Lecture 0C

Math Foundations Team

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Section Overview

Continuity

Derivatives

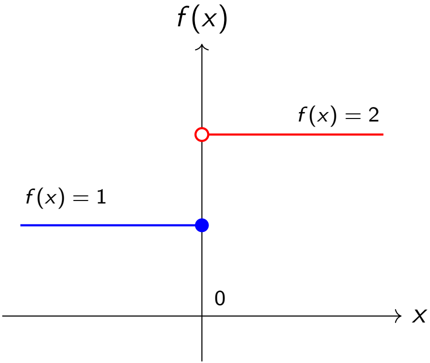
Continuity: Intuition

We explore continuity using two informal examples.

Example 1: Piecewise Function

$$f(x) = \begin{cases} 1, & x \leq 0 \\ 2, & x > 0 \end{cases}$$

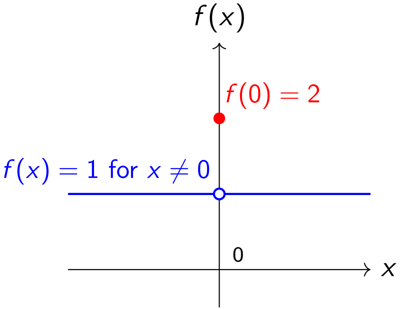
- The function is defined for all real numbers.
- Left-hand limit at $x = 0$ is 1.
- Right-hand limit at $x = 0$ is 2.
- Hence, f is **discontinuous** at $x = 0$.



Example 2: Another Function

$$f(x) = \begin{cases} 1, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

- LHL $=$ RHL $= 1$
- $f(0) = 2 \Rightarrow$ Discontinuity at $x = 0$



Note: Left and right limits at $x = 0$ exist and are equal to 1, but $f(0) = 2$, so the function is **not continuous** at $x = 0$.

Naive Idea of Continuity

- A function is continuous at a point if its graph can be drawn without lifting the pen.
- Discontinuity means a break or jump in the graph.

Definition 1: Continuity at a Point

A function f is **continuous at** $x = c$ if:

$$\lim_{x \to c} f(x) = f(c)$$

That is:

- Left-hand limit exists
- Right-hand limit exists
- Both limits are equal to $f(c)$

Rephrased Definition

Continuity at $x = c$ requires:

- $f(x)$ is defined at $x = c$
- $\lim\limits_{x \to c} f(x)$ exists
- $\lim\limits_{x \to c} f(x) = f(c)$

If any of these fails, f is **discontinuous** at $x = c$.

Summary: What We've Learned

- Concept of continuity using graphs
- Formal definition of continuity at a point
- Examples of discontinuous functions

Section Overview

Continuity

Derivatives

Importance of Derivatives in Real-Life Scenarios

In many real-world situations, it is crucial to understand how one quantity changes with respect to another. Some examples include:

- **Reservoir Management:** To predict when a reservoir will overflow based on water depth measured at different times.
- **Rocket Science:** To compute the precise velocity for launching a satellite, knowing the rocket's height at various times.
- **Financial Analysis:** To forecast changes in stock value by analyzing its current and past values.

Conclusion: These scenarios require understanding how a parameter changes with respect to another.

Definition of Derivative

Definition 1: Suppose f is a real-valued function and a is a point in its domain of definition. The derivative of f at a is defined by:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists.

Notation: The derivative of $f(x)$ at a is denoted by $f'(a)$.

Observation: $f'(a)$ quantifies the rate of change of $f(x)$ at a with respect to x.

Geometric Interpretation of Derivative

Let $y = f(x)$ be a function and let $P = (a, f(a))$ and $Q = (a + h, f(a + h))$ be two points close to each other on the graph of this function shown in Fig1.

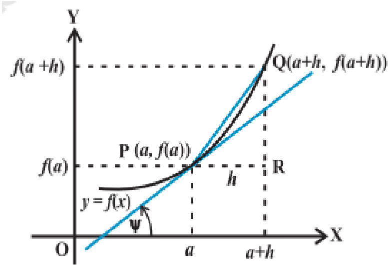


Figure 1: Geometric Interpretation

Limit Definition of Derivative

We know that the derivative of a function f at a point a is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

From triangle $\triangle PQR$, it is clear that the ratio whose limit we are taking is precisely equal to:

$$\tan(\angle QPR)$$

which represents the slope of the chord PQ.

From Chord to Tangent

As $h \to 0$, the point Q moves closer to P, and:

$$\lim_{h \to 0} \frac{QR}{PR} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Thus, the chord PQ tends to the tangent at point P on the curve $y = f(x)$.

From Chord to Tangent

As $h \to 0$, the point Q moves closer to P, and:

$$\lim_{h \to 0} \frac{QR}{PR} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Thus, the chord PQ tends to the tangent at point P on the curve $y = f(x)$.
Therefore, the limit gives us the slope of the tangent line:

$$f'(a) = \tan \psi$$

where ψ is the angle the tangent makes with the x-axis.

Derivative as a Function

Generalization:
For a given function f, if the derivative exists at every point in its domain, it defines a new function:

$$f'(x)$$

This function describes how $f(x)$ changes with respect to x at every point, and is known as the *derivative function*.

Definition 2: First Principle of Derivative

Suppose f is a real-valued function. The function defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists, is called the **derivative of f at x**.

This definition is also referred to as the **first principle of derivative**.

Note: The domain of $f'(x)$ is the set of all x for which the above limit exists.

Example

Problem: Find the derivative at $x = 2$ of the function $f(x) = 3x$.

Solution:

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{3(2+h) - 3(2)}{h} = \lim_{h \to 0} \frac{6 + 3h - 6}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$$

Conclusion: The derivative of the function $f(x) = 3x$ at $x = 2$ is $\boxed{3}$.

Notations for Derivative

- **General notation:**

$$f'(x), \quad \frac{d}{dx}f(x), \quad D(f(x))$$

- **If** $y = f(x)$, **then:**

$$\frac{dy}{dx}$$

- **At a specific point** $x = a$:

$$f'(a), \quad \left.\frac{df}{dx}\right|_{x=a}, \quad \left.\frac{d}{dx}f(x)\right|_{x=a}$$

These notations all represent the **derivative of a function with respect to** x.

Algebra of Derivatives: Linearity Rules

Let $f(x)$ and $g(x)$ be differentiable functions, and c be a constant. Then:

- **Sum Rule:**

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

- **Difference Rule:**

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$$

- **Constant Multiple Rule:**

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$$

Algebra of Derivatives: Product and Quotient Rules

- **Product Rule:**

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)g'(x) + f'(x)g(x)$$

- **Quotient Rule:**

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

These rules help us differentiate combinations of functions efficiently.

Linearity Rules with Examples

Let $f(x)$ and $g(x)$ be differentiable functions.

- **Sum Rule:** $\frac{d}{dx}(x^2 + \sin x) = 2x + \cos x$
- **Difference Rule:** $\frac{d}{dx}(e^x - \ln x) = e^x - \frac{1}{x}$
- **Constant Multiple Rule:** $\frac{d}{dx}(5x^3) = 5 \cdot 3x^2 = 15x^2$

Product and Quotient Rules with Examples

- **Product Rule:** *Example:*

$$\frac{d}{dx}(x^2 \cdot \ln x) = x^2 \cdot \frac{1}{x} + 2x \cdot \ln x = x + 2x \ln x$$

- **Quotient Rule:**

$$\frac{d}{dx}\left(\frac{x^2}{\sin x}\right) = \frac{2x \cdot \sin x - x^2 \cdot \cos x}{\sin^2 x}$$

Product Rule with Examples

Product Rule:

- $f(x) = x^2, \quad g(x) = \ln x$

$$\frac{d}{dx}(x^2 \ln x) = x^2 \cdot \frac{1}{x} + 2x \cdot \ln x = x + 2x \ln x$$

- $f(x) = e^x, \quad g(x) = \cos x$

$$\frac{d}{dx}(e^x \cos x) = e^x(-\sin x) + e^x \cos x = e^x(\cos x - \sin x)$$

- $f(x) = x^3, \quad g(x) = \sin x$

$$\frac{d}{dx}(x^3 \sin x) = x^3 \cos x + 3x^2 \sin x$$

Quotient Rule with Examples

Quotient Rule:

- *Example 1:* $\frac{x^2+1}{\cos x}$

$$\frac{d}{dx} = \frac{2x \cdot \cos x + (x^2+1) \cdot \sin x}{\cos^2 x}$$

- *Example 2:* $\frac{e^x}{x^2}$

$$\frac{d}{dx} = \frac{e^x \cdot x^2 - e^x \cdot 2x}{x^4} = \frac{e^x(x-2)}{x^3}$$

- *Example 3:* $\frac{\ln x}{x}$

$$\frac{d}{dx} = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

Composite Function Derivative: Illustrative Example

Let $f(x) = (2x+1)^3$

Method 1: Expand and Differentiate

$$f(x) = 8x^3 + 12x^2 + 6x + 1$$

$$\frac{df}{dx} = 24x^2 + 24x + 6 = 6(2x+1)^2$$

Method 2: Use Function Composition

$$g(x) = 2x+1, \quad h(t) = t^3, \quad f(x) = h(g(x))$$

$$\frac{df}{dx} = h'(g(x)) \cdot g'(x) = 3(2x+1)^2 \cdot 2 = 6(2x+1)^2$$

Observation: Using composition is more efficient, especially for $(2x+1)^{100}$

The Chain Rule

Chain Rule (General Form): If $f(x) = h(g(x))$, then

$$\frac{df}{dx} = h'(g(x)) \cdot g'(x)$$

Why is it useful?

- Avoids expanding high-degree expressions
- Essential when working with nested functions like $\sin(x^2)$, $\ln(3x + 1)$, etc.

Example: For $f(x) = (5x - 4)^{10}$, using the Chain Rule:

$$f'(x) = 10(5x - 4)^9 \cdot 5 = 50(5x - 4)^9$$

Chain Rule for Composite Functions (Two Functions)

Let $f(x)$ be a real-valued function which is a composite of two functions u and v, i.e., $f = v \circ u$.
Suppose $t = u(x)$, and both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist. Then, the Chain Rule states:

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

Example: For $f(x) = \sin(2x+1)$, let:

$$u(x) = 2x + 1, \quad v(t) = \sin(t)$$

Thus,

$$\frac{df}{dx} = \cos(2x+1) \cdot 2$$

Chain Rule for Composite Functions (Three Functions)

The Chain Rule can be extended to composite functions of three functions.
Suppose $f(x)$ is a composite of three functions u, v, and w, i.e., $f = (w \circ u) \circ v$.
Let $t = v(x)$ and $s = u(t)$, then:

$$\frac{df}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Example: For $f(x) = \sin(3x+2)^2$, we have:

$$t = 3x + 2, \quad s = t^2, \quad w(s) = \sin(s)$$

$$Thus, \frac{df}{dx} = 2 \cdot (3x+2) \cdot \cos((3x+2)^2) \cdot 3$$

Generalization and Extension of Chain Rule

The Chain Rule can be generalized for composite functions of more than three functions.
If $f(x) = w(u(v(x)))$, and you continue composing more functions, the derivative is:

$$\frac{df}{dx} = \frac{dw}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Example Extension: For a composite of four functions:

$$f(x) = w(u(v(t(x))))$$

The derivative is computed by applying the Chain Rule iteratively for each composition.
Practice: Try to compute the derivative for functions such as $\cos(2x^3 + 3x)$ or $\ln(5x^2 + 2x)$ using the Chain Rule.

Logarithmic Differentiation: Introduction

Use case: When both the base and the exponent are functions of x, i.e.,

$$y = [u(x)]^{v(x)}$$

Steps:

1. Take natural logarithm of both sides:

$$\log y = v(x) \cdot \log u(x)$$

2. Differentiate both sides using Chain Rule and Product Rule.
3. Multiply both sides by y to isolate $\frac{dy}{dx}$.

Note: This method only applies where $u(x) > 0$.

Logarithmic Properties (Recap)

Useful properties of logarithms:

- **Product Rule:** $\log(ab) = \log a + \log b$
- **Quotient Rule:** $\log\left(\frac{a}{b}\right) = \log a - \log b$
- **Power Rule:** $\log(a^n) = n \log a$
- **Root Rule:** $\log\left(\sqrt[n]{a}\right) = \frac{1}{n} \log a$

Note: These are valid only when the arguments are positive real numbers.

General Formula from Logarithmic Differentiation

Given:

$$y = [u(x)]^{v(x)}$$

Taking logs:

$$\log y = v(x) \cdot \log[u(x)]$$

Differentiating:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{v(x)u'(x)}{u(x)} + v'(x) \cdot \log[u(x)]$$

Multiplying both sides by y:

$$\frac{dy}{dx} = [u(x)]^{v(x)} \left(\frac{v(x)u'(x)}{u(x)} + v'(x) \cdot \log[u(x)] \right)$$

Illustrative Example: Setup

Differentiate:

$$y = \sqrt{\frac{(x-3)^2(x^2+4)^2}{3x^2+4x+5}}$$

Take logarithm on both sides:

$$\log y = \frac{1}{2} \cdot \log\left(\frac{(x-3)^2(x^2+4)^2}{3x^2+4x+5}\right)$$

Using log properties:

$$\log y = \frac{1}{2}\left[\log(x-3)^2 + \log(x^2+4)^2 - \log(3x^2+4x+5)\right]$$

Continue Example: Simplify Log Expression

Simplify the expression:

$$\log y = \log(x-3) + \log(x^2+4) - \frac{1}{2}\log(3x^2+4x+5)$$

Differentiate both sides:

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{2(3x^2+4x+5)}$$

Now multiply by y:

$$\frac{dy}{dx} = y\cdot\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{2(3x^2+4x+5)}\right]$$

Final Answer and Simplified Form

Recall:

$$y = \sqrt{\frac{(x-3)^2(x^2+4)^2}{3x^2+4x+5}}$$

So the final derivative is:

$$\frac{dy}{dx} = \sqrt{\frac{(x-3)^2(x^2+4)^2}{3x^2+4x+5}} \cdot \left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{2(3x^2+4x+5)}\right]$$

Note: You can leave it like this or simplify further algebraically if required.

Practice Problems for Students

Differentiate the following using logarithmic differentiation:

1. $y = (x^2 + 1)^{\tan x}$
2. $y = \sqrt{\frac{(x+1)^3(x^2-1)^2}{x^4+2}}$
3. $y = \left(\frac{\sin x}{x^2+3}\right)^x$
4. $y = \sqrt{\sin(x^2 + 1)}$

Hints:

- Use log properties wisely.
- Don't forget Chain Rule.
- Keep track of function domains.

Thank You!

Questions or Queries?

Feel free to ask. I am happy to help!

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Lecture (0)C

Math Foundations Team

27th, April, 2025

Section Overview

Maxima and Minima

Linear Programming

Introduction: Maxima and Minima

We use derivatives to determine **maximum** or **minimum** values of functions by identifying the *turning points* of their graphs. These points are crucial in understanding the function's behavior and solving real-life problems.

Applications:

- The profit from a grove is given by $P(x) = ax + bx^2$. How many trees per acre maximize the profit?
- A ball is thrown from a 60 m high building. Its path is $h(x) = -x^2 + 60x + 60$. What is the **maximum height** reached?

Introduction: Maxima and Minima

- A helicopter follows the path $f(x) = x^2 + 7$. A soldier at (1, 2) wants to know the **nearest distance** to the helicopter.

To solve such problems, we first define:

- Local and absolute maxima/minima
- Turning points using derivatives
- Tests to identify such points

Definition: Maxima and Minima

Let f be a function defined on an interval I. Then:

(a) **Maximum Value:** f has a *maximum value* in I if there exists a point $c \in I$ such that

$$f(c) > f(x), \quad \text{for all } x \in I.$$

The number $f(c)$ is called the maximum value of f in I and c is called a point of maximum value.

(b) **Minimum Value:** f has a *minimum value* in I if there exists a point $c \in I$ such that

$$f(c) < f(x), \quad \text{for all } x \in I.$$

The number $f(c)$ is called the minimum value of f in I and c is called a point of minimum value.

(c) **Extreme Value:** f has an *extreme value* in I if $f(c)$ is either a maximum or a minimum value. The number $f(c)$ is then called an extreme value, and c is called an extreme point.

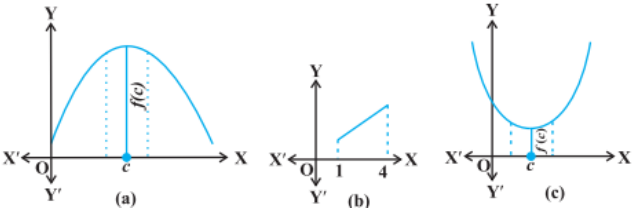


Figure 1: Graphs of Functions

Remark: Maxima and Minima through Graphs

- In Fig 1 (a), (b), and (c), the graphs of certain functions illustrate how maximum and minimum values can be identified visually.
- Graphs offer an intuitive way to understand the behavior of functions near a given point.
- They help us locate:
  - Local maxima
  - Local minima
  - Points of extreme values
- **Important:** Maxima and minima can sometimes be identified graphically even at points where the function is *not differentiable* (see Figure 2).

Non-Differentiable function

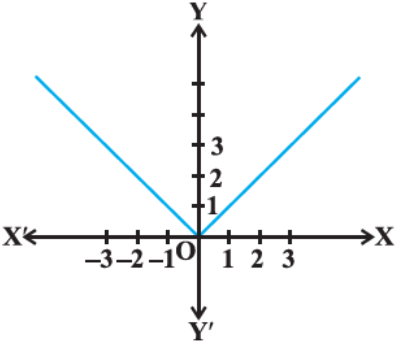


Figure 2: $f(x) = |x|$

Turning Points of a Function

- Let us examine the graph of a function as shown in Fig 3.
- Observe that at points A, B, C, and D, the function changes its nature from:
  - Decreasing to increasing or,
  - Increasing to decreasing.
- These points are called **turning points** of the function.

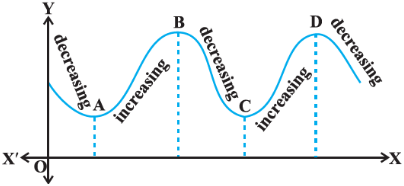


Figure 3: Turning Points of a Function

Nature of Turning Points

- At turning points, the graph has either:
  - A little hill, or
  - A little valley.
- **Points A and C** are at the bottom of valleys:
  - Function attains minimum value in their neighborhood.
- **Points B and D** are at the top of hills:
  - Function attains maximum value in their neighborhood.

Local Maxima and Minima

- Points A and C: **Local Minima** (or Relative Minima)
- Points B and D: **Local Maxima** (or Relative Maxima)
- These are the values the function attains in a small interval around each point.
- Such local behavior helps us understand how the function behaves in parts of its domain.

Definition: Local Maxima and Minima

Definition Let f be a real-valued function and let c be an interior point in the domain of f. Then:

(a) c is called a **point of local maxima** if there exists $h > 0$ such that

$$f(c) \geq f(x), \quad \text{for all } x \in (c-h, c+h),\ x \neq c$$

The value $f(c)$ is called the **local maximum value** of f.

(b) c is called a **point of local minima** if there exists $h > 0$ such that

$$f(c) \leq f(x), \quad \text{for all } x \in (c-h, c+h)$$

The value $f(c)$ is called the **local minimum value** of f.

Geometrical Meaning of Local Extrema

- Geometrically, if $x = c$ is a point of **local maxima**, then:
  - $f'(x) > 0$ in $(c - h, c)$ (function is increasing)
  - $f'(x) < 0$ in $(c, c + h)$ (function is decreasing)
- This behavior suggests that the slope changes sign at $x = c$.

$$\Rightarrow f'(c) = 0$$

(Refer to Fig 4 for visual representation)

Geometrical Meaning of Local Extrema

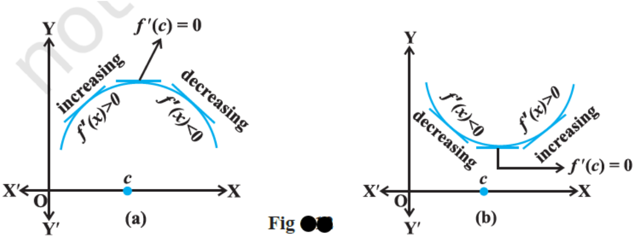


Figure 4: Geometrical Interpretation

First Derivative Test (Part 1)

Let f be a function defined on an open interval I, and let f be continuous at a critical point $c \in I$. Then:

(i) If $f'(x)$ changes sign from **positive to negative** as x increases through c, i.e.,

$$f'(x) > 0 \text{ for } x < c, \quad f'(x) < 0 \text{ for } x > c,$$

then c is a point of **local maxima**.

(ii) If $f'(x)$ changes sign from **negative to positive** as x increases through c, i.e.,

$$f'(x) < 0 \text{ for } x < c, \quad f'(x) > 0 \text{ for } x > c,$$

then c is a point of **local minima**.

First Derivative Test (Part 2)

(iii) If $f'(x)$ does **not change sign** as x increases through c, then

- c is neither a point of local maxima nor local minima.
- Such a point is called a **point of inflection**.

(Refer to Fig 5 for graphical illustration)

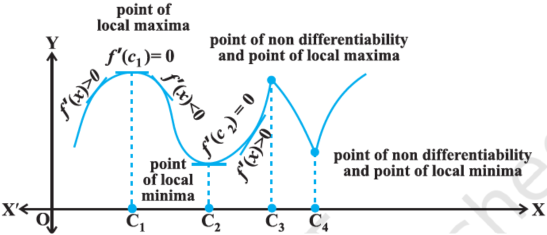


Figure 5: Derivative Test

Second Derivative Test

Let f be a function defined on an interval I and $c \in I$. Suppose f is twice differentiable at c. Then:

(i) $x = c$ is a point of **local maxima** if

$$f'(c) = 0 \quad \text{and} \quad f''(c) < 0$$

Then $f(c)$ is the **local maximum value** of f.

(ii) $x = c$ is a point of **local minima** if

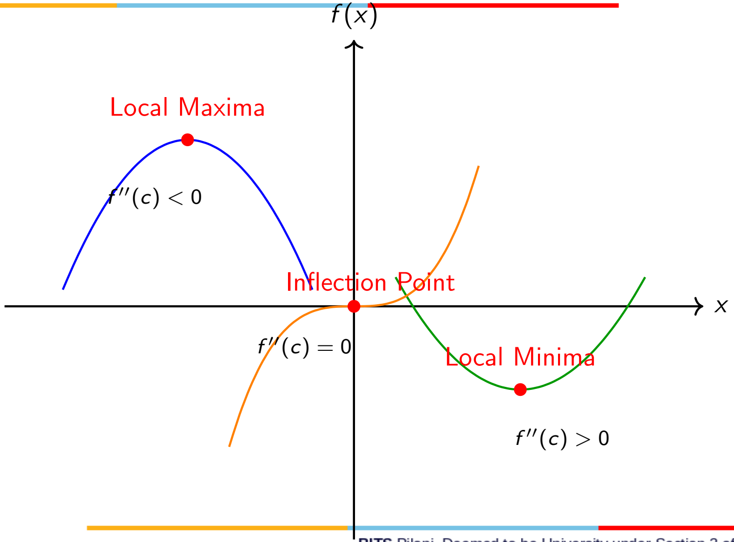
$$f'(c) = 0 \quad \text{and} \quad f''(c) > 0$$

Then $f(c)$ is the **local minimum value** of f.

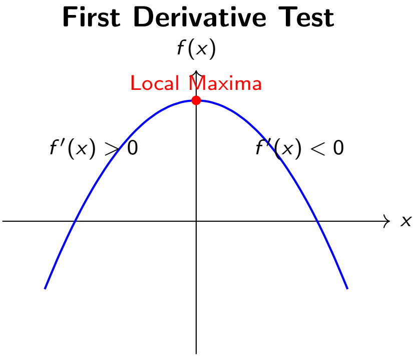
Second Derivative Test

(iii) If $f'(c) = 0$ and $f''(c) = 0$, the test **fails**.
In this case, apply the **First Derivative Test** to determine the nature of c.

Graphical Illustration: Second Derivative Test



Visual Comparison: First and Second Derivative Tests



First Derivative Test checks sign change in $f'(x)$; Second Derivative Test checks curvature via $f''(x)$.

Problem Statement

Find the local maxima and minima of the function:

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

Step 1: First Derivative

$$f'(x) = \frac{d}{dx}(3x^4 + 4x^3 - 12x^2 + 12) = 12x^3 + 12x^2 - 24x$$

$$f'(x) = 12x(x-1)(x+2)$$

Critical Points: $x = 0, 1, -2$

Step 2: Second Derivative

Differentiate again:

$$f''(x) = \frac{d}{dx}(12x^3 + 12x^2 - 24x) = 36x^2 + 24x - 24$$

$$f''(x) = 12(3x^2 + 2x - 2)$$

Evaluate at critical points:

$$f''(0) = 12(0 - 2) = -24 \Rightarrow \text{Local Maxima}$$
$$f''(1) = 12(3 + 2 - 2) = 36 \Rightarrow \text{Local Minima}$$
$$f''(-2) = 12(12 - 4 - 2) = 72 \Rightarrow \text{Local Minima}$$

Step 3: Evaluate Values at Critical Points

$$f(0) = 3(0)^4 + 4(0)^3 - 12(0)^2 + 12 = \boxed{12}$$

$$f(1) = 3(1)^4 + 4(1)^3 - 12(1)^2 + 12 = 3 + 4 - 12 + 12 = \boxed{7}$$

$$f(-2) = 3(-2)^4 + 4(-2)^3 - 12(-2)^2 + 12 = 48 - 32 - 48 + 12 = \boxed{-20}$$

Conclusion

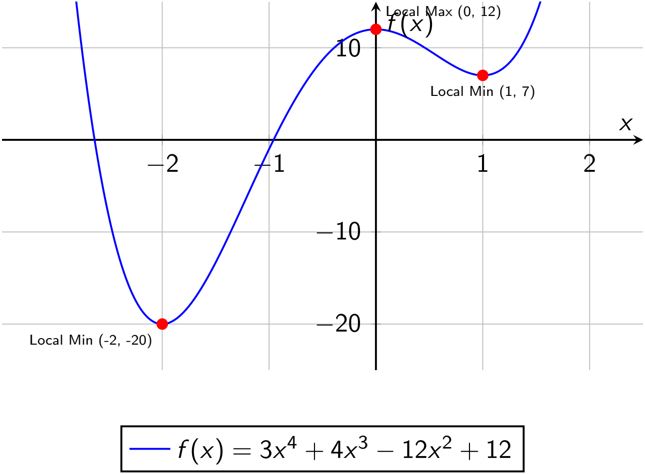
- **Local Maximum:**
  - At $x = 0$, value $f(0) = 12$
- **Local Minima:**
  - At $x = 1$, value $f(1) = 7$
  - At $x = -2$, value $f(-2) = -20$
- **Method Used:** Second Derivative Test

Conclusion

- **Local Maximum:**
  - At $x = 0$, value $f(0) = 12$
- **Local Minima:**
  - At $x = 1$, value $f(1) = 7$
  - At $x = -2$, value $f(-2) = -20$
- **Method Used:** Second Derivative Test

The graph visualizing this function and its turning points.

Graph of the Function



Problem Statement

Find all the points of local maxima and local minima of the function:

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

Step 1: First Derivative We begin by finding the first derivative of the function:

$$f'(x) = \frac{d}{dx}(2x^3 - 6x^2 + 6x + 5) = 6x^2 - 12x + 6$$

Step 2: Find Critical Points Set $f'(x) = 0$ to find critical points:

$$6x^2 - 12x + 6 = 0$$

$$x^2 - 2x + 1 = 0 \quad \Rightarrow \quad (x-1)^2 = 0$$

Thus, $x = 1$ is a critical point.

Step 3: Second Derivative

Now, we find the second derivative:

$$f''(x) = \frac{d}{dx}(6x^2 - 12x + 6) = 12x - 12$$

Evaluating at $x = 1$:

$$f''(1) = 12(1) - 12 = 0$$

Since $f''(1) = 0$, the second derivative test fails.

Step 4: First Derivative Test

To determine the nature of the critical point, use the first derivative test. We examine the sign of $f'(x)$:
For $x < 1$, choose $x = 0$:

$$f'(0) = 6(0)^2 - 12(0) + 6 = 6 > 0$$

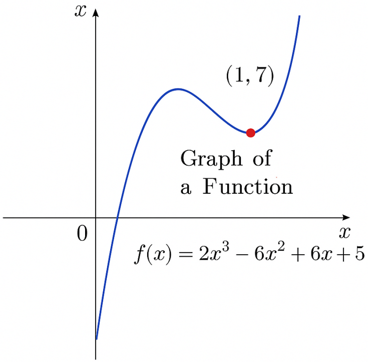
So, the function is increasing for $x < 1$.
For $x > 1$, choose $x = 2$:

$$f'(2) = 6(2)^2 - 12(2) + 6 = 6(4) - 24 + 6 = 24 - 24 + 6 = 6 > 0$$

So, the function is still increasing for $x > 1$.
Since $f'(x)$ does not change sign, $x = 1$ is a point of **inflection**.
The function does not have local maxima or minima, as $x = 1$ is a **point of inflection**.

Graph of $f(x) = 2x^3 - 6x^2 + 6x + 5$

Graph of a Function



Section Overview

Maxima and Minima

Linear Programming

What is Linear Optimization?

A furniture dealer deals in:

- **Tables** (Cost: Rs 2500, Profit: Rs 250)
- **Chairs** (Cost: Rs 500, Profit: Rs 75)

Resources:

- Investment: Rs 50000
- Storage space: 60 items

Let:

- x: Number of tables
- y: Number of chairs

Objective: Maximise $Z = 250x + 75y$

Optimisation Problems:

- Problems that aim to **maximise** or **minimise** a quantity such as profit, cost, or use of resources.
- These are commonly found in real-life situations involving economics, business, and operations.
- Example goals include:
  - Maximising profit
  - Minimising cost
  - Minimising the use of limited resources

Linear Programming:

- A special and very important class of optimisation problems.
- All relationships are linear in nature — objective function and constraints.
- The furniture dealer problem is a classic example of a linear programming problem.

Mathematical Formulation

Let x be the number of tables and y the number of chairs the dealer buys.

Constraints:

- $x \geq 0, \;\; y \geq 0$ (Non-negativity)
- $5x + y \leq 100$ (Investment constraint)
- $x + y \leq 60$ (Storage constraint)

Optimization Problem:

$$\text{Maximise} \quad Z = 250x + 75y \quad \text{subject to:}$$

$$5x + y \leq 100$$
$$x + y \leq 60$$
$$x, y \geq 0$$

Components of a Linear Programming Problem

- Decision Variables
- Objective Function
- Constraints
- Non-negativity Restrictions

Formulation and Objective

Constraints:

$$
\begin{aligned}
2500x + 500y &\le 50000 \qquad && \text{(Investment)} \\
x + y &\le 60 && \text{(Storage)} \\
x \ge 0,\ \ y &\ge 0 && \text{(Non-negativity)}
\end{aligned}
$$

Objective: Maximise Profit

$$Z = 250x + 75y$$

This is a classic **Linear Programming Problem (LPP)** used for **optimisation**.

General Form of a LPP

Maximize (or Minimize)

$Z = ax + by$

Subject to constraints:

$$c_1x + d_1y \leq e_1$$
$$c_2x + d_2y \leq e_2$$
$$x, y \geq 0$$

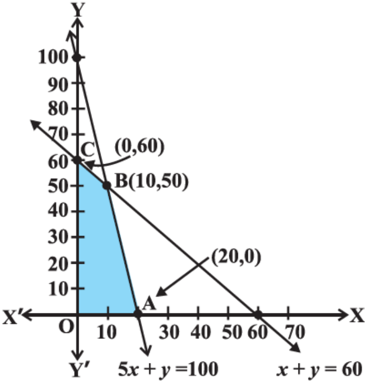
Graphical method of solving LPP

- The graph of the system of inequalities defines a **shaded region** representing points common to all half-planes.
- Each point in this region is a **feasible choice** for investing in tables and chairs.
- This region is called the **feasible region**.
- Every point in the feasible region is a **feasible solution**.
- **Feasible Region:** The common region determined by all the constraints (including $x \geq 0$, $y \geq 0$) of an LPP.
- In the graph, the region OABC (shaded) represents the feasible region.
- The area outside the feasible region is called the **infeasible region**.

Graph of System of Equations

Feasible Solutions

- Points **within and on the boundary** of the feasible region represent feasible solutions.
- In Figure, points like $(10, 50)$, $(0, 60)$, and $(20, 0)$ are feasible solutions.
- Any point **outside** the feasible region is an **infeasible solution**, e.g., $(25, 40)$.



Optimal Solutions

Optimal (Feasible) Solution

- A point in the feasible region that **optimizes (maximizes or minimizes)** the objective function.
- The region OABC satisfies all constraints (1) to (4), but contains **infinitely many points**.
- It's not obvious which point gives the **maximum value** of $Z = 250x + 75y$.
- We rely on certain **fundamental theorems** to identify the optimal point (proofs omitted).

Corner Point Method

Steps:

1. **Determine the feasible region** of the linear programming problem and identify its **corner points (vertices)**, either by inspection or by solving equations of intersecting lines.
2. **Evaluate** the objective function $Z = ax + by$ at each corner point. Let M and m denote the maximum and minimum of these values.
3. **If the feasible region is bounded:**
   - M is the maximum value of Z.
   - m is the minimum value of Z.
4. **If the feasible region is unbounded:**
   - M is the maximum value of Z, only if the half-plane $ax + by > M$ has no point in common with the feasible region.
   - m is the minimum value of Z, only if the half-plane $ax + by < m$ has no point in common with the feasible region.

Applications of Linear Programming

- Resource allocation
- Diet problems
- Transportation and logistics
- Manufacturing and production

Summary

- Linear programming helps optimize outcomes under constraints.
- The graphic method is effective for 2-variable problems.
- Solutions occur at the vertices of the feasible region.

Thank You!

Questions or Queries?

Feel free to ask. I'm happy to help!