Numerical and Statistical Methods

03 February 2021 09:37

Module No	Description	Lecture Hours
1	**Numerical Methods** Approximation in numerical computation: Truncation and rounding errors, Fixed and floating-point arithmetic, Propagation of errors. Interpolation: Newton forward/backward interpolation, Lagrange's and Newton's divided difference Interpolation. Numerical integration: Trapezoidal rule, Simpson's 1/3 rule, Expression for corresponding error terms. Numerical solution of a system of linear equations: Gauss elimination method, LU Factorization method, Gauss-Seidel iterative method. Numerical solution of Algebraic equation: Bisection method, Regula-Falsi method, Newton-Raphson method. Numerical solution of ordinary differential equation: Euler's method, Runge-Kutta methods	14

Numerical Methods K. Das

2	**Bivariate Probability Distributions** Bivariate distributions and their properties (discrete & continuous), marginal distribution, distribution of sums and quotients, conditional densities & independence. Related problems.	5
3	**Regression** Concept of Regression. Regression Lines. To find the regression equations. Properties of Regression coefficients. Principle of Least Squares, Method of fitting a straight line & a parabola to a given set of observations. Related Problems.	5

→ st. deviations.

4	**Sampling Theory** Random sampling (SRSWR & SRSWOR), parameter, stastistic and its sampling distribution. Standard error of statistic. Sampling distribution of sample mean & variance in random sampling from a normal distribution (statement only). Related problems.	6
5	**Statistical Inference** Estimation of parameters: Unbiased & Consistent estimators. Point & interval estimations. Maximum likelihood estimation of parameters (Binomial, Poisson, & Normal).Confidence intervals & Related problems. Testing of Hypotheses: Its definition-Null & Alternative Hypothesis, Critical Region, Level of significance, Type I and Type II errors, Best Critical Region. Related problems. Large sample tests: Large sample test for single mean, difference of means, single proportion, difference of proportions, standard deviations. Small sample tests: Small sample test for single mean, difference of means & correlation coefficient. Test for ratio of variances - Chi-square test for goodness of fit & independence of attributes.	12
Total		42

Learning Resources:

1. S. Ross, A First Course in Probability, Pearson Education India
2. Miller & Freund's, Probability and Statistics for Engineers, Pearson Education.
3. Spiegel M R., Schiller J.J. and Srinivasan R.A.: Probability and Statistics (Schaum's Outline Series), TMH
4. Gupta & Kapoor, Fundamentals of Mathematical Statistics, Sultan Chand & Sons
5. John E. Freund, Ronald E. Walpole, Mathematical Statistics, Prentice Hall.
6. B.S. Grewal, Numerical Methods, Khanna Publishers
7. Jain, Iyengar, & Jain: Numerical Methods (Problems and Solution), New Age International
8. Balagurusamy: Numerical Methods, Scitech.
9. Baburam: Numerical Methods, Pearson Education
10. Veerarajan, Numerical Methods, Tata McGraw Hill

□ Approximation :-

$\frac{1}{3} = 0.3333\ldots$

approx value of $\frac{1}{3} \approx 0.3333$

0.3333

π

3.14 is an approx^n π

□ Significant digits

$100 = 1 \times 10^2$ — significant

$101 = 1\times10^2 + 0\times10^1 + 1\cdot10^0$

Significant digits are 1, 0, 1 (zero in the middle of two digits is significant)

$0.001 = 1\times10^{-3}$ → significant

2 3 0 4 8 0

Significant

□ Rounding off:-

$0.2357|8$ → rounding upto 4 digits after decimal — add 1, more than 5 — 0.2358

$0.2357|2$ — less than 5 — 0.2357

Problem:

$0.2357|5$ — previous digit is odd, digit is five — Rounding upto 4 significant digits. 0.2358 (add 1 if the previous digit is odd)

$0.2358|5$ → 0.2358 — previous digit is even

□ Actual value is x_T → true value

Aprox value is x_A

E_A = Absolute error $= |x_A - x_T|$

Relative error $= E_R = \frac{|x_A - x_T|}{x_T}$

$x_T = \frac{1}{3}$, $x_A = 0.33$, $E_A = |\frac{1}{3} - 0.33|$, $E_R = \frac{|\frac{1}{3} - 0.33|}{\frac{1}{3}}$

$E_P = \frac{|\frac{1}{3} - 0.33|}{\ldots} \times 100$

$x_T = \frac{1}{3}$, $x_A = 0.33$, $E_A = |\frac{1}{3} - 0.33|$, $E_R = \frac{\ }{\frac{1}{3}}$

Percentage Error $= E_p = \frac{|x_A - x_T|}{x_T} \times 100\%$ $\quad E_p = \frac{|\frac{1}{3} - 0.33|}{\frac{1}{3}} \times 100$

e.g. 4.6285 is rounded off upto four significant digs.
Find E_A, E_R, E_p

$x_T = 4.6285$
$x_A = 4.628$

$E_A = |4.6285 - 4.628| = 0.0005$

$E_R = \frac{E_A}{x_T} = \frac{0.0005}{4.6285} = 1.0804 \times 10^{-4}$

$E_p = E_R \times 100\% = 1.0804 \times 10^{-2}\%$
$= 0.01084\%$ (small error)

- Finding Roots:

Algebraic Eqn: $x^3 - 3x + 1 = 0$

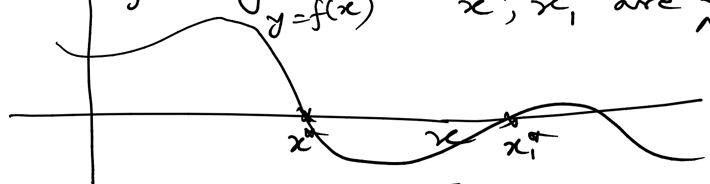
Transcendental Eqn: $xe^x - \cos x + \tan x = 0$

Bisection	Newton-Raphson	Regula-Falsi
(Slow)	(Best)	(Moderate)

- ** Intermediate Value Theorem (IVT)

If $f(x)$ be a real-valued continuous function on $[a,b]$ & $\underbrace{f(a)f(b)}_{f(a)\ \&\ f(b) \text{ are of opposite sign } (+,-)} < 0$, then $f(x) = 0$ has a root (real) in (a,b).

What we mean by finding roots.



x^*, x_1^* are real roots

Here we are finding one real root.

If $f(x) = x^7 - 5x + 1$ degree 7
Classical theorem of Number Theory says "every polynomial of degree n has atmost n roots.

some complex / some real

- Application of IVT → To locate a root of $f(x) = 0$.

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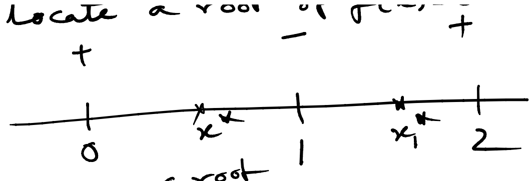
□ Application of IVT → To locate a root of



e.g. $f(x) = x^3 - 3x + 1$

$f(0) = 1 > 0$

$f(1) = 1^3 - 3.1 + 1$

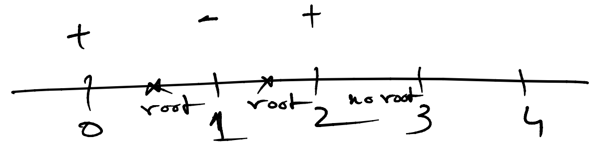
$= -1 < 0$ $\quad f(0)f(1) < 0$

a root should be lying in $(0,1)$

One more root in $(1,2)$

$f(2) = 2^3 - 3.2 + 1$

$= 8 - 6 + 1 > 0$



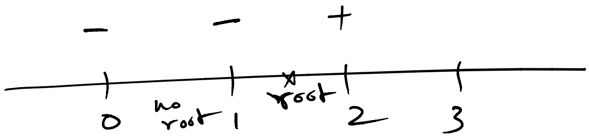
e.g. $f(x) \equiv x^2 - 5x + 1$

$f(0) = 1 > 0$

$f(1) = 1 - 5 + 1 < 0$

$f(2) = 2^2 - 5.2 + 1 > 0$

$f(3) = 3^2 - 5.3 + 1 > 0$



e.g. $f(x) = x^4 - x - 10 = 0$

$f(0) = -10 < 0$

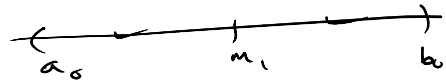
$f(1) = 1 - 1 - 10 < 0$

$f(2) = 2^4 - 2 - 10 > 0$

□ Bisection Method:-

$I_0 = (a_0, b_0)$ Step 1 : Identify an initial interval $I_0 = (a_0, b_0)$ containing a root of $f(x) = 0$.

$I_1 = (a_1, b_1)$ Step 2. Calculate $m_1 = \frac{a_0 + b_0}{2}$



If $f(m_1)f(a_0) < 0$, then the next interval $I_1 = (a_0, m_1)$

Otherwise $I_1 = (m_1, b_0)$. Call $I_1 = (a_1, b_1)$

$I_2 = (a_2, b_2)$ Step 3 : $m_2 = \frac{a_1 + b_1}{2}$

If $f(m_2) f(a_1) < 0$, then $I_2 = (a_1, m_2)$

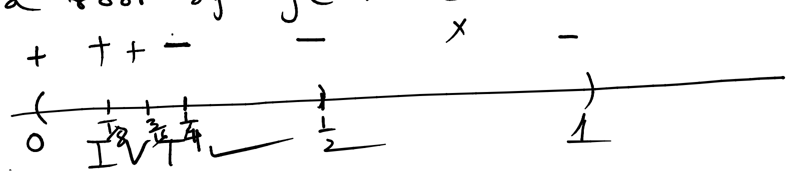
Otherwise $I_2 = (m_2, b_1)$

Call it $I_2 = (a_2, b_2)$

⋮

□ Q. Perform 4 iterations ($\to I_4$) of Bisection method to find a root of $f(x) = x^3 - 5x + 1 = 0$.

How to find $I = (a_0, b_0)$



How to find $I_0 = (a_0, b_0)$

$f(0) = 1 > 0$ So there is a root in $(0,1)$

$f(1) = -2 < 0$

We consider $I_0 = (0,1)$, $m_1 = \frac{0+1}{2} = \frac{1}{2}$

$f(m_1) = f(\frac{1}{2}) = -1.375$

So $I_1 = (0, \frac{1}{2})$, $m_2 = \frac{0+\frac{1}{2}}{2} = \frac{1}{4}$

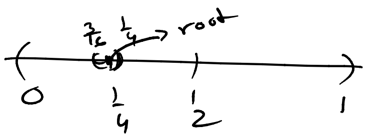
$f(\frac{1}{4}) = f(m_2) = -0.2343 < 0$ (check) $(\frac{1}{4})^3 - 5\frac{1}{4} + 1 > 0$

So $I_2 = (0, \frac{1}{4})$, $m_3 = \frac{0+\frac{1}{4}}{2} = \frac{1}{8}$

$f(m_3) = f(\frac{1}{8}) > 0$

So $I_3 = (\frac{1}{8}, \frac{1}{4})$, $m_4 = \frac{\frac{1}{8}+\frac{1}{4}}{2} = \frac{3}{16}$

$f(\frac{3}{16}) = f(m_4) > 0$



So $I_4 = (\frac{3}{16}, \frac{1}{4})$ Stop (4 iterations)

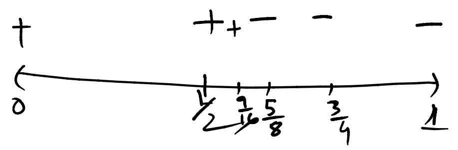
Approx root $= \frac{\frac{3}{16}+\frac{1}{4}}{2} = m_5$

Q. For $x^3 - 5x + 3 = 0$, Perform 4 iterations of Bisection method & then approximate the root.

Sol^n

$f(0) = 3 > 0$

$f(1) = 1 - 5 + 3 < 0$



$I_0 = (0,1)$

$m_1 = \frac{1}{2}$, $f(m_1) > 0$ (check)

$I_1 = (\frac{1}{2}, 1)$, $m_2 = \frac{\frac{1}{2}+1}{2} = \frac{3}{4}$

$f(m_2) < 0$ (check)

$I_2 = (\frac{1}{2}, \frac{3}{4})$, $m_3 = \frac{\frac{1}{2}+\frac{3}{4}}{2} = \frac{5}{8}$

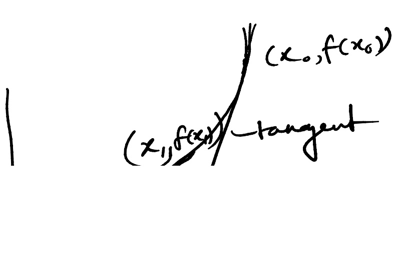
$f(m_3) < 0$

$I_3 = (\frac{1}{2}, \frac{5}{8})$, $m_4 = \frac{\frac{1}{2}+\frac{5}{8}}{2} = \frac{9}{16}$, $f(m_4) > 0$

$I_4 = (\frac{9}{16}, \frac{5}{8})$ (stop 4 iterations)

Approx root is $\frac{\frac{9}{16}+\frac{5}{8}}{2} = \frac{19}{32}$.

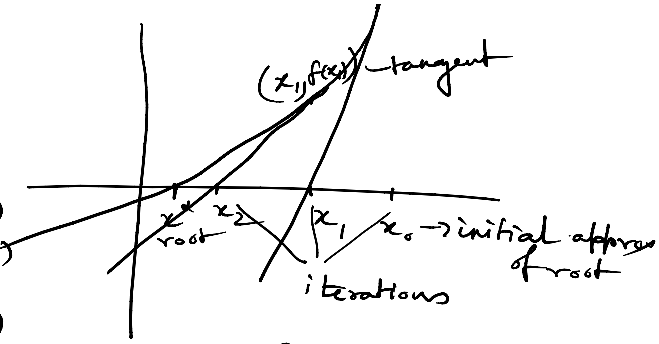
(Real ans is 0.6060)



Newton-Raphson's Method:-

Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



$f'(x) = \frac{df}{dx}$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \ldots$$

e.g. $x_1 = 0.23486$
$x_2 = 0.23457$
$x_3 = 0.23458$ stop

Q. $f(x) = x^3 - 5x + 1 = 0$. Find a root correct to 4 digits after decimals

$f'(x) = 3x^2 - 5$

How to find initial approx x_0.

$f(0) = 1 > 0$, $f(1) = -3 < 0$

So there is a root in $(0,1)$.

So take $x_0 = \frac{0+1}{2} = 0.5$

$f'(0.5)$
$= 3.(0.5)^2 - 5$
$f(x) = x^3 - 5x + 1$
$f'(x) = 3x^2 - 5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.176471$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.176471 - \frac{f(0.17671)}{f'(0.17671)}$$

$$= 0.201568$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.201640 \ldots$$

3 digs same but we need 4 digs same.

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.201640 \quad \text{Stop}$$

Correct to 4 digits.

Final Ans.

Approx root is 0.201640

Try for $x^2 - 5x + 3 = 0 \to 0.6060$

$x^4 - x - 10 = 0 \to 0.1837$

Q Perform ~~two~~ 4 iterations of Bisection method for $x^4 - x - 10 = 0$ —