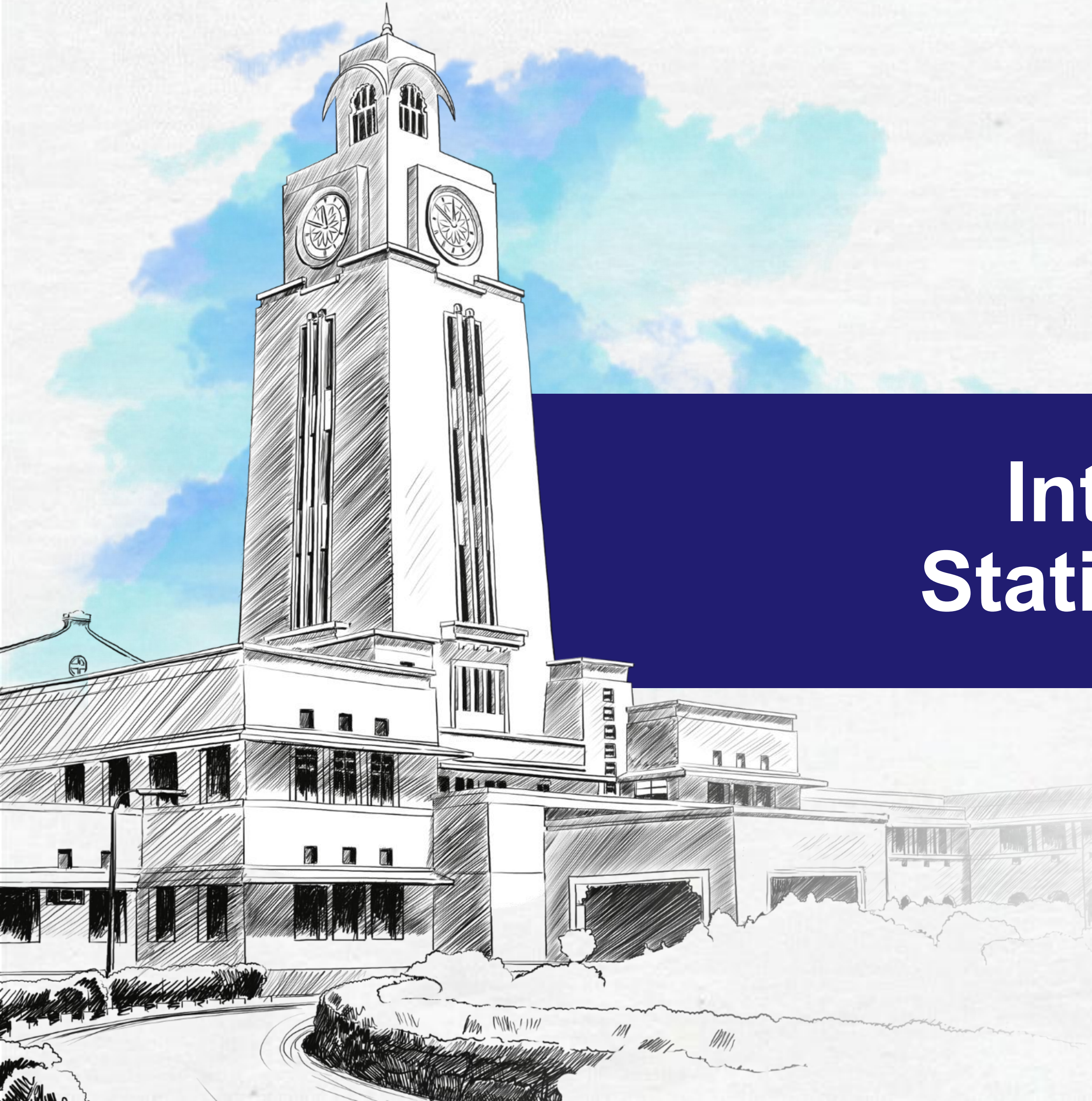

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Introduction to Statistical Methods

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Session 5
(1st June 2025)

Random Variables

Contact Session 5

Contact Session 5: Module 3: Probability Distributions

Contact Session	**List of Topic Title**	**Reference**
CS - 5	Random variables - Discrete & continuous Expectaion of a random variable,mean and variance of a random variable – Sinlge random random variable & Joint distributions	T1 & T2
HW	Problems on random variables	T1 & T2
Lab	Probability Distributions & Sampling	Lab 3

Agenda

- Random variables
- Discrete & continuous random variable,
- Probability Distribution Function
- Mean(Expectation) and variance of a random variable
- Joint distributions

Random Variables

- A **random variable** is a variable that assumes numerical values associated with the random outcome of an experiment, where one (and only one) numerical value is assigned to each sample point.

- In mathematical language, a random variable is a function whose domain is the sample space and whose range is the set of real numbers.

How many Two-wheeler

$X = 0$
$= 1$
$= 2$
$= 3$

$X = 1$
$= 2$
$= 3$
$= 4$
$= 5$
$= 6$

No. of Customers

$X = 6$
$= 1$
$= 2$
$= 2$
$= 4$

Types of Random Variables

- A **discrete random variable** can assume a countable number of values.
  - Number of steps to the top of the Eiffel Tower
  - Number of Heads obtain after tossing two coins simultaneously

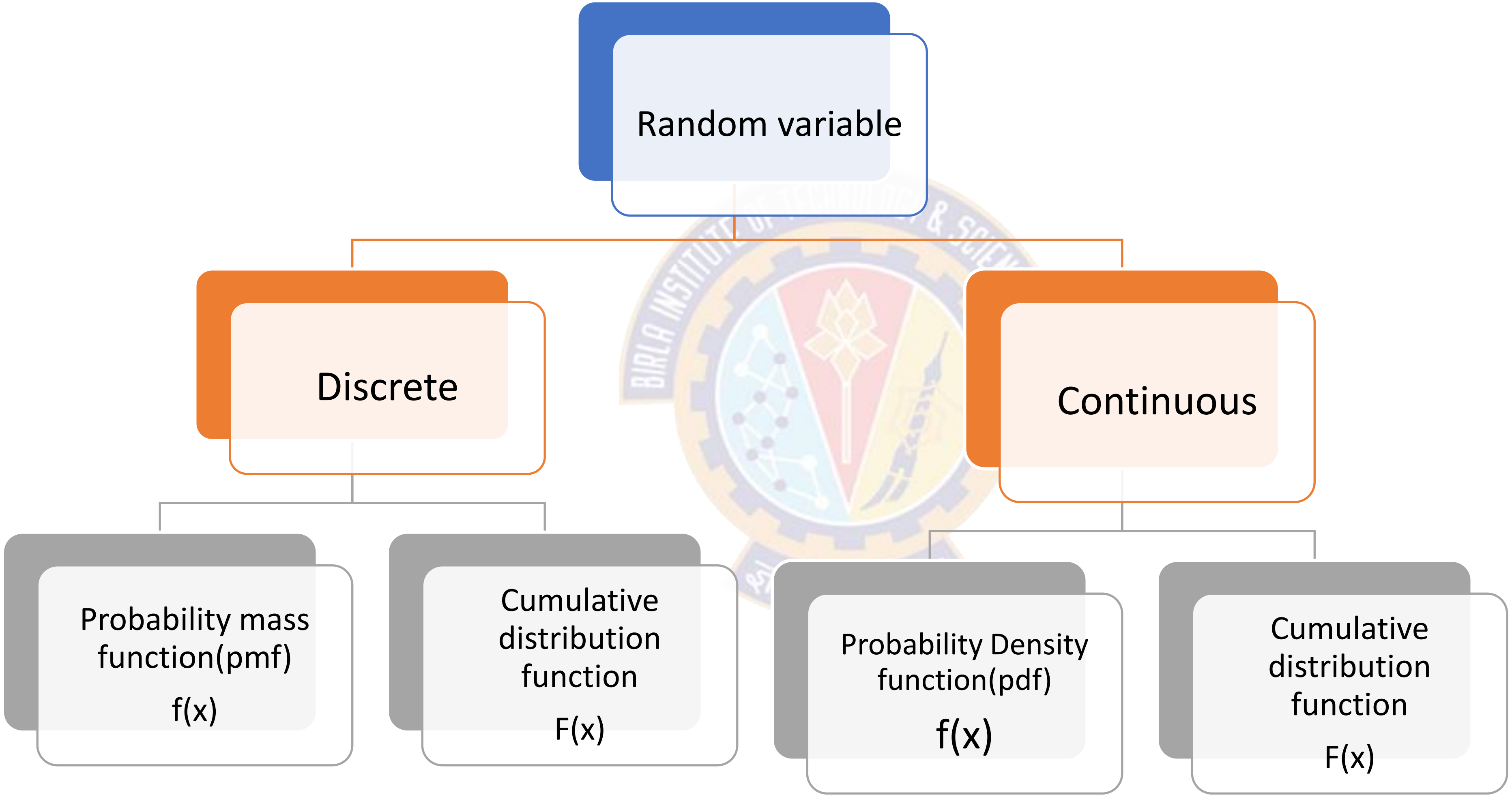
- A **continuous random variable** can assume any value along a given interval of a number line.
  - The time a tourist stays at the top once s/he gets there.
  - The height of a person can take any value within a certain range (e.g., 150 cm to 200 cm),

Examples of Random Variables

- **Discrete random variables**
  - Number of sales
  - Number of calls
  - Number of Shares of a stock
  - Number of People in a line
  - Number of Mistakes per page

- **Continuous random variables**
  - Length of each iron bar
  - Depth of each bore well
  - Volume of each cylinder
  - Time to complete each project
  - Weight of each newly born baby

Classification of Random Variables



- **Identify which of the following variables are discrete and which are continuous:**

  - The number of books in a library. (Discrete / Continuous)
  - The weight of a watermelon in kilograms. (Discrete / Continuous)
  - The amount of rainfall in a day (in millimeters). (Discrete / Continuous)
  - The number of passengers on a bus. (Discrete / Continuous)
  - The temperature of a cup of coffee (in °C). (Discrete / Continuous)
  - The number of goals scored in a football match. (Discrete / Continuous)
  - The length of a bridge (in meters). (Discrete / Continuous)
  - The number of steps you take in a day. (Discrete / Continuous)
  - The time it takes to bake a cake (in minutes). (Discrete / Continuous)

Discrete Probability Distributions

- The probability distribution for a random variable describes how probabilities are distributed over the values of the random variable.

- The probability distribution is defined by a probability function, denoted by $f(x)$ or $p(x)$, which provides the probability for each value of the random variable.

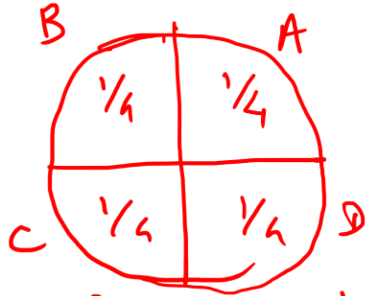
- The required conditions for a discrete probability function are:

x	$P(x)$
1	2/3
2	1/3
3	1/3
4	2/3

$$1 \quad > \quad f(x) \geq 0$$

$$\Sigma f(x) = 1$$

$A = \frac{2}{3}$, $B = \frac{1}{3}$, $C = \frac{1}{3}$



$A = 1/4$ $B = 1/4$ $C = 1/4$ $D = 1/4$

- We can describe a discrete probability distribution with a table, graph, or equation.

- Advantage: once the probability distribution is known, it is relatively easy to determine the probability of a variety of events that may be of interest to the decision maker.

$X = 1$, $X = 2$, $X = 3$, $X = 4$, $X = 5$, $X = 6$

$P(X = 1) = 1/6$, $P(X = 2) = 1/6$, $1/6$, $1/6$, $1/6$, $1/6$

Example:

- Discrete random variable with a finite number of values
  - Let *x* = number of TV sets sold at the store in one day where *x* can take on 5 values (0, 1, 2, 3, 4)

- Discrete random variable with an infinite sequence of values
  - Let *x* = number of customers arriving in one day where *x* can take on the values 0, 1, 2, . . .
  - We can count the customers arriving, but there is no finite upper limit on the number that might arrive.

Probability Distributions for Discrete Random Variables

- Suppose you toss a fair coin 2 times, and you're interested in the number of heads obtained.
- Let the random variable X represent the number of heads.
- Here, X is a discrete random variable because it can take only integer values (0, 1, 2).
- The Probability distribution function for X is:

$$f(0) = P\{TT\} = \frac{1}{4}$$

$$f(1) = P\{HT, TH\} = \frac{2}{4} = \frac{1}{2}$$

$$f(2) = P\{HH\} = \frac{1}{4}.$$

Here we can observe that $f(x) \geq 0\ \forall x$ and $\sum f(x) = 1$.

x	$f(x)$	$x \cdot f(x)$
0	1/4	
1	1/2	
2	1/4	
		$\Sigma x f(x)$

x	f	$x \cdot f$
2	3	
4	2	
5	3	

Discrete Uniform Probability Distribution

- The discrete uniform **probability** distribution is the simplest example of a discrete probability distribution given by a formula.
- The discrete uniform probability function is

$$f(x) = 1/n$$

where:

n = the number of values the random variable may assume

- Note that the values of the random variable are equally likely.
- Example: A discrete RV X: No. on a dice has the uniform probability

$f(x) = 1/6 \quad for\ x = 1,2,3,4,5,6$

Example

- Suppose a student takes **1, 2, or 3 attempts** to pass a driving test. Let the probability function be:

$$f(x) = \frac{4-x}{6}, \quad x = 1, 2, 3$$

- Is this a valid probability function?

X	f(x)	x f(x)
1	$\frac{4-1}{6} = \frac{1}{2}$	$\frac{1}{2}$
2	$\frac{4-2}{6} = \frac{1}{3}$	$\frac{2}{3}$
3	$\frac{4-3}{6} = \frac{1}{6}$	$\frac{1}{2}$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{3+2+1}{6} = \frac{1}{1}$$

$$E(x) = \frac{1}{2} + \frac{2}{3} + \frac{1}{2} = \frac{5}{3} = 1.6$$

interpretation: A student on an average will pass the test in 1.6 attempts → 1 or 2

Mathematical Expectation

- Mean is a measure of location or central tendency in the sense that it roughly locates a middle or average value of the random variable

- The mean or Expected value of a discrete random variable:

$$E(X) = \mu = \sum x\, f(x)$$

$\frac{\sum x f(x)}{\sum f(x)}$

Mean → Expectation

# Heads	$f(x)$	$x \cdot f(x)$
0	1/4	0
1	1/2	1/2
2	1/4	1/2
		1

$\boxed{E(X) = 1}$ → Whenever we toss two coins on an average there will be 1 head.

Example:

A game is played by two person A and B as follows:

- "A throws two dice. If the sum is 7 he wins \$3 from B. If the sum is 8, he loses \$2 and if the sum is 3, he loses \$4, otherwise no money changes hand. Is this a fair game?"
- A game is said to be fair if the expected gain is zero. In order to check it, let the random variable X = gain of A from B which takes values with probability distribution as follows:

(1,2), (2,1)

(3,4), (4,3)
(2,5), (5,2)
(6,1), (1,6)

(4,4), (2,6), (6,2)
(3,5), (5,3)

Sum	2	3	4	5	6	7	8	9	10	11	12	
X(in $) = x	0	-4	0	0	0	3	-2	0	0	0	0	
f(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36	
x*f(x)	0	-8/36	0	0	0	18/36	-10/36	0	0	0	0	SUM = 0 = $\sum x\, f(x) = E(X) = \mu$
												Hence, the game is fair

$\Sigma x f(x) = 0 \rightarrow$ On an average gain to A is zero \$

Mathematical Expectation

$E(ax) = \sum ax \cdot f(x)$

$= a \sum x f(x)$

$= a \cdot E(X)$

- Multiplying RV by a constant a, $E(aX) = aE(X)$ ✓
- Adding a constant b, $E(X+b) = E(X) + b$ ✓
- Therefore, E(aX + b) = ? $E(ax) + b = aE(X) + b$
- E(X/Y)$\neq \dfrac{E(X)}{E(Y)}$ Note
- E(X.Y)$\neq E(X).E(Y)$ unless they are independent

$E(X \cdot Y) = E(X) \cdot E(Y)$ iff X, Y are independent

x	f(x)
1	1/4
2	1/2
3	1/4

E(x)

4X+3	f(x)
7	1/4
11	1/2
15	1/4

$E(4X+3)$
$= 4E(X) + 3$

Variability of Discrete Random Variables

$E(X) = \sum x \cdot f$

- The **variance** of a **discrete random variable** *x* is

$$var(X) = \sigma^2 = E(X-\mu)^2 = \sum (x-\mu)^2 f(x)$$

$x \cdot f$	x	$f(x)$	x^2	$x^2 f(x)$

$\sum x^2 f(x)$

- Or

$$var(X) = \sigma^2 = E(X^2) - \big(E(X)\big)^2$$
$$= E(X^2) - \mu^2$$
$$= \sum x^2 f(x) - \mu^2$$

- **The <u>standard deviation</u>(σ),** is defined as the positive square root of the variance.

x	$f(x)$	$(x-\mu)$	$(x-\mu)^2$	$f(x)\cdot(x-\mu)^2$

$Var(x) = \dfrac{\sum f \cdot (x-\mu)^2}{\sum f}$

$= \sum f \cdot (x-\mu)^2$

Rules of variability

T_2

- $V(b) = 0$, b is the constant
- Multiplying RV by a constant $a, V(aX) = a^2V(X)$
- Therefore, V(aX+b) = ? $= V(ax) + Var(b) = a^2 V(x)$
- $V(aX + bY) = a^2V(X) + b^2V(Y)$, whenever X and Y are independent
- $\sigma_{aX} = |a|\sigma_x$

$V(X) = 2$

$V(3 \cdot X) = 18$

$V(3x+4) = 18$

$Var(X) = 4$ $\quad$ $Var(Y) = 3$

$Var(3x + 2y) = 36 + 12 = 48$

Example:

A fast-food restaurant manager analyzes how many cars are typically in line at the drive-thru at lunch hour. Let the **random variable X** represent the number of cars in line when a customer arrives. Find the mean, standard deviation and variance. Historical data shows:

interpret

$E(x^2) = 4.05$

$E(x) = 1.75$

$Var(x) = 4.05 - (1.75)^2$

$= 0.9875$

Number of Cars (x)	Probability P(X=x)	$x f(x)$	$x^2 f(x)$
0 (no wait)	0.10		0
1 car	0.30		0.3
2 cars	0.40		1.6
3 cars	0.15		1.35
4 cars	0.05		0.80
			4.05

Expected value & SD:

x	$P(X=x)$	$x \cdot P(X)$	$(x-1.75)^2 \cdot P(X)$
0	0.10	0.00	$(0-1.75)^2 \cdot 0.10 = 3.0625 \cdot 0.10 = 0.30625$
1	0.30	0.30	$(1-1.75)^2 \cdot 0.30 = 0.5625 \cdot 0.30 = 0.16875$
2	0.40	0.80	$(2-1.75)^2 \cdot 0.40 = 0.0625 \cdot 0.40 = 0.025$
3	0.15	0.45	$(3-1.75)^2 \cdot 0.15 = 1.5625 \cdot 0.15 = 0.234375$
4	0.05	0.20	$(4-1.75)^2 \cdot 0.05 = 5.0625 \cdot 0.05 = 0.253125$
Total	**1.00**	$\mu = E(X)$ **1.75**	**0.9875** (Variance σ^2)

$$\text{Standard Deviation} = \sqrt{0.9875} \approx \boxed{0.9937}$$

Cumulative Probability distribution Function

If X is a discrete random variable which assumes values $x = x_1, x_2, x_3, ..x_i, \dots x_n$ with their corresponding probabilities $f(x) = f_1, f_2, f_3, \dots f_i, \dots f_n$ respectively then the cumulative distribution function (CDF) of the RV X is defined as $F(x) = P(X \leq x)$.

Specifically,

$F(x_1) = P(X \leq x_1) = P(X=x_1)=f_1$

$F(x_2) = P(X \leq x_2) = P(X=x_1)+ P(X=x_2) =f_1+f_2$

$F(x_3) = P(X \leq x_3) = P(X=x_1)+ P(X=x_2) + P(X=x_3) \;\; =f_1+f_2+f_3$

………………………………………………………………………….

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$F(x_n)=P(X\leq x_n) = P(X=x_1)+ P(X=x_2) +..+ P(X=x_n) \;\; =f_1+f_2+\dots+f_n = 1$ (=Total probability)

Cumulative Distribution for Discrete Data

Cumulative probability Distribution:

It is given by $F(x) = P(X \leq x)$.

Example: If two dices are rolled. Find the probability distribution and cumulative probability distribution. Also represent in graph.

Solution: X- random variable is sum of the two numbers

Sum

X	1	2	3	4	5	6	7	8	9	10	11	12
P(x)	-	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
F(x) C.D.F	0	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	1

Example: 1

- Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, variance and standard deviation of the number of queens.

Solution: Let X denote the number of queens in a draw of two cards. X is a random variable which can assume the values 0, 1 or 2.

$$f(0) = P(no\ queen) = 48_{C_2}/52_{C_2} = \frac{188}{221}.$$

$$f(1) = P(1\ queen\ ) = \frac{4_{C_1}.48_{C_1}}{52_{C_2}} = \frac{32}{221}.$$

$$f(2) = P(2\ queens) = \frac{4_{C_2}}{52_{C_2}} = \frac{1}{221}$$

# Queens	$f(x)$	$x f(x)$
0	188/221	0
1	32/221	32/221
2	1/221	2/221
	1	34/221

0.15

Solution:

- Mean of X:

$$E(X) = \sum x\, f(x) = 0 \times \frac{188}{221} + 1 \times \frac{32}{221} + 2 \times \frac{1}{221} = \frac{34}{221}$$ ✓

- Variance of X:

$$E(X^2) = \sum x^2 f(x) = 0^2 \times \frac{188}{221} + 1^2 \times \frac{32}{221} + 2^2 \times \frac{1}{221} = \frac{36}{221}.$$ ✓

- $var(X) = E(X^2) + \left(E(X)\right)^2 = \frac{36}{221} - \left(\frac{34}{221}\right)^2 = \frac{6800}{(221)^2}$
- $\sigma_x = \sqrt{\frac{6800}{(221)^2}} = \underline{\underline{0.37}}.$

mean 0.15

SD 0.37

Example: 2

A random variable X has the following probability function:

X	0	1	2	3	4
$P(X=x)$	k	$2k$	$3k$	$2k$	k
	$1/9$	$2/9$	$1/3$	$2/9$	$1/9$
$\to$ $x \cdot f(x) =$	0	$2/9$	$2/3 = \frac{6}{9}$	$6/9$	$4/9$

Find

i) *the value of* k $= 1/9$

ii) $P(X<3)$ $= \frac{1}{9} + \frac{2}{9} + \frac{1}{3} =$

iii) $P(0<X<4)$ $= 2/9 + 1/3 + 2/9$

iv) the distribution function of X

v) $E(X)$, vi) $Var(X)$

$= 2$ $\quad =$

$E(x^2) = 0 + \frac{2}{9} + \frac{12}{9} + \frac{18}{9} + \frac{16}{9}$

$= 48/9 = 16/3$

$Var = E(X^2) - [E(X)]^2$

$= \frac{16}{3} - 4 = \frac{4}{3}$

Solution:

i) Find the value of k

Since the total probability must equal 1:

$$k + 2k + 3k + 2k + k = 9k = 1 \Rightarrow k = \frac{1}{9}$$

ii) Find $P(X < 3)$

$$P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = k + 2k + 3k = 6k = 6 \cdot \frac{1}{9} = \frac{2}{3}$$

iii) Find $P(0 < X < 4)$

$$P(0 < X < 4) = P(X = 1) + P(X = 2) + P(X = 3) = 2k + 3k + 2k = 7k = 7 \cdot \frac{1}{9} = \frac{7}{9}$$

iv) Find the distribution function (CDF) of X

Let $F(x) = P(X \leq x)$

x	$F(x)$
$x < 0$	0
0	$k = \frac{1}{9}$
1	$k + 2k = \frac{3}{9}$
3	$k + 2k + 3k + 2k = \frac{8}{9}$
4 or more	1

v) Find the **Expected Value** $E(X)$

$$E(X) = \sum x \cdot P(X = x) = 0 \cdot k + 1 \cdot 2k + 2 \cdot 3k + 3 \cdot 2k + 4 \cdot k$$

$$= 0 + 2k + 6k + 6k + 4k = 18k = 18 \cdot \frac{1}{9} = 2$$

vi) Find the **Variance** $\mathrm{Var}(X)$

First, compute $E(X^2)$:

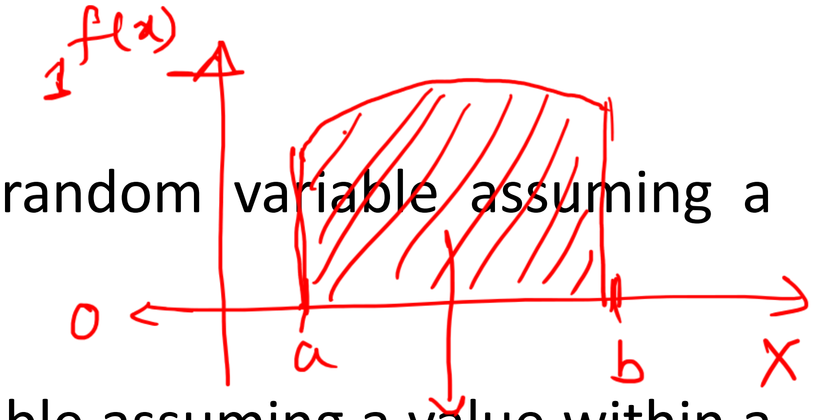
$$E(X^2) = \sum x^2 \cdot P(X = x) = 0^2 \cdot k + 1^2 \cdot 2k + 2^2 \cdot 3k + 3^2 \cdot 2k + 4^2 \cdot k$$

$$= 0 + 2k + 12k + 18k + 16k = 48k = 48 \cdot \frac{1}{9} = \frac{16}{3}$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \frac{16}{3} - 2^2 = \frac{16}{3} - 4 = \frac{4}{3}$$

Continuous Probability Distributions

- A <u>continuous random variable</u> can assume any value in an interval on the real line or in a collection of intervals.



- It is not possible to talk about the probability of the random variable assuming a particular value.
- Instead, we talk about the probability of the random variable assuming a value within a given interval.

$1 = \text{Area} = \int_a^b f(x)\, dx$

- The probability of the random variable assuming a value within some given interval from x_1 to x_2 is defined to be the <u>area under the graph</u> of the <u>probability density function</u> between x_1 and x_2.

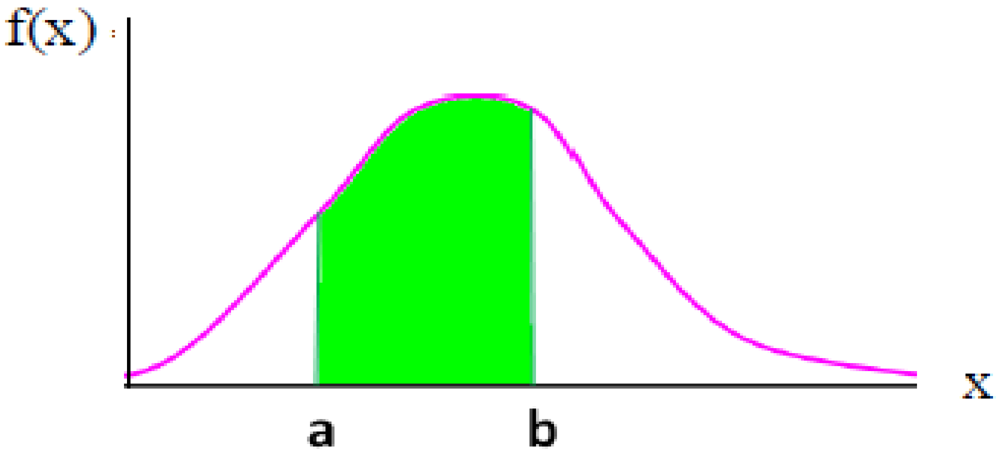
$\sum f(x) = 1$

$\int f(x) = 1$

Continuous Random Variables

A continuous random variable can assume any value in an interval on the real line or in a collection of intervals.

The probability of the random variable assuming a value within some given interval from x1 to x2 is defined to be the area under the graph of the **probability density function** between x1 and x2



Random Var. is cont.

$\Downarrow$

$$P(a \le X \le b) = \int_a^b f(x)\, dx.$$

Example:

- Height of students in a class
- Amount of ice tea in a glass
- Change in temperature throughout a day
- Price of a car in next year

Continuous Random Variables

Probability Density Function

For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx$ = area under $f(x)$ from a to b for any a and b (4.1)

Continuous Random Variables

Mean and Variance

Suppose that X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2 \quad \checkmark$$

$$E(X^2) - [E(X)]^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

Continuous Random Variables

EXAMPLE 1 **Calculating probabilities from the probability density function**

If a random variable has the probability density

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

find the probabilities that it will take on a value

(a) between 1 and 3;

(b) greater than 0.5.

Solution Evaluating the necessary integrals, we get

(a) $P(1 < x < 3) = \int_1^3 2e^{-2x}\,dx = e^{-2} - e^{-6} = 0.133$

(b) $P(X \geq 0.5)$

$P(X > 0.5)$ $\int_{0.5}^{\infty} 2e^{-2x}\,dx = e^{-1} = 0.368$

$= \frac{1}{e} = 0.368$

$\int_1^3 2e^{-2x}\,dx = 2\int_1^3 e^{-2x}\,dx$

$= 2\left[\frac{e^{-2x}}{-2}\right]_1^3$

$= -\left[e^{-6} - e^{-2}\right]$

$= \frac{1}{e^2} - \frac{1}{e^6}$

$= 0.133$

$\int_{0.5}^{\infty} 2e^{-2x} = 2\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty}$

$= -\left[e^{-\infty} - e^{-1}\right]$

$= -\left[0 - \frac{1}{e}\right]$

With reference to the preceding example, find the distribution function and use it to determine the probability that the random variable will take on a value less than or equal to 1.

Performing the necessary integrations, we get

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \displaystyle\int_0^x 2e^{-2t}\,dt = 1 - e^{-2x} & \text{for } x > 0 \end{cases}$$

and substitution of $x = 1$ yields

$$F(1) = 1 - e^{-2} = 0.865$$

Example 2

- The Pdf of a random variable is given by $f(x) = x$ for $0<x<1$

  $= 2-x$ for $1 \leq x < 2$

  $= 0$, elsewhere

- Determine, mean, variance, standard deviation
- P(x lies in between 0.2 and 0.8)
- P(x lies in between 0.6 and 1.2)

Solution

- The Pdf of a random variable is given by f(x) = x for 0<x<1
  = 2-x for 1≤ x <2
  = 0, elsewhere
  - Determine, mean, variance, standard deviation
  - P(x lies in between 0.2 and 0.8)
  - P(x lies in between 0.6 and 1.2)

Solution

Mean = μ = E(X) = $\int_{-\infty}^{\infty} xf(x)\,dx$ = =$\int_0^1 xx\,dx + \int_1^2 x(2-x)\,dx$

$$=\int_0^1 x^2\,dx + 2\int_1^2 x\,dx - \int_1^2 x^2\,dx = =\frac{x^3}{3}\Big|_0^1 + 2\frac{x^2}{2}\Big|_1^2 - \frac{x^3}{3}\Big|_1^2 = = \frac{1}{3} + 3 - \frac{7}{3} = 1$$

$$\mathbf{E(X^2)} = \int_{-\infty}^{\infty} x^2 f(x)\,dx \;= \int_0^1 x^2 x\,dx + \int_1^2 x^2(2-x)\,dx = \int_0^1 x^3\,dx + 2\int_1^2 x^2\,dx - \int_1^2 x^3\,dx \;= \frac{x^4}{4}\Big|_0^1 + 2\frac{x^3}{3}\Big|_1^2 - \frac{x^4}{4}\Big|_1^2$$

$$= \frac{1}{4} + \frac{14}{3} - \frac{15}{4} = \frac{7}{6}$$

$$\mathbf{Variance} = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2 = \frac{7}{6} - (1)^2 = \frac{1}{6}$$

$$\text{SD}=\sigma = +\sqrt{variance} = +\sqrt{\frac{1}{6}} = \frac{1}{\sqrt{6}} = 0.4082$$

- P(x lies in between 0.2 and 0.8) = $P(0.2 < X < 0.8) = \int_{0.2}^{0.8} f(x)\, dx$=$\int_{0.2}^{0.8} x\, dx$=$\left.\frac{x^2}{2}\right|_{0.2}^{0.8}$= 0.3
- P(x lies in between 0.6 and 1.2)= $P(0.6 < X < 1.2) = = P(0.6 < X < 1) + P(1 < X < 1.2) =$

=$\int_{0.6}^{1} x\, dx + \int_{1}^{1.2}(2-x)\, dx$= $\left.\frac{x^2}{2}\right|_{0.6}^{1} + 2x|_{1}^{1.2} - \left.\frac{x^2}{2}\right|_{1}^{1.2}$ =0.5.

Example 3:

A probability density function assigns probability one to $(-\infty, \infty)$

Find k so that the following can serve as the probability density of a random variable:

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ kxe^{-4x^2} & \text{for } x > 0 \end{cases}$$

Solution

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} kxe^{-4x^2}\,dx = \int_0^{\infty} \frac{k}{8} \cdot e^{-u}\,du = \frac{k}{8} = 1$$

so that $k = 8$.

Practice Problems

Problem:1

If the probability density of a random variable is given by

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 \le x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

find the probabilities that a random variable having this probability density will take on a value

(a) between 0.2 and 0.8; (b) between 0.6 and 1.2.

Problem:2

Given the probability density $f(x) = \dfrac{k}{1+x^2}$ for $-\infty < x < \infty$, find k.

Problem:3

The length of satisfactory service (years) provided by a certain model of laptop computer is random variable having the probability density function :

$$\boldsymbol{f(x)} = \begin{cases} \frac{1}{4.5} e^{-\frac{x}{4.5}} & x > 0 \\ 0 & otherwise \end{cases}$$

Find the probabilities that one of these laptops will provide satisfactory service for

a) At most 2.5 years (Ans: 0.4262)
b) Anywhere from 4 to 6 years (Ans: 0.1475)
c) At least 6.75 years (Ans:0.2231)

Problem:4

For following probability density function :

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x < 2 \\ 0 & otherwise \end{cases}$$. Find the mean and variance.

Ans: Mean =1, Variance $=\frac{1}{6}$

Problem:5

If $P(X = x) = \begin{cases} kx & x = 1,2,3,4,5 \\ 0 & otherwise \end{cases}$ represents a probability function , find $i)\ k$, $ii)\ P(X\ \ being\ a\ prime\ number)$, $iii)\ P\left(\frac{1}{2} < X < \frac{5}{2}\right)$.

(Answer: $i)\ k = \frac{1}{15}$, $ii)\ \frac{11}{15}$ $iii)\ \frac{1}{5}$)

Problem:6

From a lot of 10 items containing 3 defective items, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Answer the following when the sample is drawn without replacement:

a) Find the probability distribution of X.

b) Find $P(X \leq 1)$

c) Find $P(0 < X < 2)$

Ans: a)

x	0	1	2	3
$P(X)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

b) $\frac{2}{3}$ c) $\frac{1}{2}$

Joint Probability Distribution Function

- Let $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ be two discrete random variables. Then $P(x, y) = J_{ij}$ is called joint probability function of X and Y if it satisfies the conditions:

$$(i)\ J_{ij} \geq 0 \quad (ii) \sum_{i=1}^{m} \sum_{j=1}^{n} J_{ij} = 1$$

- Set of values of this joint probability function J_{ij} is called joint probability distribution of X and Y.

	y_1	y_2	$\dots$	y_n	Sum
x_1	J_{11}	J_{12}	$\dots$	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	$\dots$	J_{2n}	$f(x_2)$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
x_m	J_{m1}	J_{m2}	$\dots$	J_{mn}	$f(x_m)$
Sum	$g(y_1)$	$g(y_2)$	$\dots$	$g(y_n)$	$Total = 1$

- If X and Y are discrete random variables, the joint probability distribution of X and Y is a description of the set of points (x,y) in the range of (X,Y) along with the probability of each point.

- The joint probability distribution of two discrete random variables is sometimes referred to as the **bivariate probability distribution** or **bivariate distribution.**

- Thus, we can describe the joint probability distribution of two discrete random variables is through a **joint probability mass function**

$$f(x,y)=P(X=x,Y=y)$$

➢We often want to determine the joint probability of two variables, such as X and Y. Suppose we are able to determine the following information for education (X) and age (Y) for all Indian citizens based on the census.

Education (X) \ Age (Y):		30	45	70
None	0	.01	.02	.05
Primary	1	.03	.06	.10
Secondary	2	.18	.21	.15
College	3	.07	.08	.04

- Each cell is the relative frequency (f/N).
- We can define the joint probability distribution as:p(x, y)= Pr(X = x and Y = y)

Example: what is the probability of getting a 30 year old college graduate?
$p(x,y) = Pr(X=3 \text{ and } Y=30) = .07$
We can see that: $p(x) = \Sigma_y \, p(x,y)$
$p(x=1) = .03 + .06 + .10 = .19$

Age (Y): Education (X)		30	45	70
None	0	.01	.02	.05
Primary	1	.03	.06	.10
Secondary	2	.18	.21	.15
College	3	.07	.08	.04

Marginal Probability

- We call this the **marginal probability** because it is calculated by summing across rows or columns and is thus reported in the margins of the table.
- We can calculate this for our entire table.

Age (Y): Education (X)	30	45	70	p(x)
None: 0	.01	.02	.05	**.08**
Primary: 1	.03	.06	.10	**.19**
Secondary: 2	.18	.21	.15	**.54**
College: 3	.07	.08	.04	**.19**
p(y)	**.29**	**.37**	**.34**	**1**

Joint Density Function

When X and Y are continuous random variables, the **joint density function** $f(x, y)$ is a surface lying above the xy plane, and $P[(X, Y) \in A]$, where A is any region in the xy plane, is equal to the volume of the right cylinder bounded by the base A and the surface.

The function $f(x, y)$ is a **joint density function** of the continuous random variables X and Y if

1. $f(x, y) \geq 0$, for all (x, y),
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$,
3. $P[(X, Y) \in A] = \int \int_A f(x, y)\, dx\, dy$, for any region A in the xy plane.

Marginal Distributions

The **marginal distributions** of X alone and of Y alone are

$$g(x) = \sum_y f(x, y) \quad \text{and} \quad h(y) = \sum_x f(x, y)$$

for the discrete case, and

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

for the continuous case.

Independent Random Variables

- Let X and Y are two random variables with joint probability function $f(x, y)$ are said to be independent if following condition satisfied:
- $f(x, y) = g(x).h(y) for\ all\ x\ and\ y$, where $g(x)$ is marginal probability function of X and $h(y)$ is marginal probability function of Y.

Example: Suppose $f(x, y) = e^{-(x+y)}, x \geq 0, y \geq 0$. For this probability function, marginal probability function for X is $g(x) = e^{-x}, x \geq 0$ and

marginal probability function for Y is $h(y) = e^{-y}, y \geq 0$.

Clearly $f(X, Y) = g(X).h(Y)$. So X and Y are independent variables.

Conditional Probability Distribution Function

- **Discrete Case:**

$$P[X = x_i / Y = y_j] = \frac{P[X = x_i / Y = y_j]}{P[Y = y_j]} = \frac{p_{ij}}{p_{.j}}$$

The set $\{x_i, p_{ij} / p_{.j}\}$, $i = 1, 2, 3, \ldots\ldots$is called the conditional probability distribution of X given $Y = y_j$.

The conditional probability function of Y given $X = x_i$ is given by

$$P[Y = y_i / X = x_j] = \frac{P[Y = y_i / X = x_j]}{P[X = x_j]} = \frac{p_{ij}}{p_{i.}}$$

The set $\{y_i, p_{ij} / p_{i.}\}$, $j = 1, 2, 3, \ldots\ldots$is called the conditional probability distribution of Y given $X = x_i$.

Continuous Case

Conditional Probability Density Function

$$f(y \mid x) = \frac{f(x, y)}{f_X(x)} \qquad \textbf{or} \qquad f(x \mid y) = \frac{f(x, y)}{f_Y(y)}$$

- the continuous version of Bayes' theorem

$$f(y \mid x) = \frac{f(x \mid y) f_Y(y)}{f_X(x)}$$

- another expression of the marginal pdf

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_{-\infty}^{\infty} f(x \mid y) f_Y(y) dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_{-\infty}^{\infty} f(y \mid x) f_X(x) dx$$

Example 1:

You want to compute the probability that a patient has the **flu** given they have a **fever**.

Let:

- $D \in$ {Flu, No Flu} : Disease (target variable)
- $S \in$ {Fever, No Fever} : Symptom (observed variable)

You are given the **joint distribution** from past data:

Disease	Symptom	$P(D = d, S = s)$
Flu	Fever	0.30
Flu	No Fever	0.10
No Flu	Fever	0.10
No Flu	No Fever	0.50

Compute the **conditional probability distribution**: P(D|S=Fever)

Solution

- **Compute the marginal probability P(S=Fever):**

$$P(S = \text{Fever}) = P(\text{Flu}, \text{Fever}) + P(\text{No Flu}, \text{Fever}) = 0.30 + 0.10 = 0.40$$

- **Compute the Conditional probability:**

$$P(D = d \mid S = \text{Fever}) = \frac{P(D = d, S = \text{Fever})}{P(S = \text{Fever})}$$

Final Answer:

- $P(\text{Flu} \mid \text{Fever}) = \frac{0.30}{0.40} = 0.75$
- $P(\text{No Flu} \mid \text{Fever}) = \frac{0.10}{0.40} = 0.25$

Example 2:

Consider the joint distribution of X and Y.

Compute the following probabilities:

(i) $P(X = 1, Y = 2)$ (ii) $P(X \geq 1, Y \geq 2)$

(iii) $P(X \leq 1, Y \leq 2)$ (iv) $P(X + Y \geq 2)$ (v) $P(X \geq 1, Y \leq 2)$.

X \ Y	0	1	2	3
0	0	1/8	1/4	1/8
1	1/8	1/4	1/8	0

Solution:

(i) $X = \{0, 1\}, Y = \{0, 1, 2, 3, 4\}$

$$P(X = 1, Y = 2) = P(1, 2) = \frac{1}{8}$$

(ii) If $X \geq 1, X = \{1\}$. If $Y \geq 2, Y = \{2, 3\}$

$$P(X \geq 1, Y \geq 2) = P(1, 2) + P(1, 3) = \frac{1}{8} + 0 = \frac{1}{8}$$

(iii) If $X \leq 1, X = \{0, 1\}$. If $Y \leq 2, Y = \{0, 1, 2\}$

$$P(X \leq 1, Y \leq 2) = P(0, 0) + P(0, 1) + P(0, 2) + P(1, 0) + P(1, 1) + P(1, 2)$$

$$= 0 + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

Cont.

(iv) If $X + Y \geq 2$ then

$$X + Y = 0 + 2 \; or \; 0 + 3 \; or \; 1 or \; 1 + 2 \; or \; 1 + 3$$

$$P(X + Y \geq 2) = P(0,2) + P(0,3) + P(1,1) + P(1,2) + P(1,3)$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + 0 = \frac{3}{4}$$

(v) If $X \geq 1, X = \{1\}$. If $Y \leq 2, Y = \{0,1,2\}$

$$P(X \geq 1, Y \leq 2) = P(1,0) + P(1,1) + P(1,2)$$

$$= \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$$

Example 3:

- Two machines in a factory produce defective items. Let:

X: Number of defective items from **Machine A** (can be 0 or 1)

Y: Number of defective items from **Machine B** (can be 0, 1, or 2)

The joint probability mass function of X and Y is given below:

$X \setminus Y$	**0**	**1**	**2**
0	0.10	0.20	0.10
1	0.10	0.30	0.20

Compute:

1. Find the **marginal distributions** of X and Y
2. Find $P(X = 1, Y = 2)$ Ans: 0.20
3. Find $P(Y = 1 \mid X = 0)$ Ans: 0.5
4. Find the **expected values** $E(X)$ and $E(Y)$ Ans: 0.6 and 1.10
5. Are X and Y **independent**?

Solution:

3. $P(Y = 1 \mid X = 0)$

$$P(Y = 1 \mid X = 0) = \frac{P(X = 0, Y = 1)}{P(X = 0)} = \frac{0.20}{0.40} = 0.5$$

5. Check: is $P(X = 0, Y = 1) = P(X = 0) \cdot P(Y = 1)$?

- $P(X = 0, Y = 1) = 0.20$
- $P(X = 0) = 0.40$, $P(Y = 1) = 0.50$
- $P(X = 0) \cdot P(Y = 1) = 0.40 \cdot 0.50 = 0.20$

✅ For this pair, they match. Try another:

- $P(X = 1, Y = 0) = 0.10$
- $P(X = 1) \cdot P(Y = 0) = 0.60 \cdot 0.20 = 0.12$

❌ They don't match. So X **and** Y **are not independent.**

Example 4:

Find the joint distribution of X and Y which are the independent random variables with the following respective distributions.

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

Solution:

Since X and Y are independent random variables,

$$J_{ij} = f(x_i)g(y_j)$$

$x\backslash y$	-2	5	8	$f(x)$
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
$g(y)$	0.3	0.5	0.2	$Total = 1$

Example 5:

A machine learning model is trained to classify images of **animals**. In the dataset:

- Each image is labeled based on whether the animal is **wild** or **domestic**
- Each image is also tagged for visual features like **fur**, **claws**, and **ears**

➢ Let:

X: Proportion of **wild animal images** in which **claws are detected**

Y: Proportion of **domestic animal images** in which **claws are detected**

Suppose the **joint probability density function** for the proportions X and Y is given by:

Suppose the **joint probability density function** for the proportions X and Y is given by:

$$f(x, y) = \begin{cases} 8xy & \text{for } 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- ✓ Verify that f(x, y) is a valid joint probability density function
- ✓ Find the marginal densities fX(x) and fY(y).
- ✓ Compute $P(X<0.5, Y>0.5)$ — i.e., probability that fewer wild images but more domestic images show claws
- ✓ Are X and Y independent?
- ✓ Compute the expected values E(X), E(Y), and E(XY).

Solution:

1. **Verify $f(x, y)$ is a valid joint PDF**

We need to check:

$$\iint_{\text{all } x,y} f(x,y)\,dx\,dy = 1$$

2. Find marginal densities

$$f_X(x) = \int_0^1 f(x,y)dy = \int_0^1 4xy\,dy = 4x \times \frac{y^2}{2}\bigg|_0^1 = 4x \times \frac{1}{2} = 2x, \quad 0 \le x \le 1$$

$$f_Y(y) = \int_0^1 f(x,y)dx = \int_0^1 4xy\,dx = 4y \times \frac{x^2}{2}\bigg|_0^1 = 4y \times \frac{1}{2} = 2y, \quad 0 \le y \le 1$$

Solution:

3. Compute $P(X < 0.5, Y > 0.5)$

$$P = \int_0^{0.5} \int_{0.5}^{1} 4xy \, dy \, dx$$

= 0.1875

4. Are X and Y independent?

Check if

$$f(x, y) \stackrel{?}{=} f_X(x) \cdot f_Y(y)$$

Compute:

$$f_X(x) \cdot f_Y(y) = (2x)(2y) = 4xy$$

Which **equals** $f(x, y)$.

Thus, X **and** Y **are independent.**

Solution:

5. **Calculate $E(X)$, $E(Y)$, and $E(XY)$**

- $E(X)$:

$$E(X) = \int_0^1 x f_X(x) dx = \int_0^1 x \cdot 2x dx = 2\int_0^1 x^2 dx = 2 \times \frac{1}{3} = \frac{2}{3} \approx 0.6667$$

- $E(Y)$:

$$E(Y) = \int_0^1 y f_Y(y) dy = \int_0^1 y \cdot 2y dy = 2\int_0^1 y^2 dy = 2 \times \frac{1}{3} = \frac{2}{3} \approx 0.6667$$

Since independent,

$$E(XY) = E(X)E(Y) = \left(\frac{2}{3}\right) \times \left(\frac{2}{3}\right) = \frac{4}{9} \approx 0.4444$$

Example 6:

The joint probability distribution of X and Y is given by $f(x,y) = c(x^2 + y^2)$ for $x = -1, 0, 1, 3.$ and $y = -1, 2, 3.$ (i) Find the value of $c.$ (ii) $P(x = 0, y \le 2)$ (iii) $P(x \le 1, y > 2)$ (iv) $P(x \ge 2 - y)$

Solution:

By data, $X = \{-1, 0, 1, 3\}$ and $Y = \{-1, 2, 3\}$

$f(x,y) = c(x^2 + y^2)$

The joint probability distribution of X and Y:

X \ Y	-1	2	3	$f(X)$
-1	2c	$5c$	$10c$	$17c$
0	c	$4c$	$9c$	$14c$
1	$2c$	$5c$	$10c$	$17c$
3	$10c$	$13c$	$18c$	$41c$
g(Y)	$15c$	$27c$	$47c$	89c

(i) **Find c:** $1 = \Sigma f(x,y) = 89c$

$c = \frac{1}{89}$

(ii) $x = 0, y = \{-1, 2\}$

$P(x = 0, y \le 2)$

$= P(0,-1) + P(0,2)$

$= c + 4c = 5c$

$= 5/89$

(iii) $x = \{-1, 0, 1\},\ y = \{3\}$

$P(x \le 1, y > 2)$

$= P(-1,3) + P(0,3) + P(1,3)$

$= 10c + 9c + 10c$

$= 29c = 29/89$

Cont.

By data, $X = \{-1, 0, 1, 3\}$ and $Y = \{-1, 2, 3\}$

(iv) $P(x \geq 2 - y) = P(x + y \geq 2)$

$$= P(-1,3) + P(0,2) + P(0,3) + P(1,2) +$$

$$P(1,3) + P(3,-1) + P(3,2) + P(3,3)$$

$$= 10c + 4c + 9c + 5c + 10c + 10c + 13c + 18c$$

$$= 79c = 79/89$$

Practice Problems-set 2

- **Problem:1**

The joint probability mass function of (X, Y) is given by $P(x, y) = K(2x + 3y)$, $x = 0,1,2, y = 1,2,3$. Find i) the value of K, ii) The marginal probability function of X and Y, iii) $P(X = 2, Y \leq 2)$, iv) $P(X = 2)$.

Ans : i) $\frac{1}{72}$ ii) $P(X = 0) = \frac{18}{72}$, $P(X = 1) = \frac{24}{72}$, $P(X = 2) = \frac{30}{72}$, $P(Y = 1) = \frac{15}{72}$, $P(Y = 2) = \frac{24}{72}$, $P(Y = 3) = \frac{33}{72}$ ii) $\frac{17}{72}$ iv) $\frac{30}{72}$.

- Show that X and Y are independent variables.

Problem: 2

Let X and Y have the following joint probability distribution

Y \ X	2	4
1	0.10	0.15
3	0.20	0.30
5	0.10	0.15

- Show that X and Y are independent variables.

Problem: 3

For the bivariate probability distribution of (X, Y) given below.

Find i) $P(X \leq 1)$, ii) $P(Y \leq 3)$, iii) $P(X \leq 1, Y \leq 3)$

Y / X	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Ans : i) $\frac{7}{8}$, ii) $\frac{23}{64}$ iii) $\frac{9}{32}$

Problem: 4 If $f(x, y) = e^{-(x+y)}, x \geq 0, y \geq 0$ is the joint probability density function of X and Y , find $P(X < 1)$. (Ans: 0.6321)

Problem: 5

The joint probability density function of X and Y given by

$$f(x,y) = \begin{cases} Kxy & 0 < x < 2, 0 < y < 2 \\ 0 & otherwise \end{cases}.$$

$i)$ Find constant K $ii)$ $P(\frac{1}{2} < X < \frac{3}{2}, 1 < Y < 2)$.

$Ans: i)\frac{1}{4}$ ii) $\frac{3}{8}$

Problem: 6

The joint probability density function of X and Y given by

$$f(x,y) = \begin{cases} x+y & 0 < x < 1, 0 < y < 1 \\ 0 & otherwise \end{cases}.$$

Find marginal probability function of X and Y. Also, check whether X and Y are independent or not.

Ans: Marginal prob function for X: $x + \frac{1}{2}$,

Marginal probability function of Y : $\frac{1}{2} + Y$; Not independent .

Thank You