Homework 3

1) I $20 = 8x_1 + 3x_2$

$30 = 12x_2 + 6x_3$

$10 = x_1 + 10x_3$

$$= \left[\begin{array}{ccc|c} 8 & 3 & 0 & 20 \\ 0 & 12 & 6 & 30 \\ 1 & 0 & 10 & 10 \end{array}\right]$$

a) Gaussian Elimation to compute inverse of A.

$$\left[\begin{array}{ccc|ccc} 8 & 3 & 0 & 1 & 0 & 0 \\ 0 & 12 & 6 & 0 & 1 & 0 \\ 1 & 0 & 10 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 = R_1/8,\ R_3 = R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 3/8 & 0 & 1/8 & 0 & 0 \\ 0 & 12 & 6 & 0 & 1 & 0 \\ 0 & -3/8 & 10 & -1/8 & 0 & 1 \end{array}\right]$$

$R_2 = R_2/12$, $R_3 = R_3 + 3/8 R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 3/8 & 0 & 1/8 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/12 & 0 \\ 0 & 0 & 10+3/16 = 163/16 & -1/8 & 3/32 = 1/32 & 16/163 \end{array}\right] \xrightarrow{R_3 = R_3 \times 16/163} \left[\begin{array}{ccc|ccc} 1 & 3/8 & 0 & 1/8 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/12 & 0 \\ 0 & 0 & 1 & -2/163 & 1/326 & 16/163 \end{array}\right]$$

Using backward Substitution.

$$\left[\begin{array}{ccc|c} 1 & 3/8 & 0 & 1/8 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & -2/163 \end{array}\right] \rightarrow \begin{bmatrix} 20/163 \\ 1/163 \\ -2/163 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3/8 & 0 & 0 \\ 0 & 1 & 1/2 & 1/12 \\ 0 & 0 & 1 & 1/326 \end{array}\right] \longrightarrow \begin{bmatrix} -5/163 \\ 40/489 \\ 1/326 \end{bmatrix} \quad \left[\begin{array}{ccc|c} 1 & 3/8 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 16/163 \end{array}\right] \rightarrow \begin{bmatrix} 3/163 \\ -8/163 \\ 16/163 \end{bmatrix}$$

Hence,

$$A^{-1} = \begin{bmatrix} 20/163 & -5/163 & 3/163 \\ 1/163 & 40/489 & -8/163 \\ -2/163 & 1/326 & 16/163 \end{bmatrix}$$

b) $Ax = C$

$A^{-1}Ax = A^{-1}C$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20/163 & -5/163 & 3/163 \\ 1/163 & 40/489 & -8/163 \\ -2/163 & 1/326 & 16/163 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20/163(20) + 30(-5/163) + 3/163(10) \\ 20/163 + 30(40/489) + -8(10)/163 \\ -2(20)/163 + 30(326 + 10(16)/163 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 400 - 150 + 30/163 \\ 20 - 80/163 + 4600/163 \\ -40/163 + 15/163 + 160/163 \end{bmatrix} = \begin{bmatrix} 280/163 \\ 340/163 \\ 135/163 \end{bmatrix}$$

Hence, $x_1 = 280/163$; $x2 = \frac{340}{163}$, $x_3 = 135/163$

c) No, LU decomposition cannot be computed w/o pivoting

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 48 & 1/32 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 8 & 3 & 0 \\ 0 & 12 & 6 \\ 0 & 0 & 163/16 \end{bmatrix}$$

a) $4x_1 + x_2 = 2$

$2x_1 + 3x_2 + 2x_3 = 1$

$2x_1 + x_2 + 5x_3 = 1$

a) Gaussian Elimination to compute A^{-1}

$$\left[\begin{array}{ccc|ccc} 4 & 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 2 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

$R_1 = R_1/4$

$$\left[\begin{array}{ccc|ccc} 1 & 1/4 & 0 & 1/4 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 2 & 1 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 1/4 & 0 & 1/4 & 0 & 0 \\ 0 & 5/2 & 2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 5 & -1/2 & 0 & 1 \end{array}\right]$$

$R_2 = 2/5 R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 1/4 & 0 & 1/4 & 0 & 0 \\ 0 & 1 & 4/5 & -1/5 & 2/5 & 0 \\ 0 & 1/2 & 5 & -1/2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - 1/2 R_2} \left[\begin{array}{ccc|ccc} 1 & 1/4 & 0 & 1/4 & 0 & 0 \\ 0 & 1 & 4/5 & -1/5 & 2/5 & 0 \\ 0 & 0 & 46/10 & -2/5 & -1/5 & 1 \end{array}\right]$$

$R_3 = \frac{10R_3}{46}$

$$\left[\begin{array}{ccc|ccc} 1 & 1/4 & 0 & 1/4 & 0 & 0 \\ 0 & 1 & 4/5 & -1/5 & 2/5 & 0 \\ 0 & 0 & 1 & -4/46 & -2/46 & 10/46 \end{array}\right]$$

Backward Substitution

$$\left[\begin{array}{ccc|c} 1 & 1/4 & 0 & 1/4 \\ 0 & 1 & 4/5 & -1/5 \\ 0 & 0 & 1 & -4/46 \end{array}\right] \rightarrow \begin{bmatrix} 13/46 \\ -3/23 \\ -2/23 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1/4 & 0 & 0 \\ 0 & 1 & 4/5 & 2/5 \\ 0 & 0 & 1 & -2/46 \end{array}\right] \rightarrow$$

$$\begin{bmatrix} -5/46 \\ 10/23 \\ -1/23 \end{bmatrix} \quad \left[\begin{array}{ccc|c} 1 & 1/4 & 0 & 0 \\ 0 & 1 & 4/5 & 0 \\ 0 & 0 & 1 & 10/46 \end{array}\right] \rightarrow \begin{bmatrix} 1/23 \\ -4/23 \\ 5/23 \end{bmatrix}$$

Hence

$$A^{-1} = \begin{bmatrix} 13/46 & -5/46 & 1/23 \\ -3/23 & 10/23 & -4/23 \\ -2/23 & -1/23 & 5/23 \end{bmatrix}$$

b) $$\begin{bmatrix} 13/46 & -5/46 & 1/23 \\ -3/23 & 10/23 & -4/23 \\ -2/23 & -1/23 & 5/23 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 13/46(2) & -5/46 + 1/23 \\ -3/23(2) & + 10/23 - 4/23 \\ -2/23(2) & - 1/23 + 5/23 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

c) LU Decomposition

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/5 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 5/2 & 2 \\ 0 & 0 & 23/5 \end{bmatrix}$$

$$x_1 + x_2 - x_3 = -3$$
$$6x_1 + 2x_2 + 2x_3 = 2$$
$$-3x_1 + 4x_2 + x_3 = 1$$

a) Gaussian Elimination to compute inverse A

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 6 & 2 & 2 & 0 & 1 & 0 \\ -3 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = R_2 - 6R_1$
$R_3 = R_3 + 3R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -4 & 8 & -6 & 1 & 0 \\ 0 & 7 & -2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = \frac{R_2}{(-4)}} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & \\ 0 & 1 & -2 & 3/2 & -1/4 & \\ 0 & 7 & -2 & 3 & 0 & \end{array}\right]$$

$R_3 = R_3 - 7R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3/2 & -1/4 & 0 \\ 0 & 0 & 12 & -15/2 & +7/4 & 4/12 \end{array}\right]$$

$R_3 = \frac{R_3}{12}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3/2 & -1/4 & 0 \\ 0 & 0 & 1 & -15/24 & 7/48 & 1/12 \end{array}\right]$$

Backward Substitution

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 3/2 \\ 0 & 0 & 1 & -15/24 \end{array}\right] \rightarrow \begin{bmatrix} 1/8 \\ 1/4 \\ -5/8 \end{bmatrix} \quad \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & -1/4 \\ 0 & 0 & 1 & 7/48 \end{array}\right] \rightarrow \begin{bmatrix} 5/48 \\ 1/24 \\ 7/48 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \begin{bmatrix} -1/12 \\ 1/6 \\ 1/12 \end{bmatrix}$$

Hence,

$$A^{-1} = \begin{bmatrix} 1/8 & 5/48 & -1/12 \\ 1/4 & 1/24 & +1/6 \\ -5/8 & 7/48 & 1/12 \end{bmatrix}$$

b) $$\begin{bmatrix} 1/8 & 5/48 & -1/12 \\ 1/4 & 1/24 & 1/6 \\ -5/8 & 7/28 & 1/12 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} -3/8 & + 10/48 & - 1/12 \\ -3/4 & + 1/12 & + 1/6 \\ -15/8 & + 14/48 & + 1/12 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} -1/4 \\ -1/2 \\ 9/4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

c) LU Decomposition

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0.166667 & 0.133337 & 1 \end{bmatrix} \quad U = \begin{matrix} 6 & 2 & 2 \\ 0 & 5 & 2 \\ 0 & 0 & -8/5 \end{matrix}$$

1e) $8x_1 + 3x^2 = 20$

$12x_2 + 6x_3 = 30$

$x_1 + \quad 10x_3 = 10$

$$\begin{cases} x_1 = -3/8 x^2 + \frac{20}{8} \\ x_2 = -\frac{6}{12} x_3 + \frac{30}{12} \\ x_3 = \frac{-x_1}{10} + \frac{10}{10} \end{cases}$$

$$x = \begin{pmatrix} 0 & -3/8 & 0 \\ 0 & 0 & -6/12 \\ -1/10 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 20/8 \\ 30/12 \\ 10/10 \end{pmatrix}$$

$x_1^{k+1} = -3/8 x_2^{(k)} + 20/8$

$x_2^{k+1} = + \frac{6}{12} x_3^{(k)} + \frac{30}{12}$

$x_3^{k+1} = -1/10 x^{k+1} + 10/10$

* Start w/

$x_1^{(0)}, x_2^{(0)}, x_3^{(0)} = 0$

$x_1^1 = -3/8 x_2^{(0)} + 20/8 = 20/8 = 2.5$

$x_2^1 = -6/12 x_3^{(0)} + 30/12 = 30/12 = 2.5$

$x_2^1 = -1/10 x_1^{(1)} + 10/10$

$= -1/10 x_1^{(1)} + 10/10$

$-1/4 + 1 = .75$

$= G.x + c$

$x_1^1 = 1.55 \qquad x_2^2 = 2.1 \qquad x_3^2 = 0.84$

$x_3^1 = 1.68 \qquad x_2^3 = 2.1 \qquad x_3^3 = 0.8$

so,

$x_1 = 1.6 \qquad x_2 = 2.1 \qquad x_3 = 0.8$

2e) $2x_1 + x_2 + 5x_3 = 1$

$2x_1 + 3x_2 + 2x_3 = 1$

$4x_1 + x_2 = 2$

$$\left\{\begin{array}{llll} x_1 = & -\frac{x_2}{2} & -\frac{5x_3}{2} & +\frac{1}{2} \\ x_2 = -\frac{2}{3}x_1 & & -\frac{2x_3}{3} & +\frac{1}{3} \\ x_3 = & & & \end{array}\right\}$$

$$x = \begin{pmatrix} 0 & -1/2 & -5/2 \\ -2/3 & 0 & -2/3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/3 \\ 0 \end{pmatrix}$$

$x_1^{k+1} = -1/2\, x_2^k - 5/2\, x_3^k + 1/2$

$x_2^{k+1} = -1/3(x_1^{k+1}) - 2/3(x_3^k) + 1/3$

$x_3^{k+1} = 0$

$x_1^k = 0,\ x_2^k = 0,\ x_3^k = 0$

$x_1^1 = -1/2(0) - 5/2(0) + 1/2 = 1/2$

$x_2^1 = (-1/3)(1/2) - 2/3(0) + 1/3 = .4967$

$.1667 + .333$

$x_3^1 = 0$

so, $x_1 = 1/2$ $x_2 = .4967$ $x_3 = 0$

3e) $x_1 + x_2 - x_3 = -3$

$6x_1 + 2x_2 + 2x_3 = 2$

$-3x_1 + 4x_2 + x_3 = 1$

$$\left\{\begin{array}{llll} x_1 = & -x_2 & + x_3 & -3 \\ x_2 = -\frac{6x_1}{2} & & -2x_3/6 & +2/6 \\ x_3 = 3x_1 & -4x_2 & & +1 \end{array}\right\}$$

$$x_1 = \qquad x_2 = \qquad x_3 = \begin{Bmatrix} 0 & -1 & 1 \\ -6/2 & 0 & 2/6 \\ 3 & -4 & 0 \end{Bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} -3 \\ 1/3 \\ 1 \end{pmatrix}$$

$x_1^{k+1} = -1(x_2)^k + 1(x_3)^k - 3$

$x_2^{k+1} = -6/2 x_1^{k+1} + 0 - 2/6\ x_3^k + 1/3$

$x_3^{k+1} = 3x_1^{k+1} - 4x_2^{k+1} + 1$

$x_1^k = 0 \quad x_2^k = 0 \quad , \quad x_3^k = 0$

$x_1^{k+1} = -1(0) + 1(0) - 3 = -3$

$x2^1 = -6/2(-3) - 2/6(0) + 1/3$

$= 9.$

$x3 = 3(-3) - 4(9) + 1$

$= -44$

$x_1 = -3 \ , x_2 = 9 \ , \ x_3 = -44$

substituting in ①

$-3x_1 + 4x2 + x_3 = 1$

$-3(-3) + 4(9) + (-44)$

$= 9 + 36 - 44$

$= 1.$

1d) $x_1 = \frac{1}{8}(20-3x_2)$

$x_2 = \frac{1}{2}(30-6x_3)$

$x_3 = \frac{1}{10}(10-8x_1)$

iteration	x_1	x_2	x_3	error
1	2.725	2	0.9	—
2	1.75	2.05	0.7778	7e-4
3	1.7328	2.10625	0.825	
4	1.710186	2.0878	0.82083	
5	1.717786 ~~1.717539~~	2.086582	0.82904	
6	1.717539	2.08888	0.8285	
7	1.717935	2.08888	0.8253	0.60035

2d) $x_1 = \frac{1}{4}(2-x_2)$

$x_2 = 1/3(1-2x_1-2x_3)$

$x_3 = 1/5(1-2x_1-x_2)$

iteration	x_1	x_2	x_3	error
1	0.25	$0.3\overline{33}$	-0.4	
2	0.4166667	0.433333 ~~0.033333~~	0.0333333	
3	0.3716667	0.833333 ~~0.107778~~	-0.0523333	
4	0.49166667 ~~0.4730886~~	0.147778 ~~0.188889~~	0.6366667	
5	0.47308886 ~~0.50422222~~	0.988889 ~~0.0301111~~	0.01522223	
6	0.504222222 ~~0.49242222~~	0.0301114 ~~-0.0128848~~	0.044552	
7	0.49242222	-0.01288848	-0.00711	

3d) $x_1 = 1(-3-x_2+x_2)$

$x_2 = 1/2(2-6x_1-4x_3)$

$x_3 = 1(1+3x_1-4x_2)$

iteration	x_1	x_2	x_3	error
1	-3	-3	0	
2	0	10	4	
3	-9	-3	-15	
4	-15	43	-14	
5	-60	60	-216	
6	-279	397	-449	
7	-819	1257	-2424	