1 Solving for Square wave function

The objective is to solve the differential equation

$$LC\frac{d^2v_0}{dt^2} + \frac{L}{R}\frac{dv_0}{dt} + v_0 = v_p(t)$$

where $v_p(t)$ is a square wave function defined as follows:

$$v_p(t) = \begin{cases} V_{\mathrm{DC}} & \text{if } 0 \leq t < D \cdot T_{\mathrm{SW}} \\ 0 & \text{if } D \cdot T_{\mathrm{SW}} \leq t < T_{\mathrm{SW}} \end{cases}$$

This function is periodic with period T_{SW}.

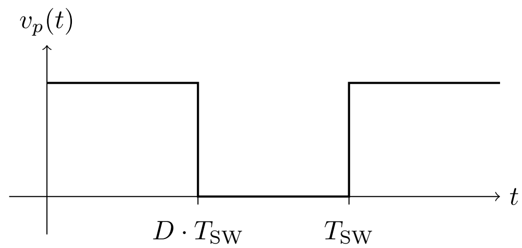

Figure 1: Square wave function $v_p(t)$ with period T_{SW} and amplitude V_{DC}

We have the following initial conditions for the differential equation:

$$v_0(0) = 0$$

$$\frac{dv_0}{dt}(0) = 0$$

We convert the square wave to its Fourier series, which is equal to:

$$D \cdot V_{\mathrm{DC}} + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{2\pi nt}{T_{\mathrm{SW}}}\right) + B_n \sin\left(\frac{2\pi nt}{T_{\mathrm{SW}}}\right) \right)$$

where A_n and B_n are the Fourier coefficients, and V_{DC} is the amplitude of the function. A_n and B_n are given by the following equations:

$$A_n = \frac{2}{T_{\mathrm{SW}}} \int_0^{T_{\mathrm{SW}}} v_p(t) \cos\left(\frac{2\pi nt}{T_{\mathrm{SW}}}\right) dt = V_{\mathrm{DC}} \frac{\sin(2\pi nD)}{\pi n}$$

$$B_n = \frac{2}{T_{\mathrm{SW}}} \int_0^{T_{\mathrm{SW}}} v_p(t) \sin\left(\frac{2\pi nt}{T_{\mathrm{SW}}}\right) dt = V_{\mathrm{DC}} \frac{1 - \cos(2\pi nD)}{\pi n} = V_{\mathrm{DC}} \frac{2\sin^2(\pi nD)}{\pi n}$$

We approximate this by the truncated Fourier series:

$$v_{p,N}(t) = D \cdot V_{\mathrm{DC}} + \sum_{n=1}^{N} \left(A_n \cos\left(\frac{2\pi nt}{T_{\mathrm{SW}}}\right) + B_n \sin\left(\frac{2\pi nt}{T_{\mathrm{SW}}}\right) \right)$$

where N is the number of terms used in the approximation.

The equation to solve is:

$$LC\frac{d^2v_0}{dt^2} + \frac{L}{R}\frac{dv_0}{dt} + v_0 = v_{p,N}(t)$$

with initial conditions:

$$v_{0,N}(0) = 0, \quad \frac{dv_{0,N}}{dt}(0) = 0$$

where $v_{p,N}(t)$ is the truncated Fourier series of $v_p(t)$ with N terms, and $v_{0,N}(t)$ is the solution to the differential equation with $v_{p,N}(t)$.

We have the equation:

$$v_{0,N}(t) = v_{0,N,\text{particular}}(t) + v_{0,N,\text{homogeneous}}(t) \tag{1}$$

Now, let's consider the homogeneous solution $v_{0,N,\text{homogeneous}}(t)$.

The homogeneous equation is given by:

$$LC\frac{d^2v_0}{dt^2} + \frac{L}{R}\frac{dv_0}{dt} + v_0 = 0$$

Consider the solution space of this equation. Let $v_{01}(t)$ and $v_{02}(t)$ be solutions. Then $v_{01}(t) - v_{02}(t)$ is a solution of the homogeneous equation.

To find solutions of the homogeneous equation, we consider the characteristic equation:

$$LCk^2 + \frac{L}{R}k + 1 = 0$$

Depending on the roots of this equation, the solutions of the homogeneous equation vary:

1. If both roots are real and distinct (α and β), then $e^{\alpha t}$ and $e^{\beta t}$ are solutions.
2. If both roots are real and equal (r), then e^{rt} and te^{rt} are solutions.
3. If the roots are complex ($\alpha \pm i\beta$), then $e^{\alpha t}\sin(\beta t)$ and $e^{\alpha t}\cos(\beta t)$ are solutions.

These solutions form a vector space with dimension 2, and the basis vectors (solutions) depend on the values of LC, L/R, and 1.

Now, we have to solve for $v_{0,N,\text{particular}}(t)$, also called the particular solution of $v_{0,N}$.

In the context of our problem, the particular solution $v_{0,N,\text{particular}}$ is expressed as the sum of $\widetilde{v}_0$ and a series of terms $\widetilde{v}_n$ for $n = 1, 2, \ldots, N$.

$$v_{0,N,\text{particular}} = \widetilde{v}_0 + \sum_{n=1}^{N} \widetilde{v}_n$$

where $v_{0,N,\text{particular}}(t)$ is the particular solution of $LC\frac{d^2v_0}{dt^2} + \frac{L}{R}\frac{dv_0}{dt} + v_0 = v_{p,N}(t)$, $\widetilde{v}_0$ is the particular solution of $LC\frac{d^2v_0}{dt^2} + \frac{L}{R}\frac{dv_0}{dt} + v_0 = D\cdot V_{\text{DC}}$, and $\widetilde{v}_n$ is the particular solution of $LC\frac{d^2v_0}{dt^2} + \frac{L}{R}\frac{dv_0}{dt} + v_0 = A_n\cos\left(\frac{2\pi nt}{T_{\text{SW}}}\right) + B_n\sin\left(\frac{2\pi nt}{T_{\text{SW}}}\right)$.

Calculating the particular solution of $LC\frac{d^2v_0}{dt^2} + \frac{L}{R}\frac{dv_0}{dt} + v_0 = D\cdot V_{\text{DC}}$, we have the following cases:

- $LC \neq 0$

$$\widetilde{v}_0(t) = D\cdot V_{\text{DC}}$$

Now, Calculating the particular solution of $LC\frac{d^2v_0}{dt^2} + \frac{L}{R}\frac{dv_0}{dt} + v_0 = A_n\cos\left(\frac{2\pi nt}{T_{\text{SW}}}\right) + B_n\sin\left(\frac{2\pi nt}{T_{\text{SW}}}\right)$
$\widetilde{v}_n = \alpha_n\sin\left(\frac{2\pi nt}{T_{\text{SW}}}\right) + \beta_n\cos\left(\frac{2\pi nt}{T_{\text{SW}}}\right)$ where

$$\alpha_n = \frac{\begin{vmatrix} B_n & -\frac{L}{R}\cdot 2\pi n \\ A_n & LC - \left(\frac{2\pi n}{T_{\text{SW}}}\right)^2 \end{vmatrix}}{\begin{vmatrix} LC - \left(\frac{2\pi n}{T_{\text{SW}}}\right)^2 & -\frac{L}{R}\cdot 2\pi n \\ \frac{L}{R}\cdot 2\pi n & LC - \left(\frac{2\pi n}{T_{\text{SW}}}\right)^2 \end{vmatrix}}, \beta_n = \frac{\begin{vmatrix} LC - \left(\frac{2\pi n}{T_{\text{SW}}}\right)^2 & B_n \\ -\frac{L}{R}\cdot 2\pi n & A_n \end{vmatrix}}{\begin{vmatrix} LC - \left(\frac{2\pi n}{T_{\text{SW}}}\right)^2 & -\frac{L}{R}\cdot 2\pi n \\ \frac{L}{R}\cdot 2\pi n & LC - \left(\frac{2\pi n}{T_{\text{SW}}}\right)^2 \end{vmatrix}}.$$