Department of Computer Science
Ashoka University

CS-1110 Discrete Mathematics - Spring 2025

QUESTION BANK

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DISCLAIMER

This document consists almost all the problems expressed through the likes of Practice Problem Sets, Flash Quizzes, Quizzes and Programming Assignments, during the span of the **CS1110 - Discrete Mathematics** course in the Spring 2025 semester at Ashoka University.

Although the course was designed to be a gateway course, i.e., intended to lay the basic mathematical foundations of an undergraduate CS major, nevertheless a significant programming component was also incorporated in the syllabus unlike any traditional undergraduate course on Discrete Mathematics. The goal was to cultivate and strengthen a proper programming attitude amongst the students. **C** progarmming language was used as the primary language and all relevant code files can be found at this GitHub repo.

Further the ultimate section of this document, named **Extras**, contains some sort of hints/-solutions to seleted questions from the indicated Problem Sets. The answers are not crisp/rigorous in any 'absolute' sense, rather they have been designed to guide the students to approach proof wrting in an appropriate manner. Some proofs have been left *incomplete*, with scope of formal expansion, so that this document does not end up becoming an academic opium for undergrads.

Any instance of typos and conflicting content is requested to be reported over an email to shubhajit.acad[@]icloud[dot]com.

INTENTION SEPARATES PEOPLE. SMALL ACTIONS. SMALL CHOICES. THE SMALL EFFORT THAT MEANS EVERYTHING. DOING THE RIGHT THING FOR THE RIGHT REASON IS INCREDIBLY RARE. IT'S GIVING TO GIVE, INSTEAD OF GIVING TO GET. IT'S DOING THE WORK, FOR THE WORK ITSELF. IT'S CARING WITHOUT CONSTRAINT, WITHOUT EXPECTATION, WITHOUT AN AGENDA. INTENTION CAN'T BE FAKED. AND IT CAN'T BE BEAT.

Contents

MODULE 01 - MODELS & PROOFS

1.1 Problem Set 01 - Propositional logic

1. Use $\neg$, $\rightarrow$, $\vee$ and $\wedge$ to express the following declarative sentences in propositional logic; in each case state what your respective propositional variables p, q, etc, mean:

   (a) If the sun shines today, then it won't shine tomorrow

   (b) If Bob has installed central heating, then he has sold his car, or he has not paid his mortgage.

   (c) Today it will rain or shine, but not both

2. Consider the following situation and an argument for it.

   Situation: Reason about whether a given number n is prime.
   Argument:
   #1: If n is not divisible by any number other than 1 and itself, then n is a prime number.
   #2: n is divisible by 1 and itself only.
   Therefore,
   #3: n is a prime number.

   Introduce propositional variables, represent the entire argument as a semantic entailment relation, and show that it holds true.

3. Consider the following situation and an argument for it.

   Situation: You are debugging a program and want to conclude that the input file format is correct.
   Argument:
   #1: If there is an error in the input file format and the error-checking module is disabled, the program crashes.
   #2: The program did not crash.
   #3: The error-checking module was disabled.
   Therefore,
   #4: The input file format is correct.

   Introduce propositional variables, represent the entire argument as a semantic entailment relation, and show that it holds true.

4. Construct a truth table for each of these propositional formulas. Be mindful of the precedence of logical connectives, as it may affect the evaluation. Refer to the slides for the correct precedence order.

   (a) $p \vee q \wedge s$

   (b) $p_1 \wedge \neg p_2 \leftrightarrow p_3 \vee p_4$

   (c) $p \vee q \rightarrow r$

5. Compute the truth table for $p \to q \to r$. Unsure about the order of evaluation? Should it be $(p \to q) \to r$ or $p \to (q \to r)$? Compute the truth table for both orders of evaluation and compare the results to see if they yield the same truth table.

   Do the same for the following formulas as well: $p \lor q \lor r$ and $p \land q \land r$.

6. Construct a truth table for each of these propositional formulas:

   (a) $(q \to \neg p) \lor (\neg p \to \neg q)$

   (b) $(p \lor \neg t) \land (p \lor \neg s)$

   (c) $(((p \land \neg q) \to r) \to (\neg r \to (p \to q)))$

   (d) $(p \lor q) \to (p \oplus q)$ [Refer to slides for the connective "$\oplus$"]

   (e) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$ [Refer to slides for the connective "$\leftrightarrow$"]

7. Let p, q and r be the propositional variables such that:

p	You get an A on the final exam
q	You do every exercise in this book
r	You get an A in this class

   Write the following declarative statements (propositions) using p, q, and r and logical connectives

   (a) You get an A in this class, but you do not do every exercise in this book.

   (b) You get an A on the final, you do every exercise in this book, and you get an A in this class

   (c) To get an A in this class, it is necessary for you to get an A on the final

   (d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class

   (e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class

8. For the semantic entailment proofs provided below, determine which ones hold and which ones do not.

   (a) $p \to q, s \to t \vDash p \lor s \to q \land t$

   (b) $p \lor q, \neg q \lor r \vDash p \lor r$

   (c) $p \to (q \lor r), \neg q, \neg r \vDash \neg p$

   (d) $q \to r \vDash (p \to q) \to (p \to r)$

9. $\boxed{\text{Definition}}$ Let ϕ and ψ be propositional formula. We say that ϕ and ψ are semantically equivalent ($\equiv$) iff $\phi \vDash \psi$ and $\psi \vDash \phi$ hold.

   Show the semantic equivalence of the following formulas:

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative Laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive Laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \rightarrow q \equiv \neg p \vee q$ $p \rightarrow q \equiv \neg q \rightarrow\neq p$ $p \vee q \equiv \neg p \rightarrow q$ $p \wedge q \equiv \neg(q \rightarrow \neg p)$ $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$ $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$ $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$ $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$ $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Equivalences involving $\rightarrow$

10. Alice and Bob have each written an algorithm for a function that takes two sorted lists, List1 and List2, of lengths m and n, respectively, and merges them into a third list, List3. Part of Alice'c code and the corresponding part of Bob's code are given below:

Alice	Bob
1 if $((i+j \leq m+n)$ && $(i \leq m)$ && $((j > n) \parallel (\text{List1}[i] \leq \text{List2}[j])))$	1 if $(((i+j \leq m+n)$ && $(i \leq m)$ && $(j > n)) \parallel ((i+j \leq m+n)$ && $(i \leq m)$ &&$(\text{List1}[i] \leq \text{List2}[j])))$
2 $\quad$ $\text{List3}[k] = \text{List1}[i]$	2 $\quad$ $\text{List3}[k] = \text{List1}[i]$
3 $\quad$ $i = i + 1$	3 $\quad$ $i = i + 1$
4 else	4 else
5 $\quad$ $\text{List3}[k] = \text{List2}[j]$	5 $\quad$ $\text{List3}[k] = \text{List2}[j]$
6 $\quad$ $j = j + 1$	6 $\quad$ $j = j + 1$
7 $k = k + 1$	7 $k = k + 1$

Do these parts of the code do the same thing ? Notice that both the codes are exactly the same except for **line 1** (assuming, both have used the same local variables)

Using the propositional variables given below, express line1 in both codes as a propositional formula. Then, demonstrate that they achieve the same result.

p to stand for $i + j \leq m + n$
q to stand for $i \leq m$
r to stand for $j > n$, and
s to stand for $\text{List1}[i] \leq \text{List2}[j]$

1.2 Problem Set 02 - Predicate logic

1. Formalise the following arguments and then use Natural Deduction to infer the conclusion from the given premises. Do not forget to clearly mention the rules of inference you use in each step.

   (a) **Premises :** "All dementors are fierce."; "Some dementors do not drink coffee."
   Conclusion : "Some fierce creatures do not drink coffee."

   (b) **Premises :**"All crows are richly coloured."; "No large bird live on honey."; "Birds that do not live on honey are dull in color."
   Conclusion :"Crows are small."

2. Prove the following logical arguments using Natural Deduction. State the rules of inference used at each step.

   (a)

$$\begin{array}{l} \forall x\ (P(x) \to \neg Q(x)) \\ \exists x\ (Q(x) \land R(x)) \\ \hline \therefore\ \exists x\ (\neg P(x) \land Q(x)) \end{array}$$

   (b)

$$\begin{array}{l} (\exists x\ P(x)) \to (\forall x\ \neg Q(x)) \\ \hline \therefore\ (\exists x\ Q(x)) \to (\forall x\ \neg P(x)) \end{array}$$

3. Mention which variables are free and bound in the following statements.

   (a) $\exists x\ P(x) \lor \exists x\ Q(y) \land \exists y Q(x)$
   (b) $\forall y\ \exists x\ P(x) \lor P(y)$
   (c) $\forall y\ \exists x\ (P(x) \lor Q(y))$
   (d) $\forall x\ P(x) \to \exists y\ Q(x)$

4. Negate the following encoded mathematical statement using concepts seen in the module for predicate logic. Which concept is being primarily used here?

$$\forall \epsilon > 0 \Big(\exists \delta > 0 \big(\forall x \in \mathbb{R} \ni (|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon) \big) \Big).$$

5. The TAs of this Discrete Mathematics course have been suspected of unauthorised access to the AC02-216 lab at Ashoka. They have made the following statements to PPD. Rudransh said, Elvia did it." Monu said, "I did not do it." Elvia said, "Vedant did it." Vedant said, "Elvia lied when she said that I did it." $\star$

   (a) If professor knows that exactly one of them is lying, who did it?
   (b) If professor knows that exactly one of them is telling the truth, who did it?

   Explain your reasoning.

6. Consider the following predicates. Using these predicates (and these predicates only), write a predicate that describes the following relationship $N(x, y)$: "x is the nephew of y's spouse". ⋆⋆

   $P(x, y)$: "x is a parent of y"
   $S(x, y)$: "x and y are siblings"
   $M(x, y)$: "x is married to y"
   $F(x)$: "x is a female"

1.3 Problem Set 03 - Propositional logic-II

1. Consider the following argument. Demonstrate its validity by encoding it in propositional logic as $\phi_1, \phi_2, \phi_3 \vdash \psi$ and providing a proof of its correctness.

   ▶ Alice is from Vienna
   ▶ It isn't the case that both Alice and Bob are Viennese
   ▶ The same goes for Alice and Russell: they aren't both from Vienna
   Therefore,
   ▶ Both Bob and Russell are not Viennese

2. Consider the following argument. Demonstrate its validity by encoding it in propositional logic sequent and providing a proof of its correctness.

   ▶ It isn't true that Alice is a logician while Bob isn't
   ▶ Also, it isn't the case that both Eve and Bob are logicians
   Therefore,
   ▶ It isn't true that both Alice and Eve are logicians

3. Prove the validity of the following sequents:

   (a) $(p \land q) \land r, s \land t \vdash q \land s$

   (b) $q \to (p \to r), \neg r, q \vdash \neg p$

   (c) $(p \to r) \land (q \to r) \vdash p \land q \to r$

   (d) $p \to q, r \to s \vdash p \land r \to q \land s$

4. Prove the validity of the following well-known derived rules:

   (a) $\boxed{\text{Hypothetical Syllogism}}$ $\dfrac{p \to q \quad q \to r}{p \to r}$ HS

   (b) $\boxed{\text{Disjunctive Syllogism}}$ $\dfrac{p \lor q \quad \neg p}{q}$ DS

   (c) $\boxed{\text{Resolution}}$ $\dfrac{p \lor q \quad \neg p \lor r}{q \lor r}$ DS

5. Consider the following argument. Demonstrate its validity by encoding it in propositional logic sequent and providing a proof of its correctness.

   ▶ Discrete Mathematics is not tough and it is a gate course in CS
   ▶ We will attend Discrete Mathematics classes only if it is tough. That is, We will attend Discrete Mathematics classes means that it is tough
   ▶ If we do not attend Discrete Mathematics classes, then we will stage a play
   ▶ If we stage a play, then we will have fun
   Therefore,
   ▶ We will have fun

6. Consider the following argument. Demonstrate its validity by encoding it in propositional logic sequent and providing a proof of its correctness.

   ▶ If Virat Kohli scores a century, then he will win the match of the match
   ▶ If Virat Kohli does not scores a century, then he will be dropped from India team
   ▶ If Virat Kohli is dropped from India team, then he will become a commentator
   Therefore,
   ▶ If Virat Kohli does not win the match of the match, then he will become a commentator.

7. Alice and Bob are two individuals, each of whom is either a knight or a knave. It is known that knights always tell the truth, while knaves always lie. Alice states: "At least one of us is a knave." Determine the identities of Alice and Bob.

   Answer. We introduce the following propositional variables to represent Alice's and Bob's identities:

p	Alice is knave
q	Bob is knave

   Alice states - at least one of us is a knave. This is given to us as premise. While the part "at least one of us is a knave" can be encoded as $(p \vee q)$, we must encode the whole statement "Alice states: at least one of us is a knave". Since Alice can be either a knave or a knight, we must account for both possibilities in our premise. Thus, the valid encoding is as follows:

$$\phi = (p \to \neg(p \vee q)) \wedge (\neg p \to (p \vee q))$$

   The above is correct because if Alice is a knave, her statement must be false, and if she is a knight, her statement must be true.

   With the premise correctly encoded, our task is to determine the exact identities of Alice and Bob. There are four possible conclusions:: $(p \wedge q)$ or $(\neg p \wedge \neg q)$ or $(\neg p \wedge q)$ or $(p \wedge \neg q)$. All that remains is to determine which conclusion follows from the premise.

   We can verify this using semantic entailment by constructing a truth table.

p	q	$p \vee q$	$\neg(p \vee q)$	$p \to \neg(p \vee q)$	$(\neg p \to (p \vee q))$	ϕ
F	F	F	T	T	F	F
F	T	T	F	T	T	T
T	F	T	F	F	T	F
T	T	T	F	F	T	F

   Thus, the validity of premise leaves us with the choice that $(p, q) = (F, T)$.
   Basically, Alice is a knight and Bob is a knave. *qed.*

8. Provide a valid construction history for each of the following propositional formulas, adhering to the precedence rules.

   (a) $(p \to q) \to \neg r \vee (q \wedge p \to r)$
   Answer:

$$\begin{array}{l} p,\ q,\ r \\ (p \to q) \\ q \wedge p \\ (q \wedge p \to r) \\ \neg r \\ \neg r \vee (q \wedge p \to r) \\ (p \to q) \to \neg r \vee (q \wedge p \to r) \end{array}$$

   (b) $p \to \neg q \vee r \to p \vee s$
   Answer:

$$\begin{array}{l} p,\ q,\ r,\ s \\ \neg q \\ \neg q \vee r \\ p \vee s \\ \neg q \vee r \to p \vee s \\ p \to \neg q \vee r \to p \vee s \end{array}$$

   (c) $p \wedge \neg q \to \neg p$
   (d) $(p \to \neg q \vee (p \wedge r) \to s) \vee \neg r$

9. Suppose a propositional formula ψ does not follow from the given set of premises $\phi_1, \phi_2, \ldots, \phi_n$. If you were to prove this, which approach would you prefer: $\phi_1, \phi_2, \ldots, \phi_n \nvdash \psi$ or $\phi_1, \phi_2, \ldots, \phi_n \nvDash \psi$? Justify your choice. ⋆

10. The table below represents the complete truth table for some propositional formula ϕ involving three propositional variables: p, q and r.

p	q	r	ϕ
F	F	F	T
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	F
T	T	T	T

   (a) Construct a disjunctive clause D_1 such that D_1 evaluates to F at the valuation (F, T, F) and evaluates to T for all other valuations.
   (b) Similarly, construct disjunctive clauses D_2 and D_3 that evaluate to F at the valuations (T, F, F) and (T, T, F) respectively, while evaluation to T otherwise.
   (c) Finally, show that $\phi = D_1 \wedge D_2 \wedge D_3$

MODULE 02 - PROOF TECHNIQUES

2.1 Problem Set 04 - Proofs

Part-A

1. We call a number p prime if it has exactly two factors. Suppose we define $Three - Prime$ as those numbers which have exactly three factors. Derive the general form of a $Three - Prime$ and **prove** that all $Three - Prime$s must be of your derived form.

2. The prime numbers p and q are called *twin primes* if $|p - q| = 2$. Let p and q be primes. **Prove** that $pq + 1$ is a square if and only if p and q are twin primes.

3. Let $F_0, F_1, F_2, \ldots F_n$ denotes the the Fibonacci sequence given by $F_0 = 0, F_1 = F_2 = 1$ and satisfying $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. We call F_n the n^{th} *Fibonacci number*. Answer the following based on this.

   (a) Show that $\gcd(F_n, F_{n+1}) = 1$, $\forall n \geq 1$.
   (b) Prove **Cassini's Identity**: $F_{n-1} \cdot F_{n+1} - (F_n)^2 = (-1)^n$, $\forall n \geq 2$.
   (c) Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Show that

   $$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \qquad \forall n \geq 0.$$

4. Choose a proof method of your choice and prove the following. State the method(s) you used to seal the deal.

   (a) If n is an even integer, then n^2 is an even integer.
   (b) If $m \in \mathbb{N}$ and $n \in \mathbb{N}$ are both perfect squares, then nm is also a perfect square. $p \in \mathbb{N}$ is a perfect square if there exists $q \in \mathbb{N}$ such that $p = q^2$.
   (c) If $n = ab$, where a and b are positive integers, then $a \geq n^{1/2}$ or $b \geq n^{1/2}$.
   (d) The sum of two rational numbers is rational.
   (e) If n is a perfect square, then $n + 2$ is not a perfect square.

5. Prove the following using Principles of Mathematical Induction.

   (a) $\sum_{k=1}^{n} k = \frac{1}{2} \cdot n(n+1)$.
   (b) $\sum_{k=1}^{n} k^2 = \frac{1}{6} \cdot n(n+1)(2n+1)$.
   (c) $3^{n/3} > 2^{n/2}$, $\forall n \in \mathbb{N}$.

6. Prove that $a^{2^n} - 1$ is divisible by 4×2^n for all odd integers a, and for all integers $n \in \mathbb{N}$. $\star\star$

Part-B

1. Provide a proof by **contraposition** for the following statements.

   (a) If n is an integer and n^2 is even, then n is even.
   (b) Let $a \geq 0$. If for every $\epsilon > 0$, we have $0 \leq a < \epsilon$, then $a = 0$.
   (c) If m, n are natural numbers such that $m + n \geq 40$, then either $m \geq 20$ or $n \geq 20$.

2. Provide a proof by **contradiction** for the following statements.

(a) Let $a > 0$ be a real number. If $a > 0$, then $\frac{1}{a} > 0$.
(b) There are infinitely many prime numbers. $\star$

3. Provide a proof using mathematical induction for the following statements.

   (a) Let $a \in \mathbb{R} \setminus \{1\}$. For all $n \geq 1$, $\sum_{k=0}^{n} a^k = \frac{1-a^{k+1}}{1-a}$.
   (b) Let $a, b \in \mathbb{N}$ be distinct. For all $n \geq 1$, $(a-b)$ divides $(a^n - b^n)$.
   (c) For all $n \neq 1$ and for all $a_1, \ldots, a_n \in \mathbb{R}$, we have the AM-GM inequality:
   $$\frac{a_1 + \cdots + a_n}{n} \geq (a_1 \cdots a_n)^{1/n}.$$
   (d) For an $a(\neq 0) \in \mathbb{R}$, if $(a + \frac{1}{a}) \in \mathbb{Z}$, then $a^n + \frac{1}{a^n} \in \mathbb{Z}$ for all $n \geq 1$. $\star$
   (e) Let S be a set such that $|S| = n$. Prove that $|\mathcal{P}(S)| = 2^n$. $\star$
   (f) Show that if n is a positive integer, then $\sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} \frac{1}{\prod_{i \in I} i} = n$. $\star$
   (g) Let $x > -1$ be a real number. Prove that $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

MODULE 03 - MATHEMATICAL STRUCTURES

3.1 Problem Set 05 - Sets

3.1.1 Problems

1. Prove the following set identities.

   (a) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$

2. Let A and B be sets. Show that

   (a) $(A \cap B) \subseteq A$

   (b) $A \subseteq (A \cup B)$

   (c) $A \setminus B \subseteq A$

   (d) $A \cap (B \setminus A) = \phi$

   (e) $A \cup (B \setminus A) = A \cup B$

   (f) $(B \setminus A) \cup (C \setminus A) = (B \cup C) \setminus A$

3. Let $A, B \subseteq \Omega$. Show that $(A \cap B) \cup (A \cap \bar{B}) = A$, where $\bar{B} = \Omega \setminus B$.

4. The symmetric difference of A and B, denoted by $A \Delta B$, is the set containing those elements in either A or B, but not both A and B. Clearly, $A \Delta B = (A \cup B) \setminus (A \cap B)$.

5. What can you say about the sets A and B if $A \Delta B = A$? $\star$

6. Suppose $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$. Express each of the following sets with binary strings of length 5 where the ith bit (left to right) in the string is 1 if i is in the set and 0 otherwise.

   (a) $\{a_1, a_3, a_5\}$

   (b) $\{a_1, a_3, a_4, a_5\}$

   (c) ϕ

7. Suppose $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$. Determine the sets specified by the following three strings: $01100, 01010$ and $(01100) \vee (01010) = (0 \vee 0, 1 \vee 1, 1 \vee 0, 0 \vee 1, 0 \vee 0)$.

8. Suppose $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$. Determine the sets specified by the following three strings: $01100, 01010$ and $(01100) \wedge (01010) = (0 \wedge 0, 1 \wedge 1, 1 \wedge 0, 0 \wedge 1, 0 \wedge 0)$.

9. Show that $|A \cup B| = |A| + |B| - |A \cap B|$.

10. We know the **De Morgan's Law** for the case of two sets. Now we attempt to prove the generalized De Morgan's Law. Prove the following where A^c is the complement of a set A: $\star$

    (a) $\bigcup_{i=1}^{n} A_i = (\bigcap_{i=1}^{n} A_i^c)^c$.   (b) $\bigcap_{i=1}^{n} A_i = (\bigcup_{i=1}^{n} A_i^c)^c$.

3.1.2 Dilworth's Lemma : An example of robust proof-writing

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This is just to present an example as to how to write robust proofs.
It looks intimidating at the first look, but clearly it is simple after
a logical break-down.
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Definition 3.1.1. ***(Chain.)*** *For any poset $R = (S, \preceq)$, a chain is a sequence,*

$$a_1 \preceq a_2 \preceq \cdots \preceq a_n$$

where $a_i \neq a_j, \forall i \neq j$ such that each item is comparable to the next one in the chain and is smaller with respect to $\preceq$.

Definition 3.1.2. ***(Antichain.)*** *An antichain in a poset is a set of elements such that no two elements in the set are comparable. Or for any distinct x, y in an antichain set, we have $x \not\preceq y$ and $y \not\preceq x$.*

Theorem 3.1.3. *If the largest chain in a partial order on a set A is of size t, then A can be partitioned into t antichains.*

1. Prove the Dilworth's Lemma stated below using the above given facts.

   (Dilworth's Lemma.) For all $t > 0$, every partially ordered set with n elements must have either a chain of size at least t or an antichain of size at least $\frac{n}{t}$.

   Proof. Assume that Dilworth's lemma is false, that is $\exists$ a poset $R = (S, \preceq)$ with $t > 0$ where all chains are of size $< t$ and all antichains are of size strictly less than $\frac{n}{t}$. Consider the smallest such t, with the poset's largest chain being size $t - 1$. Theorem(3.1.3) is used to find $t - 1$ antichains that partition the set. Now, using the fact that chains must have size strictly less than $\frac{n}{t}$ by our assumption and that the antichains A_i form a partition, their sizes must sum to the size of S.

   $$\sum_{i=1}^{t-1} |A_i| \leq \sum_{i=1}^{t-1} \frac{n}{t} = \frac{n(t-1)}{t} < n. \qquad \perp$$

   Since all antichains form a partition of the set, sum of the sizes of all the antichains must exactly be n. A contradiction and hence the assumption of Dilworth's lemma to be false is incorrect .

   This completes the proof.

   qed.

3.2 Problem Set 06 - Functions

1. Why is $f : \mathbb{R} \to \mathbb{R}$ is not a function if

   (a) $f(x) = 1/x$ (b) $f(x) = -\sqrt{x}$

2. Determine whether each of these function $f : \mathbb{Z} \to \mathbb{Z}$ is injective

   (a) $f(n) = n - 1$
   (b) $f(n) = n^2 + 1$
   (c) $f(n) = n^3$
   (d) $f(n) = \lceil n/2 \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

3. Determine whether the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto?

   (a) $f(m, n) = 2m - n$ (b) $f(m, n) = m^2 - n^2$

4. Give an explicit formula for a function $f : \mathbb{Z} \to \mathbb{N}$ such that it is

   (a) injective, but not surjective
   (b) surjective, but not injective
   (c) bijective
   (d) neither injective not surjective

5. Let $f : A \to B$ and $g : B \to C$ be functions.

   (a) Show that if $g \circ f$ is injective, then f is injective.
   (b) Show that if $g \circ f$ is surjective, then g is surjective

6. Let $f : A \to B$ be a function. Let $E, F \subseteq A$. Then prove the following:

   (a) $f(E \cup F) = f(E) \cup f(F)$ (b) $f(E \cap F) \subseteq f(E) \cap f(F)$

7. Consider a function $f : A \to B$. Let S and T be subsets of B. Show that

   (a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$
   (b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

8. Let $A = \{-1, 0, 2, 4, 7\}$. Find $f(A)$ whenever f is defined as:

   (a) $f(x) = \lceil x/5 \rceil$ (b) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$

9. Let $L = \{0, 1\}^n$, the set of all binary strings of length exactly n. Define a function $f : \{0, 1\}^n \to \mathbb{N}$ such that $\sum_{w \in L} f(w) = 2^n$. $\star$

10. Let $L = \{0, 1\}^n$, the set of all binary strings of length exactly n. Define a function $f : \{0, 1\}^n \to \mathbb{N}$ such that $\sum_{w \in L} f(w) = 2^{n-1}$. $\star$

11. Determine the number of functions $f : \{1, 2, \ldots, 7\} \to \{1, 2, \ldots, 7\}$ such that $f(x) \neq x$ for all x. $*$

12. Determine the total number of functions $f : A \to B$ that can be defined when $|A| = m$ and $|B| = n$. $*$

3.3 Problem Set 07 - Mixed Bag

1. Consider any relation R on a set S. We define R^{-1} as the inverse relation of R. $(a,b) \in R^{-1} \leftrightarrow (b,a) \in R$. Show the following :
   (a) $(R^{-1})^{-1} = R$.
   (b) Let T be another relation on S. Show the De Morgan's Law for relations, that is, $(R \circ T)^{-1} = T^{-1} \circ R^{-1}$.
   (c) $R^{-1} \circ R$ is symmetric and reflexive over some subset of S.
2. Show the following are equivalence relations.
   (a) Let two sets A, B be related if there exists functions $f : A \to B$, $g : B \to A$ such that f, g are injections.
   (b) R is a relation over $\mathbb{Z}$, defined by $R = \{(a,b) : a, b \in \mathbb{Z} \wedge n|(a-b)\}$ for some fixed $n \in \mathbb{Z}, n \neq 0$.
3. For matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, prove that for any $n \in \mathbb{N}$, $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.
4. For $C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, prove that for any $n \in \mathbb{N}$, $C^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$, where F_n is the n^{th} Fibonacci number. Further also prove that the determinant of C^n is $(-1)^n$. $\star$
5. A recurrence of the form $f(n) = h(n)f(n-1) + g(n)$ is called a first-order linear recurrence. Determine the general solution (closed-form of f) when $h(n)$ is a constant, say r. $\star$
6. Find a closed-form representation using unrolling technique, for the solution to the recurrence
$$f(n) = \begin{cases} b & \text{if } n = 0 \\ rf(n-1) + a & \text{if } n \geq 1 \end{cases},$$
where r and a are constants.
7. Show that the closed-form expression for the linear-order recurrence $f(n) = 4f(n-1) + 2^n$, with $f(0) = 3$ is $4^{n+1} - 2^n$. $\star\star$
8. Show that the closed-form expression for the linear-order recurrence $f(n) = 3f(n-1) + n$, with $f(0) = 10$ is $\frac{43}{4}3^n - \frac{n+1}{2} - \frac{1}{4}$.
[Hint: for any real number $a \neq 1$, $\sum_{i=1}^{n} ia^i = \frac{n \cdot a^{n+2} - (n+1) \cdot a^{n+1} + a}{(1-a)^2}$]
9. The recurrence given in question 6 is first-order linear recurrence. With $r \neq 1$, show that $f(n) = r^n b + a\frac{1-r^n}{1-r}$.
10. Consider the following function f. For every $n \in \mathbb{N}$, $f(n)$ be number of functions one can define with domain as the set $\{1, 2, \ldots, n\}$ and codomain as the set $\{1, 2, \ldots, m\}$. Give a recursive description of f. $\star\star\star$

MODULE 04 - GROWTH OF FUNCTIONS

4.1 Problem Set 08 - Asymptotics-I

Formal Notations of Asymptotics

Consider the following definitions for functions $f : \mathbb{Z}^+ \to \mathbb{R}^+$ and $g : \mathbb{Z}^+ \to \mathbb{R}^+$.

Notation	.	Definition	Example
		Tight Bounds	
$O(g(n))$	:	$\exists c > 0, \exists n_0 > 0$ s.t. $\forall n > n_0\ f(n) \leq c.g(n)$	$f(n) = 5n^2 + 2n - 17 \in O(n^2), O(n^3); \notin O(n)$
$\Omega(g(n))$	:	$\exists c > 0, \exists n_0 > 0$ s.t. $\forall n > n_0\ f(n) \geq c.g(n)$	$f(n) = 5n^2 + 2n - 17 \in \Omega(n^2), \Omega(n); \notin \Omega(n^3)$
$\Theta(g(n))$	:	$\exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0$ s.t. $\forall n > n_0$ $c_1.g(n) \leq f(n) \leq c_2.g(n)$	$f(n) = 5n^2 + 2n - 17 \in \Theta(n^2), \Theta(n^3); \notin \Theta(n)$
		Loose Bounds	
$o(g(n))$	:	$\forall c > 0, \exists n_0 > 0$ s.t. $\forall n > n_0\ \lvert f(n)\rvert \leq c.\lvert g(n)\rvert$	$f(n) = 7n - 17 \in o(n^2)$
$\omega(g(n))$	:	$\forall c > 0, \exists n_0 > 0$ s.t. $\forall n > n_0\ \lvert f(n)\rvert \geq c.\lvert g(n)\rvert$	$f(n) = 7n - 17 \in \omega(1)$
		Approximation	
$\sim (g(n))$	:	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$	$f(n) = n(n-1) \sim n^2$

1. Find the $\Theta-$bounds for the following recurrences.
   (a) $T(n) = 4T(n/2) + c$ where $T(1) = c_0$
   (b) $T(n) = T(n/4) + T(n/2) + n \cdot c$ where $T(1) = c_0$
   (c) $T(n) = T(n-2) + T(n-4)$, where $T(0) = T(1) = T(2) = T(3) = c_0$.
   (d) $T(n) = T(n-1) + T(n-2) + k$, where $T(0) = 0$ and $T(1) = 1$.

2. Solve the following linear-homogeneous recurrences and comment on their $O-$bounds.
   (a) $F(n) = 7F(n-1) - 12F(n-2),\ n \geq 2$ and $F(0) = 5, F(1) = -5$.
   (b) $F(n) = F(n-1) + 2F(n-2),\ n \geq 3$ and $F(1) = 0, F(2) = 6$.
   (c) $F(n) = -F(n-1) + 4F(n-2) + 4F(n-3),\ n \geq 3$ and $F(0) = 8, F(1) = 6, F(2) = 26$.
   (d) $F(n) = 4F(n-1) - 4F(n-2),\ n \geq 3$ and $F(1) = 1, F(2) = 3$.
   (e) $F(n) = 8F(n-1) - 16F(n-4),\ n \geq 4$ and $F(0) = 1, F(1) = 4, F(2) = 28, F(3) = 32$.
   (f) $F(n) = -3F(n-1) - 3F(n-2) - F(n-3),\ n \geq 3$ and $F(0) = 1, F(1) = -2, F(2) = -1$.

3. For function f, g and h mapping from $\mathbb{N}$ to $\mathbb{R}^+$, prove the following:
   (a) If $f = O(h)$ and $g = O(h)$, then $f + g = O(h)$, where $f + g : \mathbb{N} \to \mathbb{R}^+$ defined as $(f + g)(x) = f(x) + g(x)$.

(b) If $f = O(h)$ and $g = O(h)$, then $f \cdot g = O(h)$, where $f \cdot g : \mathbb{N} \to \mathbb{R}^+$ defined as $\big(f \cdot g\big)(x) = f(x)g(x)$.

4. For $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, show that $p(x) = O(x^n)$.

5. Prove the following asymptotic identities for $n \in \mathbb{N}$:

   (a) $n! = O(n^n)$.

   (b) $\log n! = O(n \log n)$.

   (c) $\ln n = O(n)$.

6. Prove that for some $k \in \mathbb{N}$, $\sum_{i=1}^{k} i^8 = O(k^9)$. $\star$

7. Prove that for some $k \in \mathbb{N}$, $\sum_{i=1}^{k} i^8 = \Omega(k^9)$. $\star\star\star$

8. Prove that $n^2 + 17n = n^2 + o(n \ln n)$. $\star\star\star$

9. Prove the result that $n^2 = o(n^2 \ln n)$. As a corollary to this result, can we show that $n^2 = O(n^2 \ln n)$? $\star\star\star$

4.2 Problem Set 09 - Asymptotics-II

1. Is it true that $f = O(g)$ implies $\log f = O(\log g)$?
2. Show that if $f(n) = \log n!$, then $f(n)$ is $O(n \log n)$.

   [Hint: $n! = n \cdot (n-1) \cdots 2 \cdot 1 \leq n \cdot n \cdots n \cdot n$]
3. Show that if $f(n) = \frac{n^2+1}{n+1}$, then $f(n)$ is $O(n)$. Provide explicit values for c and n_0.

   [Hint: $\frac{n^2+1}{n+1} \leq \frac{n^2+n^2}{n+1} \leq \frac{2n^2}{n} = 2n$]
4. Show that if $f(n) = n^{2.5}$, then $f(n)$ is $2^{O(\log n)}$.
5. A function $f(n)$ is said to have a quasi-linear growth rate if $f(n) = \Theta(n \log n)$. Show that if $f(n) = n \log n - 10n + 3$, then $f(n)$ exhibits quasi-linear growth.
6. Show that if $f(n) = \sqrt{n} - \log n$, then $f(n)$ exhibits sub-linear growth.
7. Show that if $f(n) = logn - \log \log n + 2$, then $f(n)$ exhibits sub-linear growth.
8. Show that if $f(n) = 1 + 2 + \cdots + n$, then $f(n)$ is $\Omega(n^2)$. Provide explicit values for c and n_0.
9. Suppose $f(n)$ is $O(g(n))$. Does it follow that $2^{f(n)}$ is $O(2^{g(n)})$?
10. Show that if $f(n) = 2n^3 + 4n^2 \log n$, then $f(n)$ is $O(n^3)$.
11. Show that, for $0 \leq a < b$, $n^a = o(n^b)$.
12. Show that, for $a > 0$, $\log n = o(n^a)$.
13. Show that if $f(n)$ is $o(g(n))$, then $f(n)$ cannot be $\Omega(g(n))$.
14. Show that if $f(n) = \frac{n}{\log n}$, then $f(n)$ is $O(n)$ but it is not $\Omega(n)$.
15. Show the following:
    (a) $\sqrt{n}$ is $o(n)$.
    (b) n is $o(n \log \log n)$.
    (c) $n \log n$ is $o(n^2)$.

MODULE 05 - THEORY OF DISCRETE PROBABILITY

5.1 Problem Set 10 - Probability distributions-I

1. Consider $\mathbb{P} : \Omega \to [0,1]$ to be a probability distribution on a non-empty sample space Ω. For events $A, B \subseteq \Omega$ prove the **identities**:

   (a) $\mathbb{P}(\bar{A}) = 1 - \mathbb{P}(A)$.

   (b) $\mathbb{P}(A \cap \bar{B}) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$.

2. Consider the **single-coin tossing experiment** with sample space $\Omega = \{H, T\}$. Give a probability distribution explicitly for the followign situtaions:

   (a) The coin is biased and after running the experiment sufficiently large number of times it is establishes that H appears thrice as many times as T does.

   (b) The coins are biased and it is statistically established that for every three appearances of H, we get two appearances to T.

3. For a fixed $n \in \mathbb{N}$, let $\Omega = \{0,1\}^n$, the set of all n-length binary strings. Further any $\omega \in \Omega$ can be represented as $\omega = b_1 b_2 b_3 \dots b_n$, where each b_i is either 0 or 1.
   Let $p, q \in \mathbb{R}$ such that $0 \leq p, q \leq 1$ and $p + q = 1$. Show that the function $\mathbb{P} : \Omega \to [0,1]$ defined as:
   $$\mathbb{P}(\omega) = \prod_{i=1}^{n} p^{b_i} \cdot q^{1-b_i},$$
   is a probability distribution on Ω. ⋆⋆

4. Let $\Omega = \{0,1\}^4$ be the set of all binary strings of length 4. Consider the **experiment of generating binary strings of length 4** which realises the uniform probability distribution. Assume appearances of 0s and 1s to be independant and answer the following:

   (a) What is the probability that the generated binary string will contain at least two consecutive 0s, given the fact that the binary string starts with 0?

   (b) What is the probability that the generated binary string will contain at least two consecutive 0s, given the fact that the binary string starts with 1?

5. Consider $\mathbb{P} : \Omega \to [0,1]$ to be a probability distribution on a non-empty sample space Ω. Answer the following using the fact that a probability distribution distributes over disjoint events.

   (a) For events $A, B \subseteq \Omega$, show that $\mathbb{P}(\bar{A}|B) = 1 - \mathbb{P}(A|B)$.

   (b) For events $A, B \subseteq \Omega$, it is given that $\mathbb{P}(A) = 1/5$, $\mathbb{P}(A|B) = 1/3$ and $\mathbb{P}(B|A) = 1/7$. Calculate $\mathbb{P}(B)$.

6. For $\Omega = \{1,2,3,4\}$, consider $\mathbb{P} : \Omega \times \Omega \to [0,1]$ to be the uniform probability distribution on $\Omega \times \Omega$. Let $A, B \subseteq \Omega \times \Omega$ be the event where $A := \{(s,t) \in \Omega \times \Omega \mid s + t = 6\}$ and $B := \{(s,t) \in \Omega \times \Omega \mid s + t = 0 \ mod\ 2\}$. Compute $\mathbb{P}_B(A)$.

7. Consider $\mathbb{P} : \Omega \to [0,1]$ to be a probability distribution on a non-empty sample space Ω. For independant events $A, B \in \Omega$ and event $C \in \Omega$ show that:

$$\mathbb{P}(C) \cdot \mathbb{P}(A \cap B) \leq \mathbb{P}(A) \cdot \mathbb{P}(B).$$

8. Let $\Omega = \{0,1\}^4$ be the set of all binary strings of length 4. Consider the **experiment of generating binary strings of length 4** which realises the uniform probability distribution. Let $A \subseteq \Omega$ be the event that the generated binary string starts with 1 and $B \subseteq \Omega$ be the event that the generated binary string contains even number of 1s. Under the assumption that appearances of 0s and 1s are independant, determine whether A and B are independant or not.

9. Consider $\mathbb{P} : \Omega \to [0,1]$ to be a probability distribution on a non-empty sample space Ω. If $A, B, C \subseteq \Omega$ such that $A \cap \bar{C} = B \cap \bar{C}$, then show that $|\mathbb{P}(A) - \mathbb{P}(B)| \leq \mathbb{P}(C)$. $\star$

10. Consider $\mathbb{P} : \Omega \to [0,1]$ to be a probability distribution on a non-empty sample space Ω. Show that if $A, B \subseteq \Omega$ are independant events, then events $\bar{A} \subseteq \Omega$ and B are also independant.

11. Consider probability distribution $\mathbb{P}_1 : \Omega \to [0,1]$ and $\mathbb{P}_2 : \Omega \to [0,1]$ on a non-empty sample space Ω, show that

$$\sum_{\omega \in \Omega} |\mathbb{P}_1(\omega) - \mathbb{P}_2(\omega)| \leq 2.$$

► HINT: Triangle inequality: $|a + b| \leq |a| + |b|$, might come handy.

5.2 Problem Set 11 - Probability distributions-II

1. Let events $A,\ B,\ C \subseteq \Omega$ form a *cover* of the sample space Ω. If $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C)$ and these events are mutually independant then compute $\mathbb{P}(A \cap B \cap C)$. Consider $\mathbb{P}$ realises the uniform probability distribution.

2. Let A and B be two mutually disjoint events. Further, let A and B together be independent of event C. If $\mathbb{P}[A] + P[B] = a$ and $\mathbb{P}[C] = b$, and $\mathbb{P}$ realises the uniform probability distribution then compute $\mathbb{P}[A \cup B \cup C]$. ⋆

   ► HINT : Use the Principle of Inclusion-Exclusion, i.e., $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$.

3. Let $A,\ B$ and C be three events such that the events A and B are mutually disjoint events. Further it is given that $\mathbb{P}[A \cup B] = 1$, $\mathbb{P}[A \cap C] = 1/4$ and $\mathbb{P}[C] = 7/12$. Compute $\mathbb{P}[B \cap C]$. ⋆

4. Let $A,\ B,\ C$ be any three events with $\mathbb{P}[A] = 0.3$, $\mathbb{P}_A[B] = 0.2$, $\mathbb{P}_A[C] = 0.1$ and $P_A[(B \cap C)] = 0.05$. Then compute $P[A \setminus (B \cup C)]$.

5. Consider a random variable X such that $\mathbb{E}[X^2] = 10$ and $\mathbb{E}[X]^2 = 6$. Compute $\mathbb{E}[(X - \mathbb{E}[X])^2]$.

6. **Joint distribution.** Suppose Π be a random experiment with sample space $\Omega = \{(a, b) \mid 1 \leq a, b \leq 4\}$ and it realises the uniform probability distribution P on Ω. Consider the following two random variables $X,\ Y$ from Ω to the set $\{1, 2, 3, 4\}$ defined as:
$$X(a, b) = \max(a, b);\ Y(a, b) = \min(a, b) \qquad \forall (a, b) \in \Omega.$$
   Now, consider the joint random variable (X, Y). Clearly, $(X, Y) : \Omega \times \Omega \to \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$. Compute $\mathbb{P}_{X\times Y}\big((3, 2)\big)$.

7. Consider the joint random variable (X, Y) with domain $\{0, 1\} \times \{0, 1, 2\}$. It is given $\mathbb{P}_{(X,Y)}[(a, b)] = \frac{a+b}{9},\ \forall (a, b) \in \{0, 1\} \times \{0, 1, 2\}$.

   (a) **Marginal distributions.** Determine marginal distributions $\mathbb{P}_X$ and $\mathbb{P}_Y$.

   (b) **Independence of distributions.** Are random variables X and Y independent?

8. Let $X : \Omega \to S$ and $Y : \Omega \to T$ be two random variables such that the joint random varible (X, Y) realises the uniform distribution $S \times T$. Determine the distributions $\mathbb{P}_X$ and $\mathbb{P}_Y$, qualitatively. Further, are random variables X and Y independent?

9. Let $\mathbb{P}$: $\{0, 1\}^3 \to [0, 1]$ be the Uniform probability distribution. Let $X : \{0, 1\}^3 \to \{0, 1\}$ be a random variable that maps elements $(b_1 b_2 b_1) \in \{0, 1\}^3$ to 1 if and only if $b_1 = b_3$. Let $Y : \{0, 1\}^3 \to 0, 1$ be a random variable that maps elements $(b_1 b_2 b_1) \in \{0, 1\}^3$ to 1 if and only if $(b_1 + b_2 + b_3) = 2$. Compute $\mathbb{P}_{(X,Y)}(1, 1)$. ⋆

10. Let X be a random variable with the probability distribution $\mathbb{P}_X : \{-2, -1, 0, 1, 2\} \to [0, 1]$ defined as $\mathbb{P}_X(x) = k(1 + |x|)^2$, where $k \in \mathbb{R}$ is a constant. Compute $\mathbb{P}_X(0)$.

11. Let X be a random variable such that $Range(X) = \{0, 1, 2, \ldots, n\}$. Then show that $\sum_{i=1}^{n} \mathbb{P}_X[X \geq i] = E[X]$. ⋆

12. Consider Π, the experiment of rolling two unbiased dice. Let X and Y be random variables where X encodes sum of the two faces and Y encodes the absolute value of the difference of the two faces. Show that $E[XY]$ is 0.

    ▶ HINT: Use linearity of expectations, i.e., $E[X + Y] = E[X] + E[Y]$.

MODULE 06 - THEORY OF GRAPHS

6.1 Problem Set 12 - Graphs-I

To test command over basic definitions & notations

1. Consider a *directed* graph G. Prove that graph G being strongly connected implies that G is weakly connected, and not the other way around.

2. Consider a *directed* graph G. Prove that graph G being a Null graph implies that G is also an Empty graph. Provide a counter-example to show that the other way around is not always true.

3. The general graph shown in the following figure(1) goes by the name of *GraphBuster* in standard literature. Count and determine the cardinality of V and E in *GraphBuster*.

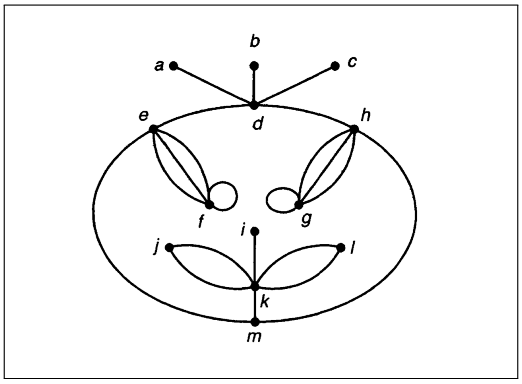


Figure 1: GraphBuster

4. A graph of order n is called *complete*, denoted by K_n provided that each pair of distinct vertices forms an edge. Show that a complete graph of order n has $n(n-1)/2$ edges.

Problems of type $\neg$ (simple)

5. Let G be a general graph. Show that the sum of the degrees of all the vertices of G is an even number, and consequently, the number of vertices of G with odd degree is even.

6. If G is a simple graph of order $n \geq 3$, such that for all pairs of distinct vertices x and y in G that are not adjacent, we have $deg(x) + deg(y) \geq n$, then show that G must be connected. $\star$

7. Let G be the graph such that elements of $\{1, 2, 3, \cdots, 20\}$ form its vertices. In G, two vertices (integers) are joined by an edge if and only if their difference is an odd integer. Show that G is a bipartite graph.

8. Prove that if a multigraph G is bipartite, then each of its cycles has even length. *Note that: length of any cycle/path is the number of edges it is composed of.*

9. For a fixed $n \in \mathbb{N}$, let $\mathrm{G_n}$ be the graph such that elements of $\{0, 1\}^n$, the set of all $n-$length binary strings, form its vertices. In $\mathrm{G_n}$ any two vertices are joined by an edge if and only if they differ in exactly one 1-bit. Show that G is a bipartite graph. $\star$

10. Prove that a graph of order n with at least

$$\frac{(n-2)(n-1)}{2} + 1$$

edges must be connected.

11. Prove that a graph of order n with every vertex having degree at least $\frac{n}{2}$ must be connected.

12. In a simple graph if two vertices x and y are joined by a path then, show that they are also joined by a simple path.

6.2 Problem Set 13 - Graphs-II

Problems of type (simple)

1. The two graphs shown below in figure (2) have the same number of vertices and edges. Prove that despite these they are not isomorphic.

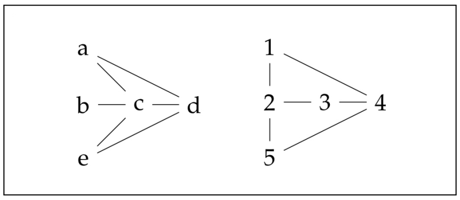


Figure 2: Sample graphs 01

2. Consider the two graphs shown below in figure (3), where both the graphs have same degree sequence $(3, 3, 3, 3, 3, 3)$. Show that despite this, they are not isomorphic.

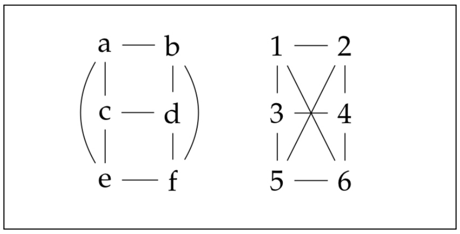


Figure 3: Sample graphs 02

3. A graph has 26 vertices and 58 edges. There are five vertices of degree 4, six vertices of degree 5, and seven vertices of degree 6. If the remaining vertices all have the same degree, what is this degree?

4. A graph has 24 vertices and 30 edges. It has five vertices of degree 4, seven pendant vertices, and seven vertices of degree 2. All other vertices have degree 3 or 4. How many vertices of degree 4 are there?

Problems of type $\neg$ (simple)

5. Prove or Disprove, whether a bipartite graph can have K_3 as its subgraph?

6. Let $d_0(G)$ be the least among the degrees of the vertices of an $n-$vertex graph G. Prove that if $d_0(G) \geq (n-1)/2$, then the graph G is connected. $\star$

7. Prove that a graph G with v vertices and e edges has at least $v - e$ connected components. [Hint : use induction on e.]

8. Prove that a connected graph G with n vertices contains at least $n - 1$ edges. [Hint : the proof might be an application of the result in the previous question, when proved!]

9. If G is a connected graph with v vertices and e edges, then $v \leq e + 1$.

10. If G is a connected graph, then removing an edge from a cycle will not make G a disconnected graph. $\star$

6.3 Problem Set 14 - Graphs-III

1. A graph $G = (V, E)$ is called k-regular if $\mathsf{deg}(v) = k$ for all $v \in V$. A graph is called regular if it is k-regular for some k. Give example of a regular bipartite graph.

2. Prove that every induced subgraph of a complete graph is complete.

3. Prove that every subgraph of a bipartite graph is bipartite.

4. If two graphs G_1 and G_2 are isomorphic then their degree sequences are the same.

5. What is the sum of the entries in a row of the adjacency matrix of an undirected simple graph?

6. Let u, v, and w be three distinct vertices in a graph. There is a path between u and v and also there is a path between v and w. Prove that there is a path between u and w.

7. Suppose $(d_1, \ldots, d_n)$ be a degree sequence of a tree. Determine $\sum_{i=1}^{n} d_i$.

8. Show that the number of vertices n in a full binary tree is always odd.

9. Let p be the number of pendant vertices in a binary tree T with n vertices. Show that
$$p = \frac{n+1}{2}.$$

10. Let $k \in \mathbb{N}$ be the height of a binary tree T. Determine the maximum number of leaf nodes of T.

11. Consider the graph defined by the adjacency matrix provided below.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 1 & 0 & 1 & 0 \\ & & & 0 & 0 & 1 & 0 & 1 \\ & & & & 0 & 0 & 1 & 1 \\ & & & & & 0 & 1 & 1 \\ & & & & & & 0 & 1 \\ & & & & & & & 0 \end{bmatrix}$$

   (a) Determine if it is an Euler graph.
   (b) Determine if it admits a Hamiltonian circuit.
   (c) Give a spanning tree of this graph.

MODULE 07 - COUNTING PRINCIPLES

7.1 Problem Set 15 - Sum & Product Rule

1. How many different licence plates with exactly 6 characters (numbers and lowercase letters) can be made given the following specifications?
   (a) No restrictions. This plate can have any arrangement of digits and letters and repetition is allowed.
   (b) The first two characters are digits and the last four are letters. Repetition is not allowed.
   (c) The characters alternate between letters and digits and no digit may be repeated.
   (d) The license plate includes no more than one digit.
   Hint: Consider all possible cases.
   (e) The first character must be either "T" or 0 and the last character must be either "J" or "Q".
2. A university student is looking to take out a book on either frogs or fireflies from their campus library. There are 45 books available covering frogs, and 13 discussing fireflies. How many books does this student have to choose from?
3. Joselyn stops by a sandwich shop on her way home from class. The shop sells 4 types of potato chips, 3 types of cookies, 7 different drinks and 10 different sandwiches. She is interested in determining how many different ways there are to order if she'd either like a drink and a cookie, or a meal which includes a sandwich, a drink, and chips.
4. How many nonempty sets of letters can be formed from 3 X's and 5 Y's?
   Hint: As these are sets, the order of the letters is irrelevant.
5. How many ternary sequences (sequences using only the digits 0,1, and 2) of length 10 exist such that no consecutive digits are the same?
6. How many integers, x, between 100 and 999 are divisible by 5?
7. How many integers, x, between 100 and 999 are divisible by 5?
8. Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. How many functions $f : A \to B$ are there such that:
   (a) $f(a_1) = f(a_2)$
   (b) $f(a_1) = b_1$ and $f(a_2) \neq b_1$
   (c) $f(a_1) \in \{b_1, b_2, b_3\}$
   (d) $f(a_1) = b_k$, for some $k \in \{1, 2, \ldots, n\}$ and for all other $a_i, i \in \{2, 3, \ldots, m\}$ $f(a_i) \neq b_k$
   (e) $f(a_1) \neq f(a_2)$
9. How mnay functions are there from a set of 5 elements to a set with 3 elements?
10. How many different ways are there to answer a true or false test with 25 questions, assuming every question is answered?

11. The CS department is hosting an event. They are randomly inviting one professor and one student to give a speech together. If there are 1500 students and 50 professors, how many different pairs could give a speech? What about if only one person gives a speech and it could be a student or a professor?

12. Jamie is buying a combination lock to lock up her work-out gear at the gym. Jamie would like to pick the most secure lock to protect her valuables. Lock 1 advertises that its combination is an ordered sequence of numbers between 1 and 35 such that the first number cannot be the third number. Lock 2 advertises that its combination is an ordered sequence of 4 numbers between 1 and 25 where the first three numbers are all distinct and the fourth number must be the same as one of the previous three numbers. Which lock should Jamie purchase?

13. How many words (strings of letters) exist that are length 1, 2 or 3?

7.2 Problem Set 16 - Permutations & Combinations

1. How many different 6 letter permutations of the word 'COFFEE' are there?
   Hint: Be aware of repeated letters.

2. (a) You and seven friends dine at a circular table at a fancy restaurant. How many different ways can the eight of you seat yourselves around this table?
   (b) What if two people insist on sitting together?
   Hint: The arrangement is considered the same if everyone sits next to the same two people.

3. There are 25 people competing in the school swim race including Elvia, Monu, and Poulomi.
   (a) At the race, the first, second, third, fourth, and fifth fastest swimmers receive medals. How many possible ways can these medals be distributed?
   (b) How many possible ways can these medals be distributed if Elvia, Monu, and Poulomi always place in the top three positions?

4. A group of eight TAs would sit in a row at the movie theater, how many ways can arrange themselves if Poulomi and Monu refuse to sit beside each other?

5. In how many ways can the numbers 3, 4, 4, 5, 6, 7, 8 be arranged to create numbers less than 6000000?

6. Aryika has 20 books in her room. Her three friends each want to borrow two books from her. Tomorrow they're all coming over to pick them up, in how many different ways can Aryika loan out the books such that the order she gives each friend their books is the order in which they read them?

7. Aaryan lost the last two digits of his friend's phone number. How many different phone numbers will Aaryan potentially have to call before calling his friend?

8. In a group of teenagers m of them are naturally brunette and n of them were not born with brown hair. How many different ways can these teenagers be arranged in a line such that the m brunette's are all together?

9. Using the definition of a permutation, show that ${}^nP_n = n!$.

10. How many ways can the letters of MISSISSIPPI be permuted?

11. A K-pop fan has 10 different posters to arrange (in a line) on their wall. Three posters are from one band, four from a different group, and three from a third group. How many ways can the posters be lined up such that posters from the same group are together?

12. Prove that for an integer $n \geq 2$ that $P(n+1, 2) - P(n, 2) = 2 \cdot P(n, 1)$.

13. (a) How many ways can the letters in BOOKKEEPER be rearranged?
    (b) What if the E's cannot be consecutive?
    (c) What if the E's had to be consecutive?
    (d) What if the vowels had to occur consecutively?

14. A lottery ticket consists of five unordered, distinct numbers between 1 and 69 and one letter. A winning ticket must contain all the numbers and the letter drawn by the lottery company. If the prize is USD 10,000,000 and the tickets cost USD 0.50 is it worth buying all the tickets to ensure a win?

15. The local college's intramural basketball team accepts 21 players. This year 80 students tried out. They want to arbitrarily decide who to let on the team. In each scenario, determine how many different possible teams there are.

    (a) No further restrictions.

    (b) The school boasts about the opportunities available for first year students so, the team wants to make 10 out of the 21 team-members first year students. Out of the 80 players who tried out, 40 of them are first year students.

    (c) While the intramural team is non-competitive, they enjoy beating the neighbouring college's team, so they guarantee the two highest scoring players from last year's team a spot.

    (d) The school wants to have a mix of students who played last year and students who didn't. 65 of the students who registered did not play last year, while 15 students did. The school wants 10 students who did not play last year and 11 who did.

    (e) The coach wants to make sure there is a good mix of types of players on the team. Each student tells the coach which position they play: 20 students play centre, 15 play shooting guard, 10 play point guard, 20 play small forward, and 15 play power forward. The coach wants to ensure the team has 5 people who play shooting guard and 4 people of every other position.

    (f) There are 5 students who are graduating this year. The coach wants to ensure at least 3 of them get to play.

16. A teacher randomly selects 4 numbers from 1 to n. There are exactly 2672670 possible sets of 4 numbers that can be chosen. Determine n.

17. (a) If there are 12 students in a class and the teacher would like to create groups of 6, how many ways can the groups be arranged?

    (b) What if two students refuse to work together?

18. Shubho wakes up every morning and makes himself a smoothie with frozen fruit. He picks 3 fruits everyday to make his smoothie with out of the 10 options types of fruit in his freezer. He likes any combination of fruit in his smoothie except banana with apple. How many ways are there for Shubho to make his smoothie?

19. Robert is picking the group from his dance class to perform the opening act at the upcoming show. The opening act will have 8 students out of a class of 20, how many possible groups of dancers are there given each of the following scenarios:

    (a) No further restrictions.

    (b) The opening act must be half advanced dancers and half beginner dancers. There are ten students of each level in the class.

    (c) Charlotte and Mohammad do not want to dance together.

    (d) The opening act has a solo at the end that one of the 8 dancers will perform.

20. Give an algebraic and a combinatorial proof of:

$$m\binom{n}{m} = n\binom{n-1}{m-1}.$$

Recall: A combinatorial proof is an arbitrary scenario where the same thing can be counted two different ways.

21. Give a combinatorial proof of the identity:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

22. Give a combinatorial proof of Pascal's Identity:
For any integer $n \geq 2$ and each integer k such that $0 < k < n$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

7.3 Problem Set 17 - Pigeonhole Principle

1. Apply the pigeonhole principle to solve the following problems. Describe the 'pigeons' and the 'pigeonholes'.
   (a) There are 367 individuals attending a mathematics seminar, is it possible that everyone has a different birthday? Explain.
   (b) Consider a subset of the positive integers with 29 elements. Prove that at least two elements in this set will have the same remainder when divided by 28.
   (c) You are handed a bag with 9 pairs of shoes in it. If you take shoes one at a time, how many shoes must you take out to guarantee that you have found a pair?
   (d) You are given a list of 17,500 three letter "words" (strings of letters of length 3, repetition is allowed). Are all of these words distinct? Explain.
2. At a party there are n people, where $n \geq 2$. Prove that it is guaranteed that two people will speak to the exact same number of people.
3. Prove that in any set of exactly 13 integers 12 divides the difference of two numbers from that set.
4. Farmer Mary has 32 cows in a rectangular paddock measuring 15 metres by 24 metres. Show that at any given moment, there are two cows that are no more than 5 metres apart.
5. How many integers must you pick in $A = \{1, 2, \ldots, 200\}$ to ensure that there is at least one number divisible by 5?
6. How many integers in $X = \{0, \ldots, 60\}$ must be chosen to ensure that an odd integer is selected?
7. How many people must attend a conference to ensure that at least two attendees share the same first and last initial?
8. While trying to apply for scholarships to pay for college, Brynn spends six weeks sending out applications. She sends out at least one application daily, but less than 60 were sent out over the course of these six weeks. Prove that there was a period of consecutive days where Brynn applied for 23 scholarships.
9. Show that any subset of the positive integers with more than three elements, will contain two distinct elements whose sum is even.
10. The local library has 12 computers available. There are 42 people who signed up to use them today. Each person may only use one computer, and to minimize the strain on the computers, the library does not allow more than six people to use a single computer in a day. Show that there are at least five computers used by three or more individuals.
11. An ice cream parlour sells 15 different ice cream flavours. A parent brings 8 children to the parlour and lets them each get a double-scoop of ice cream with the requirement that each scoop must be different flavor. Is it possible for no flavour to be ordered more than once?

12. Prove that if more than 1001 integers are selected from $\{1, \ldots, 2000\}$ then:

    (a) there are two integers with the property that one number divides the other.

    (b) there are two integers that are relatively prime (i.e. there exist two integers, say m and n, such that $\gcd(m, n) = 1$).
    Hint: Every pair of consecutive integers are relatively prime.

13. Prove that any subset of $A = 1, 2, ..., 9$ with 6 or more elements contains two elements whose sum is 10.

7.4 Problem Set 18 - Mixed Bag I & II

Part-I

1. The total number of ways you can rearrange the letters in "ARRANGE" so that all the vowels always appear together, is *(choose one)*:
A. 5040 B. 1260 **C. 180** D. 60 E. None of these.

2. The houses in a locality are numbered with two different letters (out of A, B, . . . , Z) followed by two different digits (out of 0, 1, 2, . . . , 9). The number of possible house number-plates, is *(choose one)*:
A. 67600 **B. 58500** C. 7760 D. 7400 E. None of these.

3. Which statement among the following, is TRUE *(choose one)*?
$C(n, r)$ denotes the number of r-combinations out of n objects, same as ${}^nC_r$.
(i) $C(15, 5) = 2 \times C(15, 10)$;
(ii) $C(15, 5) = (1/3) \times C(15, 10)$;
(iii) $C(15, 5) = C(15, 10)$;
(iv) $C(15, 5) = (1/2) \times C(15, 10)$;
(v) None of these.

4. In a test, Rimi has to answer exactly 10 questions out of 15 questions with the constraint that she has to answer 2 questions from the first 5 questions, and 8 from the remaining 10 questions. The number of choices Rimi can have, is *(choose one)*:
A. 3003 **B. 450** C. 55 D. 1287 E. None of these.

5. We have three dice colored red, blue, and green, which are being simultaneously rolled. What is the number of outcomes such that all values are distinct? *(choose one)*:
A. 216 B. 210 C. 180 **D. 120** E. none of these

6. A binary string comprises 0's and 1's only. We consider all 8-bit binary strings, each of which contains exactly five 0's and three 1's. For example, 11100000 is a valid string whereas 10101110 is not. The number of such strings is *(choose one)*:
A. 6 B. 32 C. 128 D. 256 **E. None of these**

7. In a DS of Discrete Mathematics, there are 10 students and Shubhajit wants to make one team that comprises at least two students. The number of the ways the team can be formed *(choose one)*:
A. 45 B. 100 **C. 1013** D. 1024 E. None of these.

8. From the alphabet $A, B, C, \cdots, Y, Z$, we want to construct three-letter strings such that letters are alphabetically ordered from the left with repetition allowed, and the last letter is always Z. For example, BBZ and CMZ are valid strings, whereas MCZ or CMY or CCY are not. The total number of such valid strings is (choose one):
A. 326 B. 650 **C. 351** D. 300 E. 325

Part-II

9. How many distinct positive divisors do each of the following numbers have: 620, 10^{10}?

10. How many sets of 3 numbers each can be formed from the numbers $\{1, 2, 3, \ldots, 10\}$ if no 2 consecutive numbers are to be in a set?

11. Prove that $\binom{n}{r} = \binom{n}{n-r}$ by using a combinatorial argument.

12. Determine the number of 11-permutation of the multiset $S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$.

13. List all 3-combinations of the multiset $S = \{2 \cdot a, 1 \cdot b, 3 \cdot c\}$.

14. How many integral solutions of $x_1 + x_2 + x_3 + x_4 = 30$ satisfy $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, and $x_4 \geq 8$.

15. Consider the multiset $S = \{n \cdot a, 1, 2, \ldots, n\}$ of size $2n$. Show that the number of its n-combinations is 2^n.
Hint: $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$

16. Show that a nonempty set has the same number of odd subsets (i.e., subsets with an odd number of elements) as even subsets.
Hint: $0 = (1-1)^n$

17. What is the coefficient of $x^8 y^{15}$ in the expansion of $(3x - 2y)^{23}$?

18. Determine the value of the sum: $\sum_{k=0}^{10} \binom{10}{k} 2^k$.

19. Show that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k} = 2^n$.

20. Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ be the sequence of numbers satisfying: $h_n = h_{n-1} + h_{n-2} (n \geq 2)$, with $h_0 = 2$ and $h_1 = -1$. Show that $h_n = \frac{\sqrt{5}-2}{\sqrt{5}} (\frac{1+\sqrt{5}}{2})^n + \frac{\sqrt{5}+2}{\sqrt{5}} (\frac{1-\sqrt{5}}{2})^n$ is the solution.

21. Consider the recurrence relation $h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3} (n \geq 3)$, subject to the initial values $h_0 = 1, h_1 = 2$, and $h_2 = 0$. Show that $h_n = 2 - \frac{2}{3}(-1)^n - \frac{1}{3} 2^n$ is the solution.

7.5 Problem Set 19 - Generating Functions

1. Given the following sequences, determine the corresponding generating function as a summation and in closed form (as a formula).

   (a) $1, 2, 3, 4, \ldots$

   (b) $5, 4, 3, 0, 0, \ldots$

   (c) $1, -1, 1, -1, 1, -1, \ldots$

   (d) $\binom{10}{10}, \binom{11}{10}, \binom{12}{10}, \binom{13}{10}, \ldots$

   (e) $\binom{10}{10}, -\binom{11}{10}, \binom{12}{10}, -\binom{13}{10}, \ldots$

   (f) $1, 0, 1, 0, 1, \ldots$

   (g) $1, -2, 4, -8, 16, -32, 0, 0, 0, \ldots$

2. Given the following generating functions, determine the sequence that represents it.

   (a) $f(x) = 0$

   (b) $f(x) = x$

   (c) $f(x) = 4 + 3x - 10x^2 + 55x^3$

   (d) $f(x) = (3x - 4)^3$

   (e) $f(x) = \frac{3x}{1-x}$

   (f) $f(x) = \frac{1}{(1-3x)^2}$

3. Determine the coefficient of the specified term in the expansion of the given function.

   (a) x^3 in $\frac{1}{1-x}$.

   (b) x^2 in $\frac{1}{(1-2x)^3}$.

   (c) x^5 in $\frac{(1-x^5)}{1-x}$.

   (d) x^3 in $\frac{1}{(1+3x)^{10}}$.

4. In how many ways can 1000 identical pamphlets be distributed to five different counselling centers, where pamphlets are put in stacks of 50, such that each center receives at least 50 but no more than 500 pamphlets?

5. In how many ways can 20 identical balls be distributed between 3 distinct boxes such that, ...

   (a) ... there are at least two balls assigned to box?

   (b) ... there are at least three, but no more than 10 balls assigned to each box?

   (c) ... using the same condition as in part b, how many distributions are possible if there were 25 balls instead of 20?

6. Determine the number of ways that USD 12 in loonies can be distributed between a father's three children so that the eldest gets at least four dollars, the middle and youngest child are both guaranteed at least two dollars, but the youngest cannot receive any more than USD 5 since he will spend it all on candy and rot his teeth.

7. In how many ways can n balls be selected from a supply of pink, orange and black balls such that the number of black balls selected must be even?
   Hint: Partial fractions may come in handy.

8. A restaurant just closed for the night and they had an extra 12 orders of fries and 16 mini-desserts left over. The restaurant manager decides to split this left over food between the four employees closing that night. How can the manager do this so that the head chef receives at least one order of fries and exactly three mini-desserts, while the three other closing-staff are guaranteed at least two orders of fries but less than 5 desserts?

9. Use generating functions to determine the number of four-element subsets of the set A, given by $A := \{1, 2, \ldots, 15\}$ that contain no consecutive integers.

10. A student is picking out a handful of gummy bears from a large container. There are red, yellow, and green gummy bears in the container. The student wishes to pick out an even number of red gummy bears, an odd number that is at least 3 of yellow gummy bears, and either 4 or 6 green gummy bears.

    (a) Determine the appropriate generating function that models this situation.

    (b) How many ways can the student pick out gummy bears if they pick out:

        i. 15?
        ii. 22?

11. Someone buys a chocolate bar and receives 50 cents in change. Create a generating function that could determine the number of ways they could receive their change in any combination of pennies, nickles, dimes, and quarters? The coefficient of which term will give the desired solution?
    Note: You are *not* being asked to determine how many ways this is possible.

12. A deck of cards has 52 cards in total. Half of the deck is red and half is black. A quarter of the deck has the symbol hearts, a quarter has the symbol diamonds, a quarter has the symbol spades, and a quarter has the symbol clubs. How many ways are there to pick 15 cards if:

    (a) You wish to pick an even number of black cards and an odd number of red cards?

    (b) You wish to pick at least two of each symbol, but no more than 5 hearts and 6 spades?

13. Three students are running for student body president: Krishna, and Jamar, and Bonnie. Find the generating function used to determine the possible distribution of n students' votes

    (a) with no further restrictions?

    (b) if every student running votes for themselves?

14. How many ways are there to obtain a sum of 7 if 2 distinct 6-sided dice, having faces numbered $1, 2, 3, 4, 5, 6$ are thrown?

ASSESSMENTS

8.1 Flash Quiz 01

1. Code the following two statements in propositional logic and then find the *negation*, *converse*, *inverse* and *contrapositive* for each statement if applicable. If for any of the statements the *negation*, *converse*, *inverse* or *contrapositive* is / are not possible to apply to the question, mention the same. [**2 * 5 = 10**]

   (a) *To be in Gryffindor, it suffices that Harry is courageous.*
   (b) *Every student of CS-1110-01 is rational.*

   Part (a)
   p: Harry is courageous.
   q: Harry is in Gryffindor.
   Formal Statement: $p \to q \equiv (\neg p \vee q)$
   Negation: $\neg(\neg p \vee q) \equiv p \wedge \neg q$
   Converse: $q \to p$
   Inverse: $\neg p \to \neg q$
   Contrapositive: $\neg q \to \neg p$

   Part (b)
   P(x) : x is a student of CS-1110-01.
   Q(x) : x is a rational.
   Formal Statement: $\forall x(P(x) \to Q(x)) \equiv \forall x(Q(x) \vee \neg P(x))$
   Negation: $\neg(\forall x(Q(x) \vee \neg P(x)) \equiv \exists x(\neg Q(x) \wedge P(x))$
   Converse: $\forall x(Q(x) \to P(x))$
   Inverse: $\forall x(\neg P(x) \to \neg Q(x))$
   Contrapositive: $\forall x(\neg Q(x) \to \neg P(x))$

2. Formalise the following argument and use both Semantic Entailment and Resolution to show its validity: [**4 + 3 + 3 = 10**]

   - **Premises**
     (a) *It is not wintry or Rita has her jacket*;
     (b) *Rita does not have her jacket or she does not catch a cold*;
     (c) *It is wintry or Rita does not catch a cold.*
   - **Conclusion**: *Rita does not catch a cold*

   Formalisation - 4 marks
   p: it is wintry
   q: Rita has her jacket
   r: Rita catches a cold
   premise 1: $\neg p \vee q$
   premise 2: $\neg q \vee \neg r$
   premise 3: $p \vee \neg r$
   conclusion: $\neg r$

   Sematic Entailment - 3 marks
   The complete truth table must consist of each of the three propositional variables, all

three premises and the conclusion. The truth table must contain eight distinct rows.

Resolution - 3 marks

$$\dfrac{\dfrac{(\neg p \vee q) \qquad (p \vee \neg r)}{\therefore (q \vee \neg r)}\ (\text{RES}) \qquad (\neg q \vee \neg r)}{\therefore \neg r}\ (\text{RES})$$

which is "Rita does not catch a cold".

3. Formalize the following using the given coded atomic statements. Justify the steps for your final answer. [**10**]

For walking on the path to be safe, it is necessary but not sufficient that grapes not be ripe along the path and for foxes not to have been seen in the area.

w: the walk along the path is safe.
f: there are foxes in the area.
g: the grapes are ripe.

The only correct answer: $\Big[w \rightarrow (\neg f \wedge \neg g)\Big] \wedge \neg\Big[(\neg f \wedge \neg g) \rightarrow w\Big]$
Reasoning: If seen as $[\phi] \wedge \neg[\psi]$, then the clause ϕ takes care of the 'necessary' condition and the clause ψ takes care of the 'not sufficient' condition imposed. Rest of the formalisation is starightforward.

8.2 Flash Quiz 02

1. Consider the recursive function $T(n) = 4T(n/2) + 3$ where $T(1) = 1$.
   [5 + 10 = 15 marks]

   (a) Unroll and find the closed form of this recursive function.

   (b) Find (with proof) a $\Theta-$bound for this function.

$$\begin{aligned} T(n) &= 4T(n/2) + c \\ &= 16T(n/4) + 4c + c \\ &= 64T(n/8) + 16c + 4c + c \\ &= 256T(n/16) + 64c + 16c + 4c + c \\ &\vdots \\ &= 4^k T(n/2^k) + c \cdot \sum_{0}^{k-1} 4^i \end{aligned}$$

   Let n be such that, $n/2^k = 1$. Then $n = 2^k$ and $k = \log_2 n$. Thus we get:
   $T(n) = 4^{\log_2 n} + c(1 + 4 + 16 + 64 + \cdots + 4^{\log_2 n - 1})$
   $\Rightarrow T(n) = n^{\log_2 4} + 3(\frac{(4^{\log_2 n} - 1)}{4-1})$
   $\Rightarrow T(n) = n^2 + 3(\frac{n^{\log_2 4} - 1}{3})$
   $\Rightarrow T(n) = n^2 + n^2 - 1$
   $\Rightarrow T(n) = 2n^2 - 1$
   Considering the given definition of $\Theta-$bound, for $c_1 = 1,\ c_2 = 3,\ n_0 = 2$ we have:

$$T(n) \in \Theta(n^2).$$

2. Show that $\log n! = O(n \log n)$. **[5 marks]**
   $1 \leq n;\ 2 \leq n;\ 3 \leq n;\ 4 \leq n; \cdots;\ n \leq n$. Thus we get :
   $1 \cdot 2 \cdot 3 \cdot \cdots \cdot n \leq \underbrace{n \cdot n \cdot n \cdot \cdots \cdot n}_{n\ times}$
   $\Rightarrow n! \leq n^n$.
   $\Rightarrow \log n! \leq n \log n$.
   Thus for $c = 1$ and $n_0 = 1$ we have $\log n! = O(n \log n)$.

8.3 Flash Quiz 03

1. Let A and B be two mutually disjoint events. Further, let A and B be events independent of event C. If $\mathbb{P}[A] + \mathbb{P}[B] = a$ and $\mathbb{P}[C] = b$, and $\mathbb{P}$ realises the uniform probability distribution then compute $\mathbb{P}[A \cup B \cup C]$. [**6 marks**]

   ANSWER : $\mathbf{a + b - ab}$.

   Let Ω be the sample space. We consider the Inclusion-Exclusion Principle:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$
$$\implies \frac{|A \cup B \cup C|}{|\Omega|} = \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} + \frac{|C|}{|\Omega|} - \frac{|A \cap B|}{|\Omega|} - \frac{|B \cap C|}{|\Omega|} - \frac{|C \cap A|}{|\Omega|} + \frac{|A \cap B \cap C|}{|\Omega|}$$

   ...due to $\mathbb{P}$ being the Uniform probability distribution...

$$\implies \mathbb{P}[A \cup B \cup C] = \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] - \mathbb{P}[A \cap B] - \mathbb{P}[B \cap C] - \mathbb{P}[C \cap A] + \mathbb{P}[A \cap B \cap C]$$
$$\implies \mathbb{P}[A \cup B \cup C] = a + b - 0 - ab + 0$$
$$\implies \mathbb{P}[A \cup B \cup C] = a + b - ab.$$

2. Consider probability distribution $\mathbb{P}_1 : \Omega \to [0, 1]$ and $\mathbb{P}_2 : \Omega \to [0, 1]$ on a non-empty sample space Ω. Show that [**4 marks**]

$$\sum_{\omega \in \Omega} |\mathbb{P}_1(\omega) - \mathbb{P}_2(\omega)| \le 2.$$

   Due to the Triangle inequality: $|a + b| \le |a| + |b|$, we have:

$$\begin{aligned}\sum_{\omega \in \Omega} |\mathbb{P}_1(\omega) - \mathbb{P}_2(\omega)| &\le \sum_{\omega \in \Omega} |\mathbb{P}_1(\omega)| + |\mathbb{P}_2(\omega)| \\ &= \sum_{\omega \in \Omega} |\mathbb{P}_1(\omega)| + \sum_{\omega \in \Omega} |\mathbb{P}_2(\omega)| \\ &= 2 \qquad \text{[since each are probability distributions.]}\end{aligned}$$

   This completes the proof. *qed*

3. Consider the joint random variable (X, Y) with domain $\{0, 1\} \times \{0, 1, 2\}$, corresponding to some random variables X and Y. It is given $\mathbb{P}_{(X,Y)}[(a, b)] = \frac{a+b}{9}$, $\forall (a, b) \in \{0, 1\} \times \{0, 1, 2\}$.

   (a) Determine marginal distributions $\mathbb{P}_X$ and $\mathbb{P}_Y$. [**7 marks**]

   (b) Are random variables X and Y independent? [**3 marks**]

   (a) We have $\mathbb{P}_X[x_0] = \sum_{t \in \{0,1,2\}} \mathbb{P}_{(X,Y)}[(x_0, t)]$ and $\mathbb{P}_Y[y_0] = \sum_{s \in \{0,1\}} \mathbb{P}_{(X,Y)}[(s, y_0)]$. Thus,

   - $\mathbb{P}_X[0] = \frac{0+0}{9} + \frac{0+1}{9} + \frac{0+2}{9} = 1/3.$
   - $\mathbb{P}_X[1] = \frac{1+0}{9} + \frac{1+1}{9} + \frac{1+2}{9} = 2/3.$
   - $\mathbb{P}_Y[0] = \frac{0+0}{9} + \frac{1+0}{9} = 1/9.$
   - $\mathbb{P}_Y[1] = \frac{0+1}{9} + \frac{1+1}{9} = 1/3.$
   - $\mathbb{P}_Y[2] = \frac{0+2}{9} + \frac{1+2}{9} = 5/9.$

   (b) **No.** If random variable X and Y are independant, than we have

$$\mathbb{P}_{(X,Y)}[(a, b)] = \mathbb{P}_X[a] \times \mathbb{P}_Y[b], \qquad \forall (a, b) \in \{0, 1\} \times \{0, 1, 2\}.$$

   Here, $\mathbb{P}_{(X,Y)}[(0, 0)] \ne \mathbb{P}_X[0] \times \mathbb{P}_Y[0]$. Thus, X and Y are not independant.

8.4 Flash Quiz 04

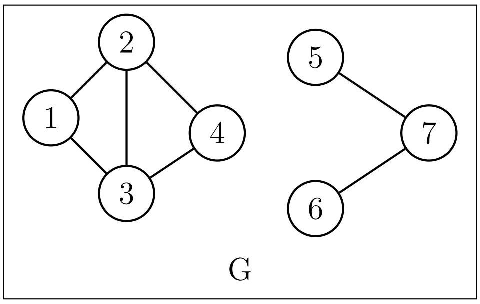
1. Proof that if a simple undirected graph G is Euler, then any degree sequence of G must be comprised of even natural numbers only. [**8 marks**]

   Proof. Since G is Euler, let one of its Eulerian cycle be of the following form:

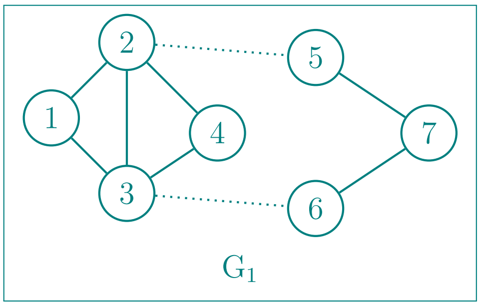
   $$[x_0 \xrightarrow{e_0} x_1 \xrightarrow{e_1} x_2 \xrightarrow{e_2} \ldots \xrightarrow{e_{n-1}} x_n \xrightarrow{e_n} x_0].$$

   with possible repetition amongst the vertices x_i, but not amongst the edges e_i. Arbitrarily, pick a vertex x. For this vertex x, there must be two distinct edges $e_1 = \{a, x\}$ and $e_2 = \{y, b\}$. Hence, for each vertex x, if there is one edge *incident* on it, then there is one edge to *outgoing* from it too. Each such pairing introduces 2 to $\deg(x)$, so $\deg(x)$ is even. Since vertex x was chosen arbitrarily, the proof is complete. □

2. Consider the disconnected graph G given the following figure. Draw the graph G_1 after adding the minimum number of edges in G so that the result becomes a simple Euler Graph. Justify your answer. [**3 marks**]



G



G_1

   We use the result we have proved in Question 1 to identify that vertices $2, 3, 5, 6$ are the only vertices with odd degrees. Further, for a graph to be Euler, it is necessary that it is connected. These leads to the above given G_1. **Observe that, one can also come up with an existential proof, nevertheless in all scenarios G_1 remains unique!**

3. Solve the following recurrence relations and write their closed forms where $n \in \mathbb{N}$:

   (a) $a_n = 6a_{n-1} - 9a_{n-2}, \quad \forall n \geq 2$. Given $a_0 = 1, a_1 = 6$. [**3 marks**]
   $a_n = (1 + n) \cdot 3^n$.
   The characterisitic equation in this case is : $r^2 - 6r + 9 = 0$.

   (b) $a_n = 7a_{n-1} - 12a_{n-2}, \quad \forall n \geq 2$. Given $a_0 = 5, a_1 = -5$. [**3 marks**]
   $a_n = 25 \cdot 3^n - 20 \cdot 4^n$.
   The characterisitic equation in this case is : $r^2 - 7r + 12 = 0$.

   (c) $a_n = 4a_{n-1} - 4a_{n-2}, \quad \forall n \geq 3$. Given $a_1 = 1, a_2 = 3$. [**3 marks**]
   $a_n = (1 + n) \cdot 2^{n-2}$.
   The characterisitic equation in this case is : $r^2 - 4r + 4 = 0$.

8.5 Quiz 01

Question 01 **[4 + 6 = 10 marks]**

Solve the following **'Exciting Life Problem'** using **natural deduction**.

Those people who read are not stupid. Poulomi can read and is wealthy. All people who are not poor and are smart are happy. Happy people have exciting lives. Can anyone be found with an exciting life?

1. Formalize the statements into a set of logical clauses.
2. Provide a correct and complete proof by natural deduction.

Answer to Question 01

▶ **Set of Predicates**

$\mathrm{READ}(x)$:	x reads
$\mathrm{SMART}(x)$:	x is smart (also, x is not stupid)
$\mathrm{WEALTHY}(x)$:	x is wealthy (also, x is not poor)
$\mathrm{HAPPY}(x)$:	x is happy
$\mathrm{EXCITING}(x)$:	x has an exciting life

1. **Formalisation** The above statements become the following when formalised.

   - $\forall x\ \big(\mathrm{READ}(x) \to \mathrm{SMART}(x)\big)$
   - $\big(\mathrm{READ}(Poulomi) \wedge \mathrm{WEALTHY}(Poulomi)\big)$
   - $\forall x\ \big((\mathrm{WEALTHY}(x) \wedge \mathrm{SMART}(x)) \to \mathrm{HAPPY}(x)\big)$
   - $\forall x\ \big(\mathrm{HAPPY}(x) \to \mathrm{EXCITING}(x)\big)$

2. **Natural Deduction** Each of the above formalised statements become premises. For obvious reasons this proof is not unique. **Yes, there is someone with an exciting life.**

1.	$\forall x\ \big(\mathrm{READ}(x) \to \mathrm{SMART}(x)\big)$	premise.
2.	$\big(\mathrm{READ}(Poulomi) \wedge \mathrm{WEALTHY}(Poulomi)\big)$	premise.
3.	$\forall x\ \big((\mathrm{WEALTHY}(x) \wedge \mathrm{SMART}(x)) \to \mathrm{HAPPY}(x)\big)$	premise.
4.	$\forall x\ \big(\mathrm{HAPPY}(x) \to \mathrm{EXCITING}(x)\big)$	premise.
5.	$\big(\mathrm{READ}(Poulomi) \to \mathrm{SMART}(Poulomi)\big)$	Universal Instantiation of 1.
3.	$\big((\mathrm{WEALTHY}(Poulomi) \wedge \mathrm{SMART}(Poulomi)) \to \mathrm{HAPPY}(Poulomi)\big)$	Universal Instantiation of 3.
6.	$\big(\mathrm{HAPPY}(Poulomi) \to \mathrm{EXCITING}(Poulomi)\big)$	Universal Instantiation of 4.
7.	$\mathrm{WEALTHY}(Poulomi)$	Simplification or $\wedge_{e,2}$ 2.
8.	$\mathrm{READ}(Poulomi)$	Simplification or $\wedge_{e,1}$ 2.
9.	$\mathrm{SMART}(Poulomi)$	Modus Ponens on 5, 8.
10.	$\mathrm{WEALTHY}(Poulomi) \wedge \mathrm{SMART}(Poulomi)$	Conjunction or $\wedge_i$ 7, 9.
11.	$\mathrm{HAPPY}(Poulomi)$	Modus Ponens on 3, 10.
12.	$\mathrm{EXCITING}(Poulomi)$	Modus Ponens on 6, 11.
13.	$\exists x\ \big(\mathrm{EXCITING}(x)\big)$ *(conclusion)*	Existantial Instatiation of 12.

– OR –

Question 02 **[5 + 5 = 10 marks]**

Solve the following mandatory sub-parts.

a. Let $\phi = \exists x(P(y,z) \wedge (\forall y(\neg Q(y,x) \vee P(y,z))))$. Consider the terms $t_1 = w$ (w is a variable), $t_2 = f(x)$ and $t_3 = g(y,z)$, where f and g are function symbols with arity 1 and 2 respectively.

   i. Compute $\phi[t_1/x]$, $\phi[t_1/y]$.

   ii. Which of the terms t_2, t_3 are free for x in ϕ.

   iii. Which of the terms t_2, t_3 are free for y in ϕ.

   [**Hint:** Scope of $\exists x$ in ϕ is $P(y,z)$]

b. Consider the predicate formula ϕ given by

$$\phi := \forall x\ (P(x) \to Q(x, f(x))),$$

where P and Q are predicates and f is a function symbol. Suppose we fix the domain of discourse or the ground set $\Omega = \{000, 001, 010, 011, 100, 101, 110, 111\}$. Define P, Q and f such that ϕ is true.

[**Hint:** Appropriate subset of Ω for P, Q stays same; and f reverses! Palindrome - that reads the same forward and backward!]

Answer to Question 02

a. Here, we have the following logical compound statement.

$$\phi = \exists x\Big(\underbrace{P(y,z)}_{I} \wedge \big(\forall y(\neg \underbrace{Q(y,x)}_{II} \vee \underbrace{P(y,z)}_{III})\big)\Big)$$

x ... II and III are scope of the quantifier $\forall y$ in ϕ, so x is free in II.
y ... Since I is the scope of the quantifier term $\exists x$ in ϕ, y is free in I and binded in II, III.
z ... ϕ has no quantifier on z, hence z is free throughout ϕ.

   i. $\phi[t_1/x] = \exists x(P(y,z) \wedge (\forall y(\neg Q(y,w) \vee P(y,z))))$,
   $\phi[t_1/y] = \exists x(P(w,z) \wedge (\forall y(\neg Q(y,x) \vee P(y,z))))$.

   ii. $\mathbf{t_2}$ — t_2 is free for x in ϕ; t_3 is not free for x in ϕ.

   iii. $\mathbf{t_3}$ — t_2 is not free for y in ϕ; t_3 is free for y in ϕ.

b. Although the following is a natural answer, there might be other correct answers too!

$P(x)$:	$\mathbf{x} \in \{000, 010, 101, 111\}$
$Q(x,y)$:	$\mathbf{x} = \mathbf{y}$ as strings.
$f(x)$:	the bit-reversal function (since we only have 3-bit binary strings here, under $\mathbf{f}$ the first bit gets replaced by the third and the third by first bit).

Question 03 **[5 + 5 = 10 marks]**

Solve the following questions.

a. Prove using the principle of mathematical induction that $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{N}, n \geq 1$.

b. Rudransh has a stock of Rs. 2/- notes and Rs. 5/- notes, only. Prove using the principle of mathematical induction, that Rudransh can dispense any amount of money, say Rs. x, where x is a positive integer ≥ 4, using these two denominations.
[**Hint:** The statement to be proved is as follows: For all $n \geq 4$, there exists $a, b \in (\mathbb{N} \cup \{0\})$ such that $n = 2a + 5b$.]

Answer to Question 03

a. We need to prove that $\forall n \in \mathbb{N},\ P(n) : 3 \mid n^3 + 2n$.
BASE CASE: $(1)^3 + 2(1) = 3$, which clearly is divisible by 3.
INDUCTIVE HYPOTHESIS: We assume, for some $k \in \mathbb{N}$ and $k > 1$, $P(k)$ is true.
INDUCTIVE STEP: We inspect the formulation for $k + 1 \in \mathbb{N}$.

$$\begin{aligned}
(k+1)^3 + 2(k+1) &= (k+1)[(k+1)^2 + 2] \\
&= (k+1)[k^2 + 2k + 3] \\
&= (k+1)[3m + 3] \qquad \text{[using IH, where } m \in \mathbb{N}] \\
&= 3(k+1)(m+1).
\end{aligned}$$

Therefore, for any $k \in \mathbb{N}$, $P(k+1)$ holds whenever $P(k)$ is true. This completes the proof by induction. *qed.*

b. We need to prove that $\forall n \in \mathbb{N}$ *and* $n \geq 4$, $P(n) :\ \exists n_1, n_2 \in \mathbb{N} \cup \{0\} \ni n = 2n_1 + 5n_2$.
BASE CASE: Observe $P(4)$ holds where $n_1 = 2$ and $n_2 = 0$.
INDUCTIVE HYPOTHESIS: We assume, for some $k \in \mathbb{N}$ and $k > 4$, $P(k)$ is true.
INDUCTIVE STEP: We inspect the formulation for $k + 1 \in \mathbb{N}$. Due to the IH we have, $\exists m_1, m_2 \in \mathbb{N} \cup \{0\} \ni k + 1 = 2m_1 + 5m_2 + 1$.

i. CASE I : $k + 1$ is even.
Observe that any even integer greater than 4 will have at least two 2s involved in its sum. Basically any even integer greater than 4 can be written as $4 + p$, where p is an appropriate positive integer.

$$\begin{aligned}
k + 1 &= 2m_1 + 5m_2 + 1 \\
&= 2(n_1 + 2) + 5m_2 + 1 \\
&= 2n_1 + 5m_2 + 5 \\
&= 2n_1 + 5(m_2 + 1) \\
&= 2n_1 + 5n_2
\end{aligned}$$

ii. CASE II : $k + 1$ is odd.
Observe that any odd integer greater than 4 will have at least one 5s involved in its sum. Basically any odd integer greater than 4 can be written as $5 + q$, where q is an appropriate positive integer.

$$\begin{aligned}
k + 1 &= 2m_1 + 5m_2 + 1 \\
&= 2m_1 + 5(n_2 + 1) + 1 \\
&= 2m_1 + 5n_2 + 6 \\
&= 2(m_1 + 3) + 5n_2 \\
&= 2n_1 + 5n_2
\end{aligned}$$

Therefore, for any $k \in \mathbb{N}$, $P(k+1)$ holds whenever $P(k)$ is true. This completes the proof by induction. *qed.*

Question 04 **[2 + 5 + 8 = 15 marks]**

Welcome to the **'Game of Logic'** which has the following two assumptions:

i. Logic is difficult or not many students like logic.
ii. For logic to be not difficult it is sufficient that mathematics is easy.

Formalise the above two assumptions (**2 marks**) and then answer the following questions on logical argument by considering these two assumptions to be two premises. The first and second assumption becomes `premise 1` and `premise 2` respectively.

a. Consider "Mathematics is not easy, if many students like logic", to be the conclusion and show that the argument `premise 1`, `premise 2` $\vdash$ `conclusion` is valid. **[1 + 4 = 5 marks]**

b. Consider "Not many students like logic, if mathematics is not easy.", to be the conclusion and show that the argument `premise 1`, `premise 2` $\vDash$ `conclusion` is not valid. **[1 + 7 = 8 marks]**

Answer to Question 04

▶ Formalisation

p : logic is difficult
q : mathematics is easy
r : many students like logic

a. **To prove :** $(p \vee \neg r);\ (q \to \neg p) \vdash (r \to \neg q)$.

1.	$p \vee \neg r$	... premise
2.	$q \to \neg p$	... premise
3.	$\neg r \vee p$	... Equivalent to 1.
4.	$r \to p$	... Definition of $\to$
5.	$p \to \neg q$	... Contrapositive of 2.
6.	$r \to \neg q$	... Hypothetical Syllogism on 4. & 5. [***conclusion***]

b. **To prove :** $(p \vee \neg r);\ (q \to \neg p) \nvDash (\neg q \to \neg r)$.

.	p	q	r	$(p \vee \neg r)$ premise	$(q \to \neg p)$ premise	$(\neg q \to \neg r)$ conclusion
1.	F	F	F	T	T	T
2.	F	F	T	F	T	F
	⋮	⋮	⋮	⋮	⋮	⋮
6.	T	F	T	**T**	**T**	**F**

In the last row of the Truth Table above, for a certain combination of semantic values of the propositional variables, all both the premises are TRUE but the conclusion is FALSE. Hence, the proof by semantic entailment is complete. *qed.*

▶ *You can use the code in 'semantic_entailment_01.c' uploaded in the Google Classroom to verify the above proof.*

– OR –

Question 05 **[5 + 5 + 5 = 15 marks]**

Solve the following mandatory sub-parts.

a. The table below represents the complete truth table for some propositional formula ϕ involving three propositional variables: p, q and r. Provide an example of a formula ϕ in conjunctive normal form (CNF) whose truth table matches the given one.

p	q	r	ϕ
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

b. The following is a proof of the sequent $p \vee q, \neg q \vee r \vdash p \vee r$. Carefully examine the proof and identify the each of the inference rules used below and labelled as Rule 1 through Rule 5. If any of these rules involve an assumption, explicitly state so.

1.	$p \vee q$	... premise
2.	$\neg q \vee r$	... premise
3.	p	... Rule1
4.	$p \vee r$	... Rule2
5.	q	... Rule3
6.	$\neg\neg q$	... Rule4
7.	r	... Rule5
8.	$\vdots$	$\vdots$

[**Hint:** $\frac{p_1 \vee p_2, \quad \neg p_1}{p_2}$ DS (Disjunctive Syllogism)]

c. Consider the following statement and express it as a predicate formula using appropriate predicates. The numbers a and b should be treated as constants.

The numbers a and b are bigger than their common factors.

Answer to Question 05

a. **An answer** is $\phi = (p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r)$, this can be made compact.
Step 1 Identify rows where ϕ is False.
Step 2 To ensure ϕ is False in the rows identified above, we construct clauses that eliminate these cases. Each clause must be False in at least one of the rows where ϕ is False. For each row where ϕ is False, we construct a clause which is False only in that row. So we get clauses $(p \vee q \vee r)$, $(p \vee \neg q \vee r)$ and $(\neg p \vee q \vee r)$.
Step 3 Combine the above clauses using conjunction. This formula ensures that ϕ is False in exactly the rows where $\phi =$ False in the given truth table, and True otherwise.

b. **Rule1** is *assumption*.
Rule2 is *introduction of disjunction on 3* **or** $\vee i_1$ 3 **or** *addition on 3*.
Rule3 is *assumption*.
Rule4 is *introduction of double-negation on 5* **or** $\neg\neg i$ 5.
Rule5 is *disjunctive syllogism on 2, 6*.

c. $\mathbf{C(x)}$: x is a common factor of a and b
$\mathbf{G(x,y)}$: x is greater than y
Encoding : $\forall x \Big(C(x) \rightarrow (G(a,x)) \wedge G(b,x) \Big)$.

8.6 Quiz 02

Question 01 **[5 marks]**

Prove that: A natural number n is composite if and only if it is divisible by a natural number less than or equal to $\sqrt{n}$.

Answer to Question 01

($\Rightarrow$) We need to prove that: *if n is divisible by a natural number less than or equal to $\sqrt{n}$, then n is composite.* Whenever $\exists a \in \mathbb{N}$ such that

$$\begin{aligned} & a|n \ \wedge \ a \leq \sqrt{n} \\ \implies & a|n \ \wedge \ a < n. \end{aligned}$$

By definition it follows that, n is a composite.

($\Leftarrow$) We need to prove that: *if n is composite, then n is divisible by a natural number less than or equal to $\sqrt{n}$.* Since n is composite, $\exists a \in \mathbb{N},\ 1 < a < n$ such that $a|n$. Here we have either of the following two cases:

$$\begin{aligned} & a^2 = n \ \vee \ a \cdot b = n && \text{......for some } b \in \mathbb{N} \\ \implies & a = \sqrt{n} \ \vee \ a \cdot a < a \cdot b = n && \text{......wlog } a < b \\ \implies & a = \sqrt{n} \ \vee \ a^2 < n \\ \implies & a = \sqrt{n} \ \vee \ a < \sqrt{n}. \end{aligned}$$

Hence, a is the required divisor of n which is less than or equal to $\sqrt{n}$.
This completes the proof. *qed*

Question 02 **[5 marks]**

Prove that: For finite sets A, B, there exists a function $f : A \cap B \to A \cup B$ such that f is injective.

Answer to Question 02

Here, we can provide an existential proof. For arbitrary but non-empty sets A, B, we always have $(A \cap B) \subseteq (A \cup B)$. Consider $f : (A \cap B) \to (A \cup B)$ to be the identity function.
For $x_1, x_2 \in A \cap B$ $f(x_1) = f(x_2) \implies x_1 = x_2$, hence this considered f is injective.
This completes the proof. *qed*

Question 03 **[5 marks]**

Let $\{0,1\}^n$ be the set of all binary strings of exactly length n. Define a function $f : \{0,1\}^n \to \mathbb{Z}$ such that:

1. $f(w) \neq 0$ for all $w \in \{0,1\}^n$, and
2. $\sum_{w \in \{0,1\}^n} f(w) = 0$.

Answer to Question 03

The function $f : \{0,1\}^n \to \mathbb{Z}$ can be defined as follows:

$$f(w) = \begin{cases} 1, & \text{if } w \in \{0,1\}^n \text{ starts with } 0, \\ -1, & \text{if } w \in \{0,1\}^n \text{ starts with } 1. \end{cases}$$

Note that there can be other correct answers too!

Question 04 **[5 marks]**

Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x^2$. Let $A = \{0, 1, 2^2, 3^2, \ldots, n^2\}$ and $B = \{0, 1, 2^2, 3^2, \ldots, n^2, (n+1)^2\}$ for some fixed $n \in \mathbb{N}$. Determine the set $f^{-1}(A)\Delta f^{-1}(B)$.

▶ Notation: Δ denotes the set symmetric difference.

Answer to Question 04

Note that the given function f is not necessarily an injection. Hence we have:

$$f^{-1}(A) = \{-n,\ -n+1, \ldots,\ -1,\ 0,\ 1, \ldots,\ n-1,\ n\}$$
$$f^{-1}(B) = \{-n-1,\ -n,\ -n+1, \ldots,\ -1,\ 0,\ 1, \ldots,\ n-1,\ n,\ n+1\}$$

Note that $f^{-1}(A) \subset f^{-1}(B)$. Hence we have:

$$\begin{aligned} f^{-1}(A)\Delta f^{-1}(B) &= \big(f^{-1}(A) \cup f^{-1}(B)\big) \setminus \big(f^{-1}(A) \cap f^{-1}(B)\big) \\ &= f^{-1}(B) \setminus f^{-1}(A) \\ &= \{-n-1,\ n+1\} \end{aligned}$$

$$\boxed{\therefore\ f^{-1}(A)\Delta f^{-1}(B) = \{-n-1,\ n+1\}}$$

Question 05 **[5 marks + 1 bonus]**

Consider the following function f defined on the set $\{0,1\}^n$ as: for $x \in \{0,1\}^n$, $f(x) = \mathsf{wt}(x)$, where $\mathsf{wt}(x)$ denotes the number of 1's in the string x.

1. Determine the range of f
2. Is f injective ?
3. Determine $f^{-1}(1)$.
4. Determine the size of $f^{-1}(A)$ where $A = \{1, n\}$.

Answer to Question 05

1. There can be $n-$bit binary string which can have no 1s, one 1, two 1s and so on until the case where the string has all n bits to 1. Hence, $Range(f) = \{0, 1, 2, \ldots, n\}$.
2. Clearly, $f(1000\ldots0) = f(0000\ldots1) = 1$. Hence, f is not injective.
3. For $i = 1, 2, 3, \ldots, n$, we define $w_i := \{x \mid x \in \{0,1\}^n \ \wedge \text{only the } i^{th} \text{ bit in } x \text{ is } 1\}$. Observe, that $\forall x \in \{0,1\}^n,\ f(x) = 1$ if and only if $x = w_i$. Hence, $f^{-1}(1) = \{w_i \mid i = 1, 2, 3, \ldots, n\}$.
4. Since $A = \{1\} \cup \{n\}$ and $\{1\}$ and $\{n\}$ are disjoint subsets of $\mathbb{N}$ we have, $f^{-1}(A) = f^{-1}(\{1\}) \cup f^{-1}(\{n\})$. Since, $f^{-1}(\{n\}) = \{111\cdots11\}$ continuing from part(3) we have
$$f^{-1}(A) = \{w_i \mid i = 1, 2, \ldots, n\} \cup \{111\cdots11\}.$$

Question 06 **[5 marks]**

Consider the following recurrence

$$f(n) = \begin{cases} 1 & \text{if } n = 1 \\ m \cdot f(n-1) & \text{if } n \geq 2 \end{cases},$$

where $m \in \mathbb{N}$ is a fixed natural number. Determine a closed form expression for f.

Answer to Question 06

First of all, we perform unrolling to claim a closed form for f.

$$\begin{aligned} f(n) &= m \cdot f(n-1) \\ &= m[m \cdot f(n-2] = m^2 \cdot f(n-2) \\ &= m^3 \cdot f(n-3) \\ &= m^4 \cdot f(n-4) \\ &\vdots \\ &= m^k \cdot f(n-k) \\ &= m^{n-1} \cdot f(1) = m^{n-1} \qquad [\textit{substitution } n-k=1] \end{aligned}$$

So, after performing unrolling we claim that the closed form of f is $f(n) = m^{n-1}$. Now we use induction to prove our claim.

BASE CASE : $m^{1-1} = 1 = f(1)$. Hence, the claim is valid for the base case.

IH : We assume that $\exists k \in \mathbb{N}$ such that $f(k) = m^{k-1}$.

Inductive Step : $f(k+1) = m \cdot f(k) = m\big(\underbrace{m^{k-1}}_{IH}\big) = m^{(k+1)-1}$.

This completes the proof and the determined closed form is $f(n) = m^{n-1}$.

Question 07 **[5 marks]**

Determine the number of functions

$$f : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5, 6\}$$

such that $f(1) = 2$.

Answer to Question 07

For each $n = 2, 3, 4, 5, 6$ observe that $f(n)$ has 6 mapping possibilities to the elements in the given codomain. Hence, the required number of plausible functions is 6^5.

OR

Question 08 **[2 + 2 + 1 = 5 marks]**

The Set Symmetric Difference Δ is defined as follows:

$$A\Delta B = (A \cup B) \setminus (A \cap B).$$

1. What can be said about sets A and B if $A\Delta B = A$?
2. What can be said about sets A and B if $A\Delta B = A \setminus B$?
3. For any non-empty set A, what is $A\Delta A$?

▶ Notation: $A \setminus B$ denotes A set minus B.

Answer to Question 08

1. $\mathbf{B} = \phi$.
   We have $A\Delta B = A$. It follows that

$$\begin{aligned}
& (A \cup B) \setminus (A \cap B) && = A && \\
\implies & (A \cup B) \cap (A \cap B)^c && = A && \\
\implies & (A \cup B) \cap (A^c \cup B^c) && = A && \ldots(De\ Morgan's\ Law) \\
\implies & \big((A \cup B) \cap A^c\big) \cup \big((A \cup B) \cap B^c\big) && = A && \ldots(Distributive\ Law) \\
\implies & \underbrace{(B \cap A^c)}_{I} \cup \underbrace{(A \cap B^c)}_{II} && = A && \ldots(on\ simplifying)
\end{aligned}$$

   Since, $I \subseteq A^c$ and $II \subseteq A$ we have:

$$\begin{aligned}
& I = \phi \ \wedge \ II = A \\
\implies & B = \phi.
\end{aligned}$$

2. $\mathbf{B} \subseteq \mathbf{A}$.
   We have $A\Delta B = A$. It follows that

$$\begin{aligned}
& (A \cup B) \setminus (A \cap B) && = A \setminus B && \\
\implies & (A \cup B) \cap (A \cap B)^c && = A \setminus B && \\
\implies & (A \cup B) \cap (A^c \cup B^c) && = A \setminus B && \ldots(De\ Morgan's\ Law) \\
\implies & \big((A \cup B) \cap A^c\big) \cup \big((A \cup B) \cap B^c\big) && = A \setminus B && \ldots(Distributive\ Law) \\
\implies & \underbrace{(B \cap A^c)}_{I} \cup \underbrace{(A \cap B^c)}_{II} && = A \setminus B && \ldots(on\ simplifying)
\end{aligned}$$

   Since, $I \subseteq A^c$ and by definition $II = A \setminus B$ we have:

$$\begin{aligned}
& I = \phi \\
\implies & A^c \cap B = \phi \\
\implies & B \subseteq A.
\end{aligned}$$

3. $\boldsymbol{\phi}$.

$$\begin{aligned}
A\Delta A &= (A \cup A) \setminus (A \cap A) \\
&= A \setminus A \\
&= A \cap A^c \\
&= \phi.
\end{aligned}$$

8.7 Quiz 03

Question 01 **[5 + 3 + 4 + 3 = 15 marks]**

1. Show that if $f(n) = n^2 \log_2 n - 5n^2 + 3n \log_2 n$, then for $c_1 = 1/2$ and $c_2 = 3$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$c_1 \cdot n^2 \log_2 n \leq f(n) \leq c_2 \cdot n^2 \log_2 n.$$

Determine the smallest computable value of n_0, so that the above inequalities establish $f(n)$ is $\Theta(n^2 \log_2 n)$.

First Inequality: $c_1 \cdot n^2 \log_2 n \leq f(n)$.
It suffices to show that,
$\frac{1}{2} n^2 \log_2 n \leq n^2 \log_2 n - 5n^2$,
implies that $1/2 \leq 1 - \frac{5}{\log_2 n}$.
Solving further we get $2^{10} \leq n$. Thus $n_1 = 1024$.

Second Inequality: $f(n) \leq c_2 \cdot n^2 \log_2 n$.
$\forall n \geq 1$, we have $n^2 \log_2 n \leq n^2 \log_2 n$; $-5n^2 \leq n^2 \log_2 n$ and $3n \log_2 n \leq n^2 \log_2 n$.
Adding these we get, $f(n) \leq 3 \cdot n^2 \log_2 n$, where $n_2 = 1$

Considering both calculations above, we get $n_0 = max\{n_1, n_2\} = 1024$.

2. If $f(n) = \log_2(n!) + n^2 \log_2 n$, then determine whether $f(n)$ is $O(n \log_2 n)$. Provide a clear justification.

[Hint: If $f_1 \in O(g)$ and $f_2 \in O(g)$, then $f_1 + f_2 \in O(g)$.]
Forall $n \in \mathbb{N}$ we have $n! \leq n^n \implies \log_2(n!) \leq n \log_2 n$. Thus, we have the result $\log_2(n!) \in O(n \log_2 n)$ for $c = 1; n_0 = 1$. Further note that for all $n \in \mathbb{N}$, we have $n \log_2 n < n^2 \log_2 n$, leading to the fact that $n^2 \log_2 n$ can never be $O(n \log_2 n)$.
Proof by contradiction : Suppose $\log(n!) + n^2 \log n = O(n \log n)$.
Hence $n^2 \log n = O(n \log n) - \log(n!)$. As $\log(n!) \in O(n \log n)$, it follows that $n^2 \log n$ is $O(n \log n)$, which is a contradiction.

3. Show that if $f(n) = \log_2 n$ and $g(n) = n^{1/4}$, then $f(n)$ cannot be $\Omega\big(g(n)\big)$.
We claim that $\log_2 n \in o(n^{1/4})$. Consider

$$\begin{aligned}\lim_{n\to\infty} \frac{\log_2 n}{n^{1/4}} &= \lim_{n\to\infty} \frac{k \cdot 1/n}{1/n^{3/4}} && \text{[L'Hôpital's rule]} \\ &= \lim_{n\to\infty} \frac{4k}{n^{1/4}} = 0. \\ \therefore\ \log_2 n \in o(n^{1/4}) & && \ldots \text{claim proved.}\end{aligned}$$

By first principle, $\log_2 n \in o(n^{1/4})$ implies that

$$\begin{aligned}&\forall c > 0,\ \exists\ n_0 > 0\ s.t.\ \forall n > n_0\ |\log_2 n| \leq c \cdot |n^{1/4}| \\ \implies &\forall c > 0,\ \exists\ n_0 > 0\ s.t.\ \forall n > n_0\ \log_2 n \leq c \cdot n^{1/4} \qquad (1)\end{aligned}$$

Proof by contradiction : Let us assume that $\log_2 n \in \Omega(n^{1/4})$. It follows that $\big(\exists c > 0,\ \exists n_0 > 0\ s.t.\ \forall n > n_0\ \log_2 n \geq c \cdot n^{1/4}\big)$. But this is a clear contradiction to inference (1) derived above. So our assumption is incorrect and hence $\log_2 n$ can never be $\Omega\big(n^{1/4}\big)$. *qed*

4. Show that if $f(n) = n^{\frac{3}{\log_2 n}}$, then $f(n)$ is $O(1)$.
$f(n) = n^{\frac{3}{\log_2 n}} = n^{3 \cdot \log_n 2} = 2^3$. Clearly, for $\forall n \geq n_0$ where $n_0 = 1$ and $c = 10$ we have $f(n) \leq c \cdot 1$. Hence, $f(n) \in O(1)$. *qed*

OR

Question 02 **[5 + 5 + 5 = 15 marks]**

For some fixed $n \in \mathbb{N}$, $p(x) : \mathbb{R} \to \mathbb{R}$ defined as $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is called a real-valued **n-degree polynomial**, where $a_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$.

For the following three questions consider the cost of binary multiplication `a * b` to be $O(1)$, and ignore any complexity due to any binary addition, variable increment and variable initiation.

1. The conventional algorithm for evaluating a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ at $x = c$ can be expressed in the following pseudocode (Algorithm-I) :

Algorithm-I

```
1:  procedure EVAL.POLYNOMIAL(c, a_0, a_1, ..., a_n: reals)
2:      y := 0                                   ▷ initiation
3:      for i := 1 to n do
4:          power := 1                           ▷ initiation
5:          for j := 1 to i do
6:              power = power * c
7:          end for
8:          y := y + a_i * power
9:      end for
10:     return y        ▷ y = a_n c^n + a_{n-1} c^{n-1} + ··· + a_1 c + a_0
11: end procedure
```

(a) What is the total number of multiplications and additions done by this algorithm to evaluate a n-degree polynomial $p(x)$ at a given point $x = c$?
The multiplication at `line 6` is happening once for each iteration of the loop at `line 5`. Further for each iteration of the loop at `line 3`, the loop at `line 5` is restarting and the multiplication at `line 8` is happening once.
The addition at `line 8` is happening one for each iteration of the loop at `line 3`. Hence we have,

$$\text{total no. of multiplications} = (1+1) + (2+1) + (3+1) + \cdots + (n+1)$$
$$= \sum_{i=1}^{n} i + n = \frac{n^2 + 3n}{2},$$
$$\text{total no. of additions} = n.$$

(b) What is the complexity of this conventional algorithm (Algorithm-I) in Θ-bound?
Since binary multiplication is the only cost inducing operation, observe that $\forall n \geq 1$ we have
$$\frac{1}{10}n^2 \leq \frac{1}{2}n^2 + \frac{3}{2}n \leq 2n^2.$$
Hence, complexity of Algorithm-I is $\Theta\left(n^2\right)$.

2. The **Horner's method** of evaluating a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ at $x = c$ can be expressed in the following pseudocode (Algorithm-II) :

Algorithm-II

1: **procedure** HORNER($c, a_0, a_1, \ldots, a_n$: reals)
2: $\quad y := a_n$ $\qquad\qquad\qquad$ ▷ initiation
3: $\quad$ **for** $i := 1$ to n **do**
4: $\qquad y = y * c + a_{n-i}$
5: $\quad$ **end for**
6: $\quad$ **return** y $\qquad$ ▷ $y = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0$
7: **end procedure**

(a) What is the total number of multiplications and additions done by this algorithm to evaluate a n-degree polynomial $p(x)$ at a given point $x = c$?
Both the multiplication and the addition at `line 4` is happening once for each iteration of the loop at `line 3`. Hence we have,

$$\text{total no. of multiplications} = n,$$
$$\text{total no. of additions} = n.$$

(b) What is the complexity of this Horner's algorithm (Algorithm-II) in Θ-bound?
Since binary multiplication is the only cost inducing operation, observe that $\forall n \geq 1$ we have

$$\frac{1}{2} n \leq n \leq 2n.$$

Hence, complexity of Algorithm-II is $\Theta\big(n\big)$.

3. An optimised algorithm for evaluating a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ at $x = c$ can be expressed in the following pseudocode (Algorithm-III) :

Algorithm-III

1: **procedure** EVAL.POLYNOMIAL($c, a_0, a_1, \ldots, a_n$: reals)
2: $\quad power := 1$ $\qquad\qquad\qquad$ ▷ initiation
3: $\quad y := a_0$ $\qquad\qquad\qquad$ ▷ initiation
4: $\quad$ **for** $i := 1$ to n **do**
5: $\qquad power = power * c$
6: $\qquad y = y + a_1 * power$
7: $\quad$ **end for**
8: $\quad$ **return** y $\qquad$ ▷ $y = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0$
9: **end procedure**

(a) What is the total number of multiplications and additions done by this algorithm to evaluate a n-degree polynomial $p(x)$ at a given point $x = c$?
Observe that for each iteration of the loop at `line 4`, we have two multiplications (one each from `line 5` and `line 6`) and one addition (`line 6`). Hence we have,

$$\text{total no. of multiplications} = 2n,$$
$$\text{total no. of additions} = n.$$

(b) What is the complexity of this optimised algorithm (Algorithm-III) in Θ-bound?
Since binary multiplication is the only cost inducing operation, observe that $\forall n \geq 1$ we have

$$n \leq 2n \leq 3n.$$

Hence, complexity of Algorithm-III is $\Theta(n)$.

Question 03 **[10 marks]**

Consider the problem $\mathsf{Select}_{n,i}$ described as follows:

$\mathsf{Select}_{n,i}$
Input: An n-size array B of distinct integers; An i in $1 \leq i \leq n$
Output: The ith smallest element in B

Suppose we have an algorithm $\mathcal{A}$ that finds the smallest element in a given array of integers, i.e., $\mathcal{A}$ is an algorithm for solving $\mathsf{Select}_{n,1}$. Using $\mathcal{A}$, we can design an algorithm $\mathcal{A}'$ to solve $\mathsf{Select}_{n,i}$ as follows:

Algorithm $\mathcal{A}'$

```
1: for j := 1 to i do
2:     Find a_j by running A on B
3:     Remove a_j from B
4: end for
5: return a_i
```

Show that if $f(n) = n - 1$ is the computational complexity of $\mathcal{A}$ for solving $\mathsf{Select}_{n,1}$, then computational complexity of $\mathcal{A}'$ is $\frac{2ni-i^2-i}{2}$.
The removal operation at `line 3` has no cost, but it reduces the size of array B by 1 in each iteration of the loop at `line 1`. The mechanics of the operation at `line 1` in the loop at `line 1` is as follows:

Iteration no.	Size of problem at `line 2`	Cost, i.e., $f(size)$
1	n	$n-1$
2	$n-1$	$n-2$
3	$n-2$	$n-3$
$\vdots$	$\vdots$	$\vdots$
i	$n-i+1$	$n-i$

Thus the total cost of $\mathcal{A}'$ gives its complexity to be

$$(n-1) + (n-2) + \cdots + (n-i) = \frac{2ni - i^2 - i}{2}.$$

Question 04 **[10 marks]**

Draw a recursion tree table for the following function, in the format given below.

$$f(n) = \begin{cases} 8f(n/2) + n^3 & \text{if } n \geq 2 \\ 1 & \text{if } n = 1 \end{cases}$$

Use the table to show that $f(n) = \Theta(n^3 \log n)$. Assume n is always a power of 2.

Format for Recursion Tree Table

Level	No. of subproblems	Size of subproblems	Cost per subproblem	Cost per level

Observe that the given recurence unrolls as follows:

$$\begin{aligned}
f(n) &= 8 \cdot f(n/2) + n^3 \\
8 \cdot f(n/2) &= 64 \cdot f(n/4) + 8 \cdot (n/2)^3 \\
64 \cdot f(n/4) &= 512 \cdot f(n/8) + 64 \cdot (n/4)^3 \\
&\vdots \qquad\qquad \vdots \\
8^k \cdot f(n/2^k) &= 8^{k+1} \cdot f(n/2^{k+1}) + 8^k \cdot (n/2^k)^3.
\end{aligned}$$

For the given recurence we the following at level k:

$$\boxed{8^k \cdot f(\underbrace{n/2^k}_{\text{II}}) = 8^{k+1} \cdot f(n/2^{k+1}) + \overbrace{\underbrace{8^k}_{\text{I}} \cdot \underbrace{(n/2^k)^3}_{III}}^{IV}.}$$

I : no. of subproblems
II : size of subproblems
III : cost per subproblems
IV : cost per level = no. of subproblems $\times$ cost per subproblems

Level	No. of subproblems	Size of subproblems	Cost per subproblem	Cost per level
0	1	n	n^3	$1 \cdot n^3$
1	8	$\frac{n}{2}$	$\left(\frac{n}{2}\right)^3$	$8 \cdot \left(\frac{n}{2}\right)^3 = n^3$
2	8^2	$\frac{n}{4}$	$\left(\frac{n}{4}\right)^3$	$8^2 \cdot \left(\frac{n}{4}\right)^3 = n^3$
3	8^3	$\frac{n}{8}$	$\left(\frac{n}{8}\right)^3$	$8^3 \cdot \left(\frac{n}{8}\right)^3 = n^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$8^{\log_2 n} = n^3$	$\frac{n}{2^{\log_2 n}} = 1$	$\left(\frac{n}{2^{\log_2 n}}\right)^3 = 1$	$8^{\log_2 n} \cdot \left(\frac{n}{2^{\log_2 n}}\right)^3 = n^3$

The required recursion tree table.

Using the table above, we get the total cost of the recurence

$$\begin{aligned}
&= n^3 + 8 \cdot \left(\frac{n}{2}\right)^3 + 8^2 \cdot \left(\frac{n}{4}\right)^3 + \cdots + 8^{\log_2 n} \cdot \left(\frac{n}{2^{\log_2 n}}\right)^3 \\
&= n^3 \cdot \left[1 + 8 \cdot \left(\frac{1}{2}\right)^3 + 8^2 \cdot \left(\frac{1}{4}\right)^3 + \cdots + 8^{\log_2 n} \cdot \left(\frac{1}{2^{\log_2 n}}\right)^3\right] \\
&= n^3 \cdot [\underbrace{1 + 1 + 1 + \cdots + 1}_{\log_2 n \;+\; 1 \text{ times}}] \\
&= n^3 \cdot \log_2 n \;+\; n^3 \\
&= \Theta(n^3 \log_2 n) \qquad\qquad \ldots\ldots [c_1 = 1, c_2 = 5; n_0 = 3]
\end{aligned}$$

8.8 Quiz 04

Question 01 **[10 marks]**

Let $\Omega = \{0,1\}^4$ be the set of all binary strings of length 4. Consider the *experiment of generating binary strings of length* 4 which realises the uniform probability distribution. Let $A \subseteq \Omega$ be the event that the generated binary string starts with 1 and $B \subseteq \Omega$ be the event that the generated binary string contains even number of 1s. Under the assumption that appearances of 0s and 1s are independant and uniform, determine whether A and B are independant or not.

► Note: The string 0000 is considered as having even number of 1-bits.

Answer: yes.

$|\Omega| = 2^4 = 16$;

$A = \{1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$ and $\mathbb{P}[A] = 1/2$;

$B = \{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\}$ and $\mathbb{P}[B] = 1/2$;

$A \cap B = \{1001, 1010, 1100, 1111\}$ and $\mathbb{P}[A \cap B] = 1/4$.

Since $\mathbb{P}[A] \cdot \mathbb{P}[B] = \mathbb{P}[A \cap B]$, we have events A and B to be independant.

OR

Question 02 **[5 + 5 = 10 marks]**

1. It is given that A, B, C are pairwise independent, with $\mathbb{P}[A] = \mathbb{P}[B] = \mathbb{P}[C] = 1/2$. Further if events A and $B \cup C$ are also independent, then compute $\mathbb{P}[A \cap (B \cup C)]$.
   Answer : 3/8.

$$\begin{aligned}
\mathbb{P}[A \cap (B \cup C)] &= \mathbb{P}[A] \cdot \mathbb{P}[(B \cup C)] && \text{[due to independence]} \\
&= \frac{1}{2} \cdot \Big[\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[(B \cap C)\Big] && \text{[Inclusion-Exclusion principle]} \\
&= \frac{1}{2} \cdot \Big[\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[B] \cdot \mathbb{P}[C]\Big] && \text{[due to independence]} \\
&= \frac{1}{2} \cdot \left[\frac{1}{2} + \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2}\right] \\
&= 3/8.
\end{aligned}$$

2. Random variable X takes values in the set $\{-3, -2, 1, 2, 3\}$ with $\mathbb{P}_X[2] = \mathbb{P}_X[-2]$ and $\mathbb{P}_X[3] = \mathbb{P}_X[-3]$. If $E[X] = 1/4$, then compute $\mathbb{P}_X[1]$.
   Answer : 1/4.

$$\begin{aligned}
E(X) &= -3 \cdot \mathbb{P}_X[-3] - 2 \cdot \mathbb{P}_X[-2] + \mathbb{P}_X[1] + 2 \cdot \mathbb{P}_X[2] + 3 \cdot \mathbb{P}_X[3] \\
&= \underbrace{-3 \cdot \mathbb{P}_X[3] - 2 \cdot \mathbb{P}_X[2] + 2 \cdot \mathbb{P}_X[2] + 3 \cdot \mathbb{P}_X[3]}_{\textit{equals } 0} + \mathbb{P}_X[1] \\
&= \mathbb{P}_X[1].
\end{aligned}$$

It is given that $E(X) = 1/4$, hence $\mathbb{P}_X[1] = 1/4$.

Question 03 **[7 + 3 = 10 marks]**

1. For a fixed $n \in \mathbb{N}$, let X be a random variable such that $Range(X) := \{1, 2, 3, \cdots, n\}$ and $\mathbb{P}_X$ is the Uniform probability distribution. Show that $\sum_{i=1}^{n} \mathbb{P}_X[X \geq i] = \frac{n+1}{2}$.
   Since probability distributes over disjoint events, we begin as follows:

$$
\begin{aligned}
\sum_{i=1}^{n} \mathbb{P}_X[X \geq i] &= \mathbb{P}_X[X \geq 1] + \mathbb{P}_X[X \geq 2] + \cdots + \mathbb{P}_X[X \geq n] \\
&= \left(\sum_{i=1}^{n} \mathbb{P}_X[i]\right) + \left(\sum_{i=2}^{n} \mathbb{P}_X[i]\right) + \cdots + \left(\sum_{i=n}^{n} \mathbb{P}_X[i]\right) \\
&= 1 \cdot \mathbb{P}_X[1] + 2 \cdot \mathbb{P}_X[2] + \cdots + n \cdot \mathbb{P}_X[n] \\
&= 1 \cdot \left(\frac{1}{n}\right) + 2 \cdot \left(\frac{1}{n}\right) + \cdots + n \cdot \left(\frac{1}{n}\right) \qquad [\because \mathbb{P}_X \sim U] \\
&= \frac{1}{n} \cdot \frac{n(n+1)}{2} \\
&= \frac{n+1}{2}.
\end{aligned}
$$

2. Consider $\mathbb{P} : \Omega \to [0, 1]$ to be a probability distribution on a non-empty sample space Ω. If $A, B, C \subseteq \Omega$ be events such that $A \cap \bar{C} = B \cap \bar{C}$, then show that $|\mathbb{P}(A) - \mathbb{P}(B)| \leq \mathbb{P}(C)$.
   Note that, $|\mathbb{P}(A) - \mathbb{P}(B)| \leq \mathbb{P}(C) \Leftrightarrow -\mathbb{P}[C] \leq \mathbb{P}[A] - \mathbb{P}[B] \leq \mathbb{P}[C]$. Hence, consider both of the following calculations.

$$
\begin{aligned}
& A \cap \bar{C} = B \cap \bar{C} \\
\Rightarrow & \mathbb{P}[A \cap \bar{C}] = \mathbb{P}[B \cap \bar{C}] \\
\Rightarrow & \mathbb{P}[A] - \mathbb{P}[A \cap C] = \mathbb{P}[B] - \mathbb{P}[B \cap C] \\
\Rightarrow & \mathbb{P}[A] - \mathbb{P}[B] = \mathbb{P}[A \cap C] - \mathbb{P}[B \cap C] \\
\Rightarrow & \mathbb{P}[A] - \mathbb{P}[B] \leq \mathbb{P}[A \cap C] \\
\Rightarrow & \mathbb{P}[A] - \mathbb{P}[B] \leq \mathbb{P}[A \cap C] \leq \mathbb{P}[C]
\end{aligned}
$$

$$
\begin{aligned}
& A \cap \bar{C} = B \cap \bar{C} \\
\Rightarrow & \mathbb{P}[A \cap \bar{C}] = \mathbb{P}[B \cap \bar{C}] \\
\Rightarrow & \mathbb{P}[A] - \mathbb{P}[A \cap C] = \mathbb{P}[B] - \mathbb{P}[B \cap C] \\
\Rightarrow & \mathbb{P}[A] - \mathbb{P}[B] = \mathbb{P}[A \cap C] - \mathbb{P}[B \cap C] \\
\Rightarrow & \mathbb{P}[A] - \mathbb{P}[B] \geq -\mathbb{P}[A \cap C] \\
\Rightarrow & \mathbb{P}[A] - \mathbb{P}[B] \geq -\mathbb{P}[A \cap C] \geq -\mathbb{P}[C]
\end{aligned}
$$

   This completes the proof. *qed.*

Question 04 **[15 marks]**

An experiment Π has sample space Ω such that $|\Omega| = m$, where m is a fixed natural number. Let X and Y be random variables on Ω such that $Range(X) := \{1, 2, 3, \cdots, m\}$ and $Range(Y) := \{\mathbb{P}_X(1), \mathbb{P}_X(2), \mathbb{P}_X(3), \cdots, \mathbb{P}_X(m)\}$, where $P_X(i)$, $\forall i = 1, 2, 3 \cdots, m$ are all distinct. Show that the expected value, $E(Y) = 1/m$ when it is given that $\mathbb{P}_Y$ is the Uniform probability distribution.

$$
\begin{tikzcd}
\{\mathbb{P}_X(1), \mathbb{P}_X(2), \dots, \mathbb{P}_X(m)\} \arrow[d, "\mathbb{P}_Y \sim U"] & \underset{\{\omega_1, \omega_2, \dots, \omega_m\}}{\Omega} \arrow[l, "Y"'] \arrow[r, "X"] \arrow[d, "P"] & \{1, 2, 3, \dots, m\} \arrow[d, "\mathbb{P}_X"] \\
{[0,1]} & {[0,1]} & {[0,1]}
\end{tikzcd}
$$

The given scenario can be represented using the block diagram given above. Thus, we consider the following:

$$
\begin{aligned}
E(Y) &= \sum_{i=1}^{m} Y(\omega_i) \cdot \mathbb{P}_Y[Y(\omega_i)] && \dots [\text{by definition}] \\
&= \sum_{i=1}^{m} \left(\mathbb{P}_X(i) \cdot \frac{1}{m} \right) && \dots [\mathbb{P}_Y \sim U] \\
&= \frac{1}{m} \cdot \underbrace{\left(\sum_{i=1}^{m} \mathbb{P}_X(i) \right)}_{=1} && \dots [\mathbb{P}_X \text{ is a prob. dist.}] \\
&= \frac{1}{m}.
\end{aligned}
$$

8.9 Quiz 05

Question 01 **[6 marks]**

Prove that if G is a graph with n vertices and n edges with no vertices of degree 0 or 1, then the degree of every vertex is 2.

Proof. Let G be a graph with n vertices and n edges. By Euler's Theorem we know,

$$\sum_{i=1}^{n} \deg(v_i) = 2 \cdot |E(G)| = 2n.$$

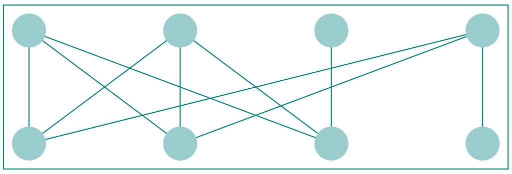
By assumption there are no vertices of degree 0 or 1, so $\delta(G) \geq 2$. Suppose for contradiction that there is at least one vertex with degree more than 2. By Euler's Theorem we have $\sum_{i=1}^{n} \deg(v_i) \geq 2(n-1) + 3 > 2n$, a contradiction. Therefore we can conclude that the degree of every vertex must be exactly 2. $\square$

Question 02 **[3 + 3 = 6 marks]**

Draw the following simple graphs, or give a formal explaination as to why they cannot exist.

2.1 A bipartite graph with 5 vertices and 7 edges.
Impossible.
The only way to partition 5 vertices would either be with partite sets of size 1 and 4, or partite sets of size 2 and 3. $K_{1,4}$ has $1 \cdot 4 = 4$ edges, while $K_{2,3}$ has $2 \cdot 3 = 6$ edges. Even a complete bipartite graph on 5 vertices can have more than 6 edges. So 7 edges in this scenario is not possible.

2.2 A bipartite graph with 8 vertices and 10 edges.
There are several graphs with these properties, here is one.



Question 03 **[6 marks]**

Prove that if two graphs are isomorphic, they must contain the same number of triangles. A triangle in a graph G is a 3-tuple of vertices (u, v, w) such that (u, v), (v, w), (u, w) are edges in G.

Proof. We prove this using the definition of isomorphism; if two graphs G and H are isomorphic, there exists a bijection, $f : V(G) \to V(H)$, between them that maintains adjacencies.

Suppose that G has a triangle with vertices a, b, c. The mapping of these vertices to H maintain that they are all pairwise adjacent, that is, $f(a), f(b), f(c)$ forms a triangle in H. Thus, each triangle in G corresponds to a triangle in H. Since an isomorphism is a bijection, it has an inverse. If a triangle with vertices u, v, w exists in H, then $f^{-1}(u), f^{-1}(v), f{-}1(w)$ form a triangle in G. The arbitrary selection of vertices u, v, w completes the proof. $\square$

Question 04 **[6 marks]**

Prove that in a graph G = (V, E) of order 9, if for every pair of distinct vertices $u, v \in V$ we have $deg(v) + deg(u) \geq 8$ then G is connected.

Proof. Suppose that G is a disconnected graph of order 9 such that for every pair of distinct vertices u, v, $deg(u) + deg(v) \geq 8$. Since G is disconnected there must exist vertices, say x, y, such that no x to y path exists in G. If no x to y path exists then certainly $xy \notin E(G)$ and vertices x, y do not share neighbours. Let $deg(x) = k$ for some $k = 0, ..., 8$. Since y is not adjacent to x and they share no neighbours, $deg(y) \leq 8 - k - 1$. Together we have $deg(x) + deg(y) \geq k + 8 - k - 1 = 7$, a contradiction.

Therefore our assumption of G to be disconnected was false, hence G is connected. □

Question 05 **[6 marks]**

Prove that if G is a bipartite graph with a Hamiltonian path, the orders of the partite sets differ by at most one.

Proof. Let us assume that there exists a Hamiltonian path in a bipartite graph G, where one partite set contains at least two vertices more than the other. Let us call the partite sets A and B and assume without loss of generality that $|A| + 2 \leq |B|$.

Certainly if a Hamiltonian path existed it will begin in B and alternate between the two partite sets. Once the path is of length $2|A| + 1$ all vertices in A will be visited, while at least one vertex in B will not be visited. This means it is not possible for a Hamiltonian path to exist. □

Question 06 **[5 marks]**

Consider a graph G formed by the addition of an edge to a tree T, while keeping G a simple graph. Prove that G must have a cycle.

Proof. Let T be a tree, then T is connected with $n - 1$ edges. Consider adding an edge e between vertices u, v in T, but by definition of a tree, there already existed a path between u, v, before adding e. Clearly, e is not a part of this path. Thus, adding e we get a new path between u, v, creating a cycle in resultant G. □

8.10 Quiz 06

Question 01 **[4 + 4 + 4 = 12 marks]**

Solve the following recurrence relations using characteristic equations only. Please write your answers clearly.

1.1 $h_n = 3h_{n-1} - 4n$ for $n \geq 1$, with $h_0 = 2$.

The solution to the recurrence is $h_n = a_n + b_n$, i.e., generic + particular. The homogeneous equation is $a_n = 3h_{n-1}$ whose characteristic equation is $(x - 3) = 0$. Hence, $a_n = c3^n$ is a generic solution to the homogeneous equation. Suppose $b_n = rn + s$ is a particular solution, then

$$rn + s = 3(r(n-1) + s) - 4n = (3r - 4)n + (3s - 3r).$$

Hence, we have the system of equations :

$$r = 3r - 4$$
$$s = 3s - 3r.$$

Solving, we get $r = 2, s = 3$ so that $b_n = 2n + 3$. Hence, the general solution is given by $h_n = c3^n + 2n + 3$. If $n = 0$, we get $2 = c + 3$ so that $c = -1$. Therefore, $h_n = -3^n + 2n + 3$ is a solution.

1.2 $h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}$ for $n \geq 3$, with $h_0 = 1,\ h_1 = 2,\ h_2 = 0$.

The characteristic equation is $x^3 - 2x^2 - x + 2 = 0$. Factoring, we get $x(x^2 - 1) - 2(x^2 - 1) = (x - 2)(x - 1)(x + 1)$. So the characteristic roots are $1, -1$ and 2. The solution to the LRR is given by $h_n = c_1(1)^n + c_2(-1)^n + c_3(2)^n$.

The initial conditions give

$$c_1 + c_2 + c_3 = 1$$
$$c_1 - c_2 + 2c_3 = 2$$
$$c_1 + c_2 + 4c_3 = 0$$

This yields $c_1 = 2, c_2 = -\frac{2}{3}, c_3 = -\frac{1}{3}$. The final answer becomes $h_n = 2 - \frac{2}{3}(-1)^n - \frac{1}{3}2^n$.

1.3 $h_n = 2h_{n-1} + 3^n$ for $n \geq 1$, with $h_0 = 2$.

The solution to the recurrence is $h_n = a_n + b_n$, i.e., generic + particular. The homogeneous equation is $a_n = 2h_{n-1}$ whose general solution is $a_n = c2^n$. For a particular solution, we try $b_n = p3^n$. Then,

$$p3^n = 2p3^{n-1} + 3^n$$
$$\Rightarrow 3p = 2p + 3$$
$$\Rightarrow p = 3.$$

Therefore, a general solution is given by $h_n = c2^n + 3^{n+1}$. At $n = 0, h_0 = 2$, so $c = -1$. Therefore, $h_n = -2^n + 3^{n+1}$ is a solution.

OR

Question 02 **[6 + 6 = 12 marks]**

Answer each of the following sub-questions.

2.1 Let $m, n \in \mathbb{N}$. Show that the number of functions from the domain $\{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ is equal to the total number of r-permutations of a multiset S. Determine r, determine S and then compute the total number of functions.

Consider the multiset $S = \{m \cdot 1, m \cdot 2, \ldots, m \cdot n\}$. Then, any m permutation gives a unique function $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ - the first member of the permutation defines $f(1)$, second member defines $f(2)$, and so on with the m^{th} member defining $f(m)$.

Conversely, any function $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ gives an m-permutation of S - which is, $(f(1), f(2), \ldots, f(m))$.

Therefore, the total number of function is equal to the total number of m permutations on the set S. Therefore $r = m$. Finally, the total number of m permutations on the set S is n^m.

2.2 Determine the sequence of numbers $h_0, h_1, h_2, \ldots, h_n$ whose generating function is given by $(1 + x + x^2 + x^3 + \cdots)(1 - x + x^2 - x^3 + \cdots)$.

We have, $(1 + x + x^2 + x^3 + \cdots)(1 - x + x^2 - x^3 + \cdots) = \frac{1}{1-x} \cdot \frac{1}{1+x} = \frac{1}{1-x^2}$. However,

$$\frac{1}{1-x^2} = 1 + x^2 + (x^2)^2 + (x^2)^3 + \cdots = 1 + x^2 + x^4 + x^6 + \cdots$$

Therefore, $h_0, h_1, h_2, h_n, \ldots = 1, 0, 1, 0, 1, 0, \ldots$, and in compact notation we have,

$$h_n = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Question 03 **[6 marks]**

How many 7-digit numbers are there such that the digits are distinct integers taken from $\{1, 2, \cdots, 9\}$ and such that the digits 5 and 6 do not appear consecutively in either order?

▶ Trivia : $7! = 5040$.

Answer : 151,200.

Method I : Consider the following cases.

a If the number has neither 5 nor 6: We can count the number of 7 permutations of the set $\{1, 2, 3, 4, 7, 8, 9\}$. There are $P(7, 7) = 7!$ such numbers.

b If only the number 5 occurs, but 6 does not occur: The six remaining digits (other than 5) can be obtained as a 6-permutation of the set $\{1, 2, 3, 4, 7, 8, 9\}$. There are $P(7, 6)$ ways of doing this. Then, there are 7 places where the 5 can go, so there are $7 \times P(7, 6) = 7 \times 7!$ such numbers. If only 6 occurs and 5 does not: Again, there are $7 \times 7!$ such numbers.

c If both 5 and 6 occur: The 5 remaining digits can be obtained as a 5-permutation of the set $\{1, 2, 3, 4, 7, 8, 9\}$. There are $P(7, 5)$ ways of doing this. Then, there are 6 places for the 5, and 5 places for the 6, so there are $6 \times 5 \times P(7, 5) = 30 \times \frac{7!}{2!}$ such numbers.

So the total number of such numbers is

$$\boxed{7! + 2 \times 7 \times 7! + (30 \times \frac{7!}{2!}) = 30 \times 7! = 151,200.}$$

Method II : To begin with, there are ${}^9P_7$ seven-digit numbers without the constraint of $5-6$ appearing together. Now we find the number of 7-digit numbers where 5 and 6 are consecutive.

a Block "56" appears together : We treat the sequence "56" as a single unit or block. Now we need to form an arrangement of length 6 (since "56" takes two original positions but acts as one unit). We must choose $7-2=5$ additional digits from the remaining $9-2=7$ digits available in $\{1,2,\cdots,9\}\setminus\{5,6\}$. The number of ways to choose these 5 digits is ${}^7C_5$. Once we have the block "56" and the 5 chosen digits, we have a total of $1+5=6$ items to arrange. The number of ways to arrange these 6 items is 6!. Thus, the number of arrangements containing the block "56" is:

$$ {}^7C_5 \times 6! = \frac{7!}{5!2!} \times 6! = \frac{7\times 6}{2} \times 720 = 21 \times 720 = 15,120. $$

b Block "65" appears together : WLOG, the count is exactly same as the previous case. It is 15, 120.

Since the digits must be distinct, an arrangement cannot contain both "56" and "65". Therefore, the total number of arrangements where 5 and 6 are consecutive is the sum of the numbers from the above cases (using sum rule). Total forbidden arrangements = 30, 240. Using the principle of Inclusion-Exclusion we get **Allowed arrangements = Total permutations - Total forbidden arrangements**. Thus the required count is

$$ \boxed{181,440 - 30,240 = 151,200.} $$

OR

Question 04 **[6 marks]**

Consider the multiset $S=\{10\cdot a_1, a_2, a_3, \ldots, a_{11}\}$. Determine number of 10-combinations of S.

Answer : 1024.

We count the number of 10-combinations by considering how many times the element a_1 appears in each combination.

- The number of 10-combinations with no a_1 included is 1. The combination is given by $\{a_2,\ldots,a_{11}\}$.
- If a 10-combination includes a_1 exactly once, then the remaining 9 elements must be chosen from $a_2,\ldots,a_{11}$, which is ${}^{10}C_9$.
- If a 10-combination includes a_1 exactly twice, then the remaining 8 elements must be chosen from $a_2,\ldots,a_{11}$, which is ${}^{10}C_8$.

$$ \cdots \quad \text{and so on} \quad \cdots $$

- The number of 10-combinations with all members equal to a_1 is 1.

Thus, using the Binomial Theorem we get the total number of required 10-combinations to be

$$ 1 + {}^{10}C_9 + {}^{10}C_8 + \ldots + {}^{10}C_1 + 1 = {}^{10}C_{10} + {}^{10}C_9 + {}^{10}C_8 + \ldots + {}^{10}C_1 + {}^{10}C_0 = 2^{10} = 1024. $$

Question 05 **[4 marks]**

Prove using PHP, that if more than 1001 integers are selected from $\{1, ..., 2000\}$ then there are at least two integers that are relatively prime, i.e., there exist two integers, say m and n, such that $gcd(m, n) = 1$. Clearly mention what are the *pigeons* and which are the *pigeonholes* in your setup.

We can partition the numbers into sets of size two, where the second digit is one less than the first: $\{1, 2\}, \{3, 4\}, \ldots, \{1997, 1998\}, \{1999, 2000\}$. Then, there are exactly 1000 of these disjoint subsets, which represent our 'pigeonholes'. Choose any 1001 integers and let them represent our 'pigeons'. Then by PHP, we will have two integers from the same disjoint subset. Hence, two integers are relatively prime.

OR

Question 06 **[4 marks]**

Let $1, 1, 5, 17, h_4, h_5, \ldots$ be a sequence of numbers satisfying a homogeneous linear recurrence with constant coefficients such that the recurrence is of order 2. Determine h_4 and h_5.

Answer : $h_4 = 61$; $h_5 = 217$.

Let $h_n = ah_{n-1} + bh_{n-2}$ $(n \geq 2)$. Then $5 = h_2 = ah_1 + bh_0 = a + b$ and $17 = ah_2 + bh_1 = 5a + b$. Thus, we have two linear equations:

$$a + b = 5$$
$$5a + b = 17.$$

On solving, we get $a = 3$, $b = 2$. Thus $h_4 = 3h_3 + 2h_2 = 3 \times 17 + 2 \times 5 = 61$, and on same tracks $h_5 = 3h_4 + 2h_3 = 3 \times 61 + 2 \times 17 = 217$.

Question 07 **[4 marks]**

A combinatorial proof is an arbitrary scenario where the same thing can be counted in two different ways. Give a combinatorial proof of the following : $m \cdot {}^{n}C_m = n \cdot {}^{n-1}C_{m-1}$.

Consider a group of n people who all apply to be on a committee of m people that requires a leader. There are two possible ways we can form the committee. We can either first choose from the larger group of n people our committee of m individuals, and then within that committee chose a leader, m possibilities. This is $m \cdot {}^{n}C_m$.

Alternatively, we can pick from the leader from the larger group of n people first, and then from the remaining $n - 1$ select the remaining $m - 1$ non-leader committee members. This is $n \cdot {}^{n-1}C_{m-1}$. Since we counted the same scenario in two different ways, these expressions are equivalent.

Question 08 **[4 + 5 = 9 marks]**

Rooks on a chessboard can only attack another piece if they travel in the same row or in the same column. So a set of rooks are said to be *non-attacking* if no two rooks share the same row or the same column. A typical example is shown in the figure, where Φ represents a Rook.

	Φ						
Φ							
			Φ				
		Φ					
					Φ		
				Φ			
							Φ
						Φ	

8.1 How many possible arrangements are there for eight non-attacking rooks on an 8×8 board?

Answer : $8!$

Each square is labelled (i, j) as usual. There are eight rooks, so they must occupy positions of the form $(1, j_1), (2, j_2), \ldots (8, j_8)$. Now the set $\{j_1, j_2, \ldots, j_8\}$ is a permutation of $\{1, 2, \ldots, 8\}$ since no two j_i's can be equal. Therefore, there are precisely $8!$ ways of arranging the rooks.

8.2 Now suppose each rook is coloured by a different colour. How many such arrangements are there?

Answer : $(8!)^2$

In the previous sub-question, we have chosen the positions for eight unlabelled rooks. When we label them, then there are a further $8!$ ways of permuting the colours, so we get $(8!)^2$ such arrangements.

8.3 **Practice Question.** What if there are four yellow, three blue and one red rook. How many arrangements are there now?

Answer : $\frac{(8!)^2}{1!3!4!}$

This is now a multiset $\{1 \cdot R, 3 \cdot B, 4 \cdot Y\}$. The number of permutations of the multiset is $\frac{8!}{1!3!4!}$. So the number of ways to arrange the rooks is $\frac{(8!)^2}{1!3!4!}$.

OR

Question 09 **[6 + 3 = 9 marks]**

Answer the following sub-questions. Please be clear while writing your answers.

9.1 Determine the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20,$$

where $x_1 \geq 5$, $x_2 \geq -2$, $x_3 \geq 0$ and $x_4 \geq 5$.

Answer : 455.

Note the given equation can be realised as:

$$(x_1 - 5) + (x_2 + 2) + x_3 + (x_4 - 5) = 12.$$

Respecting the constraints given on the variables x_1, x_2, x_3 and x_4 we can perform the following variable substituions in the given equation.

$$\begin{aligned} (x_1 - 5) &\leftrightarrow y_1 \\ (x_2 + 2) &\leftrightarrow y_2 \\ x_3 &\leftrightarrow y_3 \\ (x_4 - 5) &\leftrightarrow y_4 \end{aligned}$$

Thus we are left with the equation

$$y_1 + y_2 + y_3 + y_4 = 12,$$

where all variable y_i's are non-negative. Thus count of all non-negative integer equation of this new equation solves the problem. Following the *theory of combinations with repetitions* we get this count to be ${}^{15}C_3 = {}^{15}C_{12} = 455$.

9.2 Determine the value of $\sum_{k=0}^{10}(-1)^k\binom{10}{k}3^{10}$. Please **do not** do a brute-force calculation by computing each term of the summation.

Answer : 0.

Calculation I : Using the Binomial expantion appropriately we get,

$$\begin{aligned}0 = (3+(-3))^{10} = \sum_{k=0}^{10} {}^{10}C_k \cdot 3^{10-k}(-3)^k &= \sum_{k=0}^{10} {}^{10}C_k \cdot 3^{10-k}(-1)^k 3^k \\ &= \sum_{k=0}^{10} {}^{10}C_k \cdot 3^{10}(-1)^k.\end{aligned}$$

Calculation II : In an alternative perspective, we have

$$\begin{aligned}\sum_{k=0}^{10}(-1)^k\, {}^{10}C_k \cdot 3^{10} &= 3^{10}\sum_{k=0}^{10}(-1)^k \cdot {}^{10}C_k \\ &= 3^{10}\sum_{k=0}^{10}(-1)^k\, {}^{10}C_k \cdot (1)^{10-k} \\ &= 3^{10}(1+(-1))^{10} \\ &= 0.\end{aligned}$$

8.11 End-Term Assessment

Note : Please note that, the End-Term Assessment was a 03-hrs examination and the question paper for the same consisted questions worth 70 marks. Some of the questions which were originally part of the first draft, but later got ruled out, has also been mentioned here in this section, at the end. Although the difficulty level of the paper might subjectively appear to be easier than the other assessmnets listed above, the distribution of questions spans the entire course contents.

Question 01 **[4 marks]**

Consider the sets A and B defined as follows in the set-builder form:

$$A = \{a \in \mathbb{Z} \mid a = 2k, \text{ for some integer } k\},$$

$$B = \{b \in \mathbb{Z} \mid b = 2j - 2, \text{ for some integer } j\}.$$

Formally prove or disprove whether $A = B$?

Answer : Yes, the sets are equal.

Proof methodology goes by showing that $(A \subseteq B) \wedge (B \subseteq A)$.

Proof. We begin by showing that $A \subseteq B$. Let $a \in A$, then we know that $a = 2k$ for some integer k. Letting $k = j - 1$, where j is an integer, we see that $a = 2(j-1) = 2j - 2$, so $a \in B$, hence $A \subseteq B$.

Now we show $B \subseteq A$. Let $b \in B$. This means that $b = 2j - 2$ for some integer j. Picking $j = k + 1$, for some integer k, we can see that $b = 2(k+1) - 2 = 2k + 2 - 2 = 2k$, therefore $b \in A$. Thus $B \subseteq A$ and we can conclude that $A = B$. This complete the proof. $\square$

Question 02 **[6 marks]**

Consider the sets $A = \{1, 3, 7, 9, 11\}$ and $B = \{2, 4, 5, 7, 8, 10, 12\}$. Determine the count of total number of injective functions $f : A \to B$ such that $f(1) + f(3) = 14$.

Solution. $f(1) + f(3) = 14$ leads to the following choices for the tuple $(f(1), f(3))$:

$$(2, 12); (4, 10); (12, 2); (10, 4).$$

So, there are 4 choices for $f(1)$, and for each choice, there is exactly 1 unique choice of $f(3)$. Thus as tuple $(f(1), f(3))$, there are four choices. Note that for each choice of $f(1), f(3)$, we have $(7 - 2) = 5$ choices left for $f(7)$, so by multiplication rule, we have $4 \times 1 \times 5$ choices for the triplet $(f(1), f(3), f(7))$. Naturally, for each of these choices, we have $7 - 3 = 4$ choices left for $f(9)$, which leads to $4 \times 1 \times 5 \times 4$ choices for the quadruplet $(f(1), f(3), f(7), f(9))$. Finally, for each of these choices, we have $7 - 4 = 3$ choices left for $f(11)$ leading to $4 \times 1 \times 5 \times 4 \times 3 = 240$ number of functions we were looking for. This completes the count.

Question 03 **[4 + 4 + 4 = 12 marks]**

The following exercises relate to inhabitants of the island of knights and knaves created by Smullyan, where knights always tell the truth and knaves always lie. You encounter two people, A and B. Determine, if possible, what A and B are if they address you in the ways described. If you cannot determine what these two people are, can you draw any conclusions? Formulate ypur answer using propositional logic.

3a. A says "At least one of us is a knave" and B says nothing.

Let p be the proposition "A is a knight", and q be the proposition "B is a knight". If what A says is true, i.e., the truth value of p is T, then $\neg p \vee \neg q$ has the truth value T, else the truth value of p is F and so is the truth value of the proposition $\neg p \vee \neg q$. So in the following truth table we are interested in the row(s) where the entries in the first and the third columns match. Clearly, only the second row satisfies the desired property and hence we can conclude that A is a knight whereas B is a knave.

p	q	$\neg p \vee \neg q$
T	F	F
T	F	T
F	T	T
F	F	T

3b. A says "If I am a knight, then so is B" and B says nothing.

Let p be the proposition "A is a knight", and q be the proposition "B is a knight". If what A says is true, i.e., the truth value of p is T, then $p \to q$ has the truth value T, else the truth value of p is F and so is the truth value of the proposition $p \to q$. So in the following truth table we are interested in the row(s) where the entries in the first and the third columns match. Clearly, only the second row satisfies the desired property and hence we can conclude that both A and B are knights.

p	q	$p \to q$
T	F	F
T	T	T
F	T	T
F	F	T

3c. A says "We are both knights" and B says "Either A is a knight, or I am a knight, but not both".

Let p be the proposition "A is a knight", and q be the proposition "B is a knight". If what A says is true, i.e., the truth value of p is T, then $p \wedge q$ has the truth value T, else the truth value of p is F and so is the truth value of the proposition $p \wedge q$. Further, if what B says is true, i.e., the truth value of q is T, then the truth value of $p \oplus q$ is T, else the truth values of both q and $p \oplus q$ are F. So in the following truth table we are interested in the row(s) where the entries in the first and the third columns match and simultaneously the entries in the second and the fourth columns match. In this case we have two rows, viz., third and fourth, that satisfy the desired properties and hence we can only conclude that A is a knave. B could be either knight or knave.

p	q	$p \wedge q$	$p \oplus q$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	F

Question 04 **[2 + 6 = 8 marks]**

For a fixed $n \in \mathbb{N}$, let $\Omega = \{0,1\}^n$, the set of all n-length binary strings. Further any $\omega \in \Omega$ can be represented as $\omega = b_1 b_2 b_3 \dots b_n$, where each b_i is either 0 or 1. Let $p, q \in \mathbb{R}$ such that $0 \leq p, q \leq 1$ and $p + q = 1$. Show that the function $\mathbb{P} : \Omega \to [0,1]$ defined as:

$$\mathbb{P}(\omega) = \prod_{i=1}^{n} p^{b_i} \cdot q^{1-b_i},$$

is a probability distribution on Ω.

- **Property I :** We have $p \in [0,1] \land q \in [0,1]$. It follows that $p^{b_i} \in [0,1] \land q^{1-b_i} \in [0,1]$ for any positive integer b_i. Further, $p^{b_i} \cdot q^{1-b_i} \in [0,1]$, which leads to $\prod_{i=1}^{n} p^{b_i} \cdot q^{1-b_i} \in [0,1]$. Thus $\forall \omega \in \Omega \mathbb{P}(\omega) \in [0,1]$.

- **Property II :** Note that in Ω, there is ${}^nC_0 = 1$ binary string which is entirely composed of 0-bits. Next, there are ${}^nC_1$ binary strings which has exactly one 1-bit and rest all 0-bits. So on, there are ${}^nC_k$ binary strings which has exactly k number of 1-bits and rest 0-bits. Finally, there is again just ${}^nC_n = 1$ binary string which is entirely composed of 1-bits. Therefore, we have

$$\begin{aligned}\sum_{\omega \in \Omega} \mathbb{P}(\omega) &= [{}^nC_0 \cdot p^0 \cdot q^n] + [{}^nC_1 \cdot p^1 \cdot q^{n-1}] + [{}^nC_2 \cdot p^2 \cdot q^{n-2}] + \cdots + [{}^nC_n \cdot p^n \cdot q^0] \\ &= (p+q)^n \qquad \text{[By Binomial Theorem]} \\ &= 1 \qquad \text{[Given].}\end{aligned}$$

Thus, both the defining properties of a probability distribution are satisfied.

Question 05 **[10 marks]**

Consider the function $f(n) = \frac{n^3+2n^2+5}{n^2+3n}$. Prove that $f(n)$ is $O(n)$. Note that you are expected to provide explicit values of c and n_0.

Answer : $\mathbf{c = 2; n_0 = 1.}$

Proof. The objective is to show that $f(n) = \frac{n^3+2n^2+5}{n^2+3n}$ is $O(n)$. Observe that, for $n \geq 1$, $n^3 + 2n^2 + 5 > 0$ and $n^2 + 3n > 0$, so $f(n) > 0$. Hereby, we consider $n \geq 1$.

For the numerator: $n^3 + 2n^2 + 5 \leq n^3 + 2n^2 + 5n^2$ (since $1 \leq n^2$ for $n \geq 1$, so $5 \leq 5n^2$). Thus, $n^3 + 2n^2 + 5 \leq n^3 + 7n^2 = n^2(n+7)$.

For the denominator: $n^2 + 3n = n(n+3)$. Since $n > 0$, we have $n(n+3) > 0$. Also, $n^2 + 3n \geq n^2$.

Combining the above results we get : For $n \geq 1$,

$$\begin{aligned}f(n) = \frac{n^3+2n^2+5}{n^2+3n} &\leq \frac{n^2(n+7)}{n(n+3)} \\ &\leq \frac{n(n+7)}{n+3}.\end{aligned}$$

We have, $\frac{n+7}{n+3} = \frac{(n+3)+4}{n+3} = 1 + \frac{4}{n+3}$. For $n \geq 1$, $n + 3 \geq 4$. It follows that $\frac{4}{n+3} \leq \frac{4}{4} = 1$. Therefore, $1 + \frac{4}{n+3} \leq 2$. Substituting this into the inequality for $f(n)$, we get for $n \geq 1$

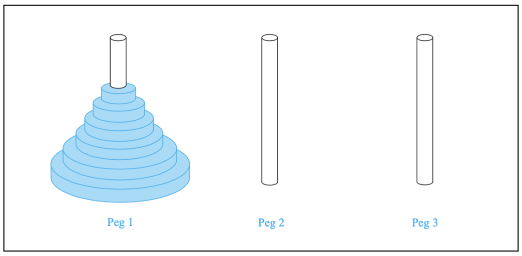
$$f(n) \leq n \cdot \left(\frac{n+7}{n+3}\right) \leq n \cdot 2 = 2n.$$

Thus, the choice for $c = 2$ and $n_0 = 1$ solves our problem and the proof is complete. □

Question 06 **[10 marks]**

Consider the puzzle called the Tower of Hanoi, which consists of three pegs mounted on a board together with disks of different sizes. Initially these disks are placed on the first peg in order of size, with the largest on the bottom (as shown in the figure). The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest on the bottom.

Let H_n denote the least number of moves needed to solve the Tower of Hanoi problem with n disks. Deduce a recurrence relation for the sequence $\{H_n\}$. Just derive the recurence, you do not have to prove it using induction or any other method.



The initial position of Tower of Hanoi

Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves. We keep the largest disk fixed during these moves. Then, we use one move to transfer the largest disk to the second peg. We can transfer the $n-1$ disks on peg 3 to peg 2 using H_{n-1} additional moves, placing them on top of the largest disk, which always stays fixed on the bottom of peg 2. Observe that the puzzle cannot be solved using fewer steps. This shows that

$$H_n = 2H_{n-1} + 1.$$

The initial condition is $H_1 = 1$, because one disk can be transferred from peg 1 to peg 2, according to the rules of the puzzle, in one move. Therefore the recurrence is:

$$\boxed{H(n) = \begin{cases} 1 & \text{if } n = 1; \\ 2 \cdot H(n-1) + 1 & \text{if } n \geq 2. \end{cases}}$$

Question 07 **[8 marks]**

If G is a simple graph of order $n \geq 3$, such that for all pairs of distinct vertices x and y in G that are not adjacent, we have $deg(x) + deg(y) \geq n$, then show that G must be connected.

Proof by contradiction.

Proof. Suppose that G is not connected. The idea is to show that that G cannot satisfy the Ore property : $\forall$ pairs of distinct vertices x and y that are not adjacent, $deg(x)+deg(y) \geq n$. Since G is not connected, its vertices can be partitioned into two parts, U and W, in such a way that there are no edges joining a vertex in U with a vertex in W. Let r be the number of vertices in U and let s be the number of verticed in W. Then $r + s = n$, and each vertex in U has degree at most $r-1$, and each vertex in W has degree at most $s-1$. Let x be any vertex in U and let y be any vertex in W. Then x and y are not adjacent, but the sum of their degrees is, at most, $(r-1)+(s-1) = r+s-2 = n-2$, and this contradicts the Ore property. We conclude that if G satisfies the Ore property, then G must be connected. □

Question 08 **[3 + 3 = 6 marks]**

Given the following sequences, determine the corresponding generating function both as a formal sum (summation) and in closed form (as a formula).

8a. $1, -1, 1, -1, 1, -1, \ldots$
$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots = \sum_{k=0}^{\infty} (-x)^k = \frac{1}{1+x}$.

8b. $1, 0, 1, 0, 1, \ldots$
$1 + x^2 + x^4 + \cdots = \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2}$.

Question 09 **[6 marks]**

Let $\mathbb{N}$ be the set of natural numbers and two functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be defined as:

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases} \quad \text{and} \quad g(n) = n - (-1)^n.$$

Then determine if the function $f \circ g$ is injective? Determine if $f \circ g$ is surjective?

When n is even,

$$\begin{aligned}(f \circ g)(n) &= f(g(n)) \\ &= f(n - (-1)^n) \\ &= f(n-1) \\ &= \frac{n-1+1}{2} \quad [n-1 \text{ is odd}] \\ &= \frac{n}{2}.\end{aligned}$$

Similarly, when n is odd,

$$\begin{aligned}(f \circ g)(n) &= f(g(n)) \\ &= f(n - (-1)^n) \\ &= f(n+1) \\ &= \frac{n+1}{2}. \quad [n+1 \text{ is even}]\end{aligned}$$

This implies, $(f \circ g)$ maps $1 \to 1$, $2 \to 1$, $3 \to 2$, $4 \to 2$, $5 \to 3$, $6 \to 3$, and so on. As $(f \circ g)(1) = (f \circ g)(2)$, it is evident that $(f \circ g)$ is not injective. Further, both the range and codomain for $(f \circ g)$ is $\mathbb{N}$, hence it is surjective.

Note : Please consider the following questions for practice, which were part of the initial drafts of the question paper, but were later ruled out. Some of them are *attention-worthy* repeats from the Problem Sets.

Question 02* **[6 marks]**

The TAs have identified 5 students who have tried copying off of each other in past quizzes. They do not want to report these students but also want to ensure no copying. They decide to place these students into 3 separate rooms such that no room remains empty. How many such ways can the students be seated? Note that the students and the rooms are both identifiable.

Answer : 150.

Solution. Note that this counting problem is equivalent of counting the number of Onto functions from set of cardinality 5 to set of cardinality 3. Total number possible functions from a cardinality 5 set to a cardinality 3 set is $3^5 = 243$. Now we shall count all scenarios in this context where the function is not Onto.

- Step-I : Firstly, count of functions whose range misses at least one element from the cardinality 3. There are ${}^3C_1$ possibilities of such element in the cardinality 3 set and for each possibility, the rest $3 - 1$ elements each have 5 mapping choices. Using rule of multiplication the count is ${}^3C_1 \cdot 2^5 = 96$. Due to Inclusion-Exclusion Principle we substract this count.

- Step-II : Unfortunately, a function whose range misses at least two elements gets counted twice in the previous step. It should be counted only once. Thus, we have to add back in the count of such functions. There are ${}^3C_2$ element choices and the rest $3 - 2$ element will have 1^5 mappings. Thus the count is ${}^3C_2 \cdot 1^5 = 3$.

- Finally : The count is $243 - 96 + 3 = 150$. Inclusion- Exclusion Principle it is!

This completes the counting.

Question 03* **[4 + 8 = 12 marks]**

Consider the following premises. Formalise each of the premises and the conclusion [**4 marks**] and then conclude *"All TAs are bad."*, using Natural Deduction only [**8 marks**].

1. "All the TAs are sleep deprived."
2. "No good TA is easily annoyed."
3. "TAs who are sleep deprived are easily annoyed."

- **Set of predicates**

$\text{TA}(x)$:	x is TA
$\text{SLEEPDEPRIVED}(x)$:	x is sleep deprived
$\text{GOOD}(x)$:	x is good (also, x is not bad)
$\text{EASILYANNOYED}(x)$:	x is easily annoyed
$\text{EXCITING}(x)$:	x has an exciting life

- **Formalisation**
  - $\forall x\ \big(\text{TA}(x) \to \text{SLEEPDEPRIVED}(x)\big)$
  - $\forall x\ \big((\text{GOOD}(x) \land \text{TA}(x)) \to \neg\text{EASILYANNOYED}(x)\big)$
  - $\forall x\ \big((\text{TA}(x) \land \text{SLEEPDEPRIVED}(x)) \to \text{EASILYANNOYED}(x)\big)$
  - $\forall x\ \big(\text{TA}(x) \to \neg\text{GOOD}(x)\big)$ ... *conclusion*

- For obvious reasons this proof is not unique.

1.	$\forall x\ \big(\text{TA}(x) \to \text{SLEEPDEPRIVED}(x)\big)$	premise.
2.	$\forall x\ \big((\text{GOOD}(x) \land \text{TA}(x)) \to \neg\text{EASILYANNOYED}(x)\big)$	premise.
3.	$\forall x\ \big((\text{TA}(x) \land \text{SLEEPDEPRIVED}(x)) \to \text{EASILYANNOYED}(x)\big)$	premise.
4.	$\text{TA}(Uttkarsh) \to \text{SLEEPDEPRIVED}(Uttkarsh)$	Uni. Inst. of 1.
5.	$(\text{GOOD}(Uttkarsh) \land \text{TA}(Uttkarsh)) \to \neg\text{EASILYANNOYED}(Uttkarsh)$	Uni. Inst. of 2.
6.	$(\text{TA}(Uttkarsh) \land \text{SLEEPDEPRIVED}(Uttkarsh)) \to \text{EASILYANNOYED}(Uttkarsh)$	Uni. Inst. of 3.
7.	$\neg\text{TA}(Uttkarsh) \lor \text{SLEEPDEPRIVED}(Uttkarsh)$	Def. of 4.
8.	$\neg\text{GOOD}(Uttkarsh) \lor \neg\text{TA}(Uttkarsh)) \lor \neg\text{EASILYANNOYED}(Uttkarsh)$	Def. of 5. & DML
9.	$\neg\text{TA}(Uttkarsh) \lor \neg\text{SLEEPDEPRIVED}(Uttkarsh)) \lor \text{EASILYANNOYED}(Uttkarsh)$	Def. of 6. & DML
10.	$\neg\text{TA}(Uttkarsh) \lor \text{EASILYANNOYED}(Uttkarsh)$	*RES* 7, 9.
11.	$\neg\text{TA}(Uttkarsh) \lor \neg\text{GOOD}(Uttkarsh)$	*RES* 8, 10.
12.	$\text{TA}(Uttkarsh) \to \neg\text{GOOD}(Uttkarsh)$	Definition 11.
13.	$\forall x\ \big(\text{TA}(x) \to \neg\text{GOOD}(x)\big)$ (*conclusion*)	Uni. Gen. of 12.

- Abbreviations used in the Natural Deduction above.

Abbreviation	Meaning
Uni. Inst.	Universal Instantiation
Uni. Gen.	Universal Generalisation
Def.	Definition
DML	De Morgan's Law
RES	Resolution

Question 05* **[3 + 7 = 10 marks]**

Solve the following subquestions. Please write your explainations/calculations clearly.

5.1 Prove that for some $k \in \mathbb{N}$, $\sum_{i=1}^{k} i^8 = O(k^9)$.

Solution. $1^8 \le k^8;\ 2^8 \le k^8;\ 3^8 \le k^8;\ 4^8 \le k^8; \cdots;\ k^8 \le k^8$.

$\Rightarrow 1^8 + 2^8 + \cdots + k^8 \le \underbrace{k^8 + k^8 + \cdots + k^8}_{k \text{ times}} = k^9$.

Thus for $c = 1$ and $n_0 = 1$ we have $\sum_{i=1}^{k} i^8 = O(k^9)$.

5.2 Prove that for some $k \in \mathbb{N}$, $\sum_{i=1}^{k} i^8 = \Omega(k^9)$.

Solution. Observe that $\sum_{i=1}^{k} i^8 = \underbrace{\sum_{i=1}^{\lceil k/2 \rceil - 1} i^8}_{I} + \underbrace{\sum_{i=\lceil k/2 \rceil}^{k} i^8}_{II}$.

In summation II, each term is at least $\left(\lceil k/2 \rceil\right)^8$. Further the number of terms in summation II is $k - \lfloor k/2 \rfloor$ which is approximately $k/2$.

Hence, $\sum_{i=1}^{k} i^8 \ge \frac{k}{2} \cdot \left(\lceil \frac{k}{2} \rceil\right)^8 \ge \frac{k}{2} \cdot \left(\frac{k}{2}\right)^8 = \frac{k^9}{2^9}$.

Thus for $c = 2^{-9}$ and $n_0 = 1$ we have $\sum_{i=1}^{k} i^8 = \Omega(k^9)$.

Question 07* **[10 marks]**

Prove that in any gathering of 6 people, you will always find 3 mutual acquaintances or 3 mutual strangers.

Solution. We can model this problem using graph theory. Let the 6 people be represented by the vertices of a complete graph K_6. For any two people (represented as vertices), they either know each other (acquaintances) or they do not (strangers). We color the edge between two vertices Red if they are acquaintances and Blue if they are strangers. Now the problem is equivalent to proving that any 2-coloring of the edges of K_6 must contain a monochromatic triangle (K_3), either a Red triangle (3 mutual acquaintances) or a Blue triangle (3 mutual strangers).

Pick an arbitrary vertex P. There are 5 edges connecting P to the other 5 vertices. We consider these 5 edges as 'pigeons' and the two colors (Red, Blue) as 'pigeonholes'. By the Pigeonhole Principle, since $5 = 2 \times 2 + 1$, at least $2 + 1 = 3$ of these edges must have the same color. We have two cases:

Case 1: At least 3 edges connected to P are Red. Let these edges connect P to vertices A, B, C. Now consider the edges connecting A, B, and C. If any of the edges $(A, B), (B, C)$, or (C, A) is Red, say (A, B) is Red, then vertices P, A, B form a Red triangle, representing 3 mutual acquaintances. If none of the edges $(A, B), (B, C)$, or (C, A) are Red, then all three must be Blue. In this case, vertices A, B, C form a Blue triangle, representing 3 mutual strangers.

Case 2: At least 3 edges connected to P are Blue. Let these edges connect P to vertices X, Y, Z. Now consider the edges connecting X, Y, and Z. If any of the edges $(X, Y), (Y, Z)$, or (Z, X) is Blue, say (X, Y) is Blue, then vertices P, X, Y form a Blue triangle, representing 3 mutual strangers. If none of the edges $(X, Y), (Y, Z)$, or (Z, X) are Blue, then all three must be Red. In this case, vertices X, Y, Z form a Red triangle, representing 3 mutual acquaintances.

In both cases, we are guaranteed to find either a Red triangle or a Blue triangle. Therefore, in any gathering of 6 people, there must be either 3 mutual acquaintances or 3 mutual strangers.

PROGRAMING PROBLEM SETS

9.1 Problem Set 01 - Intro to loops & conditional statements

1. Print the factors of a positive integer given as input.
2. Print the sum of digits in an input positive integer.
3. Count the number of digits in an integer given as input.
4. Check if a input number is a Perfect Number.
5. Check if an input positive integer is an Armstrong Number.
6. Reverse the digits of a positive integer given as input.

9.2 Problem Set 02 - Nested loops-I

1. Write a program in C programming language to evaluate the logical expression

$$(A \wedge B) \vee \neg C$$

   for given boolean values of A, B and C as inputs. Your code must take three integers (either 0 or 1) as inputs from the user and shall return the truth value of the given logical expression as result.

2. Write a program in C programming language to check whether a given number is divisible by 5, 13, both or neither. Your code must take an integers as input from the user and shall return the status of divisibility mentioned as result.

3. Write a program in C programming language which prints the complete truth table of the following logical expressions:

   (a) $(A \vee B) \wedge \neg C$
   (b) $(A \wedge \neg B) \rightarrow (B \vee \neg(A \wedge B))$
   (c) $(A \wedge B) \vee \neg C$

4. By now each one of you must be familiar about semantic entailment and how it can be used to prove any particular logical argument to be invalid. Write a program in C programming language, which uses semantic entailment to prove invalidity of the following logical arguments.

   (a) $(r \rightarrow (\neg p \vee q)),\ (p \wedge \neg r) \models \neg(\neg r \vee q)$
   (b) $(p \vee (q \wedge \neg r)),\ (r \rightarrow (\neg q \vee p)) \models (q \wedge \neg p)$
   (c) $(\neg p \vee q),\ (\neg r \vee \neg q),\ (\neg r \vee p) \models \neg r$

► 0 is the boolean value for **False** in C.

Logical Operators	Symbol in theory	Symbol in C syntax
neg	$\neg$	!
conjunction	$\wedge$	&&
disjunction	$\vee$	||

9.3 Problem Set 03 - Nested loops-II

1. Write a program in C programming language to check if a positive integer given as input is a Palindrome or not. Your code must take a positive integer as input from the user and shall print as output if it is a palindrome number or not.

2. Write a program in C programming language to check if a positive integer given as input is a Prime or not. Your code must take a positive integer as input from the user and shall print as output if it is a prime number or not. 1 is not a prime, by definition.

3. Write a program in C programming langauge to print the Fibonacci Series up to N Terms. Your code must take a positive integer as input from the user - **N** and shall print as output the Fibonacci Series as per the mentioned requirement.

   (a) Try to code the above **without** using arrays.

   (b) Try to code the above using 1-Dimensional arrays.

4. Write a program in C programming language to print all possible 4-bit binary strings, that is, binary strings of length 4. **Hint**: at best it only requires nested for loops.

9.4 Problem Set 04 - Intro to 1D arrays & more on nested loops

1. Write a program in C programming language which takes two binary strings as inputs from the user and performs the following bitwise binary operations and gives the result as output. The concept of loops, nested loops and 1D arrays shall suffice.

   (a) Bitwise AND

   (b) Bitwise OR

   (c) Bitwise XOR

   (d) Bitwise NAND

   (e) Bitwise NOR

   (f) Bitwise XNOR

2. Write a program in C, which approximates the value of $\sin(\pi/2)$ using the following Taylor Series for sine function. Do the approximation with a 0.01% precision involving the least number of Taylor series terms and print the approximated value, the actual value and the error in approximation. Do not create any implicit custom functions in your code. ⋆

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$

   - Use the header file `math.h`, which has in-built function `sin()` to get the actual value of $\sin(\pi/2)$.
   - The function `fabs()` from `math.h` is used for modulus or the absolute value of its argument.
   - The constant π is stored as `M_PI` in `math.h`.
   - error in approximation = |approximated value - actual value| .

3. Write a program in C, which approximates the value of $\cos(\pi/2)$ using the following Taylor Series for cosine function. Do the approximation with a 0.01% precision involving the least number of Taylor series terms and print the approximated value, the actual value and the error in approximation. Do not create any implicit custom functions in your code. ⋆

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

   - Use the header file `math.h`, which has in-built function `cos()` to get the actual value of $\cos(\pi/2)$.
   - The function `fabs()` from `math.h` is used for modulus or the absolute value of its argument.
   - The constant π is stored as `M_PI` in `math.h`.
   - error in approximation = |approximated value - actual value| .

4. Write a program in C, which takes members of two sets as inputs and stores them in two 1D-arrays, and gives all the members of their cross-product as output. Cardinality of each input set is 4.

5. Write a program in C, which takes members of a set as inputs and stores them in a 1D-array, and gives all the members of its power set as output. Cardinality of the input set is 3. ⋆⋆

Logical Operators	Symbol in theory	Symbol in C syntax
Bitwise NOT	$\neg$	~
Bitwise AND	$\wedge$	&
Bitwise OR	$\vee$	|
Bitwise XOR	$\oplus$	^

9.5 Problem Set 05 - Nested loops and theory related computation problems

1. For positive real numbers a and r (where $0 < r < 1$), the sum of the infinite geometric series is given by:

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}.$$

   Write a program in C that takes three real numbers a, r, c as inputs, where $a > 0$, $0 < r < 1$, and $0 < c < 1$. The program should output the smallest integer k such that

$$\left(\frac{a}{1-r}\right) - (a + ar + ar^2 + \cdots + ar^k) < c.$$

   Additionally, the program must print the values of $(\frac{a}{1-r})$ and the partial sum $(a+ar+ar^2 + \cdots + ar^k)$. ⋆

2. Write a program in C programming language which takes two positive integers k and n as inputs from the user, and outputs the equivalence class of $\mathbb{Z}_n$ in which k belongs. For example, for $n = 7$, $\mathbb{Z}_7 = \{\hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5}, \hat{6}\}$ has 7 equivalence classes and $k = 25 \in \hat{4}$. Do not use the in-built '%' operator, even once in the entire program.

3. For $x \in \mathbb{R}$, the Taylor Series for exponential function is given by:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Write a program in C that takes c $(0 < c < 1)$ and $x \in \mathbb{R}^+$ as inputs and computes the above series for until the following happens for the smallest possible integer k:

$$\Big| \underbrace{(e^x)}_{actual} - \underbrace{(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!})}_{approx.} \Big| < c.$$

As output, print the actual and the approximated value for e^x. Use the header file `math.h`, which has in-built function `exp()` to get the actual value of e^x and the function `fabs()` for the absolute value of its argument. Both needs argument of double data type.

4. Write a program in C programming language which takes a binary string of length 5 as input, stores it in a 1D array and checks whether it is palindrome or not. *Can this be extended such that the program takes a positive integer as input, and checks if it is a palindrome or not?* $\star$

9.6 Problem Set 06 - Introduction to functions in C

Definition 9.6.1. *($\mathcal{O}-$ notation) For functions $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$ we say that*

$$f(x) = \mathcal{O}\big(g(x)\big),$$

if there exists a constant $C \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that we have $|f(n)| \leq C \cdot |g(n)|$, $\forall n \geq n_0$.

Lemma 9.6.2. *All polynomials are $\mathcal{O}$ of their term with the highest power. Let $f : \mathbb{N} \to \mathbb{R}$ be a polynomial of degree k, i.e, $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$, where $a_i's$ are real numbers, and $g : \mathbb{N} \to \mathbb{R}$ be the $k-$degree monomial, i.e, $g(x) = x^k$, then $\forall k \in \mathbb{N}$ we have*

$$f(x) = \mathcal{O}\big(g(x)\big).$$

Proof. *left as homework.*

1. Write a program in C programming language which:

   step i. takes positive integer k as input,
   step ii. takes the real constants $a_0, a_1, \cdots, a_k$ as inputs and stores them in an 1D array,
   step iii. takes a real number $r > |a_k| + k$ as input and then finally

   gives the smallest natural number n_0 as output for which $a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$ is $\mathcal{O}(x^k)$.

2. The following given are the recursive and the closed forms of some functions from $\mathbb{N}$ to $\mathbb{N}$. Write programs in C programming language for each of them which takes a positive integer k as input and computes the image of the function using their recursive forms and closed forms. Define and use custom made implicit functions in your program appropriately.

.	Function	Recursive Form	Closed Form	Comment
1.	$S(n)$	$S(n-1)+n$	$\frac{n(n+1)}{2}$	sum of first n natural no.s
2.	$S_0(n)$	$S_0(n-1)+(2n-1)$	n^2	sum of first n odd natural no.s
3.	$S_s(n)$	$S_s(n-1)+n^2$	$\frac{n(n+1)(2n+1)}{6}$	sum of squares of first n natural no.s

3. Write a program in C programming language which takes a positive integer n as input and gives as output the n^{th} term of the Fibonacci sequence, computed both recusively and iteratively. Define and use custom made implicit functions in your program for each of the two ways of computation. **Note:** for Fibonacci sequence $F_1 = F_2 = 1$.

4. Write a program in C programming language which takes a positive integer n as input and gives as output the $n!$, computed both recusively and iteratively. Define and use custom made implicit functions in your program for each of the two ways of computation. **Note:** for factorials, $0! = 1$.

9.7 Problem Set 07 - Recursive functions in C-I

1. **Comparisons in Binary Search**
   Let $B(n)$ denote the number of comparisons needed to search an element in a sorted array of size n. The recurence relation $\forall n \in \mathbb{N}$ of this function is:

   $$B(n) = \begin{cases} 1 & n = 1, \\ B(n/2) + 1 & otherwise. \end{cases}$$

   Also, the closed form for the same function is:

   $$B(n) = \log_2 n + 1, \qquad \forall n \in \mathbb{N}.$$

   Write a program in C programming language which takes a positive integer k as input from the user and computes $B(k)$ both using the recursive relation and the closed form and prints them as output.
   ▶ *specifications & notes* :

   1. Define two custom functions, one each for the recurence computation and closed form evaluation.
   2. Use `log2(` _*double*_ `)` from `math.h` for calculating $\log_2 n()$.

2. **Babylonian Method** ⋆
   A square root approximation for any positive rational number r has the following recurence relation:

   $$a_i = \begin{cases} r/2 & i = 0, \\ \frac{1}{2}\left(a_{i-1} + \frac{r}{a_{i-1}}\right) & otherwise. \end{cases}$$

   Write a program in C programming language which takes a positive reals r and c $(0 < c < 1)$ as inputs from the user and recursively approximates $\sqrt{r}$ untill we have

   $$\left|\sqrt{r} - a_k\right| \leq c.$$

As output we need the approximated square root and the smallest integer k for which the above happens.

▶ *specifications & notes* :

1. Define custom function for recursive computation.
2. Use `fabs( _double_ )` from `math.h` for absolute value.
3. Use `sqrt( _double_ )` from `math.h` for square root.

3. **Moves in Tower of Hanoi**

Let $H(n)$ denote the number of moves needed to solve the Tower of Hanoi problem with n disks. The recurence relation $\forall n \in \mathbb{N}$ of this function is:

$$H(n) = \begin{cases} 1 & n = 1, \\ 2H(n-1)+1 & otherwise. \end{cases}$$

Also, the closed form for the same function is:

$$B(n) = 2^n - 1, \qquad \forall n \in \mathbb{N}.$$

Write a program in C programming language which takes a positive integer k as input from the user and computes $H(k)$ both using the recursive relation and the closed form and prints them as output.

▶ *specifications & notes* :

1. Define two custom functions, one each for the recurence computation and closed form evaluation.

4. **Newton-Raphson Method** ⋆

A cube root approximation for any positive rational number r has the following recurence relation:

$$a_i = \begin{cases} r/2 & i = 0, \\ \frac{1}{3}\left(2a_{i-1} + \frac{r}{(a_{i-1})^2}\right) & otherwise. \end{cases}$$

Write a program in C programming language which takes a positive reals r and c ($0 < c < 1$) as inputs from the user and recursively approximates $\sqrt[3]{r}$ untill we have

$$\left|\sqrt[3]{r} - a_k\right| \leq c.$$

As output we need the approximated cube root and the smallest integer k for which the above happens.

▶ *specifications & notes* :

1. Define custom function for recursive computation.
2. Use `fabs( _double_ )` from `math.h` for absolute value.
3. Use `cbrt( _double_ )` from `math.h` for cube root.

9.8 Problem Set 08 - Recursive functions in C-II

1. **Sum of Digits**
   By now, you are well aware about the problem of computing the sum of digits of a positive integer. Write a program in C, which takes a positive integer as an input and computes the sum of its digits both recursively & iteratively.
   ► *specifications & notes* :

   1. Define two custom functions, one each for the recurence computation and closed form evaluation.

2. **Viete Method** ⋆
   In 1593, François Viète published a way to express the reciprocal of π as the following infinite product of nested radicals:

$$\frac{1}{\pi} = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots = \frac{1}{2}\prod_{n=0}^{\infty} \frac{a_i}{2},$$

$$a_i = \begin{cases} \sqrt{2} & i = 0, \\ \sqrt{2 + a_{i-1}} & otherwise. \end{cases}$$

   Write a program in C programming language which takes a positive real c $(0 < c < 1)$ as input from the user and recursively approximates $\frac{1}{\pi}$ (which will give π_{approx}) untill we have

$$\left|\pi_{actual} - \pi_{approx}\right| \leq c.$$

   As output we need the approximated π and the smallest last index i for which the above happens.
   ► *specifications & notes* :

   1. Define custom function for recursive computation.
   2. Use `fabs(` *_double_* `)` from `math.h` for absolute value.
   3. Use `sqrt(` *_double_* `)` from `math.h` for square root.
   4. Use `M_PI` from `math.h` for actual π.
   5. Since the recursion formula is being used for each $i = 0, 1, 2 \ldots k$, is there a way to use the value from the pervious step in the current step?

3. **Reverse an array**
   By now, you are well aware about the problem of reversing an integer. Write a program in C, which takes a positive integer as an input and computes its reverse recursively only.
   ► *specifications & notes* :

   1. Define a custom function, for the recurence computation.
   2. **Do not** use `pow(`*_double_*, *_double_*`)` from `math.h` .

9.9 Problem Set 09 - Simulating probability distributions in C using rand( )

1. Write a program in C programming language to simulate the **'single-coin tossing experiment'** for an unbiased coin. Clearly, the sample space is $\Omega = \{H, T\}$ and if this experiment realises the probability distribution $\mathbb{P} : \Omega \to [0, 1]$ then your code must statistically verify that $\mathbb{P}$ is the uniform probability distribution.

2. Write a program in C programming language to simulate the **'single-coin tossing experiment'** for a *biased coin*. The *bias* is such that, H appears twice the number of times T appears. Clearly, the sample space is $\Omega = \{H, T\}$ and if this experiment realises the probability distribution $\mathbb{P} : \Omega \to [0, 1]$ then your code must statistically verify that $\mathbb{P}$ is **not** the uniform probability distribution.

3. Write a program in C programming language to simulate the **'single-dice throwing experiment'** for an unbiased dice. Clearly, the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$.

   (a) If this experiment realises the probability distribution $\mathbb{P} : \Omega \to [0, 1]$ then your code must statistically verify that $\mathbb{P}$ is the uniform probability distribution.

   (b) If $A \subseteq \Omega$ is the event defined as $A \coloneqq \{\omega \in \Omega \mid \omega = 0 \ mod \ 2\}$, then compute $\mathbb{P}(A)$ using your code.

   (c) If $B \subseteq \Omega$ is the event defined as $B \coloneqq \{\omega \in \Omega \mid \omega \leq 4\}$, then compute $\mathbb{P}_A(B)$ and $\mathbb{P}_B(A)$.

   (d) If $C \subseteq \Omega$ is the event defined as $C \coloneqq \{\omega \in \Omega \mid \omega = 1 \ mod \ 2\}$, then compute $\mathbb{P}(A \cap C)$ using your code and thus verify that events A and C are mutually disjoint events.

4. Consider the **'double-die throwing experiment'** for two unbiased die, i.e., two unbiased die are thrown simultaneously. If $\Omega = \{1, 2, 3, 4, 5, 6\}$, then the sample space is $\Omega \times \Omega$. Write a program in C programming language to simulate this experiment and then use your code to compute the following, when this experiment realises the probability distribution $\mathbb{P} : \Omega \times \Omega \to [0, 1]$. $\star$

   (a) If $A \subseteq \Omega \times \Omega$ is an event such that $A \coloneqq \{(s, t) \in \Omega \times \Omega \mid s + t = 0 \ mod \ 2\}$, then compute $\mathbb{P}(A)$.

   (b) If $B \subseteq \Omega \times \Omega$ is an event such that $B \coloneqq \{(s, t) \in \Omega \times \Omega \mid s + t = 0 \ mod \ 4\}$, then compute $\mathbb{P}(B)$ and numerically verify that $\mathbb{P}(B) \leq \mathbb{P}(A)$. Any theory explaining as why this is happening?

5. Consider experiment Π named **'Chausar'**, where two 4-sided die are thrown simultaneously. Die-01 has marked 1, 2, 3, 4 whereas die-02 has faces marked 5, 6, 7, 8. The sample space is $\Omega_1 \times \Omega_2$ where $\Omega_1 = \{1, 2, 3, 4\}$ and $\Omega_2 = \{5, 6, 7, 8\}$. Write a program in C programming language to simulate Π and then use your code to compute the following, when Π realises the probability distribution $\mathbb{P} : \Omega_1 \times \Omega_2 \to [0, 1]$. In Π, we play Chausar using two unbiased die. $\star\star$

   (a) If $A \subseteq \Omega_1 \times \Omega_2$ is an event such that
   $A \coloneqq \{(s, t) \in \Omega_1 \times \Omega_2 \mid s \text{ and } t \text{ differs at most by } 3\}$, then compute $\mathbb{P}(A)$.

   (b) If $B \subseteq \Omega \times \Omega$ is an event such that $B \coloneqq \{(s, t) \in \Omega_1 \times \Omega_2 \mid s + t = 1 \ mod \ 2\}$, then compute $\mathbb{P}(B)$.

6. Consider Π_1, a **'random walk'** experiment, where a walk is performed along the X-axis, starting from 0. At each step:

   - the walker can either move forward by a unit distance with probability $3/4$, or
   - the walker can move backward by a unit distance with probability $1/4$.

   Write a program in C programming language which simulates this random walk Π_1, such that it takes a positive integer n as input and determines as on to which direction the walker ends after this random walk? What is the distance traversed? ⋆

7. Consider Π_2, a **'random walk'** experiment, where a walk is performed in the 1^{st} quadrant of the XY-plane, starting from 0. At each step:

   - the walker can either move along the Y-axis by a unit distance with probability $1/2$, or
   - the walker can move along the X-axis by a unit distance with probability $1/2$.

   Write a program in C programming language which simulates this random walk Π_2, such that it takes a positive integer n as input and determines on which point in the XY-plane the walker ends after this random walk? The output must be coordinates of a point in the XY-plane. Is it counter-intuitive that the x and y coordinate of the final point is always the same? ⋆

9.10 Problem Set 10 - Matrices & Graphs in C

Problems based on graph categorisation

1. Write a program in C programming language which takes the *adjacency matrix* of a general undirected graph G as input (element-wise). The program should detect the presence of any loops in G, and if any it must count the number of loops.

2. Write a program in C programming language which takes the *adjacency matrix* of a simple undirected graph G as input (element-wise). The program must count the total number of edges and the sum of the degrees of all the vertices in G. Incorporate relevant results you have seen in the Theory of Graphs.

3. Write a program in C programming language which takes the *adjacency matrix* of an undirected multigraph G as input (element-wise). The product must return the count of the total number of multi-edges present in G.

4. Write a program in C programming language which takes the *adjacency matrix* of a general graph G as input (element-wise). The program must detect whether the corresponding graph G is a directed graph or an undirected graph. ⋆

Problems based on DFS

1. Write a program in C programming language which takes the number vertices, the number of edges and the edges (vertex pairs) of a tree as inputs. For a given starting vertex, the program must perform a DFS to find the last vertex of the tree. The output must be the path traversed during the search. ⋆

2. Write a program in C which counts the number of connected components for a given graph, using a DFS. ⋆

3. Write a program in C which detects cycles in a given undirected graph, using a DFS approach. ⋆⋆

Ideas

Now that you have seen how to generate *pseudo-random* integers in C using the `rand()` function, consider the following program (9.10) which generates a random integer matrix, equivalent to a typical *adjacency matrix* for an undirected graph G, which does not have multi-edges. Can this concept be implemented in the above listed programs?

```c

#include <stdio.h>
#include <time.h>
#include <stdlib.h>

int main()
{
        srand(time(NULL));

        int n;
        printf("Enter␣the␣order␣of␣the␣graph:␣");
        scanf("%d", &n);

        int G[n][n];

        for(int i=0; i < n; i++)
        {
                for(int j=0; j < n; j++)
                {
                        //...scope for meaningful experimentations
                        G[i][j] = rand() % 2;

                }
        }

        printf("\n");
        printf("The␣generated␣random␣matrix␣is␣:\n");

        for(int i=0; i < n; i++)
        {
                for(int j=0; j < n; j++)
                {
                        printf("%d␣", G[i][j]);
                }
                printf("\n");
        }

        return 0;
}
```

EXTRAS

10.1 Problem Set 08 - Asymptotics-I (solutions)

1. Find the $\Theta-$bounds for the following recurrences.

   (a) $T(n) = 4T(n/2) + c$ where $T(1) = c_0$

$$\begin{aligned}
T(n) &= 4T(n/2) + c \\
4T(n/2) &= 16T(n/4) + 4c \\
16T(n/4) &= 64T(n/8) + 16c \\
64T(n/8) &= 256T(n/16) + 64c \\
&\vdots \\
4^{k-2}T(n/2^{k-2}) &= 4^{k-1}T(n/2^{k-1}) + 4^{k-2}c \\
4^{k-1}T(n/2^{k-1}) &= 4^{k}T(n/2^{k}) + 4^{k-1}c
\end{aligned}$$

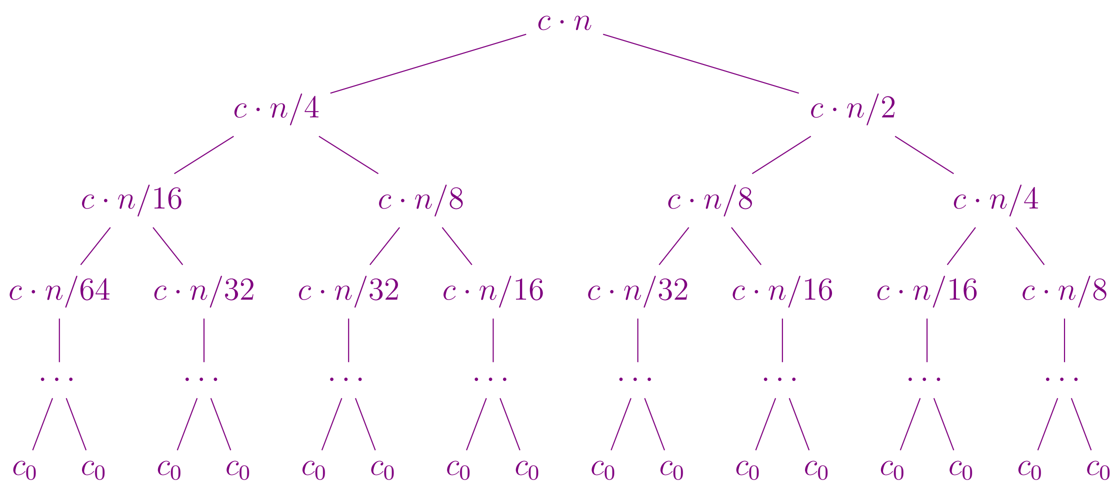
Let n such that, $n/2^k = 1$. Then $n = 2^k$ and,

$$\rightarrow \log_2 n = k \times \log_2 2 = k$$

$$\rightarrow 4^{k-1}T(n/2^{k-1}) = 4^k c_0 + 4^{k-1}c$$

Now, add all the recurrences, and cancel out all terms except T(n), $4^k c_0$ and the terms accounting for the constant cost c for each branching of the recurrence tree.
$T(n) = 4^{\log_2 n}c_0 + c(4 + 16 + 64 + \cdots + 4^{\log_2 -1})$
$\Rightarrow T(n) = n^{\log_2 4}c_0 + c(\frac{4(4^{\log_2 n-1}-1)}{4-1})$
$\Rightarrow T(n) = n^2 c_0 + c(\frac{n^{\log_2 4}-4}{3})$
$\Rightarrow T(n) = n^2 c_0 + \frac{c(n^2)}{3} - \frac{4c}{3}$
$\Rightarrow T(n) = n^2(\frac{3c_0+c}{3}) - \frac{4c}{3}$
$\Rightarrow T(n) \in \Theta(n^2)$

   (b) $T(n) = T(n/4) + T(n/2) + n \cdot c$ where $T(1) = c_0$



The cost of each level in the recursion tree is as follows:

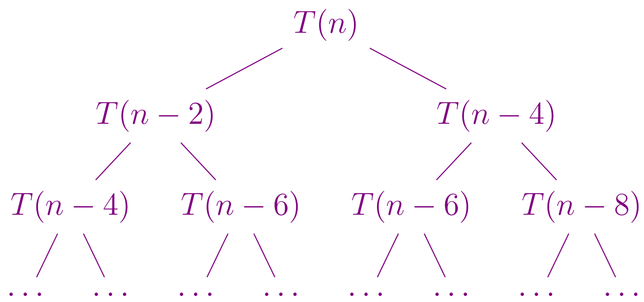
1. $c \cdot n$

2. $c \cdot \frac{n}{4} + c \cdot \frac{n}{2} = \frac{3cn}{4}$
3. $c \cdot \frac{n}{16} + c \cdot \frac{n}{8} + c \cdot \frac{n}{8} + c \cdot \frac{n}{4} = \frac{9cn}{16}$
4. $c \cdot \frac{n}{16} + c \cdot \frac{n}{32} + c \cdot \frac{n}{32} + c \cdot \frac{n}{16} + c \cdot \frac{n}{32} + c \cdot \frac{n}{16} + c \cdot \frac{n}{16} + c \cdot \frac{n}{8} = \frac{27cn}{64}$

We see that at each level the work is being multiplied by $\frac{3}{4}$.
(**Exercise** : Prove this with Induction).
Thus, the total work done is,

$$\begin{aligned}
\sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i c \cdot n &= cn + \frac{3cn}{4} + \frac{9cn}{16} + \dots \\
&= \frac{cn}{1-\frac{3}{4}} && \text{(GP Sum)} \\
&= 4cn \\
&\in \Theta(n) && (c_0 = \tfrac{1}{4(c+1)},\ c_1 = 5(c+1))
\end{aligned}$$

(c) $T(n) = T(n-2) + T(n-4)$, where $T(0) = T(1) = T(2) = T(3) = c_0$.



We have : $T(n) = T(n-2) + T(n-4)$.
$n = 2k \ \Rightarrow T(2k) = T(2(k-1)) + T(2(k-2))$
Let $a_k = T(2k)$
The resulting recurrence is: $a_k = a_{k-1} + a_{k-2}$
Using the fibonacci recurrence : $\Rightarrow a_k \in \Theta(\varphi^n)$
Now, $T(0) = T(1) = T(2) = T(3) = c_0$
$\Rightarrow \frac{\varphi^n}{1+c_0} \leq T(n) \leq (1+c_0)\varphi^n$
$\Rightarrow T(n) \in \Theta(\varphi^n)$

(d) $T(n) = T(n-1) + T(n-2) + k$, where $T(0) = 0$ and $T(1) = 1$.

2. Solve the following linear-homogeneous recurrences and comment on their $O-$bounds.

(a) $F(n) = 7F(n-1) - 12F(n-2),\ n \geq 2$ and $F(0) = 5, F(1) = -5$.
$F(n) = 25.3^n - 20.4^n$

(b) $F(n) = F(n-1) + 2F(n-2),\ n \geq 3$ and $F(1) = 0, F(2) = 6$.
$F(n) = 2.(-1)^n + 2^n$

(c) $F(n) = -F(n-1) + 4F(n-2) + 4F(n-3),\ n \geq 3$ and $F(0) = 8, F(1) = 6, F(2) = 26$.
$F(n) = 2.(-1)^n + (-2)^n + 5.2^n$

(d) $F(n) = 4F(n-1) - 4F(n-2),\ n \geq 3$ and $F(1) = 1, F(2) = 3$.
$F(n) = (1+n).2^{n-2}$

(e) $F(n) = 8F(n-1) - 16F(n-4)$, $n \geq 4$ and $F(0) = 1, F(1) = 4, F(2) = 28, F(3) = 32$.
$F(n) = (1+2n).2^n + n.(-2)^n$

(f) $F(n) = -3F(n-1) - 3F(n-2) - F(n-3)$, $n \geq 3$ and $F(0) = 1, F(1) = -2, F(2) = -1$.
$F(n) = (1+3n-2n^2).(-1)^n$

3. For function f, g and h mapping from $\mathbb{N}$ to $\mathbb{R}^+$, prove the following:

   (a) If $f = O(h)$ and $g = O(h)$, then $f+g = O(h)$, where $f+g : \mathbb{N} \to \mathbb{R}^+$ defined as $\big(f+g\big)(x) = f(x) + g(x)$.

   (b) If $f = O(h)$ and $g = O(h)$, then $f \cdot g = O(h)$, where $f \cdot g : \mathbb{N} \to \mathbb{R}^+$ defined as $\big(f \cdot g\big)(x) = f(x)g(x)$.

4. For $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, show that $p(x) = O(x^n)$.

5. Prove the following asymptotic identities for $n \in \mathbb{N}$:

   (a) $n! = O(n^n)$.
   $1 \leq n;\ 2 \leq n;\ 3 \leq n;\ 4 \leq n; \cdots;\ n \leq n$. Thus we get :
   $1 \cdot 2 \cdot 3 \cdots \cdot n \leq \underbrace{n \cdot n \cdot n \cdots \cdot n}_{n\ times}$
   $\Rightarrow n! \leq n^n$.
   Thus for $c = 1$ and $n_0 = 1$ we have $n! = O(n^n)$.

   (b) $\log n! = O(n \log n)$.
   From part(a) we borrow $n! \leq n^n$
   $\Rightarrow \log n! \leq n \log n$.
   Thus for $c = 1$ and $n_0 = 1$ we have $\log n! = O(n \log n)$.

   (c) $\ln n = O(n)$.

6. Prove that for some $k \in \mathbb{N}$, $\sum_{i=1}^{k} i^8 = O(k^9)$. ⋆
   $1^8 \leq k^8;\ 2^8 \leq k^8;\ 3^8 \leq k^8;\ 4^8 \leq k^8; \cdots;\ k^8 \leq k^8$.
   $\Rightarrow 1^8 + 2^8 + \cdots + k^8 \leq \underbrace{k^8 + k^8 + \cdots + k^8}_{k\ times} = k^9$.
   Thus for $c = 1$ and $n_0 = 1$ we have $\sum_{i=1}^{k} i^8 = O(k^9)$.

7. Prove that for some $k \in \mathbb{N}$, $\sum_{i=1}^{k} i^8 = \Omega(k^9)$. ⋆⋆⋆
   Observe that $\sum_{i=1}^{k} i^8 = \underbrace{\sum_{i=1}^{\lceil k/2 \rceil - 1} i^8}_{I} + \underbrace{\sum_{i=\lceil k/2 \rceil}^{k} i^8}_{II}$.
   In summation II, each term is at least $\big(\lceil k/2 \rceil\big)^8$. Further the number of terms in summation II is $k - \lfloor k/2 \rfloor$ which is approximately $k/2$.
   Hence, $\sum_{i=1}^{k} i^8 \geq \frac{k}{2} \cdot \big(\lceil \frac{k}{2} \rceil\big)^8 \geq \frac{k}{2} \cdot \big(\frac{k}{2}\big)^8 = \frac{k^9}{2^9}$.
   Thus for $c = 2^{-9}$ and $n_0 = 1$ we have $\sum_{i=1}^{k} i^8 = \Omega(k^9)$.

8. Prove that $n^2 + 17n = n^2 + o(n \ln n)$. ⋆⋆⋆
   To prove that $n^2 + 17n = n^2 + o(n \ln n)$ boils down to show that $n = o(n \ln n)$.
   Observe that $\lim_{n \to \infty} \frac{n}{n \ln n} = \lim_{n \to \infty} \frac{1}{\ln n} = 0$. By definition, for $c = 1$ and $n_0 = 1$ we have $n = o(n \ln n)$. This completes the proof. *qed*

10.2 Problem Set 12 - Graphs-I (solutions)

To test command over basic definitions & notations

1. Consider a *directed* graph G. Prove that graph G being strongly connected implies that G is weakly connected, and not the other way around.
Exercise.

2. Consider a *directed* graph G. Prove that graph G being a Null graph implies that G is also an Empty graph. Provide a counter-example to show that the other way around is not always true.
Exercise.

3. The general graph shown in the following figure(4) goes by the name of *GraphBuster* in standard literature. Count and determine the cardinality of V and E in *GraphBuster*.

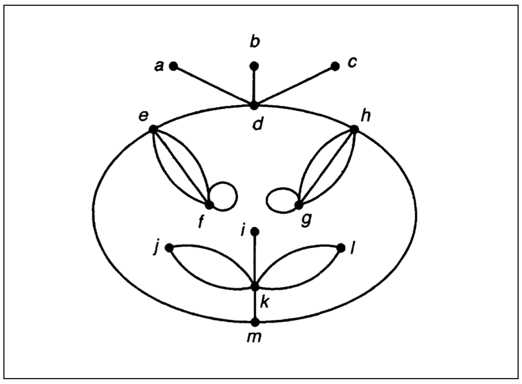


Figure 4: GraphBuster

GraphBuster has order order 13 with 21 edges.

4. A graph of order n is called *complete*, denoted by K_n provided that each pair of distinct vertices forms an edge. Show that a complete graph of order n has $n(n-1)/2$ edges.
Number edges equals total posible ways to choose 2 vertices from the n vertices in k_n, i.e., $\binom{n}{2} = \frac{n(n-1)}{2}$. □

Problems of type ¬ (simple)

5. Let G be a general graph. Show that the sum of the degrees of all the vertices of G is an even number, and consequently, the number of vertices of G with odd degree is even.
Each edge in G contributes 2 to the overall sum of the degrees in G. Hence, sum of all the degrees in G is $2e$, where is the total number of edges in G. Clearly, $2e$ is even. Further, let e_e and e_o be the number of vertices with even and odd degrees respectively. e_o is difference of two even numbers, hence it is even. □

6. If G is a simple graph of order $n \geq 3$, such that for all pairs of distinct vertices x and y in G that are not adjacent, we have $deg(x) + deg(y) \geq n$, then show that G must be connected. ⋆
Proof by contradiction. Suppose that G is not connected. We then show that G cannot satisfy the Ore property : ∀ pairs of distinct vertices x and y that are not adjacent, $deg(x)+deg(y) \geq n$. Since G is not connected, its vertices can be partitioned

into two parts, U and W, in such a way that there are no edges joining a vertex in U with a vertex in W. Let r be the number of vertices in U and let s be the number of verticed in W. Then $r + s = n$, and each vertex in U has degree at most $r - 1$, and each vertex in W has degree at most $s - 1$. Let x be any vertex in U and let y be any vertex in W. Then x and y are not adjacent, but the sum of their degrees is, at most, $(r - 1) + (s - 1) = r + s - 2 = n - 2$, and this contradicts the Ore property. We conclude that if G satisfies the Ore property, then G must be connected. $\square$

7. Let G be the graph such that elements of $\{1, 2, 3, \cdots, 20\}$ form its vertices. In G, two vertices (integers) are joined by an edge if and only if their difference is an odd integer. Show that G is a bipartite graph.
Let $X = \{k \in \mathbb{N} \mid k = 1 \ mod\ 2; k \leq 20\}$ and $Y = \{k \in \mathbb{N} \mid k = 0 \ mod\ 2; k \leq 20\}$. Observe that X and Y together is a bipartition for the graph G. $\square$

8. Prove that if a multigraph G is bipartite, then each of its cycles has even length. *Note that: length of any cycle/path is the number of edges it is composed of.*
G is a bipartite multigraph with bipartition X, Y. The vertices of a walk of G must alternate between X and Y. Since a cycle is closed, this implies that a cycle contains as many left vertices as it does right vertices and hence has even length. $\square$

9. For a fixed $n \in \mathbb{N}$, let $\mathrm{G_n}$ be the graph such that elements of $\{0, 1\}^n$, the set of all $n-$length binary strings, form its vertices. In $\mathrm{G_n}$ any two vertices are joined by an edge if and only if they differ in exactly one 1-bit. Show that G is a bipartite graph. $\star$
Proof. Let n be a positive integer. We consider the set of all $n-$tuples of 0s and 1s as the vertices of a graph Q_n with two vertices joined by an edge if and only if they differ in exactly one coordinate. If $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n)$ are joined by an edge, then the number of 1s in Y is either one more or one less than the number of 1s in X. Let X consist of those $n-$tuples that have an even number of 1s; let Y consist of those $n-$tuples that have an odd number of 1s. Then two distinct vertices in X differ in at least two coordinates and hence are not adjacent. Similarly, two distinct vertices in Y are not adjacent. Hence, Q_n is a bipartite graph with bipartition X, Y. $\square$

10. Prove that a graph of order n with at least

$$\frac{(n-2)(n-1)}{2} + 1$$

edges must be connected.
Proof by contradiction. Assume that G is a disconnected graph of order n such that the number of edges is at least $\frac{(n-2)(n-1)}{2} + 1$. WLOG, say there are only two components in G such that one component is a singleton vertex and the other is of order $n - 1$. It follows that, the non-trivial component can have $\frac{(n-1)(n-2)}{2}$ edges at max whenever it is complete. Observe that, in that case the number of edges that G would have is $\frac{(n-1)(n-2)}{2}$, which is a contradiction to the primal assumption of our proof. Therefore, G is connected. $\square$

11. Prove that a graph of order n with every vertex having degree at least $\frac{n}{2}$ must be connected.
Proof by contradiction. We borrow the construction/set-up of a graph G exactly from the previous proof. Observe that, once again the non-trivial component would have a degree $(n - 2)/2$ at max, which is lesser than $n/2$. Hence, a contradiction. $\square$

12. In a simple graph if two vertices x and y are joined by a path then, show that they are also joined by a simple path.
Exercise.

10.3 Problem Set 13 - Graphs-II (solutions)

Problems of type (simple)

1. The two graphs shown below in figure (5) have the same number of vertices and edges. Prove that despite these they are not isomorphic.

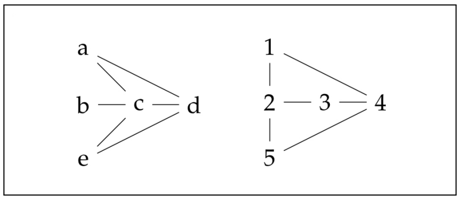


Figure 5: Sample graphs 01

$deg(b) = 1$ in the first graph, but is no vertex in the second graph with this property.

2. Consider the two graphs shown below in figure (6), where both the graphs have same degree sequence $(3, 3, 3, 3, 3, 3)$. Show that despite this, they are not isomorphic.

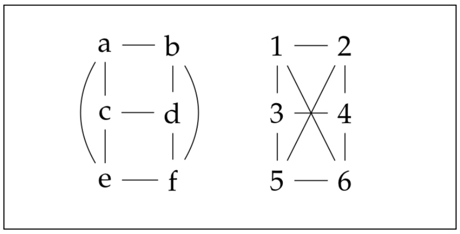


Figure 6: Sample graphs 02

Note that (a, c, e) forms a 'triangle' in that any two pairs are adjacent. There is no such triangle in the second graph, so these graphs are not isomorphic.

3. A graph has 26 vertices and 58 edges. There are five vertices of degree 4, six vertices of degree 5, and seven vertices of degree 6. If the remaining vertices all have the same degree, what is this degree?
There are $26 - 5 - 6 - 7 = 8$ vertices of degree x. Applying Euler's Theorem, we get $5 \cdot 4 + 6 \cdot 5 + 7 \cdot 6 + 8 \cdot x = 2 \cdot 58$. On solving, $x = 3$.

4. A graph has 24 vertices and 30 edges. It has five vertices of degree 4, seven pendant vertices, and seven vertices of degree 2. All other vertices have degree 3 or 4. How many vertices of degree 4 are there?
Let x be the number of vertices of degree 3, and y the number of vertices of degree 4. The order of the graph is 24 therefore : $5 + 7 + 7 + x + y = 24$. After applying Euler's Theorem, we get $5 \cdot 4 + 7 \cdot 1 + 7 \cdot 2 + 3 \cdot x + 4 \cdot y = 3 \cdot 30$. On solving, $y = 4$.

Problems of type $\neg$ (simple)

5. Prove or Disprove, whether a bipartite graph can have K_3 as its subgraph?
A bipartite graph cannot have a K_3 subgraph, as there would be three mutually adjacent vertices in the graph, requiring three different partite sets, contradicting that the graph is bipartite.

6. Let $d_0(G)$ be the least among the degrees of the vertices of an $n-$vertex graph G. Prove that if $d_0(G) \geq (n-1)/2$, then the graph G is connected. ⋆
Proof by contradiction : Assume that graph G is not connected and $d_0(G) \geq (n-1)/2$. Let one component have m vertices, where $1 \leq m \leq \lfloor n/2 \rfloor$. Then the maximum degree of any vertex in that component is at most $m-1$, since it is only connected to vertices within that component. Therefore, at least one vertex in G has degree $\lfloor n/2 \rfloor - 1 < (n-1)/2$. This contradicts the fact that $d_0(G) \geq (n-1)/2$, hence G is connected.

7. Prove that a graph G with v vertices and e edges has at least $v-e$ connected components. [Hint : use induction on e.]
BASE CASE : $e = 0$; then each of the v vertices of G are connected components in their own right. Hence, number of connected components $= v - 0$.
INDUCTIVE HYPOTHESIS : $e = k$, the graph G of order v has $v-k$ connected components.
INDUCTIVE STEP : $e = k+1$; consider the Graph G from the IH, where all k edges have been encompases by its connected components. Hence, for the additional edge, we will have to joing two connected components. It follows, that due to the additional edge there will be a decrease in the number of connected components, so the resultant graph has $v - k - 1 = v - (k+1)$ connected components.

8. Prove that a connected graph G with n vertices contains at least $n-1$ edges. [Hint : the proof might be an application of the result in the previous question, when proved!]
Any connected graph G, trivially has 1 connected component. Hence the proof is a direct application of the result in question 7.

9. If G is a connected graph with v vertices and e edges, then $v \leq e+1$.
Proof by contradiction : Let us assume that graph G is a connected graph of order v with e edges, such that $v > e+1$. Since G is connected, there must be at least one edge corresponding to each vertex. Considering this, $v > e+1$ constraint is a contradiction to G being connected. Hence $v \leq e+1$.

10. If G is a connected graph, then removing an edge from a cycle will not make G a disconnect graph. ⋆
Arbitrarily select a vertices v_i, v_j, part of a cycle in G. Any cycle starting from and ending on a vertex v_i, which includes v_j is of the form :

$$v_i - v_{k_1}, v_{k_1} - v_{k_2}, v_{k_2} - v_{k_3}, \cdots, v_{k_m} - v_j, v_j - v_{l_n}, \cdots, v_{l_1} - v_i,$$

such that $[v_i - v_{k_1}, \cdots, v_{k_m} - v_j]$ and $[v_j - v_{l_n}, \cdots, v_{l_1} - v_i]$ are two distinct simple paths between the vertices v_0 and v_n. Thus removing an edge from the cycle is equivalent to remove an edge from either of the two simple paths. The other one remains intact connecting v_0 and v_n. This completes the proof. □

10.4 Problem Set 14 - Graphs-III (solutions)

1. A graph $G = (V, E)$ is called k-regular if $\mathsf{deg}(v) = k$ for all $v \in V$. A graph is called regular if it is k-regular for some k. Give example of a regular bipartite graph.
   All complete bipartite graphs are regular, whenever the order of the bipartitions are equal. **Food for thought:** can be there other examples?

2. Prove that every induced subgraph of a complete graph is complete.
   A *complete graph* K_n is a graph where every pair of distinct vertices is connected by an edge. An *induced subgraph* H of a graph G is formed by taking a subset of vertices $V(H) \subseteq V(G)$, and including all edges from G that exist between these vertices.

   Let $G = K_n$ be a complete graph with vertex set V. Let H be an induced subgraph of G, formed by a subset $V(H) \subseteq V$. By the definition of an induced subgraph:

   > For every pair of vertices $u, v \in V(H)$, the edge $uv \in E(H)$ if and only if $uv \in E(G)$.

   Since G is complete, every pair $u, v \in V(G)$ is connected by an edge in G. Therefore, for any $u, v \in V(H)$, the edge $uv \in E(G)$, and thus it must also be in $E(H)$. Hence, every pair of distinct vertices in H is connected, and so H is a complete graph. $\square$

3. Prove that every subgraph of a bipartite graph is bipartite.
   Proof by contradiction. Let G be a bipartite graph with bipartitions X, Y. We consider a subgraph $G_1 \subseteq G$, such that G_1 is not complete. We borrow the result that subgraph of a bipartite graph is again bipartite. It follows that, $\exists\ v_x \in X$ and $v_y \in Y$ which are not adjacent. But this is a contradiction as G is assumed to be complete. This is forms *good-enough* sketch of the complete proof. $\square$

4. If two graphs G_1 and G_2 are isomorphic then their degree sequences are the same. Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic graphs. Let $f : V_1 \to V_2$ be the isomorphism between them. Since f preserves adjacency, the number of neighbors of any vertex $v \in V_1$ is equal to the number of neighbors of $f(v) \in V_2$. That is,

   $$\deg_{G_1}(v) = \deg_{G_2}(f(v)).$$

   Thus, for every vertex in G_1, there exists a corresponding vertex in G_2 with the same degree. Therefore, the set (or multiset) of degrees in G_1 is identical to that in G_2. Therefore, the degree sequences of G_1 and G_2 are the same. $\square$

5. What is the sum of the entries in a row of the adjacency matrix of an undirected simple graph?
   Degree of the vertex corresponding to that row.

6. Let u, v, and w be three distinct vertices in a graph. There is a path between u and v and also there is a path between v and w. Prove that there is a path between u and w.
   Let p_1 be the path connecting u and v; p_2 be the path connecting v and w. Then the resultant path $p_1 \cup p_2$ connects u and w. $\square$

7. Suppose $(d_1, \ldots, d_n)$ be a degree sequence of a tree. Determine $\sum_{i=1}^{n} d_i$.
   Exercise.

8. Show that the number of vertices n in a full binary tree is always odd.
   A binary tree is a tree in which every node has at most two children. For a *full binary tree* (i.e., every internal node has exactly two children), there is a well-known relationship, $n = 2i + 1$ where i is the number of internal nodes.

   Each internal node has exactly 2 children. So, the total number of children is $2i$. Each node, except the root, is a child of some node. Therefore, the total number of nodes n is : $n = \text{internal nodes} + \text{leaves} = i + (i + 1) = 2i + 1$. Since $2i + 1$ is always odd for any integer i, we conclude that the number of vertices n in a full binary tree is always odd. □

9. Let p be the number of pendant vertices in a binary tree T with n vertices. Show that

$$p = \frac{n+1}{2}.$$

   Exercise.

10. Let $k \in \mathbb{N}$ be the height of a binary tree T. Determine the maximum number of leaf nodes of T.
   Claim: The number of nodes at level k of a full binary tree is 2^k, where root node is present at level 0. Prove the claim using induction. Hence, the maximum number of leaf nodes of T, whenever $k \in \mathbb{N}$ is the height of a binary tree T equals to 2^k. □

11. Consider the graph defined by the adjacency matrix provided below.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 1 & 0 & 1 & 0 \\ & & & 0 & 0 & 1 & 0 & 1 \\ & & & & 0 & 0 & 1 & 1 \\ & & & & & 0 & 1 & 1 \\ & & & & & & 0 & 1 \\ & & & & & & & 0 \end{bmatrix}$$

   (a) Determine if it is an Euler graph.
   Exercise.

   (b) Determine if it admits a Hamiltonian circuit.
   Exercise.

   (c) Give a spanning tree of this graph.
   Exercise.

10.5 Problem Set 19 - Generating Functions (solutions)

1. Given the following sequences, determine the corresponding generating function as a summation and in closed form (as a formula).

   (a) $1, 2, 3, 4, \ldots$

   (b) $5, 4, 3, 0, 0, \ldots$

   (c) $1, -1, 1, -1, 1, -1, \ldots$

   (d) $\binom{10}{10}, \binom{11}{10}, \binom{12}{10}, \binom{13}{10}, \ldots$

   (e) $\binom{10}{10}, -\binom{11}{10}, \binom{12}{10}, -\binom{13}{10}, \ldots$

   (f) $1, 0, 1, 0, 1, \ldots$

   (g) $1, -2, 4, -8, 16, -32, 0, 0, 0, \ldots$

   The pattern for both this question and the one below it is, starting from 0, the i^{th} term in the sequence is the coefficient of x^i.

   (a) $1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n+1} + \cdots = \sum_{k=0}^{\infty} \binom{k+1}{1} x^k = \frac{1}{(1-x)^2}$

   (b) $5 + 4x + 3x^2 = \sum_{k=0}^{2} \binom{5-k}{1} x^k$.

   (c) $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots = \sum_{k=0}^{\infty} (-x)^k = \frac{1}{1+x}$.

   (d) $\binom{10}{10} + \binom{11}{10} x + \binom{12}{10} x^2 + \cdots = \sum_{k=0}^{\infty} x^k \binom{10+k}{10} = \frac{1}{(1-x)^{11}}$

   (e) $\binom{10}{10} - \binom{11}{10} x + \binom{12}{10} x^2 - \binom{13}{10} \cdots = \sum_{k=0}^{\infty} (-x)^k \binom{10+k}{10} = \frac{1}{(1+x)^{11}}$

   (f) $1 + x^2 + x^4 + \cdots = \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2}$

   (g) $1 - 2x + 4x^2 - 8x^3 + 16x^4 - 32x^5 = \sum_{k=0}^{5} (-2x)^k = \frac{1-(-2x)^6}{1-(-2x)} = \frac{1-64x^6}{1+2x}$

2. Given the following generating functions, determine the sequence that represents it.

   (a) $f(x) = 0$

   (b) $f(x) = x$

   (c) $f(x) = 4 + 3x - 10x^2 + 55x^3$

   (d) $f(x) = (3x - 4)^3$

   (e) $f(x) = \frac{3x}{1-x}$

   (f) $f(x) = \frac{1}{(1-3x)^2}$

   (a) $0, 0, 0, 0, \ldots$.

   (b) $0, 1, 0, 0, 0, \ldots$.

   (c) $4, 3, -10, 55$.

   (d) $-64, 144, -108, 27$. Since,

   $$(3x - 4)^3 = 27x^3 - 108x^2 + 144x - 64.$$

   (e) $0, 3, 3, 3, 3, \ldots$. Since,

   $$\frac{3x}{1-x} = 3x \cdot \sum_{k=0}^{\infty} x^k = 3x(1) + 3x(x) + 3x(x^2) + \ldots.$$

   (f) $1, 6, 27, 108, \ldots$. Since,

   $$\frac{1}{(1-3x)^2} = \sum_{k=0}^{\infty} \binom{k+1}{1} (3x)^k = \sum_{k=0}^{\infty} (k+1)(3x)^k.$$

3. Determine the coefficient of the specified term in the expansion of the given function.

(a) x^3 in $\frac{1}{1-x}$.

(b) x^2 in $\frac{1}{(1-2x)^3}$.

(c) x^5 in $\frac{(1-x^5)}{1-x}$.

(d) x^3 in $\frac{1}{(1+3x)^{10}}$.

(a) **1.** We know, $\frac{1}{1-x} = \sum_{r=0}^{\infty} x^r$. Therefore x^3 occurs when $r = 3$ which has a coefficient of 1.

(b) **24.** We know,

$$\frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} \binom{r+n-1}{n-1} x^r.$$

So we can determine that,

$$\frac{1}{(1-2x)^3} = \sum_{r=0}^{\infty} \binom{r+3-1}{3-1} (-2x)^r = \sum_{r=0}^{\infty} \binom{r+2}{2} (-2)^r x^r.$$

Thus the coefficient of x^2 occurs when $r = 2$ which gives a coefficient 24.

(c) **0.** We know,

$$\frac{1-x^{n+1}}{1-x} = \sum_{r=0}^{n} x^r.$$

This gives $n+1 = 5$ hence $n = 4$. The form $\frac{1-x^5}{1-x} = \sum_{r=0}^{4} x^r = 1+x+x^2+x^3+x^4$. Simply we notice that the coefficient of x^5 in this expansion will be 0.

(d) **5940.** We know,

$$\frac{1}{(1+x)^n} = \sum_{r=0}^{\infty} (-1)^r \binom{r+n-1}{n-1} x^r.$$

Therefore,

$$\frac{1}{(1+3x)^{10}} = \sum_{r=0}^{\infty} (-1)^r \binom{r+10-1}{10-1} (3x)^r = \sum_{r=0}^{\infty} (-1)^r \binom{r+9}{9} 3^r x^r.$$

Thus the coefficient of x^3 occurs when $r = 3$ and is 5940.

4. In how many ways can 1000 identical pamphlets be distributed to five different counselling centers, where pamphlets are put in stacks of 50, such that each center receives at least 50 but no more than 500 pamphlets?

3246. We first notice that thinking of this problem in terms of stacks of pamphlets rather than the pamphlets themselves reduces it to: "In how man ways can $\frac{1000}{50} = 20$ stacks be distributed to five different counselling centers such that each center receives at least $\frac{50}{50} = 1$ but no more than $\frac{500}{50} = 10$ stacks?" The generating function that represents this set up is,

$$g(x) = (x^1 + x^2 + \cdots + x^{10})^5,$$

and we are interested in determining the coefficient of x^{20}. Alternatively, we can identify the coefficient of x^{15} in, $g'(x) = (1 + x + \cdots + x^9)^5$, which was obtained by factoring out an x. We now rewrite this using what we know about series,

$$\begin{aligned}
g'(x) &= \left(\frac{1-x^{10}}{1-x}\right)^5 \\
&= (1-x^{10})^5\left(\frac{1}{1-x}\right)^5 \\
&= \left(\binom{5}{0} - \binom{5}{1}x^{10} + \binom{5}{2}x^{20} - \binom{5}{3}x^{30} + \binom{5}{4}x^{40} - \binom{5}{5}x^{50}\right)\left(\frac{1}{(1-x)^5}\right) \\
&= (1-5x^{10}+10x^{20}-10x^{30}+5x^{40}-x^{50}) \cdot \sum_{r=0}^{\infty}\binom{r+5-1}{5-1}x^r \\
&= (1-5x^{10}+10x^{20}-\dots) \cdot \sum_{r=0}^{\infty}\binom{r+4}{4}x^r.
\end{aligned}$$

When this expression is expanded, we are interested in the coefficients of x when $r = 15, 5$, which correspond to the coefficients of x^{15}. When $r = 15$, $\binom{15+4}{4} = \binom{19}{4} = 3\ 876$. When $r = 5$, $\binom{5+4}{4} = \binom{9}{4} = 126$. Therefore the coefficient of x^{15} in $g'(x)$ is $(1)\binom{19}{4} + (-5)\binom{9}{4} = (1)3\ 876 - 5(126) = 3\ 876 - 630 = 3\ 246$, which is the number of ways these stacks of pamphlets can be distributed.

5. In how many ways can 20 identical balls be distributed between 3 distinct boxes such that, ...

   (a) ... there are at least two balls assigned to box?

   (b) ... there are at least three, but no more than 10 balls assigned to each box?

   (c) ... using the same condition as in part b, how many distributions are possible if there were 25 balls instead of 20?

   Exercise.

6. Determine the number of ways that USD 12 in loonies can be distributed between a father's three children so that the eldest gets at least four dollars, the middle and youngest child are both guaranteed at least two dollars, but the youngest cannot receive any more than USD 5 since he will spend it all on candy and rot his teeth.
   14.
   We represent the eldest child's potential share of the money by $x^4 + x^5 + x^6 + \dots$.
   The middle child's, $x^2 + x^3 + x^4 + \dots$.
   The youngest child's, $x^2 + x^3 + x^4 + x^5$.
   To determine the number of ways the loonies can be distributed, we are looking for the coefficient of x^{12} in the product,

$$g(x) = (x^4 + x^5 + x^6 + \dots)(x^2 + x^3 + x^4 + \dots)(x^2 + x^3 + x^4 + x^5).$$

   We can simplify $g(x)$,

$$\begin{aligned}
g(x) &= (x^4 + x^5 + x^6 + \dots)(x^2 + x^3 + x^4 + \dots)(x^2 + x^3 + x^4 + x^5) \\
&= x^4(1 + x + x^2 + \dots)x^2(1 + x + x^2 + \dots)x^2(1 + x + x^2 + x^3) \\
&= x^8(1 + x + x^2 + \dots)^2(1 + x + x^2 + x^3).
\end{aligned}$$

Alternatively we can reduce this problem to identifying the coefficient of x^4 in,

$$g'(x) = (1 + x + x^2 + \dots)^2(1 + x + x^2 + x^3).$$

Using identities and some substitutions we rewrite $g'(x)$ as,

$$\begin{aligned}
g'(x) &= (1 + x + x^2 + \dots)^2(1 + x + x^2 + x^3) \\
&= \left(\frac{1}{1-x}\right)^2 \cdot \frac{1-x^4}{1-x} \\
&= \frac{1}{(1-x)^2} \cdot \frac{1-x^4}{1-x} \\
&= \frac{1-x^4}{(1-x)^3} \\
&= (1-x^4)\sum_{r=0}^{\infty}\binom{r+3-1}{3-1}x^r \\
&= (1-x^4)\sum_{r=0}^{\infty}\binom{r+2}{2}x^r.
\end{aligned}$$

The coefficient of x^4 will occur when $r = 0, 4$. When $r = 0$, $\binom{0+3-1}{3-1} = \binom{2}{2} = 1$. When $r = 0$, the coefficient from the sum is $\binom{0+2}{2} = \binom{2}{2} = 1$. When $r = 4$, $\binom{4+3-1}{3-1} = \binom{6}{2} = 15$. When $r = 4$, the coefficient from the sum is $\binom{4+2}{2} = \binom{6}{2} = 15$. Putting this together, there are $1 \times (\text{coeff for } r = 4) - 1 \times (\text{coeff for } r = 0) = 1(15) - 1(1) = 14$ ways to distribute the loonies. The coefficient of x^4 will occur when $r = 0, 4$. When $r = 0$, $\binom{0+3-1}{3-1} = 1$, when $r = 4$, $\binom{4+3-1}{3-1} = 15$. Putting this together, there are $15(1) - 1 = 14$ ways to distribute the loonies.

7. In how many ways can n balls be selected from a supply of pink, orange and black balls such that the number of black balls selected must be even?
Hint: Partial fractions may come in handy.

$$\boxed{\frac{1}{8} + \frac{1}{4}(n+1) + \frac{1}{2}(n+2) + \frac{1}{8}(-1)^n.}$$

The expression, $(1 + x + x^2 + \dots)$, will help keep track of the pink and orange balls, while the expression, $(1 + x^2 + x^4 + \dots)$, will keep track of the even black balls. We are interested in determining the coefficient of x^n in the product,

$$g(x) = (1 + x + x^2 + x^3 + \dots)^2(1 + x^2 + x^4 + x^6 + \dots).$$

From our identities we see that,

$$\begin{aligned}
g(x) &= \left[\frac{1}{1-x}\right]^2 \cdot \frac{1}{1-x^2} \\
&= \frac{1}{(1-x)^2} \cdot \frac{1}{1-x^2} \\
&= \frac{1}{(1-x)^2(1-x)(1+x)} \\
&= \frac{1}{(1-x)^3(1+x)}.
\end{aligned}$$

We need a different, simpler, way to express $g(x)$, so we use a partial fraction expansion,

$$g(x) = \frac{-1}{(x-1)^3(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}.$$

Multiplying both the left and right hand sides by the common denominator, $(x-1)^3(x+1)$, we obtain,

$$-1 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3.$$

Expanding and applying the binomial theorem where necessary we see that,

$$\begin{aligned}
-1 &= A(x^3 - x^2 - x + 1) + B(x^2 - 1) + C(x+1) + D(x^3 - 3x^2 + 3x - 1) \\
&= Ax^3 - Ax^2 - Ax + A + Bx^2 - B + Cx + C + Dx^3 - 3Dx^2 + 3Dx - D \\
&= (A+D)x^3 + (-A+B-3D)x^2 + (-A+C+3D)x + (A-B+C-D).
\end{aligned}$$

We know the coefficient of x^3, x^2, x are 0, so we match the coefficients with each other and solve for the unknowns:

$$\begin{aligned}
A + D &= 0 \\
-A + B - 3D &= 0 \\
-A + C + 3D &= 0 \\
A - B + C - D &= -1.
\end{aligned}$$

We must now solve this system of equations. Clearly from the first equation, $D = -A$. Plugging this into the second equation,

$$-A + B - 3(-A) = -A + B + 3A = B + 2A = 0,$$

which implies that $B = -2A$. Plugging these into the fourth equation we obtain,

$$A - (-2A) + C - (-A) = A + 2A + C + A = 4A + C = -1,$$

which implies that $C = -1 - 4A$. Finally we can plug everything into the third equation and solve for A,

$$-A + (-1 - 4A) + 3(-A) = -A - 1 - 4A - 3A = -8A - 1 = 0.$$

Certainly from this, $A = -\frac{1}{8}$. We further see that $D = \frac{1}{8}$, $B = -2 \cdot -\frac{1}{8} = \frac{1}{4}$ and $C = -1 - 4 \cdot -\frac{1}{8} = -1 + \frac{1}{2} = -\frac{1}{2}$. Hence,

$$\begin{aligned}
g(x) &= \frac{-1}{8(x-1)} + \frac{1}{4((x-1)^2)} + \frac{-1}{2((x-1)^3)} + \frac{1}{8(x+1)} \\
&= \frac{1}{8(1-x)} + \frac{1}{4(1-x)^2} + \frac{1}{2(1-x)^3} + \frac{1}{8(1+x)}.
\end{aligned}$$

We may now use our identities to express $g(x)$ in terms of sums,

$$\begin{aligned}
g(x) &= \frac{1}{8}\sum_{r=0}^{\infty} x^r + \frac{1}{4}\sum_{r=0}^{\infty}\binom{r+2-1}{2-1}x^r + \frac{1}{2}\sum_{r=0}^{\infty}\binom{r+3-1}{3-1}x^r + \frac{1}{8}\sum_{r=0}^{\infty}(-1)^r x^r \\
&= \frac{1}{8}\sum_{r=0}^{\infty} x^r + \frac{1}{4}\sum_{r=0}^{\infty}\binom{r+1}{1}x^r + \frac{1}{2}\sum_{r=0}^{\infty}\binom{r+2}{2}x^r + \frac{1}{8}\sum_{r=0}^{\infty}(-1)^r x^r.
\end{aligned}$$

The coefficient of x^n occurs when $n = r$, hence there are,

$$\frac{1}{8} + \frac{1}{4}\binom{n+1}{1} + \frac{1}{2}\binom{n+2}{2} + \frac{1}{8}(-1)^n,$$

ways to select n balls.

8. A restaurant just closed for the night and they had an extra 12 orders of fries and 16 mini-desserts left over. The restaurant manager decides to split this left over food between the four employees closing that night. How can the manager do this so that the head chef receives at least one order of fries and exactly three mini-desserts, while the three other closing-staff are guaranteed at least two orders of fries but less than 5 desserts?
Exercise.

9. Use generating functions to determine the number of four-element subsets of the set A, given by $A \coloneqq \{1, 2, \ldots, 15\}$ that contain no consecutive integers.
Exercise.

10. A student is picking out a handful of gummy bears from a large container. There are red, yellow, and green gummy bears in the container. The student wishes to pick out an even number of red gummy bears, an odd number that is at least 3 of yellow gummy bears, and either 4 or 6 green gummy bears.

    (a) Determine the appropriate generating function that models this situation.

    (b) How many ways can the student pick out gummy bears if they pick out:

        i. 15?
        ii. 22?

    Exercise.

11. Someone buys a chocolate bar and receives 50 cents in change. Create a generating function that could determine the number of ways they could receive their change in any combination of pennies, nickles, dimes, and quarters? The coefficient of which term will give the desired solution?
Note: You are *not* being asked to determine how many ways this is possible.
We are interested in determining the coefficient of x^{50} in the product,

$$(1 + x + x^2 + \ldots)(1 + x^5 + x^{10} + x^{15} + \ldots)(1 + x^{10} + x^{20} + \ldots)(1 + x^{25} + x^{50} + \ldots).$$

The first term of the product represents the pennies used, the second term the nickels used, third the dimes used, and the last term stands in for the quarters. We can rewrite this product as,

$$\frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} \cdot \frac{1}{1-x^{25}},$$

which is our desired generating function.

12. A deck of cards has 52 cards in total. Half of the deck is red and half is black. A quarter of the deck has the symbol hearts, a quarter has the symbol diamonds, a quarter has the symbol spades, and a quarter has the symbol clubs. How many ways are there to pick 15 cards if:

    (a) You wish to pick an even number of black cards and an odd number of red cards?

    (b) You wish to pick at least two of each symbol, but no more than 5 hearts and 6 spades?

    Exercise.

13. Three students are running for student body president: Krishna, and Jamar, and Bonnie. Find the generating function used to determine the possible distribution of n students' votes

    (a) with no further restrictions?

    (b) if every student running votes for themselves?

    (a) Each student can receive any number of votes, so the generating function would be as follows.

    $$g(x) = (1 + x + x^2 + x^3 + \dots)^3.$$

    To find the number of distributions of n votes, the above generating function would be used to find the coefficient of x^n.

    (b) If every student votes for themselves then we know that each student receives at least one vote. Thus the generating function is:

    $$g(x) = (x + x^2 + x^3 + \dots)^3 = x^3(1 + x + x^2 + x^3 + \dots)^3.$$

    Similar to in (a), to find the number of distributions of n votes, the above generating function would be used to find the coefficient of x^n.

14. How many ways are there to obtain a sum of 7 if 2 distinct 6-sided dice, having faces numbered $1, 2, 3, 4, 5, 6$ are thrown?
    If two distinct dice are rolled, then we can form the following generating function:

    $$g(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^2.$$

    If we are looking to obtain a sum of 7, then we are looking to find the coefficient of x^7 in $g(x)$. We can do so by expanding $g(x)$:

    $$\begin{aligned} g(x) &= (x + x^2 + x^3 + x^4 + x^5 + x^6)^2 \\ &= x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12} \end{aligned}$$

    The coefficient of x^7 is 6 and thus there are 6 possible ways to throw two dice in order to obtain a sum of 7.

Notes : Course Review

Qualitative Input

1. To begin with the course was already designed to cater a standard batch of CS enthusiasts, nevertheless the sudden reordering in the Modules (module on Discrete Probability Theory was suddenly shifted ahead) hampered the pace and natural trajectory of the course.

2. Contents of important modules, in particular modules on Discrete Probability Theory, Predicate Logic and Counting Arithmetic were made rigorously formal to extent that it exceeded the maturity level of an average first year CS undergrad. This lead to the scenario that students had very shallow understanding of the core concepts essential for their growth but were content with mathematical jargons for no meaningful reason.

3. The above two points combined, resulted in an event which was inevitable but could have a nicer turn. *Hampered pace + broken studying* methodology lead to a complete compromise of the module on Graph Theory and Counting. These modules still remain one of the most important topics in Discrete Mathematics, but this time they really got very less time and focus. Furthermore, Graph Theory requires a course in its own right.

4. A point worth mentioning is that, concepts of Generating Functions did not get enough limelight.

5. Most importantly, keeping in mind that this is a gateway course, it is of paramount importance that a uniform teaching methodology and a unitary teaching philosophy governs both the sections of this course. It is an experiential conclusion that every aspect of a course, starting from academic logistics to scientific teaching, gets compromised to a certain degree when there is a disparity between two sections of the same course.

Quantitative Input

6. The amount of practice problem sets on theory were spot on, nothing less to deprive the students of important concepts and nothing more to overburden them with too much to go through.

7. The programming component should have been much more continued with more number of graded assessments and lesser weightage in the final letter grade calculation. Only then a proper programming ethos can be cultivated.

Qualitative Output

8. Course like this often tends to become very heavy near to the end of the semester. Having said this, it is equally a good academic sign and an intriguing event that al most 10 percent of the running class had kept their curiosity and efforts throughout the semester till the very end. Continued and consistent efforts are extremely difficult to maintain, but many students have done it this time. Cheers to the curious minds!