PRACTICAL : 1

Topic : Limits & Continuity

① $\lim_{x \to a} \left[\dfrac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

② $\lim_{x \to 0} \left[\dfrac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

③ $\lim_{x \to \pi/6} \left[\dfrac{\cos x - \sqrt{3}\sin x}{\pi - 6x} \right]$

④ $\lim_{x \to \infty} \left[\dfrac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \right]$

⑤ Examine the continuity of the following function at given points.

(i) $$f(x) = \begin{cases} \dfrac{\sin 2x}{\sqrt{1-\cos 2x}} & \text{for } 0 < x \le \dfrac{\pi}{2} \\ \dfrac{\cos x}{\pi - 2x} & \text{for } \dfrac{\pi}{2} < x < \pi \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

(ii) $$f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & 0 < x < 3 \\ x+3 & 3 \le x < 6 \\ \dfrac{x^2-9}{x+3} & 6 \le x < 9 \end{cases} \quad \text{at } x=3 \ \& \ x=6$$

⑥ Find value of k so that the function f(x) is cts at the indicated point.

(i) $$f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2} & x < 0 \\ k & x = 0 \end{cases} \quad \text{at } x = 0$$

(ii) $f(x) = (\sec^2 x)^{\cot^2 x}$ $\quad x \neq 0$
$\quad = k$ $\quad x = 0$ $\Big\}$ at $x = 0$

(iii) $f(x) = \dfrac{\sqrt{3} - \tan x}{\pi - 3x}$ $\quad x \neq \dfrac{\pi}{3}$
$\quad = k$ $\quad x = \pi/3$. $\Big\}$ at $x = \pi/3$

(7) Discuss the continuity of the following functions, if any of these functions have a removable discontinuity, Redefine the function so as to remove the discontinuity.

(i) $f(x) = \dfrac{1 - \cos 3x}{x \tan x}$ $\quad x \neq 0$
$\quad = ,$ $\quad x = 0$ $\Big\}$ at $x = 0$

(ii) $f(x) = \dfrac{(e^{3x} - 1)\sin x^\circ}{x^2}$ $\quad x \neq 0$
$\quad = \dfrac{\pi}{60}$ $\quad x = 0$ $\Big\}$ at $x = 0$

(8) If $\cancel{f(x)} = \dfrac{e^{x^2} - \cos x}{x^2}$ for $x \neq 0$ is cts at x
Find $f(0)$

(9) If $f(x) = \dfrac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$ for $x \neq \dfrac{\pi}{2}$ is cts
$x = \dfrac{\pi}{2}$. find $f\left(\dfrac{\pi}{2}\right)$

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(1) Sol.n :

$$\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \times \frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}} \right]$$

$$\lim_{x \to a} \frac{(a+2x-3x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a+x-4x)(\sqrt{a+2x} + \sqrt{3x})}$$

$$\lim_{x \to a} \frac{(a-x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a-3x)(\sqrt{a+2x} + \sqrt{3x})}$$

$$\frac{1}{3} \lim_{x \to a} \frac{(a-x)(\sqrt{3a+x} + 2\sqrt{x})}{(a-x)(\sqrt{a+2x} + \sqrt{3x})}$$

$$\frac{1}{3} \cdot \frac{\sqrt{3a+2a} + 2\sqrt{a}}{\sqrt{a+2a} + \sqrt{3a}}$$

$$\frac{1}{3} \times \frac{\sqrt{4a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$$

$$\frac{1}{3} \times \frac{2\sqrt{a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$$

$$\frac{1}{3} \times \frac{\cancel{4}^{2} \cdot \cancel{\sqrt{a}}}{\cancel{2}_{1} \sqrt{3\cancel{a}}}$$

$$= \frac{2}{3\sqrt{3}}$$

(2) Soln: $\lim\limits_{y\to 0}\left[\dfrac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$

$$\lim_{y\to 0}\left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\times\frac{\sqrt{a+y}+\sqrt{a}}{\sqrt{a+y}+\sqrt{a}}\right]$$

$$\lim_{y\to 0}\frac{a+y-a}{y\sqrt{a+y}\left(\sqrt{a+y}+\sqrt{a}\right)}$$

$$\lim_{y\to 0}\frac{\cancel{y}}{\cancel{y}\sqrt{a+y}\left(\sqrt{a+y}+\sqrt{a}\right)}$$

$$\frac{1}{\sqrt{a+0}\left(\sqrt{a+0}+\sqrt{a}\right)}$$

$$\frac{1}{\sqrt{a}\left(\sqrt{a}+\sqrt{a}\right)}$$

$$\frac{1}{\sqrt{a}\left(2\sqrt{a}\right)}=\frac{1}{2a}$$

(3) Soln: $\lim\limits_{x\to \pi/6}\dfrac{\cos x-\sqrt{3}\sin x}{\pi-6x}$

By substituting. $x-\dfrac{\pi}{6}=h$

$$x=h+\frac{\pi}{6}$$

where $h \to 0$

$$\lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{6}\right) - \sqrt{3}\sin\left(h+\frac{\pi}{6}\right)}{\pi - 6\left(h+\frac{\pi}{6}\right)}$$

$$\lim_{h\to 0} \frac{\cos h \cdot \cos\frac{\pi}{6} - \sin h \cdot \sin\frac{\pi}{6} - \sqrt{3}\sin h\cos\frac{\pi}{6} + \cosh\sin\frac{\pi}{6}}{\pi - 6\left(\frac{6h+\pi}{6}\right)}$$

$$\lim_{h\to 0} \frac{\cos h \cdot \frac{\sqrt{3}}{2} - \sin h\frac{1}{\sqrt{2}} - \sqrt{3}\left(\sin h\frac{\sqrt{3}}{2} + \cos h\frac{1}{\sqrt{2}}\right)}{\cancel{\pi} - 6h + \cancel{\pi}}$$

$$\lim_{h\to 0} \frac{\cancel{\cos h \cdot \frac{\sqrt{3}}{2}} h - \sin\frac{h}{2} - \sin\frac{3h}{2} - \cancel{\cos\frac{\sqrt{3}}{2}} h}{-6h}$$

$$\lim_{h\to 0} \frac{\cancel{-}\sin 4h/2}{\cancel{-}6h}$$

$$\lim_{h\to 0} \frac{\sin \cancel{4}h}{3\cancel{12}h}$$

$$\frac{1}{3} \cancel{\lim_{h\to 0}} \frac{\sin h}{h}$$

$$\frac{1}{3} \times 1 = \frac{1}{3}$$

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④ Soln : $\lim_{n\to\infty}\left[\frac{\sqrt{n^2+5}-\sqrt{n^2-3}}{\sqrt{n^2+3}-\sqrt{n^2+1}}\right]$

By rationalizing numerator & denominator both

$\lim_{n\to\infty}\left[\frac{\sqrt{n^2+5}-\sqrt{n^2-3}}{\sqrt{n^2+3}-\sqrt{n^2+1}}\right]\times\left[\frac{\sqrt{n^2+5}+\sqrt{n^2-3}}{\sqrt{n^2+5}+\sqrt{n^2-3}}\right]\times\frac{\sqrt{n^2+3}+\sqrt{n^2+1}}{\sqrt{n^2+3}+\sqrt{n^2+1}}$

$\lim_{n\to\infty}\left[\frac{(n^2-5-n^2+3)(\sqrt{n^2+3}+\sqrt{n^2+1})}{(n^2+3-n^2-1)(\sqrt{n^2+5}+\sqrt{n^2-3})}\right]$

$\lim_{n\to\infty}\left[\frac{\cancel{8}^{4}(\sqrt{n^2+3}+\sqrt{n^2+1})}{\cancel{2}_{1}(\sqrt{n^2+5}+\sqrt{n^2-3})}\right]$

$4\lim_{n\to\infty}\frac{\sqrt{n^2(1+3/n^2)}+\sqrt{n^2(1+1/n^2)}}{\sqrt{n^2(1+5/n^2)}+\sqrt{n^2(1-3/n^2)}}$

After applying limit
we get,
$= 4$

⑤ $f(n) = \begin{cases}\frac{\sin 2n}{\sqrt{1-\cos 2n}}, & \text{for } 0<n\le \pi/2 \\ \frac{\cos n}{\pi-2n}, & \text{for } \pi/2<n<\pi\end{cases}$ at $n=\pi/2$

$$\lim_{x \to \pi/2^+} f(x) = \lim_{x \to \pi/2} + \frac{\cos x}{\pi - 2x}$$

By substituting method,

$$x - \frac{\pi}{2} = h$$

$$x = h + \frac{\pi}{2}$$

where $h \to 0$

$$\lim_{h \to 0} \frac{\cos\left(h + \frac{\pi}{2}\right)}{\pi - 2\left(h + \frac{\pi}{2}\right)}$$

$$\lim_{h \to 0} \frac{\cos\left(h + \frac{\pi}{2}\right)}{\pi - \cancel{2}\left(\frac{2h + \pi}{\cancel{2}}\right)}$$

$$\lim_{h \to 0} \frac{\cos\left(h + \frac{\pi}{2}\right)}{-2h}$$ using $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\lim_{h \to 0} \frac{\cos h \cdot \cos\frac{\pi}{2} - \sin h \cdot \sin\frac{\pi}{2}}{-2h}$$

$$\lim_{h \to 0} \frac{\cos h \cdot 0 - \sin h}{-2h}$$

$$\lim_{h \to 0} \frac{\cancel{-}\sin h}{\cancel{-}2h}$$

$$\frac{1}{2} \lim_{h \to 0} \frac{\sin h}{h} = \frac{1}{2}$$

b] $\lim_{x \to \pi/2^-} f(x) = \lim_{x \to \pi/2} \frac{-\sin 2x}{\sqrt{1-\cos 2x}}$ using $\sin 2x = 2\sin x \cdot \cos x$

$$\lim_{x \to \pi/2^-} \frac{2\sin x \cdot \cos x}{\sqrt{2\sin^2 x}}$$

$$\lim_{x \to \pi/2^-} \frac{2\cancel{\sin x} \cdot \cos x}{\sqrt{2}\ \cancel{\sin x}}$$

$$\lim_{x \to \pi/2^-} \frac{2\cos x}{\sqrt{2}}$$

$$\frac{2}{\sqrt{2}} \lim_{x \to \pi/2^-} \cos x$$

$\therefore$ L.H.L = R.H.L

$\therefore$ f is not continuous at $x = \pi/2$

5 (i) $f(x) = \cancel{\frac{x^2-9}{x-3}} = 0$

$\cancel{f}$ at $x = 3$ define

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x+3$

$f(3) = x+3 = 3+3 = 6$

f is define at $x=3$

No.

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x+3) = 6$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{(x-3)}$$

$\therefore$ L.H.L = R.H.L

f is continuous at $x = 3$

for $x = 6$

$$f(6) = \frac{x^2-9}{x+3} = \frac{36-9}{6+3} = \frac{27}{9} = 3$$

(2) $\lim_{x \to 6^+} \frac{x^2-9}{x+3}$

$$\lim_{x \to 6^+} \frac{(x-2)\cancel{(x+3)}}{\cancel{(x+3)}}$$

$$\lim_{x \to 6^+} (x-2) = 6-3 = 3$$

$$\lim_{x \to 6^-} x + 3 = 3+6 = 9$$

$\therefore$ L.H.L $\neq$ R.H.L

function is not continuous

(6) (i) $\cancel{f(x) = \frac{1-\cos 4x}{x^2}}$ $\quad x < 0$

$= k \quad x = 0$ $\Big\}$ at $x = 0$

Soln: f is continuous at $x = 0$

$$\lim_{x \to 0} f(x) = f(0)$$

$\lim\limits_{x\to 0} \dfrac{1-\cos 4x}{x^2} = k$

$\lim\limits_{x\to 0} \dfrac{2\sin^2 2x}{x^2} = k$

$2\lim\limits_{x\to 0} \dfrac{\sin^2 x}{x^2} = k$

$2\lim\limits_{x\to 0} \left[\dfrac{\sin 2x}{x}\right]^2 = k$

$2(2)^2 = k$

$\therefore \quad k = 8$

(ii) $f(x) = (\sec^2 x)^{\cot^2 x} \quad x \neq 0$
$\quad = k \quad x = 0$ $\}$ at $x = 0$

Soln: $f(x) = (\sec^2 x)^{\cot^2 x}$

$\therefore \lim\limits_{x\to 0} (\sec^2 x)^{\cot^2 x}$

$\lim\limits_{x\to 0} (1+\tan^2 x)^{1/\tan^2 x}$

~~We know that,~~
$\cancel{\lim\limits_{x\to 0} (1+px)^{1/px} = e}$

$\therefore \quad = e$

$\therefore \quad k = e$

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(iii) $f(x) = \dfrac{\sqrt{3} - \tan x}{\pi - 3x}$ $x \neq \dfrac{\pi}{3}$

$= k$ $x = \pi/3$ $\Big\}$ at $x = \pi/3$

$x - \dfrac{\pi}{3} = h$

$x = h + \dfrac{\pi}{3}$

where $h \to 0$

$f\left(\dfrac{\pi}{3} + h\right) = \dfrac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$

$\lim_{h \to 0} \dfrac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$

$\lim_{h \to 0} \dfrac{\dfrac{\sqrt{3} - \tan\frac{\pi}{3} + \tan h}{1 - \tan\frac{\pi}{3} \cdot \tan h}}{\cancel{\pi} - \pi - 3h}$

$\lim_{h \to 0} \dfrac{\dfrac{\sqrt{3}\left(1 - \tan\frac{\pi}{3} \cdot \tan h\right) - \left(\tan\frac{\pi}{3} + \tan h\right)}{1 - \tan\frac{\pi}{3} \cdot \tan h}}{-3h}$

$$\lim_{h\to 0} \frac{\left(\dfrac{\sqrt{3} - \sqrt{3} \times \sqrt{3}.\tan h}{1 - \tan \pi/3 . \tan h}\right) - (\sqrt{3} + \tan h)}{-3h}$$

$$\lim_{h\to 0} \frac{\left(\dfrac{\sqrt{3} - 3 . \tan h}{1 - \sqrt{3} . \tan h}\right) - (\sqrt{3} + \tan h)}{-3h}$$

$$\lim_{h\to 0} \frac{\left(\dfrac{\cancel{\sqrt{3}} - 3\tan h - \cancel{\sqrt{3}} - \tan h}{1 - \sqrt{3}\tan h}\right)}{-3h}$$

$$\lim_{h\to 0} \frac{\cancel{-}4\tan h}{\cancel{-}3h(1 - \sqrt{3}\tan h)}$$

$$\lim_{h\to 0} \frac{4\tan h}{3h(1 - \sqrt{3}\tan h)}$$

$$\frac{4}{3} \lim_{h\to 0} \frac{\tan h}{h} \lim_{h\to 0} \frac{1}{(1 - \sqrt{3}\tan h)}$$

$$= \frac{4}{3} \frac{1}{(1 - \sqrt{3}\cosh)}$$

$$= \frac{4}{3}\left(\frac{1}{1}\right) = \frac{4}{3}$$

7 (i)

$f(x) = \frac{1-\cos 2x}{x \tan x}$

$\lim_{x\to 0} \frac{2\sin^2 3/2\, x}{x \tan x}$

$\lim_{x\to 0} \frac{\frac{2\sin^2 3x/2}{x^2} \times x^2}{x \cdot \frac{\tan x}{x^2} \times x^2}$

$2 \lim_{x\to 0} \frac{\left(\frac{3}{2}\right)^2}{1} = 2 * \frac{9}{4} = \frac{9}{2}$

$\lim_{x\to 0} f(x) = \frac{9}{2}$ $\qquad 9 = f(0)$

$\therefore$ f is not continuous at $x=0$

Redefine function

$$f(x) = \begin{cases} \frac{1-\cos 2x}{x\tan x} & x \neq 0 \\ \frac{9}{2} & x = 0 \end{cases}$$

Now, $\lim_{x\to 0} f(x) = f(0)$

f has removable discontinuity at $x=0$

7 (ii) $f(x) = \begin{cases} \dfrac{(e^{3x}-1)\sin x^\circ}{x^2} & x \neq 0 \\ = \pi/6 & x = 0 \end{cases}$ at $x=0$

$\lim_{x\to 0} \dfrac{(e^{3x}-1)\sin(\pi x/180)}{x^2}$

$\lim_{x\to 0} \dfrac{e^{3x}-1}{x} \quad \lim_{x\to 0} \dfrac{\sin(\pi x/180)}{x}$

$\lim_{x\to 0} \dfrac{3.e^{3x}-1}{3x} \quad \lim_{x\to 0} \dfrac{\sin(\pi x/180)}{x}$

$3\lim_{x\to 0} \dfrac{e^{3x}-1}{3x} \quad \lim_{x\to 0} \dfrac{\sin(\pi x/180)}{x}$

$\cancel{3} \log e \dfrac{\pi}{\cancel{180}_{60}} = \dfrac{\pi}{60} = f(0)$

$\therefore$ f is continuous at $x=0$

(8) $f(x) = \dfrac{e^{x^2}-\cos x}{x^2}$ $\quad x = 0$

is continuous at $x=0$

$\lim_{x\to 0} f(x) = f(0)$

$= \lim_{x\to 0} \dfrac{e^{x^2}-\cos x - 1 + 1}{x^2}$

$= \lim_{x\to 0} \dfrac{(e^{x^2}-1)+(1-\cos x)}{x^2}$

$\lim_{x\to 0} \dfrac{e^{x^2}-1}{x^2} + \lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

$\log e + \lim_{x\to 0} \dfrac{2\sin^2 x/2}{x^2}$

$\log e + 2\lim_{x\to 0} \left(\dfrac{\sin x/2}{x}\right)^2$

Multiply with 2 on Num & Denominator

$= 1 + 2 \times \dfrac{1}{4} = \dfrac{3}{2} = f(0)$

(9) $f(x) = \dfrac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x}$ $\quad x \neq \dfrac{\pi}{2}$

f (0) is continuous at $x = \dfrac{\pi}{2}$

$\lim_{x\to \pi/2} \dfrac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x} \times \dfrac{\sqrt{2}+\sqrt{1+\sin x}}{\sqrt{2}+\sqrt{1+\sin x}}$

$\lim_{x\to \pi/2} \dfrac{2-1+\sin x}{\cos^2 x(\sqrt{2}+\sqrt{1+\sin x})}$

$\lim_{x\to \pi/2} \dfrac{1+\sin x}{(1-\sin x)(1+\sin x)(\sqrt{2}+\sqrt{1+\sin x})}$

$\lim_{x\to \pi/2} \dfrac{1}{(1-\sin x)(\sqrt{2}+\sqrt{1+\sin x})}$

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$$= \frac{1}{2(\sqrt{2}+\sqrt{2})}$$

$$= \frac{1}{2(2\sqrt{2})} = \frac{1}{4\sqrt{2}}$$

$$\therefore f(\pi/2) = \frac{1}{4\sqrt{2}}$$

Topic : Derivative

Q. Show that the following function defined from IR to IR are differentiable.

(i) $\cot x$

$$f(x) = \cot x$$

$$f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$$

$$= \lim_{x \to a} \frac{1/\tan x - 1/\tan a}{x - a}$$

$$= \lim_{x \to 0} \frac{\tan a - \tan a}{(x-a)\tan x \cdot \tan a}$$

put $x - a = h$

$x = a + h$

as $x \to a, h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(\cancel{a} + h - \cancel{a})\tan(a+h)\tan a}$$

$$= \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{h \times \tan(a+h)\tan a}$$

$$= \lim_{h \to 0} \frac{\tan(a-a-h) - (1+\tan a + \tan(a+h))}{h \times \tan(a+h)\tan a}$$

$$= \lim_{h \to 0} \frac{-\tan h}{h} \times \frac{1 + \tan a \tan(a+h)}{\tan(a+h)\tan a}$$

$= -1 \times \frac{1+\tan^2 a}{\tan^2 a}$

$= \frac{-\sec^2 a}{\tan^2 a}$

$= \frac{-1}{\cos^2 a} \times \frac{\cos^2 a}{\sin^2 a}$

$= -\operatorname{cosec}^2 a$

$Df(a) = -\cos^2 a$

$\therefore f$ is differentiable $\forall a \in R$

(ii) cosec x

$f(x) = \operatorname{cosec} x$

$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$\lim_{x \to a} \frac{\frac{1}{\sin x} - \frac{1}{\sin a}}{x - a}$

$\lim_{x \to a} \frac{\sin a - \sin x}{(x-a)\sin a . \sin x}$

put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$Df(h) = \lim_{h \to 0} \frac{\sin a - \sin(a+h)}{(a+h-a)\sin a . \sin(a+h)}$

$= \lim_{h \to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right) . \sin\left(\frac{a-a-h}{2}\right)}{h . \sin a . \sin(a+h)}$

$= \lim_{h \to 0} \frac{-\sin h/2}{h/2} \times \frac{1}{2} \times \frac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a . \sin(a+h)}$

$= -\frac{1}{2} \times \frac{2\cos\left(\frac{2a+0}{2}\right)}{\sin(a+0)}$

$= -\frac{\cos a}{\sin^2 a} = -\cot a . \operatorname{cosec} a$

iii] sec x

$f(x) = \sec x$

$Df(a) = \lim_{x \to 0} \frac{f(x) - f(a)}{h - a}$

$= \lim_{x \to 0} \frac{\sec x - \sec a}{x - a}$

$= \lim_{x \to 0} \frac{1/\cos x - 1/\cos a}{x - a}$

$= \lim_{x \to 0} \frac{\cos a - \cos x}{(x-a)\cos a . \cos x}$

Put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\cos a - \cos(a+h)}{h \times \cos a \cos(a+h)}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right)\sin\frac{-h}{2}}{\cos a \cos(a+h) \times -\frac{h}{2}}$$

$$= -\frac{1}{2} \times \frac{-2\sin\left(\frac{2a+h}{2}\right)\sin\frac{-h}{2}}{\cos a \cos(a+h) \times \frac{-h}{2}} \times -\frac{1}{2}$$

$$= -\frac{1}{2} \times \frac{-2\sin\left(\frac{2a+0}{2}\right)}{\cos a \cos(a+0)}$$

$$= -\frac{1}{2} \times -2 \frac{\sin a}{\cos a \times \cos a}$$

$$= \tan a \sec a$$

Q.2 If $f(x) = 4x+1, x \le 2$
$= x^2+5, x > 0$, at $x = 2$, then find function is differentiable or not.

Soln:
L.H.D:

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-(4 \times 2+1)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-9}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x-8}{x+2}$$

$$= \lim_{x \to 2^-} \frac{4(x-2)}{(x-2)} = 4$$

$Df(2^-) = 4$

R.H.D:

$$Df(2^+) = \lim_{x \to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x \to 2^+} \frac{x^2-4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(x+2)(x-2)}{(x-2)}$$

$$= 2+2 = 4$$

$Df(2^+) = 4$

$\therefore$ R.H.D = L.H.D

f is differentiable at $x = 2$

Q.3 If $f(x) = 4x+7, x < 3$
$= x^2+3x+1, x \ge 3$ at $x = 3$, then find f is differentiable or not?

Soln:

R.H.D: $Df(3^+) = \lim_{x \to 3^+} \frac{f(x) - f(3)}{x-3}$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-(3^2+3\times3+1)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-19}{x-3}$$

$$= \lim_{x\to 3^+} \frac{x^2+6x-3x-18}{x-3}$$

$$= \lim_{x\to 3^+} \frac{x(x+6)-3(x+6)}{x-3}$$

$$= \lim_{x\to 3^+} \frac{(x+6)(x-3)}{(x-3)} = 3+6 = 9$$

$Df(3^+) = 9$

L.H.D $= Df(3^-)$

$$= \lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x\to 3^-} \frac{4x+7-19}{x-3}$$

$$= \lim_{x\to 3^-} \frac{4x-12}{x-3}$$

$$= \lim_{x\to 3} \frac{4(x-3)}{(x-3)}$$

$Df(3^+) = 4$

R.H.D $\neq$ L.H.D

f is not differentiable at $x=3$

Q.4 If $f(x) = 8x-5$, $x \le 2$;
$= 3x^2-4x+7$, $x>2$ at $x=2$. Then find f is differentiable or not.

Soln:

$f(2) = 8\times 2-5 = 16-5 = 11$

R.H.D:

$$Df(2^+) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x^2-4x+7-11}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x^2-4x-4}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x^2-6x+2x-4}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x(x-2)+2(x-2)}{x-2}$$

$$= \lim_{x\to 2^+} \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$= 3\times 2+2 = 8$

$Df(2^+) = 8$

L.H.D:

$$Df(2^-) = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^-} \frac{8x-5-11}{x-2}$$

$$= \lim_{x\to 2^-} \frac{8x-16}{x-2}$$

$$= \lim_{x\to 2^-} \frac{8\cancel{(x-2)}}{\cancel{(x-2)}}$$

$= 8$

$Df(2^-) = 8$

L.H.D = R.H.D

f is differentiable at $x=3$

Alv
05/01/2022

PRACTICAL : 3

Title : Application of Derivatives

Q.1 Find the intervals in which function is increasing or decreasing.

a] $f(x) = x^3 - 5x - 11$

Soln : f is increasing if & only if

$f'(x) > 0$

$\therefore f(x) = x^3 - 5x - 11$

$\therefore f'(x) = 3x^2 - 5$

$\therefore 3x^2 - 5 > 0$

$x = \pm\sqrt{\frac{5}{3}}$

$\therefore$

$\therefore x \in \left(-\infty, -\sqrt{\frac{5}{3}}\right) \cup \left(\sqrt{\frac{5}{3}}, \infty\right)$

Now f is decreasing if & only if

$f'(x) < 0$

$\therefore 3x^2 - 5 < 0$

$\therefore \cancel{x = \pm\sqrt{\frac{5}{3}}}$

$\therefore x \in \left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$

b] $f(x) = x^2 - 4x$

Soln : f is increasing if & only if

$f'(x) > 0$

$\therefore f(x) = x^2 - 4x$

$\therefore f'(x) = 2x - 4$

$\therefore 2x - 4 > 0$

$\therefore 2(x-2) > 0$

$\therefore x - 2 > 0$

$\therefore x = 2$

$\therefore x \in (2, \infty)$

Now f is decreasing if & only if

$f'(x) < 0$

$\therefore 2x - 4 < 0$

$\therefore 2(x-2) < 0$

$\therefore \cancel{x - 2 < 0}$

$\therefore \cancel{x = 2}$

$\therefore x \in (-\infty, 2)$

c] $f(x) = 2x^3 + x^2 - 20x + 4$

Soln : f is increasing if & only if

$f'(x) > 0$

$\therefore f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

$\therefore 6x^2 + 2x - 20 > 0$

No.

$\therefore \quad 6x^2 + 12x - 10x - 20 > 0$

$\therefore \quad 6x(x+2) - 10(x+2) > 0$

$\therefore \quad (x+2)(6x-10) > 0$

$\therefore \quad x = -2, 5/3$

$\therefore \quad x \in (-\infty, -2) \cup (5/3, \infty)$

Now f is decreasing if & only if

$f'(x) < 0$

$\therefore \quad 6x^2 + 2x - 20 < 0$

$\therefore \quad (x+2)(6x-10) < 0$

$\therefore \quad x = -2, 5/3$

$x \in (-2, 5/3)$

d] $f(x) = x^3 - 27x + 5$

Sol.n: ~~f is~~ increasing if & only if

$f'(x) > 0$

$\therefore \quad f(x) = x^3 - 27x + 5$

$\therefore \quad f'(x) = 3x^2 - 27$

$\therefore \quad 3x^2 - 27 > 0$

$\therefore \quad 3(x^2 - 9) > 0$

$\therefore \quad x^2 - 9 > 0$

$\therefore \quad x = 3, -3$

$\therefore \quad x \in (-\infty, -3) \cup (3, \infty)$

Now f is decreasing if & only if

$f'(x) < 0$

$\therefore \quad 3x^2 - 27 < 0$

$\therefore \quad 3(x^2 - 9) < 0$

$\therefore \quad x^2 - 9 < 0$

$\therefore \quad x = 3, -3$

$\therefore \quad x \in (-3, 3)$

e] $f(x) = 69 - 24x - 9x^2 + 2x^3$

Sol.n: f is increasing if & only if $f'(x) > 0$

$\therefore \quad f(x) = 69 - 24x - 9x^2 + 2x^3$

$\therefore \quad f'(x) = -24 - 18x + 6x^2$

$\therefore \quad -24 - 18x + 6x^2 > 0$

$\therefore \quad 6(-4 - 3x + x^2) > 0$

$\therefore \quad x^2 - 3x - 4 > 0$

$\therefore \quad x^2 - 4x + x - 4 > 0$

$\therefore \quad (x-4)(x+1) > 0$

$\therefore \quad x = 4, -1$

$x \in (-1, 4)$

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Q.2 Find ~~the~~ the intervals in which function is concave upwards & concave downwards.

a] $y = 3x^2 - 2x^3$

Sol.n : $\because y = f(x)$

$\therefore f(x) = 3x^2 - 2x^3$

$\therefore f'(x) = 6x - 6x^2$

$\therefore f''(x) = 6 - 12x$

$\because$ f is concave upward if & only if

$f''(x) > 0$

$\therefore 6 - 12x > 0$

$\therefore 6(1-2x) > 0$

$\therefore (1-2x) > 0$

$\therefore -(2x-1) > 0$

$x \in (-\infty, \frac{1}{2})$

$\because$ f is concave downward if & ~~downward~~ only if

$f''(x) < 0$

$\therefore 6(1-2x) < 0$

$\therefore -(2x-1) < 0$

$x \in (\frac{1}{2}, \infty)$

b] $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

Sol.n : $\therefore y = f(x)$

$\therefore f(x) = x^4 - 6x^3 + 12x^2 + 5x$

$\therefore f'(x) = 4x^3 - 18x^2 + 24x + 5$

$\therefore f''(x) = 12x^2 - 36x + 24$

$\therefore$ f is concave upward if & only if

$f''(x) > 0$

$\therefore 12x^2 - 36x + 24 > 0$

$\therefore 12(x^2 - 3x + 2) > 0$

$\therefore x^2 - 3x + 2 > 0$

$\therefore (x-2)(x-1) > 0$

$\therefore x = 2, 1$

$x \in (-\infty, 1) \cup (2, \infty)$

$\therefore$ f is concave downward if & only if

$f''(x) < 0$

$\therefore 12x^2 - 36x + 24 < 0$

$\therefore 12(x^2 - 3x + 2) < 0$

$\therefore x^2 - 3x + 2 < 0$

$\therefore (x-2)(x-1) < 0$

$\therefore x = 2, 1$

$x \in (1, 2)$

c) $y = x^3 - 27x + 5$

Sol.n :

$\therefore y = f(x)$

$\therefore f(x) = x^3 - 27x + 5$

$\therefore f'(x) = 3x^2 - 27$

$\therefore f''(x) = 6x$

$\therefore$ f is concave upward if & only if

$f''(x) > 0$

$\therefore 6x > 0$

$\therefore x > 0$

$\therefore x = 0$

$\therefore x \in (0, \infty)$

$\therefore$ f is concave downward if & only if

$f''(x) < 0$

$\therefore 6x < 0$

$\therefore x < 0$

$\therefore x = 0$

$\therefore x \in (-\infty, 0)$

d) $y = 69 - 24x - 9x^2 + 2x^3$

Sol.n :

$\therefore y = f(x)$

$\therefore f(x) = 69 - 24x - 9x^2 + 2x^3$

$\therefore f'(x) = -24 - 18x + 6x^2$

$\therefore f''(x) = -18 + 12x$

$\therefore$ f is concave upward if & only if

$f''(x) > 0$

$\therefore -18 + 2x > 0$

$\therefore 6(2x - 3) > 0$

$\therefore 2x - 3 > 0$

$\therefore x = 3/2$

$\therefore x \in (+\frac{3}{2}, \infty)$

$\therefore$ f is concave downwards if & only if

$f''(x) < 0$

$\therefore -18 + 12x < 0$

$\therefore 6(2x - 3) < 0$

$\therefore 2x - 3 < 0$

$\therefore x = 3/2$

$\therefore x \in (-\infty, \frac{3}{2})$

e) $y = 2x^3 + x^2 - 20x + 4$

Sol.n :

$\therefore y = f(x)$

$\therefore f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

$\therefore f''(x) = 12x + 2$

$\therefore$ f is concave upwards if & only if

$f''(x) > 0$

$\therefore 12x + 2 > 0$

$\therefore 2(6x+1) > 0$

$\therefore 6x+1 > 0$

$\therefore x > -1/6$

$\therefore x \in (-1/6, \infty)$

$\therefore$ f is concave downward if & only if

$f''(x) < 0$

$\therefore 12x + 2 < 0$

$\therefore 2(6x+1) < 0$

$\therefore 6x+1 < 0$

$x < -1/6$

$\therefore x \in (-\infty, -1/6)$

$\therefore \cancel{x \in (-\infty, -1/6)}$

$\cancel{x \in (-\tfrac{1}{6}, \pm\infty)}$

18/12/19

PRACTICAL : 04

Topic : Application of Derivatives
&
Newton's method

Q.1 Find maximum & minimum values of following functions :

(i) $f(x) = \cancel{3 - 5x^3 + 3x^5}\ x^2 + \frac{16}{x^2}$

(ii) $f'(x) = 2x - \frac{32}{x^3}$

For maxima/minima

$f'(x) = 0$

$2x - \frac{32}{x^3} = 0$

$2x = \frac{32}{x^3}$

$\therefore x^4 = 16$

$x = \pm 2$

$f''(x) = 2 + \frac{96}{x^4}$

$f''(2) = \cancel{f''(-2)} = 2 + \frac{96}{(\pm 2)^4} = 2 + \frac{96}{16} = 8 > 0$

$\therefore f(x)$ is minimum at $x = \pm 2$

$\therefore f(2) = 8$ is the minimum value.

(ii) $f(x) = 3 - 5x^3 + 3x^5$

$f'(x) = 15x^4 - 15x^2$

For maxima / minima

$f'(x) = 15x^4 - 15x^2 = 0$

$x^4 - x^2 = 0$

$x^2(x^2 - 1) = 0$

$x = 0, -1, 1$

$f''(x) = 60x^3 - 30x$

$f(0) = 0$

$f''(-1) = -60 + 30 = -30 < 0$

$f^{-1}(1) = 60 - 30 = 30 > 0$

$f(x)$ is minimum at -1 and

$f(-1) = 3 + 5 - 3 = 5$

$f(1) = 3 - 5 + 3 = 1$

(iii) $f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$

For maxima/minima

$f'(x) = 0$

$3x^2 - 6x = 0$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0, 2$

$f''(x) = 6x - 6$

$f''(0) = -6 < 0$

$f''(2) = 6 > 0$

$\therefore f(x)$ is maxima at $x=0$ & minimum at $x=2$

$f(0) = 1$

$f(2) = -3$

(iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$ (ans)

$f'(x) = 6x^2 - 6x - 12$

For maxima / minima,

$f'(x) = 0$

$f(x^2 - 6x - 12) = 0$

$x^2 - x - 2 = 0$

$x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0$

$(x+1)(x-2) = 0$

$x = -1, 2$

$f''(x) = 12x - 6$

$f''(-1) = -12 - 6 = -18 < 0$

$f''(2) = 24 - 6 = 18 > 0$

$f(x)$ is maximum at $x = 1$ and minimum at $x = 2$

$\therefore f(-1) = 8$

$\therefore f(2) = -19$

2) (i) $f(x) = x^3 - 3x^2 - 55x + 95$

$f'(x) = 3x^2 - 6x - 55$

$x_1 = 0$

$f(x_0) = 9.5$

$f'(x_0) = -55$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 0 - \frac{9.5}{-55}$

$x_1 = 0.1727$

$f(x_1) = -0.0128$

$f'(x_1) = -55.9467$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.1727 - \frac{0.0828}{-55.9467}$

$x_2 = 0.1712$

$f(x_2) = 0.0011$

$f'(x_2) = -55.9393$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.172 - \frac{0.0011}{-55.9391}$$

$\therefore x_3 = 0.1712$

0.1712 is the root of the equation.

(ii) $f(x) = x^3 - 64x - 9$

$f'(x) = 3x^2 - 4$

$f(2) = -9$

$f(3) = 6$

f is closer to 0 in the number line

$x_0 = 3$

$f(x_0) = 6$

$f'(x_0) = 23$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$x_1 = 2.7391$

$f(x_1) = 0.5942$

$f'(x_1) = 18.508$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.7391 - \frac{0.5942}{18.508}$$

$\therefore x_2 = 2.707$

$\therefore f(x_2) = 0.0085$

$\therefore f'(x_2) = 17.9835$

$$\therefore x_3 = x_3 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.707 - \frac{0.0085}{17.9835}$$

$x_3 = 2.7065$

$f(x_3) = -0.005$

$f'(x_3) = 17.9757$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 2.7065 - \frac{0.0005}{17.9757}$$

$x_4 = 2.7065$

2.7065 is the root of the equation.

(iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$

$f'(x) = 3x^2 - 36x - 10$

$f(1) = 1 - 18 - 10 + 17 = 6.2$

$f(2) = -2.2$

$\therefore$ -2.2 is closer to 0 in the number line.

$\therefore x_0 = 2$

$f(x_0) = -2.2$

$f'(x_0) = -5.2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{2.2}{-5.2}$$

$x_1 = 1.5769$

$f(x_1) = (1.5764)^3 - 1.8(1.5764)^2$

$= 0.6762$

$f'(x_1) = -8.217$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.5764 - \frac{0.6762}{-8.217}$

$x_2 = 1.6592$

$f(x_1) = 0.0204$

$f'(x_2) = -7.7143$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 1.6592 - \frac{0.0204}{-7.7143}$

$x_3 = 1.6618$

$\therefore f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 16(1.6618) + 17$

$\therefore f(x_3) = 6$

$\therefore$ 1.6618 is the root of the equation.

PRACTICAL : 05

Topic : Integration

Q.1 Solve the following integration:

i) $\int \frac{dx}{\sqrt{x^2+2x-3}}$

$I = \int \frac{1}{\sqrt{x^2+2x-3}} dx$

$= \int \frac{1}{\sqrt{x^2+2x+1-4}} dx$

$= \int \frac{1}{\sqrt{(x+1)-2^2}} dx$

$\therefore I = x+1 + \sqrt{x^2+2x-3} + c$

ii) $\int (4e^{3x}+1) dx$

$I = \cancel{\int 4e} \; 4\int e^{3x} dx + 1 dx$

$I = \frac{4e^{3x}}{3} + x + c$

iii) $\int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$

$\rightarrow I = \int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$

$= 2\int x^2\,dx - 3\int \sin x\,dx + 5x^{1/2}\,dx$

$= \frac{2x^3}{3} + 3\cos x + \frac{5x^2}{3} \cdot x^{3/2} + c$

$I = \frac{2x^3}{3} + 3\cos x + \frac{10}{3} x\sqrt{x} + c$

iv) $\int \frac{x^3 + 3x + 4}{\sqrt{x}}\,dx$

Soln : Put $\sqrt{x} = t$

$\therefore \frac{1}{2\sqrt{x}} = \frac{dt}{dx}$

$\therefore \frac{dx}{\sqrt{x}} = 2\,dt$

$I = \int \frac{(\sqrt{x})^6 + 3(\sqrt{x})^2 + 4}{\sqrt{x}}\,dx$

$= \cancel{2\int t^6 + 3t^2 + 4\,dt}$

$= 2\left[\frac{t^7}{7} + \frac{3t^3}{3} + 4t\right] + c$

$= 2\left[\frac{x^{3/2}}{7} + 3^{3/2} + 4x^{1/2}\right] + c$

v) $\int t^7 \sin(2t^4)\,dt$

$I = \int t^7 \sin(2t^4)\,dt$

$= \int t^7 \sin(2t^4)\,dt$

Put $t^4 = x$

$4t^3 = \frac{dx}{dt}$

$\therefore t^4\,dt = \frac{1}{4}\,dx$

$\therefore I = \frac{1}{4}\int x \cdot \sin(2x)\,dx$

$\therefore I = \frac{1}{4}\left[x\int \sin 2x\,dx - \int\left[\frac{d}{dx}x \int \sin 2x\,dx\right]dx\right]$

$= \frac{1}{4}\left[-\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x \cdot dx\right]$

$= \frac{1}{16}\sin 2x - \frac{x\cos 2x}{8} + c$

$I = \frac{1}{16}\sin 2t^4 - \frac{t^4 - \cos 2t^4}{8} + 6$

vi) $\int \sqrt{x}\,(x^2 - 1)\,dx$

$\rightarrow I = \int \sqrt{x}\,(x^2 - 1)\,dx$

$I = \int x^2\sqrt{x}\,dx - \int \sqrt{x}\,dx$

$= \int x^{5/2}\,dx - \int x^{1/2}\,dx$

$= \cancel{\frac{x^{7/2}}{7/2}} - \frac{x^{3/2}}{3/2} + c$

$= \frac{2}{7}x^3\sqrt{x} - \frac{2}{3}x\sqrt{x} + c$

(vii) $\int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

$\rightarrow I = \int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

$\frac{1}{x^2} = t$

$\therefore \frac{-2}{x^3} = \frac{dt}{dx}$

$\therefore \frac{dx}{x^3} = \frac{-1}{2} dt$

$I = \frac{-1}{2} \int \sin t \, dt$

$= \frac{-1}{2} x - \cos t + c$

$= \frac{\cos t}{2} + c$

$I = \frac{\cos\left(1/x^2\right)}{2} + c$

(viii) $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}} dx$

Put $\sin x = t$

$\therefore \cos x \, dx = dt$

$\therefore I = \int \frac{1}{t^{2/3}} dt$

$= \int t^{2/3} dt$

$= \frac{t^{-2/3+1}}{-2/3+1} + c$

$= 3t^{1/3} + c$

$I = 3\sqrt[3]{\sin x} + c$

(ix) $\int e^{\cos^2 x} \sin 2x \, dx$

$I = \int e^{\cos^2 x} \cdot \sin 2x \, dx$

Put $\cos^2 x = t$

$\therefore -2\cos^2 x = t$

$\therefore \sin 2x \, dx = -dt$

$I = -\int e^t \, dt$

$= -e^t + c$

$\therefore I = -e^{\cos^2 x} + c$

(x) $\int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1}\right) dx$

$\therefore I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1}\right) dx$

$x^3 - 3x^2 + 1 = t$

$\therefore 3x^2 - 6x = \frac{dt}{dx}$

$(x^2 - 2x) \, dx = \frac{dt}{3}$

$\therefore I = \frac{1}{3} \int \frac{1}{t} dt$

$= \frac{1}{3} \log|t| + c$

$\therefore I = \frac{1}{3} \log|x^3 - 3x^2 + 1| + c$

PRACTICAL : 6

Topic : Application of Integration & Numerical Integration.

Q.1] Find the length of the following curve

① $x = t \sin t,\ y = 1 - \cos t \quad t \in [0, 2\pi]$

Solution :

arc length $= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\ dt$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1}\ dt$

$= \int_0^{2\pi} \sqrt{2 - \cos t}\ dt$

$= \int_0^{2\pi} 2 \left| \sin \pi/2 \right| dt$

$= \int_0^{2\pi} 2 \sin \frac{1}{2}\, dt$

$= \left[-4 \cos \left(\frac{1}{2} \right) \right]_0^{2\pi}$

$= \left[-4 \cos \pi \right] - \left(-4 \cos 0 \right) = 4 + 4 = 8$

② $y = \sqrt{4 - x^2} \quad x \in [-2, 2]$

$\rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{4 - x^2}} \times (-2x)$

$= \frac{-x}{\sqrt{4 - x^2}}$

$L = \int_{-2}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_{-2}^{2} \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx$

$= \int_{-2}^{2} \sqrt{\frac{4 - x^2 + x^2}{4 - x^2}}\, dx$

$= \int_{-2}^{2} \sqrt{\frac{4}{4 - x^2}}\, dx$

$= 2 \int_{-2}^{2} \frac{1}{\sqrt{2^2 - x^2}}\, dx$

$= 2 \left[\sin^{-1}(1) - \sin^{-1}(-1) \right]$

$= 2 \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right]$

$L = 2\pi$

(3) $y = x^{3/2}$ in $[0,4]$

$\rightarrow \dfrac{dy}{dx} = \dfrac{3}{2} x^{1/2}$

$\therefore L = \displaystyle\int_0^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= \displaystyle\int_0^4 \sqrt{1+\frac{9}{4}x}\, dx$

$= \dfrac{1}{2}\displaystyle\int_0^4 \sqrt{4+9x}\, dx$

$= \dfrac{1}{2}\left[\dfrac{(4+9x)^{3/2}}{3/2} \times \dfrac{1}{9}\right]_0^4$

$= \dfrac{1}{27}\left[(4+9x)^{3/2}\right]_0^4$

$= -\dfrac{1}{27}\left[(4+0)^{3/2} - (4+36)^{3/2}\right]$

$\therefore L = \dfrac{1}{27}\left(40^{3/2} - 8\right)$ units

(4) $x = 3\sin t, y = 3\cos t, \quad t \in (0, 2\pi)$

$\dfrac{dx}{dt} = 3\cos t \qquad \dfrac{dy}{dt} = -3\sin t$

$L = \displaystyle\int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$

$= \displaystyle\int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\, dt$

$= \displaystyle\int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$

$= \displaystyle\int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$

$= \displaystyle\int_0^{2\pi} 3\sqrt{x}\, dt$

$= 3\displaystyle\int_0^{2\pi} dt$

$= 3\left[x\right]_0^{2\pi}$

$= 3(2\pi - 0)$

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$\therefore \quad L = 6\pi$ units

(5) $x = \frac{1}{6}y^3 + \frac{1}{2y}$ on $y \in [1, 2]$

$\rightarrow \quad \therefore \frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$

$\frac{dx}{dy} = \frac{y^4 - 1}{2y^2}$

$$L = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{1 + \frac{(y^4 - 1)}{4y^4}}\, dy$$

$$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y^2)^2}}\, dy$$

$$= \int_1^2 \frac{y^4+1}{2y^2}\, dy$$

$$= \frac{1}{2}\int_1^2 y^2\, dy + \frac{1}{2}\int_1^2 y^{-2}\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} - \frac{y^{-2}}{1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} - \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{17}{6}\right]$$

$L = \frac{17}{12}$ units

Q.2]

(i) $\int_0^2 e^{x^2}\, dx$ with $n=4$

$L = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

x	0	0.5	1	1.5	2
y	1	1.284	2.7183	9.4875	54.5982

$$\int_0^2 e^{x^2}\, dx = \frac{L}{3}\left[(y_0 + y_2) + 4(y_1 + y_3) + (y_e)\right]$$

$$= \cancel{\frac{0.5}{3}\left[(1 + 54.5482) + 5(1.285 + 9.487) + 2x\right]}$$

$$= \frac{0.5}{3}\left[55.5982 + 43.0868 + 5.416\right]$$

$$\int_0^2 e^{x^2}\, dx = 17.3535$$

(ii) $\int_0^4 x^2\,dx \quad n = 4$

$$L = \frac{4-0}{4} = 1$$

x	0	1	2	3	4
y	0	1	4	9	16

$$\int_0^4 x^2\,dx = \frac{L}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right]$$

$$= \frac{1}{3}\left[0 + 16 + 4(1+9) + 2\times4\right]$$

$$= \frac{1}{3}\left(16 + 4(10) + 8\right)$$

$$= \frac{64}{3}$$

$$\int_0^4 x^2\,dx = 21.3333$$

(iii) $\int_0^{\pi/3} \sqrt{\sin x}\,dx$ with $n = 6$

$$L = \frac{\pi/3 - 0}{6} = \frac{\pi}{18}$$

x	0	$\pi/18$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$
y	0	0.4167	0.5848	0.7071	0.8017	0.8752	0.4306

$$\int_0^{\pi/3} \sqrt{\sin x}\,dx = \frac{L}{3}\left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{\pi/18}{3}\left[0.4167 + 0.9306 + 2(0.4167 + 0.7071 + 0.8752) + 2(0.5848 + 0.847)\right]$$

$$= \frac{\pi}{54}\left[1.3473 + 4(1.999) + 2(1.3865)\right]$$

$$= \frac{\pi}{54}\left[1.3473 + 7.996 + 2.773\right]$$

$$= \frac{\pi}{54} \times 12.1163$$

$$\int_0^{\pi/3} \sqrt{\sin x}\,dx = 0.7049$$

08/01/2020

PRACTICAL : 07

Q.1 Solve the following differential equation :

(i) $x \frac{dy}{dx} + y = e^x$

(ii) $e^x \frac{dy}{dx} + 2e^x y = 1$

(iii) $x \frac{dy}{dx} = \frac{\cos x}{x} - 2y$

(iv) $x \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

(v) $e^{2x} \frac{dy}{dx} + 2e^{2x} y = 2x$

(vi) $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

(vii) $\frac{dy}{dx} = \sin^2(x - y + 1)$

(viii) $\frac{dy}{dx} = \frac{2x + 3y - 1}{6x + 9y + 6}$

(i) Sol.n :

$x \frac{dx}{dy} + y = e^x$

$\frac{dy}{dx} + \frac{1}{x} y = \frac{e^x}{y}$

$P(x) = 1/x$. & $Q(x) = e^x/x$

$I.F = e^{\int P(x) dx}$

$= e^{\int 1/x \, dx}$

$= e^{\ln x}$

$I.F = x$

$y.(I.F) = \int (I.F) dx + c$

$y.x = \int \frac{e^x}{x} \times x \, dx + c$

$= \int e^x dx + c$

$xy = e^x + c$

(ii) Sol.n :

$$e^x \frac{dy}{dx} + 2e^x y = 1$$

$$\frac{dy}{dx} + 2y = e^{-x}$$

$P(x) = 2$ $\quad Q(x) = e^{-x}$

$$I.F = e^{\int P(x)\,dx}$$
$$= e^{\int 2\,dx}$$
$$= e^{2x}$$

$$y\,(I.F) = \int Q(x)(I.F)\,dx + c$$

$$y \cdot e^{2x} = \int e^{-x} \times e^{2x}/x + c$$
$$= \int e^{-x+2x}\,dx + c$$
$$= \int e^{x}\,dx + c$$

$$y \cdot e^{2x} = e^{x} + c$$

(iii) Sol.n :

$$x\frac{dy}{dx} = \frac{\cos x}{x} - 2y$$

$$\therefore \frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}$$

$P(x) = 2/x$ $\quad Q(x) = \frac{\cos x}{x^2}$

$$I.F = e^{\int P(x)\,dx}$$
$$= e^{\int 2/x\,dx}$$
$$= e^{2\ln x}$$
$$= e^{\ln x^2}$$
$$I.F = x^2$$

$$y\,(I.F) = \int Q_1(x) * (I.F)\,dx + c$$

$$x^2 y = \int \frac{\cos x}{\cancel{x^2}} \times x^2\,dx + c$$

$$= \int \cos x + c$$

$$x^2 y = \sin x + c$$

(iv) Sol.n :

$$x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$$

$$\therefore \frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^3}$$

$P(x) = 3/x$ $\quad$ $Q(x) = \sin x / x^3$

$$I.F. = e^{\int P(x)dx}$$
$$= e^{\int 3/x\, dx}$$
$$= e^{3 \ln x}$$
$$I.F. = x^3$$

$$y(I.F) = \int Q(x)(I.F.)dx + c$$

$$x^3 y = \int \frac{\sin x}{x^3} \times x^3 dx + c$$

$$= \int \sin x\, dx + c$$

$$x^3 y = -\cos x + c$$

(v) Sol.n :

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$$

$$\therefore \frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$$

$P(x) = 2$ $\quad$ $Q(x) = 2x e^{-2x}$

$$I.F. = e^{\int P(x)dx}$$
$$= e^{\int 2dx}$$
$$= e^{2x}$$

$$y(I.F.) = \int Q(x)(I.F.)dx + c$$

$$y \cdot e^{2x} = \int 2x \times e^{-2x} \times e^{2x} dx + c$$

$$ye^{2x} = \int 2x\, dx + c$$

$$\therefore ye^{2x} = x^2 + c$$

STD

(vi) Sol.n:
$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$\sec^2 x \cdot \tan y \, dx = -\sec^2 y \cdot \tan x \, dy$

$\frac{\sec^2 x \, dx}{\tan x} = \frac{-\sec^2 y \, dy}{\tan y}$

$\int \frac{\sec^2 x \cdot dx}{\tan x} = -\int \frac{\sec^2 y \, dy}{\tan y}$

$\therefore \log|\tan x| = -\log|\tan y| + c$

$\log|\tan x - \tan y| = c$

$\tan x \cdot \tan y = e^c$

(vii) Sol.n:
$\frac{dy}{dx} = \sin^2(x-y+1)$

Put $x-y+1=v$

Differentiating both sides,

$x-y+1 = v$

$\therefore 1 - \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore 1 - \frac{dv}{dx} = \frac{dy}{dx}$

$\therefore 1 - \frac{dv}{dx} = \sin^2 v$

$\frac{dv}{dx} = 1 - \sin^2 v$

$\frac{dv}{dx} = \cos^2 v$

$\therefore \frac{dv}{\cos^2 v} = dx$

$\therefore \int \sec^2 v \, dv = \int dx$

$\therefore \tan v = x + c$

$\therefore \tan(x+y-1) = x + c$

(viii) Sol.n: $\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$

Put $2x+3y = v$

$\therefore 2 + 3\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\frac{v-1}{(v+2)}$

$\therefore \frac{dv}{dx} = \frac{v-1}{v+2} + 2$

$\therefore \frac{dv}{dx} = \frac{v-1+2v+4}{v+2}$

$= \frac{3v+3}{v+2}$

$$= 3\left(\frac{v+1}{v+2}\right)$$

$$\therefore \int\left(\frac{v+2}{v+1}\right) dv = 3\int dx$$

$$\int \frac{v+1}{v+1}\, dv + \int \frac{1}{v+1}\, dv = 3x + c$$

$$\therefore v + \log|v+1| = 3x + c$$

$$\therefore 2x + 3y + \log|2x+3y+1| = 3x + c$$

$$3y = x - \log|2x+3y+1| + c$$

PRACTICAL : 08

Topic : Euler's Method

Q.1 $\frac{dy}{dx} = y + e^x - 2$ $y(0) = 2$ $h = 0.5$. Find $y(2) = ?$

Sol.n : $f(x) = y + e^x - 2$ $x_0 = 0$ $y(0) = 2$ $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.5743
2	1	3.5743	4.2925	5.7205
3	1.5	5.7205	8.2021	9.8215
4	2	9.8215		

$y(2) = 9.8215$

Q2 $\frac{dy}{dx} = 1 + y^2$, $y(0) = 1$ $h = 0.2$ find $y(1) = ?$

$y_0 = 0$, $y_0 = 0$ $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.408
2	0.4	0.408	1.1664	0.6412
3	0.6	0.6412	1.4111	0.9234
4	0.8	0.9234	1.8526	1.2939
5	1	1.2939		

$y(1) = 1.2939$

Q3 $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$ $y(0)=1$ $h=0.2$ find $y(1)=2$

$x_0 = 0$ $y(0)=1$ $h=0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1	0	1
1	0.2	1	0.4472	1.0894
2	0.4	1.0894	0.6059	1.2105
3	0.6	1.2105	0.7040	1.3513
4	0.8	1.3513	0.7696	1.5051
5	1	1.5051		

$\therefore y(1) = 1.5051$

Q4 a) $\frac{dy}{dx} = 3x^2+1$ $y(1)=2$ find $y(2)$ $h=0.5$

$h=0.25$

$y_0 = 2$ $x_0 = 1$ $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	4
1	1.5	4	7.75	7.875
2	2	7.875		

$y(2) = 7.875$

b) $y_0 = 2$ $x_0 = 1$ $h = 0.25$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4218
2	1.5	4.4218	59.6564	19.3360
3	1.75	19.3360	1122.4426	299.9960
4	2	299.9960		

$\therefore y(2) = 299.9960$

Q5 $\frac{dy}{dx} = \sqrt{xy} + 2$ $y(1)=1$ $h=0.2$

$\rightarrow f(x) = \sqrt{xy+2}$ $x_0 = 1$ $y_0 = 1$ $h=0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	3.6
1	1.2	3.6		

$y(1.2) = 3.6$

Practical - 9

Topic : Limits & Partial Order derivatives

Q1 Evaluate the following limits

① $\lim_{(x,y)\to(-4,-1)} \frac{x^3-3x+y^2-1}{xy+5}$

$= \lim_{(x,y)\to(-4,-1)} \frac{x^3-3x+y^2-1}{xy+5}$

Apply limit,

$= \frac{(-4)^3-3(-4)+(-1)^2-1}{(-4)(-1)+5} = \frac{-64+12+1-1}{4+5} = \frac{-52}{9}$

② $\lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$

$= \lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$ # Apply limit

$= \frac{(0+1)((2)^2+(0)^2-4(2))}{2+3(0)} = \frac{1(4+0-8)}{2} = \frac{4-8}{2} = \frac{-4}{2} = -2$

③ $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2y^2}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2y^2}$

Apply limit,

$= \frac{(1)^2-(1)^2(1)^2}{(1)^3-(1)^2(1)(1)} = \frac{1-1}{1-1} = \frac{0}{0}$ ∴ Limit does not exist

Q2 Find f_x, f_y for each of the following of :

① $f(x,y) = xye^{x^2+y^2}$

$f_x = y(1.e^{x^2+y^2}) + xy(e^{x^2+y^2}.2x)$

$= ye^{x^2+y^2} + 2x^2ye^{x^2+y^2}$

$f_y = x(1.e^{x^2+y^2}) + xy(e^{x^2+y^2})$

$= x.e^{x^2+y^2} + 2xy^2.e^{x^2+y^2}$

∴ $f_x = y.e^{x^2+y^2} + 2x^2ye^{x^2+y^2}$

∴ $f_y = xe^{x^2+y^2} + 2xy\,e^{x^2+y^2}$

② $f(x,y) = e^x \cos y$

$f_x = \cos y\, e^x$

$f_y = e^x - \sin y$

∴ $f_y = -\sin y\, e^x$

③ $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f_x = y^2\,3x^2 - 3y^2x + 0 + 0$

$= 3x^2y^2 - 6xy$

$f_y = x^3\,2y - 3x^2 + 3y^2$

$= 2x^3y - 3x^2 + 3y^2$

Q3 Using definition find values of f_x, f_y at (0,0) for

$f(x,y) = \frac{2x}{1+y^2}$

$f_x(a,b) = \lim_{h\to0} \frac{f(a+h,b) - f(a,b)}{h}$

$f_y(a,b) = \lim_{h\to0} \frac{f(a,b+h) - f(a,b)}{h}$

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According to given $(a,b) = 0,0$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{2h - 0}{h} = 2$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\therefore f_x = 2 \quad , \quad f_y = 0$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\therefore f_x = 2 \quad f_y = 0$

Q4 Find all second order partial derivatives of f. Also verify whether $f_{xy} = f_{yx}$

(i) $f(x,y) = \dfrac{y^2 - xy}{x^2}$

$\therefore f_{xx} = \dfrac{d^2f}{dx^2} \qquad f_{yy} = \dfrac{d^2f}{dy^2}$

Applying $\dfrac{u}{v}$ rule

$$f_x = \frac{x^2(0-y) - (y^2-xy)2x}{x^4}$$

$$= \frac{-x^2y - 2xy^2 + 2x^2y}{x^4}$$

$$\therefore f_x = \frac{x^2y - 2xy^2}{x^4}$$

$$\therefore f_{xx} = \frac{x^4(2xy - 2y^2) - (x^2y - 2xy^2)(4x^3)}{x^8}$$

$$= \frac{2x^5y - 2x^4y^2 - (4x^5y - 8x^4y^2)}{x^8}$$

$$= \frac{2x^5y - 2x^4y^2 - (4x^5y - 8x^4y^2)}{x^8}$$

$$= \frac{-2x^5y + 6x^4y^2}{x^8}$$

$$= \frac{6x^4y^2 - 2x^5y}{x^8}$$

$$f_{xx} = \frac{6y^2 - 2xy}{x^4}$$

$f_y = \dfrac{1}{x^2}(2y - x) \qquad \therefore f_y = \dfrac{2y-x}{x^2}$

$$\therefore f_{yy} = \frac{1}{x^2} \, 2 = \frac{2}{x^2}$$

$$\therefore f_{xy} = \frac{2y-x}{x^2}$$

$$= \frac{x^2(-1) - (2y-x)(2x)}{x^4}$$

$$= \frac{-x^2 - 4xy + 2x^2}{x^4}$$

$$= \frac{x^2 - 4y}{x^3}$$

$$\therefore f_{xy} = \frac{x - 4y}{x^3}$$

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$\therefore f_{yx} = \dfrac{x^2y - 2xy}{x^4}$

$= \dfrac{x^2 - 4xy}{x^4}$

$= \dfrac{x - 4y}{x^3}$

$= f_{xy}$

$\therefore f_{yx} = f_{xy}$

Hence, verified.

Q.S ① $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

② $f(x,y) = 1 - x + y\sin x$ at $(\pi/2, 0)$

③ $f(x,y) = \log x + \log y$ at $(1,1)$

① Sol.n: $f(x,y) = \sqrt{x^2+y^2}$ $(a,b) = (1,1)$

$\therefore f_x = \dfrac{1}{2\sqrt{x^2+y^2}} \times 2x$ $f_y = \dfrac{2y}{2\sqrt{x^2+y^2}}$

$f_x = \dfrac{x}{\sqrt{x^2+y^2}}$ $\therefore f_y = \dfrac{y}{\sqrt{x^2+y^2}}$

$\therefore f_x(1,1) = \dfrac{1}{\sqrt{2}}$ $\therefore f_y(1,1) = \dfrac{1}{\sqrt{2}}$

$\therefore L(x,y)\ f(1,1) + f_x(1,1)(x-1)\ f_y(1,1)(y-1)$

$= \sqrt{2} + \dfrac{x}{\sqrt{2}}.1 + \dfrac{y.1}{\sqrt{2}}$

$= \dfrac{\sqrt{2} + \dfrac{x + y \cdot 2}{\sqrt{2}}}{\sqrt{2}}$

$L(x,y) = \dfrac{x+y}{\sqrt{2}}$

② Sol.n: $f(x,y) = 1 - x + y\sin x$ $(a,b) = [\pi/2, 0)$

$f(\pi/2, 0) = 1 - \dfrac{\pi}{2} + 0 + \sin\pi/2$

$f(\pi/2, 0) = \dfrac{2}{2} - \pi$

$f_x = -1 + y\cos x$ $f_y = \sin x$

$f_x(\pi/2, 0) = -1 + 0 \cdot \cos\pi/2$

$f_x(\pi/2, 0) = -1$ $f_y(\pi/2, 0) = \sin\pi/2$

$f_y(\pi/2, 0) = 1$

$L(x,y) = f(\pi/2, 0) + f_x(\pi/2, 0)(x - \pi/2) + f_y(\pi/2, 0)(y - 0).$

$\therefore L(x,y) = \dfrac{2}{2} - \pi + (-1)(x - \pi/2) + 1(y)$

$= 1 - \pi/2 - x + \pi/2 + y$

$L(x,y) = 1 - x + y$

③ Sol.n: $f(x,y) = \log x + \log y$ $(a,b) = (1,1)$

$f(1,1) = \log 1 + \log 1$

$f(1,1) = 0$

$f_x = 1/x$

$f_x(1,1) = 1$

$\therefore L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= 0 + 1(x-1) + 1(y-1)$

$L(x,y) = x + y - 2$

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Practical 10 :

Q.1 Find the directional derivative of the given vector out of the points.

(i) $f(x,y) = x + 2y - 3$ at $\bar{u} = 3\hat{i} - \hat{j}$ ~ $a(1,-1)$
(ii) $f(x,y) = y^2 - 6x + 1$ at $\bar{u} = \hat{i} + 5\hat{j}$, $a(3,4)$
(iii) $f(x,y) = 2x + 3y$ at $\bar{u} = 3\hat{i} + 4\hat{j}$, $a(7,2)$

Q.2 Find Gradient vector for the following function at a given point:

(i) $f(x,y) = x^y + y^x$, $a \in (1,1)$
(ii) $f(x,y) = (\tan^{-1} x) - y^2$; $a \in (1,-1)$
(iii) $f(x,y,z) = xyz - e^{x+y+z}$, $a = (1,-1,0)$

Q.3 Find the eq.n of following tangent & normal of each of the following curves :

(i) $x^2 \cos y + e^{xy} = 2$ at $(1,0)$
(ii) $x^2 + y^2 - 2x + 3y + 2 = 0$ at $(1,-2)$

Q.4 Find the eq.n of tangent & normal of line to each of following surface :

(i) $x^2 - 2y^2 - 3xy + xz = 7$ at $(2,10)$
(ii) $3xyz - x - y + z = -4$ at $(1,-1,2)$

Q.5 Find the local maxima & minimum for following eq.n :

(i) $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$
(ii) $f(x,y) = 2x^3 + 3x^2y^4 - y^2$

Q.1 (i) Sol.n : $\bar{u} = 3\hat{i} - \hat{j}$

$\therefore \hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{1}{\sqrt{3^2+(-1)^2}}(3\hat{i} - \hat{j})$

$\therefore \hat{u} = \frac{1}{\sqrt{10}}(3\hat{i} - \hat{j})$

$\hat{u} = \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$a = (1,-1)$

$\therefore f(a) = 1 + 2(-1) - 3$
$= 1 + (-2) - 3$
$= -4$

$f(a+hu) = f\left((1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)\right)$

$= f\left(\left(1 + \frac{3}{\sqrt{10}}h\right), \left(-1 - \frac{h}{\sqrt{10}}\right)\right)$

$= 1 + \frac{3}{\sqrt{10}}h + 2\left(-1 - \frac{h}{\sqrt{10}}\right) - 3$

$= 1 + \frac{3}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$

$f(a+hu) = \frac{h}{\sqrt{10}} - 4$

$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$

$= \lim_{h \to 0} \frac{h/\sqrt{10} - 4 - (-4)}{h}$

$= \frac{1}{\sqrt{10}} \lim_{h \to 0} \frac{h}{h}$

$= \frac{1}{\sqrt{10}}$

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(ii) Sol.n: $f(x,y) = y^2 - 4x + 1$ $\qquad a = (3,4)$

$\bar{u} = \hat{i} + 5\hat{j}$

$\therefore \hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{\hat{i} + 5\hat{j}}{\sqrt{1^2 + 5^2}} = \frac{1}{\sqrt{26}}(\hat{i} + 5\hat{j})$

$\hat{u} = \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$f(a) = 4^2 - 4(3) + 1$

$= 16 - 12 + 1$

$f(a) = 5$

$f(a + hu) = f\left((3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)\right)$

$= f\left(\left(3 + \frac{h}{\sqrt{26}}\right), \left(4 + \frac{5h}{\sqrt{26}}\right)\right)$

$= \left(4 + \frac{5h}{\sqrt{26}}\right)^2 - 4\left(3 + \frac{h}{\sqrt{26}}\right) + 1$

$= 16 + \frac{40h}{\sqrt{26}} + \frac{25h^2}{26} - 12 - \frac{4h}{\sqrt{26}} + 1$

$= \frac{25h^2}{26} - \frac{36h}{\sqrt{26}} + 5$

$D_u f(a) = \lim_{h \to 0} \frac{f(a + hu) - f(a)}{h}$

$= \lim_{h \to 0} \frac{\frac{25h^2}{26} - \frac{36h}{\sqrt{26}} + 5 - 5}{h}$

$= \lim_{h \to 0} \frac{h\left(\frac{25h}{26} - \frac{36}{\sqrt{26}}\right)}{h}$

$= \lim_{h \to 0} \frac{25h}{26} - \frac{36}{\sqrt{26}}$

$D_u f(a) = \frac{36}{\sqrt{26}}$

(iii) Sol.n: $f(x,y) = 2x + 3y$ $\qquad a \in (1,2)$

$\bar{u} = 3\hat{i} + 4\hat{j}$

$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{1}{\sqrt{3^2 + 4^2}}(3\hat{i} + 4\hat{j})$

$= \frac{1}{\sqrt{25}}(3\hat{i} + 4\hat{j})$

$\hat{u} = \left(\frac{3}{3}, \frac{4}{3}\right)$

$f(a) = 2(1) + 3(2)$

$= 2 + 6$

$= 8$

$f(a + hu) = f\left((1,2) + h\left(3/5, 4/5\right)\right)$

$= f\left(\left(1 + \frac{3h}{5}\right), \left(2 + \frac{4h}{5}\right)\right)$

$= 2\left(1 + \frac{3h}{5}\right) + 3\left(2 + \frac{4h}{3}\right)$

$= 2 + 6h + 6 + \frac{12h}{5}$

$= 18h/5 + 8$

$D_u f(a) = \lim_{h \to 0} \frac{f(a + hu) - f(a)}{h}$

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$$D. f(x) = \lim_{h \to 0} \frac{\frac{18h}{5} + 8 - 8}{h}$$

$$= \lim_{h \to 0} \frac{18 h/5}{h}$$

$$= \frac{18}{5}$$

Q.2 (i) Sol.n: $f(x,y) = x^y + y^x$

$f_x = \frac{d}{dx}(x^y + y^x)$

$f_x = yx^{y-1} + y^x \cdot \log y$

$f_y = \frac{d}{dy}\, x^y + y^x$

$f_y = xy^{x-1} + x^y \cdot \log x$

$\nabla f(x,y) = (f_x, f_y)$

$\nabla f(x,y) = (yx^{y-1} + y^x \cdot \log y,\ xy^{x-1} + x^y \log x)$

$\nabla f(1,1) = (1(1)^{1-1} + 1' \log 1,\ 1(1)^{x-1} + 1' \log 1)$

$\nabla f(1,1) = (1,1)$

(ii) Sol.n: $f(x,y) = (\tan^{-1} x)\, y^2$

$f_x = \frac{d}{dx}(\tan^{-1} x) y^2$

$f_x = \frac{y^2}{1+x^2}$

$f_y = \frac{d}{dy}(\tan^{-1} x) y^2$

$= 2y \tan^{-1} x$

$\nabla f(x,y) = f(f_x, f_y)$

$= f\left(\frac{y^2}{1+x^2}, 2y \tan^{-1} x\right)$

$\nabla f(1,-1) = \left(\frac{1^2}{1+(-1)^2}, 2(-1)\tan^{-1}(1)\right)$

$= \left(\frac{1}{1+1}, -2 < \frac{\pi}{4}\right)$

$\nabla f(1,-1) = \left(\frac{1}{2}, -\frac{\pi}{2}\right)$

(iii) Sol.n: $f(x,y,z) = xyz - e^{x+y+z}$

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$\nabla f(x,y,z) = (f_x, f_y, f_z)$

$= (yz - e^{x+y+z}, xz - e^{x+y+z}, e^{x+y+z})$

$\nabla f(1,-1,0) = (-1, \cancel{x0} - e^{-1+10}, 1(0) - e^{1+1+0}, 1(-1) - e^{1-1+x})$

$\nabla f(1,-1,0) = (-1,-1,-2)$

Q.3] (i) Sol.n:

$x^2 \cos y + e^{xy} = 2 = 0$

$f_x = 2x \cos y + y e^{xy}$

$f_y = -x^2 \sin y + x.e^{xy}$

Tangent :-

$f_x (x - x_0) + f_y (y - y_0) = 0$

$(2x (\cos y + y \cdot e^{xy})(x - 1) + (-x^2 \sin y + x e^{xy})$

$(y - 0) = 0$

$2x^2 \cos y + xy e^{xy} - 2x \cos y - y e^{xy}$

$- x^2 \sin y - x e^{xy}) y = 0$

(ii) $f(x, y) = x^2 + y^2 - 2x + 3y + 2 = 0$

$f_x = 2x - 2$; $f_x = 2$

$f_y = 2y + 3$; $f_y(2, -1) = 1$

Tangent:-

$f_x (x_0, y_0)(x - x_0) + f_y (x_0 y_0)$

$(x - x_0) = 0$

$2 (x - 2) + 1 (y + 2) = 0$

$2x - y + y + 2 = 0$

$2x + y - 2 = 0$

Normal: $x - 2y + 4 = 0$

$2 - 2(-2) + d = 0$

$\therefore d = 2$

$\therefore 2x - 2y + 2 = 0$

(i) Sol.n:

$f(x, y, z) = x^2 - 2y^2 + 3y + x^2 - 7$

$f_x = 2x + z$ $\quad f_x (x_0, y_0, z_0) = 2(2) + 0 = 4$

$f_y = -2z + 3$ $\quad f_y (x_0, y_0, z_0) = -2(0) + 3 = 3$

$f_z = -2y + x$ $\quad f_z (x_0, y_0, z_0) = -2(1) + 0 = 0$

$\therefore$ Tangents

$f_x (x - x_0) + f_y (x_0, y_0, z_0)(y - y_0) + f_z (x_0, y_0, z_0)(z - z_0) = 0$

$\therefore 4(x - 2) + 3(y - 1) + 0(z - 0) = 0$

$4x - 8 + 3y - 3 + 0 = 0$

$4x + 3y - 11 = 0$

$\therefore$ Normal

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}$$

$$\therefore \frac{x - 2}{4} = \frac{y - 1}{1} = \frac{z - 0}{6}$$

(ii) Sol.n: $f(x, y, z) = 3xyz - x - y + z + 6$

$f_x = 3y^2 - 1$ $\quad f_x (x_0, y_0, z_0) = 3(-1) 2(-1) = -2$

$f_y = 3x^2 - 1$ $\quad f_y (x_0, y_0, z_0) = 3(1)(2) - 1 = 5$

$f_z = 3xy + 1$ $\quad f_z (x_0, y_0, z_0) = 3(1)(-1) + 1 = -2$

Tangent:

$f_z(x_0, y_0, z_0)(1 - l_0) + f_y(x_0, y_0, z_0)(y - y_0)$
$f_z(x_0, y_0, z_0)(z - z_0) = 0$
$+ y_z(x_0, y_0, z_0)$

$\therefore -7(x-1) + 5(y+1) - 2(z-2) = 0$
$-7x + 7 + 5y + 5 - 2z + 4 = 0$
$-7x + 5y - 2z + 10$
$7x - 5y + 2z - 16 = 0$

Normal:

$$\frac{x - x_0}{f(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}$$

$$\therefore \frac{x-1}{-7} = \frac{y+1}{45} = \frac{z-2}{-2}$$

Q.5 (i) $f(x,y) = 3x^2 + y^2 - 3xy + 6x + 4y$

$f_x = 6x - 3y + 6$
$f_y = 2y - 3x - 4$
$f_x = 0$ $\quad\quad \therefore f_y = 0$
$6x - 3y - 6$ $\quad\quad 3x - 2xy = -4$
$6x - 4y = -8$
$\therefore y = 2$
$\therefore x = 0$
$\therefore (x,y) = (0, 2)$
$r = f_{xx} = 6$
$s = f_{yy} = 2$
$rt - s^2 = 6(2) - (-3)^2 = 12 - 9 = 3 > 0$

$r > 0$

$\therefore f'$ is minima at $(0,2)$

$f(0,2) = 3(0)^2 + (2)^2 - 3(0)(6) + 6(0) + 4(2)$
$= 4 + 0 - 0 + 0 - 8$

$f(0,2) = -4$

(ii) Sol.n: $f(x,y,z) = 2x^4 + 5x^2y - y^2$

$f_x = 8x^3 + 6xz$
$f_y = -2y + 3x^2$
$f_x = 0$

$\therefore 8x^3 + 6xy = 0$ $\quad\quad f_y = 0$
$x(8x^2 + 6y) = 0$ $\quad\quad -2y + 3(0) = 0$
$\therefore x = 0$ $\quad\quad \therefore y = 0$

$(x,y) = (0,0)$ is a root

$8x^2 + 6y = 0$
$x^2 = -\frac{2}{3}y$

$f_y = -2y - \frac{2}{3} + 3y = 0$

$-6y = 0$
$y = 0$
$x^2 = 0$
$x = 0$

$\therefore (x,y) = (0,0)$ is the only root.

$y = f_{xx} = 23x^2 + 6y = 0$

$s = f_{xy} = 6x = 0$

$t = f_{yy} = -2 = -2$

$r = 0$

$rt - t^2 = 0(0) - (2)^2$

$rt - s^2 = -6 < 0$

$(0,0)$ is middle point..

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