(1-u)^{\frac{1}{2}-1} = \frac{1}{2}B\left(\frac{m+1}{2}, \frac{1}{2}\right)$

$= \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma(1/2)}{\Gamma\left(\frac{m}{2}+1\right)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$

(e) $\int_0^1 \sqrt{1-x^4}\,dx$

$x^4 = u \Rightarrow 4x^3 dx = du$

$$\int_0^1 \sqrt{1-u}\,\frac{du}{4u^{3/4}} = \frac{1}{4}\int_0^1 u^{-3/4}(1-u)^{1/2}\,du$$

$$= \frac{1}{4}\int_0^1 u^{1/4-1}(1-u)^{3/2-1}\,du$$

$$= \frac{1}{4} B(1/4, 3/2) = \frac{1}{4}\,\frac{\Gamma(1/4)\Gamma(3/2)}{\Gamma(7/4)}$$

$\frac{1}{4}+\frac{3}{2}$

$$= \frac{1}{4}\,\frac{\Gamma(1/4)\cdot \frac{1}{2}\sqrt{\pi}}{\frac{3}{4}\Gamma(3/4)}$$

$\frac{7}{4}-1 = \frac{3}{4}$

$$\Gamma 1/4\ \Gamma 3/4 = \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{1/\sqrt{2}} = \sqrt{2}\pi$$

$$= \frac{1}{6}\,\frac{(\Gamma(1/4))^2\sqrt{\pi}}{\sqrt{2}\pi} = \frac{\{\Gamma(1/4)\}^2}{6\sqrt{2\pi}}$$

(f) $\int_0^{\pi/2} \sqrt{\tan x}\,dx$

$\sin^2 x = u \Rightarrow 2\sin x\cos x\,dx = du$

$dx = \frac{du}{2\sqrt{u}\sqrt{1-u}}$

$-\frac{1}{4}-\frac{1}{2\times 2}$

$$\int_0^1 \sqrt{\frac{\sqrt{u}}{\sqrt{1-u}}}\,\frac{du}{2\sqrt{u}\sqrt{1-u}}$$

$$= \int_0^1 \frac{u^{1/4}(1-u)^{-1/4}}{2u^{1/2}(1-u)^{1/2}}\,du = \frac{1}{2}\int_0^1 u^{-1/4}(1-u)^{-3/4}\,du$$

$$= \frac{1}{2}\int_0^1 u^{3/4-1}(1-u)^{1/4-1}\,du$$

$$= \frac{1}{2}\Gamma(3/4)\Gamma(1/4) = \frac{1}{2}\cdot\pi\sqrt{2} = \frac{\pi}{\sqrt{2}}$$

(g) $\beta(m+1, n) = \int_0^1 x^m (1-x)^{n-1} dx$

$$= -x^m \frac{(1-x)^n}{n}\Big|_0^1 + \int_0^1 m x^{m-1} \frac{(1-x)^n}{n} dx$$

$$= \int_0^1 \frac{m}{n} x^{m-1} [(1-x)(1-x)^{n-1} dx$$

$$= \frac{m}{n}(\beta(m,n) - \beta(m+1, n))$$

$$\Rightarrow \left(1+\frac{m}{n}\right)\beta(m+1,n) = \frac{m}{n}\beta(m,n)$$

$$\Rightarrow \beta(m+1,n) = \frac{m}{m+n}\beta(m,n).$$

h) $\int_a^b (x-a)^{m-1}(b-x)^{n-1} dx$

$x = (b-a)u + a$

$b-x = b-(b-a)u-a$

$= (b-a)(1-u)$

$$\frac{x-a}{b-a} = u$$

$$\Rightarrow \frac{1}{b-a} dx = du$$

$$\int_0^1 \underline{(b-a)^{m-1} u^{m-1}} (b-a) \underline{(b-a)^{n-1}(1-u)^{n-1}} du$$

$$= (b-a)^{m+n-1} \int_0^1 u^{m-1}(1-u)^{n-1} du$$

$$= (b-a)^{m+n-1} B(m,n)$$

(i) $\int_0^1 \frac{x^n}{\sqrt{1-x^2}} dx$, $x^2 = u \Rightarrow 2x dx = du$.

$$\int_0^1 u^{n/2}(1-u)^{-1/2} \frac{du}{2\sqrt{u}}$$

$$= \frac{1}{2}\int_0^1 u^{\frac{n}{2}-\frac{1}{2}}(1-u)^{-1/2} du = \frac{1}{2}\int_0^1 u^{\frac{n+1}{2}-1}(1-u)^{\frac{1}{2}-1} du$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

(j) $\int_0^1 x^{p-1}(1-x^r)^{q-1}dx$

$x^r = u \Rightarrow r x^{r-1} dx = du$

$\int_0^1 u^{\frac{p-1}{r}} (1-u)^{q-1} \frac{du}{r\, u^{\frac{r-1}{r}}}$

$= \frac{1}{r}\int_0^1 u^{\frac{p-r}{r}} (1-u)^{q-1} du$

$= \frac{1}{r}\int_0^1 u^{\frac{p}{r}-1} (1-u)^{q-1} du = \frac{1}{r}\beta\left(\frac{p}{r}, q\right).$

(k) $\int_0^1 x^{p-1}\left(\ln\frac{1}{x}\right)^{\alpha-1} dx$

$\ln 1/x = u$
$\Rightarrow x = e^{-u}$
$\Rightarrow dx = -e^{-u} du.$

$\cancel{\ln 1/x = u}$
$\Rightarrow \cancel{\frac{1}{x} = e^{u}}$
$\cancel{\Rightarrow x = e^{-u}}.$

$\int_\infty^0 (e^{-u})^{p-1} u^{\alpha-1}(-e^{-u}) du$

$= \int_0^\infty (e^{-u})^{p} u^{\alpha-1} du = \frac{\Gamma(\alpha)}{p^\alpha}$

(l) $\int_0^1 \frac{dx}{(1-x^n)^{1/n}}$

$x^n = u \Rightarrow n x^{n-1} dx = du$

$\int_0^1 (1-u)^{-1/n} \frac{du}{n\, u^{\frac{n-1}{n}}}$

$= \frac{1}{n}\int_0^1 (1-u)^{-1/n} u^{1/n-1} du = \frac{1}{n}\int_0^1 (1-u)^{\frac{n-1}{n}-1} u^{\frac{1}{n}-1} du$

$= \frac{1}{n}\beta\left(\frac{n-1}{n}, \frac{1}{n}\right) = \frac{1}{n}\frac{\Gamma\left(\frac{n-1}{n}\right)\Gamma(1/n)}{\Gamma(1)}$

$= \frac{1}{n}\Gamma(1/n)\Gamma(1-1/n).$

37 a) $\int_0^\infty x^m e^{-ax^n} dx$

$ax^n = u \Rightarrow anx^{n-1}dx = du$

$dx = \frac{du}{an\left(\frac{u}{a}\right)^{\frac{n-1}{n}}} = \frac{du}{a^{1/n}\, n\, u^{\frac{n-1}{n}}}$

$$\int_0^\infty \left(\frac{u}{a}\right)^{\frac{m}{n}} e^{-u} \frac{du}{a^{\frac{1}{n}}\, n\, u^{\frac{n-1}{n}}}$$

$$= \frac{1}{n}\left(\frac{1}{a}\right)^{\frac{m+1}{n}} \int_0^\infty u^{\frac{m+1}{n}-1} e^{-u} du = \frac{1}{n}\left(\frac{1}{a}\right)^{\frac{m+1}{n}} \Gamma\left(\frac{m+1}{n}\right)$$

b) $\int_0^1 x^m \left(\ln \frac{1}{x}\right)^n dx$

$\ln \frac{1}{x} = u \Rightarrow x = e^{-u}$

$dx = -e^{-u} du.$

$$\int_0^\infty e^{-mu} u^n e^{-u} du$$

$$= \int_0^\infty e^{-(m+1)u} u^n du = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$$

u^{n+1-1}

c) $\int_0^\infty x^m n^{-x} dx$ $\qquad n^{-x} = e^{-x\ln n}$

$x \ln n = u \Rightarrow (\ln n)dx = du$

$$\int_0^\infty \frac{u^m}{(\ln n)^m} e^{-u} \frac{du}{(\ln n)} = \int_0^\infty \frac{e^{-u} u^{m+1-1}}{(\ln n)^{m+1}} du = \frac{\Gamma(m+1)}{(\ln n)^{m+1}} = \frac{m!}{(\ln n)^{m+1}}$$

Tutorial-11 (Continued).

4) P.T.,
$$\sqrt{\pi}\,\Gamma(2m+1) = 2^{2m}\,\Gamma(m+\tfrac{1}{2})\,\Gamma(m+1).$$

$$\Gamma(m+1) = m\Gamma m$$

$$\Gamma(m+\tfrac{1}{2}) = (m-\tfrac{1}{2})\Gamma(m-\tfrac{1}{2})$$
$$= (m-\tfrac{1}{2})(m-\tfrac{3}{2})\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}\sqrt{\pi}$$
$$= \frac{1}{2^m}(2m-1)(2m-3)\cdots 3\cdot 1\sqrt{\pi}$$

$$\Gamma(m+1) = m(m-1)(m-2)\cdots 2\cdot 1$$

$$\Gamma(2m+1) = 2m(2m-1)(2m-2)\cdots 3\cdot 2\cdot 1$$
$$= 2^m\,\Gamma(m+1)\,2^m\,\Gamma(m+\tfrac{1}{2})\frac{1}{\sqrt{\pi}}$$

$$\Rightarrow \sqrt{\pi}\,\Gamma(2m+1) = 2^{2m}\,\Gamma(m+\tfrac{1}{2})\,\Gamma(m+1).$$

$$\Rightarrow \sqrt{\pi}\,(2m)\,\Gamma(2m) = 2^{2m}\,\Gamma(m+\tfrac{1}{2})\,m\,\Gamma(m)$$

$$\Rightarrow \sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\,\Gamma(m)\,\Gamma(m+\tfrac{1}{2}).$$

5) $\beta(n,1-n) = \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$\int_0^1 \frac{x^n + x^{-n}}{1+x^2}\,dx$$

$x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$

$$\int_0^{\pi/4} \frac{(\tan\theta)^n + (\tan\theta)^{-n}}{\cancel{\sec^2\theta}}\cancel{\sec^2\theta}\,d\theta$$

$$= \int_0^{\pi/4} (\tan\theta)^n + (\cot\theta)^n\,d\theta = \int_0^{\pi/2} (\tan\theta)^n\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 2\sin^n\theta\cos^{-n}\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2} 2\sin^{2\left(\frac{n+1}{2}\right)-1}\theta\cos^{2\left(\frac{-n+1}{2}\right)-1}\theta\,d\theta$$

$$= \frac{1}{2}\beta\left(\frac{n+1}{2}, \frac{-n+1}{2}\right) = \frac{1}{2}\beta\left(\frac{n+1}{2}, 1-\frac{n+1}{2}\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\frac{n+1}{2}\right)$$

$$= \frac{1}{2}\frac{\pi}{\sin\left(\frac{n+1}{2}\pi\right)}$$

$$= \boxed{\frac{\pi}{2}\sec\frac{n\pi}{2}}$$

6) $\int_0^\infty \frac{x^m}{x^n+a}\,dx$

$x^n = a\tan^2\theta$

$nx^{n-1}\,dx = a\,2\tan\theta\sec^2\theta\,d\theta$

$\frac{x^m}{x^{n-1}} = x^{m-n+1}$

$$\int_0^{\pi/2} \frac{1}{\cancel{a}\cancel{\sec^2\theta}} \frac{2\cancel{a}\tan\theta\cancel{\sec^2\theta}\,d\theta}{n}\left(a\tan^2\theta\right)^{\frac{m-n+1}{n}}$$

$$= \frac{2a^{\frac{m-n+1}{n}}}{n}\int_0^{\pi/2} (\tan\theta)^{\frac{2m+2-n}{n}}\,d\theta$$

$\frac{2m+2-n}{n}+1 \over 2$ $= \frac{2m+2}{2n}$

$$= \frac{2a^{\frac{m+1}{n}-1}}{n}\,\frac{1}{2}\,\Gamma\left(\frac{m+1}{n}\right)\Gamma\left(1-\frac{m+1}{n}\right)$$

$$= \boxed{\frac{a^{\frac{m+1}{n}-1}}{n}\Gamma\left(\frac{m+1}{n}\right)\Gamma\left(1-\frac{m+1}{n}\right)}$$

7) a) b)] from def^n $\Gamma(n+1) = n\Gamma(n)$.

$e^{\alpha \ln x}$

8) $$\phi(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx$$

$$\phi'(\alpha) = \int_0^1 \frac{x^\alpha \ln x}{\ln x}\,dx = \frac{1}{\alpha+1}$$

$$\Rightarrow \phi(\alpha) = \ln(\alpha+1) + C$$

$$\phi(0) = 0 \Rightarrow C = 0$$

$$\therefore \boxed{\phi(\alpha) = \ln(\alpha+1)}$$

9) (i) $$\int_{-\pi/2}^{\pi/2} \frac{\ln(1+b\sin x)}{\sin x}\,dx = \phi(b)$$

$$\Rightarrow \phi'(b) = \int_{-\pi/2}^{\pi/2} \frac{1}{\sin x}\,\frac{\sin x}{1+b\sin x}\,dx$$

$\sin x = \dfrac{2\tan x/2}{1+\tan^2 x/2}$ $\cancel{= \int_0^{\pi/2} \left[\frac{1}{1+b\sin x} + \frac{1}{1-b\sin x}\right] dx}$

$\tan\frac{x}{2} = t$ $\cancel{= \int_0^{\pi/2} \frac{2}{1-b^2\sin^2 x}\,dx}$

$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$dx = \dfrac{2dt}{1+t^2}$

$$\tan^{-1}\left(\frac{1+b}{\sqrt{1-b^2}}\right) + \tan^{-1}\left(\frac{1-b}{\sqrt{1-b^2}}\right) = \tan^{-1}\left(\frac{\sqrt{1+b}}{\sqrt{1-b}}\right) + \tan^{-1}\left(\frac{\sqrt{1-b}}{\sqrt{1+b}}\right) = \pi/2$$

$$\therefore \phi'(b) = \int_{-1}^{1} \frac{1}{1+b\left(\frac{2t}{1+t^2}\right)}\,\frac{2dt}{1+t^2}$$

$$= \int_{-1}^{1} \frac{2dt}{1+t^2+2bt} = 2\int_{-1}^{1} \frac{dt}{(t+b)^2 + (1-b^2)}$$

$$= 2\,\frac{1}{\sqrt{1-b^2}}\tan^{-1}\left(\frac{t+b}{\sqrt{1-b^2}}\right)\Bigg|_{-1}^{1}$$

$$= \frac{2}{\sqrt{1-b^2}}\,\frac{\pi}{2} = \frac{\pi}{\sqrt{1-b^2}}$$

$$\Rightarrow \phi(b) = \pi\sin^{-1} b + C$$

$\phi(0) = 0 \Rightarrow C = 0 \Rightarrow \phi(b) = \boxed{\pi\sin^{-1} b}$

(d) $\displaystyle\int_0^\infty \frac{\tan^{-1}\alpha x \tan^{-1}\beta x}{x^2}\,dx = \phi(\alpha)$

$\displaystyle\Rightarrow \phi'(\alpha) = \int_0^\infty \frac{\tan^{-1}\beta x}{x \cancel{x}} \frac{1}{1+\alpha^2x^2} \cancel{x}\, dx$ ~~$= \psi(\beta)$~~

$= \psi(\beta)$

~~$\psi'(\beta) = \int_0^\infty \frac{1}{(1+\alpha^2x^2)(1+\dots)}$~~

~~$= \tan^{-1}\beta x$~~

$\displaystyle\psi'(\beta) = \int_0^\infty \frac{1 \cdot \cancel{x}}{\cancel{x}(1+\alpha^2x^2)(1+\beta^2x^2)}\,dx$

$\displaystyle= \int_0^\infty \frac{\beta^2(\alpha^2x^2+1) - \alpha^2(\beta^2x^2+1)}{(\beta^2-\alpha^2)(1+\alpha^2x^2)(1+\beta^2x^2)}\,dx$

$\displaystyle= \frac{\beta^2}{\beta^2-\alpha^2}\int_0^\infty \frac{dx}{1+\beta^2x^2} - \frac{\alpha^2}{\beta^2-\alpha^2}\int_0^\infty \frac{dx}{1+\alpha^2x^2}$

$\displaystyle= \frac{\beta^2}{\beta^2-\alpha^2}\cdot\frac{1}{\beta}\tan^{-1}(\beta x)\Big|_0^\infty - \frac{\alpha^2}{\beta^2-\alpha^2}\cdot\frac{1}{\alpha}\tan^{-1}(\alpha x)\Big|_0^\infty$

$\displaystyle= \frac{\beta}{\beta^2-\alpha^2}\left(\frac{\pi}{2}\right) - \frac{\alpha}{\beta^2-\alpha^2}\,\pi/2$

$\displaystyle\psi'(\beta) = \frac{\pi}{2(\beta+\alpha)}$

$\displaystyle\Rightarrow \psi(\beta) = \frac{\pi}{2}\ln(\beta+\alpha) + C$

$\displaystyle\psi(0) = 0 \Rightarrow \frac{\pi}{2}\ln(\alpha) + C = 0$

$\Rightarrow C = -\pi/2 \ln\alpha$

$\displaystyle\frac{d}{d\alpha}\left(\frac{\alpha+\beta}{\alpha}\right) = \frac{d}{d\alpha}\left(1+\frac{\beta}{\alpha}\right) = -\beta/\alpha^2$

$\displaystyle\therefore \psi(\beta) = \frac{\pi}{2}\ln\left(\frac{\alpha+\beta}{\alpha}\right) = \phi'(\alpha)$

$\displaystyle\Rightarrow \phi(\alpha) = \frac{\pi}{2}\int \ln\left(\frac{\alpha+\beta}{\alpha}\right)d\alpha$

$\displaystyle= \frac{\pi}{2}\left[\alpha\ln\left(\frac{\alpha+\beta}{\alpha}\right) - \int \frac{\alpha}{\alpha+\beta}\left(-\frac{\beta}{\cancel{\alpha^2}}\right)\cancel{\alpha}\,d\alpha\right]$

$\displaystyle\phi(\alpha) = \frac{\pi}{2}\left[\alpha\ln\left(\frac{\alpha+\beta}{\alpha}\right) + \beta\ln(\alpha+\beta)\right] + C$

$\phi(0) = 0$

$\displaystyle\Rightarrow \phi(\alpha) = \boxed{\frac{\pi}{2}\left[\alpha\ln(1+\beta/\alpha) + \beta\ln(1+\alpha/\beta)\right]}$

(iii) $\phi(\alpha)=\int_0^{\pi/2}\ln(\alpha\cos^2\theta+\beta\sin^2\theta)\,d\theta$

$$\Rightarrow \phi'(\alpha)=\int_0^{\pi/2}\frac{\cos^2\theta}{\alpha\cos^2\theta+\beta\sin^2\theta}\,d\theta$$

$$=\int_0^{\pi/2}\frac{d\theta}{\alpha+\beta\tan^2\theta}$$

$\tan\theta=t$
$\Rightarrow \sec^2\theta\,d\theta=dt$

$$=\int_0^{\infty}\frac{1}{\alpha+\beta t^2}\,\frac{dt}{(1+t^2)}$$

$$=\int_0^{\infty}\frac{(\alpha+\beta t^2)-\beta(1+t^2)}{(\alpha+\beta t^2)(1+t^2)(\alpha-\beta)}\,dt$$

$$=\frac{1}{\alpha-\beta}\left[\int_0^{\infty}\frac{dt}{1+t^2}-\beta\int_0^{\infty}\frac{dt}{\alpha+\beta t^2}\right]$$

$$=\frac{1}{\alpha-\beta}\left[\tan^{-1}t\Big|_0^{\infty}-\int_0^{\infty}\frac{dt}{t^2+\alpha/\beta}\right]$$

$$=\frac{1}{\alpha-\beta}\left[\frac{\pi}{2}-\frac{\sqrt{\beta}}{\sqrt{\alpha}}\,\frac{\pi}{2}\right]$$

$$=\frac{\pi}{2}\,\frac{1}{\alpha-\beta}\left(\frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}}\right)=\frac{\pi}{2\sqrt{\alpha}\,(\sqrt{\alpha}+\sqrt{\beta})}$$

$\phi(\alpha)=\cancel{\frac{\pi}{2\sqrt{\beta}}\int\frac{(\sqrt{\alpha}+\sqrt{\beta})-(\sqrt{\alpha})}{\sqrt{\alpha}(\sqrt{\alpha}+\sqrt{\beta})}\,d\alpha}$

$=\cancel{\frac{\pi}{2\sqrt{\beta}}\int\left(\frac{1}{\sqrt{\alpha}}-\frac{1}{\sqrt{\alpha}+\sqrt{\beta}}\right)d\alpha}$

$\Rightarrow\cancel{\frac{\pi}{2\sqrt{\beta}}\left[2\sqrt{\alpha}\right.}$

$$=\frac{\pi}{\cancel{2}}\int\frac{d\alpha}{2\sqrt{\alpha}\,(\sqrt{\alpha}+\sqrt{\beta})}$$

$\sqrt{\alpha}=u$
$\frac{1}{2\sqrt{\alpha}}\,d\alpha=du$

$$=\pi\int\frac{du}{u+\sqrt{\beta}}$$

$$=\pi\ln(\sqrt{\alpha}+\sqrt{\beta})+c$$

$\cancel{\phi(\beta)=0\Rightarrow\pi\ln(2\sqrt{\beta})+c=0}$

$\cancel{\Rightarrow c=-\pi\ln(2\sqrt{\beta})}$

$\phi(\beta)=(\ln\beta)\,\frac{\pi}{2}$

$\pi\ln(2\sqrt{\beta})+c=\frac{\pi}{2}\ln\beta$

$\Rightarrow\pi\ln 2+\frac{\pi}{2}\cancel{\ln\beta}+c=\frac{\pi}{2}\cancel{\ln\beta}$

$\Rightarrow c=-\pi\ln 2$

$$\phi(\alpha)=\boxed{\pi\ln\left(\frac{\sqrt{\alpha}+\sqrt{\beta}}{2}\right)}$$

10) $f(x,t) = (2x+t^3)^2$

$$\int_0^1 f(x,t)\,dx = \int_0^1 (2x+t^3)^2\,dx$$

$$= \frac{1}{2}\frac{(2x+t^3)^3}{3}\Big|_0^1$$

$$= \frac{1}{6}\left[(t^3+2)^3 - t^9\right]$$

$$\frac{d}{dt}\int_0^1 f(x,t)\,dx = \frac{1}{6}\left[3(t^3+2)^2\,3t^2 - 9t^8\right]$$

$$= \frac{3}{2}\left[t^2(t^3+2)^2 - t^8\right]$$

$$= 6t^2(1+t^3).$$

$$\frac{\partial}{\partial t} f(x,t) = 2(2x+t^3)\,3t^2$$

$$= 6t^2(2x+t^3)$$

$$6t^2\int_0^1 (2x+t^3)\,dx$$

$$= 6t^2\left[1+t^3\right) \quad \therefore \frac{d}{dt}\int_0^1 f(x,t)\,dx = \int_0^1 \frac{\partial f}{\partial t}\,dx$$

11)(i) $f(x,t) = \begin{cases} \dfrac{\sin xt}{t} & \text{if } t \neq 0 \\ x & \text{if } t = 0 \end{cases}$

$$F(x) = \int_0^{\pi/2} f(x,t)\,dt$$

$$F'(x) = \int_0^{\pi/2} \frac{\partial}{\partial x} f(x,t)\,dt$$

$$= \int_0^{\pi/2} \frac{1}{\cancel{t}}\cos xt \cdot \cancel{t}\,dt$$

$$= \frac{1}{x}\sin xt\Big|_0^{\pi/2} = \frac{1}{x}\sin\frac{\pi x}{2}$$

$$F'(0) = \pi/2$$

(ii) $f(x) = \int_0^{x^2} \tan^{-1}\frac{t}{x^2}\, dt$

$$f'(x) = \int_0^{x^2} \frac{\partial}{\partial x}\tan^{-1}\frac{t}{x^2}\, dt$$

$$+ \tan^{-1}\frac{x^2}{x^2}\cdot 2x$$

$x\cdot\frac{\pi}{4/2}$

$$= \frac{x\pi}{2} + \int_0^{x^2} \frac{1}{1+\left(\frac{t}{x^2}\right)^2}\left(\frac{-2t}{x^3}\right) dt$$

$\frac{\partial}{\partial x}\left(\frac{t}{x^2}\right)$

$= t\left(\frac{-2}{x^3}\right)$

$$= \frac{x\pi}{2} - \frac{1}{x^3}\int_0^{x^2} \frac{2t\, dt}{1+\frac{t^2}{x^4}}$$

$$= \frac{x\pi}{2} - \frac{x^4}{x^3}\int_0^{x^2} \frac{2t\, dt}{t^2+x^4}$$

$$= \frac{x\pi}{2} - x\int_0^{x^2} \frac{2t\, dt}{t^2+x^4}$$

$$= \frac{x\pi}{2} - x\ln(t^2+x^4)\Big|_0^{x^2}$$

$$= \frac{x\pi}{2} - x\ln\left(\frac{x^4+x^4}{x^4}\right).$$

$$f'(x) = \boxed{\frac{x\pi}{2} - x\ln(2)}$$

13) $\phi(a,b) = \int_0^\infty \frac{e^{-bx}\sin ax}{x}\,dx$

$$\frac{\partial\phi}{\partial a} = \int_0^\infty \frac{e^{-bx}}{\cancel{x}}\cos ax\,dx$$

$$= \frac{b}{b^2+a^2} \Rightarrow \phi = \tan^{-1}\left(\frac{a}{b}\right) + c$$

$\phi(0,b) = 0 \Rightarrow c = 0.$

$$I = \int e^{-bx}\cos ax\,dx = Re\left[\int e^{-bx}e^{iax}\,dx\right]$$

$b - ia$

$$= Re\left[\frac{e^{(ia-b)x}}{(ia-b)}\right]$$

$$= e^{-bx}\,Re\left[\frac{(\cos ax + i\sin ax)(b+ia)}{b^2+a^2}\right]$$

$$= \frac{e^{-bx}}{b^2+a^2}\left[b\cos ax - a\sin ax\right]\Big|_{\infty}^{0}$$

$$= \frac{1}{b^2+a^2}(b)$$

$$\therefore \phi(a,b) = \boxed{\tan^{-1}(a/b)}$$

$$\cancel{\frac{\partial\phi}{\partial b} = \int_0^\infty e^{-bx}(-x)}$$

$$\int_0^\infty \frac{\sin ax}{x}\,dx = \phi(a,0) = \boxed{\pi/2}$$

14) (i) $\phi(x) = \int_0^{\pi/2} \ln(1-x^2\sin^2\theta)\,d\theta$

$$\Rightarrow \phi'(x) = \int_0^{\pi/2} \frac{-2x\sin^2\theta}{1-x^2\sin^2\theta}\,d\theta$$

$$= -2x\int_0^{\pi/2} \frac{\sin^2\theta}{1-x^2\sin^2\theta}\,d\theta$$

$\tan\theta = u$
$\sec^2\theta\,d\theta = du$

$$= -2x\int_0^{\pi/2} \frac{\tan^2\theta}{\sec^2\theta - x^2\tan^2\theta}\,d\theta$$

$$= -2x\int_0^\infty \frac{u^2}{1+u^2-x^2u^2}\frac{du}{1+u^2} = -\frac{2x}{x^2}\int_0^\infty \frac{x^2u^2\,du}{(1+u^2)(1+(1-x^2)u^2)}$$

$$= -\frac{2}{x}\int_0^\infty \left(\frac{1}{1+(1-x^2)u^2} - \frac{1}{1+u^2}\right)du = \frac{\pi}{x}\left(1 - \frac{1}{\sqrt{1-x^2}}\right)$$

$$\therefore \phi'(x) = \pi\left(\frac{1}{x} - \frac{1}{x\sqrt{1-x^2}}\right)$$

$$\Rightarrow \phi(x) = \pi\left(\ln x - \ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)\right) + c$$

$$= \pi \ln\left(\frac{x^2}{1-\sqrt{1-x^2}}\right) + c$$

$$= \pi \ln(1+\sqrt{1-x^2}) + c$$

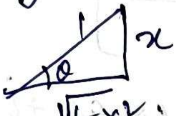
$$\phi(0) = 0 \Rightarrow \pi \ln 2 + c = 0$$

$$\Rightarrow \phi(x) = \boxed{\pi \ln\left(\frac{1+\sqrt{1-x^2}}{2}\right)}$$

$$\int \frac{1}{x\sqrt{1-x^2}}\,dx \qquad x = \sin\theta$$

$$\int \frac{1}{\sin\theta\cos\theta}\cos\theta\, d\theta$$

$$= \int \operatorname{cosec}\theta\, d\theta = \ln|\operatorname{cosec}\theta - \cot\theta|$$



$$= \ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right|$$

$$= \ln\left(\frac{1-\sqrt{1-x^2}}{x}\right).$$

(ii) $$\int_0^\infty \frac{e^{-px}\cos qx - e^{-ax}\cos bx}{x}\,dx$$

$$I(p) = \int_0^\infty \frac{e^{-px}\cos qx}{x}\,dx$$

$$\Rightarrow I'(p) = \int_0^\infty \frac{\cos qx}{\cancel{x}}\, e^{-px}(-\cancel{x})\,dx$$

$$= -\frac{2p}{2(p^2+q^2)}$$

$$\Rightarrow I(p) = -\frac{1}{2}\ln(p^2+q^2) + c$$

$$I(p,q) - I(a,b) = \boxed{\frac{1}{2}\ln\left(\frac{a^2+b^2}{p^2+q^2}\right)}$$

(iii) $$\phi(a) = \int_0^\infty e^{-x^2}\cos 2ax\,dx$$

$$\phi'(a) = \int_0^\infty e^{-x^2}(-\sin 2ax)\cdot 2x\,dx$$

$$= \int_0^\infty (\sin 2ax)\left(-e^{-x^2}2x\right)dx$$

$$= \sin 2ax\, e^{-x^2}\Big|_0^\infty - \int_0^\infty 2a\cos 2ax\, e^{-x^2}\,dx$$

$$= -2a\,\phi(a)$$

$$\frac{d\phi}{da} = -2a\phi$$

$$\Rightarrow \ln\phi = -a^2 + \ln c$$

$$\phi(a) = ce^{-a^2}, \quad \phi(0) = \frac{\sqrt{\pi}}{2} \Rightarrow \phi(a) = \boxed{\frac{\sqrt{\pi}}{2}e^{-a^2}}$$

12) $f(x,t) = \begin{cases} \dfrac{xt^3}{(x^2+t^2)^2} & \text{if } x \neq 0, t \neq 0 \\ 0 & \text{if } x = 0, t = 0 \end{cases}$

$$\left.\frac{d}{dt}\int_0^1 f(x,t)\,dx\right|_{t=0} = \lim_{h\to 0} \frac{\int_0^1 f(x,h)\,dx - \int_0^1 f(x,0)\,dx}{h}$$

$$= \lim_{h\to 0} \frac{\int_0^1 f(x,h)\,dx}{h}$$

$$\left.\frac{\partial}{\partial t} f(x,t)\right|_{t=0} = \lim_{h\to 0} \frac{f(x,h) - f(x,0)}{h}$$

$$= \lim_{h\to 0} \frac{xh^3}{(x^2+h^2)^2\, h} = 0$$

$\therefore$ at $t=0$, $\frac{d}{dt}\int_0^1 f(x,t)\,dx \neq \int_0^1 \frac{\partial}{\partial t} f(x,t)\,dx$