Tutorial 5

1) $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$

$P_A(x) = \det(xI - A) = \begin{vmatrix} x-3 & -1 \\ -1 & x-2 \end{vmatrix} = x^2 - 5x + 5$

$A^2 - 5A + 5I = 0$

$\Rightarrow A^2 = 5A - 5I$

$A^4 = (5A - 5I)^2$

$= 25A^2 + 25I - 50A$

$= 25(5A - 5I) + 25I - 50A$

$= 75A - 100I$

$A^5 = (75A - 100I)A$

$= 75(5A - 5I) - 100A$

$= 275A - 375I$

125
$\underline{50}$
75

-125
$+25$

$2A^5 - 3A^4 + A^2 - 5I$

$= 2(275A - 375I) - 3(75A - 100I) + 5A - 5I - 5I$

$= 330A - 460I$

2) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$P_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & 0 & 0 \\ -1 & x & -1 \\ 0 & -1 & x \end{vmatrix}$

$= (x-1)[x^2 - 1]$

$= (x-1)^2(x+1) = x^3 - x^2 - x + 1$

$A^3 - A^2 - A + I = 0$

$\Rightarrow A^3 = A^2 + A - I$

$$A^4 = A^3 + A^2 - A$$
$$= A^2 + A - I + A^2 - A$$
$$= 2A^2 - I = A^{4-2} + A^2 - I$$

$$A^k = A^{k-2} + A^2 - I$$
$$A^{k+1} = A^{k-1} + A^3 - A$$
$$= A^{k-1} + A^2 + \cancel{A} - I - \cancel{A}$$
$$A^{k+1} = A^{k-1} + A^2 - I$$

hence proved by mathematical induction.

$$A^{50} - A^{48} = A^2 - I$$
$$A^{48} - A^{46} = A^2 - I$$
$$\vdots$$
$$A^4 - A^2 = A^2 - I$$

$$A^{50} - A^2 = (A^2 - I) \cdot 24$$
$$= 24A^2 - 24I$$
$$\Rightarrow A^{50} = 25A^2 - 24I$$

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A^{50} = \begin{pmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{pmatrix} - \begin{pmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{pmatrix}$$

3) $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$

$$D = P^{-1}AP$$
$$P_p(x) = \det(xI - P) = \begin{vmatrix} x-1 & -1 \\ 2 & x-1 \end{vmatrix} = (x-1)^2 + 2$$
$$= x^2 - 2x + 3$$

$$P^2 - 2P + 3I = 0$$

$$\Rightarrow P - 2I + 3P^{-1} = 0$$

$$\Rightarrow P^{-1} = \frac{1}{3}(2I - P) = \frac{1}{3}\left[\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$D = P^{-1}AP = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -2 \\ 10 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 6 & 0 \\ 0 & 15 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$$

eigenvalues of A or roots of $P_A(x)$.

$$\lambda_1 + \lambda_2 = tr(A)$$
$$2 + 5 = 4 + 3$$
$$det(A) = \lambda_1\lambda_2 = 10 = 12 - 2$$

Trick

$$\Sigma\lambda_i = tr(A)$$
$$\Pi\lambda_i = det(A)$$
$$x^3 - (\Sigma\lambda_i)x^2 + cx - (\Pi\lambda_i) = 0$$

4b) $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

$$x^3 - 12x^2 + cx - 32 = 0$$
$$\Rightarrow P_A(x)$$

Step 1: Find eigenvalues of A

$$P_A(x) = det(xI - A) = \begin{vmatrix} x-6 & 2 & -2 \\ 2 & x-3 & 1 \\ -2 & 1 & x-3 \end{vmatrix}$$

$$= x^3 - 12x^2 + 36x - 32$$

2, 2, 8

$$P_A'(0) = c$$
$$= \begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 1 \\ -2 & 1 & -3 \end{vmatrix} + \begin{vmatrix} -6 & 2 & -2 \\ 2 & 1 & 1 \\ -2 & 1 & -3 \end{vmatrix} + \dots$$

Step 2: Find eigenvectors corresponding to each eigenvalue.

$$A\vec{v} = \lambda\vec{v}$$

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 4 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 - x_2 + x_3 = 0$$

$x_1 = a$, $x_3 = b$, $x_2 = 2a + b$

$$\left\{ \begin{pmatrix} a \\ 2a+b \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

basis $\to \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\therefore$ Eigen vector corresponding to $\lambda = 2$ is

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} - \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$A\vec{v} = 8\vec{v}$$

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -3 & -3 \\ 0 & -6 & -6 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2 + x_3 = 0 \Rightarrow x_2 = k,\ x_3 = -k$$

$$2x_1 - x_2 - 5x_3 = 0 \Rightarrow 2x_1 - k + 5k = 0$$

$$\Rightarrow 2x_1 = -4k \Rightarrow x_1 = -2k$$

for $\lambda = 8$: $\left\{ \begin{pmatrix} -2k \\ k \\ -k \end{pmatrix} : k \in \mathbb{R} \setminus \{0\} \right\}$

Step 3

$$D = P^{-1} A P$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad P = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & v_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

VERIFYING

$$P^{-1} = \frac{\operatorname{adj} P}{\det P} = \frac{\operatorname{adj} P}{-2 + (-2)(2)} = \frac{\operatorname{adj} P}{-6}$$

$$\operatorname{adj} P = \begin{pmatrix} -2 & -2 & 2 \\ 2 & -1 & -5 \\ 2 & -1 & 1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{6} \begin{pmatrix} 2 & 2 & -2 \\ -2 & 1 & 5 \\ -2 & 1 & -1 \end{pmatrix}$$

$$P^{-1}A = \frac{1}{6}\begin{pmatrix} 2 & 2 & -2 \\ -2 & 1 & 5 \\ -2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 4 & 4 & -4 \\ -4 & 2 & 10 \\ -16 & 8 & -8 \end{pmatrix}$$

$$P^{-1}AP = \frac{1}{6}\begin{pmatrix} 4 & 4 & -4 \\ -4 & 2 & 10 \\ -16 & 8 & -8 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 48 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

5) $b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

$$= x_3\hat{i} + \hat{j}(0) + \hat{k}(-x_1)$$

$$L(x) = \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix}$$

$L : M_{3\times1} \to M_{3\times1}$

$\to E_1, E_2, E_3$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$E_1 \quad E_2 \quad E_3$

$$L(\vec{x}) = M\vec{x}$$

$$L(E_1) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0 \cdot E_1 + 0 \cdot E_2 + (-1) E_3$$

$$L(E_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot E_1 + 0 \cdot E_2 + 0 \cdot E_3$$

$$L(E_3) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \cdot E_1 + 0 \cdot E_2 + 0 \cdot E_3$$

$$M_L = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} p_M(x) &= \det(xI - M) \\ &= \begin{vmatrix} x & 0 & -1 \\ 0 & x & 0 \\ 1 & 0 & x \end{vmatrix} \\ &= x^3 + x = x(x^2+1) \end{aligned}$$

$\therefore$ eigenvalues are $0, \pm i$

6) (a) A and B are similar if there exist a non-singular matrix P, such that

$$\boxed{B = P^{-1}AP} \Rightarrow \boxed{AP = PB}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & -1 \\ 4 & -1 \end{pmatrix}$$

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$AP = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$$

$$PB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 6 & -1 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 6a+4b & -a-b \\ 6c+4d & -c-d \end{pmatrix}$$

$$\begin{aligned} a+2c &= 6a+4b & &\Rightarrow & 5a+4b-2c &= 0 \\ b+2d &= -a-b & &\Rightarrow & a+2b+2d &= 0 \\ 3a+4c &= 6c+4d & &\Rightarrow & 3a-2c-4d &= 0 \\ 3b+4d &= -c-d & &\Rightarrow & 3b+c+5d &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 0 \\ 3 & 0 & -2 & -4 & 0 \\ 5 & 4 & -2 & 0 & 0 \\ 0 & 3 & 1 & 5 & 0 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 5R_1$

$-4-6$
$4-10$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 0 \\ 0 & -6 & -2 & -10 & 0 \\ 0 & -6 & -2 & -10 & 0 \\ 0 & 3 & 1 & 5 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 2 & 0 \\ 0 & \boxed{3} & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\text{rank} A = 2 = \text{rank} \tilde{A} \neq 4 \Rightarrow \infty$ sol^n

$a + 2b + 2d = 0$
$3b + c + 5d = 0$

$b = 1 = d$
$c = -8, a = -4$

$\therefore$ take $P = \begin{pmatrix} -4 & 1 \\ -8 & 1 \end{pmatrix}$

8) (b) Algebraic multiplicity : r where $P_A(x) = (x-\lambda)^r q(x)$

$\text{alg}(\lambda) = r$

Geometric multiplicity : $\boxed{\dim(E_\lambda)}$

$\text{geo}(\lambda) = \dim(E_\lambda)$

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

$$P_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & -3 & -3 \\ 3 & x+5 & 3 \\ -3 & -3 & x-1 \end{vmatrix}$$

$$= x^3 + 3x^2 - 4$$

$$= (x-1)^1 (x+2)^2$$

$\text{alg}(1) = 1$
$\text{alg}(-2) = 2$

$$A\vec{v} = \lambda \vec{v}$$

$$\begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 3 & 3 \\ 0 & -3 & -3 \\ 3 & 3 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_2 = k$

$3x_1 + 3x_2 = 0 \Rightarrow x_1 + x_2 = 0$ $\quad x_1 = -k$

$3x_2 + 3x_3 = 0 \Rightarrow x_2 + x_3 = 0 \Rightarrow x_3 = -k$

For $\lambda = 1$, $\left\{ \begin{pmatrix} -k \\ k \\ -k \end{pmatrix} : k \in \mathbb{R} \setminus \{0\} \right\}$

$\text{geo}(1) = 1$

$$\begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + x_2 + x_3 = 0$

For $\lambda = -2$, $\left\{ \begin{pmatrix} a \\ b \\ -a-b \end{pmatrix} : a, b \in \mathbb{R} \right\}$

$\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\} - \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\text{geo}(-2) = 2$

9) Always $\text{geo}(\lambda) \leq \text{alg}(\lambda)$

A is diagonalizable if $\forall \lambda$, $\text{geo}(\lambda) = \text{alg}(\lambda)$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 0 & 2 \end{pmatrix}$$

$$P_A(x) = \det(xI - A) = \begin{vmatrix} x-2 & 0 & 0 \\ -4 & x-2 & 0 \\ -6 & 0 & x-2 \end{vmatrix}$$

$$= (x-2)^3$$

$\text{alg}(2) = 3$

$A\vec{v} = 2\vec{v}$

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 6 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\Rightarrow 4x_1 = 0 \Rightarrow x_1 = 0$

$\left\{ \begin{pmatrix} 0 \\ a \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\}$

$\text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} - \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$geo(2) = 2 \neq alg(2)$

$\therefore$ not diagonalizable.

10) $A_{n\times n}$, $\lambda_1, \lambda_2, \dots, \lambda_n$ distinct

$alg(\lambda_1) + alg(\lambda_2) + \dots + alg(\lambda_n) = n$

$\therefore alg(\lambda_1) = 1 = \dots = alg(\lambda_n)$

$1 \leq geo(\lambda_i) \leq alg(\lambda_i) \Rightarrow geo(\lambda_i) = 1 = alg(\lambda_i) \quad \forall i$

$\therefore$ A is diagonalizable.