CS 440 Homework Assignment 3 Part 2
Probabilistic Reasoning

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Problem 1

A	B	C	D	E	F

$X_t = (A, B, C, D, F, F)$

$E_t = (Hot, nothot)$

Transition Model
$P(X_{t+1}|X_t)$

	A	B	C	D	E	F
A	0.2	0.8	0	0	0	0
B	0	0.2	0.8	0	0	0
C	0	0	0.2	0.8	0	0
D	0	0	0	0.2	0.8	0
E	0	0	0	0	0.2	0.8
F	0	0	0	0	0	1

Observational Model
$P(E_t|X_t)$

	hot	cold
A	1	0
B	0	1
C	0	1
D	1	0
E	0	1
F	0	1

$$P(X_1)= \begin{bmatrix} P(X_1) = A \\ P(X_1) = B \\ P(X_1) = C \\ P(X_1) = D \\ P(X_1) = E \\ P(X_1) = F \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1) Filtering $E_1 = h, E_2 = C, E_3 = C$

$\mathbf{P}(X_3|h1, C_1, C_2) = ?$

For this question we will use this equation,

P$(X_t|E_{1:t}) = \alpha * P(E_t|X_t) \sum_{X_{t-1}} P(X_t|X_{t-1}) * P(X_{t-1}|E_{1:t-1})$

P$(X_1|E_1 = h)$=P(X_1)= $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (given)

P$(X_2|E_1 = h, C_2 = C)$=$\alpha * P(E_2|X_2) * \sum_{X_1=(A,B,C,D,E,F)} P(X_2|X_1) * P(X_1|E_1)$

X_1 is 1 at A and 0 at rest. So, we need to only sum over A

$\sum_{X_1} P(X_2|X_1) * P(X_1|E_1)$= $\begin{bmatrix} 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $*$1+0= $\begin{bmatrix} 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Now as sensor is always accurate

P$(E_2 = C|X_2)$= $\begin{bmatrix} P(E_2 = C|X_2 = A) \\ P(E_2 = C|X_2 = B) \\ P(E_2 = C|X_2 = C) \\ P(E_2 = C|X_2 = D) \\ P(E_2 = C|X_2 = E) \\ P(E_2 = C|X_2 = F) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

P$(X_2|E_1 = h, E_2 = C) = \alpha * \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} * \begin{bmatrix} 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \alpha * \begin{bmatrix} 0 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\alpha = \frac{1}{0.8} = 1.25$

therefore,

$$P(X_2|E_1 = h, E_2 = C) = 1.25 * \begin{bmatrix} 0 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$P(X_3|e_1 = h, E_2 = C, E_3 = C)$
$= \alpha * P(E_3|X_3) * \sum_{X_2=(A,B,C,D,E,F)} P(X_3|X_2) * P(X_2|E_1, E_2)$

$Now, \mathrm{E}_3 = C$

therefore,

$$P(E_3 = C|X_3) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

X_2 is only 1 at B and 0 rest so we only need to sum over B.

$$\sum_{X_2=B} P(X_3|X_2) * P(X_2|E_1, E_2) = 0 + \begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 = \begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore

$$P(X_3|h_1, C_2, C_3) = \alpha * \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} * \begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \alpha * \begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \text{Answer} = \begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\alpha = \frac{1}{0.8+0.2} = 1$

2) Smoothing: $\mathbf{P}(X_2|h_1,C_2,C_3)$

The equation we will use for this question is

P$(X_k|E_{1:t}) = \alpha * P(E_{k+1:t}|X_k).P(X_k|E_{1:k})$

$Therefore,$
$P(\mathrm{X}_2|h_1,C_2,C_3) = \alpha * P(E_{3:3}|X_2).P(X_2|E_{1:2})$
$Andalso,$
$P(\mathrm{E}_{t+1:t}|X_t) = \sum_{X_{t+1}} *P(X_{t+1}|X_t) * P(E_{t+1}|X_{t+1}) * P(E_{t+2:t}|X_{t+1})$
Therefore
$P(E_{3:3} = cold|X_2) = \sum_{X_3} *P(X_3|X_2) * P(E_3|X_3) * P(E_{4:3}|X_3)$
Here,
t = T-1 = 3-1 = 2
Therefore, we can set
$P(E_{4:3}|X_3) = 1$

Also, $E_3 = C$ and therefore $P(E_3 = C|X_3)$= $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ (From Observational Model)

Therefore,
P$(E_{3:3} = C|X_2) = \sum_{X_3} P(X_3|X_2) * P(E_3|X_3) * 1$

$$= \begin{bmatrix} P(X_3 = B|X_2 = A) + P(X_3 = C|X_2 = A) + P(X_3 = E|X_2 = A) + P(X_3 = F|X_2 = A) \\ P(X_3 = B|X_2 = B) + P(X_3 = C|X_2 = B) + P(X_3 = E|X_2 = B) + P(X_3 = F|X_2 = B) \\ P(X_3 = B|X_2 = C) + P(X_3 = C|X_2 = C) + P(X_3 = E|X_2 = C) + P(X_3 = F|X_2 = C) \\ P(X_3 = B|X_2 = D) + P(X_3 = C|X_2 = D) + P(X_3 = E|X_2 = D) + P(X_3 = F|X_2 = D) \\ P(X_3 = B|X_2 = E) + P(X_3 = C|X_2 = E) + P(X_3 = E|X_2 = E) + P(X_3 = F|X_2 = E) \\ P(X_3 = B|X_2 = F) + P(X_3 = C|X_2 = F) + P(X_3 = E|X_2 = F) + P(X_3 = F|X_2 = F) \end{bmatrix}$$

$$= \begin{bmatrix} 0.8+0+0+0 \\ 0.2+0.8+0+0 \\ 0+0.2+0+0 \\ 0+0+0+0.8+0.0 \\ 0+0+0.8+0.2 \\ 0+0+0+1 \end{bmatrix} = \mathrm{P}(\mathrm{E}_{3:3} = C|X_2) = \begin{bmatrix} 0.8 \\ 1 \\ 0.2 \\ 0.8 \\ 1 \\ 1 \end{bmatrix}$$

And $P(X_2|E_1 = h, E_2 = C)$= $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (From Question 1)

Therefore,

$$P(X_2|E_{1:3}) = \alpha * \begin{bmatrix} 0.8 \\ 1 \\ 0.2 \\ 0.8 \\ 1 \\ 1 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \text{Answer} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

3) Prediction problem P($h_4|h_1,c_2,c_3$)

For this question we will following equations,

P($E_{t+1}|E_{1:t}$)=$\sum_{X_t} P(\mathrm{X}_t|E_{1:t}) * P(E_{t+1}|X_t)$

and

P($E_{t+1}|X_t$)=$\sum_{X_{t+1}} P(\mathrm{E}_{t+1}|X_{t+1}) * P(X_{t+1}|X_t)$

P($h_4|h_1,c_2,c_3$)=$\sum_{X_3} P(\mathrm{X}_3|h_1,c_2,c_3) * P(h_4|X_3)$

Here

P($X_3|h_1,c_2,c_3$)=$\begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (From Question 1)

and

P($h_4|X_3$)=$\sum_{X_4} P(h_4|X_4) * P(X_4|X_3)$

$$= \begin{bmatrix} \sum_{X_4} P(h_4|X_4) * P(X_4|X_3 = A) \\ \sum_{X_4} P(h_4|X_4) * P(X_4|X_3 = B) \\ \sum_{X_4} P(h_4|X_4) * P(X_4|X_3 = C) \\ \sum_{X_4} P(h_4|X_4) * P(X_4|X_3 = D) \\ \sum_{X_4} P(h_4|X_4) * P(X_4|X_3 = E) \\ \sum_{X_4} P(h_4|X_4) * P(X_4|X_3 = F) \end{bmatrix}$$

$$= \begin{bmatrix} 1*0.2+0*0.8+0*0+1*0+0*0+0*0 \\ 1*0+0*0.2+0*0.8+1*0+0*0+0*0 \\ 1*0+0*0+0*0.2+1*0.8+0*0+0*0 \\ 1*0+0*0+0*0+1*0.2+0*0.8+0*0 \\ 1*0+0*0+0*0+1*0+0*0.2+0*0.8 \\ 1*0+0*0+0*0+1*0+0*0+0*1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 \\ 0 \\ 0.8 \\ 0.2 \\ 0 \\ 0 \end{bmatrix} = \mathrm{P}(h_4|X_3)$$

Now, P$(h_4|h_1, c_2, c_3) = \sum_{X_3}(X_3|h_1, c_2, c_3) * P(h_4|X_3)$
= 0*0.2+0.2*0+0.8*0.8+0*0.2+0*0+0*0

= 0.64 = Answer

4) Prediction P($X_4|h_1,C_2,C_3$)

For this question we will use the equation
P($X_{t+1}|E_{1:t}$)= P($X_t|E_{1:t}$)*P($X_{t+1}|X_t$)

Therefore,
P($X_4|h_1,C_2,C_3$)= P($X_3|h_1,C_2,C_3$)*P($X_4|X_3$)

Here,

$$P(X_3|h_1,C_2,C_3)=\begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sum_{X_3} P(X_3|h_1,C_2,C_3) * P(X_4|X_3)$$

$$\begin{bmatrix} \sum_{X_3} P(X_4=A|X_3) * P(X_3|E_{3:1}) \\ \sum_{X_3} P(X_4=B|X_3) * P(X_3|E_{3:1}) \\ \sum_{X_3} P(X_4=C|X_3) * P(X_3|E_{3:1}) \\ \sum_{X_3} P(X_4=D|X_3) * P(X_3|E_{3:1}) \\ \sum_{X_3} P(X_4=E|X_3) * P(X_3|E_{3:1}) \\ \sum_{X_3} P(X_4=F|X_3) * P(X_3|E_{3:1}) \end{bmatrix}$$

$$=\begin{bmatrix} 0+0+0+0+0+0 \\ 0+0.2*0.2+0+0+0+0 \\ 0+0.2*0.2+0.8*0.2+0+0+0 \\ 0+0+0.8*0.8+0+0+0 \\ 0+0+0+0+0+0 \\ 0+0+0+0+0+0 \end{bmatrix}$$

$$= \text{Answer} = \begin{bmatrix} 0 \\ 0.04 \\ 0.32 \\ 0.64 \\ 0 \\ 0 \end{bmatrix}$$

5) P($h_4, h_5, c_6|h_1, c_2, c_3$)

For this question we will use this equation

P($E_{t+1:T}|E_{1:t}) = \sum_{X_t} P(X_t|E_{1:t}) * P(E_{t+1:T}|X_t)$

Here t=3, T=6 and t+1=4

P($E_{4:6}|E_{1:3}) = \sum_{X_3} P(X_3|E_{1:3}) * P(E_{4:6}|X_3)$

P($X_3|h_1, c_2, c_3$)= $\begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ From Question 1

Now

P($E_{t+1:T}|X_t) = \sum_{X_{t+1}} P(X_{t+1}|X_t) * P(E_{t+1}|X_{t+1}) * P(E_{t+2:T}|X_{t+1})$

Therefore

P($E_{4:6}|X_3) = \sum_{X_4} P(X_4|X_3) * P(E_4|X_4) * P(E_{5:6}|X_4)$

P($E_{5:6}|X_4) = \sum_{X_5} P(X_5|X_4) * P(E_5|X_5) * P(E_{6:6}|X_5)$

P($E_{6:6}|X_5) = \sum_{X_6} P(X_6|X_5) * P(E_6|X_6) * P(E_{7:6}|X_5)$

We start by setting $P(E_{7:6}|X_6) = 1$

Now,

P($E_{6:6} = cold|X_5) = \sum_{X_6} P(X_6|X_5) * P(C_6|X_6) * 1$

$$= \begin{bmatrix} \sum_{X_6} P(X_6|X_5 = A) * P(C_6|X_6) \\ \sum_{X_6} P(X_6|X_5 = B) * P(C_6|X_6) \\ \sum_{X_6} P(X_6|X_5 = C) * P(C_6|X_6) \\ \sum_{X_6} P(X_6|X_5 = D) * P(C_6|X_6) \\ \sum_{X_6} P(X_6|X_5 = E) * P(C_6|X_6) \\ \sum_{X_6} P(X_6|X_5 = F) * P(C_6|X_6) \end{bmatrix} = \begin{bmatrix} 0+0.8+0+0+0+0 \\ 0+0.2+0.8+0+0+0 \\ 0+0+0.2+0+0+0 \\ 0+0+0+0+0.8+0 \\ 0+0+0+0+0.2+0.8 \\ 0+0+0+0+0+1 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 1 \\ 0.2 \\ 0.8 \\ 1 \\ 1 \end{bmatrix}$$

Now, P($E_{5:6}|X_4$)

$$= \begin{bmatrix} \sum_{X_5} P(X_5|X_4=A)*P(E_5=h|X_5)*P(E_{6:6}X_5) \\ \sum_{X_5} P(X_5|X_4=B)*P(E_5=h|X_5)*P(E_{6:6}X_5) \\ \sum_{X_5} P(X_5|X_4=C)*P(E_5=h|X_5)*P(E_{6:6}X_5) \\ \sum_{X_5} P(X_5|X_4=D)*P(E_5=h|X_5)*P(E_{6:6}X_5) \\ \sum_{X_5} P(X_5|X_4=E)*P(E_5=h|X_5)*P(E_{6:6}X_5) \\ \sum_{X_5} P(X_5|X_4=F)*P(E_5=h|X_5)*P(E_{6:6}X_5) \end{bmatrix} = \begin{bmatrix} 0.2+0.8+0+0+0+0 \\ 0+0+0+0+0+0 \\ 0+0+0+0.8*0.8+0+0 \\ 0+0+0+0.2*0.8,0,0 \\ 0+0+0+0+0+0 \\ 0+0+0+0+0+0 \end{bmatrix} = \begin{bmatrix} 0.16 \\ 0 \\ 0.64 \\ 0.16 \\ 0 \\ 0 \end{bmatrix}$$

Finally, P($E_{4:6}|X_3$)

$$= \begin{bmatrix} \sum_{X_4} P(X_4|X_3=A)*P(E_4=h|X_4)*P(E_{5:6}X_4) \\ \sum_{X_4} P(X_4|X_3=B)*P(E_4=h|X_4)*P(E_{5:6}X_4) \\ \sum_{X_4} P(X_4|X_3=C)*P(E_4=h|X_4)*P(E_{5:6}X_4) \\ \sum_{X_4} P(X_4|X_3=D)*P(E_4=h|X_4)*P(E_{5:6}X_4) \\ \sum_{X_4} P(X_4|X_3=E)*P(E_4=h|X_4)*P(E_{5:6}X_4) \\ \sum_{X_4} P(X_4|X_3=F)*P(E_4=h|X_4)*P(E_{5:6}X_4) \end{bmatrix} = \begin{bmatrix} 0.2*0.16+0+0+0+0+0 \\ 0+0+0+0+0+0 \\ 0+0+0+0.8*0.16+0+0 \\ 0+0+0+0.2*0.16,0,0 \\ 0+0+0+0+0+0 \\ 0+0+0+0+0+0 \end{bmatrix} = \begin{bmatrix} 0.032 \\ 0 \\ 0.128 \\ 0.032 \\ 0 \\ 0 \end{bmatrix}$$

and now,

P($E_{4:6}|E_{1:3}) = \sum_{X_3} P(x_3|E_{1:3}) * P(E_{4:6}|X_3)$

= 0*0.032+0.2*0+0.8*0.128+0*0.032+0*0+0*0

= 0.1024 = Answer

Problem 2:

s	a	$s^{'}$	$T(S,a,s^{'})$
A	1	A	1
A	1	B	0
A	2	A	0.5
A	2	B	0.5
B	1	A	0
B	1	B	1
B	2	A	0
B	2	B	1

s	a	R(s,a)
A	1	0
A	2	-1
B	1	5
B	2	0

$\gamma = 1$

1) Bellman's Equation

$V^{\pi}(s)=R(s,\pi(s))+\gamma\sum_{s' \varepsilon States} T(s,\pi(s),s^{'})V^{\pi}(s^{'})$

2) $\pi(A) = 1$ **and** $\pi(B) = 1. As \gamma = 1$**, we will not be including it in further calculations.**

State	V_0^π	V_1^π
A	**0**	**R(A,1)+T(A,1,A)***V_0^π**(A)+T(A,1,B)***V_0^π**(B)** **=0+1*0+0*0** **=0**
B	**0**	**R(B,1)+T(B,1,A)***V_0^π**(A)+T(B,1,B)***V_0^π**(B)** **=5+0*0+1*0** **=5**
State	V_1^π	V_2^π
A	**0**	**R(A,1)+T(A,1,A)***V_1^π**(A)+T(A,1,B)***V_1^π**(B)** **=0+1*0+0*5** **=0**
B	**5**	**R(B,1)+T(B,1,A)***V_1^π**(A)+T(B,1,B)***V_1^π**(B)** **=5+0*0+1*5** **=10**

$\mathbf{V}_2^\pi(A) = 0$
$V_2^\pi(B) = 0$

3) We need to find $Q^\pi(A,1), Q^\pi(A,2), Q^\pi(B,1)$ **and** $Q^\pi(B,2)$

$Q^\pi(A,s)$=**R(s,a)+**$\gamma \sum_{s' \varepsilon S} T(s,a,s^1)V^\pi s^1$
$Q^\pi(A,1)$=$R(A,1) + 1 * [T(A,1,A) * V^\pi(A) + T(A,1,B) * V^\pi(B)]$
=0+1*[1*0+0*10]
=0

$Q^\pi(A,2)$=$R(A,2) + 1 * [T(A,2,A) * V^\pi(A) + T(A,2,B) * V^\pi(B)]$
=-1+(0.5*0+0.5*10)
=4

$Q^\pi(B,1)$=$R(B,1) + 1 * [T(B,1,A) * V^\pi(A) + T(B,1,B) * V^\pi(B)]$
=5+1*(0*0+1*10)
=15

$Q^\pi(B,2)$=$R(B,2) + 1 * [T(B,2,A) * V^\pi(A) + T(B,2,B) * V^\pi(B)]$
=0+1*(0*0+1*10)
=10

$\pi_{new}(A) = 2$ **and** $\pi_{new}(B) = 1$