Mathematics-II

Assignment-I

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Attempt any four questions. [105]

Q.1 Prove that:

$$\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^{n+1}} = \frac{(2n)!}{[n!]^2} \cdot \frac{2\pi}{(2a)^{2n+1}}$$

Solution: Given LHS: $\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^{n+1}}$

In particular, the only singularity of $\frac{1}{(a^2+x^2)^{n+1}}$ is in the upper half of plane is at $x = ai$ and it is a pole of order of $n+1$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^{n+1}} = 2\pi i \text{ Residue}\left[\frac{1}{(a^2+x^2)^{n+1}}, x = ai\right]$$

$$= \frac{2\pi i}{n!} \frac{d^n}{dx^n} \frac{(x-ai)^{n+1}}{(a^2+x^2)^{n+1}} \Bigg|_{x=ai}$$

$$= \frac{2\pi i}{n!} \frac{d^n}{dx^n} (x+ai)^{n+1} \Bigg|_{x=ai} \quad \left(\because a^2+x^2 = (a+ix)(a-x)\right)$$

$$= \frac{2\pi i}{n!} (-n-1)(-n-2)(-n-3)\ldots\ldots(-n-n)(2ai)^{-2n-1}$$

Multiply & Divide by $n!$

$$= \frac{2\pi i}{n!} (-1)^n \frac{n!}{n!} \frac{(n+1)(n+2)\ldots 2n}{(2ai)^{2n+1}}$$

$$= \frac{(2n)!}{[n!]^2} \cdot \frac{2\pi}{(2a)^{2n+1}} \quad \left(\because (i)^{2n+1} = i\cdot(-1)^n\right)$$

$= RHS$

$LHS = RHS$

Hence Proved

Q.3 Find all the singularities of.

(a) $f(z) = \frac{1}{\sin z - \cos z}$

Solution for any singularity to occur →

Here, $\sin z = \cos z$, then $f(z)$ is not Analytic

$\therefore$ if $\sin z = \cos z$.

then z must be $= n\pi + \pi/4, n \in I$

Also, when z = infinite, $f(z)$ is not Analytic

$\therefore$ when $z = n\pi + \frac{\pi}{4}, n \in I$ and at ∞, $f(z)$ is not Analytic

(b) $f(z) = z^3 . e^{1-z}$

Solution Given! $f(z) = z^3 . e^{1-z}$

for e^{-z}, Expanding in Taylor Series →

$$e^{-z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \text{------}$$

$$e^{1-z} = e + ez + \frac{ez^2}{2!} + \frac{ez^3}{3!} + \text{-----}$$

$$\therefore z^3 e^{1-z} = ez^3 + ez^4 + \frac{ez^5}{2!} + \frac{ez^6}{3!} + \text{----}$$

Now →

for $f(z) = z^3 e^{1-z} = ez^3 + ez^4 + \frac{ez^5}{2!} + \frac{ez^6}{3!} + \text{----}$

$f(z)$ has NO singular Point

$\therefore$ $f(z)$ is Analytic function

Q.4 find the fixed Points of the following bilinear transformation, $w = \frac{8z+3i}{7i}$

Solution: Given! $w = \frac{8z+3i}{7i}$

for getting fixed points →

put $w = z$. where $z = x+iy$

we get →

$$z = \frac{8z+3i}{7i}$$

$$\Rightarrow i7z = 8z+3i$$

Putting $z = x+iy$, we get →

$$7i(x+iy) = 8(x+iy)+3i$$

$$\Rightarrow i7x - 7y = 8x + i(8y+3)$$

on solving →

$$(8x+7y) + i(8y-7x) = 0-3i$$

On comparing, we get →

$$8x+7y = 0 \quad \text{—— (1)}$$

$$7x-8y = 3 \quad \text{—— (2)}$$

On solving above equations, we get

$$x = \frac{21}{113}, \quad y = -\frac{24}{113}$$

$$\therefore \boxed{z = \frac{21}{113} - \frac{24}{113}i}$$

Q.5 Find the Bilinear Transformation which maps points $\{\infty, i, 0\}$ onto $\{0, 1, \infty\}$

Solution: Given: $z_1 = \infty$, $z_2 = i$, $z_3 = 0$

Also, $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$

Let the transformation be $\rightarrow$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{(w-w_1).w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3).z_1\left(\frac{z_2}{z_1}-1\right)}$$

$$\Rightarrow \frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

Now substituting all the values, we get

$$\frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(0-1)(i-0)}{(z-0)(0-1)}$$

$$\Rightarrow \frac{-w}{-1} = \frac{-i}{-z}$$

$$\Rightarrow \boxed{w = \frac{i}{z}}$$

$$\Rightarrow \boxed{z = \frac{i}{w}}$$