Tutorial 1

1.i) $(x,y,z)+(x',y',z')=(x+x',y+y',z+z')$

$k(x,y,z)=(kx,y,z) \quad \forall k \in \mathbb{R},\ (x,y,z),(x',y',z') \in \mathbb{R}^3$

$(k_1+k_2)\cdot(x,y,z) = ((k_1+k_2)x, y, z)$

$k_1(x,y,z)+k_2(x,y,z) = ((k_1+k_2)x, 2y, 2z)$

$\therefore (\alpha+\beta)\cdot v \neq \alpha\cdot v+\beta\cdot v$

hence it is not a vector space.

ii) $V=\{(x_1,x_2)\in\mathbb{R}^2 : x_1+x_2=1,\ x_1\in[0,1], x_2\in[0,1]\}$

$(x_1,x_2)+(y_1,y_2)=\left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}\right)$

$r(x_1,x_2)=(rx_1, rx_2)$

$\forall r\in\mathbb{R},\ (x_1,x_2),(y_1,y_2)\in V$

$(x_1,x_2)+((y_1,y_2)+(z_1,z_2)) = (x_1,x_2)+\left(\frac{y_1+z_1}{2}, \frac{y_2+z_2}{2}\right)$

$=\left(\frac{x_1}{2}+\frac{y_1}{4}+\frac{z_1}{4}, \frac{x_2}{2}+\frac{y_2}{4}+\frac{z_2}{4}\right)$

$((x_1,x_2)+(y_1,y_2))+(z_1,z_2) = \left(\frac{x_1}{2}+\frac{y_1}{2}, \frac{x_2}{2}+\frac{y_2}{2}\right)+(z_1,z_2)$

$=\left(\frac{x_1}{4}+\frac{y_1}{4}+\frac{z_1}{2}, \frac{x_2}{4}+\frac{y_2}{4}+\frac{z_2}{2}\right)$

$v_1+(v_2+v_3)\neq(v_1+v_2)+v_3$

hence it is not a vector space.

iii) $x\oplus x' = xx'$

$k\odot x = x^k \quad \forall k\in\mathbb{R}$

$x,x'\in\mathbb{R}^+$

$(\mathbb{R}^+(\mathbb{R}),\oplus,\odot)$

for $(\mathbb{R}^+(\mathbb{R}),\oplus,\odot)$ to be a vector space, it must satisfy these:

a) Closed under vector addition

$x \oplus x' = x \cdot x' \in \mathbb{R}^+$

b) Vector addition is associative

$\vec{v}_1 \oplus (\vec{v}_2 \oplus \vec{v}_3) = (\vec{v}_1 \oplus \vec{v}_2) \oplus \vec{v}_3$

for $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in V$ (here $\mathbb{R}^+$)

$\vec{v}_1 \oplus (\vec{v}_2 \oplus \vec{v}_3) = v_1 \oplus (v_2 v_3) = v_1 \cdot (v_2 \cdot v_3)$

$(\vec{v}_1 \oplus \vec{v}_2) \oplus \vec{v}_3 = (v_1 \cdot v_2) \oplus v_3 = (v_1 \cdot v_2) \cdot v_3$

as multiplication is associative

hence true.

c) Existence of identity element for vector addition

$\vec{v}_1 \oplus \vec{v}_2 = \vec{v}_2 \oplus \vec{v}_1 = \vec{v}_1$

$\Rightarrow v_1 v_2 = v_2 v_1 = v_1 \Rightarrow v_2 = 1$

d) Existence of inverse for vector add^n

$\vec{v}_1 \oplus \vec{v}_2 = \vec{v}_2 \oplus \vec{v}_1 = 1$

$\Rightarrow v_1 v_2 = v_2 v_1 = 1 \Rightarrow v_2 = 1/v_1$

e) Vector add^n is commutative

$\vec{v}_1 \oplus \vec{v}_2 = v_1 v_2 = v_2 v_1 = \vec{v}_2 \oplus \vec{v}_1$

true as multiplication is commutative.

f) Scalar multiplication is closed

$k \odot \vec{v}_1 = v_1^k \in \mathbb{R}^+$

g) $k_1 \odot (k_2 \odot \vec{v}_1) = k_1 \odot (v_1^{k_2}) = (v_1^{k_2})^{k_1} = v_1^{k_2 k_1}$

$= v_1^{k_1 k_2} = (k_1 \cdot k_2) \odot \vec{v}_1$

h) $1 \odot \vec{v} = v^1 = v$

i) $k \odot (\vec{v}_1 \oplus \vec{v}_2) = k \odot (v_1 v_2) = (v_1 v_2)^k = v_1^k \cdot v_2^k$

$= v_1^k \oplus v_2^k = (k \odot \vec{v}_1) \oplus (k \odot \vec{v}_2)$

j) $(k_1 + k_2) \odot \vec{v} = v^{k_1 + k_2} = v^{k_1} \cdot v^{k_2} = v^{k_1} \oplus v^{k_2}$

$= (k_1 \odot \vec{v}) \oplus (k_2 \odot \vec{v})$

$\therefore (\mathbb{R}^+(\mathbb{R}), \oplus, \odot)$ is a vector space.

iv) $V = \left\{ \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$

$(\mathbb{R}, +, \cdot)$

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} + \begin{pmatrix} a' & 1 \\ 1 & b' \end{pmatrix} = \begin{pmatrix} a+a' & 2 \\ 2 & b+b' \end{pmatrix} \notin V$$

$v_1 + v_2 \notin V \Rightarrow$ not a vector space

v) $V = \{ f \in \underbrace{C(\mathbb{R})}_{\text{set of all cont. } f^n \text{ over } \mathbb{R}} : \exists p \in \mathbb{N}, f(x+p) = f(x) \ \forall x \in \mathbb{R} \}$

Let period $(f_1) = p_1$, period $(f_2) = p_2$.

$f_1(x+p_1) = f_1(x) = f_1(x + n_1 p_1)$

$f_2(x+p_2) = f_2(x) = f_2(x + n_2 p_2)$

Let $g = f_1 + f_2$

Let $p = \text{lcm}(p_1, p_2) \Rightarrow p_1 \mid p$, $p_2 \mid p$

$\Rightarrow p = n_1 p_1 = n_2 p_2$

$\Rightarrow g(x+p) = f_1(x + n_1 p_1) + f_2(x + n_2 p_2)$

$= f_1(x) + f_2(x) = g(x)$

$\therefore g(x+p) = g(x) \Rightarrow f_1 + f_2 \in V$ for $f_1, f_2 \in V$

As add^n over $\mathbb{R}$ is associative $\therefore f_1 + (f_2 + f_3) = (f_1 + f_2) + f_3$

Consider $f_0(x) = 0 \ \forall x$. $f_0 \in V$

$f_0 + f = f + f_0 = f$

Consider $g = -f$

$g(x+p) = -f(x+p) = -f(x) = -g(x)$

$\therefore -f \in V$ if $f \in V$

$f(x) + (-f(x)) = 0 = f_0(x)$

As addⁿ over $\mathbb{R}$ is commutative, $\therefore f_1 + f_2 = f_2 + f_1$

Consider $g(x) = \alpha f(x)$

$g(x+p) = \alpha f(x+p) = \alpha f(x) = g(x)$

$\therefore \alpha f \in V$

$\alpha(f(x) + g(x)) = \alpha f(x) + \alpha g(x)$ as $+, \cdot$ follow distributivity over $\mathbb{R}$

$(\alpha+\beta) \cdot f(x) = \alpha f(x) + \beta f(x)$

$\alpha \cdot (\beta \cdot f(x)) = (\alpha \cdot \beta) \cdot f(x) \rightarrow$ as $\cdot$ follow associativity over $\mathbb{R}$

$1 \cdot f(x) = f(x) \rightarrow$ as 1 is multiplicative identity for $\mathbb{R}$.

$\therefore (V(\mathbb{R}), +, \cdot)$ is a vector space.

2 i) $(\mathbb{R}^3(\mathbb{R}), +, \cdot)$ is a vector space.

$W = \{(a, b, c) : (a, b, c) \in \mathbb{R}^3, b = a + c\}$

Let $\vec{u} = (a_1, a_1 + c_1, c_1)$

$\vec{v} = (a_2, a_2 + c_2, c_2)$

$\therefore \vec{u}, \vec{v} \in W$

$\alpha \vec{u} + \beta \vec{v} = (\alpha a_1 + \beta a_2, \alpha a_1 + \beta a_2 + \alpha c_1 + \beta c_2, \alpha c_1 + \beta c_2)$

$\therefore \alpha \vec{u} + \beta \vec{v} \in W \ \forall \alpha, \beta \in \mathbb{R}, \vec{u}, \vec{v} \in W$

$\therefore W$ is a subspace.

ii) $M_{n\times n}(\mathbb{R})$ is a vector space over $\mathbb{R}$

$$W = \{A \mid A \in M_{n\times n}(\mathbb{R}),\ A = A^T\}$$

$$C = \alpha A + \beta B$$

$$C^T = (\alpha A + \beta B)^T = (\alpha A)^T + (\beta B)^T$$

$$= \alpha A^T + \beta B^T$$

$$= \alpha A + \beta B = C$$

$$\Rightarrow \alpha A + \beta B \in W \quad \forall \alpha, \beta \in \mathbb{R},\ A, B \in W$$

$\therefore$ W is a subspace.

iii) $M_{2\times 2}(\mathbb{R})$ is a vector space over $\mathbb{R}$,

$$W = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| \ a, b, c \in \mathbb{R} \right\}$$

$$tr(\alpha A + \beta B) = tr(\alpha A) + tr(\beta B)$$

$$= 0 + 0 = 0$$

$$\therefore \alpha A + \beta B \in W \quad \forall \alpha, \beta \in \mathbb{R}, A, B \in W$$

$\therefore$ W is a subspace.

iv) $W = \{A \mid A \in M_{n\times n}(\mathbb{R}),\ \det(A) = 0\}$

$$1 \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \notin W$$

$\therefore$ W is not a subspace

v) $W = \{ (a,b,c) \mid (a,b,c) \in \mathbb{R}^3, \ ab = 0 \}$

$\alpha\vec{u} + \beta\vec{v} = (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)$

$a_1 b_1 = 0$

$a_2 b_2 = 0$

$(\alpha a_1 + \beta a_2)(\alpha b_1 + \beta b_2) = \alpha^2 a_1 b_1 + \alpha\beta a_1 b_2 + \alpha\beta a_2 b_1 + \beta^2 a_2 b_2$

$= \alpha\beta(a_1 b_2 + a_2 b_1)$

take $a_2 = 0, b_1 = 0, a_1 = 5, b_2 = 10, \alpha = 1 = \beta$

$(\alpha a_1 + \beta a_2)(\alpha b_1 + \beta b_2) = 50 \neq 0$

$\therefore \alpha\vec{u} + \beta\vec{v} \notin W$ for some $\alpha, \beta, \vec{u}, \vec{v}$

$\therefore$ W is not a subspace.

vi) $a^3 = b^3 \not\Rightarrow a = b(1)^{1/3}$

$\Rightarrow a = b, b\omega, b\omega^2$

$W = \{ (b,b,c) : b, c \in \mathbb{R} \}$

$\alpha\vec{u} + \beta\vec{v} = (\alpha b + \beta b, \alpha b + \beta b, \alpha c + \beta c)$

$\therefore \alpha\vec{u} + \beta\vec{v} \in W$

$\therefore$ W is a subspace of $\mathbb{R}^3$

$W = \{ (b,b,c), (b\omega, b, c), (b\omega^2, b, c) : b, c \in \mathbb{R} \}$

$\alpha = 1,\ \beta = 1,\ \vec{u} = (b, b, c),\ \vec{v} = (bw, b, c)$

$$\alpha\vec{u} + \beta\vec{v} = (b + bw,\ 2b,\ 2c)$$
$$= (-bw^2,\ 2b,\ 2c) \notin W$$

$\therefore$ W is not a subspace of F^3

vii) $U = \{ B \mid B \in M_{n\times n}(\mathbb{R}),\ AB = BA \}$
for a fix $A \in M_{n\times n}(\mathbb{R})$

$$AB_1 = B_1A,\quad AB_2 = B_2A$$
$$A(\alpha B_1 + \beta B_2) = \alpha AB_1 + \beta AB_2$$
$$= \alpha B_1A + \beta B_2A$$
$$= (\alpha B_1 + \beta B_2)A$$

$\therefore \alpha B_1 + \beta B_2 \in U\ \forall \alpha, \beta \in \mathbb{R},\ B_1, B_2 \in U$

$\therefore U$ is a subspace of $M_{n\times n}(\mathbb{R})$

3) a) $S = \{ f \in C[0,1] : \int_0^1 f(x)\,dx = b \}$

$C[0,1]$ over $\mathbb{R}$ is a vector space

Let $f_1, f_2 \in S$

$$g = \alpha f_1 + \beta f_2$$
$$\int_0^1 g(x)\,dx = \int_0^1 \alpha f_1(x)\,dx + \int_0^1 \beta f_2(x)\,dx$$
$$= (\alpha + \beta)b$$

$\int_0^1 g(x)\,dx = b\ \forall \alpha, \beta \in \mathbb{R}$ iff $b = 0$

$\therefore S$ is a subspace iff $b = 0$

b) $W \subset C[-4,4]$

W is the set of differentiable f^n_s on $(-4,4)$ such that $f'(-1) = 3f(2)$

Let $f_1, f_2 \in W$, $\alpha, \beta \in \mathbb{R}$

$g = \alpha f_1 + \beta f_2$

$\Rightarrow g' = (\alpha f_1 + \beta f_2)' = (\alpha f_1)' + (\beta f_2)' = \alpha f_1' + \beta f_2'$

$g'(-1) = \alpha f_1'(-1) + \beta f_2'(-1) = \alpha(3f_1(2)) + \beta(3f_2(2))$

$= 3(\alpha f_1(2) + \beta f_2(2)) = 3g(2)$

$\therefore \alpha f_1 + \beta f_2 \in W$

$\therefore$ W is a subspace of $C[-4,4]$

4. $V \to$ vector space over $\mathbb{R}$

$W_1, W_2 \to$ subspaces of V

Let $W = W_1 \cap W_2$

Let $\vec{u}, \vec{v} \in W \Rightarrow \vec{u}, \vec{v} \in W_1$ and $\vec{u}, \vec{v} \in W_2$

$\alpha\vec{u} + \beta\vec{v} \in W_1$ (as W_1 is a subspace of V)

$\alpha\vec{u} + \beta\vec{v} \in W_2$ (as W_2 is a subspace of V)

$(\alpha, \beta \in \mathbb{R})$

$\therefore \alpha\vec{u} + \beta\vec{v} \in W_1 \cap W_2 = W$

Consider $(\mathbb{R}^2(\mathbb{R}), +, \cdot)$

$W_1 = \{(x, x) \mid x \in \mathbb{R}\}$

$W_2 = \{(x, 2x) \mid x \in \mathbb{R}\}$

W_1 and W_2 are subspaces for $(\mathbb{R}^2(\mathbb{R}), +, \cdot)$

$W_1 \cup W_2 = \{(x, x), (x, 2x) \mid x \in \mathbb{R}\}$

Let $\vec{u} = (x_1, x_1)$, $\vec{v} = (x_2, 2x_2)$

$$\alpha\vec{u} + \beta\vec{v} = (\alpha x_1 + \beta x_2, \alpha x_1 + 2\beta x_2)$$

take $\alpha = 1 = \beta$, $x_1 = 1$, $x_2 = 1$

$$\alpha\vec{u} + \beta\vec{v} = (2, 3) \notin W_1 \cup W_2$$

$\therefore$ $W_1 \cup W_2$ is not a subspace of V

5) a) $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$$E = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = 2A - B + 2C$$

b) $p = 2 + 2x + 3x^2$

$p_1 = 2 + x + 4x^2$, $p_2 = 1 - x + 3x^2$,

$p_3 = 3 + 2x + 5x^2$

$$p = \alpha p_1 + \beta p_2 + \gamma p_3$$

$$\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\alpha + \beta + 3\gamma \\ \alpha - \beta + 2\gamma \\ 4\alpha + 3\beta + 5\gamma \end{bmatrix}, \quad \alpha = \frac{D_1}{D} = \frac{\begin{vmatrix} 2 & 1 & 3 \\ 2 & -1 & 2 \\ 3 & 3 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{vmatrix}}$$

$$= \frac{-10+6+18+9-10-12}{-10+8+9+12-5-12}$$

$$= \frac{1}{2}$$

$$\beta = \frac{D_2}{D} = \frac{\begin{vmatrix} 2 & 2 & 3 \\ 1 & 2 & 2 \\ 4 & 3 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{vmatrix}}$$

$$= \frac{20+16+9-24-10-12}{2}$$

$$= -1/2$$

$$2\gamma + 1 = 2 \Rightarrow \gamma = 1/2$$

$$\therefore p = \frac{1}{2} p_1 - \frac{1}{2} p_2 + \frac{1}{2} p_3$$

c) $\overline{u} \times \overline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & 4 & 0 \end{vmatrix} = (-12, 6, 6)$

$$-12x + 6y + 6z = 0$$

$$\Rightarrow 2x = y + z$$

(i) $(3,3,3)$ (ii) $(4,2,6)$ (iv) $(0,0,0)$

6) $\vec{u_1} = (1,2,1)$, $\vec{u_2} = (3,1,5)$, $\vec{u_3} = (3,-4,7)$

$\text{span}\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} = \alpha\vec{u_1} + \beta\vec{u_2} + \gamma\vec{u_3}$

$\vec{u_3} = 2\vec{u_2} - 3\vec{u_1}$ $\quad = \alpha\vec{u_1} + \beta\vec{u_2} + 2\gamma\vec{u_2} - 3\gamma\vec{u_1}$

$= (\alpha - 3\gamma)\vec{u_1} + (\beta + 2\gamma)\vec{u_2}$

$= \alpha'\vec{u_1} + \beta'\vec{u_2} = \text{span}\{\vec{u_1}, \vec{u_2}\}$

7) a) $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ spans a vector space V

$T = \{\bar{v}_1 - \bar{v}_2, \bar{v}_2 - \bar{v}_3, \bar{v}_3 - \bar{v}_4, \bar{v}_4\}$

consider an element in span (T) $c_1(\bar{v}_1 - \bar{v}_2) + c_2(\bar{v}_2 - \bar{v}_3) + c_3(\bar{v}_3 - \bar{v}_4) + c_4\bar{v}_4$

$= c_1\bar{v}_1 + (c_2 - c_1)\bar{v}_2 + (c_3 - c_2)\bar{v}_3 + (c_4 - c_3)\bar{v}_4 \in \text{span}(S)$

$\Rightarrow \text{span}(T) \subseteq \text{span}(S)$

consider an element in span(S) $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 + c_4\bar{v}_4$

$= c_1(\bar{v}_1 - \bar{v}_2 + \bar{v}_2) + c_2(\bar{v}_2 - \bar{v}_3 + \bar{v}_3) + c_3(\bar{v}_3 - \bar{v}_4 + \bar{v}_4) + c_4\bar{v}_4$

$= c_1(\bar{v}_1 - \bar{v}_2) + c_2(\bar{v}_2 - \bar{v}_3) + c_3(\bar{v}_3 - \bar{v}_4) + (c_3 + c_4)\bar{v}_4 + c_1(\bar{v}_2 - \bar{v}_3 + \bar{v}_3) + c_2\bar{v}_3$

$= c_1(\bar{v}_1 - \bar{v}_2) + (c_1 + c_2)(\bar{v}_2 - \bar{v}_3) + c_3(\bar{v}_3 - \bar{v}_4) + (c_3 + c_4)\bar{v}_4 + (c_1 + c_2)(\bar{v}_3 - \bar{v}_4 + \bar{v}_4)$

$= c_1(\bar{v}_1 - \bar{v}_2) + (c_1 + c_2)(\bar{v}_2 - \bar{v}_3) + (c_1 + c_2 + c_3)(\bar{v}_3 - \bar{v}_4) + (c_1 + c_2 + c_3 + c_4)\bar{v}_4 \in \text{span}(T)$

$\Rightarrow \text{span}(S) \subseteq \text{span}(T) \Rightarrow \text{span}(S) = \text{span}(T)$

b) $S = \{u_1, u_2, u_3\}$, $T = \{u_1, u_1+u_2, u_1+u_2+u_3\}$

$U = \{u_1+u_2, u_2+u_3, u_3+u_1\}$

To prove: $\text{span } S = \text{span } T = \text{span } U$

Consider an element in span T

$$c_1 u_1 + c_2(u_1+u_2) + c_3(u_1+u_2+u_3)$$
$$= (c_1+c_2+c_3)u_1 + (c_2+c_3)u_2 + c_3 u_3 \in \text{span } S$$
$$\therefore \text{span } T \subseteq \text{span } S$$

Consider an element in span U

$$c_1(u_1+u_2) + c_2(u_2+u_3) + c_3(u_3+u_1)$$
$$= (c_1+c_3)u_1 + (c_1+c_2)u_2 + (c_2+c_3)u_3 \in \text{span } S$$
$$\therefore \text{span } U \subseteq \text{span } S$$

Consider an element in span S

$$c_1 u_1 + c_2 u_2 + c_3 u_3 = c_1 u_1 + c_2(u_2+u_1-u_1) + c_3(u_3+u_2+u_1-u_1-u_2)$$
$$= (c_1-c_2)u_1 + (c_2-c_3)(u_1+u_2) + c_3(u_1+u_2+u_3) \in \text{span } T$$
$$\Rightarrow \text{span } S \subseteq \text{span } T$$

$$u_1 = \alpha(u_1+u_2) + \beta(u_2+u_3) + \gamma(u_3+u_1)$$
$$1 = \alpha + \gamma$$
$$\alpha + \beta = 0, \quad \beta + \gamma = 0$$
$$\therefore \alpha = \gamma = 1/2, \quad \beta = -1/2$$

$$c_1 u_1 + c_2 u_2 + c_3 u_3 = \left(\frac{c_1}{2} + \frac{c_2}{2} - \frac{c_3}{2}\right)(u_1+u_2) + \left(-\frac{c_1}{2} + \frac{c_2}{2} + \frac{c_3}{2}\right)(u_2+u_3) + \left(\frac{c_1}{2} - \frac{c_2}{2} + \frac{c_3}{2}\right)(u_3+u_1) \in \text{span } U$$

$\Rightarrow \text{span } S \subseteq \text{span } U$ $\quad\therefore \text{span } S = \text{span } T = \text{span } U$

8) a) $c_1(4,-4,8,0)+c_2(2,2,4,0)+c_3(6,0,0,2)$
$+c_4(6,3,-3,0) = (0,0,0,0)$

$$\Rightarrow \begin{aligned} 4c_1 + 2c_2 + 6c_3 + 6c_4 &= 0 \\ -4c_1 + 2c_2 \qquad + 3c_4 &= 0 \\ 8c_1 + 4c_2 \qquad - 3c_4 &= 0 \\ 2c_3 \qquad\qquad &= 0 \Rightarrow c_3 = 0 \end{aligned}$$

$$\begin{aligned} 2c_1 + c_2 + 3c_4 &= 0 \\ -4c_1 + 2c_2 + 3c_4 &= 0 \\ 8c_1 + 4c_2 - 3c_4 &= 0 \end{aligned}$$

$$\begin{vmatrix} 2 & 1 & 3 \\ -4 & 2 & 3 \\ 8 & 4 & -3 \end{vmatrix} = \begin{vmatrix} 10 & 5 & 0 \\ 4 & 6 & 0 \\ 8 & 4 & -3 \end{vmatrix} = (-3)(60-20) = -120 \neq 0$$

$\therefore c_1 = 0 = c_2 = c_3 = c_4$

$\therefore$ linearly independent

b) $C[-\pi,\pi]$ $\quad \{2, 4\sin^2 x, \cos^2 x\}$

$2c_1 + 4c_2 \sin^2 x + c_3\cos^2 x = 0$

$2c_1 + 4c_2\sin^2 x + c_3 - c_3\sin^2 x = 0$

$\Rightarrow 2c_1 + 4c_2\sin^2 x + c_3 - c_3 \sin^2 x = 0$

$\Rightarrow (2c_1 + c_3) + (4c_2 - c_3)\sin^2 x = 0$

$c_3 = 4c_2 = -2c_1$

take $c_1 = 2, c_2 = -1, c_3 = -4$

$\therefore$ linearly dependent.

c) $c_1(t^3 - 5t^2 - 2t + 3) + c_2(t^3 - 4t^2 - 3t + 4)$
$+ c_3(2t^3 - 7t^2 - 7t + 9) = 0$

$$\Rightarrow \begin{aligned} c_1 + c_2 + 2c_3 &= 0 \quad \text{—(i)} \\ -5c_1 - 4c_2 - 7c_3 &= 0 \quad \text{—(ii)} \\ -2c_1 - 3c_2 - 7c_3 &= 0 \quad \text{—(iii)} \end{aligned}$$

$$3c_1 + 4c_2 + 9c_3 = 0 \quad \text{—(iv)}$$

(i) − (iii) = (iv)

$$\begin{vmatrix} 1 & 1 & 2 \\ -5 & -4 & -7 \\ -2 & -3 & -7 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ -5 & -4 & -7 \\ 3 & 1 & 0 \end{vmatrix}$$

$$= 3(-7+8) - 1(-7+10)$$

$$= 3 - 3 = 0$$

$\therefore$ infinite sol^n, linearly dependent.

9) $f_1(t) = t, \quad t \in [-1,1]$

$$f_2(t) = \begin{cases} -t & \text{if } t \in [-1,0] \\ t & \text{if } t \in [0,1] \end{cases}$$

in $C[0,1]$

$c_1 f_1 + c_2 f_2 = 0 \Rightarrow c_1 \cdot t + c_2 \cdot t = 0 \Rightarrow c_1 + c_2 = 0$

take $c_1 = 1, c_2 = -1$

$\therefore \{f_1, f_2\}$ is linearly dependent in $C[0,1]$

in $C[-1,0]$, $c_1 f_1 + c_2 f_2 = 0$

$\Rightarrow c_1 t + c_2(-t) = 0 \Rightarrow c_1 = c_2$

take $c_1 = 1 = c_2$

$\therefore \{f_1, f_2\}$ is linearly dependent in $C[-1,0]$

in $C[-1,1]$, $c_1 f_1 + c_2 f_2 = 0$

$\Rightarrow c_1 t + c_2 |t| = 0 \Rightarrow c_1 = \begin{cases} c_2 & \text{for } t < 0 \\ -c_2 & \text{for } t > 0 \end{cases}$

$\therefore c_1 = 0 = c_2$

$\therefore \{f_1, f_2\}$ is linearly independent in $C[-1,1]$

10) $\{1+i, 1-i\} \subset \mathbb{C}$

over $\mathbb{R}$ $\quad c_1(1+i)+c_2(1-i)=0$, $c_1, c_2 \in \mathbb{R}$

$$\left.\begin{array}{l} \Rightarrow c_1+c_2=0 \\ c_1-c_2=0 \end{array}\right] \rightarrow c_1=0=c_2$$

linearly independent if $\mathbb{C}$ is taken over $\mathbb{R}$

over $\mathbb{C}$ $\quad c_1(1+i)+c_2(1-i)=0$

take $c_1=i, c_2=1$

$\therefore$ linearly dependent if $\mathbb{C}$ is taken over $\mathbb{C}$

11) $u_1, u_2, \ldots, u_k \in \mathbb{R}^n$ $\quad \underset{\hookrightarrow \text{invertible}}{A} \in M_{n\times n}(\mathbb{R})$

Let $Au_1, Au_2, \ldots, Au_k$ are linearly independent

say $\exists\, b_i \neq 0$, such that $b_1u_1+\cdots+b_iu_i+\cdots+b_ku_k=0$

$$A(b_1u_1+\cdots+b_ku_k)=A\cdot 0$$

$$\Rightarrow b_1Au_1+\cdots+b_kAu_k=0$$

$\therefore \exists\, b_i \neq 0$ such that $\sum b_jAu_j=0$

contradiction

$\therefore Au_1, Au_2, \ldots, Au_k$ linearly independent $\Rightarrow u_1, u_2, \ldots, u_k$ linearly independent

Let $u_1, u_2, \ldots, u_k$ are linearly independent

say $\exists\, c_i \neq 0$ such that $c_1Au_1+\cdots+c_iAu_i+\cdots+c_kAu_k=0$

$$\Rightarrow A^{-1}(c_1Au_1+\cdots+c_kAu_k)=A^{-1}\cdot 0$$

[as A is invertible, A^{-1} exists]

$$\Rightarrow c_1u_1+\cdots+c_ku_k=0$$

i.e. $\exists\, c_i \neq 0$ such that $\sum c_ju_j=0$ [contradiction]

$\therefore u_1, u_2, \dots u_k$ linearly independent $\Rightarrow Au_1, \dots, Au_k$ linearly independent

12) $(V(F), +, \cdot)$ is a vector space

$A \subseteq V, \; B \subseteq V$

Prove or disprove $\text{Span}(A) \cap \text{span}(B) \neq \{0\} \Rightarrow A \cap B \neq \phi$

$(\mathbb{R}^3(\mathbb{R}), +, \cdot)$

$A = \{(1,0,0), (0,1,0)\}$

$B = \{(2,0,0), (0,0,3)\}$

$\text{span}\, A = \{(x, y, 0)\}$

$\text{span}\, B = \{(x, 0, z)\}$

$\text{span}\, A \cap \text{span}\, B = \{(x, 0, 0)\} \neq \{0\}$

but $A \cap B = \phi$