Basics of Inflationary Cosmology

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1.1 Brief review of FRW cosmology

On large enough scales($\sim 100Mpc$), beyond clusters and super clusters of galaxies, our universe looks isotropic and homogeneous. General form of the metric satisfying these symmetries is given by

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + sin^2\theta d\phi^2) \right] \tag{1.1}$$

κ is a parameter which describes whether the space is flat($\kappa = 0$), sphere($\kappa = 1$) or hyperbolic($\kappa = -1$). The coordinates used here are called comoving coordinates. These are coordinates in which galaxies and other objects are stationary, i.e, they are defined with respect to expansion.

The unknown quantities in the above metric are a(t) and κ. a(t) can be obtained by solving the Einstein's equations with an energy-momentum tensor. The energy in the universe on large scales are described by a constant density and isotropic pressure in co-moving coordinates. Hence we can model this by a perfect fluid. The energy momentum tensor of a perfect fluid is given by

$$T^{\mu}_{\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{bmatrix} \tag{1.2}$$

Here ρ is the energy density and p is the pressure. Upon solving the Einstein's equation, we gets two equations involving a(t).

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \tag{1.3}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{1.4}$$

These are known as Freidmann Equations. The quantity $\frac{\dot{a}}{a}$ is also called Hubble parameter, H.

There is one more equation we can use to solve the Freidmann equations for a(t). It is the conservation equation for $T^{\mu}_{\nu}(\nabla_{\mu}T^{\mu}_{\nu} = 0)$. This gives us

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{1.5}$$

we can simplify this equation by defining an equation of state

$$p = w\rho \tag{1.6}$$

Thus we can now relate ρ with a(t) using the conservation equation.

$$\rho \propto a^{-3(1+w)} \tag{1.7}$$

For matter, w=0, for radiation, $w = \frac{1}{3}$, and for cosmological constant we have w=-1.

Cosmological parameters

A study of cosmological parameters is necessary in the field of cosmology.

Defining

$$\Omega = \frac{\rho}{\rho_c}, \text{ where } \rho_c = \frac{3H^2}{8\pi G} \tag{1.8}$$

we can write Friedmann equation (1.3) as

$$\Omega - 1 = \frac{\kappa}{a^2 H^2} \tag{1.9}$$

Ω is known as the density parameter. Energy density, ρ, can have contribution from matter(ρ_m), radiation(ρ_r) and vacuum(ρ_v).

$$\rho = \rho_m + \rho_r + \rho_v \tag{1.10}$$

The Ω can be written as

$$\Omega = \frac{\rho_m + \rho_r + \rho_v}{\rho_c} = \Omega_m + \Omega_r + \Omega_v \tag{1.11}$$

These different density parameters do not evolve in the same way because

$$\rho_m = \frac{\rho_{m0}}{a^3}, \quad \rho_r = \frac{\rho_{r0}}{a^4}, \quad \rho_v = \rho_{v0} \tag{1.12}$$

The subscript 0 denotes the present value of the parameter. The Freidmann equation (1.3) now just becomes a differential equation in a(t).

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\left[\frac{\Omega_{r0}\rho_{c0}}{a^4} + \frac{\Omega_{m0}\rho_{c0}}{a^3} + \Omega_{v0}\rho_{c0}\right] - \frac{k}{a^2} \tag{1.13}$$

A Particle Horizon refers to the maximum distance light can travel since the beginning of the universe. We can also treat this as maximum distance of causal contact. Consider the null radial curves in the FRW metric, which give

$$\frac{dt}{a(t)} = \frac{dr^2}{1 - kr^2} \tag{1.14}$$

we can integrate L.H.S from t=0 to t=t‘ and get the comoving distance(R.H.S) travelled by light in that interval.

$$\int_0^{t'} \frac{dt}{a(t)} = \int_0^R \frac{dr^2}{1 - kr^2}$$

If t=0 correspond to Big Bang, then this integral gives us the maximum distance light can travel till t=t‘, i.e, the particle horizon. If t‘ corresponds to today, we take a(t‘)=1 as reference.

$$d_H = \int_0^{t'} \frac{dt}{a(t)} = \int_0^1 \frac{da}{a(t)^2 H} \tag{1.15}$$

we can use (1.13) to write H in terms of a(t) and solve this integral.

Thermal history

T	Description
$\sim 0.1 - 10Mev$	Neutrons and protons combine to form nuclei(BBN).
$\sim 1Mev$	Neutrino decouple. Forms a separate cosmic neutrino background.
$\sim 0.5Mev$	e^- and e^+ annihilate. This slightly increases the T of photons than that of CNB.
$\sim 10ev$	Neutral atoms form. CMB photons decouple.

Table 1: Brief Thermal history of the universe.

Big Bang Nucleosynthesis

A brief discussion of thermal history is helpful in our study of cosmology. The universe, as it expanded, cooled down and we expect different physics dominating at different times. But we only have observations till Big Bang Nucleosynthesis(BBN). As is evident from the name, BBN is the time when free nucleons started to combine to form nucleus. Temperature at this point is of the order of binding energy of the nucleons(1-10Mev). After this stage, the photons did not have enough energy to rip apart nucleus to form free nucleons.

Neutrino Decoupling

At around the same temperature(T $\sim$ 1Mev), neutrino decoupled from baryonic matter. Neutrinos interact with baryons through weak interaction. After T $\sim$ 1Mev, the interaction time scale for weak interactions become greater than the age of universe at that time and neutrinos moved practically freely without interactions forming a separate Cosmic Neutrino Background.

Electron-Positron Annihilation

Electron-positron annihilation happened at Temperature around 0.5Mev. This is the mass of electron or positron.

$$\mathrm{e}^+ + \mathrm{e}^- \longleftrightarrow \gamma + \gamma \tag{1.16}$$

When T $\geq$ 0.5Mev, the electrons and positrons are in thermal equilibrium. But as the temperature fall below, 0.5Mev, the forward reaction is strongly favoured. The photons do not have enough energy for e^+e^- pair production after this. The annihilation also causes temperature of photons to increase slightly relative to neutrinos. The energy of annihilation is transferred to photons, but not to neutrinos as they have decoupled.

Photon Decoupling

At Temperature around $\sim$ 10ev, the binding energy of electrons and nucleus, neutral atoms start to form. After this, the photons do not have enough energy to ionise the atoms. Thus density of free ions fell sharply and mean free path of photons became larger than horizon distance. This resulted in decoupling of photons and matter, they were no longer in thermal equilibrium. These photons today are called CMB.

Flatness and Horizon Problem

Defining

$$\Omega = \frac{\rho}{\rho_c}, \text{ where } \rho_c = \frac{3H^2}{8\pi G} \tag{1.17}$$

we can write Friedmann equation (1.3) as

$$\Omega - 1 = \frac{\kappa}{a^2 H^2} \tag{1.18}$$

Today $|1 - \Omega| = 10^{-3}$, which is very small and suggests that our universe is very close to being a flat one. Let us work with matter dominated case and try to obtain a rough idea of the order of $|1 - \Omega|$ at Big Bang Nucleosynthesis. For matter dominated case, $\frac{1}{a^2H^2}$ and hence $|1-\Omega|$, goes as $t^{2/3}$. In units where t=1 represents today's time and t=0 refers to Big Bang, BBN happened at $t \approx 10^{-17}$($t \approx 1s$ in usual units). Thus we get

$$|1 - \Omega_{\text{BBN}}| = 10^{-3}(10^{-17})^{\frac{2}{3}} \approx 10^{-15} \tag{1.19}$$

Implying that the initial condition is Ω being extremely close to 1, which is unlikely. This is called flatness problem.

Using (1.15), Horizon size at CMB, which is at z=1100, turns out to be

$$d_{\text{bigbang-CMB}} = 0.062 H_0^{-1}$$

H_0 is the present value of Hubble parameter. Distance travelled by a photon originated at CMB to us turn out to be

$$d_{\text{CMB-today}} = 3.18 H_0^{-1}$$

The angular separation in the sky by these distances is

$$\frac{d_{\text{bigbang-CMB}}}{d_{\text{CMB-today}}} = 0.019 \approx 1^\circ$$

Thus regions in sky separated by more than 1° couldn't possibly have been in causal contact at CMB. This is in tension with the fact that temperature of CMB is homogeneous, with very little fluctuation($\frac{\Delta T}{T} \sim 10^{-5}$). This is known as horizon problem.

1.2 Solving the Flatness and the Horizon problem: Inflation

Both flatness and horizon problem can be explained by introducing an early phase of accelerated expansion.

For flatness problem. we want $|1-\Omega|$ to start from a generic initial condition, not necessarily close to zero, but reduce to a value close to zero during the Inflationary phase. This way we can explain Equation 2.10. Therefore during inflation, we need

$$\frac{d|1-\Omega|}{dt} < 0 \tag{1.20}$$

This implies that

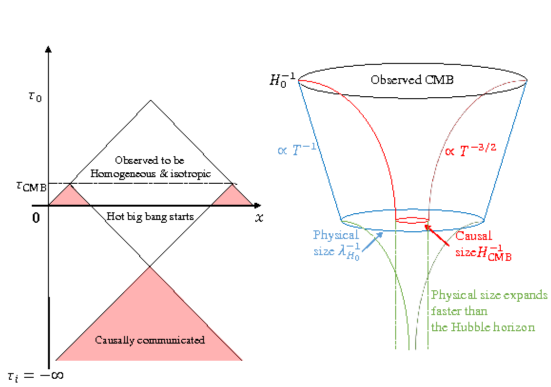
$$\frac{d}{dt}\left(\frac{\kappa}{a^2H^2}\right) = \frac{-2\kappa}{\dot{a}^3}\ddot{a} < 0 \quad \rightarrow \quad \ddot{a} > 0 \tag{1.21}$$

Thus flatness problem is solved with an early era of accelerated expansion.

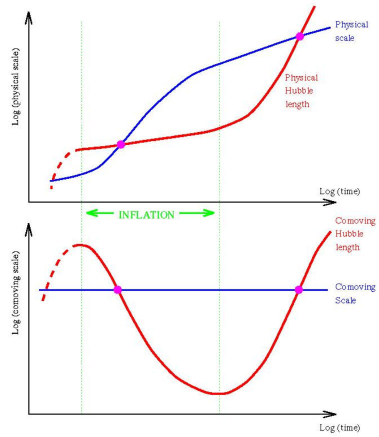
Comoving Horizon size in case of matter and radiation dominated case are $2(aH)^{-1}$ and $(aH)^{-1}$, respectively. We want the universe to begin as inhomogeneous but become homogeneous due to thermal equilibrium, during inflationary phase. For this we require them to be initially in causal contact but which breaks down during inflation and we are left with a universe which is causally disconnected yet homogeneous. Thus during inflation

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0 \tag{1.22}$$

this implies that $\ddot{a} > 0$. Hence the early accelerated expansion can also solve the horizon problem.



(a) Horizons during inflation



(b) Hubble during inflation

1.3 Single-field slow-roll inflation

We will try to derive the slow roll and other cosmological parameters with minimum number of assumptions to look for possible conditions in which expansion occurs but swampland criterion is not violated.

As we have seen in the previous section, an era of accelerated expansion solves both flatness and horizon problem.

$$\frac{\ddot{a}}{a} > 0 \tag{1.23}$$

we can relate it to the Hubble parameter H and its derivatives,

$$\dot{H} = \frac{\ddot{a}}{a} - H^2 \tag{1.24}$$

the two equations give us

$$-\frac{\dot{H}}{H^2} < 1$$

We define the quantity $-\dfrac{\dot{H}}{H^2}$ as ϵ. ϵ must be less than 1 during accelerated expansion.

$$\epsilon = -\frac{\dot{H}}{H^2} < 1 \tag{1.25}$$

We can now use the Friedmann equation to put a bound on some parameters using this equation. We can write

$$\epsilon = \frac{H^2 - \frac{\ddot{a}}{a}}{H^2} \tag{1.26}$$

Using the Friedmann equations (1.3) and (1.4) we have

$$\epsilon = \frac{3}{2}\left(\frac{\rho + p}{\rho}\right) = \frac{3}{2}(1 + w) < 1 \tag{1.27}$$

where w comes from the equation of state $p = w\rho$. The above equation also implies that for expanding universe $w < -\frac{1}{3}$.

In Inflaton model, we have

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \qquad \& \qquad p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{1.28}$$

this gives

$$\epsilon = \frac{3}{2}\left(\frac{\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}\right) < 1 \tag{1.29}$$

The condition $\epsilon < 1$ or $w < -\frac{1}{3}$ imply that

$$\dot{\phi}^2 < V(\phi) \tag{1.30}$$

We can try to express (1.29) only in terms of the potential $V(\phi)$ and its derivatives. We know the equation of motion of the Inflaton field

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = 0 \tag{1.31}$$

Differentiating the First Friedmann equation (1.3), we get

$$2H\dot{H} = \frac{\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}}{3M_p^2} \tag{1.32}$$

The ϵ, (1.25), is then given by

$$\epsilon = -\frac{\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}}{6M_p^2 H^3} \tag{1.33}$$

using (1.31), it can be simplified to give

$$\epsilon = \frac{H^2\dot{\phi}^2}{2M_p^2 H^4} = \frac{1}{2}M_p^2\left(\frac{\ddot{\phi} + V'(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}\right)^2 \tag{1.34}$$

This tells us that ϵ is always greater than 0. Thus

$$0 \leq \epsilon < 1$$

We can take $V(\phi)$ and $V'(\phi)$ out and express it in terms of the definition of ϵ which exists in literature and which is derived using some approximation which will be explained shortly

$$\epsilon = \frac{1}{2}M_p^2\left(\frac{V'(\phi)}{V(\phi)}\right)^2\left(\frac{\frac{\ddot{\phi}}{V'(\phi)}+1}{\frac{\dot{\phi}^2}{2V(\phi)}+1}\right)^2 = \epsilon_{old}\left(\frac{\frac{\ddot{\phi}}{V'(\phi)}+1}{\frac{\dot{\phi}^2}{2V(\phi)}+1}\right)^2 \tag{1.35}$$

where ϵ_{old} is $\frac{1}{2}M_p^2\left(\frac{V'(\phi)}{V(\phi)}\right)^2$. We also need to know how small ϵ needs in order to be consistent with observations. Thus we should relate ϵ with number of e-folds required.

$$N = \ln\left(\frac{a_f}{a_i}\right) = \int_i^f H dt = \int_i^f \frac{H}{\dot{\phi}}d\phi \tag{1.36}$$

From the expression of epsilon in (1.34), we get

$$N = \int_{\phi_i}^{\phi_f}\frac{1}{M_p\sqrt{2\epsilon}}d\phi \approx \frac{\Delta\phi}{M_p\sqrt{2\epsilon}} \tag{1.37}$$

Now assuming $\Delta\phi \approx M_p$, thus we have $\epsilon \approx \frac{1}{N^2}$. Given the number of e-folds required for consistency with observations(~ 60), we get $\epsilon \sim 10^{-3}$.

Not only do we want epsilon to be small, but also that it remains small for sufficiently long time for inflation to be maintained. The quantity we are interested in is $\frac{\dot{\epsilon}}{\epsilon}$.

$$\dot{\epsilon} = \frac{1}{M_p^2}\frac{\dot{\phi}\ddot{\phi}H - \dot{H}\dot{\phi}^2}{H^3} = 2\epsilon\frac{\ddot{\phi}}{\dot{\phi}} + 2\epsilon^2 H \tag{1.38}$$

The fractional change in ϵ over the inflationary period is given by

$$\int_{t_i}^{t_f}\frac{\dot{\epsilon}}{\epsilon}dt = \int_{t_i}^{t_f}\frac{\dot{\epsilon}}{\epsilon\dot{\phi}}d\phi = \int_{t_i}^{t_f}\frac{\dot{\epsilon}}{\epsilon H}dN \tag{1.39}$$

The last expression has a dimensionless integrand, thus is of interest.

$$\frac{\dot{\epsilon}}{\epsilon H} = \frac{2\ddot{\phi}}{\dot{\phi}H} + 2\epsilon \tag{1.40}$$

In literature, people usually take the approximation

$$|\ddot{\phi}| << |H\dot{\phi}|, |V'(\phi)| \tag{1.41}$$

Thus the equation of motion reduces to

$$3H\dot{\phi} + V'(\phi) = 0 \tag{1.42}$$

We can now use this equation to obtain expression for $\eta_v = \dfrac{2\ddot{\phi}}{\dot{\phi}H}$ and ϵ. From (1.34) we can see that this approximation along with equation (1.3) leads to

$$\epsilon_{old} = \frac{M_p^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \tag{1.43}$$

In order to obtain η_v, we need to differentiate (1.42),

$$3(H\ddot{\phi} + \dot{H}\dot{\phi}) + V''(\phi)\dot{\phi} = 0 \tag{1.44}$$

rearranging gives us

$$\eta_v = \frac{\ddot{\phi}}{\dot{\phi}H} = -M_p^2\frac{V''(\phi)}{V(\phi)} + 2\epsilon \tag{1.45}$$

since $|\eta_v| < 1$(because of our approximation) and $\epsilon < 1$(accelerated expansion), $|\eta| = |M_p^2\dfrac{V''(\phi)}{V(\phi)}| < 1$ must be true.

1.3.1 Example 1: $V(\phi) = \frac{1}{2}m^2\phi^2$

For quadratic potential we can calculate the ϵ using (1.43)

$$\epsilon = \frac{2M_p^2}{\phi^2} < 1 \tag{1.46}$$

this implies that $\phi > M_p$ for accelerated expansion to take place to take place. We have to start from a value of ϕ larger than M_p and as inflation ends, we have $\phi_{\text{end}} \approx M_p$.

From (1.42), we can calculate $\dot{\phi}$ for this potential under slow-roll conditions

$$\dot{\phi} = -\frac{V'(\phi)}{3H}$$

Now H can be written as

$$H = \sqrt{\frac{V(\phi)}{3M_p^2}} \tag{1.47}$$

which we get from friedmann equation (1.3) by ignoring curvature and assuming that $V(\phi)$ mostly contributes to ρ, from (1.37). By substituting the values, we get

$$\dot{\phi} = \sqrt{\frac{2}{3}}mM_p \tag{1.48}$$

Hence under slow-roll approximation, $\dot{\phi}$ is a constant in quadratic potential. We can also express ϕ in terms of e-folds. It is convenient because latter is constrained observationally. From (1.46) and (1.37), we have

$$N(\phi) = \int_{\phi_{end}}^{\phi} \frac{\phi`}{2M_p^2} d\phi` = \frac{1}{4M_p^2}[\phi^2 - \phi_{end}^2] \tag{1.49}$$

Now, we know that $N_{\phi_{in}}$ must be ~ 60 and since $\phi_{\text{end}} \approx M_p$, contribution of ϕ_{end} to N is negligible. Hence,

$$N(\phi) = \frac{1}{4M_p^2}\phi^2 \tag{1.50}$$

We can define some quantities known as "Scalar tilt(n_s)", "Tensor tilt(n_t)" and "Tensor-to-scalar ratio(r)" etc., which are associated with quantum fluctuations of the fields. We will discuss about them in detail in later sections. For now

$$n_s = 1 - 6\epsilon - 2\eta = 1 - \frac{8M_p^2}{\phi^2} = 1 - \frac{2}{N(\phi)} \tag{1.51}$$

$$n_t = 1 - 2\epsilon = 1 - 1 - \frac{4M_p^2}{\phi^2} = 1 - \frac{1}{N(\phi)} \tag{1.52}$$

$$r = 16\epsilon = \frac{32M_p^2}{\phi^2} = \frac{8}{N(\phi)} \tag{1.53}$$

$$A_s = \frac{H^4}{4\pi^2\dot{\phi}^2} = \frac{H^2}{8\pi^2 M_p^2 \epsilon} = \frac{m^2 N^2}{6\pi^2 M_p^2} \tag{1.54}$$

$$A_t = \frac{2H^2}{\pi^2 M_p^2} \tag{1.55}$$

After looking at the above expressions one can find out a relation between r and n_s,

$$n_s + \frac{r}{4} = 1 \tag{1.56}$$

1.3.2 Example 2:$V(\phi) = V_0\left(1 - \frac{\phi^2}{m^2}\right)$

The quadratic potential discussed above represents what is called large field inflation. This is because from (1.46), we see that $\phi > M_p$ for inflation to occur. Let us now consider inflation with a scalar field in Higgs potential.

$$V(\phi) = V_0\left(1 - \frac{\phi^2}{m^2}\right)^2 \tag{1.57}$$

The ϵ is given by

$$\epsilon = \frac{1}{2}M_p^2 \left(\frac{4\phi}{m^2-\phi^2}\right)^2 \tag{1.58}$$

Near $\phi = M_p$, the potential behaves like a quadratic potential(it is the minima), hence in that regime we have large field inflation. But near $\phi = 0$, i.e, the maxima, we have

$$\epsilon = \frac{1}{2}M_p^2 \left(\frac{4\phi}{m^2}\right)^2 < 1 \tag{1.59}$$

therefore we need small field excursions in order for inflation to take place, hence the name small field inflation.

1.3.3 Example 3: Axionic inflation($V(\phi) = \Lambda^4(1 - cos(\frac{\phi}{f}))$)

It is also known as natural inflation. For this potential we have

$$\epsilon = \frac{M_p^2}{2f^2}(cot^2(\frac{\phi}{2f})) \tag{1.60}$$

and

$$\eta = \frac{M_p^2}{2f^2}(1 - cot^2(\frac{\phi}{2f})) \tag{1.61}$$

With these slow roll parameters one can find that

$$n_s = 1 - \frac{M_p^2}{f^2}(2cosec^2(\frac{\phi}{2f}) - 1) \tag{1.62}$$

$$r = \frac{8M_p^2}{f^2}(cot^2(\frac{\phi}{2f})) \tag{1.63}$$

The r and n_s satisfy

$$n_s + \frac{r}{4} = 1 - \frac{M_p^2}{f^2} \tag{1.64}$$

The other parameters are

$$n_t = \frac{M_p^2}{f^2}(1 - cot^2(\frac{\phi}{2f})) \tag{1.65}$$

2 Quantum fluctuation during inflation

Our analysis of inflation so far has been classical. We assumed spatial homogeneity of the inflaton field and were unable to discuss any quantum fluctuations of the field due to the classical treatment. Such fluctuations are believed to play an important role in the formation of large scale structure and CMB temperature fluctuations. Studying and observing these fluctuations can give us some insight and provide evidence of early universe inflationary phase.

But first we need to see how inflation gives rise to these primordial fluctuations. Consider a simple case of a massless scalar field whose action is given by

$$\mathcal{S} = \int d^4x\sqrt{-g}[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi] \tag{2.1}$$

where the metric, $g^{\mu\nu}$, is given by (1.1). One can also transform the time coordinate to conformal time, which is given by

$$d\eta = \frac{dt}{a(t)} \tag{2.2}$$

in this coordinate system we have $\sqrt{-g} = det(g_{\mu\nu}) = a^4$ and $g^{\mu\nu} = \frac{\eta^{\mu\nu}}{a^2}$, where η is Minkowskian metric. The action thus becomes

$$\mathcal{S} = \int d^4xa(t)^2[\frac{1}{2}(\partial_\eta\phi)^2 - \frac{1}{2}(\partial_i\phi)^2] \tag{2.3}$$

Here we digress a little bit and try to see what we would have got by doing the above analysis on a QED Lagrangian. The QED action is given by

$$\mathcal{S} = \int d^4x\sqrt{-g}[g^{\alpha\beta}g^{\mu\nu}F_{\alpha\mu}F_{\beta\nu}] \tag{2.4}$$

Now by going to the conformal coordinates as before and substituting $\sqrt{-g}$ and $g^{\mu\nu}$, we get

$$\mathcal{S} = \int d^4x[\eta^{\alpha\beta}\eta^{\mu\nu}F_{\alpha\mu}F_{\beta\nu}] \tag{2.5}$$

The action turns out to be the same as in Minkowski. It is invariant under this transformation and hence the dynamics are the same in de-Sitter and in Minkowski. Thus it is not a good choice to study inflation.

Coming back to scalar field, we now define

$$y = a\phi \tag{2.6}$$

The action now looks like

$$\mathcal{S} = \int d^4x \left[\frac{1}{2}[y' - \frac{a'}{a}y]^2 - \frac{1}{2}(\partial_i y)^2\right] \tag{2.7}$$

where $'$ denotes ∂_η. Doing a fourier transformation of y,

$$y(x) = \int \frac{d^3k}{(2\pi)^3} e^{ik.\vec{x}} y_k(\eta) \tag{2.8}$$

where $\vec{x}$ is the spatial vector. The action becomes

$$\mathcal{S} = \int d\eta \int d^3x \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\left[\frac{1}{2}[y'(k_1) - \frac{a'}{a}y(k_1)][y'(k_2) - \frac{a'}{a}y(k_2)] + \frac{k_1k_2}{2}y(k_1)y(k_2)\right] e^{i(k_1+k_2).\vec{x}} \tag{2.9}$$

using

$$\int d^3x e^{i(k_1+k_2).\vec{x}} = (2\pi)^3\delta^3(k_1+k_2)$$

and

$$\int d^3k_1 F(k,k_1)\delta^3(k+k_1) = F(k,-k)$$

we get the action as

$$\mathcal{S} = \int d\eta \int \frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}[y'(k) - \frac{a'}{a}y(k)][y'(-k) - \frac{a'}{a}y(-k)] - \frac{k^2}{2}y(k)y(-k)\right] \tag{2.10}$$

The conjugate momentum, p(k), is given by

$$p(k) = \frac{\partial\mathcal{L}}{\partial y'(k)} = y'(k)^\dagger - \frac{a'}{a}y(k)^\dagger \tag{2.11}$$

where $\mathcal{L}$ is the Lagrangian density. For a real field $y(k)^\dagger = y(-k)$. Now we can write the Hamiltonian in terms of p(k) and y(k).

$$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3}[y'(k)p(k) - \mathcal{L}] = \int \frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}p(k)p(k)^\dagger + \frac{1}{2}k^2y(k)y(k)^\dagger + \frac{a'}{2a}[y(k)p(k)^\dagger + y(k)^\dagger p(k)]\right] \tag{2.12}$$

We can define new variables b(k), such that

$$\hat{b}(k) = \frac{1}{\sqrt{2}}(\sqrt{k}y(k) + \frac{i}{\sqrt{k}}p(k))$$

These are like creation and annihilation operators. In terms of these variables, we can write

$$y(k) = \frac{\hat{b}(k) + \hat{b}^\dagger(-k)}{\sqrt{2k}} \text{ and } p(k) = -i\sqrt{\frac{k}{2}}[\hat{b}(k) - \hat{b}^\dagger(-k)] \tag{2.13}$$

The Hamiltonian now becomes

$$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3}\left[\frac{k}{2}[\hat{b}(k)\hat{b}^\dagger(k) + \hat{b}^\dagger(k)\hat{b}(k)] - i\frac{a'}{2a}[\hat{b}^\dagger(k)\hat{b}^\dagger(-k) - \hat{b}(k)\hat{b}(-k)]\right] \tag{2.14}$$

we can now try to work out how these variables evolve with time. From (2.11) and (2.13), we can see that

$$y'(k) = \frac{\hat{b}'(k) + \hat{b}'^\dagger(-k)}{\sqrt{2k}} = -i\sqrt{\frac{k}{2}}[\hat{b}^\dagger(-k) - \hat{b}(k)] + \frac{a'}{a}\frac{\hat{b}(k) + \hat{b}^\dagger(-k)}{\sqrt{2k}} \tag{2.15}$$

and

$$p'(k) = -i\sqrt{\frac{k}{2}}[\hat{b}'(k) - \hat{b}'^\dagger(-k)] = k^2\frac{\hat{b}(k) + \hat{b}(-k)}{\sqrt{2k}} - i\sqrt{\frac{k}{2}}\frac{a'}{a}[\hat{b}^\dagger(-k) - \hat{b}(k)] \tag{2.16}$$

Here we obtained $p'(k)$ using $-\frac{\partial H}{\partial y(k)}$. Using these two equation, we obtain

$$\hat{b}'(k) = -ik\hat{b}(k) + \frac{a'}{a}\hat{b}^\dagger(-k)$$

and

$$\hat{b}'^\dagger(-k) = \frac{a'}{a}\hat{b}(k) + ik\hat{b}^\dagger(-k)$$

In matrix form it looks like

$$\begin{bmatrix} \hat{b}'(k) \\ \hat{b}'^\dagger(-k) \end{bmatrix} = \begin{bmatrix} -ik & \frac{a'}{a} \\ \frac{a'}{a} & ik \end{bmatrix}\begin{bmatrix} \hat{b}(k) \\ \hat{b}^\dagger(-k) \end{bmatrix} \tag{2.17}$$

We can see that in absence of expansion($\frac{a'}{a} = 0$), creation and annihilation operators do not mix. But they do so in an expanding universe($\hat{b}(k)$ mixes with $\hat{b}^\dagger(-k)$). This is called squeezing of states. (2.17) also allows us to decide when are allowed to neglect squeezing($k >< \frac{a'}{a}$).

$$\frac{a'}{a} = \frac{1}{a}\frac{da}{d\eta} = \frac{da}{dt} = aH$$

therefore we only need to check whether

$$\frac{k}{a} >< H \tag{2.18}$$

When a mode is inside the Hubble radius($k^{-1} < (aH)^{-1}$), we can neglect the expansion of the universe and the evolution is same as in Minkowski. In this case we have no squeezing.

In the limit when mode is outside the Hubble radius($k^{-1} > (aH)^{-1}$), we can neglect k in (2.17). We get

$$y'(k) = \frac{\hat{b}'(k) + \hat{b}'^{\dagger}(-k)}{\sqrt{2k}} = \frac{a'}{a} y(k) \tag{2.19}$$

This implies $y \propto a$. With this condition, from (2.6), we can conclude that ϕ is a constant. We can also work out how p(k) evolves

$$p'(k) = -i\sqrt{\frac{k}{2}}[\hat{b}'(k) - \hat{b}'^{\dagger}(-k)] = -\frac{a'}{a} p(k) \tag{2.20}$$

Thus$p(k) \propto \frac{1}{a}$. Thus one of the variables become large(y(k)) and one becomes small(p(k)), when the mode is outside Hubble radius.

2.0.1 2-point function in de-Sitter space

2-point functions tell us about spatial fluctuations. We assume that space-time is de-Sitter(H is constant) in calculating the 2-point function. This a good approximation because from (1.25), we can conclude that

$$\left|\frac{\dot{H}}{H}\right| < \left|\frac{\dot{a}}{a}\right| \tag{2.21}$$

Hence H changes slowly during inflation. In de-Sitter space

$$H = \frac{\dot{a}}{a} = H_0 \tag{2.22}$$

implies that $a(t) = a_0 e^{H_0 t}$. With this we can calculate η

$$d\eta = \frac{dt}{a(t)} \tag{2.23}$$

Thus $\eta = -\frac{1}{a(t)H}$. The action (2.9) now looks like

$$\mathcal{S} = \int d^4x \left(\frac{1}{\eta(t)H}\right)^2 [\frac{1}{2}(\partial_\eta \phi)^2 - \frac{1}{2}(\partial_i \phi)^2] \tag{2.24}$$

We expand $\phi(\eta, x)$ in fourier space

$$\phi(\eta, x) = \int \frac{d^3k}{(2\pi)^3} \phi_k(\eta) e^{i\vec{k}.\vec{x}} \tag{2.25}$$

and write

$$\phi_k(\eta) = \phi^{cl}(\eta)\hat{a}^{\dagger}(k) + \phi^{cl}(\eta)^* \hat{a}(-k) \tag{2.26}$$

where the superscript denotes that it is not an operator, just a function. In Minkowski space-time, $\phi^{cl}(\eta)$ is just $e^{i\omega t}$. Using a similar calculation as that in (2.9), we get the action

$$\mathcal{S} = \int d\eta \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{\eta(t)H}\right)^2 [\frac{1}{2}\partial_\eta \phi_k \partial_\eta \phi_{-k} - \frac{k^2}{2}\phi_k \phi_{-k}] \tag{2.27}$$

We must calculate the equation of motion of ϕ. Using Lagrange's equation of motion

$$\frac{\partial \mathcal{L}}{\partial(\partial_\eta \phi_k)} = \frac{\partial \mathcal{L}}{\partial \phi_k}$$

we get

$$\partial_\eta[\eta^{-2}\partial_\eta \phi_k] + \eta^{-2}k^2\phi_k = 0 \tag{2.28}$$

η dependence is there in ϕ^{cl} so the equations are just as good after replacing ϕ_k with it. Let us propose the ansatz

$$\phi^{cl}(\eta) = e^{A\eta} \tag{2.29}$$

substituting this in (2.28), we get

$$\partial_\eta[\eta^{-2}\partial_\eta \phi^{cl}] + \eta^{-2}k^2\phi^{cl} = \eta A^2 - 2A + \eta k^2 \tag{2.30}$$

If we choose A=ik, we can make first and last term cancel each other. We are just left with -2ik in R.H.S. Let us propose one more possible solution

$$\phi^{cl}(\eta) = \eta e^{ik\eta} \tag{2.31}$$

we get

$$\partial_\eta[\eta^{-2}\partial_\eta \phi^{cl}] + \eta^{-2}k^2\phi^{cl} = \eta[2ik - k^2] - 2[1 + ik\eta] + \eta k^2 = -2 \tag{2.32}$$

Thus we are left with -2 in R.H.S. So if we choose

$$\phi^{cl}(\eta) = e^{ik\eta} - ik\eta e^{ik\eta} \tag{2.33}$$

R.H.S becomes zero. Hence this is a solution of (2.28) upto a normalization.

The exact solution is given by

$$\phi^{cl}(\eta) = \frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta} \tag{2.34}$$

When a mode is well inside Hubble radius, we have $k >> aH$. Since $\eta = -\frac{1}{a(t)H}$, we have $|k\eta| >> 1$. Thus in this limit

$$\phi^{cl}(\eta) = \frac{H}{\sqrt{2k^3}} ik\eta e^{ik\eta} \tag{2.35}$$

or after making some substitutions we get

$$\phi^{cl}(\eta) = i\frac{a^{-1}}{\sqrt{2k}}e^{ik\eta} \tag{2.36}$$

As we discussed before, in this limit we can neglect the expansion of the universe as it is very slow as compared to the oscillation of the field. Thus the expression we obtained here must be similar to a field in Minkowski space-time. We have $\sqrt{k}$ in the denominator, which corresponds to $\sqrt{E_p}$ of a field in Minkowski. But we have expressed ϕ in comoving coordinates. So k is comoving momentum. In order to make it physical we divide by a(t).

$$\phi^{cl}(\eta) = i\frac{a^{-\frac{3}{2}}}{\sqrt{2\frac{k}{a}}}e^{ik\eta_0}e^{i\frac{k}{a}\Delta t} \tag{2.37}$$

Here we also expanded $e^{ik\eta}$ as some phase $e^{ik\eta_0}$ and $e^{i\frac{k}{a}\Delta t}$. We can write it this way because a(t) is almost constant during time period of oscillation(expansion is approximately negligible). The factor of $a^{-\frac{3}{2}}$ comes from different normalization of $\hat{a}$ and $\hat{a}^\dagger$ in comoving coordinates. In other words

$$\hat{a}_p = a^{\frac{3}{2}}\hat{a}_k \tag{2.38}$$

Now we can move on to calculate the two point function $\langle 0|\phi_k(\eta)\phi_{k'}(\eta)|0\rangle$. For this we assume the equal time commutation relation

$$[\hat{a}_k, \hat{a}^\dagger_{k'}] = (2\pi)^3\delta(k+k') \tag{2.39}$$

We now have from (2.26)

$$\langle 0|\phi_k(\eta)\phi_{k'}(\eta)|0\rangle = \langle 0|[\phi^{cl}(\eta)\hat{a}^\dagger(k) + \phi^{cl}(\eta)^*\hat{a}(-k)][\phi^{cl}(\eta)\hat{a}^\dagger(k') + \phi^{cl}(\eta)^*\hat{a}(-k')]|0\rangle \tag{2.40}$$

The only term which survives is of the form $\langle 0|\hat{a}(-k)\hat{a}^\dagger(k')|0\rangle$. Thus we get

$$\langle 0|\phi_k(\eta)\phi_{k'}(\eta)|0\rangle = |\phi^{cl}|^2\langle 0|\hat{a}(-k)(\eta)\hat{a}^\dagger(k')|0\rangle$$

using the commutation relation, we get

$$\langle 0|\phi_k(\eta)\phi_{k'}(\eta)|0\rangle = |\phi^{cl}|^2\langle 0|0\rangle(2\pi)^3\delta(k+k')$$

$$\langle 0|\phi_k(\eta)\phi_{k'}(\eta)|0\rangle = |\phi^{cl}|^2(2\pi)^3\delta(k+k') \tag{2.41}$$

$$\langle 0|\phi_k(\eta)\phi_{k'}(\eta)|0\rangle = \frac{H^2}{2k^3}[1+k^2\eta^2](2\pi)^3\delta(k+k') \tag{2.42}$$

On large scales or $k\eta << 1$, we have

$$\langle 0|\phi_k(\eta)\phi_{k'}(\eta)|0\rangle = \frac{H^2}{2k^3}(2\pi)^3\delta(k+k') \tag{2.43}$$

on small scales or $k\eta >> 1$, we have the Minkowski limit

$$a^6 \langle 0|\phi_k(\eta)\phi_{k'}(\eta)|0\rangle = a^6 \frac{H^2\eta^2}{2k}(2\pi)^3\delta(k+k') = \frac{a^3}{2\frac{k}{a}}(2\pi)^3\delta(k+k') = \frac{1}{2p}(2\pi)^3\delta(p+p') \tag{2.44}$$

where the a^6 is added because

$$\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3} = a^6 \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3p'}{(2\pi)^3}$$

we have also used $\delta(ax) = \frac{1}{a}\delta(x)$. Thus

$$\langle 0|\phi_p(\eta)\phi_{p'}(\eta)|0\rangle = \frac{1}{2p}(2\pi)^3\delta(p+p') \tag{2.45}$$

which is the two point function of massless scalar in Minkowski. We can calculate the two point function in real space by fourier transforming (2.43)

$$\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3} \langle 0|\phi_k(\eta)\phi_{k'}(\eta)|0\rangle\, e^{i(k.x+k'.x')} = \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\frac{H^2}{2k^3}(2\pi)^3\delta(k+k')e^{i(k.x+k'.x')}$$

This gives us

$$\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3} \langle 0|\phi_k(\eta)\phi_{k'}(\eta)|0\rangle\, e^{i(k.x+k'.x')} = \int \frac{d^3k}{(2\pi)^3}\frac{H^2}{2k^3}e^{ik.(x-x')} \tag{2.46}$$

d^3k can be written as $k^2 sin\theta dk d\theta d\phi$. Let x-x$'$ be along z axis. This lets us have

$$\vec{k}.(x-x') = k|x-x'|cos\theta$$

so we can reduce the integral and get

$$\int \frac{d^3k}{(2\pi)^3}\frac{H^2}{2k^3}e^{ik.(x-x')} = \int \frac{dk}{(2\pi)^2}\frac{H^2}{2k}\int \mathrm{sin}\theta d\theta e^{ik|x-x'|\mathrm{cos}\theta} \tag{2.47}$$

Here we have used integral over $d\phi$ to be 2π. The inner integral can be simplified to give

$$\int \mathrm{sin}\theta d\theta e^{ik|x-x'|\mathrm{cos}\theta} = \int_{-1}^{1} d\mu e^{ik|x-x'|\mu} = \frac{2\mathrm{sin}(k|x-x'|)}{k|x-x'|}$$

Thus the expression finally looks like

$$\int \frac{d^3k}{(2\pi)^3}\frac{H^2}{2k^3}e^{ik.(x-x')} = \int \frac{dk}{(2\pi)^2}\frac{H^2}{k}\frac{2\mathrm{sin}(k|x-x'|)}{k|x-x'|} \tag{2.48}$$

In de-Sitter space, the metric looks like

$$ds^2 = \frac{1}{H^2\eta^2}[-d\eta^2 + d\sigma^2] \tag{2.49}$$

where $d\sigma$ is the metric on spatial indices. We can get this by a coordinate transformation of (1.1) to coordinates with conformal time, η. This metric possesses a dilation symmetry which is when

$$\eta \rightarrow \lambda\eta \;\; and \;\; \vec{x} \rightarrow \lambda\vec{x}$$

the metric remains invariant. This symmetry must be reflected in the two point function also. When $\vec{x} \rightarrow \lambda\vec{x}$, the k transforms as $\vec{k} \rightarrow \frac{\vec{k}}{\lambda}$. From (2.25), we can see that $\phi_k(\eta) \rightarrow \lambda^3\phi_{\frac{k}{\lambda}}(\eta)$(in the inverse fourier, there is a d^3x which gives λ^3). Thus when our two point function is of the form

$$\langle 0|\phi_k(\eta)\phi_{k'}(\eta)|0\rangle = (2\pi)^3\delta(k+k')\frac{F(k\eta)}{k^3} \tag{2.50}$$

It transforms as $\langle 0|\phi_k(\eta)\phi_{k'}(\eta)|0\rangle \rightarrow \lambda^6 \langle 0|\phi_{\frac{k}{\lambda}}(\eta)\phi_{\frac{k'}{\lambda}}(\eta)|0\rangle$(one λ^3 comes from delta function and the other comes from $\frac{1}{k^3}$). The function $F(k\eta)$ is invariant under this transformation. The λ^6 is cancelled by the $\int\int d^3k d^3k'$. Thus our spectrum is scale invariant.

2.1 Helicity Decomposition and Gauges

So far we have discussed about inflaton in a de-Sitter background. We cannot just treat background as non-dynamical, we must also discuss the perturbation in the metric. The most general perturbed metric one can write is

$$ds^2 = a(\eta)^2\left[-(1+2\phi)d\eta^2 + 2(\partial_i B - S_i)d\eta dx^i + [(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E + 2\partial_i F_j + \gamma_{ij}]dx^i dx^j\right] \tag{2.51}$$

The object with indices are vectors and tensors, while those with no index are scalars. Because of rotational symmetry, we can describe the modes by how they transform under rotation, i.e, scalars, vectors or tensors.

When working with gravity one must choose an appropriate gauge for calculations. Choosing a gauge means doing a coordinate transformation to eliminate some degrees of freedom. Here, as an example, we choose what is called a spatially flat gauge. In this gauge, scalars in the spatial part of the metric are zero, i.e,

$$\psi = E = 0 \tag{2.52}$$

If we ignore the vectors and tensors, the metric is spatially flat in this gauge. In this gauge, the inflaton field can be seen as

$$\phi(t,\vec{x}) = \phi_0(t) + \epsilon\varphi(t,\vec{x}) \tag{2.53}$$

$\phi_0(t)$ is the classical unperturbed field and $\epsilon\varphi(t,\vec{x})$ is the perturbation. Our calculations involving squeezing and two point function involve φ. We can instead choose a gauge in which the inflaton is not perturbed($\varphi(t, \vec{x}) = 0$) but space-time is curved. In other words surfaces of constant inflaton are not flat. We can choose whichever gauge is convenient for our calculations.

2.2 Equations of motion and limiting solutions

Scalar field with action given by

$$\mathcal{S} = \int d^4x\sqrt{-g}[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)] \tag{2.54}$$

has a classical equation of motion which in de-Sitter space is

$$\partial_t^2\phi + 3H\partial_t\phi - \frac{1}{a^2}\nabla^2\phi + \frac{\partial V}{\partial\phi} = 0 \tag{2.55}$$

For a spatially homogeneous field it reduces to (1.31), which was used to analyse single field slow-roll inflation. In general, however it is not spatially homogeneous but we will see it is a good approximation. Consider the fourier transformation second and the third term(which gives $\frac{k}{a}3H\phi$ and $\frac{k^2}{a^2}\phi$). When $k >> aH$, i.e, inside Hubble's radius, we can neglect the second term. For V=0, we get

$$\partial_t^2\phi_k + \frac{k^2}{a^2}\phi_k = 0 \tag{2.56}$$

which has solutions which are approximately oscillatory. This is the Minkowski limit. In this limit curvature can be neglected and solutions can be approximated to those in Minkowski.

In the other limit, $k << aH$, we have

$$\partial_t^2\phi_k + 3H\partial_t\phi_k = 0 \tag{2.57}$$

which is exactly like (1.31) with V=0. Thus on scales larger than Hubble's radius we see that neglecting the gradient term in the general equation of motion is a good approximation. At these scales the effect of curvature is significant. In this limit, the equation has two solutions. One is constant and the other is a decaying solution which goes as $\frac{1}{a^3}$. The decaying solution is generally neglected because during inflation a(t) grows fast and it quickly becomes insignificant. Thus at scales larger than Hubble radius ϕ is a constant.

Just for clarity, we introduce a new variable

$$\sigma = a\phi \tag{2.58}$$

and after an analysis similar to that after (2.6), we obtain the equation of motion in terms of this new variable as

$$\sigma_k'' + (k^2 - \frac{a''}{a})\sigma_k = 0 \tag{2.59}$$

where the primes denote derivative with respect to conformal time. $\frac{a''}{a}$ in de-Sitter space just gives us $\frac{2}{\eta^2}$. The equation thus becomes

$$\sigma_k'' + (k^2 - \frac{2}{\eta^2})\sigma_k = 0 \tag{2.60}$$

This equation is convenient because we just need to compare $k\eta$ with 1 and neglect accordingly. The corresponding equation in ϕ is

$$\phi_k'' - \frac{2}{\eta}\phi_k' + k^2\phi_k = 0 \tag{2.61}$$

This differential equation, as we have seen before, has the exact solution given by

$$\phi_k = A(1 - ik\eta)e^{ik\eta} \tag{2.62}$$

where A is independent of η. But ϕ_k* can also be a solution. In the minkowski limit, the equation in σ_k looks like

$$\sigma_k'' + k^2\sigma_k = 0 \tag{2.63}$$

It has solutions

$$\sigma_k = Re^{ik\eta}, R'e^{-ik\eta} \tag{2.64}$$

σ_k would be a linear combination of both of the solutions(here we also demand σ_k to be real).

$$\sigma_k = \hat{A}_k e^{ik\eta} + \hat{A}_k^\dagger e^{-ik\eta} \tag{2.65}$$

where we have promoted the coefficients to operators. σ_{-k} can similarly be written as

$$\sigma_{-k} = \hat{A}_{-k} e^{-ik\eta} + \hat{A}_{-k}^\dagger e^{ik\eta} \tag{2.66}$$

In order to find an expression for energy, we need $\partial_\eta \sigma_k$ as well. It is given by

$$\partial_\eta \sigma_k = ik[\hat{A}_k e^{ik\eta} - \hat{A}_k^\dagger e^{-ik\eta}] \tag{2.67}$$

and

$$\partial_\eta \sigma_{-k} = -ik[\hat{A}_{-k} e^{-ik\eta} - \hat{A}_{-k}^\dagger e^{ik\eta}] \tag{2.68}$$

Energy is given by

$$E_k = \frac{1}{2}[\partial_\eta \sigma_k \partial_\eta \sigma_{-k} + k^2 \sigma_k \sigma_{-k}] \tag{2.69}$$

Substituting the previous expressions in the expression for energy we get

$$E_k = k^2[\hat{A}_k\hat{A}_{-k} + \hat{A}_k^\dagger\hat{A}_{-k}^\dagger] \tag{2.70}$$

Now since we want to find the k dependence of the operators, if we assume

$$|\hat{A}_k| = |\hat{A}_{-k}| = |\hat{A}_k^\dagger| = |\hat{A}_{-k}^\dagger| = A \tag{2.71}$$

we get for n^{th} energy level

$$(n + \frac{1}{2})k = 2k^2A^2 \tag{2.72}$$

or

$$A = \sqrt{\frac{n+\frac{1}{2}}{2k}} \tag{2.73}$$

therefore ϕ_k can be written as

$$\phi_k = \frac{1}{a}\sqrt{\frac{n+\frac{1}{2}}{2k}}e^{\pm ik\eta} \tag{2.74}$$

If we consider the minkowski limit of (2.62) and also demand it to be real, we have

$$\phi_k = ik\eta[Ce^{ik\eta} - C^*e^{-ik\eta}] \tag{2.75}$$

By comparing we get

$$|C| = H\sqrt{\frac{n+\frac{1}{2}}{2k^3}} \tag{2.76}$$

In the other limit(super-horizon) we have ϕ_k which is constant in leading order if C is purely real and decays as $\frac{1}{a^3}$ if C is purely imaginary. The decaying solution will quickly become insignificant so we choose the constant solution. Thus C is real and its magnitude is given by the above equation. In general ϕ_k is given by

$$\phi_k(\eta) = H\sqrt{\frac{n+\frac{1}{2}}{2k^3}}[(1-ik\eta)e^{ik\eta} + (1+ik\eta)e^{-ik\eta}]$$

or

$$\phi_k(\eta) = H\sqrt{\frac{4n+2}{2k^3}}[cos(k\eta) + k\eta sin(k\eta)] \tag{2.77}$$

From (2.41), we see that the amplitude of two point function is given by $|\phi_{cl}|^2$. In the subhorizon limit, it gives

$$|\phi_{cl}|^2 = H^2\frac{4n+2}{2k^3}(k\eta)^2sin^2(k\eta) = \frac{2n+1}{2k}\frac{1}{a^2} \tag{2.78}$$

where we replaced $sin^2(k\eta)$ by its average over full cycle. In the superhorizon limit, the amplitude looks like

$$|\phi_{cl}|^2 = H^2 \frac{4n+2}{2k^3} = \frac{2}{(k\eta)^2} \frac{2n+1}{2k} \frac{1}{a^2} \tag{2.79}$$

Which differs from the subhorizon case by a factor of $\frac{2}{(k\eta)^2}$. Thus amplitude of fluctuation outside the Hubble's radius is very high as compared to the subhorizon case.

Now, with the solution for classical field in hand, we must probe the perturbations of the field around it. we write

$$\phi(x,t) = \phi(t) + \epsilon\varphi(x,t) \tag{2.80}$$

putting this in (2.55), we get

$$\partial_t^2\phi(t) + 3H\partial_t\phi(t) + \frac{\partial V}{\partial \phi}_{\phi(t)} + \epsilon[\partial_t^2\varphi(x,t) + 3H\partial_t\varphi(x,t) - \frac{1}{a^2}\nabla^2\varphi(x,t) + \frac{\partial^2 V}{\partial \phi^2}_{\phi(t)}\varphi(x,t)] = 0 \tag{2.81}$$

from which we get two equation by comparing the ϵ and non-ϵ terms. These are

$$\partial_t^2\phi(t) + 3H\partial_t\phi(t) + \frac{\partial V}{\partial \phi}_{\phi(t)} = 0 \tag{2.82}$$

and

$$\partial_t^2\varphi(x,t) + 3H\partial_t\varphi(x,t) - \frac{1}{a^2}\nabla^2\varphi(x,t) + \frac{\partial^2 V}{\partial \phi^2}_{\phi(t)}\varphi(x,t) = 0 \tag{2.83}$$

If we are interested in only super-horizon scales, then we can neglect the gradient term. Taking a time derivative of the first equation gives us

$$\partial_t^2\dot{\phi(t)} + 3H\partial_t\dot{\phi(t)} + \frac{\partial^2 V}{\partial \phi^2}_{\phi(t)}\dot{\phi(t)} = 0 \tag{2.84}$$

which looks exactly like the second equation with gradient removed and $\dot{\phi(t)}$ instead of $\varphi(x,t)$. Thus we can say that

$$\varphi(x,t) = \dot{\phi}(t)F(x) \tag{2.85}$$

Thus (2.80) can be written as

$$\phi(x,t) = \phi(t) + \epsilon\dot{\phi}(t)F(x)$$

or

$$\phi(x,t) = \phi(t + \epsilon F(x)) = \phi(t + \delta t(x)) \tag{2.86}$$

Thus inflation can be thought of as a clock, it gets delayed in some places and advanced in other depending on the spatial position. We will be exploring this in the next section.

2.3 Scalar Perturbations

For the time being let us ignore the vectors and tensors in the general perturbed metric and try to obtain a formula for scalar perturbations in the metric. We choose a gauge in which the inflaton is not perturbed, but perturbations appear in the spatial part of the metric. This can be done by defining a new space dependent time variation $t + \delta t$ such that the perturbation in (2.53) can be absorbed into $\phi_0(t)$.

$$\phi(t + \delta t, \vec{x}) = \phi_0(t) \tag{2.87}$$

After doing taylor expansion and comparing we get

$$\varphi(t, \vec{x}) = \dot{\phi}_0(t)\delta t$$

therefore

$$\delta t = \frac{\varphi(t, \vec{x})}{\dot{\phi}_0(t)} \tag{2.88}$$

In this new coordinates the spatial part of the metric will look like

$$a(t + \delta t)^2 \delta_{ij} = [a(t)^2 + 2a(t)\dot{a}(t)\delta t]\delta_{ij}$$

$$a(t + \delta t)^2 \delta_{ij} = a(t)^2[1 + 2H\delta t]\delta_{ij}$$

$$a(t + \delta t)^2 \delta_{ij} = a(t)^2[1 + 2\frac{H\varphi(t, \vec{x})}{\dot{\phi}_0(t)}]\delta_{ij}$$

We basically obtained the first term in the g_{ij} part in (2.51). We refer to the extra piece in the above equation by $\zeta(t, \vec{x})$

$$\zeta(t, \vec{x}) = \frac{H\varphi(t, \vec{x})}{\dot{\phi}_0(t)} \tag{2.89}$$

Its two point function is given by

$$\langle 0|\zeta_k(\eta)\zeta_{k'}(\eta)|0\rangle = \langle 0|\varphi_k(\eta)\varphi_{k'}(\eta)|0\rangle \frac{H^2}{\dot{\phi}_o^{\,2}} = (2\pi)^3\delta(k + k')\frac{H^2}{2k^3}\frac{H^2}{\dot{\phi}_0^{\,2}} \tag{2.90}$$

We can see from the normalisation of the spectrum that fluctuations in ζ are of the order $\frac{H^2}{\dot{\phi}_0}$. It is a dimensionless quantity and it can be re-written as $\frac{H}{\dot{\phi}_0/H}$. $\dot{\phi}_0/H$ is field distance or the classical fluctuations during one Hubble time(H^{-1}). H represents the scale of spatial perturbation in de-Sitter space(H is the only scale in de-Sitter). So the quantity $\frac{H}{\dot{\phi}_0/H}$ represents the ratio of quantum fluctuations of inflaton and classical motion of the inflaton. From (1.34), we can write this quantity in terms of the slow-roll parameter ϵ.

$$\frac{H^2}{\dot{\phi}_0} = \frac{H}{M_p\sqrt{2\epsilon}} \tag{2.91}$$

ζ represents the fluctuations in the metric. We can also try to find an expression for fluctuation in the density. Since during slow-roll, density is given by potential, we have

$$\frac{\delta\rho}{\rho} = \frac{\nabla_\phi V(\phi)\delta\phi}{V(\phi)} = \frac{\nabla_\phi V(\phi)H}{V(\phi)} = \sqrt{2\epsilon}\frac{H}{M_p} \tag{2.92}$$

where we used φ in (2.53) as $\delta\phi$ whose amplitude(from (2.43)) is H. Next we used the expression of ϵ from (1.43). We observe that density fluctuations are ϵ suppressed.

2.4 Normalization and Scale dependence

Defining the normalization of the ζ spectrum as

$$P_J = \frac{H^4}{2k^3\dot{\phi_0}^2} \tag{2.93}$$

If we integrate this object over momentum space, it becomes

$$\int \frac{d^3k}{(2\pi)^3}P_J = \int dkd\theta d\phi k^2 sin\theta \frac{P_J}{(2\pi)^3} = \int \frac{dk}{k}\frac{P_J k^3}{2\pi^2} = \int d\log(k)\frac{P_J k^3}{2\pi^2} \tag{2.94}$$

Thus the integrand is the contribution per change in logarithm of mode. This is a quantity which can be observationally constrained.

$$\frac{P_J k^3}{2\pi^2} = (2.4 \pm 0.1) \times 10^{-9} \tag{2.95}$$

We can calculate this quantity for different models of inflation. Let us work with quadratic potential, $V(\phi) = \frac{1}{2}m^2\phi^2$. We can use the results derived in the quadratic potential under single-field slow-roll inflation. It is convenient to calculate square root of P_J, as it appears in quantities which are squared. From (1.48) and (1.47) we have

$$\sqrt{\frac{P_J k^3}{2\pi^2}} = \frac{H^2}{2\pi\dot{\phi_0}} = \frac{1}{4\pi\sqrt{6}}\frac{m}{M_p^2}\phi^2 \tag{2.96}$$

From (1.46), we have

$$\sqrt{\frac{P_J k^3}{2\pi^2}} = \frac{1}{\pi\sqrt{6}}\frac{m}{M_p}N \tag{2.97}$$

where N is the number of e-folds and is of order 60. Using this along with (2.95), we can estimate that the mass of the scalar field is of order $\sim 10^{13}Gev$.

We would also like to calculate how much our given spectrum deviates from scale invariance($\frac{1}{k^3}$). If H and $\dot{\phi_0}$ were constant in (2.93), then our spectrum would have been scale invariant. But as we saw in (2.97), there is dependence on N, which changes during inflation. Hence there is a deviation from scale invariance in our spectrum. It is parametrized by a quantity n_S, which is called 'Tilt' and is defined as

$$P_J \propto k^{-3+(n_S-1)} = k^{-3+(n_S-1)} F(t) \tag{2.98}$$

where $(n_S - 1)$ is the deviation from scale invariance. F(t) doesn't depend on k. We can calculate the tilt by

$$k^{-3+(n_S-1)} F(t) = \frac{H^4}{2k^3 \dot{\phi_0}^2}$$

taking log

$$(n_S - 1)\log k + \log F(t) = \log \frac{H^4}{2\dot{\phi_0}^2}$$

and then differentiating with respect to $\log k$

$$(n_S - 1) = \frac{d}{d\log k}\left(\log \frac{H^4}{\dot{\phi_0}^2}\right) \tag{2.99}$$

Let us choose a particular fourier mode k(say on which we get CMB maxima), and evaluate the above equation at the time when this mode crosses the Hubble horizon($k \sim aH$). This way we can see the k dependence of the quantity to be differentiated but we only need to do a time derivative. Assuming H to be constant during inflation and $a \sim e^{Ht}$, we have

$$d\log k = H dt$$

therefore the expression for tilt becomes

$$(n_S - 1) = H^{-1}\frac{d}{dt}\left(\log \frac{H^4}{\dot{\phi_0}^2}\right) \tag{2.100}$$

Using (1.42), we can write the above in terms of the potential

$$(n_S - 1) = H^{-1}\frac{d}{dt}\left(\log \frac{H^6}{(\nabla_\phi V)^2}\right) = -6(-\frac{\dot{H}}{H^2}) + 2\frac{\nabla_\phi^2 V \dot{\phi}}{\nabla_\phi V H} = -6\epsilon - 2\eta \tag{2.101}$$

For quadratic potential, we have

$$\epsilon = -\eta = 2\frac{M_{p^2}}{\phi^2} = \frac{1}{2N(\phi)} \tag{2.102}$$

where we used (1.46) in last part. Thus the tilt for quadratic potential becomes

$$n_S = 1 - \frac{2}{N(\phi)} = 0.97 \tag{2.103}$$

Data from Planck measurements [**?**] gives the value of tilt to be

$$n_S = 0.9649 \pm 0.0042 \tag{2.104}$$

2.5 Gravitational waves

Tensor fluctuations can be studied by considering the quadratic action similar to (2.9)

$$\mathcal{S} = c\int d\eta d^3x a(\eta)^2[(\partial_\eta \gamma_{ij})^2 - (\nabla\gamma_{ij})^2] \tag{2.105}$$

One can get this action by calculating the Einstein-Hilbert action of the perturbed metric(with the tensor perturbation) and get the constant c to be $\frac{M_p^2}{8}$. In order to simplify , we go to the fourier space and write

$$\gamma_{ij} = \int \frac{d^3k}{(2\pi)^3}\sum_{s=\pm}\epsilon^s_{ij}(k)\gamma^s_{\vec{k}}(\eta)e^{i\vec{k}.\vec{x}} \tag{2.106}$$

where s represents the two states of polarisation of gravitational waves. The polarisation tensors ϵ^s_{ij} are transverse, traceless($\epsilon^s_{ii} = k^j\epsilon^s_{ij} = 0$) and obey the normalisation $\epsilon^s_{ij}\epsilon^{s',ij} = 2\delta_{ss'}$. With these rules one can calculate the action in fourier space and get

$$\mathcal{S} = \sum_{s=\pm}\int d\eta d^3k\frac{a(\eta)^2}{4}M_p^2[(\partial_\eta\gamma^s_{\vec{k}})^2 - k^2(\gamma^s_{\vec{k}})^2] \tag{2.107}$$

Apart from the summation over polarisations, the above equation is similar to that of scalar field(to be precise, rewriting the action in terms of $v^s_{\vec{k}} = \frac{M_p}{\sqrt{2}}\gamma^s_{\vec{k}}$ gives us the exact form). Thus without much effort we write the power spectrum for tensor fluctuations

$$\langle 0|v^s_{\vec{k}}v^{s'}_{\vec{k}'}|0\rangle = \frac{H^2}{2k^3}(2\pi)^3\delta(k+k')2\delta_{ss'} \tag{2.108}$$

or equivalently, we can write

$$\langle 0|\gamma^s_{\vec{k}}\gamma^{s'}_{\vec{k}'}|0\rangle = \frac{1}{2k^3}\frac{2H^2}{M_{p^2}}(2\pi)^3\delta(k+k')2\delta_{ss'} \tag{2.109}$$

The $2\delta_{ss'}$ comes from the normalisation of ϵ^s_{ij}. The normalisation of the spectrum is

$$P_I = 2\frac{1}{2k^3}\frac{2H^2}{M_{p^2}}2 \tag{2.110}$$

where we multiplied by two because there are two polarisations. We observe that there is no ϵ dependence unlike that in scalar perturbations. Ratio of tensor to scalar amplitudes is

$$r = \frac{P_I}{P_J} = \frac{8\dot{\phi}^2}{M_{p^2}H^2} = 16\epsilon \tag{2.111}$$

where we used (1.34) in the last part.

2.6 Choice of Vacuum

Commutation relation in X and P in quantum mechanics is

$$[X, P] = i\hbar \tag{2.112}$$

Let ψ be the wavefunction. We know that the action of momentum operator, P, on ψ in x basis is

$$\langle x|P|\psi\rangle = -i\hbar\frac{\partial\psi(x)}{\partial x} \tag{2.113}$$

Where $\psi(x)$ is the wavefunction in position space, $\langle x|\psi\rangle$. Now let the wavefunction be in the momentum basis, $|p\rangle$, such that

$$P\,|p\rangle = p\,|p\rangle \tag{2.114}$$

Projecting this onto x basis we get

$$\langle x|P|p\rangle = p\,\langle x|p\rangle \tag{2.115}$$

From (2.113), the R.H.S can be written as

$$-i\hbar\frac{\partial\,\langle x|p\rangle}{\partial x} = p\,\langle x|p\rangle \tag{2.116}$$

This is a linear differential in $\langle x|p\rangle$, whose solution is

$$\langle x|p\rangle = e^{ipx/\hbar} \tag{2.117}$$

We know that momentum is the Generator of infinitesimal translations. In other words the infinitesimal translation operator T(ϵ)

$$T(\epsilon)\,|x\rangle = |x+\epsilon\rangle \tag{2.118}$$

can be written as

$$T(\epsilon) = I - \frac{i\epsilon}{\hbar}P \tag{2.119}$$

where P is the generator. Under this transformation, the wavefunction transforms as

$$T(\epsilon)\,|\psi\rangle = T(\epsilon)\int_{-\infty}^{\infty}|x\rangle\,\langle x|\psi\rangle = \int_{-\infty}^{\infty}|x+\epsilon\rangle\,\langle x|\psi\rangle = \int_{-\infty}^{\infty}|x'\rangle\,\langle x'-\epsilon|\psi\rangle \tag{2.120}$$

Therefore

$$\langle x|T(\epsilon)|\psi\rangle = \psi(x-\epsilon) \tag{2.121}$$

Taylor expanding R.H.S and using the expression of T(ϵ) on L.H.S we get

$$\langle x|\psi\rangle - \frac{i\epsilon}{\hbar}\,\langle x|P|\psi\rangle = \psi(x) - \frac{\partial\psi}{\partial x}\epsilon \tag{2.122}$$

comparing we get (2.113). In quantum field theory we have

$$|x\rangle = \phi(x)\,|0\rangle \tag{2.123}$$

and

$$|p\rangle = \sqrt{2\omega_p} a_p^\dagger\,|0\rangle \tag{2.124}$$

Mode expansion of ϕ is

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{p'}}}[a_p e^{\pm\frac{i}{\hbar}p.x} + a_p^\dagger e^{\mp\frac{i}{\hbar}p.x}] \tag{2.125}$$

Then we have

$$\langle x|p\rangle = \langle 0|\int \frac{d^3p'}{(2\pi)^3}[a_{p'}a_p^\dagger e^{\pm\frac{i}{\hbar}p'.x} + a_{p'}^\dagger a_p^\dagger e^{\mp\frac{i}{\hbar}p'.x}]|0\rangle \tag{2.126}$$

The second term is zero and on first term we use the commutation relation of annihilation and creation operators.

$$[a_p, a_{p'}^\dagger] = \delta^3(p - p') \tag{2.127}$$

we get

$$\langle x|p\rangle = e^{\pm\frac{i}{\hbar}p'.x} \tag{2.128}$$

in order to be consistent with quantum mechanics we see that annihilation operator must be associated with $e^{\frac{i}{\hbar}p'.x}$ term.