EE663, Assignment 5

Question 2

Problem Statement. Consider a 1D image (for example, a single row from a 2D image). You know that given such an image, computing its gradients is trivial. An inquisitive student frames this as a convolution problem to yield $g = h * f$ where g is the gradient image (in 1D), h is the convolution kernel to represent the gradient operation, and f is the original 1D image.

Part 1. The student tries to develop a method to determine f given g and h. What are the fundamental difficulties he/she will face in this task? Justify your answer. You may assume appropriate boundary conditions. [10 points]

Solution. Let us try to look at this problem in the fourier domain. Since $g = h * f$, we will have $G(w) = H(w).F(w)$, where $G = \mathcal{F}(g), H = \mathcal{F}(h), F = \mathcal{F}(f)$, where $\mathcal{F}$ is the Fourier transform operation. Since we already know g, h, we can determine $G(w), H(w)$ as well. Thus, we want

$$\begin{aligned} F(w) &= \frac{G(w)}{H(w)} \\ f &= \mathcal{F}^{-1}(F) \end{aligned}$$

Let us try to think of how $H(w)$ will look in the frequency domain. We can express the gradient of a 1-D image in the following way:

$$\begin{aligned} g(x) &= I(x+1) - I(x) \\ \implies G(w) &= I(w)\left(1 - e^{\frac{-j2\pi w}{L}}\right) \end{aligned}$$

where L is the length of the image. Note that, at small frequencies $G(w) \to 0$; in specific, $G(0) = 0$. Thus, unless $G(0) = 0$ and thus we can ignore the F value at this point, actually obtaining the lower frequencies will be a problem; the values will blow up. This problem is further amplified if there exists the presence of small amounts of noise. Since an image has a lot of low frequency components, thus crucial image information will be lost. ■

Part 2. Now consider that you are given the gradients of a 2D image in the X and Y directions, and you wish to determine the original image. What are the difficulties you will face in this task? Justify your answer. Again, you may assume appropriate boundary conditions. [10 points]

Solution. The solution to the 2-D part is very similar to the 1-D part. We have a similar relation in 2-D:

$$
\begin{aligned}
F(u,v) &= \frac{G_1(u,v)}{H_1(u,v)} \\
F(u,v) &= \frac{G_2(u,v)}{H_2(u,v)} \\
f &= \mathcal{F}^{-1}(F)
\end{aligned}
$$

In two dimensions, we can actually recover the image in two ways; by using $H_1(u,v), G_1(u,v)$ or by using $H_2(u,v)$ and $G_2(u,v)$ where the first are the filter and output corresponding to the derivatives along x and the second are the filter and output corresponding to derivatives along y. Let us think about the structure of $H_1(u,v)$ and $H_2(u,v)$.

$$
\begin{aligned}
h_1(x,y) &= I(x+1,y) - I(x,y) \\
\implies H_1(u,v) &= I(u,v)\left(1 - e^{\frac{-j2\pi u}{N}}\right) \\
h_2(x,y) &= I(x,y+1) - I(x,y) \\
\implies H_2(u,v) &= I(u,v)\left(1 - e^{\frac{-j2\pi v}{M}}\right)
\end{aligned}
$$

Thus, we see that $H_1(u,v) \to 0$ as $u \to 0$, and $H_2(u,v) \to 0$ as $v \to 0$. Thus, depending on values of $G_1(u,v), G_2(u,v)$, recovery at low frequencies is again a problem. Noise can perturb the filter coefficients, and therefore there is a possibility of a blowup at these low frequencies. It is known that most images contain important low frequency information, thus the loss of this information can make it hard to reconstruct the image accurately. ■