Work Integrated Learning Programmes Division
Zero Level Mathematical Foundation - M.Tech AIML(NSP1-S1-25)

Practice Problems-I

1 System of Equations

Find the nature of the solutions for the following system of equations by hand and verify using octave

1. $$\begin{aligned} x + y + z &= 6 \\ 2x + 3y + z &= 10 \\ x - y + 2z &= 5 \end{aligned}$$

2. $$\begin{aligned} x + y + z &= 6 \\ 2x + 2y + 2z &= 12 \\ x - y + z &= 0 \end{aligned}$$

3. $$\begin{aligned} x + 2y - z &= 3 \\ 2x + 4y - 2z &= 6 \\ -x - 2y + z &= -3 \end{aligned}$$

2 Matrix Operations

1. Consider the matrices

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \\ 4 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 0 & 4 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -2 & 0 \end{bmatrix}$$

For each matrix:

(a) Find the minor and cofactor of all the elements.
(b) Compute the determinant.
(c) Find the adjoint matrix.
(d) Determine whether the inverse exists. If yes, find it.

3 Linearly Dependent/Independent Vectors

Consider the following sets of vectors. For each, determine whether the vectors are linearly independent or linearly dependent. Justify your answer.

1. $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$
2. $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}$

4 Convex Combinations and System of Equations

Definition: A **convex combination** of two vectors $\vec{x}$ and $\vec{y}$ is a linear combination of the form

$$\vec{z} = \lambda\vec{x} + (1 - \lambda)\vec{y}, \quad \text{where } 0 \leq \lambda \leq 1.$$

This represents all the points on the line segment connecting $\vec{x}$ and $\vec{y}$.

Question: Explain why a system of linear equations cannot have exactly two distinct solutions. Use the concept of convex combinations in your explanation.

5 Operation Count from REF to RREF

Let A be an $n \times n$ matrix of full rank that is already in row echelon form (REF), given by:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Assuming that all pivot elements $a_{ii} \neq 0$, **estimate the number of elementary row operations, additions/subtractions, multiplications, and divisions** required to convert the above matrix from **REF** to **RREF**:

6 Discrete and continuous probability distributions

1. Consider an assignment that contains 5 questions. Let the discrete random variable x represent the number of questions a student is able to solve correctly. Let $P(X = x)$ denote the probability that a student

solves exactly x questions correctly. The probability distribution is given below:

x	0	1	2	3	4	5
$P(X = x)$	0.05	0.10	0.20	0.30	0.20	0.15

(a) Verify whether the given distribution is a valid probability distribution.

(b) Compute the **mean** (expected value) $\mu = E(X)$.

(c) Compute the **variance** $\sigma^2 = \mathrm{Var}(X)$.

(d) Interpret the mean and variance in this context

Note: Use the formulas:

$$E(X) = \sum x \cdot P(x), \quad E(X^2) = \sum x^2 \cdot P(x), \quad \mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

2. A team of professionals is working on a time-bound software module. Based on historical data, the time x (in hours) required to complete the task follows the probability density function:

$$f(x) = \begin{cases} \frac{2}{15}(x+1), & 0 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

(a) Verify that $f(x)$ is a valid probability density function (pdf).

(b) Compute the expected value (mean) $\mu = E(X)$.

(c) Compute the variance $\sigma^2 = \mathrm{Var}(X)$.

(d) Interpret the mean and variance in this context.

Useful formulas:

$$E(X) = \int_a^b x \cdot f(x)\, dx, \quad E(X^2) = \int_a^b x^2 \cdot f(x)\, dx, \quad \mathrm{Var}(X) = E(X^2) - [E(X)]^2$$