Physical simulation of a simple pendulum

8.5 Simple pendulums

It can be shown that a simple pendulum exhibits approximate simple harmonic motion if it consists of a heavy concentrated mass suspended by an inextensible cord and restricted to a swing of up to $\pm 14°$. To simulate this behaviour dynamically we must analyse the forces shown in Figure 8.15. In this figure we see that the mass of the pendulum, m, is supported by a cord of length L. If the mass is displaced from its upright position by an angle θ radians, a restoring force $m \sin \theta$ will attempt to re-establish equilibrium. The acceleration a produced by this restoring force is given by:

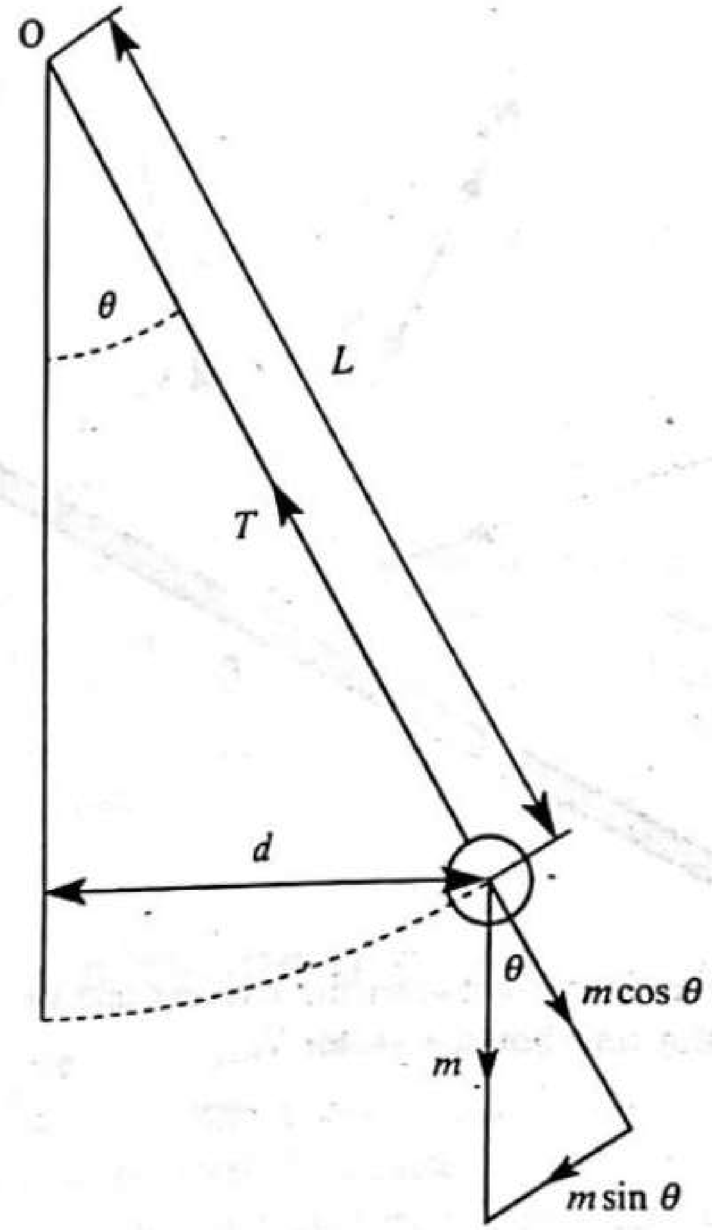

Figure 8.15 When a pendulum is displaced by an angle θ from the vertical position, a restoring force $m \sin\theta$ attempts to re-establish equilibrium. The future position of the pendulum is determined by computing its velocity and acceleration from this force.

$$a = \frac{m \sin \theta}{m/g} \tag{8.81}$$

$$a = g \sin \theta \tag{8.82}$$

If the angle of swing does not exceed ±14°, $\sin \theta$ is approximately equal to θ radians. Therefore, we can write:

$$a = g\theta \tag{8.83}$$

But for these small angles:

$$\theta = \frac{d}{L} \tag{8.84}$$

Therefore:

$$a = g\frac{d}{L} \tag{8.85}$$

At any point in time t, the pendulum has displacement d, velocity v and acceleration a. Therefore, at time $t + \Delta t$ the new displacement d' is given by:

$$d' = d - v\Delta t \tag{8.86}$$

The acceleration is given by:

$$a = g\frac{d}{L} \tag{8.87}$$

and the new velocity v' by:

$$v' = v + a\Delta t \tag{8.88}$$

If the equations for the displacement, velocity and acceleration are iterated, the pendulum's motion is simulated. The pendulum is animated by rotating the object description through and angle θ relative to the upright position, where $\theta = d/L$.

8.6 Springs

The motion of an object attached to a spring or a piece of elastic can be simulated by analysing the dynamic forces acting upon the object. Figure 8.16 shows a spring of length l hanging from a rigid fixing. If an object of mass m is attached to the free end of the spring, it will bounce up and down with simple harmonic motion. Two laws are used to describe the active forces, namely Newton's second law of motion, and Hooke's law. Hooke's law states that the tension T in a spring is given by:

$$T = \lambda \frac{e}{l} \tag{8.89}$$

where λ is the modulus of the spring and e is the extension of the spring. While the object is moving up and down, its motion is controlled by:

$$ma = mg - T \tag{8.90}$$

where m is the mass of the object, a is the acceleration of the object and g is the acceleration due to gravity.

Newton's second law

According to Newton's second law:

$$F = m \times a$$

Where:

- F is the force acting on the pendulum bob ($mg\sin(\theta)$),
- m is the mass of the pendulum bob, and
- a is the acceleration of the pendulum bob.

Hooke law

$$F = -kx$$

Where:

- F is the force applied to the material,
- k is the spring constant (also known as the force constant or stiffness constant), which represents the stiffness of the material,
- x is the displacement produced by the force.

The negative sign indicates that the force exerted by the material is opposite in direction to the displacement. This means that if you compress or stretch the material, it will exert a force in the opposite direction, trying to return to its

Springs

If at any time t, the mass has position y, velocity v and acceleration a, at a time $t + \Delta t$ its new position y' relative to the top of the spring is given by:

$$y' = y + v\Delta t \tag{8.91}$$

The extension e is given by:

$$e = y - l \tag{8.92}$$

and if this is substituted into (8.93):

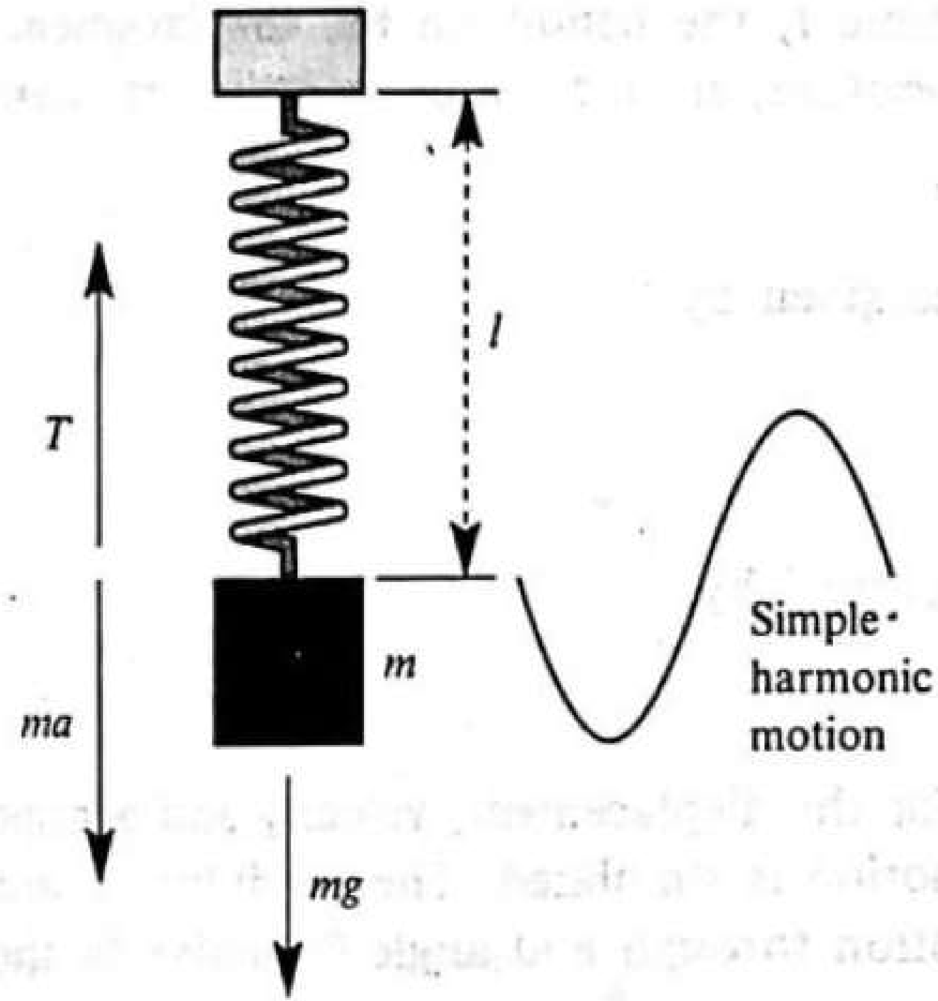

Figure 8.16 If a mass m is attached to a light spring it will oscillate with simple harmonic motion.

$$ma = mg - \lambda\frac{e}{l} \tag{8.93}$$

the acceleration a is equal to:

$$a = g - \lambda\frac{e}{lm} \tag{8.94}$$

and the new velocity v' of the mass is equal to:

$$v' = v + a\Delta t \tag{8.95}$$

If this value of v' is substituted back into (8.91) to develop a new value for y, and continuously repeated, the trace of y is simple harmonic. The value of y can then be substituted into a matrix to subject an object to this motion. Perhaps one of the most useful features of this approach is that the model can be dynamically modified while it is running. Any of the above parameters can be interactively changed, resulting in an almost instantaneous change in the motion.