Tutorial - 3

$T : V \to W$

Null space $= \{ \vec{u} \in V \mid T(\vec{u}) = \vec{0}_W \} \subseteq V$

range space $= \{ T(\vec{u}) \mid \vec{u} \in V \} \subseteq W$

3) (a) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$

$$T(f(x)) = 2f'(x) + \int_0^x 3 f(t)\,dt$$

$$f(x) = ax^2 + bx + c$$

$$T(f(x)) = 2(2ax + b) + 3\left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]$$

$$= ax^3 + \frac{3b}{2}x^2 + (3c + 4a)x + 2b$$

null space $\Rightarrow$ $a = 0,\ b = 0,\ c = 0 \ni f(x) = 0$

$\{0\} \to \dim = 0 \Rightarrow$ nullity $= 0$

range space $= \{ ax^3 + \frac{3b}{2}x^2 + (3c+4a)x + 2b \}$

$$\left(a, \frac{3b}{2}, 3c+4a, 2b\right)$$

$$= a(1, 0, 4, 0) + b\left(0, \frac{3}{2}, 0, 2\right) + c(0, 0, 3, 0)$$

range space $=$ span$\{ (1,0,4,0), (0,\frac{3}{2},0,2), (0,0,3,0) \}$

$$\begin{bmatrix} \boxed{1} & 0 & 4 & 0 \\ 0 & \boxed{3/2} & 0 & 2 \\ 0 & 0 & \boxed{3} & 0 \end{bmatrix} \to \text{REF} \quad \therefore \text{ all are linearly independent}$$

rank $= 3$

rank + nullity $= \dim(V)$

(b) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = \left(\frac{x-y-z}{2}, \frac{z}{2}\right)$

nullspace $= \{ \vec{u} \in \mathbb{R}^3 ; T(\vec{u}) = 0_W \}$

$\ni T(\vec{u}) = (0,0)$

$T(x,y,z) = (0,0)$

$\frac{x-y-z}{2} = 0 \Rightarrow \quad x-y-z=0$

$\frac{z}{2} = 0 \quad \Rightarrow \quad z = 0$

$\Rightarrow x = y = t$

null space $= \{ (t, t, 0) : t \in \mathbb{R} \}$

basis of null space $= \{ (1,1,0) \}$, nullity $= 1$

$T(x,y,z) = \left(\frac{x-y-z}{2}, \frac{z}{2} \right)$

$x\left(\frac{1}{2}, 0\right) + y\left(-\frac{1}{2}, 0\right), z\left(-\frac{1}{2}, \frac{1}{2}\right)$

$$\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1/2 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 \to R_1$

$$= \begin{bmatrix} \boxed{1/2} & 0 \\ \cancel{0} & \cancel{0} \\ 0 & \boxed{1/2} \end{bmatrix}$$ $\to$ removed

rank $= 2$

nullity + rank $= 3 = \dim(\mathbb{R}^3)$

(c) $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$

$T(A) = \frac{A - A^t}{2}$

$$\frac{A-A^t}{2}=0 \Rightarrow A=A^t$$

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, A^t=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$b=c$

$$\therefore \text{null space} = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} : a, b, d \in \mathbb{R} \right\}$$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

nullity $=3$

$$T(A)=\begin{pmatrix} 0 & \frac{b-c}{2} \\ \frac{c-b}{2} & 0 \end{pmatrix}$$

$$\frac{b-c}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

rank $=1$

nullity + rank $=4=\dim(M_{2\times 2}(\mathbb{R}))$

4Y(a) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(1,1)=(1,0,2)$

$T(2,3)=(1,-1,4)$

$T(x,y)=?$

$(x,y)=c_1(1,1)+c_2(2,3)$

$$\left.\begin{aligned} c_1+2c_2&=x \\ c_1+3c_2&=y \end{aligned}\right] \quad c_2=y-x$$

$c_1+2y-2x=x \Rightarrow c_1=3x-2y$

$$(x,y) = (3x-2y)(1,1) + (y-x)(2,3)$$

$$T(x,y) = T\left[(3x-2y)(1,1) + (y-x)(2,3)\right]$$

$$= T((3x-2y)(1,1)) + T((y-x)(2,3))$$

$$= (3x-2y)\cdot T(1,1) + (y-x)\cdot T(2,3)$$

$$= (3x-2y)(1,0,2) + (y-x)(1,-1,4)$$

$$T(x,y) = \boxed{(2x-y,\ x-y,\ 2x)}$$

(b) $T: \mathbb{R}^3 \to \mathbb{R}^2$

$$\{(1,0,0),(0,1,0),(0,0,1)\} \Rightarrow \{(1,1),(2,3),(3,2)\}$$

$$(x,y,z) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$$

$$c_1 = x,\ c_2 = y,\ c_3 = z$$

$$T(x,y,z) = x\,T(1,0,0) + y\,T(0,1,0) + z\,T(0,0,1)$$

$$= x(1,1) + y(2,3) + z(3,2)$$

$$= (x+2y+3z,\ x+3y+2z)$$

$N(T) \Rightarrow$

$$x+2y+3z=0$$
$$x+3y+2z=0$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right] \quad R_2 \to R_2 - R_1$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{1} & -1 & 0 \end{array}\right]$$

$y - z = 0$ $\quad z = t,\ y = t$

$x + 2y + 3z = 0$

$\Rightarrow x + 5t = 0 \Rightarrow x = -5t$

$$N(T) = \{(-5t, t, t) : t \in \mathbb{R}\}$$

$$\text{basis}(N(T)) = \{(-5,1,1)\} \Rightarrow \text{nullity} = 1$$

$$T(x,y,z) = (x+2y+3z,\ x+3y+2z)$$
$$x(1,1)+y(2,3)+z(3,2)$$
$$= \text{span}\{(1,1),(2,3),(3,2)\}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\begin{bmatrix} \boxed{1} & 1 \\ 0 & \boxed{1} \\ \cancel{0} & \cancel{0} \end{bmatrix} \to \text{removed} \qquad \text{rank} = 2$$

$$\text{nullity} + \text{rank} = 3 = \dim(\mathbb{R}^3)$$

T is called one-one if $T(\vec{x}) = T(\vec{y}) \Rightarrow \vec{x} = \vec{y}$

onto if $T(V) = W$

$$T(x_1,y_1,z_1) = (x_1+2y_1+3z_1,\ x_1+3y_1+2z_1)$$
$$T(x_2,y_2,z_2) = (x_2+2y_2+3z_2,\ x_2+3y_2+2z_2)$$

$$x_1+2y_1+3z_1 = x_2+2y_2+3z_2$$
$$\Rightarrow x+2y+3z = 0 \quad \text{---(i)}$$

let $x_1 - x_2 = x$
$y_1 - y_2 = y$
$z_1 - z_2 = z$

$$x_1+3y_1+2z_1 = x_2+3y_2+2z_2$$
$$\Rightarrow x+3y+2z = 0 \quad \text{---(ii)}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right] \quad R_2 \to R_2 - R_1$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{1} & -1 & 0 \end{array}\right]$$

$y - z = 0 \Rightarrow y = t, z = t$
$x = -5t$

say $t=1$, we have

$T(x_1, y_1, z_1) = T(x_1+5, y_1-1, z_1-1)$

$\therefore T$ is not one-one.

$R(T) = \text{span}\{(1,1), (2,3)\} = \mathbb{R}^2$

$\therefore T$ is onto

5) (a) Matrix repⁿ of a L.T.

$T: V \to W$

$T(\vec{v}_1) = a_{11} w_1 + a_{21} w_2 + a_{31} w_3 + \cdots$

$T(\vec{v}_2) = a_{12} w_1 + a_{22} w_2 + a_{32} w_3 + \cdots$

$T(\vec{v}_3) = a_{13} w_1 + a_{23} w_2 + a_{33} w_3 + \cdots$

$$\begin{array}{c|c} & T(\vec{v}_1) \quad T(\vec{v}_2) \quad T(\vec{v}_3) \cdots \\ \begin{matrix} w_1 \\ w_2 \\ w_3 \\ \vdots \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

5) (a) $D: \mathbb{P}_4(\mathbb{R}) \to \mathbb{P}_4(\mathbb{R})$

$D(p(x)) = 3\dfrac{d^3}{dx^3}(p(x))$, $\{1, x, x^2, x^3, x^4\}$ for $\mathbb{P}_4(\mathbb{R})$

$D(1) = 0 = 0.(1) + 0.(x) + 0.(x^2) + 0.(x^3) + 0.(x^4)$

$D(x) = 0 = 0.(1) + 0.(x) + 0.(x^2) + 0.(x^3) + 0.(x^4)$

$D(x^2) = 0 = 0.(1) + 0.(x) + 0.(x^2) + 0.(x^3) + 0.(x^4)$

$D(x^3) = 18 = 18(1) + 0\cdot(x) + 0\cdot(x^2) + 0\cdot(x^3) + 0\cdot(x^4)$

$x^3 \xrightarrow{d/dx} 3x^2 \xrightarrow{d^2/dx^2} 6x \xrightarrow{d^3/dx^3} 6$

$$D(x^4) = 72x = 0\cdot(1) + 72(x) + 0\cdot(x^2) + 0\cdot(x^3) + 0\cdot(x^4)$$

$$x^4 \xrightarrow{d/dx} 4x^3 \xrightarrow{d^2/dx^2} 12x^2 \xrightarrow{d^3/dx^3} 24x$$

$$\begin{array}{c c} & \begin{array}{ccccc} D(1) & D(x) & D(x^2) & D(x^3) & D(x^4) \end{array} \\ \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 18 & 0 \\ 0 & 0 & 0 & 0 & 72 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \rightarrow A$$

(b) $T: P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$

$$T(f(x)) = \begin{bmatrix} 2f''(0) & f(3) \\ 0 & f'(2) \end{bmatrix}$$

$$\{1, x, x^2, x^3\}, \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$T(1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} = 0\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + [illegible]\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x^2) = \begin{bmatrix} 4 & 9 \\ 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 9\cdot\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x^3) = \begin{bmatrix} 0 & 27 \\ 0 & 12 \end{bmatrix} = 0\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 27\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 12\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$3x^2 \to 6x$

$$\begin{array}{c|c} & \begin{array}{cccc} T(1) & T(x) & T(x^2) & T(x^3) \end{array} \\ \begin{array}{c} E_{11} \\ E_{12} \\ E_{21} \\ E_{22} \end{array} & \begin{bmatrix} 0 & 0 & 4 & 0 \\ 1 & 3 & 9 & 27 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 12 \end{bmatrix} \end{array}$$

6) $T: \mathbb{R}^5 \to \mathbb{R}^5$, $R(T) = N(T)$

Rank + nullity = $\dim(\mathbb{R}^5) = 5$

$x + x = 5 \Rightarrow x \notin \mathbb{Z}$ $\therefore$ not possible.

$\therefore$ Using Rank Nullity theorem, such a L.T. is not possible.

7) $A: M_{3\times1}(\mathbb{R}) \to M_{3\times3}(\mathbb{R})$

rank + nullity = 3

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & -1 \\ 7 & 3 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -7 \\ 0 & 10 & -14 \end{bmatrix}$$

$\cancel{2+3}$

$\cancel{-1-6}$

$\cancel{7-\quad 3+7}$

$$\begin{bmatrix} \boxed{1} & -1 & 2 \\ 0 & \boxed{5} & -7 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$\to$ removed

$N(T) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ -7 \end{pmatrix} \right\}$

$\therefore$ nullity $= 2$

rank$(A) = 1$

8) $T: \mathbb{R}^2 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 2x_1 \end{bmatrix}$$

$T(X) = \underset{3\times 2}{A} X_{2\times 1}$

$\mathbb{R}^2 \to \mathbb{R}^3 \to \{(1,0,0), (0,1,0), (0,0,1)\}$

$\{\underline{(1,0)}, (0,1)\}$

$$T(1,0) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T(0,1) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} T(1,0) \quad T(0,1) \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} \end{array}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}$$

$$AX = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 2x_1 \end{pmatrix}$$

$$\left.\begin{array}{l} x_1+x_2=0 \\ x_1-x_2=0 \\ 2x_1=0 \end{array}\right\} \quad x_1=0=x_2$$

$$\therefore \; N(T)=\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$$

$$R(T)=\left\{\begin{pmatrix}x_1+x_2\\x_1-x_2\\2x_1\end{pmatrix}\right\}$$

$$=\left\{x_1\begin{pmatrix}1\\1\\2\end{pmatrix}+x_2\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\}$$

rank $=2$

rank + nullity $=2=\dim(\mathbb{R}^2)$

9) $T: V\to V$

$\text{rank}(T^2)=\text{rank}(T)$

- First we prove that $\ker(T)=\ker(T^2)$

Choose $v\in\ker(T) \Rightarrow Tv=0 \Rightarrow T^2v=0 \Rightarrow v\in\ker(T^2)$

$\Rightarrow \ker(T)\subseteq\ker(T^2)$

$|\ker(T^2)|=|V|-\text{rank}(T^2)=|V|-\text{rank}(T)=|\ker(T)|$

$\therefore \; \ker(T)=\ker(T^2)$

- Choose $v\in R(T)\cap N(T)$. For some $\alpha\in V$,

$T\alpha=v$. $\quad T^2\alpha=Tv=0$ (as $v\in N(T)$)

$\therefore \; \alpha\in\ker(T^2)$, but $\ker(T)=\ker(T^2)$

$\Rightarrow T\alpha=v=0$

10) $T: V \to W$

$\{u_1, u_2, \dots, u_n\} \subseteq V$, s.t. $\{T(u_1), T(u_2), \dots, T(u_n)\}$ are linearly indep. in W.

$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0$

$\Rightarrow T(c_1 u_1 + \dots + c_n u_n) = 0$

$\Rightarrow c_1 T(u_1) + \dots + c_n T(u_n) = 0$

$\Rightarrow c_1 = 0 = \dots = c_n$

$\therefore \{u_1, u_2, \dots, u_n\}$ are linearly independent in V.