1 Order of Magnitude estimate

1.1 Time period of Gravitational wave oscillations

We use Newtonian theory to calculate order of time period of space-time oscillations generated by a binary black hole each of 30 solar masses. The signal is most pronounced when event horizon of black holes just start to merge. Schwarzchild radius of each black hole is given by

$$R = \frac{2Gm}{c^2} \tag{1}$$

Newtonian force on each mass is

$$F = \frac{Gm^2}{(2R)^2} \tag{2}$$

Equating to centripetal force we get

$$v = \sqrt{\frac{Gm}{4R}} \tag{3}$$

Now using

$$v = 2\pi R\nu \tag{4}$$

we get

$$\nu = \frac{1}{2\pi R}\sqrt{\frac{Gm}{4R}} \tag{5}$$

Using this equation we estimate the order of frequency of oscillation to be around 10^2 Hz, which is in good agreement with the frequency detected at LIGO(75Hz). This shows that Newtonian theory of Gravitation still provides a good estimate.

1.2 minimum mass of black hole to be probed

In order to probe an object using these waves, the size of the object must be comparable to wavelength of the wave. Since Gravitational waves travel at the speed of light, wavelength of the above mentioned wave must be of the order of 10^6m. This implies that schwarzchild radius of the black hole to be probed must be of this order.

$$\mathcal{O}(R) = 10^6$$

using (1) this gives

$$\mathcal{O}(m) = 10^3 M_0$$

where M_0 is mass of sun

2 Brief overview of theory Of GR waves

2.1 Linearized Weak Field Equations

Far away from a static source, the space-time metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{6}$$

where

$$|h_{\mu\nu}| << 1$$

everywhere. It describes a nearly flat space-time.
Now consider a coordinate transformation of the type

$$x^a = \Lambda^a_b x^b \tag{7}$$

where Λ^a_b is the Lorentz transformation matrix. Since we are not in Special Relativity this is only one class of transformations out of many. Under this transformation

$$g_{a'b'} = \Lambda^{\mu}_{a'}\Lambda^{\nu}_{b'}g_{\mu\nu}g_{a'b'} = \Lambda^{\mu}_{a'}\Lambda^{\nu}_{b'}\eta_{\mu\nu} + \Lambda^{\mu}_{a'}\Lambda^{\nu}_{b'}h_{\mu\nu} \tag{8}$$

but

$$\Lambda^{\mu}_{a'}\Lambda^{\nu}_{b'}\eta_{\mu\nu} = \eta_{a'b'} \tag{9}$$

defining

$$h_{a'b'} = \Lambda^{\mu}_{a'}\Lambda^{\nu}_{b'}h_{\mu\nu} \tag{10}$$

we get

$$g_{a'b'} = \eta_{a'b'} + h_{a'b'} \tag{11}$$

Thus h_{ab} transform as if it were a tensor in SR(It is not because it is just a piece of g_{ab}.The transformations under considerations are restricted.). This allows us to visualise the space-time as flat with a tensor field h_{ab} defined on it.

We have another type of coordinate transformation that leaves eqn(6) unchanged:

$$x^{a'} = x^a + \epsilon^a(x^b) \tag{12}$$

where ϵ^a is small.
We then have

$$\Lambda^a_{b'} = \delta^a_b - \epsilon^a{}_{,b} \tag{13}$$

to first order in ϵ.
Then

$$g_{a'b'} = \eta_{ab} + h_{ab} - \epsilon_{a,b} - \epsilon_{b,a} \tag{14}$$

Now if we define

$$\bar{h}^{ab} = h^{ab} - \frac{1}{2}\eta^{ab}h \tag{15}$$

where

$$h = h^a_a \tag{16}$$

with these definitions we can show that Einstein tensor is

$$G_{ab} = -\frac{1}{2}[\bar{h}_{ab,v}{}^{;v} + \eta_{ab}\bar{h}^{,cv}_{cv} - \bar{h}_{av,b}{}^{;v} - \bar{h}_{bv,a}{}^{;v} + \mathcal{O}(h^2_{ab})] \tag{17}$$

This equation can be simplified if we choose

$$\bar{h}^{av}{}_{,v} = 0 \tag{18}$$

Since we have four gauge functions ϵ^a, we can find a gauge in which eqn(18) is true.

Suppose we have an arbitrary $\bar{h}_{ab}$ in which $\bar{h}^{av}{}_{,v} \neq 0$. Then under a gauge change $\bar{h}_{ab}$ changes as

$$\bar{h}'_{ab} = \bar{h}_{ab} - \epsilon_{a,b} - \epsilon_{b,a} + \eta_{ab}\epsilon^{v}{}_{,v} \tag{19}$$

Its divergence is

$$\bar{h}'^{av}{}_{,v} = \bar{h}^{av}{}_{,v} - \epsilon^{a,v}{}_{,v} \tag{20}$$

Thus epsilon is determined by the equation

$$\Box\epsilon^{a} = \bar{h}^{av}{}_{,v} \tag{21}$$

In this gauge we have

$$G^{ab} = -\frac{1}{2}\Box\bar{h}^{ab} \tag{22}$$

and Einstein equations give(in units G=1 and c=1)

$$\Box\bar{h}^{ab} = -16\pi T^{ab} \tag{23}$$

Newtonian gravity is valid when gravitational fields are too weak to produce velocities near speed of light.

$$\mid \phi \mid << 1, \mid v \mid << 1$$

Under these conditions components of T^{ab} obey

$$\mid T^{00} \mid >> \mid T^{0i} \mid >> \mid T^{ij} \mid$$

These inequalities transfer to $\bar{h}^{ab}$ because of eqn(23).

$$\mid \bar{h}^{00} \mid >> \mid \bar{h}^{0i} \mid >> \mid \bar{h}^{ij} \mid$$

Thus we can expect the dominant Newtonian gravitational field comes from the dominant field equation

$$\Box\bar{h}^{00} = -16\pi\rho$$

where ρ is T^{00}. Moreover for fields that change only because of motion of source with velocity v, we can approximate

$$\Box = \nabla^2 + \mathcal{O}(v^2\nabla^2)$$

Therefore our equation to lowest order is

$$\nabla^2\bar{h}^{00} = -16\pi\rho \tag{24}$$

comparing with

$$\nabla^2\phi = 4\pi\rho$$

we get

$$\bar{h}^{00} = -4\phi$$

upon doing some analysis we obtain

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2 + dy^2 + dz^2) \tag{25}$$

This is the metric which gives us the correct Newtonian laws of motion. So we obtain Newtonian gravity as a limiting case of GR.

2.2 Propagation of Gravitational waves

Einstein equations in vacuum far away from source(T^{ab}=0) is given by eqn(23). This equation has a complex solution of the form

$$\bar{h}^{ab} = A^{ab} exp(ik_a x^a) \tag{26}$$

Putting this equation back in eqn(23) shows that k_a is a null vector. Now imagine a photon travelling in the direction of this null vector.

$$x^a(\lambda) = k^a \lambda + l^a$$

This gives $\bar{h}^{ab}$ a constant implying that a photon travels at the same phase with the gravitational wave. Thus gravitational waves travel at the speed of light.

We arrived at eqn(23) by imposing gauge condition given by eqn(18). Applying that condition on eqn(26) gives us

$$A^{ab} k_b = 0 \tag{27}$$

We can use our gauge freedom to impose more constraints on the amplitude. Consider a vector ϵ_a satisfying

$$\Box \epsilon_a = 0$$

This is a solution to eqn(21). Consider a solution of the form

$$\epsilon_a = B_a exp(ik_b x^b)$$

where k^b is the same null vector. Using eqn(19) we have

$$A'_{ab} = A_{ab} - iB_a k_b - iB_b k_a + i\eta_{ab} B^\mu k_\mu \tag{28}$$

The B_a can be chosen to impose two more restriction on A_{ab} :

$$A^a_a = 0 \tag{29}$$

and

$$A_{ab} U^b = 0 \tag{30}$$

where U is any constant timelike unit vector. Eqn(27),(29) and (30) are called transverse-traceless(TT) gauge conditions. Since we exhausted all our gauge freedom, the remaining quantities must be physically relevant. We choose $U^b = \delta^b_0$ and align our coordinates such that wave propagates along z direction, $\vec{\mathbf{k}} = (\omega, 0, 0, \omega)$. Thus our A_{ab} in TT gauge is

$$A^{TT}_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{31}$$

There are only two independent constants A^{TT}_{xx} and A^{TT}_{xy}.

Effect on free particles

A free particle obeys the geodesic equation

$$\frac{d}{dt}U^a + \Gamma^a_{uv}U^u U^v = 0 \tag{32}$$

where U is the four velocity. Since particle is initially at rest, $U^a = (1, 0, 0, 0)$. The initial value of acceleration in TT gauge is given by

$$-\Gamma^a_{00} = -\frac{1}{2}\eta^{ab}(h_{b0,0} + h_{0b,0} - h_{00,b}) = 0 \tag{33}$$

Hence particle remains at rest in TT coordinates. This implies TT gauge gives coordinates which stays attached to the particles and makes adjustment to the wiggles of the spacetime. To get a better measure of the effect of the wave, we need to calculate something independent of coordinates, such as proper distance. Let there be two particles at (0,0,0,0) and (0,ϵ,0,0).

$$\Delta l = \int \mid ds^2 \mid^{1/2} = \int \mid g_{ab}dx^a dx^b \mid^{1/2}$$

$$\Delta l \approx [1 + \frac{1}{2}h^{TT}_{xx}]\epsilon \tag{34}$$

This shows that proper distance does change with time. The change in distance is proportional to ϵ, hence the change is greater if original distance is big. This is the reason why modern gravitational wave detectors are built on huge scales(order of km on ground). But the effect is small, proportional to h^{TT}_{ab}(of the order of 10^{-21} or smaller).

2.3 Detection of Gravitational waves

The small amplitudes of metric perturbations gives rise to enormous technical difficulties. Due to this reason gravitational waves were discovered only recently, 100yrs after their prediction. There are several methods of detection such as bar detectors and interferometers. I will write mainly about interferometers because they are the most sensitive as of now.

The interferometer at LIGO gravitational wave observatory at Hanford has arms of length 4km. It is 360 times larger than the one used in Michelson-Morley experiment(11m). The change in length due to gravitational waves in this cavity is $\frac{1}{1000}$ the size of proton. In order to detect this change LIGO has to be more sensitive than any scientific instrument ever built. Even with 4km length, it is still not possible to detect gravitational waves. It is overcome by adding Fabry Perot cavities to basic Michelson design. It reflects parts of each laser beam back and forth within 4km long arms about 280 times before they are merged again. This increases the effective length to 1120km long, making it 144,000 times bigger than Michelson's instrument. It greatly increases LIGO's sensitivity and makes it capable of detecting changes thousands of times smaller than proton.

A gravitational wave can stretch space and hence can change the wavelength of light used to measure the stretching. A question arise,'can this cancel the effect of stretching?' The answer is 'no'. The apparatus is arranged such that light from both the arms interfere destructively. In the event of passage of gravitational wave, one arm undergoes change in length relative to other. Hence, light from both the arms now reach at different times at the site of merging(speed of light doesn't change). This breaks the sync which was initially there and we no longer have destructive interference. Hence we don't care whether wavelength has changed or not, just the introduction of extra path due to the wave enables us to detect its presence.

2.4 Analysis of LIGO discovery using Introductory Physics

Binary black hole mergers take place in three stages. Initially the black hole circle their common center of mass in circular orbits. During this they lose energy in the form of gravitational radiation and spiral inward. In second stage they coalesce to form a single black hole. In the third stage, called ring down, the black hole relaxes into its equilibrium state called a Kerr black hole.

Consider a system of binary black hole, consisting of black holes of mass M and m rotating about their common center of mass in orbits of radius R and r. We can show that angular frequency is given by

$$\omega^2 = \frac{G(M+m)}{(R+r)^3} \tag{35}$$

circular orbits are assumed for simplicity. One can show that any deviation from circular orbits die out due to emission of gravitational waves. Total energy of the system(individual kinetic and mutual potential energy) is given by

$$E_{tot} = -\frac{1}{2}\frac{GMm}{R+r} \tag{36}$$

from eqn.(35) we get

$$E_{tot} = -\frac{1}{2}\frac{G^{2/3}Mm}{(M+m)^{1/3}}\omega^{2/3} \tag{37}$$

Moment of inertia

$$\mathcal{I} = \frac{mM}{M+m}(r+R)^3 \tag{38}$$

Within Newtonian framework, circular orbits persist due to absence of gravitational radiation. In general relativity however, there is emission of gravitational waves and black holes spiral towards each other. Drawing analogy from electromagnetic dipole radiation we expect that gravitational radiation should be proportional to the quadrupole moment that sources it and hence $P_{rad} \propto \mathcal{I}^2$. P_{rad} must also depend on c, G and ω. Hence

$$P_{rad} = \alpha \mathcal{I}^2 \omega^\epsilon G^\eta c^\zeta$$

α is dimensionless. By dimension analysis we have

$$P_{rad} = \alpha \frac{G\mathcal{I}^2\omega^6}{c^5} \tag{39}$$

Analysis using general relativity gives $\alpha = 32/5$. Using eqn (38) and (35) we have

$$P_{binary} = \alpha \frac{G^{7/3}\omega^{10/3}}{c^5} \frac{m^2 M^2}{(M+m)^{2/3}} \tag{40}$$

Differentiating eqn(37) with respect to time gives us the rate of loss of orbital energy. Equating that to our expression of radiated power gives

$$\mathcal{M} = \frac{(mM)^{3/5}}{(m+M)^{1/5}} = \frac{c^3}{G}\Big(\frac{1}{3\alpha}\omega^{-11/3}\frac{d\omega}{dt}\Big)^{3/5} \tag{41}$$

Where $\mathcal{M}$ is called chirp mass. Setting $\omega = \pi f$ in (41) gives us the only equation in LIGO discovery paper.

$$\mathcal{M} = \frac{c^3}{G}\Big(\frac{1}{3\alpha}\pi^{-8/3} f^{-11/3}\frac{df}{dt}\Big)^{3/5} \tag{42}$$

3 Robertson-Walker Metric

Contemporary cosmological models are based on the idea that universe is pretty much the same everywhere. This is known as Copernicam principle and it suggests that universe is isotropic and homogenous in space an time in space and time. But observations suggest that universe is spatially homogenous and isotropic but evolving in time. This is manifested in the form of spacetime metric

$$ds^2 = -dt^2 + R^2(t)d\sigma^2 \tag{43}$$

R is the scale factor and σ is the metric on spatial part of the manifold. The coordinates used here, which makes metric free of cross terms between spacial and time differentials and in which coefficient of dt^2 is independent of spacial coordinates, are known as comoving coordinates.
We now want our spatial part of our metric to be maximally symmetric. This in turn implies it is spherically symmetric. Thus we can write

$$d\sigma^2 = \exp^{2\beta(\bar{r})} d\bar{r}^2 + \bar{r}^2 d\Omega^2 \tag{44}$$

Now setting riemann tensor, obtained from above metric, proportional to components of the metric, we obtain

$$\beta = -\frac{1}{2}\ln(1 - k\bar{r}^2) \tag{45}$$

thus metric on the spatial part is

$$d\sigma^2 = \frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\Omega^2 \tag{46}$$

sign of k sets the curvature. We can normalise it so that

$$k\epsilon\{-1, 0, +1\}$$

which correspond to negative, zero and positive curvature. It we define

$$d\chi = \frac{\bar{r}}{\sqrt{1 - k\bar{r}^2}} \tag{47}$$

this gives

$$\bar{r} = S_k(\chi) \tag{48}$$

where

$$S_k(\chi) = \begin{cases} sin(\chi) & k = +1 \\ \chi & k = 0 \\ sinh(\chi) & k = -1 \end{cases} \tag{49}$$

Thus we get the Robertson walker metric

$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\Big] \tag{50}$$

where a(t) is the scale factor at time t. Now in order to find the evolution of scale factor we need to solve the Einstein's equations. Let us model the matter and energy bey a perfect fluid. Fluid will be at rest in comoving coordinates. The four is then $U^\mu = (1, 0, 0, 0)$ and energy-momentum tensor is given by

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{51}$$

Trace is given by

$$T^\mu_\nu = -\rho + 3p \tag{52}$$

Zeroth component of conservation of energy-momentum tensor is given by

$$0 = \nabla_\mu T^\mu_0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda$$

Using the cristoffel symbols from the FRW metric we obtain

$$\partial_0 \rho = -3\frac{\dot{a}}{a}(\rho + p) \tag{53}$$

By proposing the ansatz

$$p = w\rho$$

the energy conservation equation becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \tag{54}$$

which gives

$$\rho \propto a^{-3(1+w)} \tag{55}$$

w for matter, radiation and vaccum dominated universe are $0, \frac{1}{3}$ and -1 respectively.

An alternate form of einstein's equation is

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) \tag{56}$$

The $\mu\nu = 00$ equation is given by

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p) \tag{57}$$

and $\mu\nu = ij$ equation(only one due to isotropy) is given by

$$\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) \tag{58}$$

While deriving them we used the ricci tensor for FRW metric. Simplifying we obtain

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{59}$$

and

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}(\rho + 3p) \tag{60}$$

These equations are known as Friedmann equation. Defining $H = \dot{a}/a$ and

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{crit}}$$

the first friedmann equation can be written as

$$\Omega - 1 = \frac{k}{H^2 a^2} \tag{61}$$

If we introduce a fictitious energy density

$$\rho_c = -\frac{3k}{8\pi G a^2}$$

and corresponding density paramter

$$\Omega_c = -\frac{k}{H^2 a^2}$$

then the first friedmann equation becomes

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i \tag{62}$$

Where the index i runs over matter, radiation, vacuum and curvature(ρ_c).w for the ρ_c is -1/3. Also

$$\sum_i \Omega_i = 1$$

Using both the friedmann equations, the time derivative of H can be written as

$$\dot{H} = -4\pi G \sum_i (1 + w_i)\rho_i \tag{63}$$

3.1 Redshift and Distances

As the universe expands, frequency of photon emitted with frequency ω_{em} will be observed with a lower frequency ω_{obs}.

$$\frac{\omega_{obs}}{\omega(em)} = \frac{a_{em}}{a_{obs}}$$

Redshift z between two events is defined by fractional change in wavelength

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$$

Present value of scale factor is taken to be 1. Thus

$$a_{em} = \frac{1}{1 + z_{em}} \tag{64}$$

In a general spacetime, there is no correct way to measure distance as opposed to euclidian spacetime. This mainly because spacetime is curved and because distance is measured not at a given instant of time but along a backward light cone. There is not a unique meaning of distance. But one one can construct methods on how to measure distance and define distance acording to these procedures.

Consider a body of physical diameter 2R and observed angular diameter δ. In euclidean space one measures the distance to be $D = 2R/\delta$. Accordindly one defines

$$D_{ang}(z) = 2R/\delta = aS_k(\chi) = \frac{S_k(\chi)}{1+z} \tag{65}$$

which is obtained by putting dθ = δ and ds=2R in the FRW metric. For our universe, $S_k(\chi) = \chi$. χ is also known as comoving distance. For a null geodesic(path of light) we have $ds^2 = 0$. This gives

$$d_{comoving}(z) = \chi(z) = \int_0^z \frac{dt}{a} = \int \frac{da}{a^2 H(a)} = \int \frac{dz}{H(z)} \tag{66}$$

But using equation (62) and (55), we have

$$H(z) = HoE(z) \tag{67}$$

where

$$E(z) = \Big[\sum_i \Omega_{i0}(1+z)^{3(1+w)}\Big]^{1/2}$$

Therefore d_{ang} is given by

$$d_{ang} = \frac{1}{(1+z)Ho} \int_0^z \frac{dz'}{E(z')} \tag{68}$$

4 Description of python code

Einstein radius

$$R_E = \sqrt{\frac{4GMd_{LS}d_L}{c^2 d_S}} \tag{69}$$

the d's are angular diameter distances, which do not necessarily add.

$$d_{angular} = \frac{c}{(1+z)H_o} \int_0^z \frac{dz'}{E(z')} \tag{70}$$

$$dV = \pi R_E^2 dd_{angular} \tag{71}$$

$$dd_{angular} = \frac{c}{(1+z)E(z)} \frac{dz}{H_o}$$

$$d_{LS} = \frac{1}{1+z_S} \int_{z_L}^{z_S} \frac{dz'}{E(z')}$$

combining everything we have

$$V = \int_0^{z_S} dV = \frac{4\pi GMR1}{H_o^2 R3} \tag{72}$$

where

$$R1 = \int_0^{z_S} \int_0^{z} \int_z^{z_S} \frac{1}{(1+z)^2 E(z)E(z_1)E(z_2)} dz_1 dz_2 dz$$

$$R3 = \int_0^{z_S} \frac{dz}{E(z)}$$

5 Lensing

Lensing is the deflection and time delay of light by a Gravitational field. Consider a lens of mass M and source at impact parameter b. The angulardiameter distances between lens and earth, source and earth and lens and source be d_L, d_S and d_{LS} respectively. Let β, α, θ and $\bar{\alpha}$ be angles between source and lens, image and source, image and lens and deflection angle respectively. Reduced lensing angle α is given by

$$\alpha = \theta - \beta \tag{73}$$

By simple geometry

$$\alpha = \frac{d_{LS}}{d_S} \bar{\alpha} \tag{74}$$

we therefore get the lens equation

$$\beta = \theta - \frac{d_{LS}}{d_S} \bar{\alpha} \tag{75}$$

Now using

$$\bar{\alpha} = \frac{4GM}{c^2 b}$$

and

$$b = d_L \theta$$

we get

$$\beta = \theta - \frac{d_{LS}}{d_S d_L} \frac{4GM}{c^2 \theta} \tag{76}$$

with $\beta = 0$, the source will be lensed into an Einstein ring surrounding the lens, at an angular separation given by Einstein angle

$$\theta_E = \sqrt{\frac{4GMd_{LS}}{c_2 d_L d_S}} \tag{77}$$

Corresponding Einstein radius is defined as

$$R_E = \sqrt{\frac{4GMd_Ld_{LS}}{c_2d_S}} \tag{78}$$

We can also write the lens equation as

$$\beta = \theta - \frac{\theta_E^2}{\theta} \tag{79}$$

Solving for θ we obtain

$$\theta_{\pm} = \frac{\beta \pm \sqrt{\beta_2 + 4\theta_E^2}}{2} \tag{80}$$

Thus there is one image on either side of the lens. Image on same side of the source is further away from the lens than the other image. Let $x = \theta/\theta_E$ and $y = \beta/\theta_E$. Then the two solutions take the form

$$x_{\pm} = \frac{1}{2}(y \pm \sqrt{y^2 + 4})$$

Image magnification μ is given to be

$$\mu = (1 - \frac{1}{x^4})^{-1}$$

this gives

$$\mu_{\pm} = \pm\frac{1}{4}[\frac{y}{\sqrt{y^2+4}} + \frac{\sqrt{y^2+4}}{y} \pm 2] \tag{81}$$

the total magnification of the source is given by

$$\mu_p = \mu_+ + |\mu_-| = \frac{y^2+2}{y\sqrt{y^2+4}}$$

the ratio of magnification is given by

$$R = |\frac{\mu_+}{\mu_-}| = \frac{(y^2+2) + y\sqrt{y^2+4}}{(y^2+2) - y\sqrt{y^2+4}}$$

Also time delay in terms of M,zl and y is given by

$$\Delta t = \frac{4GM}{c^3}(1 + zl)[\frac{y}{2}\sqrt{y^2+4} + ln(\frac{\sqrt{y^2+4}+y}{\sqrt{y^2+4}-y})] \tag{82}$$

R and Δt are monotonic functions of y.

In microlensing we are unable to distinguish the two images and we observe different peaks in the light curve corresponding to the two images. For the images to be detectable there must be a minimum detectable time delay $\bar{\Delta t}$ between the two images, where if $\Delta t < \bar{\Delta t}$ the time delay is too short to be resolved by autocorrelation. Similarly there exists a maximum detectable magnification $\bar{R}$, where for $R > \bar{R}$ the second image is too weak to be detected. These two factors give an upper and lower bound(they are monotonic) on possible values of y. Hence we are interested in an effective cross section which is

$$\sigma(M, zl, zs; \bar{R}, \bar{\Delta t}) = \pi\theta_E Ramp[y_{max}^2(\bar{R}) - y_{min}^2(M, zl, \bar{\Delta t})] \tag{83}$$

Ramp ensures if quantity inside is negative then it is zero. Now assuming all MACHOs to be of the same mass and uniformly distributed in the universe, we get the optical depth at redshift zs to be

$$\tau(zs, M, f_{DM}; \bar{R}, \bar{\Delta t}) = \int_0^{zs} d\chi(zl)(1+zl)^2 n_L \sigma(M, zl, zs; \bar{R}; \bar{\Delta t}) \tag{84}$$

where χ is comoving distance, n_L is the comoving MACHO number density($f_{DM} = 8\pi G n_L M c^2/3H_0^2\Omega_d$).. Total optical depth is given by

$$\tau_{tot}(M, f_{DM}; \bar{R}, \bar{\Delta t}) = \int N(zs) dzs \tau(zs, M, f_{DM}; \bar{R}, \bar{\Delta t}) \tag{85}$$

In a catalog cantaining N GRBs, if an algorithm picks N_* events as lensed with specified $(\bar{R}; \bar{\Delta t})$, then $\tau_{tot} < N_*/N$ can be put as constraint.