Double Fourier Series

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Fourier series in 1-D

- A periodic function $f(x)$ with a period of 1 and for which $\int_0^1 f(x)^2\,dx$ is finite has a Fourier series expansion:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(2\pi nx) + b_n \sin(2\pi nx))$$

- If $f(x)$ is continuously differentiable, its Fourier series converges uniformly.

Fourier series in higher dimensions (vector notation)

- Consider a function $f(x_1, x_2)$ (p_1, p_2)-periodic in variables x_1 and x_2.
- Assuming p_1 and p_2 to be 1, the condition is:

$$f(x_1 + n_1, x_2 + n_2) = f(x_1, x_2) \quad \forall x_1, x_2 \in [0, 1]$$

- Using vector notation, $x = (x_1, x_2)$ and $n = (n_1, n_2)$, the condition becomes:

$$f(x + n) = f(x) \quad n \in \mathbb{N}$$

Fourier series in higher dimensions (vector notation)

- In 2-D, the building blocks for periodic function $f(x_1, x_2)$ are the product of complex exponentials in one variable.
- The general higher harmonic is of the form:

$$e^{2\pi i n_1 x_1} e^{2\pi i n_2 x_2}$$

- We can write the Fourier series expansion as:

$$\sum_{n_1, n_2} c_{n_1, n_2} e^{2\pi i (n_1 x_1 + n_2 x_2)}$$

Fourier series in 2-D

- Let $f(x,y)$ be a continuously differentiable periodic function with a period of 1 in both variables.
- For each value of y, we can expand $f(x,y)$ in a uniformly convergent Fourier series:

$$
\begin{aligned}
f(x,y) = a_0 &+ \sum b_m \cos(2\pi mx) + \sum c_m \sin(2\pi mx) \\
&+ \sum d_n \cos(2\pi ny) + \sum e_n \sin(2\pi ny) \\
&+ \sum \sum f_{mn} \cos(2\pi mx) \sin(2\pi ny) \\
&+ \sum \sum g_{mn} \cos(2\pi mx) \cos(2\pi ny) \\
&+ \sum \sum h_{mn} \sin(2\pi mx) \sin(2\pi ny) \\
&+ \sum \sum k_{mn} \sin(2\pi mx) \cos(2\pi ny)
\end{aligned}
$$

Fourier series in 2-D

- where,

$$a_0 = \int_0^1 \int_0^1 f(x,y)\, dxdy$$

$$b_m = 2\int_0^1 \int_0^1 f(x,y)\cos(2\pi mx)dxdy$$

$$c_m = 2\int_0^1 \int_0^1 f(x,y)\sin(2\pi mx)dxdy$$

$$d_n = 2\int_0^1 \int_0^1 f(x,y)\cos(2\pi ny)dxdy$$

$$e_n = 2\int_0^1 \int_0^1 f(x,y)\sin(2\pi ny)dxdy$$

Fourier Series in 2D

$$f_{mn} = 4\int_0^1\int_0^1 f(x,y)\cos(2\pi mx)\sin(2\pi ny)dxdy$$
$$g_{mn} = 4\int_0^1\int_0^1 f(x,y)\cos(2\pi mx)\cos(2\pi ny)dxdy$$
$$h_{mn} = 4\int_0^1\int_0^1 f(x,y)\sin(2\pi mx)\sin(2\pi ny)dxdy$$
$$k_{mn} = 4\int_0^1\int_0^1 f(x,y)\sin(2\pi mx)\cos(2\pi ny)dxdy$$

- $V_m(t) = m \sin(2\pi f_1 t)$

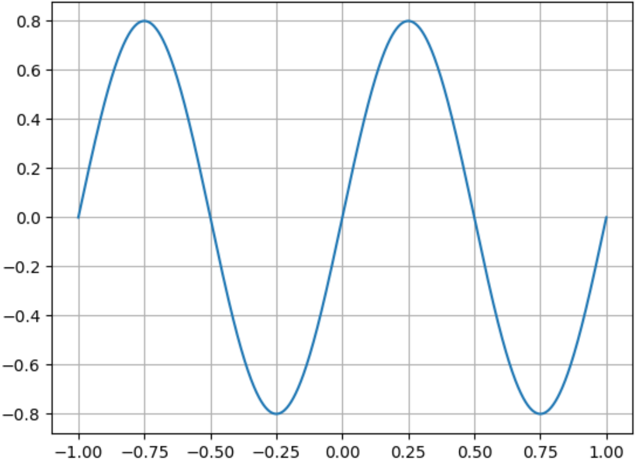


Figure: Sinusoidal modulating signal

Sine Triangle PWM

- $V_c(t) = \begin{cases} 4tf_2 - 1 & 0 \leq t \leq \frac{1}{2f_2} \\ 3 - 4tf_2 & \frac{1}{2f_2} \leq t \leq \frac{1}{f_2} \end{cases} \quad \textit{for} \quad 0 \leq t \leq \frac{1}{f_2}$

  periodic with period $= \frac{1}{f_2}$

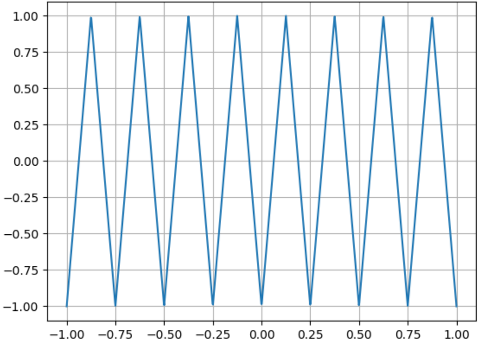


Figure: Triangular Carrier Signal

Sine Triangle PWM

- $V(t) = \begin{cases} 1 & V_m(t) \geq V_c(t) \\ 0 & otherwise \end{cases}$

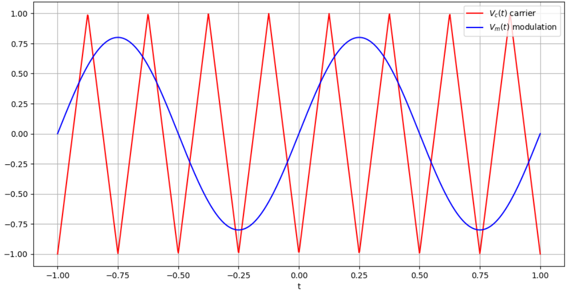


Figure: $V_m \quad and \quad V_c$

Sine Triangle PWM

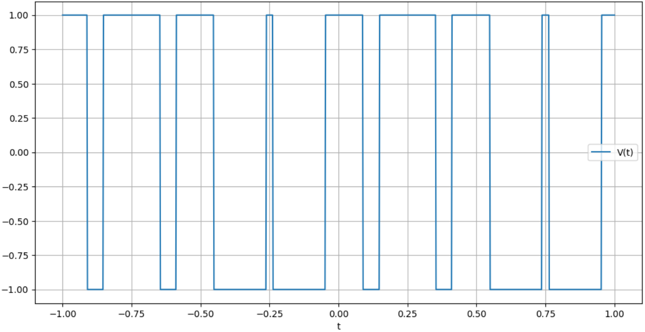
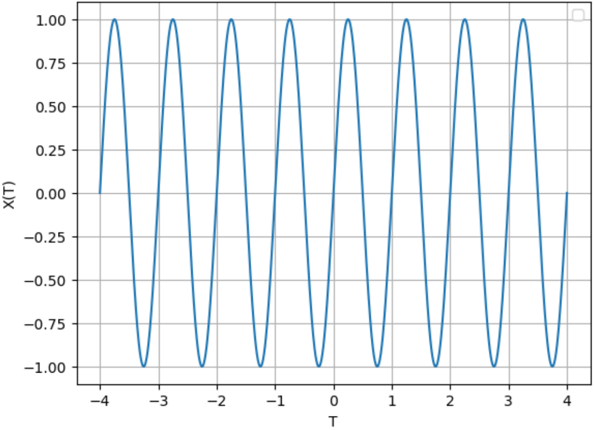


Figure: $V(t)$

Double Fourier Series of Sine Triangle PWM

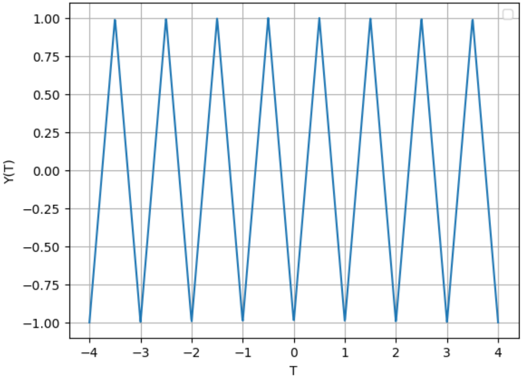
- $X(T) = sin(2\pi T)$



- $X(T+1) = X(T)$

Double Fourier Series of Sine Triangle PWM

- $Y(T) = \begin{cases} 4T - 1 & 0 \leq T \leq 1/2 \\ 3 - 4T & 1/2 \leq T \leq 1 \end{cases}$ for $0 \leq T \leq 1$



- Y(T+1) = Y(T)

Double Fourier Series of Sine Triangle PWM

- $V_m(t) = X(f_1 t)$
- $V_c(t) = Y(f_2 t)$
- $V(t) = \begin{cases} 1 & X(f_1 t) \geq Y(f_2 t) \\ 0 & otherwise \end{cases}$
- Define

$$f(x, y) = \begin{cases} 1 & X(x) \geq Y(y) \\ 0 & otherwise \end{cases}$$

- $f(x+1, y) = f(x, y+1) = f(x, y)$

Double Fourier Series of Sine Triangle PWM

- Now, $V(t) = f(f_1 t, f_2 t)$
- Fourier series of f is

$$
\begin{aligned}
f(f_1 t, f_2 t) = a_0 &+ \sum b_m (2\pi m f_1 t) + \sum c_m \sin(2\pi m f_1 t) \\
&+ \sum d_n \cos(2\pi n f_2 t) + \sum e_n \sin(2\pi n f_2 t) \\
&+ \sum \sum f_{mn} \cos(2\pi m f_1 t) \sin(2\pi n f_2 t) \\
&+ \sum \sum g_{mn} \cos(2\pi m f_1 t) \cos(2\pi n f_2 t) \\
&+ \sum \sum h_{mn} \sin(2\pi m f_1 t) \sin(2\pi n f_2 t) \\
&+ \sum \sum k_{mn} \sin(2\pi m f_1 t) \cos(2\pi n f_2 t)
\end{aligned}
$$

-

$$
\begin{aligned}
f(f_1 t, f_2 t) = a_0 &+ \sum b_m \cos(2\pi m f_1 t) + \sum c_m \sin(2\pi m f_1 t) \\
&+ \sum d_n \cos(2\pi n f_2 t) + \sum e_n \sin(2\pi n f_2 t) \\
&+ \sum_m \sum_n \sin(2\pi (n f_2 + m f_1) t) \left(\frac{f_{mn}}{2} + \frac{k_{mn}}{2} \right) \\
&+ \sum_m \sum_n \sin(2\pi (n f_2 - m f_1) t) \left(\frac{f_{mn}}{2} - \frac{k_{mn}}{2} \right) \\
&+ \sum_m \sum_n \cos(2\pi (n f_2 + m f_1) t) \left(\frac{g_{mn}}{2} - \frac{h_{mn}}{2} \right) \\
&+ \sum_m \sum_n \cos(2\pi (n f_2 - m f_1) t) \left(\frac{g_{mn}}{2} + \frac{h_{mn}}{2} \right)
\end{aligned}
$$

Double Fourier Series of Sine Triangle PWM

- $\sum b_m \cos(2\pi m f_1 t) + \sum c_m \sin(2\pi m f_1 t)$ is Harmonics from modulation signal
- $\sum d_n \cos\left(2\pi n f_2 t\right) + \sum e_n \sin\left(2\pi n f_2 t\right)$ is Harmonics from Carrier Signal
- $\sum_m \sum_n \sin\left(2\pi\left(nf_2 + mf_1\right)t\right)\left(\frac{f_{mn}}{2} + \frac{k_{mn}}{2}\right) +$
$\sum_m \sum_n \sin\left(2\pi\left(nf_2 - mf_1\right)t\right)\left(\frac{f_{mn}}{2} - \frac{k_{mn}}{2}\right) +$
$\sum_m \sum_n \cos\left(2\pi\left(nf_2 + mf_1\right)t\right)\left(\frac{g_{mn}}{2} - \frac{h_{mn}}{2}\right) +$
$\sum_m \sum_n \cos\left(2\pi\left(nf_2 - mf_1\right)t\right)\left(\frac{g_{mn}}{2} + \frac{h_{mn}}{2}\right)$ are Sideband Harmonics