Homework 5

① Given linear system

$$A = \begin{bmatrix} 3.02 & -1.05 & 2.53 \\ 4.33 & 0.56 & -1.78 \\ -0.83 & -0.54 & 1.47 \end{bmatrix} \quad \& \quad b = \begin{bmatrix} -1.61 \\ 7.23 \\ -3.38 \end{bmatrix}$$

a) In this use 1-norm of A

$$\|A\|_1 = \max_{i \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

= maximum of the coloumn sum (ignoring (-)ve signs)

In above A

$|a_{11}| + |a_{21}| + |a_{31}| = 3.02 + 4.33 + 0.83 = 8.18$

2nd column sum = $1.05 + 0.56 + 0.54 = 2.15$

3rd " " $= 2.53 + 1.78 + 1.47 = 5.78$

$\therefore \|A\|_1 = 8.18$

$\therefore \kappa(A) = \|A\| \cdot \|A^{-1}\|$

find A^{-1}

$A^{-1} = \frac{1}{\det(A)}$ (adjoint A)

$\det(A) = (3.02)((0.56 \times 1.47) - (1.78 \times 0.54))$
$+ 1.05((4.33 \times 1.47) - (0.83 \times 1.78))$
$+ 2.53((-4.33 \times 0.54) + (0.56 \times 0.83))$
$= -0.41676 + 5.132085 - 4.739702$
$= -0.024377$

adjoint $(A) = [b_{ij}]^T = (-1)^{i+j} B_{ij}$ where B_{ij} are ij minors.

$$\text{adjoint}(A) = \begin{bmatrix} -0.318 & -4.0877 & -1.8734 \\ 0.1773 & 6.5383 & +2.5023 \\ 0.4522 & 16.3305 & 6.2372 \end{bmatrix}^T$$

$$\therefore A^{-1} = \frac{-1}{0.024377}\begin{bmatrix} -0.138 & 0.1773 & 0.4522 \\ -4.8877 & 6.5393 & 16.3305 \\ -1.8734 & 2.5023 & 6.2372 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 5.66107 & -7.27325 & -18.55027 \\ 200.50457 & -268.2569638 & -669.91426 \\ 76.8513016 & -102.650039 & -255.8846454 \end{bmatrix}$$

Now to find $\|A^{-1}\|_1$

1st column sum $= 283.0167702$

2nd column sum $= 378.1802527$

3rd column sum $= 944.3491754$

$\|A^{-1}\|_1 = 944.3491753$

$\therefore K(A) = \|A\|_1 \cdot \|A^{-1}\|_1$

$= (8.18)(944.3491754)$

$= 7724.776257$

b) A is given & $\vec{b} = \begin{bmatrix} -1.61 \\ 7.23 \\ -3.38 \end{bmatrix}$

we have to find solution of $Ax = b$,

consider the Augmented matrix;

$[A : b]$

$$= \left[\begin{array}{ccc|c} 3.02 & -1.05 & 2.53 & -1.61 \\ 4.33 & 0.56 & -1.78 & 7.23 \\ -0.83 & -0.54 & 1.47 & -3.38 \end{array}\right]$$

$$R_1 \to \frac{1}{3.02} R_1 \left[\begin{array}{ccc|c} 1 & -0.347682 & 0.837748 & 0.533112 \\ 1.433775 & 0.185430 & -0.589404 & 2.394040 \\ -0.274834 & -0.1788078 & 0.486754 & 1.119205 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - (1.433775)R_1 \\ R_3 \to R_3 + (0.274834)R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -0.347682 & 0.837748 & 0.533112 \\ 0 & 0.683928 & -1.790546 & 1.629677 \\ 0 & -0.274363 & 0.716996 & 1.265722 \end{array}\right]$$

$$R_2 \to \frac{1}{0.683928} R_2 \left[\begin{array}{ccc|c} 1 & -0.347682 & 0.837748 & 0.533112 \\ 0 & 1 & -2.618033 & 2.382820 \\ 0 & -0.274363 & 0.716996 & 1.265722 \end{array}\right]$$

$R_3 \to R_3 + (0.274363)R_2$ $\left[\begin{array}{ccc|c} 1 & -0.347682 & 0.837748 & 0.533112 \\ 0 & 1 & -2.618033 & 2.382820 \\ 0 & 0 & -0.001295 & 1.919480 \end{array}\right]$

$\therefore -0.001295\, x_3 = 1.919480$

$\therefore x_3 = -1482.223938$

$x_2 - 2.618033\, x_3 = 2.382820$

$\therefore x_2 = 2.382820 + 2.618033(-1482.223938)$

$= -3878.128364$

$x_1 - 0.347682 x_2 + 0.837748\, x_3 = 0.533112$

$\therefore x_1 = 0.533112 + (0.347682) x_2 - 0.837748\, x_3$

$x_1 = -106.0921742$

$\therefore x = \begin{bmatrix} -106.092174 \\ -3878.128364 \\ -1482.223938 \end{bmatrix}$ is solution (using 6-digit arithmetic)

(c) ~~for 3-digit arithmetic, we apply the same method but by considering only 3 decimal places.~~

(c) $b = \begin{bmatrix} -1.61 \\ 7.23 \\ -3.38 \end{bmatrix}$ $\left[\begin{array}{ccc|c} 3.02 & -1.05 & 2.53 & -1.61 \\ 4.33 & 0.56 & -1.73 & 7.23 \\ -0.83 & -0.54 & 1.44 & -3.28 \end{array}\right]$ $R_1 \to \frac{1}{3.02} R_1$

$\left[\begin{array}{ccc|c} 1 & -0.348 & 0.838 & 0.531 \\ 1.434 & 0.185 & -0.579 & 2.394 \\ -0.275 & -0.179 & 0.489 & 1.119 \end{array}\right]$

$R_2 \to -1.434 R_1 + R_2$
$R_3 \to 0.275 R_1 + R_3$ $\left[\begin{array}{ccc|c} 1 & -0.348 & 0.838 & 0.531 \\ 0 & 0.604 & -1.781 & 1.633 \\ 0 & -0.275 & 0.717 & 1.265 \end{array}\right]$ $R_2 \to \frac{1}{0.604} R_2$

$\left[\begin{array}{ccc|c} 1 & -0.348 & 0.838 & 0.531 \\ 0 & 1 & -2.618 & 2.387 \\ 0 & -0.275 & 0.717 & 1.265 \end{array}\right]$ $0.275R_2 + R_3 \to R_3$

$\left[\begin{array}{ccc|c} 1 & -0.348 & 0.838 & 0.531 \\ 0 & 1 & -2.618 & 2.387 \\ 0 & 0 & -0.003 & 1.921 \end{array}\right]$ $x_3 = 640.333$, $x_2 = -1678.779$, $x_1 = 48.142$ $\to$ $\begin{matrix} 48.149 \\ -1678.779 \\ -640.333 \end{matrix}$

d) Using 3-digit arithmetic ~~leads~~ to loss of accuracy of large degree. Using absolute value for emphasis, the abs value of the solution from pt b)i) double that of part c. Simply using three less digit can lead to a large number

6

$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$

$y = \frac{1}{(ax+b)^3}$

$y = y^{-1/3}$

$y^{-1/3} = ax+b$

$y = ax+b$

$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

i	$a_i = x_i$	$y_i = y_i^{-1/3}$
0		
1		
2		
⋮		
n	x_n	y_n

$x = x$

$y = y^{-1/3}$

x & $y^{-1/3}$ are linearly realated, so $n=1$, making $P_1(x) = a_0 + a_1 x$ a linear relationship

ff $P_1(x) = y^{-1/3} = y_1$ set $a = a_1$ & $b = a_0$

given in points where $x = x$ & $y = y^{-1/3}$, use least squares

$$\begin{bmatrix} m & \sum_{i=1}^{m} x_i \\ \sum_{i=1}^{m} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} y_i^{-1/3} \\ \sum_{i=1}^{m} x_i y_i^{-1/3} \end{bmatrix}$$

$$a_0 = \frac{\sum y_i^{-1/3} \sum x_i^2 - \sum x_i \sum x_i y_i^{-1/3}}{m \sum x_i^2 - (\sum x_i)^2}$$

$$a_1 = \frac{m \sum x_i y_i^{-1/3} - \sum x_i \sum y_i}{m \sum x_i^2 - (\sum x_i)^2}$$

(3) $P_2(x) = a_0 + a_1 x + a_2 x^2 \quad m = 21 \quad n = 2$

i	x_i	y_i	x_i^2	x_i^3	x_i^4	$x_i y_i$	$x_i^2 y_i$
0	-1	7.904	1	-1	1	-7.904	7.904
1	-0.9	7.452	.81	-.729	.6561	-6.7068	6.03612
2	-0.8	5.827	.64	-.512	.4096	-4.6616	3.72928
3	-0.7	4.400	2.49	-.343	.2401	-3.08	2.156
4	-0.6	2.908	.36	-.216	.1296	-1.7448	1.04688
5	-0.5	2.144	.25	-.125	.0625	-1.072	.536
6	-0.4	0.581	.16	-.064	.0256	-0.2324	.09296
7	-0.3	0.335	.09	-.027	.0081	-0.1005	.03015
8	-0.2	-0.271	.04	-.008	.0016	0.0542	-.01086
9	-0.1	-0.903	.01	-.001	.0001	0.0903	-0.0.0891
10	0	-0.847	0	0	0	0	0
11	.1	-1.278	.01	.001	.0001	-.1278	-0.278
12	.2	-1.335	.04	.008	.0016	-.267	-0.0534
13	.3	-0.656	.09	.027	.0081	-.1968	-0.5904
14	.4	-0.711	.16	.064	.0256	-.2844	-0.11376
15	.5	0.224	.25	.125	.0625	.112	.056
16	.6	0.689	.36	.216	.1296	.4134	.24804
17	.7	0.861	.49	.343	.2401	.6027	.42189
18	.8	1.358	.64	.512	.4096	1.0864	.86912
19	.9	2.613	.81	.729	.6561	2.3517	2.11653
20	1	4.599	1	1	1	4.599	4.599
	0	35.634	7.7	0	5.0666	-17.0624	29.57255

$$\begin{bmatrix} m & \Sigma x_i & \Sigma x_i^2 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \\ \Sigma x_i^2 y_i \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} 0.1875147 & 0 & -0.16345217 \\ 0 & 0.12987013 & 0 \\ -0.1634260 & 0 & .445778421 \end{bmatrix}$$

$$\begin{array}{ccc} A & \bar{x}' & = \bar{b} \end{array}$$

$$\begin{bmatrix} 21 & 0 & 7.7 \\ 0 & 7.7 & 0 \\ 7.7 & 0 & 5.060 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 35.834 \\ -17.0629 \\ 29.5852 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} .98133 \\ 2.2159 \\ 7.33018 \end{bmatrix}$$

$$P_2(x) = -0.98123 - 2.2519x + 7.3301x^2$$

4) $\displaystyle\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = 0$

$n, m \geq 0$, $n \neq m$

$T_n(x) = \cos(n\cos^{-1}(x))$ & $T_n(x) = \cos(n\theta)$

$x = \cos(\theta)$

$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$

$\displaystyle\int_{\pi} \frac{\cos(n\theta)\cos(m\theta)}{\sqrt{1-\cos^2(\theta)}}(\sin(\theta))\,d\theta$ $\quad x = \cos\theta$

$dx = -\sin(\theta)\,d\theta$

$\cos(\theta) = 1 \rightarrow \theta = 0$

$\cos(\theta) = -1 \rightarrow \theta = \pi$

$$= \int_0^{\pi} \frac{\cos(n\theta)\cos(m\theta)}{\sqrt{1-\cos^2(\theta)}} \sin(\theta)\,d\theta$$

$$= \int_0^{\pi} \frac{\cos(n\theta)\cos(m\theta)}{\sqrt{\sin^2(\theta)}} \sin\theta\,d\theta \qquad 1-\cos^2(\theta) = \sin^2(\theta)$$

$$= \int_0^{\pi} \cos(a)\cos(b)\,d\theta$$

$$= \int_0^{\pi} \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right] d\theta$$

$$= \frac{1}{2}\left[\int_0^{\pi} \cos(a-b)\,d\theta + \int_0^{\pi} \cos(a+b)\,d\theta\right]$$

$$= \frac{1}{2}\left[\int_0^{\pi} \cos((n-m)\theta)\,d\theta + \int_0^{\pi} \cos((n+m)\theta)\,d\theta\right]$$

$$= \frac{1}{2}\left[\frac{\sin((n-m)\theta)}{n-m}\Big|_0^{\pi} + \frac{\sin((n+m)\theta)}{n+m}\Big|_0^{\pi}\right] \quad \begin{matrix}\sin 0 = 0 \\ \sin \pi = 0\end{matrix}$$

$$= \frac{1}{2}\left[\cancel{\frac{\sin((n-m)\theta)}{n-m}\Big|_0^{\pi} + \sin}\right.$$

$$= \frac{1}{2}\left[\frac{\sin(n\theta)\cos(m\theta) - \cos(n\theta)\sin(m\theta)}{n-m}\Big|_0^{\pi} + \right.$$

$$\frac{\sin(n\theta)\cos(m\theta) + \cos(n\theta)\sin(m\theta)}{n+m}\Big|_0^{\pi}$$

$$= \frac{1}{2}\left[\frac{\sin(n\pi)\cos(m\pi) - \cos(n\pi)\sin(m\pi) - \sin(n\theta)\cos(m\theta) + \cos(n\theta)\sin(m\theta)}{n-m}\right.$$

$$+ \left.\frac{\sin(n\pi)\cos(m\pi) + \cos(n\pi)\sin(m\pi) - \sin(n\theta)\cos(m\theta) + \cos(n\theta)\sin(n\theta)}{n+m}\right]$$

$$= \frac{1}{2}\left[\frac{(n+m)\left[\sin(n\pi)\cos(n\pi) - \cos(n\pi) - \cos(n\pi)\sin(m\pi)\right] + (n-m)\left[\sin(n\pi)\cos(m\pi)\right]}{(n+m)(n-m)}\right]$$

$$= \frac{1}{2}\left[\frac{n\sin(n\pi)\cos(m\pi) + m\sin(n\pi)\cos(m\pi) - n\cos(n\pi)\sin(m\pi) - m\cos(n\pi)\sin(m\pi) + n\sin(n\pi)\cos(m\pi) - m\sin(n\pi)\cos(m\pi) + n\cos(n\pi)\sin(m\pi) - m\cos(n\pi)\sin(m\pi)}{(n+m)(n-m)}\right]$$

$$= \frac{n\sin(n\pi)\cos(m\pi) - m\cos(n\pi)\sin(m\pi)}{(n+m)(n-m)} \quad \begin{matrix}\sin(n\pi) = 0 \\ \to \sin(m\pi) = 0 \\ n \neq m\end{matrix}$$

$$= 0$$

$$\boxed{\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx = 0}$$