DM- Assignment - 3

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CSG - 19

A-1 i) $a_n = 4a_{n-1} - 4a_{n-2}$ $\because a_0 = 6 , a_1 = 8$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$\therefore x = 2, 2$

General Solution :- $a_n = (c_1 + c_2 n) 2^n$

$a_0 = 6$ and $a_1 = 8$

a) $6 = c_1 + c_2(0)$

$\therefore c_1 = 6$

b) $(6 + c_2)2 = 8$

$c_2 + 6 = 4$

$\therefore c_2 = -2$

$\therefore a_n = (6-2n)2^n$

ii) $$a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$$

$$x^3 - 2x^2 - 5x + 6 = 0$$

$$(x-1)(x-2)(x+3) = 0$$

$$\therefore x = 1, 2, -3$$

$$\Rightarrow a_n = C_1(1)^n + C_2(2)^n + C_3(-3)^n$$

A-2 i) $$a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + n^2$$

$$x^3 - 6x^2 + 12x - 8 = 0$$

$$(x-2)^3 = 0$$

$$\therefore x = 2, 2, 2$$

$$\therefore a_n^{(h)} = (C_1 + C_2 n + C_3 n^2) 2^n$$

$$a_n^{(p)} = (An^3 + Bn^2 + Cn) 2^n$$

$$\text{Given} = n \cdot 2^n$$

After substituting,

$$A = \frac{1}{6}$$

$$B = 0$$

$$C = 0$$

$$\therefore a_n^{(p)} = \frac{1}{6} n^3 \cdot 2^n$$

$$\Rightarrow a_n = (C_1 + C_2 n + C_3 n^2) 2^n + \frac{1}{6} n^3 \cdot 2^n$$

ii) $$a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$$

$$x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

$$\therefore x = 1, 3$$

So, $$a_n^{(h)} = A \cdot 3^n + B$$

Non-homogeneous part :- $2^n + n + 3$

$$a_n^{(p)} = C \cdot 2^n + Dn + E$$

$$a_{n-1}^{(p)} = C \cdot 2^{(n-1)} + D(n-1) + E$$

$$a_{n-2}^{(p)} = C \cdot (2)^{(n-2)} + D(n-2) + E$$

$$C \cdot 2^n + Dn + E = 4C \cdot 2^{n-1} + 4D(n-1) + 4E - 3C2^{(n-2)} - 3D(n-2) - 3E + 2^n + n + 3$$

After calculating,

$$C = -4$$
$$D = -1$$
$$E = -3$$

$$\therefore a_n = A \cdot 3^n + B - 4(2)^n - n - 3$$

A-3 i) $a_n = 3^n$ for all $n = 0, 1, 2, \cdots$

$$\therefore G(x) = \sum_{n=0}^{\infty} 3x^n = \frac{3}{1-x}$$

ii) $a_n = 2n+3$

$$\therefore G(x) = \frac{3}{1-x} + \frac{2x}{(1-x)^2}$$

A-4 i) $(x^2+1)^3$

$$(x^2+1)^3 = x^6 + 3x^4 + 3x^2 + 1$$

Sequence is :-

$$a_n \begin{cases} 1 & \text{if } n=0 \\ 3 & \text{if } n=2 \\ 3 & \text{if } n=4 \\ 1 & \text{if } n=6 \\ 0 & \text{otherwise} \end{cases}$$

ii) $x-1 + \dfrac{1}{1-3x}$

$$G(x) = x+1 + \sum_{x=0}^{\infty} x^n = x-1+1+x+x^2+\cdots = x + \sum_{n=0}^{\infty} x^n$$

$$a_n = \begin{cases} 0 & \text{if } n=0 \\ 2 & \text{if } n=1 \\ 1 & \text{if } n \geq 2 \end{cases}$$

A-5 i) $a_k = a_{k-1} + 2a_{k-2} + 2^k \quad ; a_0 = 4$ and $a_1 = 12$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$G(x) - a_0 - a_1 x = x\left(G(x) - a_0\right) + 2x^2 G(x) + \sum_{k=2}^{\infty} 2^k . x^k$$

$$G(x) - 4 - 12x = x\left(G(x) - 4x + 2x^2 G(x) + \frac{4x^2}{1-2x}\right.$$

$$\therefore G(x) = \frac{4 + 8x + \frac{4x^2}{1-2x}}{1 - x - 2x^2}$$

ii) $a_k = 4a_{k-1} - 4a_{k-2} + k^2 \quad ; a_0 = 2$ and $a_1 = 5$

$$A(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$\sum_{k=2}^{\infty} a_k . x^k = 4 \sum_{k=2}^{\infty} a_{k-1} . x^k - 4 \sum_{k=2}^{\infty} a_{k-2} . x^k + \sum_{k=2}^{\infty} k^2 . x^k$$

$$A(x) - a_0 - a_1 . x = 4x\left(A(x) - a_0\right) - 4x^2 A(x) + \sum_{k=2}^{\infty} k^2 x^k$$

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{x(1+x)}{(1-x)^3}$$

$$\sum_{k=2}^{\infty} k^2 x^k = \frac{x(1+x)}{(1-x)^3} - x - x^2$$

$$\therefore A(x) - 2 - 5x = 4x\left(A(x) - 2\right) - 4x^2 A(x) + \frac{x(1+x)}{(1-x)^3} - x - x^2$$

A-6 i) A semigroup but not a monoid

Eg :- The set of positive integers under multiplication $(Z^+, \times)$.
$2 \times 3 = 6$ follows associativity, but no identity exists.

ii) A monoid but not group

Eg :- set of natural numbers under addition $(N, +)$.
Identity is 0, but inverse don't exist.

iii) A non-abelian group

Eg :- The symmetric group. Swapping (1,2) then (2,3) gives diff. result than swapping (2,3) first.

A-7 $Z_6 = \{0, 1, 2, 3, 4, 5\}$

$(a+b) \bmod 6 \in Z_6$ for any a, b (Closure)

Addition modulo 6 is associative.

0 is identity element : $a + 0 = a$.

Each element has inverse (eg. $2+4 \equiv 0 \bmod 6$).

Order is smallest k such that $ka \equiv 0 \bmod 6$.

Order (0) = 1, Order (1) = 6, Order (2) = 3 etc.

A-8. For any A, B in $M_2(R)$, $A+B$ and AB are also in $M_2(R)$.

Addition and Multiplication are associative.

Identity Matrix = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A(B+C) = AB + AC$

It is not an integral domain.

A-9 Closure, associativity and distributivity hold.

Find x such that $ax \equiv 1 \mod 7$

For eg:- $3^{-1} \equiv 5 \mod 7$, since $3 \times 5 = 15 \equiv 1 \mod 7$

Since every element has an inverse, Z_7 is a field.

A-10 A zero divisor is a non-zero a such that $ab \equiv 0 \mod n$ for some nonzero b.

For Z_6:-

$2 \times 3 \equiv 0 \mod 6$, so 2 and 3 are zero divisors.

For Z_{12}:-

Divisors of 12 like 2, 3, 4, 6 are zero divisors.

A-11 Distance to s is zero, all others ∞.

$$d[v] = \min(d[v], d[u] + w(u,v))$$

where $w(u,v)$ is edge weight.

Vertices : s, a, b, c, d, t

i) $s \to a$ (23)

ii) $s \to c$ (20)

iii) $a \to b$ (14)

iv) $a \to d$ (17)

v) $b \to d$ (19)

vi) $b \to t$ (26)

vii) $d \to t$ (11)

viii) $c \to d$ (22)

ix) $d \to t$ (30)

The shortest path from s to t using Dijkstra's algorithm is:-

$$s \to a \to d \to t$$

with a total distance of 51.