$B \to$ l.i.
$\text{span}(B) = V$

Tutorial 2

1 (a) $S = \{4t^2 - 2t + 3,\ 6t^2 - t + 4,\ 8t^2 - 8t + 7\}$ of $P_2(\mathbb{R})$

$ax^2 + bx + c$

$$\alpha(4t^2 - 2t + 3) + \beta(6t^2 - t + 4) + \gamma(8t^2 - 8t + 7) = 0$$

$$4\alpha + 6\beta + 8\gamma = 0 \Leftrightarrow 2\alpha + 3\beta + 4\gamma = 0$$
$$-2\alpha - \beta - 8\gamma = 0 \Leftrightarrow 2\alpha + \beta + 8\gamma = 0$$
$$3\alpha + 4\beta + 7\gamma = 0$$

Method 2

$\det(A)$
$= 14 + 72 + 32$
$- 12 - 42$
$- 64$
$= 0$

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 2 & 1 & 8 & 0 \\ 3 & 4 & 7 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to 2R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \quad R_3 \to 2R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{r} 2\alpha + 3\beta + 4\gamma = 0 \\ -2\beta + 4\gamma = 0 \end{array}\right] \Rightarrow \text{infinite sol}^n.$$

for eg. $(\alpha, \beta, \gamma) = (-5, 2, 1)$

$\therefore S$ is l.d. hence not a basis.

To check if $\text{span}(S) = V$.

To prove sets $A = B$
show that $A \subseteq B$
and $B \subseteq A$

$S = \{4t^2 - 2t + 3,\ 6t^2 - t + 4,\ 8t^2 - 8t + 7\}$

$\text{span}(S) = \{\alpha \vec{v_1} + \beta \vec{v_2} + \gamma \vec{v_3}\}$

$V = P_2(\mathbb{R}) = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$

$$\alpha(4t^2-2t+3)+\beta(6t^2-t+4)+\gamma(8t^2-8t+7)$$

$$=\underbrace{(4\alpha+6\beta+8\gamma)}_{a}t^2+\underbrace{(-2\alpha-\beta-8\gamma)}_{b}t+\underbrace{(3\alpha+4\beta+7\gamma)}_{c}\in V$$

$\therefore\ \text{span}(S)\subseteq V$

at^2+bt+c

$$t^2=p(4t^2-2t+3)+q(6t^2-t+4)+r(8t^2-8t+7)$$

$$2p+q+8r=0$$
$$3p+4q+7r=0$$
$$4p+6q+8r=1$$

$$\left[\begin{array}{ccc|c}2&1&8&0\\3&4&7&0\\4&6&8&1\end{array}\right]\quad R_2\to 2R_2-3R_1,\ R_3\to R_3-2R_1$$

$\cancel{8-3}$ $\cancel{1+-2q}$

$$\left[\begin{array}{ccc|c}2&1&8&0\\0&5&-10&0\\0&4&-8&1\end{array}\right]\quad R_3\to 5R_3-4R_2$$

$-40+40$

$$\left[\begin{array}{ccc|c}2&1&8&0\\0&5&-10&0\\0&0&0&1\end{array}\right]\to \text{no sol}^n$$

$\therefore\ V\not\subseteq \text{span}(S)$

$\therefore\ \text{span}(S)\neq V$

$B \to l.i$
$span(B) = V$

(b) $B = \{\vec{u}, \vec{v}, \vec{w}\} \to$ basis of V

$S = \{\vec{u}+\vec{v}+\vec{w}, \vec{v}+\vec{w}, \vec{w}\}$

$\alpha(\vec{u}+\vec{v}+\vec{w}) + \beta(\vec{v}+\vec{w}) + \gamma\vec{w} = 0$

$\Rightarrow \alpha\vec{u} + (\alpha+\beta)\vec{v} + (\alpha+\beta+\gamma)\vec{w} = 0$

$\alpha'\vec{u} + \beta'\vec{v} + \gamma'\vec{w} = 0$

$\vec{u}, \vec{v}, \vec{w} \in B$

as B is basis, B is $l.i.$

$\therefore \alpha' = 0 = \beta' = \gamma'$

$\Rightarrow \alpha = 0, \quad \alpha+\beta = 0 \Rightarrow \beta = 0$

$\alpha+\beta+\gamma = 0 \Rightarrow \gamma = 0$

$\therefore \alpha = 0 = \beta = \gamma \Rightarrow S$ is $l.i.$ —— (i)

Let $\vec{v} \in span(S)$

$\vec{v} = \alpha(\vec{u}+\vec{v}+\vec{w}) + \beta(\vec{v}+\vec{w}) + \gamma(\vec{w})$

$= \alpha\vec{u} + (\alpha+\beta)\vec{v} + (\alpha+\beta+\gamma)\vec{w}$

as B is basis, $span(B) = V$

$\Rightarrow span(B) \subseteq V$

$\therefore \vec{v} \in V$

$\Rightarrow span(S) \subseteq V$ —— (ii)

Let $\vec{v} \in V$

as B is basis, $span(B) = V$

$\Rightarrow V \subseteq span(B)$

$\vec{v} = a\vec{u} + b\vec{v} + c\vec{w}$

$\vec{u} = p(\vec{u}+\vec{v}+\vec{w}) + q(\vec{v}+\vec{w}) + \gamma(\vec{w})$

$p = 1$

$p + q = 0 \qquad q = -1$

$p + q + \gamma = 0 \quad \underline{\gamma = 0}$

$$\vec{u} = (\vec{u}+\vec{v}+\vec{w}) - (\vec{v}+\vec{w})$$
$$\vec{v} = (\vec{v}+\vec{w}) - (\vec{w})$$
$$\vec{w} = \vec{w}$$

$$\vec{v} = a\vec{u} + b\vec{v} + c\vec{w}$$
$$= a(\vec{u}+\vec{v}+\vec{w}) + (b-a)(\vec{v}+\vec{w})$$
$$+ (c-a-b)\vec{w} \in \operatorname{span}(S)$$

$\therefore V \subseteq \operatorname{span}(S) \longrightarrow$ (iii)

for (ii) and (iii), $\operatorname{span}(S) = V \longrightarrow$ (iv)

for (i) and (iv), S is a basis

(c) $$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

$$\boxed{\begin{array}{l} B \text{ is l.i.} \\ \operatorname{span}(B) = V \end{array}}$$

$$V = M_{2\times 2}(\mathbb{R})$$

$$\alpha\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \gamma\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$
$$+ \delta\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\alpha + \beta + \gamma + \delta = 0 \Leftarrow \Rightarrow \alpha = 0$$
$$\beta + \gamma + \delta = 0 \Leftarrow \Rightarrow \beta = 0$$
$$\gamma + \delta = 0 \Leftarrow \Rightarrow \gamma = 0$$
$$\delta = 0$$

$\therefore S$ is l.i. $\longrightarrow$ (i)

$$\alpha\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\beta\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}+\gamma\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}+\delta\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}\alpha+\beta+\gamma+\delta & \beta+\gamma+\delta\\ \gamma+\delta & \delta\end{pmatrix} \in M_{2\times2}(\mathbb{R})$$

$$\therefore \operatorname{span}(S) \subseteq V \longrightarrow \text{(ii)}$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}=p\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+q\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}+r\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}+s\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

$$p+q+r+s=a$$

$$q+r+s=b$$

$$r+s=c$$

$$s=d$$

$$r=c-d$$

$q = b - c$

$p = a - b$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a-b)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (b-c)\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + (c-d)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{span}(S)$$

$$\therefore V \subseteq \text{span}(S) \text{ — (iii)}$$

by (ii) and (iii) $\text{span}(S) = V$ — (iv)

by (i) and (iv), S is a basis.

2) (a) $M_{2\times2}(\mathbb{R})$

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \text{Basis} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\dim = n(B) = 3$$

(b) $U = \{(x, y, z, w) \in \mathbb{R}^4 : \begin{array}{l} x + 2y - z = 0 \\ 2x + y + w = 0 \end{array}\}$

$$\begin{array}{r} 2x + 4y = 2z \\ - \quad 2x + y = -w \\ \hline 3y = 2z + w \Rightarrow y = \dfrac{2z+w}{3} \end{array}$$

$$x = z - 2y = z - \frac{2}{3}(2z + w) = \frac{-z - 2w}{3}$$

$$U = \left\{\left(\frac{-z-2w}{3}, \frac{2z+w}{3}, z, w\right) : z, w \in \mathbb{R}\right\}$$

$$\text{basis} = \left\{\left(-\frac{1}{3}, \frac{2}{3}, 1, 0\right), \left(-\frac{2}{3}, \frac{1}{3}, 0, 1\right)\right\}$$

dim = 2

(c) $U = \left\{ p(t) \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p(t)\,dt = 0 \right\}$ of $\mathcal{P}_4(\mathbb{R})$

$$\int_{-1}^{1} (ax^4 + bx^3 + cx^2 + dx + e)\,dx = 0$$

$$\Rightarrow 2\int_{0}^{1} (ax^4 + cx^2 + e)\,dx = 0$$

$$\Rightarrow 2\left[a\frac{x^5}{5} + c\frac{x^3}{3} + ex\right]_0^1 = 0$$

$$\Rightarrow 2\left[\frac{a}{5} + \frac{c}{3} + e\right] = 0 \Rightarrow 3a + 5c + 15e = 0$$

$$ax^4 + bx^3 + cx^2 + dx + e$$

$$= \left(-\frac{5c - 15e}{3}\right)x^4 + bx^3 + cx^2 + dx + e.$$

$b(x^3) + c\left(-\frac{5}{3}x^4 + x^2\right) + d(x) + e(-5x^4 + 1)$

$$B = \left\{ x^3, \ -\frac{5}{3}x^4 + x^2, \ x, \ -5x^4 + 1 \right\}$$

$\underline{\underline{\dim = 4}}$.

3) (a) $U = \text{span}\{(1,2,1), (2,1,3)\}$

$W = \text{span}\{(1,0,0), (0,0,1)\}$

show that U, W are subspaces of R^3 and find $\dim(U)$, $\dim(W)$, $\dim(U+W)$, $\dim(U \cap W)$

$\underline{\underline{Th}}$ $\boxed{\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)}$

$\underline{\underline{U}}$ is a subspace iff $\alpha\vec{u} + \beta\vec{v} \in U$ for $\forall \vec{u}, \vec{v} \in U$, $\alpha, \beta \in F$

$U = \{(p+2q, \ 2p+q, \ p+3q) : p, q \in R\}$

$W = \{(p_1, 0, q_1) : p_1, q_1 \in R\}$

$\vec{u}, \vec{v} \in U$,

$\alpha(p_1 + 2q_1, \ 2p_1 + q_1, \ p_1 + 3q_1)$

$+ \beta(p_2 + 2q_2, \ 2p_2 + q_2, \ p_2 + 3q_2)$

$2p + q = 0$

$q = -2p$

$p + 2(-2p)$

$= -3p$.

$p + 3(-2p)$

$= -5p$.

$p := \alpha p_1 + \beta p_2 \,,\; q := \alpha q_1 + \beta q_2$

$$\left(p+2q,\, 2p+q,\, p+3q\right) \in U$$

$\therefore \alpha\vec{u} + \beta\vec{v} \in U \quad \therefore U$ is a subspace.

$$\alpha\,(p_1, 0, q_1) + \beta\,(p_2, 0, q_2) = (p, 0, q) \in W$$

$p := \alpha p_1 + \beta p_2$

$q := \alpha q_1 + \beta q_2$

$\therefore W$ is a subspace.

$$U = \{(p+2q,\, 2p+q,\, p+3q)\}$$

$$p\,(1, 2, 1) + q\,(2, 1, 3)$$

Basis of U is $\{(1,2,1), (2,1,3)\}$

$\dim(U) = 2$ —— (i)

$$W = \{(p, 0, q)\}$$

$$p\,(1, 0, 0) + q\,(0, 0, 1)$$

Basis of W is $\{(1,0,0), (0,0,1)\}$

$\dim(W) = 2$ —— (ii)

$$U + W = \{\vec{u}_1 + \vec{u}_2 \mid \vec{u}_1 \in U,\, \vec{u}_2 \in W\}$$

$$U+W = \{(p_1+2q_1+p_2,\ 2p_1+q_1,\ p_1+3q_1+q_2)\}$$

$$p_1, p_2, q_1, q_2 \in \mathbb{R}$$

$$p_1(1,2,1)+p_2(1,0,0)+q_1(2,1,3)$$
$$+q_2(0,0,1)$$

$$S = \{(1,2,1), (1,0,0), (2,1,3), (0,0,1)\}$$

$$(1,2,1) = -3(1,0,0)-5(0,0,1)+2(2,1,3)$$

$$\{(1,0,0), (0,0,1), (2,1,3)\}$$

$$\dim(U+W) = 3$$

$$U\cap W = \{(-3p,\ 0,\ -5p)\}$$

$$p(-3, 0, -5)$$

$\longrightarrow$ basis

$$\dim(U\cap W) = 1$$

$$\dim(U+W) = 2+2-1 = 3$$

(b) $V = \{A = (a_{ij})_{n\times n} : a_{ij} \in \mathbb{C},\ a_{ij} = -a_{ji}\}$

$(V(\mathbb{R}), +, \cdot)$

$$\begin{pmatrix} 0 & a_2 & \cdots & a_n \\ -a_2 & 0 & & b_n \\ \vdots & \vdots & \ddots & \vdots \\ -a_n & -b_n & \cdots & 0 \end{pmatrix}$$

$1+2+\cdots+(n-1) = \frac{(n-1)n}{2}$

$a_j, b_j, \cdots \to \alpha_j + \beta_j i$

$(a_2)_R \begin{pmatrix} 0 & 1 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{pmatrix} + (a_2)_I \begin{pmatrix} 0 & i & \cdots & 0 \\ -i & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots \quad \therefore \dim = n^2 - n$

(c) $(C^5[a,b](\mathbb{R}), +, \cdot)$

$V \to C^5[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f$ is 5 times continuously differentiable function$\}$

$$W = \{ f \in V : \frac{d^4 f}{dt^4} + 2\frac{d^2 f}{dt^2} - f = 0 \}$$

$(D^4 + 2D^2 - 1) f = 0$

$m^4 + 2m^2 - 1 = 0$

$m^2 = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$

$m = \pm\sqrt{\sqrt{2}-1}, \ \pm i\sqrt{\sqrt{2}+1}$

$f(x) = c_1 e^{\sqrt{\sqrt{2}-1}\,x} + c_2 e^{-\sqrt{\sqrt{2}-1}\,x} + (c_3 \cos(\sqrt{\sqrt{2}+1}\,x) + c_4 \sin(\sqrt{\sqrt{2}+1}\,x))$

$\therefore$ basis for W is $\{ e^{\sqrt{\sqrt{2}-1}\,x}, e^{-\sqrt{\sqrt{2}-1}\,x}, \cos\sqrt{\sqrt{2}+1}\,x, \sin\sqrt{\sqrt{2}+1}\,x \}$

$\therefore \dim(V) \to \infty, \ \dim(W) = 4$

4) $S = \{ p_0(t), p_1(t), \ldots, p_m(t) \} \subset \mathbb{P}_m(\mathbb{R})$

$p_j(2) = 0$ for $j = 0, 1, \ldots, m$

$c_0 p_0(t) + c_1 p_1(t) + \cdots + c_m p_m(t) = 0 \ \forall t$

$p_j(t) = (t-2) q_{j-1}(t)$

$C_0(0) + c_1(t-2)q_0(t) + \cdots + c_m(t-2)q_{m-1}(t) = 0 \quad \forall t$

$C_0\vec{(0)} + c_1 q_0(t) + \cdots + c_m q_{m-1}(t) \in \mathbb{P}_{m-1}(t)$

basis for $q_{m-1}(t)$ are $\underbrace{\{1, t, t^2, \ldots, t^{m-1}\}}$

take $c_1 = 0 = \cdots = c_m$, $C_0 \neq 0$

$\therefore \exists\, c_i \neq 0$ such that $\sum c_i p_i(t) = 0$

$\therefore$ Linearly dependent

5Y (a)

$$9x + 3y + 4z = 7$$
$$4x + 3y + 4z = 8$$
$$x + y + z = 3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 4 & 3 & 4 & 8 \\ 9 & 3 & 4 & 7 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 9R_1 \end{array}$$

$7-27$

$8-12$

$4-9$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & -5 & -20 \end{array}\right] \qquad R_3 \to R_3 - 6R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & -5 & 4 \end{array}\right]$$

$-6+6$

$-20 - 6\times(-4)$

$24-20$

$-5z = 4 \Rightarrow z = -4/5$

$-y = -4 \Rightarrow y = 4$

$x + y + z = 3 \Rightarrow x + 4 - 4/5 = 3$

$\Rightarrow x = 4/5 - 1 = -1/5$

$\left(-\frac{1}{5}, 4, -4/5\right)$

(b) $x + 2y + 3z + 2w = -1$

$-x - 2y - 2z + w = 2$

$2x + 4y + 8z + 12w = 4$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 8 & 6 \end{array}\right]$$

$4+2$

$\cancel{4-4}$

$8-6$

$12-4$

$\cancel{8-6}$

$12-4=8$

$\cancel{4-2}$

$R_3 \to R_3 - 2R_2$

$8-6$

$6-2$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right]$$

$2w = 4 \Rightarrow w = 2$

$z + 3w = 1 \Rightarrow z + 6 = 1 \Rightarrow z = -5$

$x + 2y + 3z + 2w = -1$

$\Rightarrow x + 2y - 15 + 4 = -1 \Rightarrow x + 2y = 15 - 5$
$= 10$

$(x, y, z, w) = (10 - 2t, t, -5, 2)$

6)(i) $\begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 5 \\ 2 & 0 & 6 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$

$= \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{-1} & -7 \\ 0 & 0 & 0 \end{bmatrix}$ rank = 2

$5 - 12$

(ii) $\begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 6 & 9 & -3 & 3 \end{bmatrix}$ $R_2 \to 2R_2 - 3R_1$
$R_3 \to R_3 - 3R_1$

$\begin{bmatrix} \boxed{2} & 3 & -1 & 1 \\ 0 & \boxed{-9} & \cdot & \cdot \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\cancel{9}-9$

$\cancel{6}$
$0-9$
4

rank = 2

7) (b) $\begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - 3R_2$

$\begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{bmatrix}$ $R_2 \leftrightarrow R_1$

$6-8$
$1-6$
$10-12$
$6-3$

$R_3 \to R_2 + R_1$
$R_4 \to 2R_4 + 5R_2$

$\begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 6 \end{bmatrix}$

$-10+10$
$-4+10$
6

$R_4 \leftrightarrow R_3$

$\begin{bmatrix} \boxed{1} & 3 & 2 & 4 & 1 \\ 0 & 0 & \boxed{2} & 2 & 0 \\ 0 & 0 & 0 & \boxed{6} & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ rank = 3

8) $\det\begin{pmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{pmatrix} = 0$

$\Rightarrow 1 + 2x^3 - 3x^2 = 0$

$2x^3 - 3x^2 + 1 = 0$

$(x-1)[2x^2 - x - 1] = 0$

$(x-1)^2[2x+1] = 0$

$\therefore x = 1, -1/2$

12) $x + y + z = b$
$2x + y + 3z = b + 1$
$5x + 2y + az = b^2$

for no sol^n, rank $A \neq$ rank $\tilde{A}$
for one sol^n rank A = rank $\tilde{A} = 3$
for ∞ sol^n rank = rank $\tilde{A} \neq 3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 2 & 1 & 3 & b+1 \\ 5 & 2 & a & b^2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 5R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & -1 & 1 & 1-b \\ 0 & -3 & a-5 & b^2-5b \end{array}\right]$$

$1-2$ $\quad 3-2$
$b+1-2b$
$2-5$

$R_3 \to R_3 - 3R_2$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & b \\ 0 & \boxed{-1} & 1 & 1-b \\ 0 & 0 & \boxed{a-8} & b^2-2b-3 \end{array}\right]$$

$-3+3$

$a-5-3$

$b^2-5b-3+3b$

- one sol^n, rank A = rank $\tilde{A}$ = 3

  $a-8 \neq 0 \Rightarrow \boxed{a \neq 8}$

- ∞ sol^n, $a=8$, $b^2-2b-3=0$

  $\Rightarrow (b+1)(b-3)=0$

  $\boxed{(a,b)=(8,-1),(8,3)}$

- no sol^n, rank $A \neq$ rank $\tilde{A}$

  rank $A \to 2$, rank $\tilde{A} \to 3$

  rank $A \geq 2$

  rank $\tilde{A} \geq 2$

  $\boxed{a=8,\ \cancel{b \neq} \ b \notin \{-1,3\}}$