



Lecture 0a

Math Foundations Team



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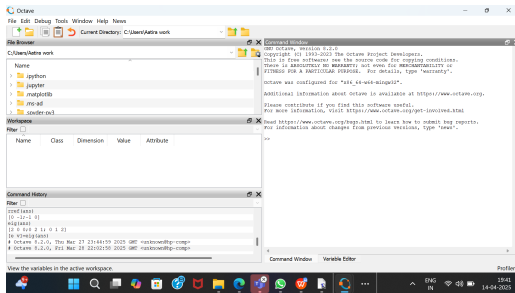


- ▶ Octave is very similar to MATLAB — often used as a free alternative.
- ▶ Ideal for linear algebra, calculus, differential equations, and data visualization.
- ▶ The syntax for octave is MATLAB compatible.
- ▶ Download from: <https://www.gnu.org/software/octave/>

Getting Started with Octave



To start, when we double click on the icon, what we get is



One can start typing commands in command prompt or script editor.

Example



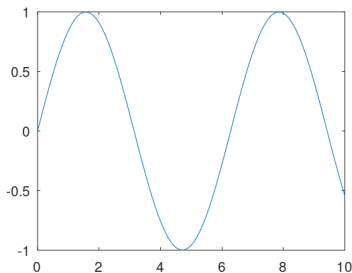
Let us type this in the command prompt.

```
x = 0:0.1:10;
```

```
y = sin(x);
```

```
plot(x, y);
```

Then, we get



- A **matrix** is a rectangular array of numbers or functions which we will enclose in brackets. For example,

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad (1)$$
$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad [a_1 \quad a_2 \quad a_3], \quad \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

- The numbers (or functions) are called **entries** or, less commonly, *elements* of the matrix.
- The first matrix in (1) has two **rows**, which are the horizontal lines of entries.

- We shall denote matrices by capital boldface letters **A**, **B**, **C**, ... , or by writing the general entry in brackets; thus **A** = $[a_{jk}]$, and so on.
- By an $m \times n$ **matrix** (read *m by n matrix*) we mean a **matrix with m rows and n columns**—rows always come first! $m \times n$ is called the **size** of the matrix. Thus an $m \times n$ matrix is of the form

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \quad (2)$$

Octave: $A=[1 \ 2 \ 3; 4 \ 5 \ 6]$



- A **vector** is a **matrix with only one row or column**. Its entries are called the **components** of the vector.
- We shall denote vectors by *lowercase* boldface letters **a**, **b**, ... or by its general component in brackets, $\mathbf{a} = [a_j]$, and so on. Our special vectors in (1) suggest that a (general) **row vector** is of the form

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}. \quad \text{For instance,} \quad \mathbf{a} = \begin{bmatrix} -2 & 5 & 0.8 & 0 & 1 \end{bmatrix}.$$

A **column vector**

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \quad \text{For instance,} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$



- Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**, written $\mathbf{A} = \mathbf{B}$, if and only if (1) they have the same size and (2) the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on.
- Matrices that are not equal are called **different**. Thus, matrices of different sizes are always different.



1. Addition of Matrices

- The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ *of the same size* is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of \mathbf{A} and \mathbf{B} . Matrices of different sizes cannot be added.

2. Scalar Multiplication (Multiplication by a Number)

- The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of \mathbf{A} by c .

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(a) $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (written $\mathbf{A} + \mathbf{B} + \mathbf{C}$)

(b) $(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$

(c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$

(c) $c(k\mathbf{A}) = (ck)\mathbf{A}$ (written $ck\mathbf{A}$)

(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

(d) $1\mathbf{A} = \mathbf{A}$.

- Here $\mathbf{0}$ denotes the **zero matrix** (of size $m \times n$), that is, the $m \times n$ matrix with all entries zero.

Multiplication of a Matrix by a Matrix

• The **product** $\mathbf{C} = \mathbf{AB}$ (in this order) of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ times an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is **defined if and only if $r = n$** and is then the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

$$(3) \quad c_{jk} = \sum_{l=1}^n a_{jl} b_{lk} = a_{j1} b_{1k} + a_{j2} b_{2k} + \cdots + a_{jn} b_{nk} \quad \begin{matrix} j = 1, \dots, m \\ k = 1, \dots, p. \end{matrix}$$

- The condition $r = n$ means that the second factor, \mathbf{B} , must have as many rows as the first factor has columns, namely n . A diagram of sizes that shows when matrix multiplication is possible is as follows:

$$\begin{matrix} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times n] & [n \times p] & = & [m \times p]. \end{matrix}$$

EXAMPLE 1

$$AB = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

• Here $c_{11} = 3 \cdot 2 + 5 \cdot 5 + (-1) \cdot 9 = 22$, and so on. The entry in the box is $c_{23} = 4 \cdot 3 + 0 \cdot 7 + 2 \cdot 1 = 14$.

• The product **BA** is not defined.

Matrix Multiplication Is *Not Commutative*, $AB \neq BA$ in General

- This is illustrated by Example 1, where one of the two products is not even defined. But it **also holds for square matrices**. For instance,

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

- It is interesting that this also shows that **$AB = 0$ does *not* necessarily imply $BA = 0$ or $A = 0$ or $B = 0$.**

- The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix \mathbf{A}^T (read *A transpose*) that has the first *row* of \mathbf{A} as its first *column*, the second *row* of \mathbf{A} as its second *column*, and so on. Thus the transpose of \mathbf{A} in (2) is $\mathbf{A}^T = [a_{kj}]$, written out

$$\mathbf{A}^T = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

- As a special case, **transposition converts row vectors to column vectors and conversely.**



- Rules for transposition are

$$\begin{aligned} \text{(a)} \quad & (\mathbf{A}^\top)^\top = \mathbf{A} \\ \text{(b)} \quad & (\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top \\ \text{(c)} \quad & (c\mathbf{A})^\top = c\mathbf{A}^\top \\ \text{(d)} \quad & (\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top. \end{aligned} \tag{5}$$

CAUTION! Note that in (5d) the transposed matrices are *in reversed order*.

```
A=[1 2 3; 4 5 6];  
B=[2 4 0; 6 7 8];  
C=[1 2; 3 4; 4 1];  
disp('The sum of 2 matrices is')  
A+B  
disp('4A is given by')  
4*A  
disp('The product AB')  
A*B  
disp('The product BA')  
B*A  
disp('The transpose of A' )  
A'
```

Special Matrices



- **Symmetric**: $a_{ij} = a_{ji}$ Eg:
$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

- **Skew Symmetric**: $a_{ij} = -a_{ji}$ Eg:
$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

Upper triangular matrix: U

$$\begin{bmatrix} 1 & 1/2 & 3 & 0 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

- **Triangular**: Upper Triangular $\rightarrow a_{ij} = 0$ for all $i > j$
Lower Triangular $\rightarrow a_{ij} = 0$ for all $i < j$

Lower triangular matrix: L

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

- **Diagonal Matrix**: $a_{ij} = 0$ for all $i \neq j$ Eg:
$$\begin{bmatrix} \text{pink square} & 0 & \dots & 0 \\ 0 & \text{blue square} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \text{green square} \end{bmatrix}$$
- **Sparse Matrix**: Many zeroes and few non-zero entities

$$\begin{bmatrix} 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 5 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 & 0 \end{bmatrix}$$



- Let **A** be a **real symmetric matrix**. Then **A** is positive definite if for any $x \neq 0$,

$$x^T A x > 0$$

- Example:

- $$A = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \quad x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= 2x_1^2 + 12x_1x_2 + 20x_2^2 = 2(x_1 + 3x_2)^2 + 2x_2^2 > 0$$

- A** is **positive semi-definite** if $x^T A x \geq 0$

- $$A = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \Rightarrow x^T A x = 2(x_1 + 3x_2)^2 = 0 \text{ when } x_1 = 3 \text{ and } x_2 = -1$$

Given a matrix A , the following operations are called Elementary Row Operations .

- *Interchange of two rows*
- *Addition of a constant multiple of one row to another row*
- *Multiplication of a row by a **non-zero** constant c*

CAUTION! These operations are for rows, *not for columns!*

Row Echelon Form (REF)



- ▶ Any rows of all zeros are below any other non zero rows.
- ▶ The first non zero entry in each row is called as the leading entry of that row.
- ▶ Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- ▶ All entries in a column below a leading entry are zeros.

Example

$$\begin{bmatrix} 3 & 2 & 0 & 7 & 9 \\ 0 & 4 & 5 & 10 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



- We say that a matrix is in Reduced Row Echelon Form if it is in Echelon form and additionally,
 1. The leading entry in each row is 1.
 2. Each leading 1 is the only non zero entry in its column

• Example

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 9 \\ 0 & 1 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



- We can transform any matrix into a matrix in reduced row echelon form by using elementary row operations.
- No matter what sequence of row operations we use each matrix is row equivalent to one and only one reduced row echelon matrix

$$\begin{bmatrix} 0 & 1 & 2 & 0 & -1 \\ -2 & 2 & 0 & 3 & 2 \\ 2 & 8 & 20 & 0 & 6 \end{bmatrix} \xrightarrow{\text{Swap rows 1 and 2}} \begin{bmatrix} -2 & 2 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 2 & 8 & 20 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 3 & 18 \end{bmatrix} \xleftarrow{\text{Replace R3 by } R3+(-10).R2} \begin{bmatrix} -2 & 2 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 10 & 20 & 3 & 8 \end{bmatrix} \xrightarrow{\text{Replace R3 by } R3+1.R1} \begin{bmatrix} -2 & 2 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 3 & 18 \end{bmatrix}$$

Row Echelon Form



- ▶ The number of non zero rows, r in the reduced row or row echelon form \mathbf{R} of matrix is called the rank of \mathbf{R} and also the rank of \mathbf{A} .
- ▶ The rank is invariant under elementary row operations.
- ▶ In the last example, number of nonzero rows is 3 and hence rank is 3.

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \quad (\text{given})$$

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \quad \begin{array}{l} \text{Row 2} + 2 \text{ Row 1} \\ \text{Row 3} - 7 \text{ Row 1} \end{array}$$

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Row 3} + \frac{1}{2} \text{ Row 2.}$$

- The last matrix is in row-echelon form and has two nonzero rows.
Hence rank $\mathbf{A} = 2$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in **A** has a minor.

Delete first row and column from **A**. The determinant of the remaining 2x2 submatrix is the minor of a_{11} which is $a_{22}a_{33} - a_{23}a_{32}$

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

And the minor for a_{13} is:

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$



The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number i and column j is even, $c_{ij} = m_{ij}$ and when $i+j$ is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1, j=1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i=1, j=2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i=1, j=3) = (-1)^{1+3} m_{13} = +m_{13}$$



The determinant of an $n \times n$ matrix \mathbf{A} can now be defined as

$$|\mathbf{A}| = \det \mathbf{A} = a_{11}c_{11} + a_{12}c_{12} + \dots + a_{1n}c_{1n}$$

The determinant of \mathbf{A} is therefore the sum of the products of the elements of the first row of \mathbf{A} and their corresponding cofactors.

(It is possible to define $|\mathbf{A}|$ in terms of any other row or column but for simplicity, the first row only is used)

Therefore the 2 x 2 matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Example 1:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Has cofactors :

$$c_{11} = m_{11} = |a_{22}| = a_{22}$$

$$|A| = (3)(2) - (1)(1) = 5$$

And:

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

And the determinant of **A** is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactors of the first row are:

$$c_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$c_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$$

$$c_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Which by substituting for the cofactors in this case is:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example 2:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$|A| = (1)(2 - 0) - (0)(0 + 3) + (1)(0 + 2) = 4$$

Properties of Determinants



1. $\det(AB) = \det(A) * \det(B)$
2. $\det(A)$ nonzero implies there exists a matrix B such that $AB=BA=I$
3. Two Rows Equal $\rightarrow \det = 0$ (Singular)
4. R_i and R_j swapped $\rightarrow \det$ gets a minus sign ($i \neq j$)
5. $\det(A) = \det(A^T)$
6. $R_i \leftarrow cR_j \rightarrow \det A \leftarrow c \det A$



The adjoint matrix of \mathbf{A} , denoted by $\text{adj } \mathbf{A}$, is the transpose of its cofactor matrix

$$\text{adj} A = C^T$$

It can be shown that:

$$\mathbf{A}(\text{adj } \mathbf{A}) = (\text{adj } \mathbf{A}) \mathbf{A} = |\mathbf{A}| \mathbf{I}$$

Example: $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

$$|A| = (1)(4) - (2)(-3) = 10$$

$$\text{adj} A = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$



$$A(adj A) = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

$$(adj A)A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

- $A^{-1} = \frac{\text{adj}(A)}{\det(A)}$ where $\det(A) \neq 0$

Reiterate $\det(A) \neq 0 \rightarrow A$ is Non singular

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$
$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

The determinant of **A** is

$$|\mathbf{A}| = (3)(-1-0) - (-1)(-2-0) + (1)(4-1) = -2$$

The elements of the cofactor matrix are

$$\begin{array}{lll} c_{11} = +(-1), & c_{12} = -(-2), & c_{13} = +(3), \\ c_{21} = -(-1), & c_{22} = +(-4), & c_{23} = -(7), \\ c_{31} = +(-1), & c_{32} = -(-2), & c_{33} = +(5), \end{array}$$

The cofactor matrix is therefore

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

so

$$\text{adj}A = C^T = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

and

$$A^{-1} = \frac{\text{adj}A}{|A|} = \frac{1}{-2} \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$



The result can be checked using

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The determinant of a matrix must not be zero for the inverse to exist as there will not be a solution

Nonsingular matrices have non-zero determinants

Singular matrices have zero determinants

So that for a 2 x 2 matrix the inverse can be constructed in a simple fashion as

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Exchange elements of main diagonal
- Change sign in elements off main diagonal
- Divide resulting matrix by the determinant



- ▶ `A = zeros(m,n)` creates a zero matrix `A` of size $m \times n$
- ▶ `A = eye(k)` creates an identity matrix `A` of size k
- ▶ `[r, c] = size(A)` gives the size of `A` and the number of rows is stored in `r` and number of columns in `c`
- ▶ `rref(A)` gives the reduced row echelon form of `A`
- ▶ `rank(A)` gives the rank of `A`
- ▶ `det(A)` gives the determinant of `A`
- ▶ `inv(A)` gives the inverse of `A`
- ▶ `issymmetric(A)` returns either 0 (not symmetric) or 1 (symmetric)

Matrix Form of the Linear System (1). (continued)

- We assume that the coefficients a_{jk} are not all zero, so that \mathbf{A} is not a zero matrix. Note that \mathbf{x} has n components, whereas \mathbf{b} has m components. The matrix

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array} \right]$$

- is called the **augmented matrix** of the system (1).
- The dashed vertical line could be omitted, as we shall do later. It is merely a reminder that the last column of $\tilde{\mathbf{A}}$ did not come from matrix \mathbf{A} but came from vector \mathbf{b} . Thus, we *augmented* the matrix \mathbf{A} .

Consider the following system of linear equations

$$x_1 + x_2 + x_3 = 3$$

$$x_1 - x_2 + 2x_3 = 2$$

$$2x_1 + 3x_3 = 1$$

Adding the first and second equations gives $2x_1 + 3x_3 = 5$ which contradicts the third equation. Thus there is no set of values for the variables x_1, x_2, x_3 such that the equations above are simultaneously satisfied.



Consider a slightly modified example

$$x_1 + x_2 + x_3 = 3$$

$$x_1 - x_2 + 2x_3 = 2$$

$$x_2 + x_3 = 2$$

In this case we can see from the first and third equations that $x_1 = 1$. Substituting this value of x_1 into equation (2), we get $-x_2 + 2x_3 = 1$. Adding this equation to equation (3), we get $3x_3 = 3$ which means $x_3 = 1$. Substituting $x_3 = 1$ into equation (3) shows $x_2 = 1$, so the overall solution is $x_1 = x_2 = x_3 = 1$. This is the unique solution to the problem



Now consider another modification to the original set of equations

$$x_1 + x_2 + x_3 = 3$$

$$x_1 - x_2 + 2x_3 = 2$$

$$2x_1 + 3x_3 = 5$$

Adding the first and second equations gives $2x_1 + 3x_3 = 5$ which is the same as the third equation. Thus the solution to the three equations is any tuple x_1, x_2, x_3 which satisfies $2x_1 + 3x_3 = 5$, and there are infinite solutions. We now express these solutions in a way whose motivation will become clear later: adding equations (1) and (2) above we get $2x_1 = 5 - 3x_3$.



- ▶ Subtracting equation (2) from (1) we get $2x_2 - x_3 = 1$, so we can write

$$\begin{aligned}x_1 &= \frac{5}{2} - \frac{3}{2}x_3 \\x_2 &= \frac{1}{2} + \frac{x_3}{2}\end{aligned}$$

- ▶ For the previous problem we can express the set of infinite solutions in terms of the free variable x_3 .
- ▶ Once x_3 is fixed, the other two variables have to take on specific values - they are known as pivot variables.



- ▶ To solve we always try to eliminate variables and then solve.
- ▶ To do that we have to add a multiple of one equation to the other etc.
- ▶ So, if we consider the matrix form of the system of transformed equations for which solutions could be obtained essentially by inspection.
- ▶ To make this work we need to preserve solutions of the original system of equations. So whatever elementary transformations we perform on coefficient matrix, same row transformation needs to be performed on the constant vector. So, we perform all elementary row transformation on the augmented matrix of the linear system.

Example



Consider the following system where we seek all solutions for some $a \in \mathbb{R}$.

$$-2x_1 + 4x_2 - 2x_3 - x_4 + 4x_5 = -3$$

$$4x_1 - 8x_2 + 3x_3 - 3x_4 + x_5 = 2$$

$$x_1 - 2x_2 + x_3 - x_4 + x_5 = 0$$

$$x_1 - 2x_2 - 3x_4 + 4x_5 = a$$

Let us consider the augmented matrix

$$\left[\begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ 1 & -2 & 1 & -1 & 1 & 0 \\ 1 & -2 & 0 & -3 & 4 & a \end{array} \right]$$

Now swap rows 1 and 3 in the augmented matrix to get

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ -2 & 4 & -2 & -1 & 4 & -3 \\ 1 & -2 & 0 & -3 & 4 & a \end{array} \right]$$

Does this change the system of equations? No, because we are swapping **both left and right hand sides of the equality sign**, so we are still dealing with the same set of equations.

Subtract rows



$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ -2 & 4 & -2 & -1 & 4 & -3 \\ 1 & -2 & 0 & -3 & 4 & a \end{array} \right] \begin{array}{l} \\ -4R_1 \\ +2R_1 \\ -R_1 \end{array}$$

The notation above is used to convey that we could like to add $-4 \times$ first row to the second row, $2 \times$ the first row to the third row, and $-1 \times$ the first row to the fourth row to get a new augmented matrix.

New Augmented Matrix



$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & -1 & -2 & 3 & a \end{array} \right] -R_2 - R_3$$

Note that the augmented matrix shown is obtained by performing the operations shown on the previous slide. To get the next augmented matrix we subtract the second and third rows of this augmented matrix from the last row.

New Augmented Matrix



$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 0 & 0 & 0 & a+1 \end{array} \right] \begin{array}{l} \\ -1 \\ -1/3 \\ \end{array}$$

Now multiply the second row by -1 and the third row by $\frac{-1}{3}$ to get the augmented matrix in its final form, known as the row-echelon form. These equations are equivalent to the original set of equations:

$$x_1 - 2x_2 + x_3 - x_4 + x_5 = 0$$

$$x_3 - x_4 + 3x_5 = -2$$

$$x_4 - 2x_5 = 1$$

$$0 = a + 1$$



- ▶ The preceding set of equations cannot be solved when $a \neq -1$.
- ▶ The last equation is consistent only for $a = -1$ i.e when rank of the coefficient matrix is same the rank of augmented matrix then the system has solution.
- ▶ In REF of a matrix, the leading entry is called as the pivot element and columns containing pivot elements are called as pivot columns. The corresponding variables are called pivot variables.
- ▶ The variables corresponding to non pivot columns are called as free variables.

Consider

$$x_1 - 2x_2 + x_3 - x_4 + x_5 = 0$$

$$x_3 - x_4 + 3x_5 = -2$$

$$x_4 - 2x_5 = 1$$

Let $x_5 = t$ and $x_2 = s$. Then $x_4 = 1 + 2t$

Then, we get $x_3 = -1 - t$ and $x_1 = 2 + 2s + 2t$

Thus, solution is $(2 + 2s + 2t, s, -1 - t, 1 + 2t, t)^T, \forall t, s \in \mathbb{R}$

- ▶ The system is inconsistent if the rank of the coefficient matrix is less than the rank of augmented matrix.
- ▶ The system is consistent if the rank of the coefficient matrix is same as the rank of augmented matrix.
- ▶ For the consistent system, if there are no free variables, the solution is unique.
- ▶ For the consistent system, if there are free variables, there are infinitely many solutions.