Solⁿ of ODE by Numerical Methods

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- IVP (Initial Value Problem)

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0, \quad < t < b$$

1st order & 1st degree

- Euler's Method*
- Runge-Kutta Method (R-K Method)
- Predictor-Corrector Method)

- Euler's Method:-

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0$$

Need to find $y(t^*) = ?$

Method: Divide $[t_0, t^*]$ into n equal sub-intervals, t_0 t^* $h = \frac{t^* - t}{n}$

$t_1 = t_0 + h, \; t_2 = t_1 + h, \ldots,$

$t^* = t_n = t_{n-1} + h$

$y_0 = y(t_0), \; y_1 = y(t_1), \; y_2 = y(t_2), \ldots, y_n = y(t_n) = y(t^*)$ goal.

$y'(t_0) = \frac{dy}{dt}\Big|_{t=t_0} = f(t_0, y(t_0)) = f(t_0, y_0)$

$y'(t_0) = \lim_{h \to 0} \frac{y(t_0+h) - y(t_0)}{h}$

$y'(t_0) \simeq \frac{y(t_0+h) - y(t_0)}{h}$

$y(t_0+h) \simeq y(t_0) + h\, y'(t_0)$

$y_1 = y_0 + h f(t_0, y_0)$

$\vdots$

$y_1 = y_0 + h f(t_0, y_0)$

$y_2 = y_1 + h f(t_1, y_1)$

$\vdots$

$y_n = y_{n-1} + h f(t_{n-1}, y_{n-1})$

e.g. $y(0) = 1$ (initial), $\frac{dy}{dt} = y$, Estimate $y(0.5)$ in 5 steps using Euler's method. in 5 steps (equal)

$f(t,y) = y$ | $t_0 = 0, \; y_0 = 1, \; t_n = 0.5$

0	0.1	0.2	0.3	0.4	0.5
t_0	t_1	t_2	t_3	t_4	t_5

Goal is to find $y_5 = y(t_5) = y(0.5)$

$h = \frac{0.5 - 0}{5} = 0.1$

$y_1 = y_0 + h f(t_0, y_0) = 1 + 0.1 \times 1 = 1.1$ $\quad f(t_0, y_0) = y_0 = 1$

$y_2 = y_1 + h f(t_1, y_1) = 1.1 + 0.1 \times 1.1$ $\quad f(t_1, y_1) = y_1 = 1.1$

$= 1.1$

$y_2 = \underline{y_1} + hf(t_1, y_1) = 1.1 + 0.1 \times 1.1$ $\qquad f(t_1, y_1) = y_1 = 1.1$

$= 1.21$

$y_3 = y_2 + hf(t_2, y_2) = 1.21 + 0.1 \times 1.21 = 1.331$

$y_4 = y_3 + hf(t_3, y_3) = 1.331 + 0.1 \times 1.331 = 1.4641$

$y_5 = y_4 + hf(t_4, y_4) = 1.4641 + 0.1 \times 1.4641$

$= 1.61051$ (Ans)

approx value

[Check! $\frac{dy}{dt} = y$

$y = e^t$, $y(0.5) = e^{0.5} \simeq 1.64872$ (exact value)]

e.g. Use Euler's method with $\underline{\text{step size } 0.1}$, to $\underline{\text{find}}$ the $\underline{\text{sol}^n}$ of the eq^n $\frac{dy}{dt} = t^2 + y^2$, $y(0) = 1$ in the range $0 \le t \le 0.4$.

Sol^n!

$f(t, y) = t^2 + y^2$, $y(0) = 1$, $t_0 = 0$, $y_0 = 1$

$h = 0.1$

$t_0 = 0$, $t_1 = 0.1$, $t_2 = 0.2$, $t_3 = 0.3$, $t_4 = 0.4$

$y_0 = \underline{1}$ $\quad y_1 = ?$ $\quad y_2 = ?$ $\quad y_3 = ?$ $\quad y_4 = ?$

$y_1 = y_0 + hf(t_0, y_0) = 1 + 0.1 \times (0^2 + 1^2)$ $\qquad f(t_0, y_0) = f(0, 1) = t_0^2 + y_0^2 = 0^2 + 1^2$

$= 1.1$

$y_2 = y_1 + hf(t_1, y_1) = 1.1 + 0.1 \times (t_1^2 + y_1^2) = 1.1 + 0.1((0.1)^2 + (1.1)^2)$

$= 1.222$

$y_3 = y_2 + hf(t_2, y_2) = 1.222 + 0.1 \times (t_2^2 + y_2^2)$

$= 1.222 + 0.1 \times ((0.2)^2 + (1.222)^2)$

$= 1.3753284$

$y_4 = y_3 + hf(t_3, y_3) = y_3 + 0.1 \times (t_3^2 + y_3^2) = 1.573481221$

e.g. $\frac{dy}{dt} = t + \sqrt{y}$, $y(0) = 1$, $0 \le t \le 0.4$. $(h = 0.1)$

$t_1 = 0.1$

$y_0 = 1$, $y_1 = y_0 + hf(t_0, y_0) = 1 + 0.1(t_0 + \sqrt{y_0}) = 1.1$

$y_2 = y_1 + hf(t_1, y_1) = 1.1 + 0.1(t_1 + \sqrt{y_1}) = 1.21488$

$y_3 = y_2 + h(t_2 + \sqrt{y_2}) = 1.34510$

$y_4 = y_3 + h(t_3 + \sqrt{y_3}) = 1.49107$.

□ Practice Problems :-

Ans: 1.01457 (1) $\frac{dy}{dt} = \ln(t+y)$, $y(0) = 1.0$, $h = 0.05$.
Find $y'(0.2)$;

Ans: 1.5067 Write $\frac{dy}{dt} = f(t,y)$ (2) $y y' = t$, $y(0) = 1.5$, $h = 0.1$, find $y(0.2)$

Ans: 2.2493 (3) $y' = \frac{1}{t+y}$, $y(0) = 2$, $h = 0.2$, Find $y(0.6)$