(v) $P(X < 2)\ Y \leq 3) = +\ P(X = 0, Y = 1) + P(X = 0, Y = 2) + P(X = 0, Y = 3) + P(X = 1, Y = 1)$
$+ P(X = 1, Y = 2) + P(X = 1, Y = 3)$

$$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} = \frac{1+2+2+4}{32} = \frac{9}{32}$$

Example 3.2. Let X and Y have joint p.d.f.

X / Y	*−1*	*0*	*1*
0	b	$2b$	b
1	$3b$	$2b$	b
2	$2b$	b	$2b$

Find marginal distribution of X and Y. Also find conditional distribution of X given $Y = 1$.

Solution.

Y / X	*−1*	*0*	*1*	$p_Y(y)$
0	b	$2b$	b	$4b$
1	$3b$	$2b$	b	$6b$
2	$2b$	b	$2b$	$5b$
$p_X(x)$	$6b$	$5b$	$4b$	$15b$

Marginal distribution of X

$$P(X = -1) = 6b, \quad P(X = 0) = 5b, \quad P(X = 1) = 4b$$

Marginal distribution of Y

$$P(Y = 0) = 4b, \quad P(Y = 1) = 6b, \quad P(Y = 2) = 5b$$

Conditional distribution of X when $Y = 1$

$$P(X = x/Y = 1) = \frac{P(X = x \cap Y = 1)}{P(Y = 1)}$$

$$= \begin{cases} \dfrac{P(X = -1 \cap Y = 1)}{P(Y = 1)} \\ \dfrac{P(X = 0 \cap Y = 1)}{P(Y = 1)} \\ \dfrac{P(X = 1 \cap Y = 1)}{P(Y = 1)} \end{cases}$$

$$= \begin{cases} \dfrac{3b}{6b} = \dfrac{1}{2} & \text{when} \quad X = -1/Y = 1 \\ \dfrac{2b}{6b} = \dfrac{1}{3} & \text{when} \quad X = 0/Y = 1 \\ \dfrac{b}{6b} = \dfrac{1}{6} & \text{when} \quad X = 1/Y = 1 \end{cases}$$

Example 3.3. The joint probability distribution of X and Y is given in the following table.

X \ Y	1	3	9
2	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$
4	$\frac{1}{4}$	$\frac{1}{4}$	0
6	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$

(*a*) Find the marginal probability distribution of Y.

(*b*) Find the conditional distribution of Y *given* $X = 4$.

(*c*) Find covariance of X and Y.

(*d*) Are X and Y independent?

Solution.

X \ Y	1	3	9	$f_X(x)$
2	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{6}{24}$
4	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{2}{4}$
6	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{6}{24}$
$f_Y(y)$	$\frac{4}{8}$	$\frac{8}{24}$	$\frac{2}{12}$	1

(*a*) Marginal probability distribution of Y

$$P(Y = 1) = \frac{4}{8} = \frac{1}{2}, \quad P(Y = 2) = \frac{8}{24} = \frac{1}{3}, \quad P(Y = 3) = \frac{2}{12} = \frac{1}{6}$$

(*b*) Conditional probability of Y when $X = 4$

$$P(Y = y/X = 4) = \frac{P(Y = y \cap X = 4)}{P(X = 4)}$$

$$P(Y = 1/X = 4) = \frac{P(Y = 1 \cap X = 4)}{P(X = 4)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

$$P(Y = 3/X = 4) = \frac{P(Y = 3 \cap X = 4)}{P(X = 4)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

$$P(Y = 9/X = 4) = \frac{P(Y = 9 \cap X = 4)}{P(X = 4)} = 0$$

(*c*) To find Cov (X, Y) we will first find $E(XY)$, $E(X)$ and $E(Y)$

$$E(X) = \Sigma x f_X(x) = 2 \times \frac{6}{24} + 4 \times \frac{2}{4} + 6 \times \frac{6}{24} = 4$$

$$E(Y) = \Sigma y f_Y(y) = 1 \times \frac{4}{8} + 3 \times \frac{8}{24} + 9 \times \frac{2}{12} = 3$$

$$E(X, Y) = \Sigma xy\, f_{X,Y}(x, y)$$

$$= \left(2\times\frac{1}{8}+4\times\frac{1}{4}+6\times\frac{1}{8}\right)+\left(6\times\frac{1}{24}+12\times\frac{1}{4}+18\times\frac{1}{24}\right)$$

$$+\left(18\times\frac{1}{12}+36\times 0+54\times\frac{1}{12}\right) = 12$$

$$\text{Cov}(X, Y) = E(XY) - E(X)\,E(Y) = 12 - 12 = 0$$

(*d*) $$f_{X,Y}(4, 3) = \frac{1}{4} \quad f_X(4) = \frac{2}{4} \quad f_Y(3) = \frac{8}{24}$$

$$f_X(4)\, f_Y(3) = \frac{2}{4}\times\frac{8}{24} = \frac{1}{6}$$

$\therefore$ $$f_{X,Y}(4, 3) \neq f_X(4)\, f_Y(3)$$

$\therefore$ X and Y are not independent.

Example 3.4. Two tetrahedra with sides numbered 1 to 4 are tossed. Let X denote the number on the downturned face of the first tetrahedron and Y the larger of the downturned numbers. Find

(*i*) the joint density function of X and Y,

(*ii*) the marginal density function of X and Y,

(*iii*) $P[X \leq 2, Y \leq 3]$,

(*iv*) the conditional distribution of Y given $X = 2$ and $X = 3$,

(*v*) $E[Y/X = 2]$ and $E[Y/X = 3]$,

(*vi*) $\rho(X, Y)$

Solution. (*i*) It is given that X denotes the number of the downturned face of the first tetrahedron and Y the larger of the numbers on the two downturned faces. Then (X, Y) takes the values:

(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)

The joint discrete density function of X and Y is given below:

(x, y)	(1, 1)	(1; 2)	(1, 3)	(1, 4)	(2, 2)	(2, 3)	(2, 4)	(3, 3)	(3, 4)	(4, 4)
$f_{X,Y}(x, y)$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{4}{16}$

We can also write the above data as follows:

Y \ X	*1*	*2*	*3*	*4*	$f_Y(x)$
1	$\frac{1}{16}$	0	0	0	$\frac{1}{16}$
2	$\frac{1}{16}$	$\frac{2}{16}$	0	0	$\frac{3}{16}$
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	0	$\frac{5}{16}$
4	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{7}{16}$
$f_X(x)$	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{4}{16}$	1

(*ii*) Marginal distribution of X. From the above table, we get

$$f_X(1) = \frac{4}{16},\quad f_X(2) = \frac{4}{16},\quad f_X(3) = \frac{4}{16},\quad f_X(4) = \frac{4}{16}$$

Marginal distribution of Y. From the above table, we get

$$f_Y(1) = \frac{1}{16},\quad f_Y(2) = \frac{3}{16},\quad f_Y(3) = \frac{5}{16},\quad f_Y(4) = \frac{7}{16}$$

(*iii*) $P[X \leq 2, Y \leq 3] = P[X = 1, Y = 1] + P[X = 1, Y = 2] + P[X = 1, Y = 3] + P[X = 2, Y = 2]$
$+ P[X = 2, Y = 3]$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{2}{16} + \frac{1}{16} = \frac{6}{16} = \frac{3}{8}$$

(*iv*) Conditional distribution of Y given $X = 2$. We have

$$P[Y = y \cap X = 2] = P[X = 2]\, P[Y = y/X = 2]$$

$$\therefore \quad P[Y = y \mid X = 2] = \frac{P[Y = y \cap X = 2]}{P[X = 2]}$$

Now $$P[Y = 2 \mid X = 2] = \frac{P[Y = 2 \cap X = 2]}{P[X = 2]} = \frac{2/16}{4/16} = \frac{1}{2},$$

$$P[Y = 3 \mid X = 2] = \frac{P[Y = 3 \cap X = 2]}{P[X = 2]} = \frac{1/16}{4/16} = \frac{1}{4},$$

$$P[Y = 4 \mid X = 2] = \frac{P[Y = 4 \cap X = 2]}{P[X = 2]} = \frac{1/16}{4/16} = \frac{1}{4}$$

Conditional distribution of Y given $X = 3$. We have

$$P[Y = 3 \mid X = 3] = \frac{P[Y = 3 \cap X = 3]}{P[X = 3]} = \frac{3/16}{4/16} = \frac{3}{4}$$

$$P[Y = 4 \mid X = 3] = \frac{P[Y = 4 \cap X = 3]}{P[X = 3]} = \frac{1/16}{4/16} = \frac{1}{4}$$

(v) By definition

$$E[Y \mid X = 2] = \Sigma y_j P[Y = y_j \mid X = 2]$$

$$= 2P[Y = 2 \mid X = 2] + 3P[Y = 3 \mid X = 2] + 4P[Y + 4 \mid X = 2]$$

$$= 2\times\frac{1}{2} + 3\times\frac{1}{4} + 4\times\frac{1}{4} = \frac{11}{4}$$ [Use part (iv)]

$$E[Y \mid X = 3] = \Sigma y_j P[Y = y_j \mid X = 3]$$

$$= 3P[Y = 3 \mid X = 3] + 4P[Y = 4 \mid X = 3]$$

$$= 3\times\frac{3}{4} + 4\times\frac{1}{4} = \frac{13}{4}$$ [Use part (iv)]

(vi) Now we shall find $\rho(X, Y)$ *i.e.*, the correlation coefficient between X and Y. From the table, we have

$$E[X] = \Sigma x f_X(x) = 1\times\frac{4}{16} + 2\times\frac{4}{16} + 3\times\frac{4}{16} + 4\times\frac{4}{16} = \frac{5}{2}$$

$$E[X^2] = \Sigma x^2 f_X(x) = 1^2\times\frac{4}{16}\times 2^2\times\frac{4}{16} + 3^2\times\frac{4}{16} + 4^2\times\frac{4}{16} = \frac{15}{2}$$

$$E[Y] = \Sigma y f_Y(y) = 1\times\frac{1}{16} + 2\times\frac{3}{16} + 3\times\frac{5}{16} + 4\times\frac{7}{16} = \frac{25}{8}$$

$$E[Y^2] = \Sigma y^2 f_Y(y) = 1^2\times\frac{1}{16} + 2^2\times\frac{3}{16} + 3^2\times\frac{5}{16} + 4^2\times\frac{7}{16} = \frac{85}{8}$$

$$E[XY] = \Sigma xy f_{X,Y}(x, y) = \left(1\times\frac{1}{16} + 2\times\frac{1}{16} + 3\times\frac{1}{16} + 4\times\frac{1}{16}\right)$$

$$+\left(4\times\frac{2}{16} + 6\times\frac{1}{16} + 8\times\frac{1}{16}\right) + \left(9\times\frac{3}{16} + 12\times\frac{1}{16}\right) + 16\times\frac{4}{16} = \frac{135}{16}$$

$$\therefore \quad \text{Var}[X] = E[X^2] - \{E[X]\}^2 = \frac{15}{2} - \frac{25}{4} = \frac{5}{4}$$

$$\text{Var}[Y] = E[Y^2] - \{E[Y]\}^2 = \frac{85}{8} - \left(\frac{25}{8}\right)^2 = \frac{55}{64}$$

$$\text{Cov}(X, Y) = E[X, Y] - E[X]\,E[Y] = \frac{135}{16} - \frac{5}{2}\times\frac{25}{8} = \frac{5}{8}$$

Hence,

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} = \frac{5/8}{\sqrt{5/4\times 55/64}} = \frac{2}{\sqrt{11}}$$

SOLVED EXAMPLES

Example 3.8. Find k so that $f(x, y) = kxy$, $1 \le x \le y \le 2$ will be a joint probability density function.

Solution. We have

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = \int_{1}^{2}\int_{1}^{2} kxy\, dx\, dy$$

$$= k\int_{1}^{2} x\left(\int_{x}^{2} y\, dy\right) dx = k\int_{1}^{2} x \cdot \frac{1}{2}(4 - x^2)\, dx$$

$$= \frac{k}{2}\int_{1}^{2}(4x - x^3)\, dx = \frac{k}{2}\left[2x^2 - \frac{x^4}{4}\right]_1^2$$

$$= \frac{k}{2}\left[2(4-1) - \frac{1}{4}(16-1)\right] = \frac{k}{2}\left(6 - \frac{15}{4}\right) = \frac{9k}{8}$$

Hence, $1 = \frac{9k}{8} \Rightarrow k = \frac{8}{9}$.

Example 3.9. Find K so that $f(x, y) = K(x + y)$, $0 < x < 1$ and $0 < y < 1$, is a joint probability density function.

Solution. We have

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = \int_{0}^{1}\int_{0}^{1} K(x + y)\, dx\, dy$$

$$= K\int_{0}^{1}\left[\left(xy + \frac{1}{2}y^2\right)\right]_0^1 dx = K\int_{0}^{1}\left(x + \frac{1}{2}\right) dx$$

$$= K\left(\frac{1}{2} + \frac{1}{2}\right) = K. \text{ Hence } K = 1.$$

Example 3.10. Let the joint p.d.f of X and Y be

$$f(x, y) = (x + y),\ 0 \le x \le 1,\ 0 \le y \le 1$$
$$= 0, \text{ otherwise.}$$

Find $P\left[0 < X < \frac{1}{2}, 0 < Y < \frac{1}{4}\right]$, $E[X]$, $E[Y]$, $E[XY]$, $E[X + Y]$, $\rho[X, Y]$.

Solution. We have

$$P\left[0 < X < \frac{1}{2}, 0 < Y < \frac{1}{4}\right] = \int_{0}^{1/2}\left[\int_{0}^{1/4}(x + y)\, dy\right] dx = \int_{0}^{1/2}\left[xy + \frac{1}{2}y^2\right]_0^{1/4} dx$$

$$= \int_{0}^{1/2}\left(\frac{1}{4}x + \frac{1}{32}\right) dx = \frac{1}{4} \times \frac{1}{8} + \frac{1}{64} = \frac{3}{64}.$$

Now $$E[X] = \int_0^1 \int_0^1 xf(x, y)\, dx\, dy = \int_0^1 \int_0^1 x(x+y)\, dx\, dy$$

$$= \int_0^1 \left[x^2 y + x\left(\frac{1}{2} y^2\right)\right]_0^1 dx = \int_0^1 \left(x^2 + \frac{1}{2}x\right) dx$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

Similarly, $$E[Y] = \int_0^1 \int_0^1 yf(x, y)\, dx\, dy = \frac{7}{12}.$$

Now $$E[XY] = \int_0^1 \int_0^1 xyf(x, y)\, dx\, dy \int_0^1 \int_0^1 xy(x+y)\, dx\, dy$$

$$= \int_0^1 \left[x^2\left(\frac{1}{2}y^2\right) + x\left(\frac{1}{3}y^3\right)\right]_0^1 dx = \int_0^1 \left(\frac{1}{2}x^2 + \frac{1}{3}x\right) dx$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

$$E[X+Y] = \int_0^1 \int_0^1 (x+y)\, f(x, y)\, dx\, dy = \int_0^1 \int_0^1 (x+y)^2\, dx\, dy$$

$$= \int_0^1 \int_0^1 .(x^2 + 2xy + y^2)\, dx\, dy$$

$$= \int_0^1 \left[x^2 y + xy^2 + \left(\frac{1}{3}y^3\right)\right]_0^1 dx$$

$$= \int_0^1 \left(x^2 + x + \frac{1}{3}\right) dx = \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{7}{6}.$$

We have $\text{cov}\,[X, Y] = E\,[XY] - E\,[X]\,E\,[Y] = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = -\frac{1}{144}.$

Now $$E\,[X^2] = \int_0^1 \int_0^1 x^2 f\,(x, y)\, dx\, dy = \int_0^1 \int_0^1 x^2\,(x+y)\, dx\, dy$$

$$= \int_0^1 \left[x^3 y + x^2\left(\frac{1}{2}y^2\right)\right]_0^1 dx$$

$$= \int_0^1 \left(x^3 + \frac{1}{2}x^2\right) dx = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

Similarly, $$E[Y^2] = \frac{5}{12}.$$

Now $$\text{var}[X] = E[X^2] - (E[X])^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}.$$

Similarly, $$\text{var}[Y] = \frac{11}{144}.$$

Hence, $$\rho[X, Y] = \frac{\text{cov}[X, Y]}{\sqrt{\text{var}[X]\,\text{var}[Y]}} = \frac{-\frac{1}{144}}{\sqrt{\frac{11}{144} \times \frac{11}{144}}} = -\frac{1}{11}.$$

Example 3.11. The joint p.d.f. of (X, Y) is given by

$$f(x, y) = 2;\ 0 < x < 1,\ 0 < y < x$$
$$= 0\ ;\ \text{elsewhere},$$

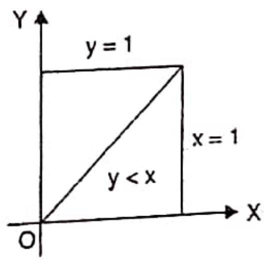
(i) Find the marginal density functions of X and Y.

(ii) Find the conditional density function of Y given $X = x$ and that of X given $Y = y$

(iii) Are X and Y independent?

Solution. It may be observed that

$$f(x, y) \geq 0 \text{ and } \int_0^1 \left(\int_0^x 2\,dy \right) dx = \int_0^1 2x\,dx = 1$$



(i) The marginal density function of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dx = \int_0^x 2dy = 2x,\ 0 < x < 1$$
$$= 0, \text{ elsewhere.}$$

The marginal density function of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,dx = \int_y^1 2dx = 2(1 - y),\ 0 < y < 1$$
$$= 0, \text{ elsewhere.}$$

(ii) The conditional density function of Y given $X = x$ is

$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x},\ 0 < x < 1.$$

The conditional density function of X given $Y = y$ is

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} = \frac{2}{2(1 - y)} = \frac{1}{1 - y},\ 0 < y < 1.$$

(iii) $f_X(x) f_Y(y) = 2x \times 2(1 - y) = 4x(1 - y) \neq f(x, y)$.

Hence X and Y are not independent.

Example 3.12. If X and Y are two random variables having joint density function.

$$f(x, y) = \frac{1}{8}(6 - x - y);\ 0 < x < 2,\ 2 < y < 4$$
$$= 0, \text{ otherwise}$$

Find (*i*) $P[X < 1 \cap Y < 3]$, (*ii*) $P[X + Y < 3]$, (*iii*) $P[X < 1 \mid Y < 3]$.
Find also the marginal and conditional distributions.

Solution. We have

(*i*) $P[X < 1 \cap Y < 3] = \int_{-\infty}^{1}\int_{-\infty}^{3} f(x, y)\,dx\,dy$

$$= \int_0^1\int_2^3 \frac{1}{8}(6 - x - y)\,dx\,dy$$

$$= \frac{1}{8}\int_0^1 \left\{(6 - x) - \frac{1}{2}(9 - 4)\right\} dx = \frac{1}{8}\int_0^1 \left(\frac{7}{2} - x\right) dx$$

$$= \frac{1}{8}\left(\frac{7}{2} - \frac{1}{2}\right) = \frac{3}{8}$$

(*ii*) $P[X + Y < 3] = \int_0^1\int_2^{3-x} \frac{1}{8}(6 - x - y)\,dx\,dy$

$$= \frac{1}{8}\int_0^1 \left[(6 - x)(1 - x) - \frac{1}{2}\{(3 - x)^2 - 4\}\right] dx$$

$$= \frac{1}{16}\int_0^1 (x^2 - 8x + 7)\,dx = \frac{1}{16}\left(\frac{1}{3} - 4 + 7\right) = \frac{5}{24}$$

(*iii*) Firstly, we calculate $P[Y < 3]$. We have

$$P[Y < 3] = \int_0^2\int_2^3 \frac{1}{8}(6 - x - y)\,dx\,dy$$

$$= \frac{1}{8}\int_0^2 \left\{(6 - x) - \frac{1}{2}(9 - 4)\right\} dx$$

$$= \frac{1}{8}\int_0^2 \left(\frac{7}{2} - x\right) dx = \frac{1}{8}\left(\frac{7}{2}\times 2 - 2\right) = \frac{5}{8}.$$

Now $P[X < 1 \mid Y < 3] = \dfrac{P[X < 1 \cap Y < 3]}{P[Y < 3]} = \dfrac{3/8}{5/8} = \dfrac{3}{5}$.

(*iv*) The marginal distribution of X is given by

$$f_X(x) = \int_2^4 f(x, y)\,dy = \frac{1}{8}\int_2^4 (6 - x - y)\,dy$$

$$= \frac{1}{8}\left[(6-x)\times 2 - \frac{1}{2}(16-4)\right]$$

$$= \begin{cases} \frac{1}{4}(3-x), & 0<x<2 \\ 0, & \text{otherwise.} \end{cases}$$

The marginal distributions of Y is given by

$$= \int_0^2 \frac{1}{8}(6-x-y)\,dx = \frac{1}{8}\left[6\times 2 - \frac{1}{2}\times 4 - 2y\right]$$

$$= \begin{cases} \frac{1}{4}(5-y), & 2<y<4 \\ 0, & \text{otherwise.} \end{cases}$$

The conditional distributions of X and Y are

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{6-x-y}{2(5-y)}, 0<x<2, 2<y<4$$

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(y)} = \frac{6-x-y}{2(3-x)}, 0<x<2, 2<y<4.$$

Example 3.13. The probability density function of a continuous bivariate distribution is given by

$$f(x, y) = x + y, \text{ where } 0 \le x \le 1, 0 \le y \le 1$$
$$= 0, \text{ otherwise.}$$

Find the marginal distributions and the correlation coefficient of x and y.

Solution. The marginal distribution, of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (x+y)\,dy$$

$$= \left[xy + \frac{1}{2}y^2\right]_0^1 = x + \frac{1}{2}.$$

$$\therefore \quad f_X(x) = x + \frac{1}{2}, \ 0 \le x \le 1$$
$$= 0, \text{ otherwise}$$

Similarly

$$f_Y(y) = y + \frac{1}{2}, 0 \le y \le 1$$
$$= 0 \quad , \text{otherwise.}$$

Now

$$E[X] = \int_0^1 x f_X(x)\,dx = \int_0^1 x\left(x + \frac{1}{2}\right)dx = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

By symmetry, $E[Y] = E[X] = \frac{7}{12}$.

Now $$E[XY] = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \int_0^1 \left(x^2 \cdot \frac{1}{2} + x \cdot \frac{1}{3}\right)dx$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

$\therefore$ $$\text{cov}(X, Y) = E[XY] - E[X]\,E[Y] = \frac{1}{3} - \frac{7}{12} \times \frac{7}{12} = -\frac{1}{144}.$$

Now $$E[X^2] = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 x^2\left(x + \frac{1}{2}\right)dx = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$\therefore$ $$\text{var}[X] = E[X^2] - \{E[X]\}^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}.$$

By symmetry, $\text{var}[Y] = \frac{11}{144}$.

The correlation coefficient of X and Y is given by

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X\,\sigma_Y} = \frac{-1/144}{11/144} = -\frac{1}{11}.$$

Example 3.14. Two random variables X and Y have the joint density

$$f(x, y) = 2 - x - y,\ 0 < x < 1,\ 0 < y < 1$$

Find the marginal and conditional density functions of X and Y, and show that $\rho[X, Y] = -1/11$: Find $E[X \mid y]$.

Solution. The marginal density function of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dy = \int_0^1 (2 - x - y)\,dy = 2 - x - \frac{1}{2} = \frac{3}{2} - x.$$

Thus $$f_X(x) = \frac{3}{2} - x,\ 0 < x < 1$$
$$= 0, \text{ otherwise.}$$

By symmetry, $$f_Y(y) = \frac{3}{2} - y,\ 0 < y < 1$$
$$= 0, \text{ otherwise.}$$

The conditional density functions of X and Y are

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} = \frac{2 - x - y}{\frac{3}{2} - y},\ 0 < x, y < 1;$$

$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)} = \frac{2 - x - y}{\frac{3}{2} - x},\ 0 < x, y < 1.$$

ution is given

Now
$$E[X \mid y] = \int_{-\infty}^{\infty} x \cdot \frac{f(x, y)}{f_Y(y)}\, dx = \int_0^1 \frac{x(2 - x - y)}{\frac{3}{2} - y}\, dx$$

$$= \frac{1}{\left(\frac{2}{3} - y\right)}\left[1 - \frac{1}{3} - \frac{1}{2}y\right] = \frac{4 - 3y}{9 - 6y}, 0 < y < 1.$$

Now we shall determine the correlation coefficient of X and Y.

$$E[X] = \int_0^1 x f_X(x)\, dx = \int_0^1 x\left(\frac{3}{2} - x\right) dx = \frac{3}{2} \cdot \frac{1}{2} - \frac{1}{3} = \frac{5}{12}.$$

By symmetry, $E[Y] = E[X] = \dfrac{5}{12}$.

Now
$$E[XY] = \int_0^1 \int_0^1 xy(2 - x - y)\, dx\, dy$$

$$= \int_0^1 \left(2x \cdot \frac{1}{2} - x^2 \cdot \frac{1}{2} - x \cdot \frac{1}{3}\right) dx = \frac{1}{2} - \frac{1}{6} - \frac{1}{6} = \frac{1}{6}.$$

$\therefore$
$$\text{cov}(X, Y) = E[XY] - E[X]\, E[Y] = \frac{1}{6} - \frac{5}{12} \cdot \frac{5}{12} = -\frac{1}{144}.$$

$$E[X^2] = \int_0^1 x^2 f_X(x)\, dx = \int_0^1 x^2\left(\frac{3}{2} - x\right) dx$$

$$= \frac{3}{2} \cdot \frac{1}{3} - \frac{1}{4} = \frac{1}{4}.$$

$\therefore$
$$\text{var}[X] = E[X^2] - \{E[X]\}^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{11}{144}.$$

Hence,
$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X\, \sigma_Y} = \frac{-1//144}{11/144} = -\frac{1}{11}.$$

EXERCISE 3.2

1. If $f(x, y) = e^{-(x+y)}$, $x \geq 0$, $y \geq 0$
   $= 0$ elsewhere

   is joint probability density function of random variable X and Y, find:

   (a) $P[X < 1]$

   (b) $P[X > 4]$

   (c) $P[X + Y < 1]$ [Ans. (a) $1 - e^{-1}$, (b) $\frac{1}{2}$, (c) $1 - 2e^{-1}$]