EE663, Assignment 4

Question 3

Problem Statement. Consider a matrix $\boldsymbol{A}$ of size $m \times n, m \leq n$. Define $\boldsymbol{P} = \boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{A}^T$. (Note: all matrices, vectors and scalars involved in this question are real-valued).

Part 1. Prove that for any vector $\boldsymbol{y}$ with appropriate number of elements, we have $\boldsymbol{y}^t\boldsymbol{P}\boldsymbol{y} \geq 0$. Similarly show that $\boldsymbol{z}^t\boldsymbol{Q}\boldsymbol{z} \geq 0$ for a vector $\boldsymbol{z}$ with appropriate number of elements. Why are the eigenvalues of $\boldsymbol{P}$ and $\boldsymbol{Q}$ non-negative? [5 points]

Solution. Note that, since the dimensions of $\boldsymbol{P}$ and $\boldsymbol{Q}$ are $n \times n$ and $m \times m$ respectively, the dimensions of $\boldsymbol{y}$ and $\boldsymbol{z}$ will be $n \times 1$ and $m \times 1$ respectively.

Expanding the expression for $\boldsymbol{P}$, we get that $\boldsymbol{y}^t\boldsymbol{P}\boldsymbol{y} = \boldsymbol{y}^t\boldsymbol{A}^t\boldsymbol{A}\boldsymbol{y} = ||\boldsymbol{A}\boldsymbol{y}||_2^2 \implies \boldsymbol{y}^t\boldsymbol{P}\boldsymbol{y} \geq 0$

An alternative way to prove this is to fully expand after getting $\boldsymbol{y}^t\boldsymbol{P}\boldsymbol{y} = \boldsymbol{y}^t\boldsymbol{A}^t\boldsymbol{A}\boldsymbol{y}$; the answer turns out to be of the form $\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\boldsymbol{y}_i\boldsymbol{A}_{ji}\right)^2$, which is clearly non-negative. Here, $\boldsymbol{A}_{ij}$ is the entry in the i^{th} row and j^{th} column of $\boldsymbol{A}$.

Similarly, for $\boldsymbol{Q}$ we get $\boldsymbol{z}^t\boldsymbol{Q}\boldsymbol{z} = \boldsymbol{z}^t\boldsymbol{A}\boldsymbol{A}^t\boldsymbol{z} = ||\boldsymbol{A}^t\boldsymbol{z}||_2^2 \implies \boldsymbol{z}^t\boldsymbol{Q}\boldsymbol{z} \geq 0$.

Let $\boldsymbol{r}$ be an eigenvector of $\boldsymbol{P}$. Using the above proof, it is true that $\boldsymbol{r}^t\boldsymbol{P}\boldsymbol{r} \geq 0$. Let the eigenvalue of $\boldsymbol{r}$ be γ. Then, we have that $\boldsymbol{r}^t\boldsymbol{P}\boldsymbol{r} = \boldsymbol{r}^t\gamma\boldsymbol{r} = \gamma\boldsymbol{r}^t\boldsymbol{r} = \gamma||\boldsymbol{r}||_2^2$. We already know that $\boldsymbol{r}^t\boldsymbol{P}\boldsymbol{r} \geq 0 \implies \gamma||\boldsymbol{r}||_2^2 \geq 0 \implies \gamma \geq 0$. A similar mirrored argument again follows through for $\boldsymbol{Q}$, exchanging roles of n, m and $\boldsymbol{A}, \boldsymbol{A}^t$. Let ζ be the eigenvalue of $\boldsymbol{s}$, an eigenvector of $\boldsymbol{Q}$. Then we have $\boldsymbol{s}^t\boldsymbol{Q}\boldsymbol{s} = \boldsymbol{s}^t\zeta\boldsymbol{s} = \zeta||\boldsymbol{s}||_2^2$. Since we also know $\boldsymbol{s}^t\boldsymbol{Q}\boldsymbol{s} \geq 0 \implies \zeta||\boldsymbol{s}||_2^2 \geq 0 \implies \zeta \geq 0$. ■

Part 2. If $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{P}$ with eigenvalue λ, show that $\boldsymbol{A}\boldsymbol{u}$ is an eigenvector of $\boldsymbol{Q}$ with eigenvalue λ. If $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{Q}$ with eigenvalue μ, show that $\boldsymbol{A}^T\boldsymbol{v}$ is an eigenvector of $\boldsymbol{P}$ with eigenvalue μ. What will be the number of elements in $\boldsymbol{u}$ and $\boldsymbol{v}$? [5 points]

Solution. Let the eigenvalue of $\boldsymbol{u}$ be λ and the eigenvalue of $\boldsymbol{v}$ be μ. Firstly, we note that since the dimensions of $\boldsymbol{P}$ and $\boldsymbol{Q}$ are $n \times n$ and $m \times m$, the dimensions of $\boldsymbol{u}$ and $\boldsymbol{v}$ are $n \times 1$ and $m \times 1$ respectively. Hence, the number of elements in $\boldsymbol{u}$ are n and the number of elements in $\boldsymbol{v}$ are m.

We know that $\boldsymbol{P}\boldsymbol{u} = \lambda\boldsymbol{u} \implies \boldsymbol{A}^t\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$. Pre-multiplying by $\boldsymbol{A}$, we get that

$$\boldsymbol{A}\boldsymbol{A}^t\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{A}\boldsymbol{u} \implies \boldsymbol{Q}(\boldsymbol{A}\boldsymbol{u}) = \lambda\boldsymbol{A}\boldsymbol{u}$$

Hence, $\boldsymbol{Au}$ is an eigenvector of $\boldsymbol{Q}$ with an eigenvalue of λ. Similarly, in the opposite direction,

$$\boldsymbol{Qv} = \mu\boldsymbol{v} \implies \boldsymbol{AA}^t\boldsymbol{v} = \mu\boldsymbol{v} \implies \boldsymbol{A}^t\boldsymbol{AA}^t\boldsymbol{v} = \mu\boldsymbol{A}^t\boldsymbol{v} \implies \boldsymbol{P}(\boldsymbol{A}^t\boldsymbol{v}) = \mu\boldsymbol{A}^t\boldsymbol{v}$$

Hence, $\boldsymbol{A}^t\boldsymbol{v}$ is an eigenvector of $\boldsymbol{P}$ with eigenvalue μ. ■

Part 3. If $\boldsymbol{v}_i$ is an eigenvector of $\boldsymbol{Q}$ and we define $\boldsymbol{u}_i \triangleq \dfrac{\boldsymbol{A}^T\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2}$. Then prove that there will exist some real, non-negative γ_i such that $\boldsymbol{Au}_i = \gamma_i\boldsymbol{v}_i$. [5 points]

Solution. Let the eigenvalue of $\boldsymbol{v}_i$ be λ. Note that, since $\boldsymbol{Q}$ is a square, symmetric matrix, it has only non-negative eigenvalues; hence, $\lambda \geq 0$. We get that

$$\boldsymbol{Au}_i = \frac{\boldsymbol{AA}^t\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2} = \frac{\boldsymbol{Qv}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2} = \frac{\lambda\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2}$$

Hence, $\boldsymbol{Au}_i$ is of the form $\gamma_i\boldsymbol{v}_i$, where $\gamma_i \geq 0$. ■

Part 4. It can be shown that $\boldsymbol{u}_i^T\boldsymbol{u}_j = 0$ for $i \neq j$ and likewise $\boldsymbol{v}_i^T\boldsymbol{v}_j = 0$ for $i \neq j$ for correspondingly distinct eigenvalues. Now, define $\boldsymbol{U} = [\boldsymbol{v}_1|\boldsymbol{v}_2|\boldsymbol{v}_3|...|\boldsymbol{v}_m]$ and $\boldsymbol{V} = [\boldsymbol{u}_1|\boldsymbol{u}_2|\boldsymbol{u}_3|...|\boldsymbol{u}_m]$. Now show that $\boldsymbol{A} = \boldsymbol{U\Gamma V}^T$ where $\boldsymbol{\Gamma}$ is a diagonal matrix containing the non-negative values $\gamma_1, \gamma_2, ..., \gamma_m$. [5 points]

Solution. We note that $\boldsymbol{U\Gamma V}^t = \sum_{i=1}^m \boldsymbol{\Gamma}_{ii}\boldsymbol{v}_i\boldsymbol{u}_i^t$, where $\boldsymbol{\Gamma}_{ii}$ is the element in the i^{th} row and i^{th} column, and similar notation for $\boldsymbol{u}$ and $\boldsymbol{v}$. We note that $\boldsymbol{\Gamma}_{ii} = \gamma_i$. Hence, we have, using that $\gamma_i\boldsymbol{v}_i = \boldsymbol{Au}_i$ from Part 3,

$$\boldsymbol{U\Gamma V}^t = \sum_{i=1}^m \gamma_i\boldsymbol{v}_i\boldsymbol{u}_i^t = \sum_{i=1}^m \boldsymbol{Au}_i\boldsymbol{u}_i^t = \boldsymbol{AUU}^t = \boldsymbol{A}$$

Note that, it is true that $\boldsymbol{UU}^t = I_{m\times m}$ since if we let $\boldsymbol{X} = \boldsymbol{UU}^t$, we have that $\boldsymbol{X}_{ij} = \boldsymbol{u}_i^t\boldsymbol{u}_j$, which is 1 if $i = j$ and 0 if $i \neq j$. Hence, $\boldsymbol{X}$ is only 1 on the diagonals $\implies \boldsymbol{X} = I_{m\times m}$.
Thus, $\boldsymbol{A} = \boldsymbol{U\Gamma V}^t$. ■