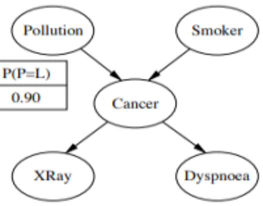
Wednesday, December 11, 2019 11:45 PM

Problem 1

A patient has been suffering from shortness of breath (called **dyspnoea**) and visits the doctor, worried that he has lung **cancer**. The doctor knows that other diseases are also possible causes. The doctor also knows that other relevant information includes whether or not the patient is a **smoker** (increasing the chances of cancer) and what sort of **air pollution** he has been exposed to. A **positive X-ray** would often indicate a lung cancer.



P(S=T)
0.30

P(P=L)
0.90

P	S	P(C=T|P,S)
H	T	0.05
H	F	0.02
L	T	0.03
L	F	0.001

Node name	Type	Values
Pollution	Binary	{*low, high*}
Smoker	Boolean	{*T, F*}
Cancer	Boolean	{*T, F*}
Dyspnoea	Boolean	{*T, F*}
X-ray	Binary	{*pos, neg*}

C	P(X=pos|C)
T	0.90
F	0.20

C	P(D=T|C)
T	0.65
F	0.30

Does not affect cancer

(They are cancer's children so we delete them)

1. What is the probability that a patient has cancer if we don't know anything about him or his symptoms ($P(cancer)$)? Use the Bayesian Network above.
2. The patient has dyspnoea and is not a smoker. If the x-ray comes positive, what is the probability that the patient has cancer?

Show in detail the calculations that take place.

(1) $P(cancer) = \sum_{S \in \{t,f\}} P(C,S)$

$= \sum_{S\in(t,f)} \sum_{P\in(t,f)} \sum_{D\in\{t,f\}} \sum_{X\in\{t,f\}} P(C,S,P,D,X)$

$= \sum_S \sum_P \sum_D \sum_X P(C|S,P)\cdot P(S)\cdot P(P)\cdot P(D|C)\cdot P(X|C)$

$= \sum_S \sum_P P(C|S,P)\cdot P(S)\cdot P(P)\cdot \sum_D \sum_X P(D|C)\cdot P(X|C)$

$= \sum_S \sum_P P(C|S,P)\cdot P(S)\cdot P(P)\cdot \left[\sum_D P(D|C)\cdot \sum_X P(X|C)\right]$

$= \sum_S P(S) \sum_P P(C|S,P)\cdot P(P)$

$= P(S=T)\cdot\left[P(C|S=T,P=T)\cdot P(P=T) + P(C|S=T,P=F)\cdot P(P=F)\right]$
$+ P(S=F)\cdot\left[P(C|S=F,P=T)\cdot P(P=T) + P(C|S=F,P=F)\cdot P(P=F)\right]$

$= 0.3\cdot[0.05\cdot 0.1 + 0.03\cdot 0.9]$
$+ 0.7\,[0.02\cdot 0.1 + 0.001 \quad 0.9]$

(2) $D=T,\ S=F,\ X=T$

$P(C=T|S=F,D=T,X=T) = \frac{P(C,S=F,D=T,X=T)}{P(S=F,D=T,X=T)} = \alpha \sum_{P=\{T,F\}} P(C,S=F,D=T,X=T,P)$

$= \alpha \sum_{P\in\{T,F\}} P(C|S=F,P)\cdot P(S=F)\cdot P(D=T|C)\cdot P(X=T|C)\cdot P(P)$

$= \alpha\left(P(C \overset{t}{=} |S=F,P=T)\cdot P(S=F)\cdot P(D=t|C\overset{t}{=})P(X=t|C\overset{t}{=})\cdot P(P=T) + P(C\overset{t}{=}|S=F,P=F)\cdot P(S=F)\cdot P(D=t|C=t)\cdot P(X=t|C=t)\cdot P(P=F)\right)$

$= \alpha\,[0.02\cdot 0.7\cdot 0.65\cdot 0.9\cdot 0.1]$

$$= \alpha \left[\begin{array}{l} 0.02 \cdot 0.7 \cdot 0.65 \cdot 0.9 \cdot 0.1 \\ + 0.001 \cdot 0.7 \cdot 0.65 \cdot 0.9 \cdot 0.9 \end{array} \right]$$

$$= \alpha (0.0012) = 0.0275$$

$$P(C=F \mid S=F, D=T, X=T) = \alpha \left[\begin{array}{l} P(C=F \mid S=f, P=t) \cdot P(S=f) \cdot P(D=t \mid C=F)\, P(P=t) \cdot P(X=t \mid C=f) + \\ P(C=F \mid S=f, P=f) \cdot P(S=f) \cdot P(D=t \mid C=f) \cdot P(P=F) \cdot P(X=t \mid C=f \end{array} \right]$$

$$= \alpha \left[\begin{array}{l} 0.98 \cdot 0.7 \cdot 0.3 \cdot 0.1 \cdot 0.2 + \\ 0.999 \cdot 0.7 \cdot 0.3 \cdot 0.9 \cdot 0.2 \end{array} \right]$$

$$= \alpha (0.479) = 0.973$$

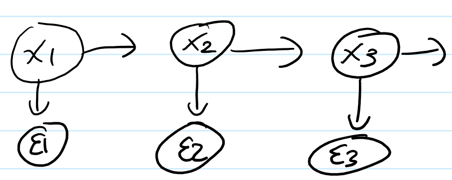
$$\alpha = \frac{1}{0.479 + 0.0012}$$

$$= 23.22$$

Problem 2

You are up in your friend's apartment building, watching cars on the street below. They are far enough away that all you can see is their color. You want to catch a taxi home, so you are trying to reason about the probability that a given car is a taxi, given its color. You know that 75% of all taxis are yellow, and that only 10% of non-taxi cars are yellow. You also know that taxis are not likely to bunch up: The car following a taxi is another taxi only 25% of the time. However, non-taxi cars are followed by taxis 50% of the time. Assume 40% of all cars are taxis and 60% are non-taxis.

1. To formulate the above problem as a Hidden Markov Model (temporal model), give the transition model and the evidence model as conditional probabilities (use any correct notation you like).
2. At time $t = 1$, you see a yellow car. What is the probability that the car you see (at $t = 1$) is a taxi?
3. What is the probability that the next car (at $t = 2$) will be a taxi?
4. You observe that the next car (at $t = 2$) is also yellow. Use this new information to update your belief that the previous car you saw (at $t = 1$) was a taxi.
5. Explain qualitatively why your new estimate in Question 4 makes sense, given your evidence and transition models (I am looking here for a short informal explanation in English).



1) $X_t \in \{Taxi, \neg Taxi\}$ $\quad E_t \in \{Yellow, \neg Yellow\}$

Transitional Model

Observational Model

Transitional Model

X_t \ X_{t+1}	Taxi	¬Taxi
Taxi	0.25	0.75 (1−0.25)
¬Taxi	0.5	0.5 (1−0.5)

Observational Model

X_t \ E_t	Yellow	¬Yellow
Taxi	0.75	0.25 (1−0.75)
¬Taxi	0.1	0.9 (1−0.1)

$P(X_1)$ = Initial Distribution

$= [0.4, 0.6]$ (Taxi, ¬taxi)

2) $P(X_1 | E_1 = \text{yellow})$ → Filtering

$= \alpha P(E_1 = \text{yellow} | X) \cdot P(X_1)$

- $P(X_1 = \text{Taxi} | E_1 = \text{yellow}) = P(E_1 = \text{yellow} | X_1 = \text{taxi}) \cdot P(X_1 = \text{taxi})$

$= \alpha\, 0.75 \cdot 0.4$

$= \alpha \cdot 0.3$

$= 0.833$

- $P(X_1 = \neg\text{taxi} | E_1 = y) = \alpha P(E_1 = y | X_1 = \neg\text{taxi}) \cdot P(X_1 = \neg\text{taxi})$

$= \alpha \cdot 0.1 \cdot 0.06$

$= 0.006\alpha = 0.167$

$\alpha = \frac{1}{0.006 + 0.3}$

- Extra. $P(X_2 | E_1 = \text{yellow}, E_2 = \neg\text{yellow}) = \alpha P(E_2 = \neg\text{yellow} | X_2) \cdot P(X_2 | E_1 = \text{yellow})$

$= \alpha P(E_2 = \neg\text{yellow} | X_2) \cdot \sum_{X_1} P(X_1 | E_1 = \text{yellow}) \cdot P(X_2 | X_1)$

→ $P(X_2 = T | E_1 = y, E_2 = \neg y) = \alpha P(E_2 = \neg y | X_2 = T) \left[P(X_1 = T | E_1 = y) P(X_2 = T | X_1 = T) + P(X_1 = F | E_1 = y) \cdot P(X_2 = T | X_1 = F) \right]$

$= \alpha \cdot 0.25 \, (0.833 \cdot 0.25 + 0.167 \cdot 0.5)$

$= \underline{\alpha \cdot 0.0729} = 0.102629$

→ $P(X_2 = F | E_1 = y, E_2 = \neg y) = \alpha P(E_2 = \neg y | X_2 = F) \left[P(X_1 = T | E_1 = y) P(X_2 = F | X_1 = T) + P(X_1 = F | E_1 = y) P(X_2 = F | X_1 = F) \right]$

$= \alpha \cdot 0.9 \, (0.833 \cdot 0.75 + 0.167 \cdot 0.5)$

$= \underline{\alpha \cdot 0.637425} = 0.8973709$

$\alpha = \frac{1}{\ }$

$0.0729 + 0.637425$

$= 1/0.710325$

$= \underline{1.4078}$

(3) $P(X_2 = \text{taxi} \mid E_1 = \text{yellow}) = \sum_{X_1} P(X_2 \overset{=T}{} \mid X_1) \cdot P(X_1 \mid E_1 = \text{yellow})$ **Prediction**

NO ALPHA

$= P(X_2=T \mid X_1=T) \cdot P(X_1=T \mid E_1=\text{yellow})$
$+ P(X_2=T \mid X_1=F) \cdot P(X_1=F \mid E_1=\text{yellow})$

$= 0.25 \cdot 0.833 + 0.5 \cdot 0.167$

$= \boxed{0.29175}$

$P(X_2=F \mid E_1=y) = 1 - 0.29175 = 0.70825$

Extra. $P(E_2=y \mid E_1=y) = \sum_{X_2} P(X_2 \mid E_1=y) P(E_2=y \mid X_2)$

$= P(X_2=T \mid E_1=y) P(E_2=y \mid X_2=T)$
$+ P(X_2=F \mid E_1=y) P(E_2=y \mid X_2=F)$

$= 0.29175 \cdot 0.75 + 0.70825 \cdot 0.10$

(4) $P(X_2 \mid E_1=T, E_2=F, E_3=T)$ **Smoothning**

$= \alpha \cdot \underbrace{P(X_2 \mid E_1=T, E_2=F)}_{\text{Filtering}} \cdot \underbrace{P(E_3=T \mid X_2)}_{\text{Prediction}}$

$\rightarrow P(X_2=T \mid E_1=T, E_2=F, E_3=T) = \alpha (X_2=T \mid E_1=T, E_2=F) \cdot P(E_3=T \mid X_2=T)$

$P(E_3=T \mid X_2=T) = \sum_{X_3} P(E_3=T \mid X_3) \cdot P(X_3 \mid X_2=T)$

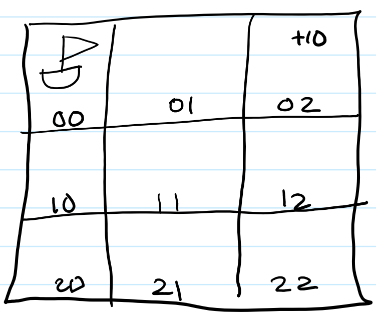
Problem 3

Consider the problem of implementing an autopilot system for a boat. Consider here a simplified version of this control problem.

- Assume the position of the boat is discret and takes values from the grid $\{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$.
- The boat's position is always known.
- The boat's actions are move-east, move-west, move-north , move-south.
- $(0,0)$ is the most west-north position, and $(2,2)$ is the most east-south position.
- The boat moves into the intended adjacent position with probability 0.9, and to a random position among the other adjacent positions with probability 0.1.
  Example 1: $P((0,1) \mid (1,1), north) = 0.9$, $P((1,0) \mid (1,1), north) = \frac{0.1}{3}$, $P((1,2) \mid (1,1), north) = \frac{0.1}{3}$, $P((2,1) \mid (1,1), north) = \frac{0.1}{3}$.
  Example 2: $P((0,1) \mid (0,0), east) = 0.9$, $P((1,0) \mid (0,0), east) = 0.1$.
- When the boat tries to move outside the grid, it remains in the same position with probability 1.
- The initial position is $(0,0)$
- The reward of all the states is 0, except for state $(0,2)$ (the goal) where the reward is 10, and for state $(0,1)$ (iceberg) where the reward is -5.

Using a discount factor $\gamma = 1$, show the policy π and value function V^π at each iteration of the Policy Iteration algorithm for 2 iterations. Run the policy evaluation part of the Policy Iteration algorithm for 2 iterations only. The initial values are set to 0, and the initial policy for all the states is move east.

MDP



$S = \{ (0,0) \ldots\ldots (2,2) \}$

$A = \{ L, R, U, D, \text{Nothing} \}$

$R([0,2], L) = +10$ $\quad$ $R([0,1], \cdot) = -5$ any

$R([0,2], \text{Nothing}) = +10$

$R([0,2], R) = +10$

State	π(state)
0,0	R
0,1	R
0,2	R
1,0	R
1,1	R
1,2	R
2,0	R
2,1	R
2,2	R

State.	V_0^π	V_1^π
0,0	0	$R((0,0),R) + \gamma \cdot [T((0,0),R,(0,1)) V_0^\pi((0,1)) + T([0,0],R,[1,0]) V_0^\pi([1,0])] = 0 + 0.9[0.9 \cdot 0 + 0.1 \cdot 0] = 0$
0,1	0	$R[(0,1),R) + \gamma \cdot [T[(0,1),R,(0,2)] V_0^\pi((0,1)] +$ $0,1 \to 0,0 + 0,1 \to 1,1 + 0,1 \to]$ $= -5 + 0.9(0.9 \cdot 0 + 0.1 \cdot 0 + 0.1 \cdot 0)$ $= -5$
0,2	0	10
1,0	0	0
1,1	0	0
1,2	0	0

1,1	0	0
1,2	0	0
2,0	0	0
2,1	0	0
2,2	0	0

	U_0	U_1	U_2
0,0	0	0	-4.05
0,1	0	-5	$-5 + 0.9[\underset{[0,2]}{0.9 \cdot 10} + \underset{[0,0]}{0.05 \cdot 0} + \underset{[1,1]}{0.05 \cdot 0}] = 3.1$
0,2	0	10	$10 + 0.9 \cdot 10 \cdot 1 = 19$
1,0	0	0	0
1,1	0	0	$0 + 0.9[0.9 \cdot 0 + 0.05 \cdot (-5) + 0.05 \cdot 0] =$
1,2	0	0	0
2,0	0	0	0
2,1	0	0	0
2,2	0	0	0

Policy Improvement

State	U^{π}(state)
0,0	7
0,1	5.3
0,2	9.6
1,0	
1,1	
1,2	
2,0	
2,1	
2,2	

Compute

$Q([0,0], R] = R([0,0], \text{Right}) + \gamma \cdot [T((0,0), R, (0,1)) \cdot U(0,1) + T((0,0), R, (1,0)) \cdot U(1,0))$

$= 0 + 0.9(0.9 \cdot 5.3 + 0.1 \cdot 8.3)$

$= \underline{5.04}$

$Q([0,0], L) =$

" " " U =

" " " D =

" " " Not =

Now go left from [0,0]

$R([0,0], L) + \gamma \cdot [T((0,0), L, (0,0)]$

$= 0 + 0.9[7 \cdot 1) = \underline{6.3}$