

EXISTENCE OF PERIODIC WAVE TRAINS FOR AN AGE-STRUCTURED MODEL WITH DIFFUSION

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ABSTRACT. In this paper, we make a mathematical analysis of an age-structured model with diffusion including a generalized Beverton-Holt fertility function. The existence of periodic wave train solutions of the age structure model with diffusion are investigated by using the theory of integrated semigroup and a Hopf bifurcation theorem for second order semi-linear equations. We also carry out numerical simulations to illustrate these results.

1. Introduction. Cannibalism (or predation within species) is an important and universal characteristics of animal species. For fishery models, cannibalism is slightly different from predator-prey interactions among potential competitors, Ricker [23] suggested

$$\beta(N) = \beta_0 e^{-\alpha N}, \quad (1.1)$$

while Beverton and Holt [2] suggested

$$\beta(N) = \frac{\beta_0}{1 + \alpha N}, \quad (1.2)$$

where β_0 is the coefficient involving compensatory density dependent predation. Heavy cannibalization of juveniles by adults of the same species shortly after birth can be explained by the above both functional forms [25]. Different forms come from different hypotheses about the distribution of the length of the juvenile period; in the Ricker equation (1.1) the juvenile period has a fixed length, while in the Beverton-Holt equation (1.2) its length is exponentially distributed, which means individuals make the transition from juvenile to adult stage at a fixed constant rate. To some extent, it is disconcerting that different assumptions about the length distribution of the juvenile period lead to these dramatically different forms. Models as the Beverton-Holt and Ricker equations [25, Chapter 9], or the well-known quadratic

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family (also known as the discrete logistic model [20, Sect. 2.3]) were investigated by a large number of authors [2, 3, 23, 25, 20, 24, 8]. In this paper, the generalized Beverton-Holt map $\beta(N) = \frac{\beta_0}{1+\alpha N^n}$ is considered as the boundary condition, where $n \geq 1$ measures the abruptness of the onset of density dependence [8]. When $n = 1$, $\beta(N)$ is the well known Beverton-Holt equation (1.2) and is an increasing, saturating, concave function. The generalized form of $n > 1$ is the hump shape function, which is very consistent with the insect population census data [1].

In population dynamics, age becomes the major way to affect the size and growth of the population. Individuals with different ages have different reproduction, survival abilities and behaviors. In many situation, population dynamical models become considerably intricate when age structure will be considered in individual interactions. In recent years, considering the significance of age structure in populations has become more and more prevalent, the literature on all aspects of the interacting populations with age structure has proliferated [4, 5, 19, 11, 27, 28, 30, 31, 32, 15].

The influence of spatial variables on population dynamics is crucial. When the spatial variables are added to the age structure model, more complicated behavior is possible, including stable patterns, spiral waves and periodic wave trains [9, 16, 21, 29, 17, 22, 26, 18, 10]. The existence of traveling wave solutions for a class of diffusive predator-prey type systems was investigated by [9] and their results can apply to various kinds of ecological models. [16] considered periodic travelling wave solutions and computed numerically the dispersion relation curves for periodic wave trains of the FitzHugh-Nagumo nerve axon equation. Authors [21] discussed time- and space-periodic solutions of reaction-diffusion systems via bifurcations when the source function parameter, the frequency and the wave vector change. The slow dispersion of nonlinear water waves and periodic wave trains in water of arbitrary depth were studied in [29]. The selection and stability of periodic wave trains behind the invasion fronts were concerned with [17] in predator-prey systems where the prey has an intraspecific nonlocal competition. The bifurcation of periodic wave trains with large wave length in reaction-diffusion systems was studied in [21]. For other literature about periodic wave trains, we refer to the references [9, 16, 21, 29, 17, 22, 26, 18, 10].

At present, a large number of existing literatures [9, 16, 21, 29, 17, 22, 26, 18, 10] mainly consider the wave train solutions of the reaction-diffusion equation. In this paper, we explore the periodic wave train solutions of the following age-structured model with diffusion:

$$\begin{cases} (\partial_t + \partial_a - \Delta_z)u(t, a, z) = -\mu u(t, a, z), & t \in \mathbb{R}, a > 0, \text{ and } z \in \mathbb{R}^N, \\ u(t, 0, z) = \frac{\beta_0 \int_0^{+\infty} \beta(a)u(t, a, z)da}{1 + \alpha \left(\int_0^{+\infty} \beta(a)u(t, a, z)da \right)^3}, \end{cases} \quad (1.3)$$

where μ , β_0 and α are positive parameters; $u(t, a, z)$ denotes the density of population of baleen whale with respect to the age of individuals a in the location z at time t ; $\beta(a)$ is an age-specific fertility function related to individual-age a and satisfies Assumption 1.1. Here Δ_z denotes the Laplace operator for the variable $z \in \mathbb{R}^N$, for some given integer $N \geq 1$.

Assumption 1.1. Assume that

$$\beta(a) := \begin{cases} \beta^*, & \text{if } a \geq \tau, \\ 0, & \text{if } a \in (0, \tau), \end{cases}$$

where $\tau > 0$ and $\beta^* > 0$.

Recall that a couple $(c, U \equiv U(x, a))$ is said to be a periodic wave train profile with speed $c \in \mathbb{R}$ if the function U is periodic with respect to its variable $x \in \mathbb{R}$, namely there exists a period $T > 0$ such that $U(T + \cdot, \cdot) = U(\cdot, \cdot)$ and such that for each direction $e \in \mathbb{S}^{N-1}$ the function $u(t, a, z) := U(z \cdot e + ct, a)$, $x := z \cdot e + ct$, where c is the speed, is an entire solution of (1.3). In other words, the profile (c, U) is a periodic-in x -solution of the following second order problem

$$\begin{cases} \partial_x^2 U(x, a) - c \partial_x U(x, a) - \partial_a U(x, a) - \mu U(x, a) = 0, & x \in \mathbb{R}, a > 0, \\ U(x, 0) = \frac{\beta_0 \int_0^{+\infty} \beta(a) U(x, a) da}{1 + \alpha \left(\int_0^{+\infty} \beta(a) U(x, a) da \right)^3}. \end{cases} \quad (1.4)$$

In order to prove the model (1.3) has periodic wave train solutions, we just need to prove (1.4) has periodic solutions. To achieve the results, we first performed a variable transform on the systems (1.4) and transformed it into a second order abstract semilinear equation

$$\varepsilon^2 \frac{d^2 v(t)}{dt^2} - \frac{dv(t)}{dt} + Av(t) + G(v(t)) = 0, \quad t \in \mathbb{R}, \quad (1.5)$$

in which $\varepsilon > 0$ is a given parameter; $A : D(A) \subset X \rightarrow X$ is a weak Hill-Yosida linear operator acting on a Banach space $(X, \|\cdot\|)$ while $G : \overline{D(A)} \rightarrow X$ is a smooth nonlinear operator (see Section 3 below). The second order equation (1.5) can be regarded as the singularly perturbed system of the first order equation

$$\frac{dv(t)}{dt} = Av(t) + G(v(t)), \quad t \in \mathbb{R}. \quad (1.6)$$

From the results in [7], one can know that non-degenerate Hopf bifurcation for (1.6) is persistent for (1.5) when ε is small enough, which means if the system (1.6) has periodic solutions emanating from the positive equilibrium, then the system (1.5) will also have periodic solutions. Therefore, if we can prove (1.4) has periodic solutions, then the age-structured model with diffusion (1.3) will have periodic wave train solutions.

The major contribution of this paper is organized as follows. Using the Hopf bifurcation theory to prove the existence of periodic solutions of the age-structured model without diffusion are presented in Section 2. The existence of periodic wave train solutions for the age-structured model with diffusion are derived in Section 3. A detail numerical simulation is included in Section 4, and finally, the paper ends with a conclusion.

2. The age-structured model without diffusion. Let $a \rightarrow u(t, a)$ be the density of population of baleen whale with respect to the age of individuals at time t . This means that for each $a_1 \leq a_2$ the number of individuals with a size in between a_1 and a_2 is

$$\int_{a_1}^{a_2} u(t, a) da.$$

In particular the total number of baleen whale is

$$U(t) = \int_0^{+\infty} u(t, a) da.$$

The baleen whale equation with an age structure can be rewrite as

$$\begin{cases} \frac{\partial u(t, a)}{\partial t} + \frac{\partial u(t, a)}{\partial a} = -\mu u(t, a), & a \geq 0, \\ u(t, 0) = \frac{\beta_0 \int_0^{+\infty} \beta(a) u(t, a) da}{1 + \alpha \left(\int_0^{+\infty} \beta(a) u(t, a) da \right)^3}, \\ u(0, \cdot) = u_0 \in L_+^1((0, +\infty), \mathbb{R}). \end{cases} \quad (2.1)$$

As shown in the literature [19, Page 7], $R = \int_0^{+\infty} \beta(a) e^{-\mu a} da$ is called the *net reproduction rate* and gives the number of the newborn that an individual is expected to produce during his reproductive life, where $e^{-\mu a}$ denotes the survival probability, i.e., the probability for an individual to survive to age a .

2.1. Rescaling time and age. In this subsection, our destination is to get a smooth dependency of the system (2.1) about τ . We first use the time-scaling $\hat{t} = \frac{t}{\tau}$, age-scaling $\hat{a} = \frac{a}{\tau}$, and the change of variables $\hat{u}(\hat{t}, \hat{a}) = \tau u(\tau \hat{t}, \tau \hat{a})$ and $\hat{\beta}(\hat{a}) = \beta(\tau \hat{a})$ to normalize τ in (2.1). For abbreviation, after the change of variables we drop the hat notation and obtain the following system

$$\begin{cases} \frac{\partial u(t, a)}{\partial t} + \frac{\partial u(t, a)}{\partial a} = -\mu \tau u(t, a), \\ u(t, 0) = \frac{\tau \beta_0 \int_0^{+\infty} \beta(a) u(t, a) da}{1 + \alpha \left(\int_0^{+\infty} \beta(a) u(t, a) da \right)^3}, & t > 0, \\ u(0, \cdot) = u_0 \in L_+^1((0, +\infty), \mathbb{R}), \end{cases} \quad (2.2)$$

where the new function $\beta(a)$ is given by

$$\beta(a) = \begin{cases} \beta^*, & \text{if } a \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta^* = \mu R e^{\mu \tau},$$

where $\tau \geq 0$, $\beta^* > 0$ and $0 < R < +\infty$.

Next we consider the following Banach space

$$X := \mathbb{R} \times L^1((0, +\infty), \mathbb{R})$$

with $\left\| \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right\| = |\psi| + \|\varphi\|_{L^1((0, +\infty), \mathbb{R})}$, $\forall \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \in X$. The linear operator $A_\tau : D(A_\tau) \subset X \rightarrow X$ is defined by

$$A_\tau \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu \tau \varphi \end{pmatrix} \quad (2.3)$$

with the domain

$$D(A_\tau) = \{0\} \times W^{1,1}((0, +\infty), \mathbb{R}).$$

Define operator $F : \overline{D(A)} \rightarrow X$ by

$$F \left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \right) = \begin{pmatrix} B(\varphi) \\ 0_{L^1((0, +\infty), \mathbb{R})} \end{pmatrix}, \quad (2.4)$$

where

$$B(\varphi) = \frac{\beta_0 \int_0^{+\infty} \beta(a) \varphi(a) da}{1 + \alpha \left(\int_0^{+\infty} \beta(a) \varphi(a) da \right)^3}.$$

Then by identifying $u(t)$ with $u(t, a)$ and $w(t) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$, system (2.1) can be rewritten as the following non-densely defined abstract Cauchy problem

$$\begin{cases} \frac{dw(t)}{dt} = A_\tau w(t) + \tau F(w(t)), & t \geq 0, \\ w_0 = \begin{pmatrix} 0 \\ u_0 \end{pmatrix} \in \overline{D(A_\tau)}. \end{cases} \quad (2.5)$$

The results of global existence and uniqueness of solution of system (2.5) can be obtained by [14] and [13].

2.2. Equilibria and linearized equation. In this subsection, we will obtain the equilibria of system (2.5) and linearized equation of (2.5) around the positive equilibrium.

2.2.1. Existence of equilibria.

Suppose that $\bar{w}(a) = \begin{pmatrix} 0 \\ \bar{u}(a) \end{pmatrix} \in X_0$ is a steady state of system (2.5). Then

$$\begin{pmatrix} 0 \\ \bar{u}(a) \end{pmatrix} \in D(A_\tau) \quad \text{and} \quad A_\tau \begin{pmatrix} 0 \\ \bar{u}(a) \end{pmatrix} + \tau F \left(\begin{pmatrix} 0 \\ \bar{u}(a) \end{pmatrix} \right) = 0,$$

which is equivalent to

$$\begin{cases} -\bar{u}(0) + \tau B(\bar{u}(a)) = 0, \\ -\bar{u}'(a) - \tau \mu \bar{u}(a) = 0. \end{cases}$$

Solving the above system, we obtain

$$\bar{u}(a) = \frac{\tau \beta_0 \int_0^{+\infty} \beta(a) \bar{u}(a) da}{1 + \alpha \left(\int_0^{+\infty} \beta(a) \bar{u}(a) da \right)^3} e^{-\mu \tau a}. \quad (2.6)$$

Integrating both sides of (2.6), we have

$$\int_0^{+\infty} \beta(a) \bar{u}(a) da = \frac{R \beta_0 \int_0^{+\infty} \beta(a) \bar{u}(a) da}{1 + \alpha \left(\int_0^{+\infty} \beta(a) \bar{u}(a) da \right)^3} \quad \text{and} \quad \int_0^{+\infty} \bar{u}(a) da = \frac{1}{\mu R} \int_0^{+\infty} \beta(a) \bar{u}(a) da.$$

Hence, we have the following lemma.

Lemma 2.1. *System (2.5) always has the boundary equilibrium*

$$\bar{w}_0(a) = \begin{pmatrix} 0 \\ 0_{L^1((0, +\infty), \mathbb{R})} \end{pmatrix}.$$

In addition, system (2.5) also has a unique positive equilibrium

$$\bar{w}_\tau(a) = \begin{pmatrix} 0 \\ \bar{u}_\tau(a) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\tau e^{-\mu \tau a}}{R} \sqrt[3]{\frac{\beta_0 R - 1}{\alpha}} \end{pmatrix}$$

if and only if

$$\beta_0 R - 1 > 0.$$

Correspondingly, the unique positive equilibrium of system (2.1) has the following form

$$\bar{u}_\tau(a) = \frac{\tau e^{-\mu \tau a}}{R} \sqrt[3]{\frac{\beta_0 R - 1}{\alpha}}$$

if and only if

$$\beta_0 R - 1 > 0.$$

In the remainder of our paper we assume that $\beta_0 R - 1 > 0$.

2.2.2. Linearized equation.

Here we will obtain the linearized equation of (2.5) around the positive equilibrium \bar{w}_τ . By applying the change of variable $y(t) := w(t) - \bar{w}_\tau$, (2.5) becomes

$$\begin{cases} \frac{dy(t)}{dt} = A_\tau y(t) + \tau F(y(t) + \bar{w}_\tau) - \tau F(\bar{w}_\tau), & t \geq 0, \\ y(0) = \begin{pmatrix} 0 \\ u_0 - \bar{u}_\tau \end{pmatrix} =: y_0 \in \overline{D(A_\tau)}. \end{cases} \quad (2.7)$$

Therefore we can write the linearized equation (2.7) around the equilibrium 0 in the form

$$\frac{dy(t)}{dt} = A_\tau y(t) + \tau DF(\bar{w}_\tau)y(t), \quad t \geq 0, \quad y(t) \in X_0, \quad (2.8)$$

where

$$\tau DF(\bar{w}_\tau) \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \tau DB(\bar{u}_\tau)(\varphi) \\ 0_{L^1((0,+\infty),\mathbb{R})} \end{pmatrix}, \quad \forall \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D(A_\tau)$$

with

$$DB(\bar{u}_\tau)(\varphi) = -\frac{\beta_0 \left[2\alpha \left(\int_0^{+\infty} \beta(a)\bar{u}(a)da \right)^3 - 1 \right]}{\left[1 + \alpha \left(\int_0^{+\infty} \beta(a)\bar{u}(a)da \right)^3 \right]^2} \int_0^{+\infty} \beta(a)\varphi(a)da.$$

Accordingly system (2.7) can be rewritten as

$$\frac{dy(t)}{dt} = B_\tau y(t) + H(y(t)) \quad \text{for } t \geq 0, \quad (2.9)$$

where

$$B_\tau := A_\tau + \tau DF(\bar{w}_\tau) \quad (2.10)$$

is a linear operator and

$$H(y(t)) = \tau F(y(t) + \bar{w}_\tau) - \tau F(\bar{w}_\tau) - \tau DF(\bar{w}_\tau)y(t) \quad (2.11)$$

satisfying $H(0) = 0$ and $DH(0) = 0$.

2.3. Characteristic equation. In this subsection, we will get the characteristic equation of (2.5) around the positive equilibrium \bar{w}_τ . Denote

$$\Omega := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > -\mu\tau\}.$$

Following the results of [12], we derive the following lemma.

Lemma 2.2. For $\lambda \in \Omega$, $\lambda \in \rho(A_\tau)$ and

$$(\lambda I - A_\tau)^{-1} \begin{pmatrix} \delta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \Leftrightarrow \varphi(a) = e^{-\int_0^a (\lambda + \mu\tau)dl} \delta + \int_0^a e^{-\int_s^a (\lambda + \mu\tau)dl} \psi(s)ds \quad (2.12)$$

with $\begin{pmatrix} \delta \\ \psi \end{pmatrix} \in X$ and $\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D(A_\tau)$. Furthermore, A_τ is a Hille-Yosida operator and

$$\|(\lambda I - A_\tau)^{-n}\| \leq \frac{1}{(\operatorname{Re}(\lambda) + \mu\tau)^n}, \quad \forall \lambda \in \Omega, \quad \forall n \geq 1. \quad (2.13)$$

Let A_0 be the part of A_τ in $\overline{D(A_\tau)}$, namely, $A_0 := D(A_0) \subset X \rightarrow X$. For $\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D(A_0)$, we get

$$A_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{A}_0(\varphi) \end{pmatrix},$$

where $\hat{A}_0(\varphi) = -\varphi' - \mu\tau\varphi$ with $D(\hat{A}_0) = \{\varphi \in W^{1,1}((0, +\infty), \mathbb{R}) : \varphi(0) = 0\}$.

Note that $\tau DF(\bar{w}_\tau) : D(A_\tau) \subset X \rightarrow X$ is a compact bounded linear operator. According to (2.13) we obtain

$$\|T_{A_0}(t)\| \leq e^{-\mu\tau t}, \quad \forall t \geq 0.$$

Thereby, we have

$$\omega_{0,ess}(A_0) \leq \omega_0(A_0) \leq -\mu\tau.$$

Combining with the perturbation results from [6], we get

$$\omega_{0,ess}((A_\tau + \tau DF(\bar{w}_\tau))_0) \leq -\mu\tau < 0.$$

Consequently we derive the following proposition.

Lemma 2.3. *The linear operator B_τ is a Hille-Yosida operator, and its part $(B_\tau)_0$ in $\overline{D(B_\tau)}$ satisfies*

$$\omega_{0,ess}((B_\tau)_0) < 0.$$

Set $\lambda \in \Omega$. Since $(\lambda I - A_\tau)$ is invertible, it follows that $\lambda I - B_\tau$ is invertible iff $I - \tau DF(\bar{w}_\tau)(\lambda I - A_\tau)^{-1}$ is invertible, and

$$\begin{aligned} (\lambda I - B_\tau)^{-1} &= (\lambda I - (A_\tau + \tau DF(\bar{w}_\tau)))^{-1} \\ &= (\lambda I - A_\tau)^{-1} (I - \tau DF(\bar{w}_\tau)(\lambda I - A_\tau)^{-1})^{-1}. \end{aligned} \quad (2.14)$$

Let

$$(I - \tau DF(\bar{w}_\tau)(\lambda I - A_\tau)^{-1}) \begin{pmatrix} \delta \\ \varphi \end{pmatrix} = \begin{pmatrix} \gamma \\ \psi \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} \delta \\ \varphi \end{pmatrix} - \tau DF(\bar{w}_\tau)(\lambda I - A_\tau)^{-1} \begin{pmatrix} \delta \\ \varphi \end{pmatrix} = \begin{pmatrix} \gamma \\ \psi \end{pmatrix}.$$

Then we obtain

$$\begin{cases} \delta - \tau DB(\bar{u}_\tau) \left(e^{-\int_0^a (\lambda + \mu\tau) dl} \delta + \int_0^a e^{-\int_s^a (\lambda + \mu\tau) dl} \varphi(s) ds \right) = \gamma, \\ \varphi = \psi, \end{cases}$$

i.e.,

$$\begin{cases} \delta - \tau DB(\bar{u}_\tau) \left(e^{-\int_0^a (\lambda + \mu\tau) dl} \delta \right) = \gamma + \tau DB(\bar{u}_\tau) \left(\int_0^a e^{-\int_s^a (\lambda + \mu\tau) dl} \varphi(s) ds \right), \\ \varphi = \psi. \end{cases}$$

Taking the formula of $DB(\bar{u}_\tau)$ into consideration, we obtain

$$\begin{cases} \Delta(\lambda) \delta = \gamma + K(\lambda, \psi), \\ \varphi = \psi, \end{cases}$$

where

$$\Delta(\lambda) = 1 - \tau DB(\bar{u}_\tau) \left(e^{-\int_0^a (\lambda + \mu\tau) dl} \right) \quad (2.15)$$

and

$$K(\lambda, \psi) = \tau DB(\bar{u}_\tau) \left(\int_0^a e^{-\int_s^a (\lambda + \mu\tau) dl} \psi(s) ds \right). \quad (2.16)$$

Whenever $\Delta(\lambda)$ is invertible, then

$$\delta = (\Delta(\lambda))^{-1}(\gamma + K(\lambda, \psi)). \quad (2.17)$$

Combining the above conclusion and the proof of Lemma 3.5 in [27], we derive the lemma as follows.

Lemma 2.4. *The following results hold*

- (i) $\sigma(B_\tau) \cap \Omega = \sigma_p(B_\tau) \cap \Omega = \{\lambda \in \Omega : \det(\Delta(\lambda)) = 0\}$;
- (ii) *If $\lambda \in \rho(B_\tau) \cap \Omega$, we obtain the formula for resolvent*

$$(\lambda I - B_\tau)^{-1} \begin{pmatrix} \delta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix}, \quad (2.18)$$

where

$$\psi(a) = e^{-\int_0^a (\lambda + \mu\tau) dl} (\Delta(\lambda))^{-1} [\gamma + K(\lambda, \varphi)] + \int_0^a e^{-\int_s^a (\lambda + \mu\tau) dl} \varphi(s) ds$$

with $\Delta(\lambda)$ and $K(\lambda, \varphi)$ given by (2.15) and (2.16).

Under Assumption 1.1, we have

$$\int_0^{+\infty} \beta(a) e^{-\int_0^a (\lambda + \mu\tau) dl} da = \frac{\beta^* e^{-(\lambda + \mu\tau)}}{\lambda + \mu\tau} = \frac{R\mu e^{-\lambda}}{\lambda + \mu\tau}. \quad (2.19)$$

It follows from (2.15) and (2.19) that the characteristic equation at the positive equilibrium \bar{w}_τ is

$$\det(\Delta(\lambda)) = \frac{\lambda + \tau p_0 + \tau q_0 e^{-\lambda}}{\lambda + \mu\tau} := \frac{\tilde{f}(\lambda)}{\tilde{g}(\lambda)}, \quad (2.20)$$

where

$$\begin{aligned} p_0 &= \mu, \\ q_0 &= \frac{\mu(2\beta_0 R - 3)}{\beta_0 R}, \\ \tilde{f}(\lambda) &= \lambda + \tau p_0 + \tau q_0 e^{-\lambda}, \\ \tilde{g}(\lambda) &= \lambda + \mu\tau. \end{aligned} \quad (2.21)$$

Let

$$\lambda = \tau\zeta.$$

Thereby we readily obtain

$$\tilde{f}(\lambda) = \tilde{f}(\tau\zeta) := \tau f(\zeta) = \tau[\zeta + p_0 + q_0 e^{-\tau\zeta}]. \quad (2.22)$$

Based on the above discussion, it is simple to prove that

$$\{\lambda \in \Omega : \det(\Delta(\lambda)) = 0\} = \{\lambda = \tau\zeta \in \Omega : f(\zeta) = 0\}.$$

2.4. Existence of Hopf bifurcation. In this section, we consider the length of the juvenile period τ as a bifurcation parameter and investigate the existence of Hopf bifurcation by applying the Hopf bifurcation theory [12] to the Cauchy problem (2.5). From (2.22), we have

$$f(\zeta) = \zeta + p_0 + q_0 e^{-\tau\zeta}, \quad (2.23)$$

where p_0 and q_0 are given by (2.21). Obviously, if $\beta_0 R - 1 > 0$, then $p_0 + q_0 = \frac{3\mu(\beta_0 R - 1)}{\beta_0 R} > 0$ and $\zeta = 0$ is not a eigenvalue of (2.23). Note that when $\beta_0 R - 1 > 0$, the all roots of (2.23) have negative real parts for $\tau = 0$.

Assume that $\zeta = i\omega$ ($\omega > 0$) is a purely imaginary root of $f(\zeta) = 0$. Thus we have

$$i\omega + p_0 + q_0 e^{-i\omega\tau} = 0.$$

Separating real and imaginary parts of the above equation gives rise to

$$\begin{cases} p_0 + q_0 \cos(\omega\tau) = 0, \\ \omega - q_0 \sin(\omega\tau) = 0. \end{cases} \quad (2.24)$$

Consequently, we obtain

$$\omega^2 = q_0^2 - p_0^2 = (q_0 + p_0)(q_0 - p_0). \quad (2.25)$$

Consequently, it is apparent from (2.25) that when $q_0 - p_0 = \frac{\mu(\beta_0 R - 3)}{\beta_0 R} > 0$ (i.e., $\beta_0 R > 3$), (2.25) has only one positive real root

$$\omega_0 = \sqrt{q_0^2 - p_0^2}. \quad (2.26)$$

According to (2.24), we can yield that $f(\zeta) = 0$ with $\tau = \tau_k$, $k = 0, 1, 2, \dots$ has a pair of purely imaginary roots $\pm i\omega_0$, where

$$\tau_k = \begin{cases} \frac{1}{\omega_0} \left(\arccos\left(-\frac{p_0}{q_0}\right) + 2k\pi \right), & \text{if } \frac{\omega_0}{q_0} \geq 0, \\ \frac{1}{\omega_0} \left(2\pi - \arccos\left(-\frac{p_0}{q_0}\right) + 2k\pi \right), & \text{if } \frac{\omega_0}{q_0} < 0, \end{cases} \quad (2.27)$$

for $k = 0, 1, 2, \dots$.

Assumption 2.1. Assume that $\beta_0 R > 3$.

Lemma 2.5. Let Assumption 1.1 and 2.1 hold, then we have

$$\left. \frac{df(\zeta)}{d\zeta} \right|_{\zeta=i\omega_0} \neq 0.$$

Thereby, (2.23) has a simple root $\zeta = i\omega_0$.

Proof. According to (2.23), we have

$$\left. \frac{df(\zeta)}{d\zeta} \right|_{\zeta=i\omega_0} = (1 - \tau q_0 e^{-\tau\zeta}) \Big|_{\zeta=i\omega_0}.$$

On the basis of $f(\zeta) = 0$, we obtain

$$(1 - \tau q_0 e^{-\tau\zeta}) \frac{d\zeta(\tau)}{d\tau} = \zeta q_0 e^{-\tau\zeta}.$$

Suppose that $\left. \frac{df(\zeta)}{d\zeta} \right|_{\zeta=i\omega_0} = 0$, then

$$i\omega_0 q_0 e^{-i\omega_0 \tau} = 0.$$

Therefore, we have

$$\begin{cases} q_0 \omega_0 \sin(\omega_0 \tau) = 0, \\ q_0 \omega_0 \cos(\omega_0 \tau) = 0. \end{cases} \quad (2.28)$$

That is,

$$(q_0 \omega_0)^2 = 0,$$

which implies

$$q_0 \omega_0 = 0.$$

Since $\omega_0 > 0$, we conclude that

$$q_0 = 0.$$

However, $q_0 = \frac{\mu(2\beta_0 R - 3)}{\beta_0 R} > 0$, which leads to a contradiction. Hence

$$\left. \frac{df(\zeta)}{d\zeta} \right|_{\zeta=i\omega_0} \neq 0.$$

□

Lemma 2.6. *Let Assumption 1.1 and 2.1 hold. $f(\zeta) = 0$ has the root $\zeta(\tau) = \alpha(\tau) + i\omega(\tau)$ satisfying $\alpha(\tau_k) = 0$ and $\omega(\tau_k) = \omega_0$, where τ_k is given by (2.27). Then*

$$\alpha'(\tau_k) = \frac{d\operatorname{Re}(\zeta)}{d\tau} \Big|_{\tau=\tau_k} > 0.$$

Proof. For simplicity, we consider $\frac{d\tau}{d\zeta}$ instead of $\frac{d\zeta}{d\tau}$. On the basis of $f(\zeta) = 0$, we obtain

$$(1 - \tau q_0 e^{-\tau\zeta}) \frac{d\zeta}{d\tau} = \zeta q_0 e^{-\tau\zeta},$$

i.e.,

$$\begin{aligned} \frac{d\tau}{d\zeta} \Big|_{\zeta=i\omega_0} &= \frac{1 - \tau q_0 e^{-\tau\zeta}}{\zeta q_0 e^{-\tau\zeta}} \Big|_{\zeta=i\omega_0} \\ &= \left(\frac{1}{-\zeta(\zeta + p_0)} - \frac{\tau}{\zeta} \right) \Big|_{\zeta=i\omega_0} \\ &= \frac{1}{-i\omega_0(i\omega_0 + p_0)} - \frac{\tau}{i\omega_0} \\ &= \frac{1}{\omega_0} \frac{\omega_0 - ip_0}{\omega_0^2 + p_0^2} - \frac{\tau}{i\omega_0} \\ &= \frac{1}{\omega_0} \frac{\omega_0 + ip_0}{\omega_0^2 + p_0^2} + \frac{i\tau}{\omega_0}. \end{aligned}$$

Therefore, we have

$$\operatorname{Re} \left(\frac{d\tau}{d\zeta} \Big|_{\zeta=i\omega_0} \right) = \frac{1}{\omega_0^2 + p_0^2}.$$

Since

$$\omega_0^2 = q_0^2 - p_0^2,$$

we can further obtain

$$\begin{aligned} \operatorname{sign} \left(\frac{d\operatorname{Re}(\zeta)}{d\tau} \Big|_{\tau=\tau_k} \right) &= \operatorname{sign} \left(\operatorname{Re} \left(\frac{d\tau}{d\zeta} \Big|_{\zeta=i\omega_0} \right) \right) \\ &= \operatorname{sign} \left(\frac{1}{\omega_0^2 + p_0^2} \right) > 0. \end{aligned}$$

□

Thus we conclude the following results.

Theorem 2.1. *Let Assumption 1.1 and 2.1 hold. Then there exists $\tau_k > 0, k = 0, 1, 2, \dots$ (τ_k is defined in (2.27)), such that*

- (i) *when $\tau \in [0, \tau_0)$, all the roots of (2.23) have negative real parts, and the positive steady state $\bar{u}(\cdot)$ of model (2.1) is locally asymptotically stable;*
- (ii) *when $\tau > \tau_0$, (2.23) has at least one root with positive real part, and the positive steady state $\bar{u}(\cdot)$ of model (2.1) is unstable;*
- (iii) *when $\tau = \tau_k$, (2.23) has a pair of purely imaginary roots $\pm i\omega_0$ (ω_0 is given by (2.26)), and the model (2.1) undergoes a Hopf bifurcation at the equilibrium $\bar{u}(\cdot)$.*

3. The age-structured model with diffusion. In this section we consider the existence of periodic wave train solutions for the age-structured model with diffusion (1.3). Let us first rewrite (1.4). Considering the Banach spaces

$$X = \mathbb{R} \times L^1(0, \infty) \quad \text{and} \quad X_0 = \{0\} \times L^1(0, \infty),$$

and the non-densely defined linear operator $A : D(A) \subset X \rightarrow X$ defined by

$$D(A) = \{0\} \times W^{1,1}(0, \infty) \quad \text{and} \quad A \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi \end{pmatrix}.$$

Observing that $X_0 = \overline{D(A)} \neq X$. We also consider the nonlinear map $G : X_0 \rightarrow X$ defined by

$$G \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \frac{\beta_0 \int_0^\infty \beta(a)U(x,a)da}{1+a(\int_0^\infty \beta(a)U(x,a)da)^3} \\ 0 \end{pmatrix}.$$

Next setting $v(x) = \begin{pmatrix} 0 \\ U(x, \cdot) \end{pmatrix}$, then system (1.4) can be rewritten as

$$\frac{d^2 v(x)}{dx^2} - c \frac{dv(x)}{dx} + Av(x) + G(v(x)) = 0, \quad x \in \mathbb{R}. \quad (3.1)$$

Setting $c = \frac{1}{\varepsilon}$, $t = \varepsilon x$ and $v(x) = \tilde{v}(\varepsilon x)$, then we drop the hat notation and system (3.1) becomes

$$\varepsilon^2 \frac{d^2 v(t)}{dt^2} - \frac{dv(t)}{dt} + Av(t) + G(v(t)) = 0, \quad t \in \mathbb{R}. \quad (3.2)$$

It is obviously that system (3.2) is the singularly perturbed system of the following system

$$\frac{dv(t)}{dt} = Av(t) + G(v(t)), \quad t \in \mathbb{R}. \quad (3.3)$$

By setting $v(t) = \begin{pmatrix} 0 \\ u(t, \cdot) \end{pmatrix}$, we obtain that system (3.3) is equivalent model (2.1).

From section 2, we know that Hopf bifurcation occurs in model (3.3). By using the corollary 7.8 in [7], we obtain the following result.

Theorem 3.1. *Let Assumption 1.1 be satisfied. Then there exists $c^* > 0$ large enough and $\varepsilon^* > 0$ such that for each $c \in (c^*, \infty)$, there exists $\tau = \tau(c) \in (\tau_0 - \varepsilon^*, \tau_0 + \varepsilon^*)$ such that $\tau = \tau(c)$ is a Hopf bifurcation point for (3.2) around $\bar{w}_\tau(a)$.*

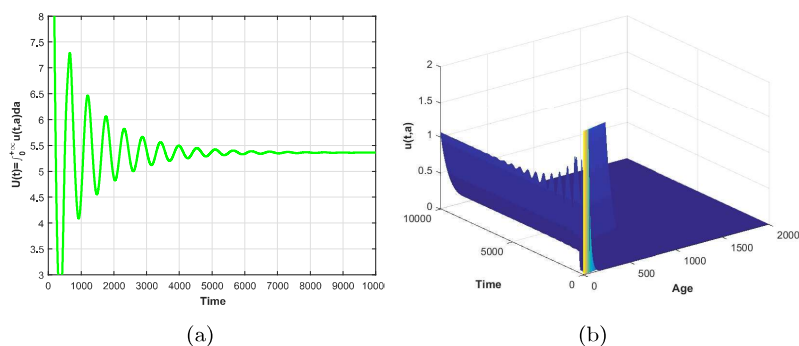
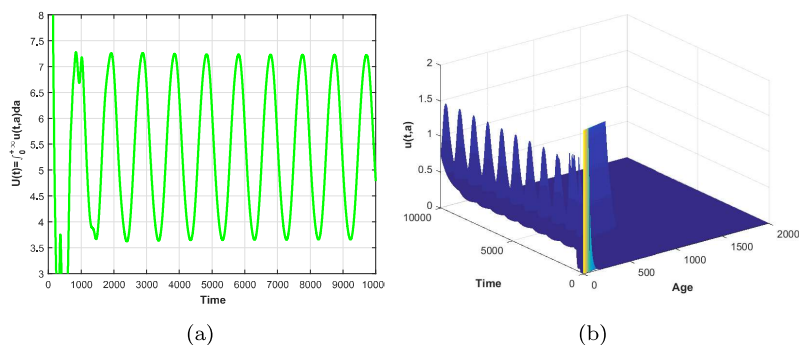
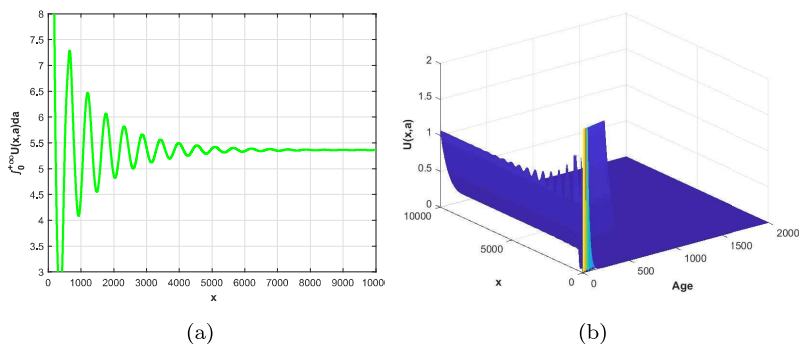
Hopf bifurcation occurs in system (3.2) and then there exists periodic solution in model (1.4), so there is a periodic wave train in model (1.3).

4. Numerical simulations. In this section, some numerical simulations about system (1.4) and (2.1) are presented to illustrate the results shown in Sections 2 and 3. We choose the parameter values $\mu = 0.2, \alpha = 2.35, \beta_0 = 4, R = 1$ and the initial values $u(0, a) = 7.5936e^{-0.25a}$. The age-specific fertility function becomes

$$\beta(a) := \begin{cases} 0.2e^{0.2\tau}, & \text{if } a \geq \tau, \\ 0, & \text{if } a \in (0, \tau). \end{cases}$$

Additionally, $\beta_0 R = 4 > 3$ satisfies the conditions of Assumption 2.1. Calculating it further, we can easily obtain that $\omega_0 \approx 0.15$ and the first critical value $\tau_0 \approx 16.6539$.

In Figure 1, we choose the bifurcation parameter $\tau = 10 < \tau_0$ and the positive equilibrium $\bar{u}_{\tau=10}(a) = 10.8480e^{-2a}$ is locally asymptotically stable. Figure 1 (a) demonstrates the solution behaviors of the population. Figure 1 (b) describes the change of the distribution function of the population $u(t, a)$ as the time and age vary. By further continuously increasing τ to $20 > \tau_0$, there appears a solution of system (2.1) around the positive equilibrium that has an oscillating behavior and gradually tends to closed orbit. $\bar{u}_{\tau=20}(a) = 10.8480e^{-0.4a}$, meanwhile the conclusion of Theorem 2.1 is also numerically demonstrated (see Figure 2). In Figure 2 (a), the solution curves illustrate an oscillation behavior gradually maintaining closed orbit. The variation of $u(t, a)$ as time and age vary at $\tau = 20 > \tau_0$ is demonstrated in Figure 2 (b).

FIGURE 1. Numerical solutions of system (2.1) when $\tau = 10 < \tau_0$.FIGURE 2. Numerical solutions of system (2.1) when $\tau = 20 > \tau_0$.FIGURE 3. Numerical solutions of system (1.4) when $\tau = 10 < \tau_0$ and $c = 1000$.

In Figure 3, we choose the bifurcation parameter $\tau = 10 < \tau_0$ and set $c = 1000$, then the solution of system (1.4) is locally asymptotically stable. In Figure 4, we choose the value of the bifurcation parameter $\tau = 20 > \tau_0$ and set the speed $c = 1000$. It is not difficult to find that oscillatory solution tending closed orbit for system (1.4) is consistent for system (2.1). From these numerical analysis results, we can demonstrate the theoretical findings in Section 3.

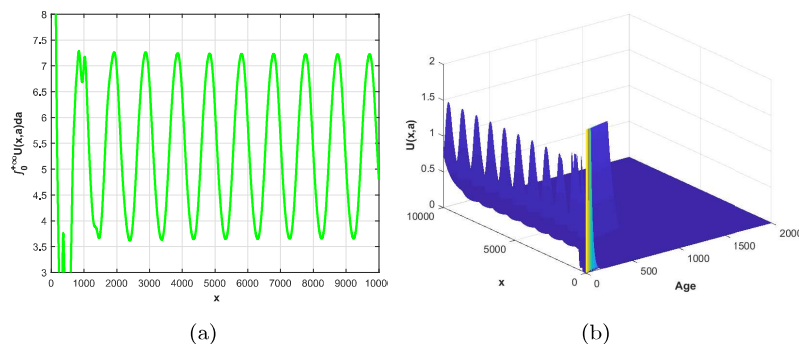


FIGURE 4. Numerical solutions of system (1.4) when $\tau = 20 > \tau_0$ and $c = 1000$.

5. Conclusions. In this paper, we introduce an age-structured model with diffusion including a generalized Bevert-Holt fertility function. The existence of the Hopf bifurcation for the age-structured model without diffusion is studied. Bifurcation analysis indicates that there exist some parameter values such that this age-structured model without diffusion has a non-trivial periodic solution which bifurcates from the positive equilibrium. In particular, periodic wave train solutions of the age-structured model with diffusion are considered by employing the methods developed in [7]. Additionally, numerical simulations of model (1.4) and (2.1) are also provided respectively to exemplify the theoretical findings.

Conflict of interest. The authors declare that there is no conflict of interest with respect to the publication of this paper.

REFERENCES

- [1] T. S. Bellows, Jr., [The descriptive properties of some models for density dependence](#), *J. Animal Ecology*, **50** (1981), 139–156.
- [2] R. J. H. Beverton and S. J. Holt, *On the Dynamics of Exploited Fish Populations*, Springer Netherlands, HMSO, 1957.
- [3] M. J. Bohner and H. Warth, [The Beverton-Holt dynamic equation](#), *Appl. Anal.*, **86** (2007), 1007–1015.
- [4] J. M. Cushing, *An Introduction to Structured Population Dynamics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1998.
- [5] J. M. Cushing and M. Saleem, [A predator prey model with age structure](#), *J. Math. Biol.*, **14** (1982), 231–250.
- [6] A. Ducrot, Z. H. Liu and P. Magal, [Essential growth rate for bounded linear perturbation of non-densely defined Cauchy problems](#), *J. Math. Anal. Appl.*, **341** (2008), 501–518.
- [7] A. Ducrot and P. Magal, [A center manifold for second order semi-linear differential equations on the real line and applications to the existence of wave trains for the Gurtin-McCamy equation](#), *Trans. Amer. Math. Soc.*, **372** (2019), 3487–3537.
- [8] W. M. Getz, [A hypothesis regarding the abruptness of density dependence and the growth rate of populations](#), *Ecology*, **77** (1996), 2014–2026.
- [9] C.-H. Hsu, C.-R. Yang, T.-H. Yang and T.-S. Yang, [Existence of traveling wave solutions for diffusive predator-prey type systems](#), *J. Differential Equations*, **252** (2012), 3040–3075.
- [10] J. P. Keener, [Waves in excitable media](#), *SIAM J. Appl. Math.*, **39** (1980), 528–548.
- [11] Z. Liu and N. Li, [Stability and bifurcation in a predator-prey model with age structure and delays](#), *J. Nonlinear Sci.*, **25** (2015), 937–957.
- [12] Z. Liu, P. Magal and S. Ruan, [Hopf bifurcation for non-densely defined Cauchy problems](#), *Z. Angew. Math. Phys.*, **62** (2011), 191–222.

- [13] P. Magal, Compact attractors for time-periodic age-structured population models, *Electron. J. Differential Equations*, **2001** (2001), 1–35.
- [14] P. Magal and S. Ruan, On semilinear Cauchy problems with non-dense domain, *Adv. Differential Equations*, **14** (2009), 1041–1084.
- [15] P. Magal and S. Ruan, *Structured Population Models in Biology and Epidemiology*, Springer-Verlag, Berlin, 2008.
- [16] K. Maginu, Geometrical characteristics associated with stability and bifurcations of periodic travelling waves in reaction-diffusion systems, *SIAM J. Appl. Math.*, **45** (1985), 750–774.
- [17] S. M. Merchant and W. Nagata, Selection and stability of wave trains behind predator invasions in a model with non-local prey competition, *IMA J. Appl. Math.*, **80** (2015), 1155–1177.
- [18] S. M. Merchant and W. Nagata, Wave train selection behind invasion fronts in reaction-diffusion predator-prey models, *Phys. D*, **239** (2010), 1670–1680.
- [19] M. Iannelli, *Mathematical Theory of Age-structured Population Dynamics*, Giardini Editori E Stampatori, Pisa, 1995.
- [20] J. D. Murray, *Mathematical Biology. I. An Introduction*, Springer-Verlag, New York, 2002.
- [21] J. D. M. Rademacher and A. Scheel, Instabilities of wave trains and Turing patterns in large domains, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **17** (2007), 2679–2691.
- [22] J. D. M. Rademacher and A. Scheel, The saddle-node of nearly homogeneous wave trains in reaction-diffusion systems, *J. Dynam. Differential Equations*, **19** (2007), 479–496.
- [23] W. E. Ricker, Stock and recruitment, *J. Fish. Res. Bd. Canada*, **11** (1954), 559–623.
- [24] S. J. Schreiber, Chaos and population disappearances in simple ecological models, *J. Math. Biol.*, **42** (2001), 239–260.
- [25] H. R. Thieme, *Mathematics in Population Biology*, Princeton University Press, Princeton, NJ, 2003.
- [26] J.-M. Vanden-Broeck and E. I. Părău, Two-dimensional generalized solitary waves and periodic waves under an ice sheet, *Philos. Trans. Roy. Soc. A*, **369** (2011), 2957–2972.
- [27] Z. Wang and Z. Liu, Hopf bifurcation of an age-structured compartmental pest-pathogen model, *J. Math. Anal. Appl.*, **385** (2012), 1134–1150.
- [28] G. F. Webb, *Theory of Nonlinear Age-dependent Population Dynamics*, Marcel Dekker, Inc., New York, 1985.
- [29] G. B. Whitham, Non-linear dispersion of water waves, *J. Fluid Mech.*, **27** (1967), 399–412.
- [30] X. Zhang and Z. Liu, Bifurcation analysis of an age structured HIV infection model with both virus-to-cell and cell-to-cell transmissions, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **28** (2018), 1850109, 20 pp.
- [31] X. Zhang and Z. Liu, Hopf bifurcation for a susceptible-infective model with infection-age structure, *J. Nonlinear Sci.*, **30** (2020), 317–367.
- [32] X. Zhang and Z. Liu, Periodic oscillations in age-structured ratio-dependent predator-prey model with Michaelis-Menten type functional response, *Phys. D*, **389** (2019), 51–63.

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