Mathematics - II

Assignment - 2

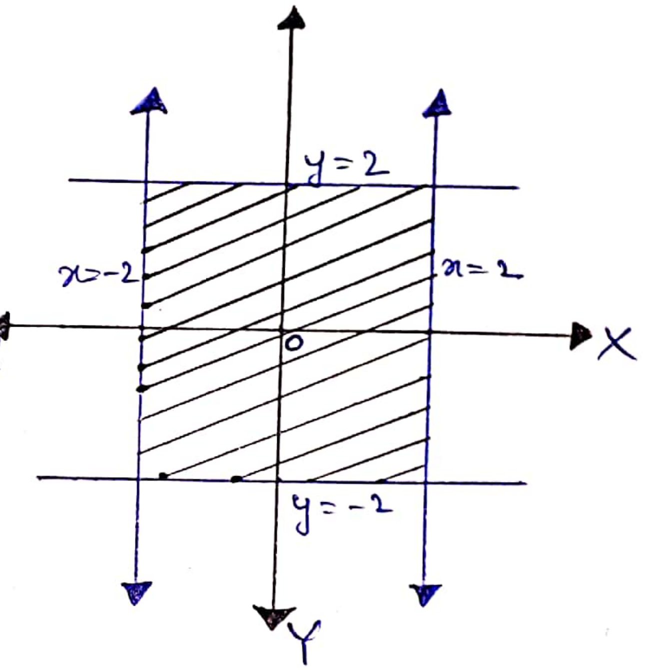
Shubham Garg
9919103057
F2

1.) As $|x_0| < 2$

and putting $(z-x_0)^2 = 0$

we get $\boxed{z = x_0, x_0}$

Thus, z lies inside the boundary



Now, According to Cauchy's Integral formula $\longrightarrow$

$$\int \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^n(a)$$

Let $f(z) = \tan(z/2)$

$$\therefore \int \frac{\tan(z/2)}{(z-x_0)^2} = \frac{2\pi i}{1!} f'(x_0)$$

$$= 2\pi i \left[\sec^2(x_0/2)/2\right]$$

$$= \boxed{\pi \sec^2(x_0/2) i}$$

2.) Here, $I = \int_0^{1+i} (x-y+ix)^2 dz$

or $I = \int_0^{i+1} (x-y+ix)(dx+idy)$ —— (A)

a) Along the line from $z=0$ to $z=1+i$

Now, the equation of line becomes-

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} = t \text{ (Say)}$$

$$\frac{y-0}{1-0} = \frac{x-0}{1-0} = t$$

$\Rightarrow y = t$ and $x = t$ —— (1)

OR $dy = dt$ and $dx = dt$ —— ②

Substituting equation ① and ② in Ⓐ →

We get → $$I = \int_{t=0}^{1} (t - t + it)(dt + i\,dt)$$

$$\Rightarrow I = \int_0^1 it\cdot(1+i)\,dt$$

$$\Rightarrow I = i(1+i)\int_0^1 t\,dt$$

$$\Rightarrow I = i(1+i)\left[t^2/2\right]_0^1$$

$$\boxed{I = \frac{-1+i}{2}}$$

b) Along the real axis from $z=0$ to $z=1$ and then along a line parallel to imaginary axis from $z=1$ to $z=1+i$

Now, Along the real axis $y=0$ and along the imaginary axis $x=0$

$$\therefore I = \int_0^1 (x + ix - y)(dx + i\,dy) + \int_1^{1+i} (x + ix - y)(dx + i\,dy)$$

$$\Rightarrow I = \int_0^1 (x + ix)(dx) + \int_0^1 (1 + i - y)(i\,dy)$$ $\left\{ \begin{array}{l} \because dy = 0 \text{ (along real axis)} \\ \& \ dx = 0 \text{ (along imaginary axis)} \end{array} \right.$

$$\Rightarrow I = (1+i)\int_0^1 x\,dx + (1+i)(i)\int_0^1 dy - i\int_0^1 y\,dy$$

$$\Rightarrow I = (1+i)\left[\frac{x^2}{2}\right]_0^1 + (-1+i)\left[y\right]_0^1 - i\left[y^2/2\right]_0^1$$

$$\Rightarrow I = \frac{1+i}{2} + (-1+i) - i/2$$

$$\Rightarrow I = \left[\frac{1}{2} - 1\right] + \left[\cancel{\tfrac{i}{2}} + i - \cancel{\tfrac{i}{2}}\right]$$

$$\boxed{I = -1/2 + i}$$

3.) Resolving $f(z)$ into partial functions, we get

$$\boxed{f(z) = 2(z-1) + 1/z + 1/z+1} \text{ ——— (A)}$$

(i) Let $f_1(z) = 2(z-1)$; $f_2(z) = 1/z$, $f_3(z) = 1/z+1$

Expanding f_1 in Taylor series about $z=i$, we get →

$$f_1(z) = f_1(i) + \sum_{n=1}^{\infty} \frac{(z-i)^n}{n!} f^n(i)$$

where, $f_1(i) = 2(i-1)$, $f_1'(i) = 2$ and $f_1^n(i) = 0$ for $n \geq 2$

∴ Taylor series' Expansion of $f_1(z)$ about $z=i$. is

$$\boxed{f_1(z)} = 2(i-1) + 2(z-i) \text{ ——— (1)}$$

Again, expanding $f_2(z)$ in Taylor series about $z=i$

$$f_2(z) = f_2(i) + \sum_{n=1}^{\infty} \frac{(z-i)^n}{n!} f_2^n(i)$$

where, $f_2(i) = 1/i$, $f_2^n(z) = (-1)^n \frac{n!}{z^{n+1}}$ so that $f_2^n(i) = (-1)^n \frac{n!}{i^{n+1}}$

∴ Taylor Series for $f_2(z)$ about $z=i$ is,

$$\boxed{f_2(z)} = \frac{1}{i} + \sum_{n=1}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}} \text{ ——— (2)}$$

Similarly $\boxed{f_3(z)} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(i+1)^{n+1}}$ (about $z=i$) ——— (3)

Hence, the Taylor Series Expansion using above eqⁿ →

$$\boxed{f(z) = 2(i-1) + 2(z-i) + \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}} + \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^{n+1}}}$$

Laurent series for $f(z)$ with $|z|<1$ is given:

$$\boxed{f(z) = 2(z-1) + \frac{1}{z}(1+z)^{-1} = 2(z-1) + \frac{1}{z} + 1 - z + z^2 - z^3 - - -}$$