CS 440: Probability Basics

01:198:440

Question 1 - Monty Hall

Monty Hall presents you with three doors, labeled A, B, and C. He has hidden a car, ***uniformly at random****, behind one of the doors. You have the opportunity to select a door to open in hope of finding it. Without prior information, there is no reason not to select a door* ***at random*** *to open. Before he opens the door you selected, however, Monty selects* ***at random*** *one of the doors you didn't select that has nothing behind it, to show you there is nothing behind it.*

Question 1.a: Complete the following table, which summarizes (in part) your probabilistic knowledge of the situation as presented above:

$\mathbf{P}(\text{behind A}) = \frac{1}{3}$	$\mathbf{P}(\text{selected A} \mid \text{behind A}) = \frac{1}{3}$	$\mathbf{P}(\text{ shown B} \mid \text{behind A} \wedge \text{selected A }) = \frac{1}{2}$
$\mathbf{P}(\text{behind B}) = \frac{1}{3}$	$\mathbf{P}(\text{selected A} \mid \text{behind B}) = \frac{1}{3}$	$\mathbf{P}(\text{ shown B} \mid \text{behind B} \wedge \text{selected A }) = 0$
$\mathbf{P}(\text{behind C}) = \frac{1}{3}$	$\mathbf{P}(\text{selected A} \mid \text{behind C}) = \frac{1}{3}$	$\mathbf{P}(\text{ shown B} \mid \text{behind C} \wedge \text{selected A }) = 1$

Question 1.b: Suppose that you select door A, and Monty shows you door B, opening it to reveal no car. What was the probability of this event happening, i.e., $\mathbf{P}(\text{selected A} \wedge \text{shown B})$? Consider marginalizing on the events { behind A, behind B, behind C }.

$\mathbf{P}(\text{selected A} \wedge \text{shown B}) = \mathbf{P}(\text{selected A} \wedge \text{shown B} \wedge \text{behind A}) + \mathbf{P}(\text{selected A} \wedge \text{shown B} \wedge \text{behind B}) + \mathbf{P}(\text{selected A} \wedge \text{shown B} \wedge \text{behind C})$
$= \mathbf{P}(\text{shown B} \mid \text{behind A} \wedge \text{selected A})\, \mathbf{P}(\text{behind A} \wedge \text{selected A}) + \mathbf{P}(\text{shown B} \mid \text{behind B} \wedge \text{selected A})\, \mathbf{P}(\text{behind B} \wedge \text{selected A}) + \mathbf{P}(\text{shown B} \mid \text{behind C} \wedge \text{selected A})\, \mathbf{P}(\text{behind C} \wedge \text{selected A})$
$= \mathbf{P}(\text{shown B} \mid \text{behind A} \wedge \text{selected A})\, \mathbf{P}(\text{selected A} \mid \text{behind A})\, \mathbf{P}(\text{behind A}) + \mathbf{P}(\text{shown B} \mid \text{behind B} \wedge \text{selected A})\, \mathbf{P}(\text{selected A} \mid \text{behind B})\, \mathbf{P}(\text{behind B}) + \mathbf{P}(\text{shown B} \mid \text{behind C} \wedge \text{selected A})\, \mathbf{P}(\text{selected A} \mid \text{behind C})\, \mathbf{P}(\text{behind C})$
$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{6}.$

Given this evidence, what is the probability that the car is behind door A, i.e., $\mathbf{P}(\text{behind A} \mid \text{selected A} \wedge \text{shown B})$? What is the probability that the car is behind door B, $\mathbf{P}(\text{behind B} \mid \text{selected A} \wedge \text{shown B})$? What is the probability that the car is behind door C, $\mathbf{P}(\text{behind C} \mid \text{selected A} \wedge \text{shown B})$?

$\mathbf{P}(\text{behind A} \mid \text{selected A} \wedge \text{shown B}) = \mathbf{P}(\text{selected A} \wedge \text{shown B} \wedge \text{behind A}) \,/\, \mathbf{P}(\text{selected A} \wedge \text{shown B}) = (\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} / \frac{1}{6}) = \frac{1}{3}.$

$\mathbf{P}(\text{behind B} \mid \text{selected A} \wedge \text{shown B}) = \mathbf{P}(\text{selected A} \wedge \text{shown B} \wedge \text{behind B}) \,/\, \mathbf{P}(\text{selected A} \wedge \text{shown B}) = (0 \cdot \frac{1}{3} \cdot \frac{1}{3} / \frac{1}{6}) = 0.$

$\mathbf{P}(\text{behind C} \mid \text{selected A} \wedge \text{shown B}) = \mathbf{P}(\text{selected A} \wedge \text{shown B} \wedge \text{behind C}) \,/\, \mathbf{P}(\text{selected A} \wedge \text{shown B}) = (1 \cdot \frac{1}{3} \cdot \frac{1}{3} / \frac{1}{6}) = \frac{2}{3}.$

Question 1.c: After opening door B, Monty gives you the opportunity to switch doors, to open door C instead of door A. Should you? What is the success rate of someone who always sticks with their original door? What is the success rate of someone who always switches to the remaining door?

Yes, you should always switch. The success rate of someone who always sticks with their original door is 33.33%, while the success rate of someone who always switches to the remaining door is 66.67%.

Question 1.d: How does your analysis change if you only selected door A initially with probability 1/4?

It does not change - you should still switch. (*Why?*)

Question 2 - CoinBot

You are given a mysterious coin. It is either Coin A, which has a probability of 0.4 of giving heads and 0.6 of giving tails, or it is Coin B, which has a probability of 0.7 of giving heads and 0.3 of giving tails. Initially, you have no reason to think that either coin is more likely than the other. However, by flipping the coin multiple times, you can use the data that you collect to reason about which coin it is more likely to be.

Let $F_1, F_2, \cdots, F_n$ be the sequence of flips collected through n flips ($F_t = 0$ indicates that the t-th flip was tails, $F_t = 1$ indicates that the t-th flip was heads). We will define two functions $p_A(n) = P(Coin = A|F_1, \cdots, F_n)$ and $p_B(n) = P(Coin = B|F_1, \cdots, F_n)$ to denote our beliefs at any given point in time.

Question 2.1. Argue that $p_A(n) + p_B(n) = 1$.

For any n, the experiment has only two possibilities: that the coin we are tossing is Coin A, or it is Coin B. Hence, our beliefs after n flips are either that the coin we are tossing is Coin A, or it is Coin B, with probabilties $p_A(n)$ and $p_B(n)$ respectively. Hence, $p_A(n) + p_B(n) = 1$.

Question 2.2. Argue that for any n, $p_A(n)$ and $p_B(n)$ depend only on the **number** of heads and tails recorded rather than the specific **order** of flips collected. Provide an explicit formula for both $p_A(n)$ and $p_B(n)$ in terms of the respective probabilities and the total number of heads recorded in n flips, $Heads(n)$. You will find that writing some probabilities in terms of $Heads(n)$ is particularly useful.

Let us consider $p_A(n)$. The treatment is the same for $p_B(n)$. In the following steps (and in probability calculations in general), since $0 \leq P(X) \leq 1$ for any event X, we can write $\alpha = 1/P(X)$.

$$\begin{aligned}
p_A(n) &= P(Coin = A|F_1, \cdots F_n) \\
&= \frac{P(X, F_1, \cdots, F_n)}{P(F_1, \cdots, F_n)} && \text{(X is the event that Coin = A)} \\
&= \alpha P(X, F_1, \cdots, F_n) \\
&= \alpha P(X)P(F_1|X)P(F_2|X, F_1) \cdots P(F_n|X, F_1, \cdots F_{n-1}) \\
&= \alpha P(X)P(F_1|X)P(F_1|X) \cdots P(F_n|X) && \text{(Since every coin flip is independent of each other)}
\end{aligned}$$

From the last step, we can see that $p_A(n)$ (or $p_B(n)$) for that matter) do not depend on the **order** of the flips, but rather, the **number** of heads and tails recorded. Since we initially have no reason to think that either coin is more likely than the other, we can assume that our **prior** probability that $P(Coin = A) = P(Coin = B) = 0.5$. Using a neat little distribution called the **binomial distribution**, we have:

$$p_A(n) = \alpha \times 0.5 \times \binom{n}{Heads(n)} \times (0.4)^{Heads(n)} \times (0.6)^{n-Heads(n)} = \beta(0.4)^{Heads(n)}(0.6)^{n-Heads(n)}.$$

$$p_B(n) = \alpha \times 0.5 \times \binom{n}{Heads(n)} \times (0.7)^{Heads(n)} \times (0.3)^{n-Heads(n)} = \beta(0.7)^{Heads(n)}(0.3)^{n-Heads(n)}.$$

Given a set of data $F_1, \cdots, F_n$, we can construct a decision rule for guessing what the coin is: Let $Guess(n) = A$ if $p_A(n) \geq p_B(n)$ or take $Guess(n) = B$ if $p_A(n) < p_B(n)$.

Question 2.3. What is $P(Guess(n) = B|CoinA)$? That is, if the coin were *actually* A, what is the probability that after n flips, we would guess that the coin is B? What is $P(Guess(n) = A|Coin = B)$? Report your answers for $n = 5, 10, 100$.

We will now compute the ratio between $p_A(n)$ and $p_B(n)$. Our guess will be B if this ratio is less than 1.

$$\begin{aligned}
\frac{p_A(n)}{p_B(n)} &= \frac{\beta(0.4)^{Heads(n)}(0.6)^{n-Heads(n)}}{\beta(0.7)^{Heads(n)}(0.3)^{n-Heads(n)}} < 1. \\
&= (\frac{4}{7})^{Heads(n)}(2)^{n-Heads(n)} < 1. \\
&= Heads(n)[\ln 4 - \ln 7] + (n - Heads(n)) \ln 2 < 0. \\
&= -1.25 Heads(n) + 0.69n < 0. \\
&\implies Heads(n) > 0.55n. && \text{(When we guess that Coin is B.)}
\end{aligned}$$

For $n = 5, 0.55n = 2.75$. Now we have:

$$\begin{aligned}
P(Guess(n) = B | Coin = A) &= P(p_B(n) > p_A(n) | Coin = A) \\
&= \frac{P(Coin = A \wedge (p_B(n) > p_A(n)))}{P(Coin = A)} \\
&= 2 \sum_{k=3}^{5} Heads(k) = 2 \sum_{k=3}^{5} \binom{5}{k} (0.4)^k (0.6)^{5-k} \\
&= 0.6348.
\end{aligned}$$

Similarly, we can show that $P(Guess(n) = A | Coin = B) = 2\sum_{k=0}^{2} Heads(k) = 2\sum_{k=0}^{2} \binom{5}{k}(0.7)^k(0.3)^{5-k} = 0.3262$. $n = 10, 100$ *cases are left as an exercise to you. Do you expect the probabilities of taking a wrong guess to increase or decrease with more flips?*

Question 2.4. What is $P(Guess(n) \neq ActualCoin)$? That is, what is the probability that after n flips, your guess is wrong? Give answers for $n = 5, 10, 100$.

We will now marginalize over the identity of the coin.

$$\begin{aligned}
P(Guess(n) \neq ActualCoin) &= P(Guess(n) \neq ActualCoin \wedge Coin = A) + P(Guess(n) \neq ActualCoin \wedge Coin = B) \\
&= P(Guess(n) = B | Coin = A) P(Coin = A) + P(Guess(n) = A | Coin = B) P(Coin = B) \\
&= 0.5 \times (0.6348 + 0.3262) = 0.4805.
\end{aligned}$$

$n = 10, 100$ *cases are left as an exercise to you. Do you expect the probabilities of taking a wrong guess to increase or decrease with more flips?*

Question 2.5. What is the smallest n such that $P(Guess(n) \neq ActualCoin) \leq 0.1$?

For different values of n, write a small script that will compute $P(Guess(n) \neq ActualCoin)$ for different values of n, using the computations provided in Questions 2.3 and 2.4 as a starting point. Plot the value of $P(Guess(n) \neq ActualCoin)$ for different values of n, and see when the value dips below 0.1. You should see this happening around $n = 30$.

Question 2.6. What is the probability that you guessed wrong *after actually having performed n flips*? i.e., what is $P(Guess(n) \neq ActualCoin | F_1, \cdots, F_n)$?