Iterated Exponential Growth

Shubham Tiwari
Section : A
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Department of Mathematics
Dyal Singh College

Introduction

For a large class of transcendental entire functions, including exponential and trigonometric functions, points in the fast escaping set are characterised as those whose orbits exhibit **iterated exponential growth**.
Let us define $F\colon [0, \infty) \to [0, \infty)$ by $F(t) = \exp(t) - 1$.
Then $F(t) > t$ for $t > 0$, and hence the sequence $F^n(t)$ tends to infinity.
We are interested in the rate at which these orbits grow.

Definition 2.1

(Iterated Exponential Growth)
A sequence $(a_n)_{n=0}^{\infty}$ of non-negative real numbers has iterated exponential growth if

$$0 < \liminf_{n\to\infty} F^{-n}(a_n) \leq \limsup_{n\to\infty} F^{-n}(a_n) < \infty.$$

The specific function F is not relevant; any exponentially growing function gives rise to the same notion of iterated exponential growth.

Proposition 2.2

(Properties of Iterated Exponential Growth)
(a) Let $\delta > 0$ and define $\Omega_\delta(t) := \exp(\delta t)$ for $t \in \mathbb{R}$. Let t_0 be such that $\Omega_\delta(t) > t$ for $t \geq t_0$. Then the sequence $(\Omega_\delta^n(t_0))_{n=0}^\infty$ has iterated exponential growth.
(b) Let $C > 1$. A sequence $(a_n)_{n=0}^\infty$ has iterated exponential growth if and only if the sequence $(a_n^C)_{n=0}^\infty$ has iterated exponential growth.