Casimir effect

Shubham Raghuvanshi
Harish Chandra Research Institute

June 7, 2021

Overview

Introduction

- Working on the stability of colloids **Theodore overbeek** arrived at the conclusion that for large distances London—van der Waals energy decreases more rapidly than $\frac{1}{R^6}$.
- Working on his suggestion **Hendrik Casimir** and **Dirk Polder** explained that this could be explained if finite speed of light is taken into account, and in their 1948 paper [1] derived the following correction to London—van der Waals energy

$$\Delta_L E = -\frac{23\hbar c}{4\pi R^7}\alpha_1(0)\alpha_2(0) \tag{1}$$

Introduction

- In this paper they also suggested that such simple form of the expression makes it possible to derive it perhaps from more elementary considerations.
- Later same year Casimir discovered that the same distance dependence of the force of attraction can be derived by considering change in zero point energy of electromagnetic field in the presence of perfect conductors. This phenomenon is known as the **Casimir effect**.

Classically interacting dipoles

- The energy of interaction between a charge Q and a fixed dipole with moment p is given by

$$h(R,\theta) = \frac{Q}{4\pi\epsilon_0}\frac{pcos\theta}{R^2} \tag{2}$$

- If the dipole is free to rotate then the average interaction energy is given by

$$< h(R) >= \lim_{T\to\infty} \frac{\int_0^\pi h(R,\theta)e^{-u(r,\theta)/kT}sin\theta d\theta}{\int_0^\pi e^{-u(r,\theta)/kT}sin\theta d\theta} \approx -\frac{1}{3kT}\left(\frac{Qp}{4\pi\epsilon_0}\right)^2\frac{1}{R^4} \tag{3}$$

- The energy of interaction between two fixed dipoles is

$$H(R,\theta_1,\theta_2) = \frac{1}{4\pi\epsilon_0 R^3}\left(\vec{p}_1(t).\vec{p}_2(t) - 3(\vec{p}_1(t).\hat{r})(\vec{p}_2(t).\hat{r})\right) \tag{4}$$

If both the dipoles are free to rotate then the average interaction energy is given by [2]

$$< H(R) >= -\frac{2}{3kT}\left(\frac{p_1 p_2}{4\pi\epsilon_0}\right)^2\frac{1}{R^6} \tag{5}$$

Zero point energy of atomic dipole and Van-der Walls force

Consider simple model of one dimensional atom where the electrons are bound to atom by harmonic oscillator force, the hamiltonian is given by

$$H = H_0 + H_1$$

$$H_0 = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_0 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0 x_2^2$$

$$H_1 = \frac{e^2}{4\pi}\left(\frac{1}{R} + \frac{1}{R + x_1 - x_2} - \frac{1}{R + x_1} - \frac{1}{R - x_2}\right) \approx -\frac{2e^2 x_1 x_2}{4\pi R^3}$$

The system can be diagonalized in terms of $x_\pm = (x_1 \pm x_2)/\sqrt{2}$, giving

$$\omega_\pm = \sqrt{m\omega_0^2 \mp \frac{2e^2}{4\pi R^3}} \approx= \omega_0 \mp \frac{e^2}{4\pi m\omega_0 R^3} - \frac{e^4}{32\pi^2 m^2 \omega_0^3 R^6} + \ldots$$

The Van-Der walls potential is simply the shift in ground state zero point energy

$$V(R) = \frac{\hbar\omega_+}{2} + \frac{\hbar\omega_-}{2} - \hbar\omega_0 = -\frac{\hbar e^4}{32\pi^2 m^2 \omega_0^3 R^6}$$

Retarted Van-Der Walls Interaction

- The Van-Der Walls Interaction can be written in more familiar form as

$$V(R) = -\frac{\hbar\alpha^2\omega_0}{8R^6}$$

Where $\alpha = \frac{2e^2}{4\pi m\omega_0^2}$ is the atomic polarizability.

- Casimir and Polder proved that the effect of retardation leads to the following interaction energy [1]

$$V_{ret}(R) = -\frac{23\hbar c\alpha^2}{4\pi R^7} \tag{6}$$

> The very simple form of Eq. (56) and the analogous formula (25) suggest that it might be possible to derive these expressions, perhaps apart from the numerical factors, by more elementary considerations. This would be desirable since it would also give a more physical background to our result, a result which in our opinion is rather remarkable. So far we have not been able to find such a simple argument.

Zero point energy of the electric field and the Casimir force

- The interaction energy between the dipoles in the presence of an electric field is given by

$$\langle H_{int}^{(2)} \rangle \propto \frac{\alpha^2}{R^3} \langle E(r_1)E(r_2) \rangle \tag{7}$$

- Treating the electric field quantum mechanically it can be shown that $\langle E(r_1)E(r_2) \rangle_0 \propto \frac{1}{R^4}$.

$$\langle H_{int}^{(2)} \rangle_0 \propto -\frac{\alpha^2}{R^7} \tag{8}$$

Longitudinal force on single bead on taut string

Consider a one dimentional string along x axis with uniform mass density μ and tension T. A bead of mass 'm' is placed at x=0 and is constrained to move only in transverse direction. The wave equation for transverse displacement $\psi(x,t)$ is

$$\mu\psi_{tt}(x,t) - T\psi_{xx}(x,t) = -m\psi_{tt}(x,t)\delta(x) \tag{9}$$

For a plane wave incident from left the solutions are

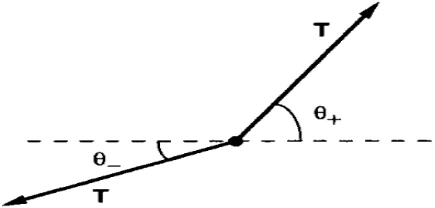
$$\psi(x,t) = \begin{cases} Ae^{ikx-\omega t} + Be^{-ikx-\omega t} & \text{for } x < 0 \\ Ce^{ikx-\omega t} & \text{for } x > 0 \end{cases} \tag{10}$$

and at $x = 0$

$$\Delta\psi = 0 \tag{11}$$

$$F_{transverse} = T(sin\theta_+ - sin\theta_-) = T\Delta(\psi_x) = m\psi_{tt} \tag{12}$$

$$F_x = T(cos\theta_+ - cos\theta_-) \approx -\frac{T}{2}\Delta\psi_x^2 \tag{13}$$



Using the boundary conditions we get

$$B = \frac{i\alpha}{1 - i\alpha}A \ , \ C = \frac{1}{1 - i\alpha}A, \text{ where } \alpha = \frac{m\omega}{2\sqrt{\mu T}} \tag{14}$$

The longitudinal force per cycle unit cycle is

$$\langle F_x \rangle = \frac{k^2 T}{4}\left(|A - B|^2 - |C|^2\right)$$

$$\langle F_x \rangle = \frac{\mu\omega^2}{1 + (4\mu T/m^2\omega^2)}|A|^2 = \frac{2cP}{1 + (4\mu T/m^2\omega^2)} \tag{15}$$

Where c is speed of wave and $P = \frac{1}{2}\mu\omega^2 c|A|^2$ is the average power carried by wave.

Longitudinal forces on two beads on a taut string

Fot two point masses located at x='0' and 'a' respectively we will have

$$\psi(x,t) = \begin{cases} Ae^{ikx-\omega t} + Be^{-ikx-\omega t} & \text{for } x < 0 \\ Ce^{ikx-\omega t} + De^{-ikx-\omega t} & \text{for } 0 < x < a \\ Ee^{ikx-\omega t} & \text{for } x > a \end{cases} \tag{16}$$

Applying boundary conditions we get the longitudinal forces on mass point 1 and 2

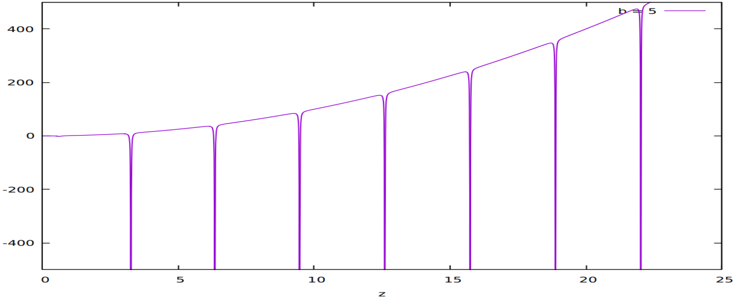
$$\langle F_1 \rangle = \frac{k^2 T}{4}\left(|A-B|^2 - |C-D|^2\right) = k^2 T \left(\frac{u^2-\alpha^2}{u^2+1}\right)|A|^2$$

$$\langle F_2 \rangle = \frac{k^2 T}{4}\left(|C-D|^2 - |E|^2\right) = k^2 T \left(\frac{\alpha^2}{u^2+1}\right)|A|^2$$

where $u = 2\alpha\left(cos(ka) - \alpha sin(ka)\right)$

Let $z = ka$ and $\beta = \frac{m}{\mu a}$, then $u = \beta z \left(cosz - \frac{\beta}{2} zsinz \right)$ then the Casimir force between the beads is

$$F_C = \frac{\langle F_1 \rangle - \langle F_2 \rangle}{2} = \frac{T}{2} \left(\frac{|A|^2}{a^2} \right) \left(\frac{u^2 - \beta^2 z^2/2}{u^2 + 1} \right) z^2 \qquad (17)$$



In the limit $\beta z >> 1$ we get

$$\frac{F_C}{T} \approx \frac{z^2 |A|^2}{2} \frac{1}{a^2} \qquad (18)$$

Casimir effect for Scalar Field in One dimention

The Hamiltonian can be written as $H = \sum_n \hbar\omega_n \left(a_n^\dagger a_n + \frac{1}{2}\right)$. Imagine a macroscopic boundaries at x=0 and x=a with the boundary conditions that $\psi(0) = \psi(a) = 0$, with these boundary conditions we have

$$\psi(x,t) = \sum_n \left(a_n e^{-i\omega t} + a_n^\dagger e^{i\omega t}\right) sin(k_n x) \text{ where } k_n = \frac{n\pi}{a} \tag{19}$$

The Vaccumm energy between the plates is

$$E_0 = \frac{1}{2}\sum_n \omega_n = \frac{1}{2}\sum_n k_n \tag{20}$$

The Vaccumm energy between without the plates

$$E_{vac} = \frac{a}{2\pi}\int_0^\infty kdk \tag{21}$$

Both of these Energies are infinite but the difference between the two can be finite, for this consider a regularised infinite sum of energies within the plates as

$$E_0 = \frac{\pi}{2a}\sum_{n=1}^{\infty} n = \lim_{s\to 0}\frac{1}{2}\sum_{n=1}^{\infty}\frac{n\pi}{a}e^{-\frac{n\pi}{a}s} = \lim_{s\to 0} -\frac{1}{2}\frac{\partial}{\partial s}\sum_{n=1}^{\infty} e^{-\frac{n\pi}{a}s}$$
$$= \lim_{s\to 0}\frac{\pi}{8a} sinh^{-2}\left(\frac{\pi s}{2a}\right)$$

Expandng in the power series of s we get

$$E_0 = \lim_{s\to 0}\frac{\pi}{2a}\left(\frac{a^2}{\pi^2 s^2} - \frac{1}{12} + O(s^2)\right) = \lim_{s\to 0}\left(\frac{a}{2\pi s^2}\right) - \frac{\pi}{24a} \tag{22}$$

Similarly

$$E_{vac} = \frac{a}{2\pi}\lim_{s\to 0}\int_0^{\infty} k e^{-ks} dk = \lim_{s\to 0}\frac{a}{2\pi s^2} \tag{23}$$

So even though both E_0 and E_{vac} are infinite the difference between them is the finite Casimir potential energy in one dimention

$$\Delta E = E_0 - E_{vac} = -\frac{\hbar c \pi}{24a} \tag{24}$$

The corresponding Casimir force of attraction is given by

$$F_C = \frac{\partial \Delta E}{\partial a} = \frac{\hbar c \pi}{24a^2} \tag{25}$$

For photons in three dimentions the above formula generalises to [6]

$$\frac{F_C}{A} = \frac{\hbar c \pi^2}{240a^4} \tag{26}$$

Experimental detection of the Casimir Effect

Precise measurement with uncertainty of about 5% was made by Lamoreaux who measured the force between 2.54 cm diam, 0.5 cm thick quartz optical flat, and a spherical lens with radius of curvature 11.3 6 0.1 cm and diameter 4 cm both mounted on a torsion balance which is connected to a feedback circuit which corrects for the change in displacements. [7] .

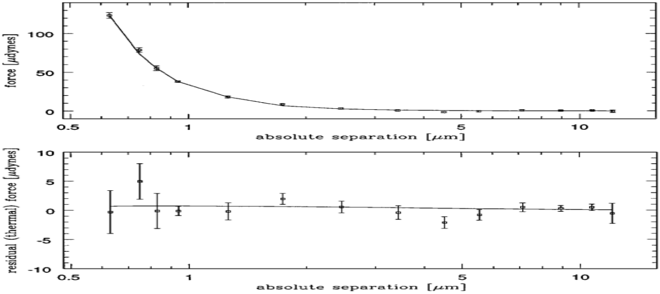
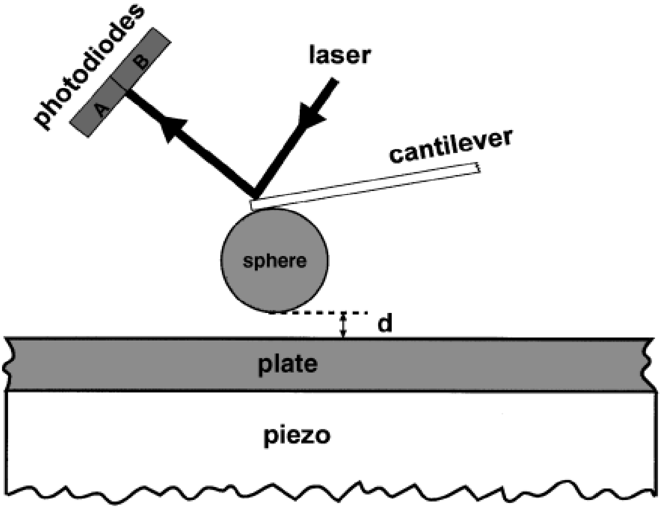


FIG. 4. Top: All data with electric force subtracted, averaged into bins (of varying width), compared to the expected Casimir force for a 11.3 cm spherical plate. Bottom: Theoretical Casimir force, without the thermal correction, subtracted from top plot; the solid line shows the expected residuals.

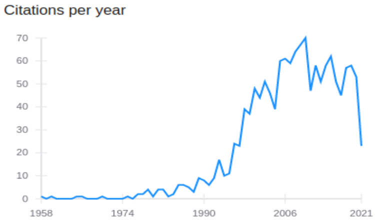
Experimental detection of the Casimir Effect



Recent theories

• The idealization of conducting plates does not hold in physical situations since metals with finite conductivity become transparent to radiation beyond a frequency. The geometry also plays an important role in calculation of Casimir force.

• The analysis was generalized to arbitrary dielectric by Lifshitz. His theory for two metal plates reduces to Casimir's idealized $1/a^4$ force law for separations much greater than the skin depth of the metal, and conversely reduces to the $1/a^3$ force law of the London dispersion force with Hamaker constant. The force for intermediate separations is determined by the dispersion of the materials.



In a famous quote by Rudi Parlos it is said that if the hard copies of physical reviews on shelf on Casimir effect keep growing at the current rate then pretty soon the velocity by which the width of pile grows will overtake the speed of light but that does not violate relativity since no information is begin conveyed.

Reality of Vaccumm energy and the cosmological constant problem

- $\epsilon_{dark} = \sum_n \frac{\hbar\omega}{2} = \frac{\hbar c}{2\pi} \int_0^{k_{max}} k^3 dk = \frac{\hbar c}{8\pi} k_{max}^4 = \frac{\hbar c}{8\pi} \left(\frac{2\pi}{l_{plank}} \right)^4$
  where $l_{plank} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-35}$

$$\Lambda = \epsilon_{dark} \approx 10^{165} \tag{27}$$

- Experimentally detected value for the cosmological constant

$$\Lambda = 3 \left(\frac{H_0}{c} \right)^2 \Omega_A \approx 10^{-52}$$

Casimir, H. B. G.; Polder, D. (15 February 1948). "The Influence of Retardation on the London-van der Waals Forces". Physical Review. 73 (4): 360–372. Bibcode:1948PhRv...73..360C. doi:10.1103/PhysRev.73.360. ISSN 0031-899X.

Molecular driving forces, Ken A Dill, page 452

Denardo, B. C.; Puda, J. J.; Larraza, A. S. (2009). "A water wave analog of the Casimir effect". American Journal of Physics. 77 (12): 1095. Bibcode:2009AmJPh..77.1095D. doi:10.1119/1.3211416.

Larraza, A. S.; Denardo, B. (1998). "An acoustic Casimir effect". Physics Letters A. 248 (2–4): 151. Bibcode:1998PhLA..248..151L. doi:10.1016/S0375-9601(98)00652-5.

Griffiths, D. J.; Ho, E. (2001). "Classical Casimir effect for beads on a string". American Journal of Physics. 69 (11): 1173. Bibcode:2001AmJPh..69.1173G. doi:10.1119/1.1396620.

Casimir, H. B. G. (1948). "On the attraction between two perfectly conducting plates" (PDF). Proc. Kon. Ned. Akad. Wet. 51: 793.

Demonstration of the Casimir Force in the 0.6 to 6 mm Range S. K