Mathematics II

PBL

Assignment

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Suppose the Bare be placed along the x-axis with its one end (which is at $10°C$) at origin and other end at $x=l$

Then we can solve given Heat eq^n

$$\frac{\partial y}{\partial t} = k\left(\frac{\partial^2 y}{\partial x^2}\right)$$

Hence non-zero sol^n $X_n(x)$ be given by

$$X_n(x) = B_n \sin\left\{(2n-1)\frac{\pi x}{2}\right\}$$

Again reduce the eq.

$$\frac{dT}{dt} = \frac{-(2n-1)^2\pi^2 k T}{4} \quad \text{or} \quad \frac{dT}{dt\,T} = -C_n^2\, dt \quad \text{—①}$$

where

$$C_n^2 = \frac{1}{4} \times (2n-1)^2 \pi^2 k^2 \quad \text{—②}$$

solving eq① $\quad T_n(t) = D_n e^{-C_n^2 t}$ —③

Thus $\quad U_n(x,t) = X_n \cdot T_n$

$$= E_n \sin\left(\frac{2n-1}{2}\right)\pi x \cdot e^{-C_n^2 t}$$

Here $E_n = B_n D_n$ is another arbitrary constant

Consider a general sol^n

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{2n-1}{2}\right)\pi x \cdot e^{-C_n^2 t} \quad \text{—(4)}$$

Putting $t=0$ in (4) we have.

$$-(x+a) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{2n-1}{2}\right)\pi x \quad \text{—(5)}$$

Multiply Both side of (5) by $\sin\left(\frac{(2m-1)\pi x}{2}\right)$

and then integrating with respect to x from 0 to 1

we get

$$-\int_0^1 (x+a)\sin\left(\frac{2m-1}{2}\right)\pi x\, dx = \sum_{n=1}^{\infty} E_n \int_0^1 \sin\left(\frac{2n-1}{2}\right)\pi x \sin\left(\frac{2m-1}{2}\right)\pi x\, dx \quad \text{—(6)}$$

$$\int_0^1 \sin\left(\frac{2n-1}{2}\right)\pi x \cdot \sin\left(\frac{2m-1}{2}\right)\pi x\, dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \quad \text{—(7)}$$

using (6) & (7)

$$-\int_0^1 (x+a)\sin\left(\frac{2m-1}{2}\right)\pi x\, dx = E_m$$

$$E_n = -\int_0^1 (x+a)\sin\left(\frac{2n-1}{2}\right)\pi x\, dx.$$

$$\text{or } E_n = -2\left[(x+a)\left\{\frac{-\cos\left(\frac{2n-1}{2}\right)\pi x}{(2n-1)\pi/2}\right\} - \left\{\frac{-\sin\left(\frac{2n-1}{2}\right)\pi x}{(2n-1)^2\frac{\pi^2}{4}}\right\}\right]_0^1$$

Hence we get the required solⁿ.

$$y(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin\left(\frac{2n-1}{2}\right)\pi x e^{-C_n^2 t}$$

where C_n and E_n are given by (2) & (8).

Q2

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Let $u = XT$

$XT'' = c^2 X'' T$ $\quad \frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T''}{T} = -k^2, 0, +k^2$

Case I $u = [c_1 \cos(kx) + c_2 \sin(kx)][c_3 \cos kct + c_4 \sin kct]$

Case II $u = (c_5 + c_6 x)(c_7 + c_8 t)$

Case III $u = (c_9 \cdot e^{kx} + c_{10} e^{-kx})(c_{11} e^{kct} + c_{12} e^{-kct})$

Only case I is acceptable with the physical Nature

B.C $\Rightarrow$ $u(0,t) = 0$ ———— cond ①

$u_x(1,t) = 0$ - - - - cond ②

I.C $\Rightarrow$ $u(x,0) = 0$ ———— cond ③

$\left(\frac{\partial u}{\partial x}\right)$ at $x = 0 = g(x)$ ———— cond ④

Applying condition 1 -

$u(0,t) = 0$.

$[c_1 \cos(0) + c_2 \sin 0][c_3 \cos(kct) + c_4 \sin(kct)] = 0$.

$c_1 [c_3 \cos(kct) + c_4 \sin(kcT)] = .0$.

$c_1 = 0$. $\quad$ $c_3 \cos(kct) + c_4 \sin(kct) = 0$.

this can't be zero as it will give zero soln

So $c_1 = 0$.

Now soln after condition 1 $\rightarrow$

$u(x,t) = c_2 \sin(kx) [c_3 \cos(kct) + c_4 \sin(kct)]$ — ③.

Applying condition 2 —

$u(1,t) = 0.$

$c_2 \sin x \left[c_3 \cos(Kct) + c_4 \sin(Kct) \right] = 0.$

$c_2 \sin(K) = 0.$

$C_2 = 0$ or $\sin K = 0.$

Not possible otherwise it will give zero soln.

$\sin K = 0.$

$\boxed{K = n\pi}$

Now soln after applying cond 1 & 2

$u(x,t) = c_2 \sin(n\pi x) \left[c_3 \cos(Kct) + c_4 \sin(Kct) \right]$

Applying cond. 3

$u(x,0) = 0.$

$c_2 \sin(n\pi x) \cdot c_3 = 0.$

$c_3 = 0.$

Now soln after applying first three cond —

$u(x,t) = c_2 \sin(n\pi x) \cdot c_4 \sin(n\pi ct)$

$= c_2 c_4 \sin(n\pi x) . \sin(n\pi ct)$

$u(x,t) = b_n \sin(n\pi x) \sin(n\pi ct)$ — 4

The general soln will be obtained after adding all such soln (i.e for different value of n)

$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) . \sin(n\pi ct)$ — 5.

Applying condition 4 —

$\left(\frac{\partial u}{\partial t} \right)_{x,0} = g(x)$

$$\left[\sum_{n=1}^{\infty} b_n \sin(n\pi x)\cdot(n\pi c)\cos(n\pi ct)\right]_{(t=0)} = g(x)$$

$$\sum_{n=1}^{\infty} (b_n n\pi c)\cdot \sin(n\pi x) = g(x)$$

$$(b_n n\pi c) = \frac{2}{L}\int_0^L g(x)\cdot \sin\left(\frac{n\pi x}{L}\right) dx.$$

(using fourier Half angle sine series)

$$b_n (n\pi c) = 2\int_0^1 g(x) \sin(n\pi x)\, dx$$

$$= \frac{2}{n\pi}\left[2\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right]$$

$$b_n = \frac{2}{n^2\pi^2 c}\left[2\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\right]$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2 c} \sin(n\pi x)\cdot \sin(n\pi ct)\left[2\sin\left(\frac{n\pi}{4}\right) - \sin\frac{n\pi}{2}\right]$$

Q3 Temperature $u(x,y)$ is steady state in two dimension plate is governed by the laplace eq^n

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{--- (1)}.$$

$u(0,y) = 0$. $\quad u(a,y) = 0$. --- (2)

$u(x,0) = 0$. --- (3) (a)

$u(x,b) = 100$ --- 3 (b)

Here $u(x,b) = f(x) = 100$.

$$E_n = \frac{2}{a \sinh\left(\frac{\pi n b}{a}\right)} \int_0^a 100 \sin\frac{n\pi x}{a}\, dx = \frac{200}{a \sinh\left(\frac{\pi n b}{a}\right)} \left[\frac{-\cos\left(\frac{n\pi x}{b}\right)}{\pi n / a} \right]_0^a$$

with the value of ~~E~~. E_m reduce to.

$$u(x,y) = \sum_{n=1}^{\infty} E_{2m-1} + \frac{\sin(2m-1)\pi x}{a} \cdot \sinh\left(\frac{2m-1}{a}\right)\pi y.$$

or

$$u(x,y) = \frac{400}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)} \sin\left(\frac{2m-1}{a}\right)\pi x \cdot \sinh\left(\frac{2m-1}{a}\right)\pi y$$

$$\operatorname{cosech} \frac{(2m-1)\pi b}{a}.$$