A reappraisal of streaklines in a shear layer perturbed by two waves

Jahanwi Singh 2018CH10218

Kanika Sheoran 2018CH70285

Sakshi Mirge 2018CH7029

Shubham Mittal 2018CH10245

Urvashi Panwar 2018CH10253

Abstract

Two waves of the form of a fundamental oscillation and subharmonic, when superimposed with various combinations of amplitude and phase-difference values, result in different kinds of shear flows. The streaklines of such flows have been examined to verify the presence of discrete vortices by Williams et.al.[1] in their paper. These results for the graphs of streaklines, streamlines and vorticity distributions in a shear layer perturbed by two waves are reproduced and the correlations, between the various plots obtained, are investigated. The physical interpretation of the numerical results are explored with reference to the idea of vortex-merging and are critically examined against the simulation results delineated in the paper.

1 Background

Streaklines and streamlines are field lines, characteristic of a fluid flow, which will be investigated in the current analysis for the case of a fluid flowing along the X-axis with a shear flow superposed by two fluctuating perturbation velocities.

A streakline is a locus of points corresponding to the positions of fluid particles that have passed through a common spatial point at different values of time. In the case of our perturbed shear layer,streakline plots are helpful for visualizing the formation of vortices by observing the roll-ups occurring in the streaklines at certain distances along the direction of the flow.

Streamlines are defined as lines drawn in the flow field so that at a given instant they are tangent to the direction of flow at every point in the flow field [3]. The cat's eye streamlines obtained in our analysis at various different initial values of the Y-coordinate of the streamline, are analyzed to reinforce the presence of roll-ups observed in the streakline plots.

Another physical quantity calculated to examine the roll-ups is the vorticity. The vorticity is defined as the degree of rotation of a fluid element as it flows in the field [4], and is therefore very closely related to the roll-ups we are investigating. Vortices are the large scale coherent structures present in the turbulent shear layers. A vortex is a region in which the fluid rotates around an axis to form a two dimensional spiral kind of structures, two of which can amalgamate to form a bigger vortex.

When the study of vortices was extended to laminar flow, it was discovered that the existence of a vortex is precludes the laminar flow of the liquid under observation. This is because vortical structures typically appear in the near-wall regions of the flow and require the presence of spiralling streamlines around them. Such streamlines cannot occur near the walls for a fluid flowing under laminar conditions since it is, by definition, irrotational. Hence the formation of a vortex in laminar flow is considered as the transition point between laminar to turbulent flow. Dye has often been used to study the formation and structure of these vortices.

2 Introduction

The velocity profiles used for the calculations in this text include:

Parallel shear flow

The basic flow of the fluid in the absence of perturbations is a shear flow given by the relation: $\bar{u} = 1 + \tanh y$

Fundamental perturbation velocity

One of the two fluctuating velocity profiles that are introduced in the parallel shear flow, causing roll-ups is of the form of a fundamental oscillation given as: $u'_1 = 2a_1 \operatorname{sech} y \tanh y \sin \alpha(x - ct)$

Subharmonic perturbation velocity

The second perturbation velocity that has been superposed has the form of a subharmonic of the fundamental perturbation velocity, which can be expressed as: $u'_2 = 2a_2 \operatorname{sech} y \tanh y \sin \frac{1}{2}\alpha(x - ct)$

Where, a_1 and a_2 are a measure of the amplitudes of the two components, respectively, and are taken to be constants, independent of space and time. $\alpha(= \frac{2\pi}{\lambda})$ is the wave number and c is the wave speed. The propagation speeds of the two waves are assumed to be identical.

The overall perturbation velocity is a combination of the fundamental and the subharmonic: $u' = u'_1 + u'_2$ and the velocity of the fluid in the X-direction is given as a superposition of all three velocity profiles as: $u = \bar{u} + u'$. The velocity profile for the normal component of the velocity has been calculated using the continuity equation in Section 3.

The numerical investigation used one frequency component in the form of a travelling wave formed by the superimposition of sine wave and mean shear flow, but it did not offer sufficient variable situations for the development of streaklines. The subharmonic oscillations are therefore overlapped with the previous fundamental oscillations. The resultant wave offers a variety of situations for the development of streaklines. Numerical methods have been used for the calculation of particle displacement and solving ODEs.The goal is to investigative the correlation between the streaklines developed for various situations with the streamlines and vorticity by studying the graphs produced.

3 Calculations

As shown in section 2, the basic flow chosen for computation is a shear flow given by:

$$\bar{u} = 1 + \tanh y, \qquad \bar{v} = 0$$

The normal component of fluctuating velocity is found from the equation of continuity.

$$\frac{\partial p}{\partial t} + \nabla.(\rho\bar{v}) = 0$$

Since, the flow is steady, $\frac{\partial p}{\partial t} = 0$
So, the equation becomes, $\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$

$$\implies \frac{\partial v'}{\partial y} = -2a_1\alpha\,\mathrm{sech}\,y\tanh y\cos\alpha(x-ct) - a_2\alpha\,\mathrm{sech}\,y\tanh y\cos\frac{\alpha}{2}(x-ct)$$

$$\implies v' = -(2a_1\alpha\cos\alpha(x-ct) + a_2\alpha\cos\frac{\alpha}{2}(x-ct)).\int \mathrm{sech}\,y\tan y\,\mathrm{d}y$$

On solving, the normal component of fluctuating velocity can be expressed as :

$$v' = 2a_1\alpha\,\mathrm{sech}\,y\cos\alpha(x-ct) + a_2\alpha\,\mathrm{sech}\,y\cos\frac{\alpha}{2}(x-ct)$$

which is the velocity of the fluid in the Y-direction

In the present computation, the wavelength(λ) is chosen to be 2π, so the wave number(α) becomes unity. The wave speed (c) is also taken to be 2π, and thus $\frac{c}{\lambda} = 1$.

3.1 Streaklines

Consider a particle released from the location x_0, y_0 in the flow field at time t_0. The motion of the particle is governed by the equations,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u(x,y,t), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = v(x,y,t)$$

In order to find the location of the particle at time T, we need to simultaneously integrate the two differential equations over time,

$$x(T) = \int_{t_0}^{T} u(x,y,t)\,\mathrm{d}t + x_0 \qquad y(T) = \int_{t_0}^{T} v(x,y,t)\,\mathrm{d}t + y_0$$

So, the location of the particle at time $(t_0 + \Delta t)$ becomes,

$$x(t_0+\Delta t) = x_0 + \Delta x = x_0 + u(x_0,y_0,t_0)\Delta t \qquad y(t_0+\Delta t) = y_0 + \Delta y = y_0 + v(x_0,y_0,t_0)\Delta t$$

The time is again increased by Δt. The new location of the particle, in the same manner, comes out to be:

$$x(t_0{+}2\Delta t) = x(t_0{+}\Delta t){+}u(x_0{+}\Delta x, y_0{+}\Delta y, t_0{+}\Delta t)\,\Delta t \qquad y(t_0{+}2\Delta t) = y(t_0{+}\Delta t){+}v(x_0{+}\Delta x, y_0{+}\Delta y, t_0{+}\Delta t)\,\Delta t$$

The process is repeated until the time T is reached and the final position of the particle is recorded. A short time later, at $(t_0 + \Delta t^*)$, a second is released at the same original location x_0, y_0. The path of this will be similarly given by:

$$x(t_0+\Delta t^*+\Delta t)x_0 + u(x_0,y_0,t_0+\Delta t^*)\Delta t \qquad y(t_0+\Delta t) = y_0 + v(x_0,y_0,t_0+\Delta t^*)\Delta t$$

and so on and its final location at time T is recorded. The same procedure is repeated for all the particles that form a streakline.

The computation was performed for 1000 particles which were released in succession from the point x_0, y_0 at equal time intervals, $\Delta t^* = 0.01$. The final location of each particle at time, T = 10 is determined by the Euler's method with time step, $\Delta t = 0.01$. The critical layer is on the horizontal axis, $y = 0$.

In the last three cases conditions, (f), (g) and (h), the subharmonic has a phase shift relative to the fundamental so that,

$$u' = 2a_1 \operatorname{sech} y \tanh y \sin \alpha(x - ct) + 2a_2 \operatorname{sech} y \tanh y \sin \frac{1}{2}\alpha(x - ct + \phi)$$

$$v' = 2a_1 \alpha \operatorname{sech} y \cos \alpha(x - ct) + a_2 \alpha \operatorname{sech} y \cos \frac{\alpha}{2}(x - ct + \phi)$$

There is no phase shift between the fundamental and subharmonic in all the previous conditions, (a), (b), (c) and (d).

In order to facilitate the computation, x and y coordinates were normalised by the wavelength of the fundamental. The new coordinate system and time variables are expressed as:

$$X = \frac{x}{\lambda} \qquad Y = \frac{y}{\lambda} \qquad T = \frac{ct}{\lambda} \qquad \text{where } \lambda = 2\pi$$

This gives:

$$U = 1 + \tanh(2\pi Y) + 2a_1 \operatorname{sech}(2\pi Y) \tanh 2\pi Y \sin(2\pi\alpha(X - T)) + 2a_2 \operatorname{sech}(2\pi Y) \tanh(2\pi Y) \sin(\pi\alpha(X - T) + \phi)$$

$$V' = 2a_1 \operatorname{sech}(2\pi Y) \cos(2\pi\alpha(X - T)) + a_2 \operatorname{sech}(2\pi y) \cos(\pi\alpha(X - T) + \phi)$$

3.2 Streamlines

The generalized equation for a streamline is given as, $\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w}$. In the current analysis, the z-component of velocity, w, is taken to be 0. So the equation becomes,

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v}$$

Upon rearrangement, we get the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v}{u} = \frac{v'}{u} = \frac{2a_1 \alpha \operatorname{sech} y \cos \alpha(x - ct) + a_2 \alpha \operatorname{sech} y \cos \frac{\alpha}{2}(x - ct)}{1 + \tanh y + 2a_1 \operatorname{sech} y \tanh y \sin \alpha(x - ct) + 2a_2 \operatorname{sech} y \tanh y \sin \frac{1}{2}\alpha(x - ct)}$$

Since the streaklines have been computed up to time $T = 10\,\mathrm{s}$, so in order to do a comparative study of streamlines and streaklines, we compute the streamlines at a constant time $T = 10\,\mathrm{s}$ as well. This makes the components of velocity functions of only the x and y coordinates.

Similar to as shown earlier for the streaklines, we use Euler's method to solve the above differential equation. To compute the streamlines of multiple particles at given specifications, we change the initial position in y coordinate i.e. y_0, keeping $x_0 = 0$, and perform the iterations.

In condition (b), the subharmonic has a phase shift relative to the fundamental so that,

$$u' = 2a_1 \operatorname{sech} y \tanh y \sin \alpha(x - ct) + 2a_2 \operatorname{sech} y \tanh y \sin \frac{1}{2}\alpha(x - ct + \phi)$$

$$v' = 2a_1 \alpha \operatorname{sech} y \cos \alpha(x - ct) + a_2 \alpha \operatorname{sech} y \cos \frac{\alpha}{2}(x - ct + \phi)$$

There is no phase shift between the fundamental and subharmonic in the conditions (a) and (c).

3.3 Vorticity distributions

The vorticity distribution has also been used as a parameter to examine the presence of roll ups in the streaklines. The mean vorticity is calculated as:

$$\bar{\zeta} = \frac{\partial \bar{u}}{\partial y} - \frac{\partial \bar{v}}{\partial x} = \frac{\partial \bar{u}}{\partial y}$$

$$\implies \bar{\zeta} = \operatorname{sech}^2 y$$

The fluctuating component is expressed as:

$$\zeta' = \frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x}$$

$$\implies \zeta' = 4a_1 \operatorname{sech}^3 y \, \sin(x - ct) + 4a_2 \operatorname{sech} y \left(\operatorname{sech}^2 y - \frac{3}{8}\right) \sin \frac{1}{2}(x - ct + \phi)$$

The mean vorticity is superposed with the fluctuating component to obtain,

$$\zeta = \operatorname{sech}^2 y + 4a_1 \operatorname{sech}^3 y \sin(x - ct) + 4a_2 \operatorname{sech} y \left(\operatorname{sech}^2 y - \frac{3}{8}\right) \sin \frac{1}{2}(x - ct + \phi)$$

Normalising x and y coordinate by the wavelength,

$$Z = \operatorname{sech}^2 2\pi Y + 4a_1 \operatorname{sech}^3 2\pi Y \sin[(2\pi(X - T)] + 4a_2 \operatorname{sech} 2\pi Y \left(\operatorname{sech}^2 2\pi Y - \frac{3}{8}\right) \sin[\pi(X - T) + \phi/2]$$

4 Discussion

The aforementioned calculations have been done for a shear flow which has been formed by the superposition of two waves in the form of a fundamental oscillation and its subharmonic.

To understand the interdependence between vorticity, streamlines and streaklines, a reference case was first created where only the fundamental oscillation wave is present in the flow field. The instantaneous vorticity distribution is sinusoidal in the downstream direction and the streamlines move with the wave speed in a uniformly sized cat's-eye pattern, the formation of which has originally been shown by Lord Kelvin [5].

Since the cat's eye streamlines in the moving reference frame are stationary about the critical layer, any fluid particle that is introduced near the critical layer gets trapped in this pattern and roll around the eye as it is swept downstream with the wave speed. This creates a spiral kind of pattern that gives the impression of a two-dimensional discrete vortex as shown in by the streaklines.

All the combinations of the amplitude-ratio and phase-shift between the fundamental oscillation and its subharmonic wave, used in the original text, have been reassessed to produce a number of streakline, streamline and instantaneous vorticity curves corresponding to the parametric variations in all the cases considered.
Here are some of the observations that can be inferred from the graphs shown in Fig.1.
Observations from the results for streaklines shown in Fig.1(i) :

- The first streakline, i(a), consists of equally sized roll-ups formed by the presence of only the fundamental oscillation wave. The subharmonic perturbation velocity is not considered in this case.
- The second streakline, i(b), gives an impression of amplification and discrete vortices as the roll-ups become larger along the critical layer. However, since it is composed of the subharmonic wave alone, the vorticity for this case (not included in the original text) is a uniform sinusoidal and fails to explain the roll-ups.
- The next two streaklines, i(c) and i(d), are observed for the cases of superposition of the fundamental wave with the subharmonic wave with equal amplitudes for both. When the amplitude is increased, the roll-up occurs more prominently and forms larger vortices. During the early stages, the two roll-ups formed are distinguishable but after the flow is well-developed, the appearance of one fundamental oscillation is observed.
- Streakline i(e), initially has the appearance of two spirals, but the smaller is absorbed into the larger one, resulting in one roll-up corresponding to the subharmonic oscillations. The amplitude of the subharmonic is twice as large as the fundamental.
- The influence of a phase-shift between the two perturbation velocities is reflected in i(f), i(g) and i(h) in terms of the spacing between the vortical structures formed.

Observations from the results for streamlines shown in Fig.1(ii), and their relationship with streaklines :

- The first streamline (Fig. ii(a)), drawn corresponding to Fig. i(c), shows a large main eye accompanied by a smaller auxiliary eye. Corresponding to the centers of the eyes, major and minor roll-ups take place in the streakline in Fig. i(c).
- In the second streamline (Fig. ii(b)), drawn corresponding to Fig. i(f), the eyes are more even and the spacing between the two eyes is also nearly one wavelength. The main and auxiliary eyes are more evenly spaced and matched.

- Third streamline (Fig. ii(c)), drawn corresponding to Fig. i(e), includes only one eye being formed per two wavelengths of the fundamental, despite the presence of two waves and accordingly only one streamline. It shows cusp-like behavior, due to extremely small u' and v' fluctuations as well as the 0 mean velocity at the cusp.

Observations regarding the relationship between instantaneous vorticity distributions along the critical layer and streaklines:

- When ζ' is positive, instantaneous vorticity has a local maximum because of addition to the mean vorticity. When ζ' is negative, the instantaneous vorticity is subtracted from the mean vorticity and thus shows a minimum.
- The roll-up of streaklines occurs around the local instantaneous vorticity maximum whereas the streaklines thin out around the region of local instantaneous vorticity minimum.
- The vorticity distribution in iii(a) corresponds to the streaklines in i(a) (fundamental harmonic alone). The positions of the points of maxima in the instantaneous vorticity distribution coincide with the roll ups observed in the streakline pattern.
- The vorticity distribution in iii(b) corresponds to the streakline pattern i(e). The major roll-up in the streakline pattern is around the large vorticity maximum whereas the minor roll-up occurs at the small vorticity peak.
- The vorticity distributions in Figs. iii(c), iii(d) and iii(e) illustrate the effect of phase shift in the subharmonic wave and correspond to the streakline patterns i(f), i(g) and i(h) respectively.
- iii(c) and iii(d) suggest that in the streakline patterns, one large roll-up is followed by a small satellite roll-up. The streaklines i(f) and i(g) conform closely with these expectations.
- The vorticity distribution in iii(e) indicates that two equal sized roll-ups should occur with a narrow gap between two successive pairs which leads to a false impression of the presence of only one fundamental wave. This deduction is also in good agreement with what is observed in the streakline pattern i(h).

Cat's eye streamlines provide the clear explanations for the roll-ups as they result from the superposition of the velocity fluctuations on a shear layer. The eyes are closely correlated with the vorticity peaks. However, this is not necessarily the case in all the cases studied. There are some examples where there is near-zero vorticity fluctuations at the critical layer, which shows that there is no relation of cat's eye pattern or the streakline roll-up with the vorticity distributions. Hence the cat's eye streamlines are better for explaining the presence of roll-ups in the streaklines.

The results of Fig.iii(c) and Fig.iii(d) are inconsistent with the plot given in original paper even though the same set of equations have been used to produce the graphs in this text. The above explanations are based on the results obtained here instead of the results of the original text being analyzed. Since the graphs of instantaneous vorticity produced here explain the streaklines more accurately than those of the original paper, the results presented here are deemed to have higher credibility.

Explanation for the inconsistency in the results for vorticity distribution: Based on the numerical solutions of the instantaneous vorticity and plotting multiple graphs for different combinations of amplitudes and phase-shift values for the perturbation velocities, the nature of the function was observed. The difference in the first local minima and the consecutive local maxima decreases with increase in the value of a_2 for a constant value of a_1 and no phase-shift. As the phase-shift goes from $-\pi$ to $+\pi$ the difference between two consecutive minima first decreases and then increases. Also, on correlating these graphs with streaklines, it is apparent that the behaviour of the roll-ups is justified using the graphs obtained in the current analysis.

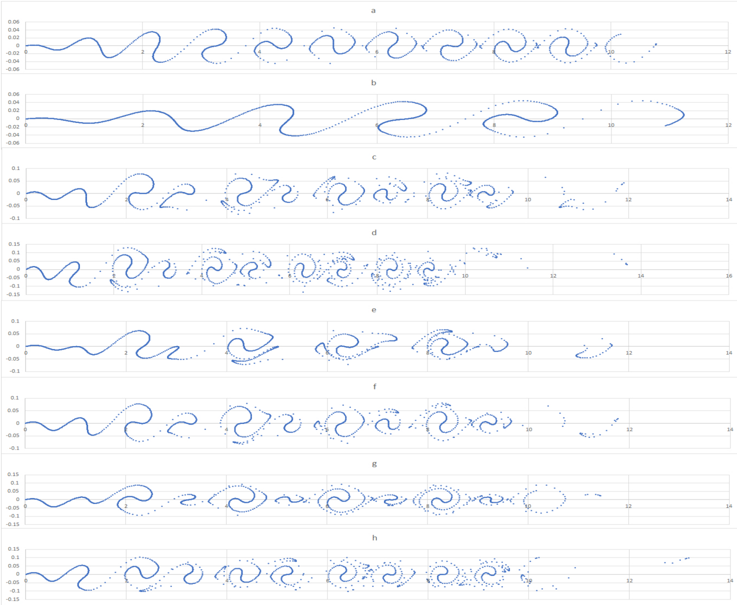
5 Conclusion

A number of different cases have been taken based on the differing values of the parameters a_1, a_2 and ϕ, and streaklines, streamlines and vorticity distribution values, for a perturbed shear flow, were calculated and plotted for each of these cases. The relationships between the patterns for every one these quantities was observed in all the different cases, along with the inter-relationships in the plots of the three quantities among themselves, in a bid to observe the effects of these changes on the vortex-formation and vortex-merging processes taking place in the flow being considered for analysis.

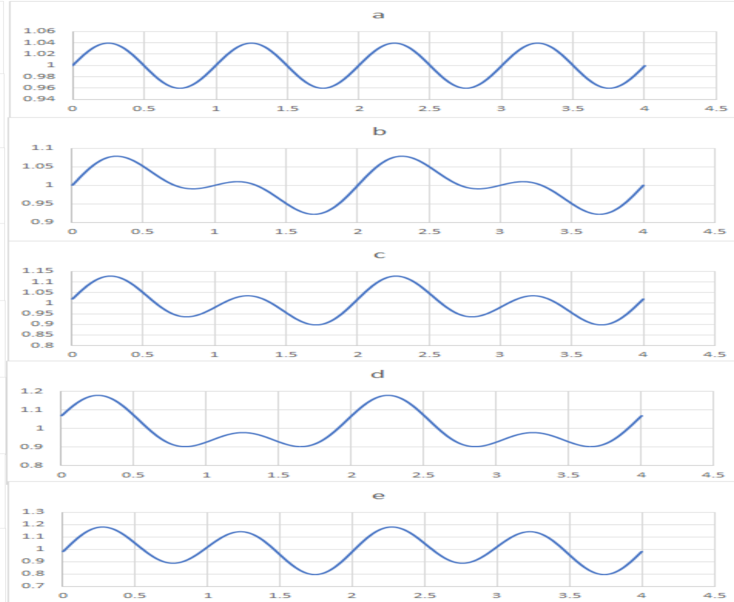
The results show that the vortical structures appearing in the plots for streaklines, are explained to a significant degree by the streamline plots. The vorticity distribution, however, fails to give an explanation for some of these results and should, thus, be withheld from use for the analysis of vortices in a fluid flow.

This result holds importance due to the fact that the computational power and the time duration required for the calculation of streamlines for a given flow is significantly lower than that required for the calculation of streaklines, and both of these quantities have been shown to be equally effective towards the analysis of a fluid flow in terms of vortices. This implies that, for further research based around the numerical or computational modelling of vortical structures in a flow, streamlines are the most suitable quantity to be calculated, due to the requirement of lesser computational power than streakline-calculations and providing a more credible explanation than the graphs for vorticity distributions.

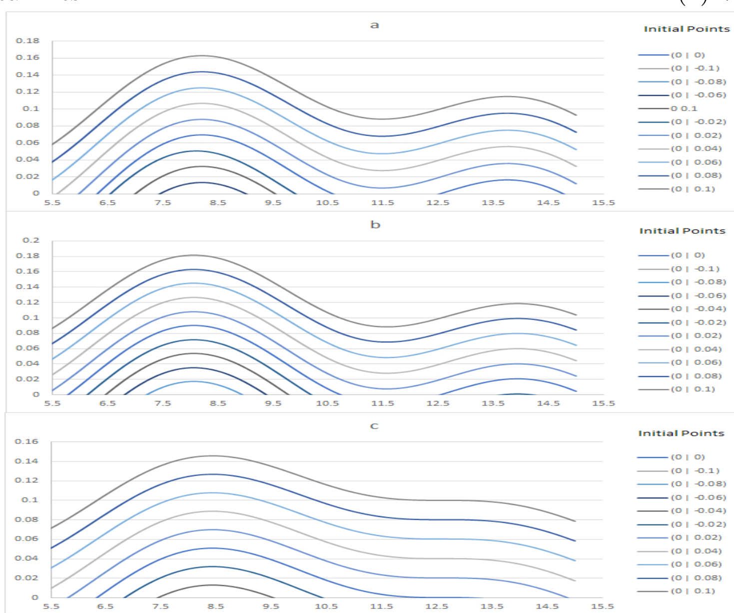
The program files used to generate the results presented here can be obtained from the GitHub repository, https://github.com/bibliophilia/shear-layer



(i) Streaklines



(ii) Vorticity



(iii) Streamlines

Figure 1: ***(i)Streaklines****:* *(a)* $a_1 = 0.01, a_2 = 0$; *(b)* $a_1 = 0, a_2 = 0.01$; *(c)* $a_1 = 0.02, a_2 = 0.02$; *(d)* $a_1 = 0.05, a_2 = 0.05$; *(e)* $a_1 = 0.02, a_2 = 0.02$; *(f)* $a_1 = 0.02, a_2 = 0.02, \phi = \frac{\pi}{4}$; *(g)* $a_1 = 0.02, a_2 = 0.04, \phi = \frac{\pi}{2}$; *(h)* $a_1 = 0.02, a_2 = 0.02, \phi = \frac{\pi}{4}$;
(ii)Vorticity*:* *(a)* $a_1 = 0.01, a_2 = 0, \phi = 0$; *(b)* $a_1 = 0.01, a_2 = 0.02, \phi = 0$; *(c)* $a_1 = 0.02, a_2 = 0.02, \phi = \frac{\pi}{4}$; *(d)* $a_1 = 0.02, a_2 = 0.04, \phi = \frac{\pi}{2}$; *(e)* $a_1 = 0.04, a_2 = 0.02, \phi = -\frac{\pi}{4}$;
(iii)Streamlines*:* *(a)* $a_1 = 0.02, a_2 = 0.02$; *(b)* $a_1 = 0.02, a_2 = 0.02, \phi = \frac{\pi}{4}$; *(c)* $a_1 = 0.01, a_2 = 0.02$;

References

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