PARTIAL DIFFERENTIATION

OBJECTIVES

1. To understand basics of Partial differentiation.
2. The Chain Rule, Euler's Theorem, Higher order partial derivatives.
3. Extrema of a function including constraint.

1 PARTIAL DIFFERENTIATION

A function may depend on several variables. The process of differentiating such function with respect to one of its variables while keeping the other variable(s) fixed is called partical differentiation. The resultant expression is called partial derivative of the function.

The function **diff**(expr, x_1, n_1, ..., x_m, n_m) returns the mixed partial derivative of expr with respect to x_1, ..., x_m. It is equivalent to **diff**(...(**diff**(expr, x_m, n_m) ...), x_1, n_1).

Find the partial derivatives f_x and f_y of $f(x, y) = x^3 y^4 + x^4 y^2$.

```
(%i1)   f(x, y): = x^3*y^4 + x^4*y^2;
        'fx(x, y) = fx:diff(f(x, y), x, 1);
        'fy(x, y) = fy:diff(f(x, y), y, 1);
```

(%o1) $f(x, y) := x^3\ y^4 + x^4\ y^2$

(%o2) $fx(x, y) = 3\ x^2\ y^4 + 4\ x^3\ y^2$

(%o3) $fy(x, y) = 4\ x^3\ y^3 + 2\ x^4\ y$

Let $Z = x^2 \sin(3xy + x^2 y^3)$. Find its partial derivative Z_x and Z_y and evaluate at (1, 1).

```
(%i4)   Z(x, y): = x^2*sin(3*x*y + x^2*y^3);
        'Zx = Zx:diff(Z(x, y), x, 1);
        'Zy = Zy:diff(Z(x, y), y, 1);
        'at('Zx, [x = 1, y = 1]) = at(Zx, [x = 1, y = 1]);float(rhs(%));
        'at('Zy, [x = 1, y = 1]) = at(Zy, [x = 1, y = 1]);float(rhs(%));
```

(%o4) $Z(x, y) := x^2 \sin(3\ x\ y + x^2\ y^3)$

(%o5) $Zx = 2\ x \sin(x^2\ y^3 + 3\ x\ y) + x^2(2\ x\ y^3 + 3\ y)\cos(x^2\ y^3 + 3\ x\ y)$

(%o6) $Zy = x^2(3\ x^2\ y^2 + 3\ x)\cos(x^2\ y^3 + 3\ x\ y)$

(%o7) $Zx|_{[x=1,\ y=1]} = 2\sin(4) + 5\cos(4)$

(%o8) - 4.781823094933916

(%o9) $Zy|_{[x=1,\ y=1]} = 6\cos(4)$

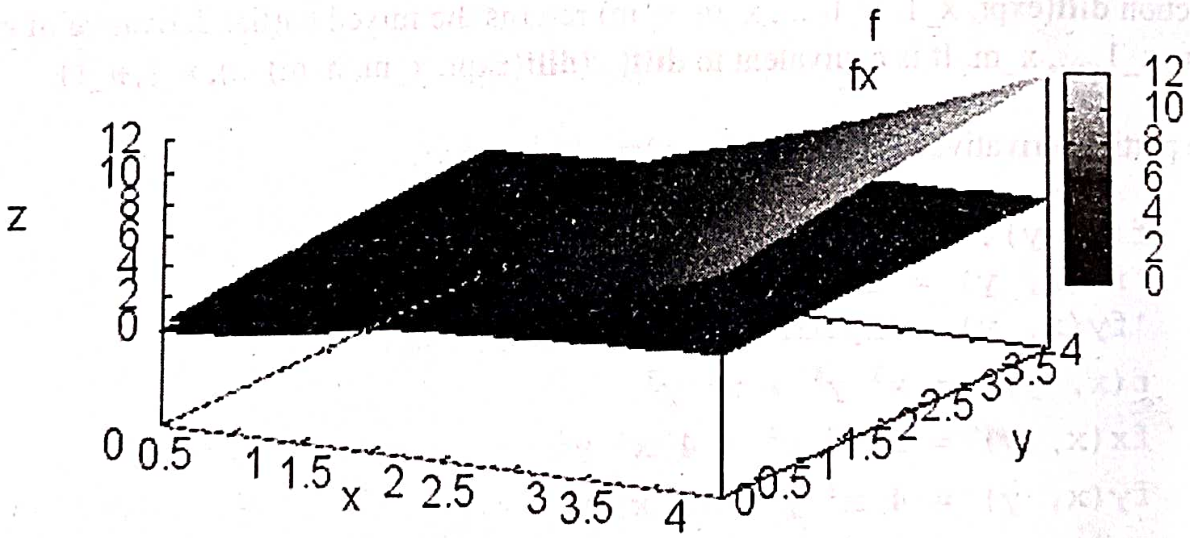
(%o10) - 3.921861725181672

Find the slope of a line that is parallel to the xz - plane and tangent to a surface $z = x\sqrt{x+y}$ at the point P(1, 3, 2).

```
(%i11) kill(all)$
       f(x, y): = x*sqrt(x + y);
       fx(x, y): = diff(f(x, y), x, 1);
       wxplot3d([fx(x, y), f(x, y), [x, 0, 4], [y, 0, 4]],
       [legend, "fx", "f"]);
       'at('fx(x, y), [x = 1, y = 3]) = at(fx(x, y),[x = 1, y = 3]);
```

(%o1) $f(x, y) := x\sqrt{x+y}$

(%o2) fx(x, y): = diff(f(x, y), x, 1)



(%o4) $fx(x, y)|_{[x=1,\ y=3]} = \frac{9}{4}$

1.1 Higher Order Partial Derivatives

As the partial derivative of a function of several variables is itself a function of several variables so it is possible to take its partial derivative again. If we take two consecutive partial derivatives with respect to the same variable, the resultant derivative is called the second order partial derivative with respect to that variable. However, if we take the partial derivative with respect to one variable and then take a second partial derivative with respect to a different variable, we get mixed second-order partial derivatives.

For $z = f(x, y) = 5x^2 - 2xy + 3y^3$, find f_{xx}, f_{xy}, f_{yy} and also f_{xy} (3, 2).

```
(%i5)  kill(all)$
       f(x, y): = 5*x^2 - 2*x*y + 3*y^3;
       'fxx = fxx:diff(f(x, y), x, 2);
       'fxy = fxy:diff(f(x, y), x, 1, y, 1);
       'fyx = fyx:diff(f(x, y), y, 1, x, 1);
       'fyy = fyy:(diff(f(x, y), y, 2));
       'at('fxy, [x = 3, y = 2]) = at(fxy, [x = 3, y = 2]);
```

(%o1) $f(x, y) := 5\,x^2 - 2\,x\,y + 3\,y^3$

(%o2) fxx = 10

(%o3) fxy = - 2

(%o4) fyx = - 2

(%o5) fyy = 18 y

(%o6) $fxy|_{[x = 3,\, y = 2]} = -2$

Note: If the function f(x, y) has mixed second-order partial derivative f_{xy} and f_{yx} that are continuous in an open disk containing (x_0, y_0), then $f_{yx}(x_0, y_0) = f_{xy}(x_0, y_0)$.

Verify that $T(x, t) = e^{-t}\cos(x/c)$ satisfies the heat equation, $\frac{{}^2T}{t^2}\ \ c^2\,\frac{{}^2T}{x^2}$.

```
(%i7)  kill(all)$
       T(x, t): = %e^ - t*cos(x/c);
       'diff(T(x, t), t) = diff(T(x, t), t);
       (c^2)*'diff(T(x, t), x, 2) = (c^2)*diff(T(x, t), x, 2);
       is(diff(T(x, t), t, 2) = (c^2)*diff(T(x, t), x, 2));
```

(%o1) $T(x, t) := \%e^{-t}\cos\left(\frac{x}{c}\right)$

(%o2) $\frac{d}{dt}\left(\%e^{-t}\cos\left(\frac{x}{c}\right)\right) = \%e^{-t}\cos\left(\frac{x}{c}\right)$

(%o3) $c^2\left(\frac{d^2}{dx^2}\left(\%e^{-t}\cos\left(\frac{x}{c}\right)\right)\right) = \%e^{-t}\cos\left(\frac{x}{c}\right)$

(%o4) true

The wave equation is $\frac{{}^2 z}{t^2} \quad \frac{c^2 \ {}^2 z}{x^2}$. Determine if, z satisfies the wave equation where $z = \sin(5ct)\cos(5x)$.

```
(%i5)  kill(all)$
       z(x, t): = sin(5*c*t)*cos(5*x);
       'diff(z(x, t), t, 2) = diff(z(x, t), t, 2);
       (c^2)*'diff(z(x, t), x, 2) = (c^2)*diff(z(x, t), x, 2);
       is(diff(z(x, t), t, 2) = (c^2)*diff(z(x, t), x, 2));
(%o1)  z(x, t): = sin(5 c t)cos(5 x)
```

(%o2) $\frac{d^2}{dt^2}(\sin(5\ c\ t)\cos(5\ x)) = -\ 25\ c^2 \sin(5\ c\ t)\cos(5\ x)$

(%o3) $c^2\left(\frac{d^2}{dt^2}(\sin(5ct)\cos(5\,x))\right) = -\ 25\ c^2 \sin(5\ c\ t)\cos(5\ x)$

(%o4) true

The Cauchy-Riemann equations are $\frac{u}{x} \quad \frac{v}{y}$ and $\frac{u}{y} \quad -\frac{v}{x}$ where $u = u(x, y)$ and $v = v(x, y$

Which of the following functions satisfies the Cauchy Riemann equations ?

$u = e^{-x}(x \cos y)$, $v = e^{-x}(\sin y)$

$u = \log(x^2 + y^2)$, $v = 2\tan^{-1}(y/x)$

```
(%i5)  kill(all)$
       u(x, y): = %e^(- x)*cos(y);v(x, y): = %e^(- x)*sin(y);
       'diff(u(x, y), x, 1) = diff(u(x, y), x, 1);
       'diff(v(x, y), y, 1) = diff(v(x, y), y, 1);
       'diff(u(x, y), y, 1) = diff(u(x, y), y, 1);
       'diff(v(x, y), x, 1) = diff(v(x, y), x, 1);
       is(diff(u(x, y), x, 1) = diff(v(x, y), y, 1));
       is(diff(u(x, y), y, 1) = - diff(v(x, y), x, 1));
```

(%o1) u(x, y): = $\%e^{-x}$ cos(y)

(%o2) v(x, y): = $\%e^{-x}$ sin(y)

(%o3) $\frac{d}{dx}(\%e^{-x}\cos(y)) = -\ \%e^{-x}\cos(y)$

(%o4) $\frac{d}{dy}(\%e^{-x}\sin(y)) = \%e^{-x}\cos(y)$

(%o5) $\frac{d}{dy}(\%e^{-x}\cos(y)) = -\ \%e^{-x}\sin(y)$

(%o6) $\frac{d}{dx}(\%e^{-x}\sin(y)) = -\ \%e^{-x}\sin(y)$

```
(%o7)  false
(%o8)  false
(%i9)  kill(all)$
       u(x, y): = log(x^2 + y^2);v(x, y): = 2*atan(y/x);
       'diff(u(x, y), x, 1) = diff(u(x, y), x, 1);
       'diff(v(x, y), y, 1) = diff(v(x, y), y, 1);
       'diff(u(x, y), y, 1) = diff(u(x, y), y, 1);
       'diff(v(x, y), x, 1) = diff(v(x, y), x, 1);
       is(diff(u(x, y), x, 1) = ratsimp(diff(v(x, y), y, 1)));
       is(diff(u(x, y), y, 1) = - ratsimp(diff(v(x, y), x, 1)));
```

(%o1) $u(x, y) := \log(x^2 + y^2)$

(%o2) $v(x, y) := 2\ \text{atan}\ (y/x)$

(%o3) $$\frac{d}{d\,x}\log(y^2 + x^2) = \frac{2x}{y^2 + x^2}$$

(%o4 $$\frac{d}{d\,y}\ 2\,\text{atan}\ \frac{y}{x} = \frac{2}{x\ \dfrac{y^2}{x^2} + 1}$$

(%o5) $$\frac{d}{d\,y}\log(y^2 + x^2) = \frac{2y}{y^2 + x^2}$$

(%o6) $$\frac{d}{d\,x}\ 2\,\text{atan}\ \frac{y}{x} = -\frac{2y}{x^2\ \dfrac{y^2}{x^2} + 1}$$

(%o7) true

(%o8) true

If $u(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}; x^2 \quad y^2 \quad z^2 \quad 0$ show that $\dfrac{{}^2u}{x^2} \quad \dfrac{{}^2u}{y^2} \quad \dfrac{{}^2u}{z^2} \quad 0$.

```
(%i9)  kill(all)$
       u(x, y, z): = 1/sqrt(x^2 + y^2 + z^2);
       'diff('u(x, y, z), x, 2) + 'diff('u(x, y, z), y, 2) + 'diff
       ('u(x, y, z), z, 2) = diff(u(x, y, z), x, 2) + diff(u(x, y, z),
       y, 2)+ diff(u(x, y, z), z, 2);
       ratsimp(rhs(%));
```

(%o1) $$u(x, y, z) := \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

(%o2) $\frac{d^2}{dz^2}\,u(x, y, z) \quad \frac{d^2}{dy^2}\,u(x, y, z) \quad \frac{d^2}{dx^2}\,u(x, y, z) \quad \frac{3}{(z^2 \quad y^2 \quad x^2)^{3/2}}$

$\frac{3\,z^2}{(z^2 \quad y^2 \quad x^2)^{5/2}} \quad \frac{3\,y^2}{(z^2 \quad y^2 \quad x^2)^{5/2}} \quad \frac{3\,x^2}{(z^2 \quad y^2 \quad x^2)^{5/2}}$

(%o3) 0

1.2 The Chain Rule

(Two- variable) If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} \quad \frac{z}{t}\frac{dx}{dt} \quad \frac{z}{y}\frac{dy}{dt}$$

Let $z = x^2$, $x = t^2$, $y = t^3$. Use chain rule to find dz/dt and check the result by expressing z as a function of t and differentiating directly.

```
(%i4)  kill(all)$
       depends(z, [x, y]);depends([x, y], t);
       'z = z:x^2*y;
       'at('z, [x = t^2, y = t^3]) = z(t): = at(z, [x = t^2, y = t^3]);
       'z(t) = diff(z(t), t, 1);
```

(%o1) [z(x, y)]

(%o2) [x(t), y(t)]

(%o3) $z = x^2\ y$

(%o4) $z|_{[x = t^2,\, y = t^3]} = z(t) := at(z, [x = t^2, y = t^3])$

(%o5) $z(t) = 7\ t^6$

Using Chain rule, we obtain:

```
(%i6)  kill(all)$
       'z = z(x, y): = x^2*y;x(t): = t^2;y(t): = t^3;
       'd = d:(diff(z(x, y), x))*(diff(x(t), t)) +
       (diff(z(x, y), y))*(diff(y(t), t));
       'at('d, [x = t^2, y = t^3]) = at(d, [x = t^2, y = t^3]);
```

(%o1) $z = z(x, y) := x^2\ y$

(%o2) $x(t) := t^2$

(%o3) $y(t) := t^3$

(%o4) $d = 4\ t\ x\ y + 3\ t^2\ x^2$

(%o5) $d|_{[x = t^2,\, y = t^3]} = 7\, t^6$

(%o3) $\Delta(y) = \frac{1}{4}$

(%o4) $\Delta(z) = -\frac{1}{3}$

```
(%o5) Cx(x, y, z): = diff(C(x, y, z), x, 1)
(%o6) Cy(x, y, z): = diff(C(x, y, z), y, 1)
(%o7) Cz(x, y, z): = diff(C(x, y, z), z, 1)
```

(%o8) $\left. \mathrm{Cz}(x_0, y_0, z_0)\ (z) + \mathrm{Cy}(x_0, y_0, z_0)\ (y) + \mathrm{Cx}(x_0, y_0, z_0)\ (x) \right|_{[x_0 = 3,\, y_0 = 1,\, z_0 = 2]} = \frac{5}{3}$

1.6 Total Differential

The total differential of the function f(x, y) is: $df = f_x(x, y)\,dx + f_y(x, y)\,dy$ where dx and dy are independent variables. Similarly, for a function of three variables $w = f(x, y, z)$, the total differential is : $df = f_x(x, y, z)\,dx + f_y(x, y, z)\,dy + f_z(x, y, z)\,dz$.

The function **diff**(expr) returns the total differential of expr, that is, the sum of the derivatives of expr with respect to each of its variable times the differential del of each variable.

Find the total differential of the given functions :

a. $f(x, y, z) = 2x^3 + 2y^4 - 16z^3$

b. $f(x, y) = x^3 \log(3y^2 - 2x^2)$.

```
(%i9) diff(2*x^3 + 2*y^4 - 16*z^3);
```

(%o9) $- 48\ z^2\ \mathrm{del}(z) + 8\ y^3\ \mathrm{del}(y) + 6\ x^2\ \mathrm{del}(x)$

and by definition :

```
(%i10) kill(all)$
       f(x, y, z): = 2*x^3 + 2*y^4-16*z^3;
       fx(x, y, z): = diff(f(x, y, z), x);
       fy(x, y, z): = diff(f(x, y, z), y);
       fz(x, y, z): = diff(f(x, y, z), z);
       'fx(x,y, z)*'diff(x) + 'fy(x, y, z)*'diff(y) + 'fz(x, y, z)*
       'diff(z) = fx(x, y, z) *diff(x) + fy(x, y, z)* diff(y) +
       fz(x, y, z) *diff(z);
```

(%o1) $f(x, y, z) := 2\ x^3 + 2\ y^4 + (-16)\,z^3$

```
(%o2) fx(x, y, z): = diff(f(x, y, z), x)
```

```
(%o3)  fy(x, y, z): = diff(f(x, y, z), y)
(%o4)  fz(x, y, z): = diff(f(x,y,z),z)
(%o5)  fz(x, y, z)del(z) + fy(x, y, z)del(y) + fx(x, y, z)del(x)
       = -48 z² del(z) + 8 y³ del(y) + 6 x² del(x)

(%i6)  kill(all)$
       f(x, y, z): = x^3*log(3*y^2-2*x^2);
       fx(x, y, z): = diff(f(x, y, z), x);
       fy(x, y, z): = diff(f(x, y, z), y);
       fz(x, y, z): = diff(f(x, y, z), z); 'fx(x, y, z)*'diff(x)
       + 'fy(x, y, z)*'diff(y) + 'fz(x, y, z)*'diff(z)
       = fx(x, y, z)*diff(x) + fy(x, y, z)*diff(y) + fz(x, y, z)*diff(z);
       diff(x^3*log(3*y^2-2*x^2));

(%o1)  f(x, y, z): = x³ log(3 y² - 2x²)

(%o2)  fx(x, y, z): = diff(f(x, y, z), x)

(%o3)  fy(x, y, z): = diff(f(x, y, z), y)

(%o4)  fz(x, y, z): = diff(f(x, y, z), z)

%o5)   fz(x, y, z)del(z) + fy(x, y, z)del(y) + fx(x, y, z)del(x)
```

$$= \frac{6\,x^3\,y\,\mathrm{del}(y)}{3\,y^2-2\,x^2} + \left(3\,x^2\log(3\,y^2-2\,x^2) - \frac{4\,x^4}{3\,y^2-2\,x^2}\right)\mathrm{del}\,(x)$$

;i6) $\frac{6\,x^3\,y\,\mathrm{del}(y)}{3\,y^2\quad 2\,x^2}\quad 3\,x^2\log(3\,y^2\quad 2\,x^2)\quad \frac{4\,x^4}{3\,y^2\quad 2\,x^2}\ \mathrm{del}(x)$

:XTREMA OF FUNCTION OF TWO VARIABLES

f be a function defined on a region containing (x_0, y_0). Then $f(x_0, y_0)$ is a relative maximum, if $y) \leq f(x_0, y_0)$ for all (x, y) in an open disk containing (x_0, y_0). $f(x_0, y_0)$ is a relative minimum if $\prime) \leq f(x_0, y_0)$ for all (x, y) in an open disk containing (x_0, y_0). The relative maxima and minima ollectively called as relative extrema. A critical point of a function f defined on an open set D oint (x_0, y_0) in D where either :

CHAPTER 15

DIFFERENTIAL EQUATIONS

OBJECTIVES

In this chapter we learn how to solve:

1. **Initial value problems.**
2. **Boundary value problems.**
3. **Higher order differential equations.**

1 DEFINITION

A differential equation is an equation that contains the derivatives or differentials of one or more dependent variables with respect to one or more independent variables. If the equation contains only ordinary derivatives (of one or more dependent variables) with respect to a single independent variable, the equation is called an ordinary differential equation.

A solution to the n^{th}order ordinary differential equation $F(x, y, y', y'', ...y^{(n)}) = 0$ on the interval $a < x < b$ is a function $f(x)$ that is continuous on the interval $a < x < b$ and has all the derivatives present in the differential equation such that $F(x, f, f', f'', ...f^{(n)}) = 0$ on $a < x < b$.

Verify that the given function is a solution to the corresponding differential equation:

$$y''' + 2y'' + y' + 2y = 0, y(x) = \sin(x).$$

```
(%i1)  kill(all)$y:sin(x)$
       is(diff(y, x, 3) + 2*diff(y, x, 2) + diff(y, x) + 2*y = 0);
(%o2)  true
```

Verify that the given implicit function $2x^2 + y^2 - 2xy + 5x = 0$ satisfies the differential equation.

$$\frac{dy}{dx} = \frac{2y - 4x - 5}{2y - 2x}$$

```
(%i3)  kill(all)$f:2*x^2 + y^2 - 2*x*y + 5*x = 0$
       depends(y, x)$derivabbrev :true$
       solve(diff(f, x), diff(y, x));
```

(%o4) $y_x = \frac{2\ y - 4\ x - 5}{2\ y - 2\ x}$

2 EULER'S METHOD

This method uses extrapolation techniques for developing solution. Given, first order differential equation with an initial value.

$$\frac{dy}{dx} = f(x, y) \quad \text{where } y(x_0) = y_0 .$$

The Euler's formula is :

$$y_{i+1} = y_i + hf(x_i, y_i)$$

The error is of the order of (h^2).

If $|1 + h(df/dy)| < 1$, then errors will damp down with successive iterations. In this case, the Euler's method is stable otherwise unstable.

Solve the differential equation $dy/dx + xy = 0$, $y(0) = 1$ from $x = 0$ to 0.5, using Euler's method.

First we check, if solution is stable :

```
(%i5)  kill(all)$'n = n:10;'h = h:.05;'x = x:.5;
       'at(1 + 'h*'diff(- x*y, y), 'x = .5)
       = at(1 + h*diff(- x*y, y), x = .5);
       is(rhs(%) < 1);
(%o1)  n = 10
(%o2)  h = 0.05
(%o3)  x = 0.5
```

(%o4) $h(-0.5\, y_y) + 1\Big|_{x=0.5} = 0.975$

```
(%o5)  true
```

As $|1 + h(df/dy)| < 1$, so Euler's solution is stable. The solution is given as :

```
(%i6)  kill(all)$
       block(h:.05, x0:0, y0:1, for i:1 thru 10 do
       at(y1:y0 - h*x0*y0, [x0 = 0, y0 = 1]), x1:x0 + .05,
       print(x1, y1), x0:x1, y0:y1));
0.05 1
0.1 0.9975
0.15 0.9925125
0.2 0.98506865625
0.25 0.9752179696875
0.3 0.963027745066406
0.35 0.94858232889041
0.4 0.931982138134828
0.45 0.913342495372132
0.5 0.892792289226259
(%o1)  done
```

This method can be re-written in general indices:

```
(%i2) kill(all)$block(
      h:.05, x[0]:0, y[0]:1, n:9,
      for i:0 thru n do
      (at(y[i + 1]:y[i] - h*x[i]*y[i], [x[i] = 0, y[i] = 1]),
      x[i + 1]:x[i] + .05,
      print('x[i + 1] = x[i + 1],", ", 'y[i + 1] = y[i + 1])));

x1 = 0.05, y1 = 1
x2 = 0.1,  y2 = 0.9975
x3 = 0.15, y3 = 0.9925125
x4 = 0.2,  y4 = 0.98506865625
x5 = 0.25, y5 = 0.9752179696875
x6 = 0.3,  y6 = 0.963027745066406
x7 = 0.35, y7 = 0.94858232889041
x8 = 0.4,  y8 = 0.931982138134828
x9 = 0.45, y9 = 0.913342495372132
x10 = 0.5, y10 = 0.892792289226259
(%o1) done
```

The analytical solution to the differential equation $\frac{dy}{dx} + xy = 0$ is $e^{-x^2/2}$. We can compare y_i and z_i, where $z_i = e^{-x^2/2}$.

```
((%i2) kill(all)$block(
      h:.05,x[0]:0,y[0]:1,z[0]:1,n:9,
      for i:0 thru n do
      (at([y[i+1]:y[i]-h*x[i]*y[i]],
      [x[0]=0,y[0]=1]),x[i+1]:x[i]+.05,print('x[i+1]=x[i+1],
      ", ",'y[i+1]=y[i+1],'z[i+1]=exp((-x[i+1]^2)/2))))$

x1 = 0.05, y1 = 1                  z1 = 0.998750780924581
x2 = 0.1,  y2 = 0.9975             z2 = 0.995012479192682
x3 = 0.15, y3 = 0.9925125          z3 = 0.988813044611233
x4 = 0.2,  y4 = 0.98506865625      z4 = 0.980198673306755
x5 = 0.25, y5 = 0.9752179696875    z5 = 0.96923323447634
x6 = 0.3,  y6 = 0.963027745066406  z6 = 0.9559974818331
x7 = 0.35, y7 = 0.94858232889041   z7 = 0.940588063364342
x8 = 0.4,  y8 = 0.931982138134828  z8 = 0.923116346386636
x9 = 0.45, y9 = 0.913342495372132  z9 = 0.90370707787319
x10 = 0.5, y10 = 0.892792289226259 z10 = 0.882496902584595
```

Thus, by inspecting above solutions obtained by Euler's method and the analytical solutions we infer that error is building up.

3 TAYLOR'S METHOD

The Euler's method of extrapolation corresponds to only first two terms of the Taylor's Series:

$y_2 = y_1 + h\, f(x_1, y_1) + (h^2/2)\,(f_x(x_1, y_1) + f(x_1, y_1)\, f_y(x_1, y_1)) + \ldots$

```
(%i2)  kill(all)$block(
       h:.05,x[0]:0,y[0]:1,z[0]:1,n:9,
       for i:0 thru n do
       (at([y[i+1]:y[i]-h*x[i]*y[i]+(.5*h^2)*(-y[i]+y[i]*x[i]^2)],
       [x[0] = 0,y[1] = 1]),x[i+1]:x[i]+.05,print('x[i+1] = x[i+1],", ",
       'y[i+1]=y[i+1],'z[i+1] = exp((-x[i+1]^2)/2))))$
```

```
x1  = 0.05,  y1  = 0.99875             z1  = 0.998750780924581
x2  = 0.1,   y2  = 0.99500780859375    z2  = 0.995012479192682
x3  = 0.15,  y3  = 0.988801447387647   z3  = 0.988813044611233
x4  = 0.2,   y4  = 0.980177244763712   z4  = 0.980198673306755
x5  = 0.25,  y5  = 0.969199259622359   z5  = 0.969233234476344
x6  = 0.3,   y6  = 0.955948488494709   z6  = 0.9559974818331
x7  = 0.35,  y7  = 0.940521869761626   z7  = 0.940588063364342
x8  = 0.4,   y8  = 0.923031102114903   z8  = 0.923116346386636
x9  = 0.45,  y9  = 0.903601297415384   z9  = 0.903707077873196
x10 = 0.5,   y10 = 0.882369490680177   z10 = 0.882496902584595
```

4 VARIOUS FUNCTIONS

4.1 Desolve

The function **desolve** solves systems of linear ordinary differential equations using Laplace transform.The general syntax is:

desolve (eqn, x)

desolve ([eqn_1, ..., eqn_n], [x_1, ..., x_n])

The eqns are differential equations in the dependent variables x_1, ..., x_n. The functional dependence must be explicitly indicated in the variables and its derivatives.

```
(%i2)  kill(all)$
       eqn:diff(y(x), x, 1) + y(x) + 2 = 0;
       desolve(eqn, y(x));
(%o1)  y(x))x + y(x) + 2 = 0
(%o2)  y(x) = (y(0) + 2)%e - x - 2
```

4.2 ODE2

The function **ode2** solves an ordinary differential equation(ODE) of first or second order. The general syntax is: **ode2** (eqn, dvar, ivar).

It takes three arguments: an ODE given by eqn, the dependent variable dvar, and the independent variable ivar. When successful, it returns either an explicit or implicit solution for the dependent variable. **%c** is used to represent the integration constant in the case of first-order equations, and **%k1** and **%k2** the constants for second-order equations. The dependence of the dependent variable on the independent variable does not have to be written explicitly, as in the case of **desolve,** but the independent variable must always be given as the third argument.

```
(%i3)  kill(all)$depends(y, x);
       eqn:diff(y, x, 2) + y = 0;
       ode2(eqn, y, x);
```

(%o1) [y(x)]

(%o2) $y_{xx} + y = 0$

(%o3) y = %k1 sin(x) + %k2 cos(x)

In order to solve initial value problems (IVP), functions ic1 and ic2 are available for first and second order equations, and to solve second-order boundary value problems (BVP) the function **bc2** can be used.

4.3 ic1

The function **ic1**(solution, xval, yval) solves initial value problems for first order differential equations. Here, solution is a general solution to the equation, as found by **ode2**, xval gives an initial value for the independent variable in the form $x = x_0$, and yval gives the initial value for the dependent variable in the form $y = y_0$.

Graph the solution of the initial-value problem $y' = x^2$; $y(0) = 1$ on the interval [0, 10]. Evaluate y(5).

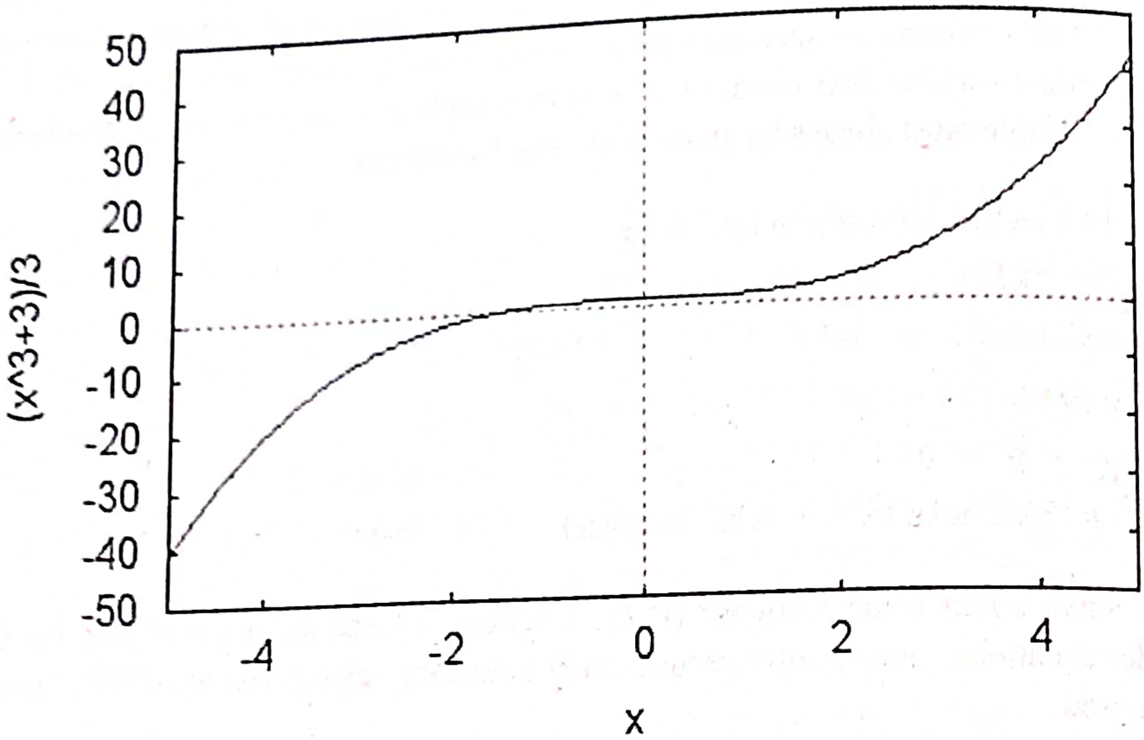
```
(%i4)  kill(all)$depends(y,x)$
       eqn:'diff(y, x) = x^2;
       sol:ode2(eqn, y, x);
       fsol:ic1(sol, x = 0, y = 1);
       ev(%, x:5);
       wxplot2d(rhs(fsol), [x, - 5, 5]);
```

(%o2) $y_x = x^2$

(%o3) $y = \frac{x^3}{3} + \%c$

(%o4) $y = \frac{x^3 + 3}{3}$

(%o5) $y = \frac{128}{3}$



4.4 ic2

The function **ic2**(solution, xval, yval, dval) solves initial value problems for second-order differential equations. Here, solution is a general solution to the equation, as found by **ode2,** xval gives the initial value for the independent variable in the form x = x_0, yval gives the initial value of the dependent variable in the form y = y_0, and dval gives the initial value for the first derivative of the dependent variable with respect to independent variable, in the form diff(y, x) = dy_0 (Note: diff need not to be quoted).

```
(%i7) kill(all)$depends(y, x);
      eqn:'diff(y, x, 2) = - sin(x);
      sol:ode2(eqn, y, x);
      fsol:ic2(sol, x = 0, y = 1, diff(y, x) = 1);
```

(%o1) [y(x)]

(%o2) $y_{xx} = - \sin(x)$

(%o3) y = sin(x) + %k2 x + %k1

(%o4) y = sin(x) + 1

4.5 bc2

The function **bc2**(solution, xval1, yval1, xval2, yval2) solves a boundary value problem for a second order differential equation. Here, solution is a general solution to the equation, as found by **ode2**; xval1 specifies the value of the independent variable in a first point, in the form $x = x_1$, and yval1 gives the value of the dependent variable in that point, in the form $y = y_1$. The expressions xval2 and yval2 give the values for these variables at a second point, using the same form.

```
(%i5) kill(all)$depends(y, x);
      eqn:diff(y, x, 2) + y^2*diff(y, x)^3 = 0;
      sol:ode2(eqn, y, x);
      bc2(sol, x = 0, y = 1, x = 1, y = 2);
```

(%o1) [y(x)]

(%o2) $y_{xx} + y^2 (y_x)^3 = 0$

(%o3) $$\frac{y^4 + 12\,\%k1\,y}{12} = x + \%k2$$

(%o4) $$\frac{y^4 - 3\,y}{12} = x - \frac{1}{6}$$

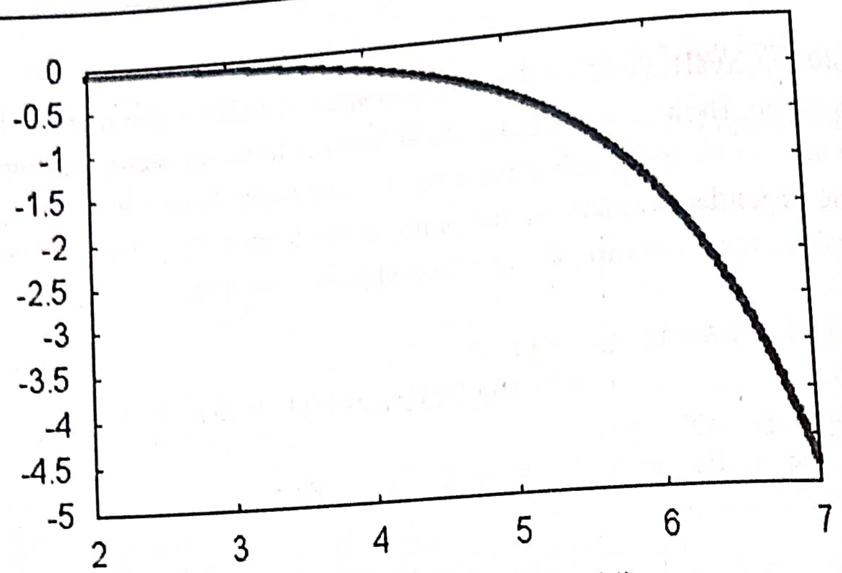
4.6 rk

The general syntax for the function **rk** (Runge-Kutta) is **rk** (ODE, var, initial, domain) and **rk** ([ODE1, ..., ODEm], [v1, ..., vm], [init1, ..., initm], domain). The first form solves numerically a first-order ordinary differential equation, and the second form solves a system of m of those equations, using the 4th order Runge-Kutta method. var represents the dependent variable. ODE must be an expression that depends only on the independent and dependent variables and defines the derivative of the dependent variable with respect to the independent variable.

The independent variable is specified with domain, which must be a list of four elements as, for instance: [t, 0, 10, 0.1], the first element of the list identifies the independent variable, the second and third elements are the initial and final values for that variable, and the last element sets the increments that should be used within that interval.

If m equations have to be solved, there should be m dependent variables v1, v2, ..., vm. The initial values for those variables will be init1, init2, ..., initm. There will still be just one independent variable defined by domain,

```
(%i5) kill(all)$points:rk(exp( - t) + u, u, - 0.1, [t, 2, 7,
      0.01])$
      wxplot2d( [discrete, points], [t, 0, 7],
      [style, [lines, 4]], [ylabel," "],
      [xlabel, "t0 = 2, u0 = - 0.1, du/dt = exp(- t) + u"])$
```

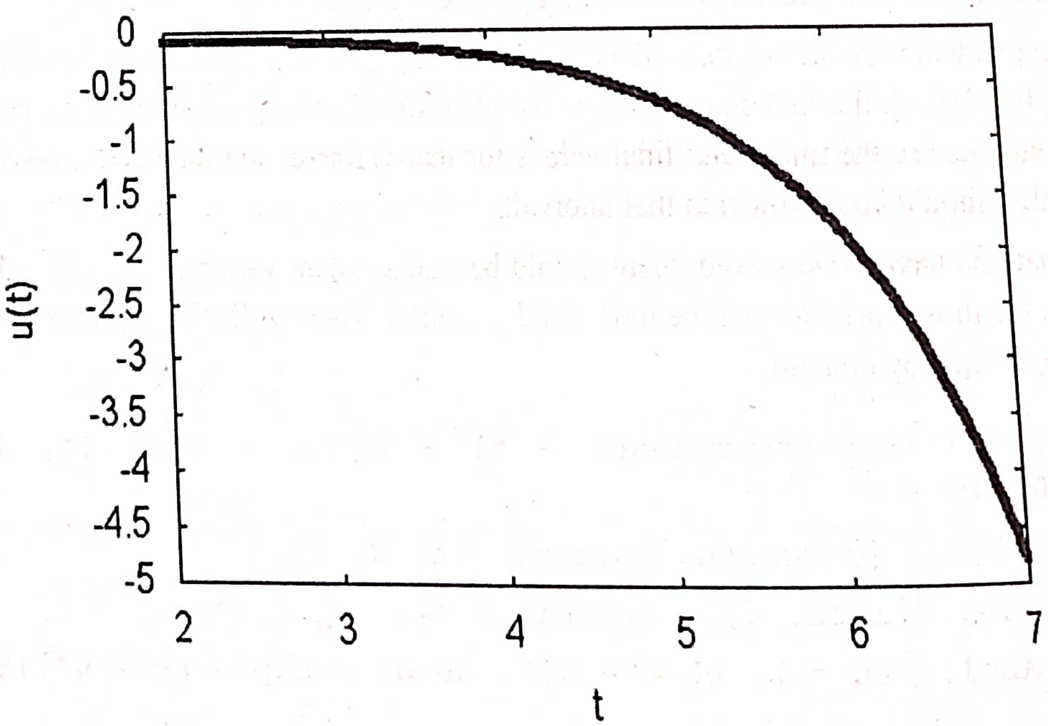


We solve the above problem again using **ode2** and **ic1**. We obtain identical graphs of solutions.

```
(%i4)  kill(all)$depends(y, x)$ratprint:false$
       eqn:'diff(u, t) = exp(- t) + u;
       sol:ode2(eqn, u, t)$
       fsol:ic1(sol, t = 2, u = - 0.1)$
       ev(%, x:5);
       wxplot2d(rhs(fsol), [t, 2, 7], [ylabel, "u(t)"],
       [style, [lines, 4]]);
```

(%o3) $u_t = u + \%e^{-t}$

(%o6) $u = \dfrac{\%e^{t-4}((\%e^2 - 5)\%e^{2t} + 5\,\%e^4)}{10}$



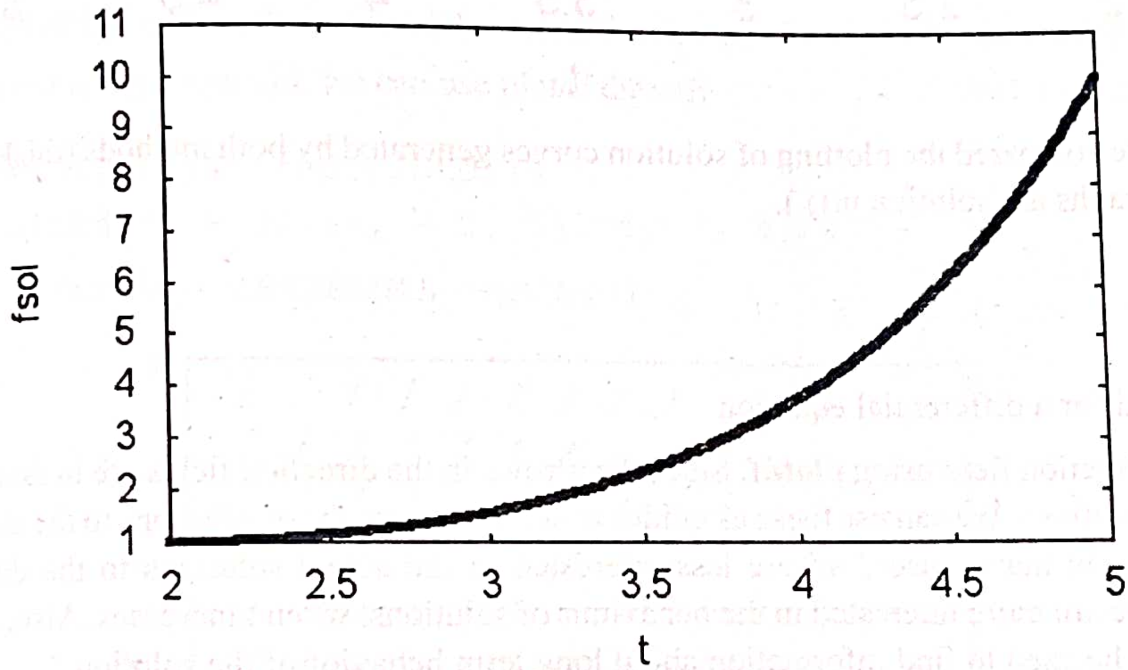
Now, we consider a second order differential equation: $\frac{d^2u}{dt^2} = u$, $u(2) = 1$ and $u'(2) = 0$;

First, we solve using **ode** and **ic2** and obtain the solutions. we obtain the graph also.

```
(%i8) kill(all)$depends(u, t)$
      eqn:'diff(u, t, 2) = u;
      sol:ode2(eqn, u, t)$
      fsol:ic2(sol, t = 2, u = 1, diff(u, t) = 0);
      wxplot2d(rhs(fsol), [t, 2, 5], [style, [lines, 4]],
      [ylabel, "fsol"]);
```

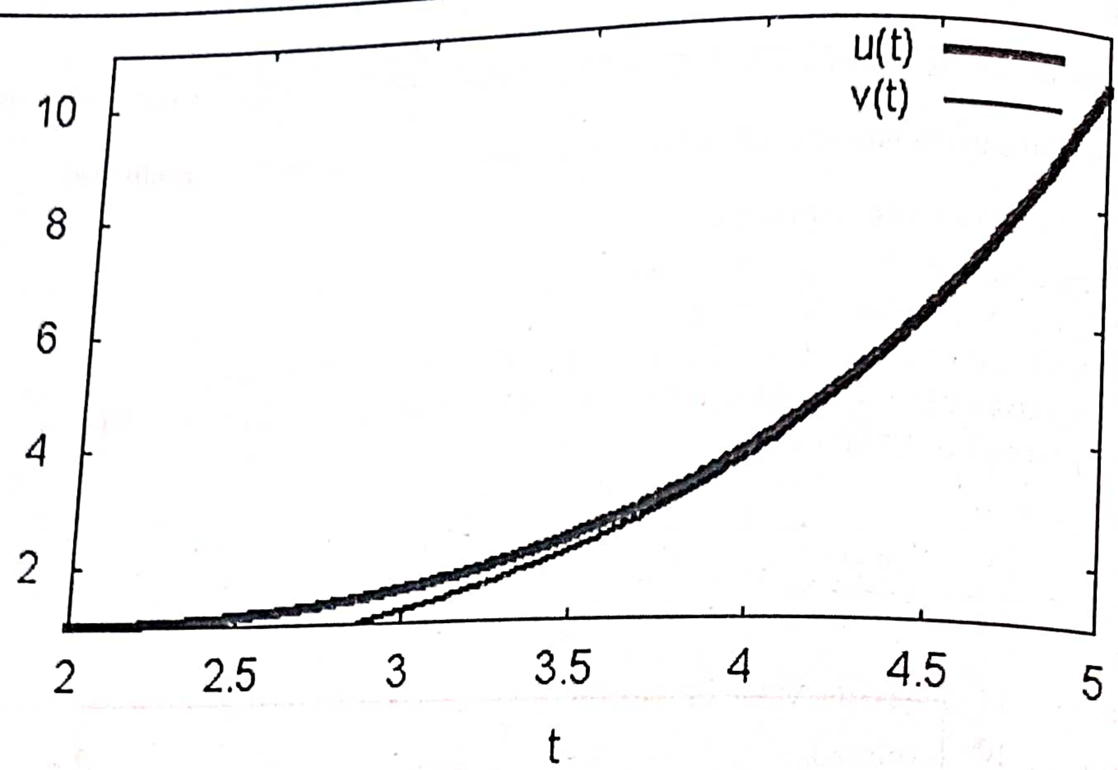
(%o2) $u_{tt} = u$

(%o4) $u = \frac{\%e^{t-2}}{2} + \frac{\%e^{2-t}}{2}$



Now, we use **rk** method for higher degree differential equations. The foremost task is to convert the second order differential equation into two first-order differential equations i.e. du/dt = v and dv/dt = u, the initial conditions are $u(2) = 1$ and $u'(2) = 0$;

```
(%i6) kill(all)$
      points:rk([v, u], [u, v], [1, 0], [t, 2, 5, 0.01])$
      uL : makelist([points[i][1], points[i][2]],i,1,
          length(points))$
      vL : makelist([points[i][1], points[i][3]],i,1,
          length(points))$
      wxplot2d([ [discrete, uL], [discrete, vL]], [t, 2, 5],
      [style, [lines, 4], [lines, 2]], [y, 1, 11], [ylabel," "],
      [xlabel, "t"], [legend, "u(t)", "v(t)"])$
```



Thus, we have compared the plotting of solution curves generated by both methods (thicker curves in both the graphs are solution u(t)).

4.7 plotdf

Direction field for a differential equation

We can plot direction field using **plotdf.** Since the arrows in the direction fields are in fact tangents to the actual solutions. We can use these as guides to sketch the graphs of solutions to the differential equation. Also, in many cases, we are less interested in the actual solutions to the differential equations as we are more interested in the behaviour of solutions, when t increases. Also, direction fields can also be used to find information about long term behavior of the solution.

Sketch the direction field for the following differential equation. Sketch the set of integral curves for the following differential equation:

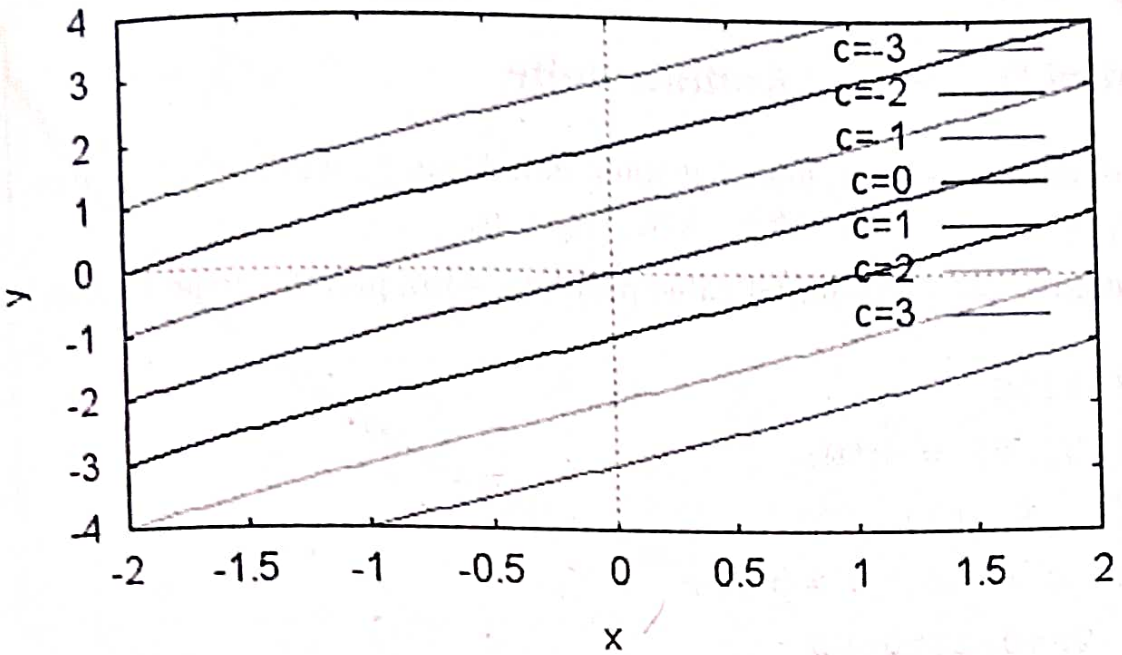
$$y' = y - x.$$

To sketch the direction field, we assume the derivative as constant and then plot its graph for

$$y - x = c \quad \text{where } c = -3, -2, -1, 0, 1, 2, 3,$$

which can be seen by using plotdf as well.

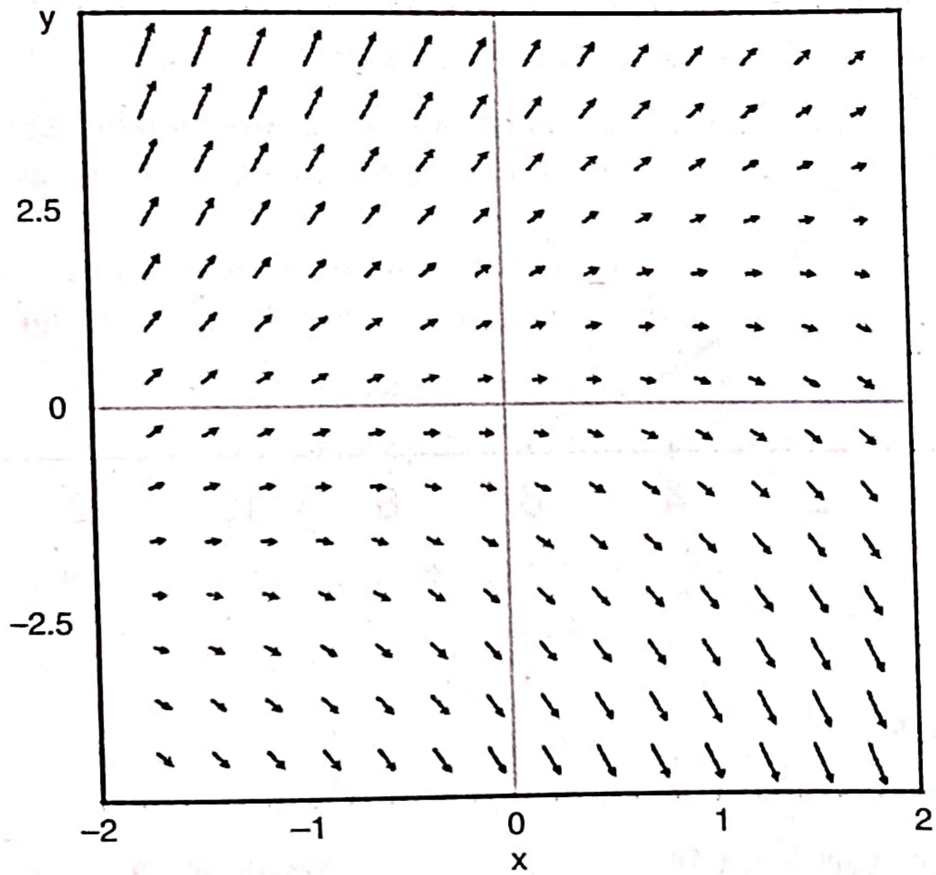
```
(%i5)  wxplot2d([x + 3, x + 2, x + 1, x, x - 1, x - 2, x - 3],
       [x, - 2, 2], [y, - 4, 4],
       [legend, "c = - 3", "c = - 2", "c = - 1", "c = 0", "c = 1",
       "c = 2", "c = 3"]);
```



On each of these lines, the value of the derivative is c e.g for c = 0, the derivative will be zero.

Instead of using wxplot2d, we can use plotdf directly.

```
%i6) kill(all)$ load(plotdf)$
     plotdf(y - x, [x, - 2, 2], [y, - 4, 4]);
%o2) Structure [EXTERNAL-PROCESS]
```



Thus, we can get the idea of solutions from directional graph.