Chapter 7

Duality for Linear Programming

7.1 Optimality condition for LP

When $f(x) = c^T x$, problems (P-POL), (P-ST), (P-GEN) studied in Section 5.9 are Linear programming problems, so that convex. Hence the KKT are necessary and sufficient for optimality. COnsider the LP

$$\begin{aligned} \min \quad & c^T x \\ & Ax \geq b \end{aligned} \tag{7.1}$$

The Lagrangian function is

$$L(x, \lambda) = c^T x + \lambda^T (b - Ax) = b^T \lambda + (c^T - \lambda^T A)x$$

Theorem 7.2 (KKT conditions for LP (7.1)**)** *A point x^* is a global solution of the LP* (7.1) *if and only if multipliers $\lambda^* \in \mathbb{R}^m$ exists such that*

(i) $Ax^* \geq b$,

(ii) $c = A^T \lambda^*$,

(iii) $\lambda^* \geq 0$,

(iv) $\lambda^{*T} (b - Ax^*) = 0$.

Condition (i) involves only varaibles x. Conditions (ii) e (iii) involves only the multipliers λ and they express feasibility of $\lambda^* \in \mathbb{R}^m$ with respect to the linear constraints

$$\begin{aligned} & A^T \lambda = c \\ & \lambda \geq 0 \end{aligned} \tag{7.2}$$

Consider now a feasible $x \in \mathbb{R}^n$ for problem (7.1), and a feasible $\lambda \in \mathbb{R}^m$ with respect to (7.2), namely a pair (x, λ) such tht

$$Ax \geq b, \quad \lambda \geq 0, \; A^T \lambda = c. \tag{7.3}$$

Since

$$\lambda^T(b - Ax) = \sum_{i=1}^{m} \overbrace{\lambda_i}^{\geq 0}(\overbrace{b_i - a_i^T x}^{\leq 0}) \leq 0$$

we can write

$$\begin{aligned} 0 &\geq \lambda^T(b - Ax) \\ &= b^T\lambda - \lambda^T Ax \\ &\overset{(A^T\lambda=c)}{=} b^T\lambda - c^T x. \end{aligned}$$

> for each pair (x, λ) satisfying $Ax \geq b,\ \lambda \geq 0,\ A^T\lambda = c$ we get
>
> $$c^T x \geq b^T\lambda.$$

In particular, since the global solution x^* is feasible for problem (7.1), it holds

$$c^T x^\star \geq b^T\lambda \tag{7.4}$$

for each λ satisfying (7.2). So that $b^T\lambda$ represents a lower bound to the optial value of the objective function.
We can improve the lower bound expressed by inequality (7.4) by solving the problem

$$\begin{aligned} \max \quad & b^T\lambda \\ & A^T\lambda = c \\ & \lambda \geq 0 \end{aligned} \tag{7.5}$$

The pair of LP problems (7.1) and (7.5) is called *primal-dual* pair.

> **(Primal and dual problems)**
>
> The problems
>
> $$\begin{aligned} \min \quad & c^T x \\ & Ax \geq br \end{aligned} \tag{7.6}$$
>
> $$\begin{aligned} \max \quad & b^T\lambda \\ & A^T\lambda = c \\ & \lambda \geq 0 \end{aligned} \tag{7.7}$$
>
> are a *primal-dual* pair of LPs.

The primal (P) - dual (D) pair (7.6) and (7.7) is only an example of primal-dual pairs. The relatinship among primal-dual objective functions in feasible points hold for any pair of primal-dual problems and it is known as Weak Duality Theorem.

Theorem 7.3 (Weak Duality Theorem) *Given any pair of primal-dual LP problems. For any feasible $\bar{x}$ of the primal LP (P) and any feasible $\bar{\lambda}$ of the dual problems (D) we have*

$$b^T \bar{\lambda} \leq c^T \bar{x}$$

Theorem 7.4 (Strong duality theorem) *The primal LP* (7.6) *hs a solution $x^* \in \mathbb{R}^n$ if and only if the ddual LP (7.7) has a solution $\lambda^* \in \mathbb{R}^m$ and it holds*

$$c^T x^\star = b^T \lambda^\star.$$

Proof. Since both problems are convexthe KKT ocnditions are necessary and sufficient for optimality. We prove that the KKT fot the primal problems coincide with the KKT for the dual.

The KKT conditions for the primal are given in Theorem 7.2. Condition (iv) can be written in an equivalent form using (ii). Indeed using $(A^T \lambda^*)^T = \lambda^{*T} A$ we get

$$0 \overset{\text{(iv)}}{=} \lambda^{*T}(b - Ax^*) = \lambda^{*T} b - \lambda^{*T} A x^* \overset{\text{(ii)}}{=} \lambda^{*T} b - x^{*T} c.$$

Hence the KKT for the primal are

(i) $Ax^* \geq b$,

(ii) $c = A^T \lambda^*$,

(iii) $\lambda^* \geq 0$,

(iv) $b^T \lambda^* = c^T x^*$.

Let us consider the dual problem (7.5). We consider multipliers $z \in \mathbb{R}^n$ for the constraints $A^T \lambda = c$ and multipliers $v \in \mathbb{R}^m$ for the constraints $\lambda \geq 0$. The Lagrangian function for the dual is

$$L_D(\lambda, z, v) = -b^T \lambda + z^T (A^T \lambda - c) - v^T \lambda.$$

The KKT conditions are

(a) $A^T \lambda^* = c$, $\lambda^* \geq 0$,

(b) $-b + Az^* - v^* = 0$,

(c) $v^* \geq 0$,

(d) $v^{*T} \lambda^* = 0$.

Condition (a) gives immediately (i) and (ii).

From condition (b) we get

$$-b + Az^* = v^*.$$

Using (c) $v^* \geq 0$ we get

$$-b + Az^* \geq 0.$$

Substituting in (d) we obtain

$$0 = v^{*T}\lambda^* = (-b + Az^*)^T\lambda^* = -b^T\lambda^* + z^{*T}A^T\lambda^* = -b^T\lambda^* + z^{*T}c$$

Letting $z^* = x^*$ we get (iii) and (iv). Hence KKT conditions for primal and dual coincide. $\square$

Fron the weak duality theorem we get:

(Lower and upper bounds on the optimal value of LP)

Let x^* be a globalsolution of the LP. For all $(\tilde{x}, \tilde{\lambda})$ such that $Ax \geq b,\ \lambda \geq 0,\ A^T\lambda = c$, we have

$$b^T\tilde{\lambda} \leq c^T x^* \leq c^T\tilde{x}.$$

We also have this result

Corollary 7.5 *If $\bar{x}$ is a feasible solution for primal (P) and $\bar{\lambda}$ is a feasible solution for dual (D) such that*

$$c^T\bar{x} = b^T\bar{\lambda} \tag{7.8}$$

then $\bar{x}$ e $\bar{\lambda}$ are optimal solutions of (P) and (D).

We can also prove the following

Theorem 7.6 (Unboundedness condition) *If primal problem (P) in unbounded (below) then the dual problem (D) is unfeasible. Vice versa if the dual problem (D) in unbounded (above) then the primal problem (P) is unfeasible.*

Proof. Assume that (P) in unbounded (below) and by contradiction let $\bar{\lambda}$ be a feasible solution of the dual (D). then by the weak duality theorem 7.3, we have $c^T x \geq b^T\bar{\lambda}$ for any primal feasible point x. But this contradicts that (P) is unbounded below.

The fundamental theorem of LP (see Theorem 8.16 in Chapter 8) states that in a LP problem we can have only one of the three cases:

(i) the problem admits an optimal solution,

(ii) the problem is unbounded

(iii) the problem in unfeasible.

Using duality results we can have only these cases

		DUAL		
		OPTIMAL SOLUTION	UNBOUNDED ABOVE	UNFEASIBLE
PRIMAL	OPTIMAL SOLUTION	YES	NO	NO
	UNBOUNDED BELOW	NO	NO	YES
	UNFEASIBLE	NO	YES	YES

7.7 Construction of the dual

Let define other primal-dual LP pairs.

$$\begin{array}{llcll} \min & c^T x & \qquad & \max & b^T \lambda \\ & Ax \geq b & & & A^T \lambda \leq c \\ & x \geq 0 & & & \lambda \geq 0 \end{array}$$

It is possible to give simple schematic rules to derive dual formulation from a given primal problem.

Consider the LP

$$\begin{array}{ll} \min & c_1^T y + c_2^T z \\ & Cy + Dz = h \\ & Ey + Fz \geq g \\ & y \geq 0 \end{array} \tag{7.9}$$

where the variables are $(y,\ z)^T \in \mathbb{R}^n$ with $y \in \mathbb{R}^{n_1}$ e $z \in \mathbb{R}^{n_2}$ con $n = n_1 + n_2$ and $c_1 \in \mathbb{R}^{n_1}$, $c_2 \in \mathbb{R}^{n_2}$. Variables $y \geq 0$ while z has no restriction on the sign.

Matrices C, D, E, F have dimensione respectively $p \times n_1$, $p \times n_2$, $q \times n_1$ and $q \times n_2$. The Lp problem has p equality constraints $Cy + Dz = h$ with $h \in \mathbb{R}^p$ e q inequality constraints $Ey + Fz \geq g$ with $g \in \mathbb{R}^q$.

REMARK

When using automatic rules for the construction of the dual problem, it is important to check the type of optimization min or max and the type of inequality $\geq$ o $\leq$. When a min problem is considered the inequality must be in the form of $\geq$; vice versa with a max problem the inequality must be in the form of $\leq$.

The dual problem is

$$\begin{array}{ll} \max & h^T\mu + g^T w \\ & C^T\mu + E^T w \leq c_1 \\ & D^T\mu + F^T w = c_2 \\ & w \geq 0 \end{array} \tag{7.10}$$

Remark

Primal variables (x, y) correspond to multipliers of the linear constraints of the dual problem. The dual variables (μ, w) correspond to multipliers of the linear constraints of the primal problem.

Some simple rule for writing a dual problem:

- the dual problem of a min primal problem is a maximization and analogously the dual problem of a max is a minimization problem;
- a dual variable associated to an equality primal constraint have no constraints on the sign;
- a dual variable associated to an inequality primal constraint must be non negative;
- non negative primal variables correspond to inequality constraints in the dual ($\leq$ if the dual is a max, $\geq$ if the dual is a min);
- primal variables without sign correspond to equality constraints in the dual.

We summarize in the table:

	PRIMAL	**DUAL**	
	min $c^T x$	max $b^T u$	
CONSTRAINTS	$= b_i, \quad i \in I$	$u_i, \quad i \in I$, libere	VARIABLES
	$\geq b_i, \quad i \in J$	$u_i, \quad i \in J, u_i \geq 0$	
VARIABLES	$x_j \geq 0, \quad j \in M$	$\leq c_j, \quad j \in M$	CONSTRAINTS
	$x_j, \quad j \in N$ libere	$= c_j, \quad j \in N$	

Example 7.8 *Let the primal be*

$$\begin{array}{ll} \max & 4x_1+3x_2+2x_3 \\ & x_1+2x_2+3x_3 \le 8 \\ & 2x_1-x_3 \le 7 \\ & 3x_1+4x_2-x_3 \le 5 \\ & x_2+x_3 \le 6 \\ & x_2 \ge 0 \end{array}$$

The dual is

$$\begin{array}{ll} \min & 8u_1+7u_2+5u_3+6u_4 \\ & u_1+2u_2+3u_3 = 4 \\ & 2u_1+4u_3+u_4 \ge 3 \\ & 3u_1-u_2-u_3+u_4 = 2 \\ & u_1 \ge 0, u_2 \ge 0, u_3 \ge 0, u_4 \ge 0. \end{array}$$

□

Example 7.9 *Let the primal be*

$$\begin{array}{ll} \min & 2x_1-3x_2+x_3 \\ & 3x_1+x_2+5x_3 \ge 7 \\ & x_1+x_2-6x_3 \le 9 \\ & 4x_1-x_2-2x_3 = 8 \\ & x_1 \ge 0, x_2 \ge 0. \end{array}$$

We change the constraint $-x_1-x_2+6x_3 \ge -9$ *and the dual is*

$$\begin{array}{ll} \max & 7u_1-9u_2+8u_3 \\ & 3u_1-u_2+4u_3 \le 2 \\ & u_1-u_2-u_3 \le -3 \\ & 5u_1+6u_2-2u_3 = 1 \\ & u_1 \ge 0, u_2 \ge 0. \end{array}$$

□

Example 7.10 *Let the primal be*

$$\begin{array}{ll} \min & x_1+3x_2 \\ & x_1+4x_2 \ge 24 \\ & 5x_1+x_2 \ge 25 \\ & x_1 \ge 0, x_2 \ge 0 \end{array}$$

Solving graphically the problem has a solution in $(x_1,x_2)=(4,5)$ *with value* 19. *The dual is*

$$\begin{array}{ll} \max & 24u_1+25u_2 \\ & u_1+5u_2 \le 1 \\ & 4u_1+u_2 \le 3 \\ & u_1 \ge 0, u_2 \ge 0; \end{array}$$

We cans olve geometrically and we obtain an optimal solution in the point $(u_1,u_2) = \left(\frac{14}{19},\frac{1}{19}\right)$ *with value vale* 19.

Example 7.11 *Let the primal be*

$$\begin{array}{ll} \max & 2x_1+3x_2 \\ & -2x_1+x_2 \leq 3 \\ & -\frac{1}{2}x_1+x_2 \leq 6 \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

Solving graphically we see that the problem is unbounded above.hence the dual must be unfeasible.Indeed the dual is

$$\begin{array}{ll} \min & 3u_1+6u_2 \\ & -2u_1-\frac{1}{2}u_2 \geq 2 \\ & u_1+u_2 \geq 3 \\ & u_1 \geq 0, u_2 \geq 0 \end{array}$$

which does not have feasible solution.

7.12 The conitnuous Knapsack problem

Let us consider the continuous Knapsack.

$$\begin{array}{ll} \max & c^T x \\ & a^T x \leq b, \\ & x \geq 0 \end{array}$$

where $x,a,c \in \mathbb{R}^n$, $c_i > 0$ e $0 < a_i < b$ for $i = 1,\ldots,n$ and b is a positive scalar.

The dual problem has a unique variable $y \in \mathbb{R}$

$$\begin{array}{ll} \min & by \\ & a_i y \geq c_i, \quad i=1,\ldots,n \\ & y \geq 0 \end{array}$$

Solving this problem is trivial. Indeed let

$$\frac{c_k}{a_k} = \max_{1\leq i\leq n}\left\{\frac{c_i}{a_i}\right\}$$

and

$$I_k = \left\{i : \frac{c_i}{a_i} = \frac{c_k}{a_k}\right\}.$$

A point y is dual feasible if and only if it satisfies $y \geq \frac{c_k}{a_k}$. Since is a minimizatin problem and $b > 0$, the optimal solution is $y^* = \frac{c_k}{a_k} > 0$.

To obtain the primal solution x^* we can use the complementarity conditions:

$$\begin{array}{ll} x_i^*(a_i y^* - c_i) = 0 & i = 1, \dots, n \\ y^*(b - a^T x^*) = 0 & \end{array}$$

For any $i \notin I_k$ we obtain $x_i^* = 0$. Further from the second we have $a^T x^* = b$ which can be written also as

$$\sum_{i \in I_k} a_i x_i^* = b.$$

If $I_k = \{k\}$, then there is a unique optimal solution which is

$$\begin{array}{l} x_k^* = \dfrac{b}{a_k}, \\ x_i^* = 0, \quad \text{per ogni} \quad i \neq k. \end{array}$$

Otherwise (if $|I_k| \geq 2$) the optimal solution exists but it is not unique; actually every solution satyisfying the constraint $\sum_{i \in I_k} a_i x_i^* = b$ is optimal. As an example

$$\begin{array}{l} x_j^* = \dfrac{b}{a_j}, \;\; \text{for one } j \in I_k \\ x_i^* = 0, \;\; \text{for all} \quad i \neq j \end{array}$$

Is optimal. □

Assume that we add bounds constraint

$$\begin{array}{ll} \max & c^T x \\ & a^T x \leq b, \\ & 0 \leq x \leq 1 \end{array}$$

and we assume that variables are ordered such that

$$\frac{c_1}{a_1} > \frac{c_2}{a_2} > \dots > \frac{c_n}{a_n}.$$

Under the assumption on the coefficients, the primal is not empty and bounded; hence there exists an optimal solution. We write the problem as

$$\begin{array}{lll} & \max & c^T x \\ (\lambda) & & a^T x \leq b, \\ (z_i) & & x \leq 1 \\ & & x \geq 0 \end{array}$$

and its dual problem is

$$\begin{array}{ll} \min & by + e^T z \\ & a_i y + z_i \geq c_i, \quad i = 1, \dots, n \\ & y \geq 0, \quad z_i \geq 0 \quad i = 1, \dots, n \end{array}$$

with $n+1$ variables $y \in \mathbb{R}$ e $z \in \mathbb{R}^n$,

Optimal primal and dual solution satisfy the complementarity conditions:

$$\begin{array}{ll} x_i^*(a_i y^* + z_i^* - c_i) = 0 & i = 1, \dots, n \\ y^*(b - a^T x^*) = 0 & \\ z_i^*(1 - x_i^*) = 0 & \end{array}$$

There exists a unique $k \in \{1, \dots, n\}$ such that $x_k^* \in (0, 1)$ (fractional component).

Then we have $y^* = \dfrac{c_k}{a_k} \neq 0$ and hence from complementarity $a^T x^* = b$.

Partitioning indeces $\{1, \dots, n\} \setminus \{k\}$ into $I_1 = \{i : \ x_i^* = 1\}$ and $I_0 = \{i : \ x_i^* = 0\}$, we can write $\sum_{i \in I_1} a_i x_i^* + \sum_{i \in I_0} a_i x_i^* + a_k x_k^* = b$, from which we get

$$x_k^* = \frac{b - \sum_{i \in I_1} a_i}{a_k}.$$

It is easy to show that $x_i^* = 1$ for $i < k$ and $x_i^* = 0$ for $i > k$. Index k is called *critical index* because it represents the item which leads to the violation of the capacity.

7.12.1 Sensitivity analysis

Both a primal and a dual problem have a intimate relationship as they both use the same data and one problem bounds the other. In this section, we explore possible economic meanings for dual linear programs and dual variables in relation to primal problems and primal variables.

7.12.2 A geometric point of view

Consider the primal-dual pair

$$(P) \quad \begin{array}{ll} \max & 2x_1 + 6x_2 \\ & x_1 + 2x_2 \leq 5 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array} \qquad \begin{array}{ll} \min & 5u_1 + 4u_2 \\ & u_1 + u_2 \geq 2 \\ & 2u_1 + u_2 \geq 6 \\ & u_1, u_2 \geq 0 \end{array} \quad (D)$$

Assume that the primal represents a simple production problem where x_1, x_2 are levels of production and the constraints are on availabilty of the resoucrces.

Graphical solution of both problems is in the picture 7.1.

The primal as a unique optimal solution $x^* = (0,\ 5/2)^T$; the dual has a unique optimal solution $u^* = (3,\ 0)^T$.

In x^* the active constraints are $I(X^*) = \{1\}$. The first resource is used up to the availability ($x_1^* + 2x_2^* = 5$); the availabilty of the second resource exceed the requirement ($x_1^* + x_2^* = 2,5 < 4$). Further $x_1^* = 0$which means that the item x_1 is not manufactured.

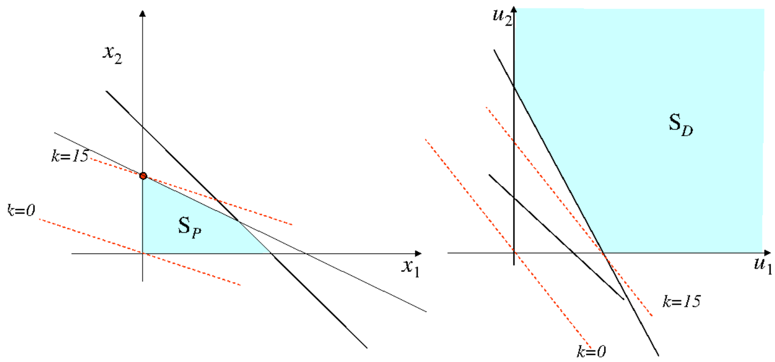

Figure 7.1: Soluzione grafica della coppia primale duale.

Dual Variables and Marginal Values: changes in the r.h.s of the constraints What happens if we change the availability of the first from 5 to $5+\delta$ with $\delta \geq 0$? The primal dual pair change as follows

$$(P)_\delta \quad \begin{array}{ll} \max & 2x_1 + 6x_2 \\ & x_1 + 2x_2 \leq 5+\delta \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array} \qquad \begin{array}{ll} \min & (5+\delta)u_1 + 4u_2 \\ & u_1 + u_2 \geq 2 \\ & 2u_1 + u_2 \geq 6 \\ & u_1, u_2 \geq 0 \end{array} \quad (D)_\delta$$

Geometrically increasing by a factor δ corresponds to move the constraint as shown in red in the picture 7.2. The optimal solution changes to $x^*_\delta = (0, \frac{5+\delta}{2})^T$ with value $c^T_\delta x^*_\delta = 6\frac{(5+\delta)}{2} = c^T x^* + 3\delta$. When

If $\delta = 3$, the constraint $x_1 + 2x_2 \leq 5+\delta$ touches the point $(0,\ 4)^T$. If $\delta > 3$, the constraint $x_1 + 2x_2 \leq 5+\delta$ does not influs any more the feasible region.

In the dual a variation of δ corresponds to a variation of the slope of the objective function whereas the dual feasible region does not change. For any value of $\delta \leq 3$, the optimal solution of the dual stays in $u^* = (3,\ 0)^T$ with value $b^T_\delta u^*_\delta = (5+\delta)3 = 5u^*_1 + \delta u^*_1$. The variation of the objective function is again 3δ.
The value $3 = u^*_1$ is the maximum amount that the producer is willing to pay for an additional unit of the first resource

If $\delta > 3$, tjhe optimal dual solution changes to $(0,\ 6)^T$ with value 24 which does not depend anymore on the value of δ.

Possiamo riassumere le considerazioni The value u^*_1 is called *shadow price* of the first resource.

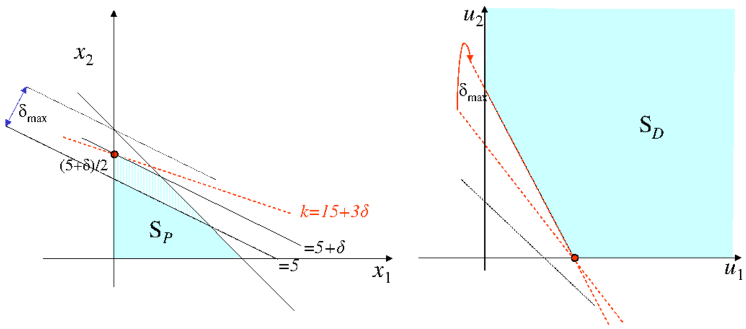

Figure 7.2: Soluzione grafica della coppia primale duale al variare di δ

7.12.3 Shadow price

Consider a primal dual pair (P) (D). Let (P_Δ) and (D_Δ) the primal dual pair obtained by modifying the r.h.s. of the primal costraint from b to $b+\Delta$.

$$
(\mathrm{P}) \quad \begin{cases} \min c^T x \\ Ax = b \\ x \geq 0 \end{cases} \qquad\qquad (\mathrm{D}) \quad \begin{cases} \max b^T u \\ A^T u \leq c \end{cases}
$$

$$
(\mathrm{P}_\Delta) \quad \begin{cases} \min c^T x \\ Ax = b + \Delta \\ x \geq 0 \end{cases} \qquad\qquad (\mathrm{D}_\Delta) \quad \begin{cases} \max (b+\Delta)^T u \\ A^T u \leq c \end{cases}
$$

Let $x^\star$ and $u^\star$ the optimal solutions of (P) and (D) and $x^\star(\Delta)$ and $u^\star(\Delta)$ the optimal solutions of(P_Δ) and (D_Δ).

The strong duality theorem applied to (P) and (D) states that

$$
c^T x^\star = b^T u^\star, \tag{7.11}
$$

whreas applied to (P_Δ)–(D_Δ) we get

$$c^T x^\star(\Delta) = (b+\Delta)^T u^\star(\Delta). \tag{7.12}$$

Under sutable assumptions on $x^\star$ (that we do not report here) and if Δ is sufficiently small then it is posisble to prove that:

$$u^\star(\Delta) = u^\star. \tag{7.13}$$

Using (7.11), (7.12) and (7.13) we get:

$$c^T x^\star(\Delta) = (b+\Delta)^T u^*(\Delta) = b^T u^\star + \Delta^T u^\star = c^T x^\star + \Delta^T u^\star, \tag{7.14}$$

which gives:

$$c^T x^\star(\Delta) - c^T x^\star = \delta_1 u_1^\star + \delta_2 u_2^\star + \ldots + \delta_m u_m^\star, \tag{7.15}$$

dove $\Delta = (\delta_1, \ldots, \delta_m)^T$.

The dual variable $u_i^\star$ can be seen as the price asscoaite dto a unit increment of b_i.

The *shadow price* associated to a constraint represents the variation in the obiective function whe the r.h.s. of the constraint change by a unit.