Exam 2

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1) $2x_1 + 3x_2 + 3x_3 = b_1$

$x_1 + 4x_2 + 3x_3 = b_2$

$2x_1 + 7x_2 + 7x_3 = b_3$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to -R_1 + R_2 \\ R_3 \to -2R_1 + R_3 \end{array} \quad \left[\begin{array}{ccc|cc} 1 & 3 & 3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & +1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

Back substitution

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right] = \begin{array}{l} x - 3 - 3 = 1 \\ y = -1 \\ z = -1 \end{array} \quad x = 7 \quad \begin{bmatrix} 7 \\ -1 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] = \begin{array}{l} x + 3 - 2 = 0 \\ y = 1 \\ z = -1 \end{array} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} x + 0 + 3 = 0 \\ y = 0 \\ z = 1 \end{array} \quad x = -3 \quad \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

2. $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{pmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{+2R_1 + R_3 \to R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-R_2 + R_3 \to R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array}\right]$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \qquad U = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. $8x_1 + 4x_2 - 3x_3 = 3$ $\quad \bar{x}^0 = (0,0,0)$

$x_1 - 10x_2 + 2x_3 = 2$

$3x_1 + 3x_2 - 7x_3 = 1$ $\quad x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 8 & 4 & -3 \\ 1 & -10 & 2 \\ 3 & 3 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

i) Jacobi $x_i^{(1)} = \dfrac{b_i - \sum_{j=1, j \neq i}^{n} a_{ij} x_j^{(0)}}{a_{ii}}$ $\quad$ error $= \|x^{(1)} - x^{(0)}\|_2 \leq 0.0001$

iteration	x_1^1	x_2^1	x_3^1	error
0	0.375	-0.2	-0.142857	0.445357
1	0.421429	-0.191071	-0.067857	0.088659
2	0.441089	-0.171429	-0.044132	0.038839
3	0.444165	-0.164318	-0.025574	0.019895
4	0.447569	-0.160698	-0.022923	0.005832
5	0.446753	-0.159828	-0.019912	0.003239
6	0.447447	-0.159307	-0.019889	0.000868
7	0.447195	-0.159233	-0.019369	0.000583
8	0.447353	-0.159153	-0.019445	0.000192
9	0.447285	-0.159153	-0.019378	0.000122
10	0.44723	-0.159140	-0.019372	0.000050
11	0.447306	-0.159142	-0.019350	0.000025
12	0.447315	-0.159139	-0.019358	0.000012
13	0.447310	-0.159140	-0.019353	0.000007

$$\bar{x} \simeq \begin{pmatrix} 0.447310 \\ -0.159140 \\ -0.019353 \end{pmatrix}$$ w/ error of $0.000007 < 10^{-5}$ in 13 iterations

3 ii) Gauss-Seidel $x_i^{(1)} = \dfrac{b_i - \left(\sum_{j}^{i-1} a_{ij} x_j^{(1)} + \sum_{j=i+1}^{n} a_{ij} x_j^{(0)}\right)}{a_{ii}}$

iteration	x_1	x_2	x_3	error
0	0.375	-0.1625	-0.017786	0.411962
1	0.436830	-0.166676	-0.022076	0.066715
2	0.440184	-0.160597	-0.019606	0.014858
3	0.447946	-0.159127	-0.019072	0.001580
4	0.447409	-0.159075	-0.019285	0.000578
5	0.447306	-0.159126	-0.019357	0.000133
6	0.447306	-0.159140	-0.019352	0.000015
7	0.447311	-0.159140	-0.019355	0.000005

$\vec{x} \approx \begin{pmatrix} 0.447311 \\ -0.159140 \\ -0.019355 \end{pmatrix}$ w/ error of 0.000005 $< 10^{-5}$ in 7 iterations

b) ~~||8||~~ $||8|| \geq ||4|| + ||-3||$ it will converge
$||-10|| \geq ||1|| + ||2||$ b/c is diagonally
$||-7|| > ||3|| + ||3||$ dominant

c) Jacobi had more iterations than Gauss Seidel. Seidel is more efficient since we use the most recent value for the variable vers the Jacobi method in which old values were used to calculate the vector components

4)

$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix}$ $U = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$y_1 + 0y_2 + 0y_3 = 1$

$y_1 = 1$

$1 + y_2 + 0(y_3) = 2$

$1 + y_2 = 2$

$y_2 = 1$

$-2(1) + 3(1) + y_3 = 3$

$-2 + 3 + y_3 = 3$

$1 + y_3 = 3$

$1 + y_3 = 3$

$y_3 = 2$

$y = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$Ux = y$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$0(x_1) + 0x_2 + 2x_3 = 2$

$x_3 = 1$

$x_2 - 1 = 1$

$x_2 = 2$

$x_1 + 2 - 1 = 1$

$x_1 + 1 = 1$

$x_1 = 0$

$x = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

solution $x = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

5.

x	y
1	-1
2	1
3	-1

$$L_1(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{x^2-5x+6}{2}$$

$$L_2(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = \frac{x^2-4x+3}{-1}$$

$$L_3(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{x^2-3x+2}{2}$$

$$P_2(x) = -1L_1(x) + 1L_2(x) - 1L_3(x)$$

$$\frac{-x^2+5x-6}{2} + \frac{x^2-4x+3}{-1} \frac{-x^2+3x-2}{2}$$

$$\frac{-x^2+5x-6-2x^2+8x-6-x^2+3x-2}{2}$$

$$\frac{-4x^2+16x-14}{2} = -2x^2+8x-7$$

$$f(x) = -2x^2+8x-7$$

6)

x	1.0	1.3	1.6	1.9
0	0.7651977	-0.4370 57	-0.108738	0.0658721
1		0.6206860	-0.5489460	-0.0494433
2		~~0.4554022~~	0.4554022	-0.5786618
3				0.28186

a) $P(0,2) = \dfrac{-0.5489460 + 0.4837057}{1.6 - 1.0} = -0.108755$

$P(0,3) = \dfrac{-0.0494433 + 0.1087338}{1.9 - 1.0} = 0.0658783$

$P(1,3) = -0.0494433 = \dfrac{f(2,3) + 0.5489460}{1.9 - 1.3}$ $= -0.02966058$ $= f(3,3) + 0.5489460$ $f(2,3) = -0.57861198$

$P(2,3) = -0.57861198 = \dfrac{f(3,3) - 0.4554022}{1.7 - 1.6}$

$= -0.173584 = f(3,3) - 0.4554022$

$f(3,3) = 0.28186$

b) $P_n(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$
$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$

$P_3(x) = 0.7651977 - 0.4837057(x - 1.0) - 0.108738(x - 1.0)(x - 1.3)$
$+ 0.0658721(x - 1.0)(x - 1.3)(x - 1.6)$

c) $P_n(1.5) = 0.5118127$

7. $x_i = 0.025, 0.4, 0.60, 0.75, 1.0 \quad y_i = \sin \pi x_i$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0 \\ 0 & 0 & h_2 & 2(h_2+h_3) & h_3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 a_0) \\ \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\ 3/h_3 (a_4 - a_3) - 3/h_2 (a_3 - a_2) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.375 & 1.15 & 0.2 & 0 & 0 \\ 0 & 0.2 & 0.7 & 0.15 & 0 \\ 0 & 0 & 0.15 & 0.8 & 0.25 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ -6.980784 \\ -4.879 \\ -3.606284 \\ 0 \end{bmatrix}$$

$$b_i = \frac{a_{i+1} - a_i}{h_i} - \frac{(2c_i + c_{i+1})h_i}{3}$$

$b_0 = 2.913$
$b_1 = 1.013335$
$b_2 = -0.975794$
$b_3 = -2.223721$

iteration	x_i	h_i	$y_i = a_i$	b_i	c_i	d
0	0.025	0.375	0.078455	2.983	0	-4.670554
1	0.4	0.2	0.951057	1.013335	-5.254373	0.9385
2	0.60	0.15	0.951057	-0.971794	-4.691273	2.362289
3	0.75	0.25	0.707107	-2.223721	-3.628243	4.83765
4	1.0		0		d	

$S_0(x) = 0.078459 + 2.483(x-0.025) - 4.670554(x-0.025)^3$

$S_1(x) = 0.951057 + 1.013335(x-0.4) - 5.254373(x-0.4)^2 + 0.9385(x-0.4)^3$

$S_2(x) = 0.951057 - 0.975794(x-0.6) - 4.691273(x-0.6)^2 + 2.362289(x-0.6)^3$

$S_3(x) = 0.707107 - 2.223721(x-0.75) - 3.628243(x-.75)^2 + 4.837657(x-.75)^3$

b) $\int_{0.025}^{0.4} S_0(x)\,dx + \int_{0.4}^{0.6} S_1(x)\,dx + \int_{.6}^{.75} S_2(x)\,dx + \int_{.75}^{1} S_3(x)\,dx$

$\int_{0}^{.4} 0.078459 + 2.483(x-0.025) - 4.670554(x-0.025)^3$

$+ \int_{.4}^{.6} 0.951057 + 1.013335(x-0.4) - 5.254373(x-0.4)^2 + .9385(x-0.4)^3$

$+ \int_{.6}^{.75} .951057 - 0.925794(x-.6) - 4.641273(x-.6)^2 + 2.362289(x-.6)^3$

$+ \int_{.75}^{1} .707107 - 2.223721(x-.75) - 3.628243(x-.75)^2 + 4.837657(x-.75)^3$

$= .217104 + .196842 + .126758 + .093113 =$

$.633817$

Result of $\int_0^1 s(x)\,dx = .633817 < \int_0^1 \sin(\pi x)\,dx)$

(c) $f'(.5) \approx 1.013335 = 2(5.254373)(x - .4)$
$+ 3(0.4385)(x - .4)^2$

$f'(0.5) \approx -0.009385$

$f''(0.5) \approx -2(5.254373) + 6(0.9385)(x - 0.4))$
$f''(.5) \approx -9.945646$

$f'(0.5) = \pi \cos(0.5\pi) = 0$
$f''(0.5) = -\pi^2 \sin(0.5\pi) = -1$

$f'(.5) = 0$
$f''(.5) = -9.869604$

f' & f'' are relatively similar based on actual & spline.