SEQUENTIAL MANN-WHITNEY U TEST FOR THE TWO SAMPLE LOCATION PROBLEM

A Research Project Report

Submitted to

Department of Statistics, Shivaji University, Kolhapur

As a Partial Fulfilment for the Degree of

Master of Science in Statistics

By

Mr. Shubham Madhukar Patil

Under the Guidance of

Prof. Dr. Shashibhushan B. Mahadik

Department of Statistics

Shivaji University, Kolhapur

April,2025

SEQUENTIAL MANN-WHITNEY U TEST FOR THE TWO SAMPLE LOCATION PROBLEM

Abstract

Statistical hypothesis testing is vital in data-driven areas, particularly when normality and equal variance assumptions are violated. The Mann-Whitney U Test is commonly used nonparametric test to compare two independent groups. Its fixed-sample nature, however, requires knowing in advance the sample size to be used, which is wasteful in practice where data is accumulated sequentially. To counter this, the Sequential Mann-Whitney U Test was created, allowing for early stopping and ongoing monitoring. This method minimizes sample size, enhances decision speed, and preserves error control, making it worthwhile in fields like clinical trials, industrial monitoring, and financial risk analysis. This work extends the Sequential Mann-Whitney U Test by specifying decision boundaries for early stopping and adapting the test statistic to sequential application. The boundaries are calibrated with simulation-based regression models so that correct significance levels are achieved. The test is assessed using simulations on different probability distributions Normal, Laplace, Lognormal, and Shifted Exponential for both symmetric and skewed data. The key performance measures, namely average sample number, power, and the Power-to-ASN Ratio (PAR), are utilized in comparing it to the conventional approach.

Results indicate that the Sequential Mann-Whitney U Test significantly minimizes sample size with or without a gain in power. The addition of PAR indicates its effectiveness in maximizing power with reduced samples. This technique is particularly beneficial in time-constrained decisions and in resource-constrained environments. Future research can generalize this technique to multivariate analysis, adaptive testing, and machine learning paradigms, making it even more useful in contemporary statistics.

1 Introduction

Making data-driven decisions across many fields, including finance and medicine, depends on statistical hypothesis testing. Often used to compare group variations are conventional parametric tests including ANOVA and the t-test. These tests, however, are predicated on some presumptions about the underlying population, such normality and variance homogeneity. Parametric tests may produce false results when these presumptions fall short, so producing erroneous conclusions. By reducing or not assuming population distribution, non-parametric tests provide a more flexible solution to solve this problem.

The following situations particularly call for these tests: Ordinal data include measurements that have some rank but do not fall on an interval or ratio scale. Any data distribution with a significant amount of skewness is termed a skewed distribution. Extreme values create the situation of outliers, which may influence parametric testing results. Small sample sizes signal situations in which it becomes quite difficult to predict if the Central Limit Theorem will hold. The Mann-Whitney U test or Wilcoxon rank-sum test is one of the most popular nonparametric procedures, which, not unlike the latter, is best applied to two independent groups not presumed to follow a normal distribution. It checks whether the values from one group are likely to be all larger or all smaller than from the other. In form of the hypothesis is H_0: $F_X(x)=F_Y(x)$ V/s H_1: $F_X(x)\neq F_Y(x)$. Although the Mann-Whitney U Test is flexible, it is very often implemented as a fixed sample test. This means that the sample size has to be fixed by the researcher before any actual testing starts. In many real-world situations, however, it is not possible to gather a pre-defined sample size before concluding. This need for sequential testing arises, a method whereby data may be continuously reviewed as it is collected.

The conventional, and fundamentally correct, way of thinking about hypothesis testing would entail the gathering of all data and then proceeding to perform a statistical analysis. Nevertheless, several serious limitations present themselves. The process of gathering large sample sizes can become expensive, time-consuming, or ethically challenging (for example, in clinical trials) in other experimental settings. A delay in decision-making: In cases where a timely decision should be made, waiting for a complete dataset could cause undue delay. A study having to specify sample size beforehand may lead to either underpowered studies resulting in inconclusive results or overpowered studies resulting in purposeless data collection.

Sequential testing allows for the intermediate evaluation of evidence as data is obtained, so judgments can be made at various points instead of waiting until a pre-specified sample size. Sequential testing decides whether to accept the null hypothesis (H_0) and terminate early, reject it if a significant difference is found, or keep sampling if the evidence is still inconclusive. Sequential analysis is highly effective in areas of rapid decision-making, including clinical trials, quality control, and financial analysis. The present research proposes a Sequential Mann-Whitney U Test, aimed at enhancing efficiency through sample size reduction while maintaining error control. Unlike the fixed-sample approach, the sequential version continuously evaluates data, enabling the test to terminate prematurely with statistical certainty. The adaptive strategy reduces data collection costs, decision lag, and unnecessary observations, leading to more efficient hypothesis testing in dynamic environments. The underlying objective of this research is to derive and verify a sequential testing method founded on the Mann-Whitney U statistic, ensuring that the proposed method is precise while improving sample efficiency. The research will utilize simulation to analyse performance measures like Type I and Type II error rates, Average Sample Number (ASN), power, and Power-to-ASN ratio (PAR).The hypothesis is; H_0:$F_X(x)$=$F_Y(x)$ V/s H_1: $F_X(x)$≠$F_Y(x)$.The distributions of two separate populations are denoted as $F_X(x)$ and $F_Y(x)$, respectively. The results will indicate how sequential testing accelerates decision-making without sacrificing strenuous statistical properties.

The literature review contains key developments and advancements related to the Mann-Whitney U test, including its theoretical foundation, statistical properties, and sequential adaptations. Mann-Whitney U test, as invented by Mann and Whitney (1947), is a non-parametric version of the t-test used to compare two independent samples. The test relies on ordering observations and identifying whether one sample consistently has higher values than another. Mann and Whitney's test applies rank sums to estimate the probability of one observation from one distribution being higher than another (*On a test of whether one of two random variables is stochastically larger than the other*, Mann & Whitney, 1947, p. 50)

Gibbons & Chakraborti (2010) describe the normalization of the U-statistic, which allows for its approximation by a normal distribution in large-sample scenarios. This transformation enables researchers to use standard normal critical values for hypothesis testing, improving the test's applicability across various disciplines (*Nonparametric Statistical Inference*, Gibbons & Chakraborti, 2010, p.261) Similarly, Rohatgi & Saleh (2001) provide a detailed mathematical foundation for the Mann-Whitney U test, including its expectation and variance. They define the test statistic as a summation of indicator functions and establish its statistical properties, making it a fundamental tool in non-parametric inference (*An Introduction to Probability and Statistics*, Rohatgi & Saleh, 2001, p. 629).

Bandyopadhyay and Biswas, in their 1999 publication entitled "*Sequential-type nonparametric test using Mann-Whitney statistics"*, proposed a sequential approach to nonparametric testing using the Mann-Whitney U statistic. The approach enables ongoing data analysis, the choice having been taken before that of fixed-sample tests. Nowak, Mütze, and Konietschke presented interim analyses in their 2022 paper, "*Group Sequential Methods for the Mann-Whitney Parameter"*, to enhance clinical trials. They proved that tests like Wilcoxon-Mann-Whitney and Brunner-Munzel, together with an adjusted win odds approach, asymptotically possess the joint canonical distribution.

In *"Introduction to Nonparametric Statistics for the Life Sciences using R",* MacFarland and Yates describe the Mann-Whitney U test as a nonparametric equivalent to the t-test for independent samples using R. They highlight its utility when parametric assumptions are violated, particularly for ranked data, skewed data, and unequal sample sizes. The authors find room to add an R-based application and helpful suggestions for biologists.

The remainder of the paper is organized as follows. Section 2 describes Mann-Whitney U Test. Section 3 introduces the proposed Sequential Mann-Whitney U Test. Section 4 evaluates the performance of the Sequential Mann-Whitney U Test in comparison with the Mann-Whitney U Test. Finally, Section 5 presents the conclusions and the potential future direction.

2 Mann-Whitney U Test

Rohatgi & Saleh (2001) provide a rigorous mathematical foundation for the Mann-Whitney U test, defining the test statistic as a summation of indicator functions and establishing its expectation and variance. This formulation is central to nonparametric inference, offering a distribution-free alternative to parametric tests (*Rohatgi & Saleh, 2001, p. 629*). Similarly, Gibbons & Chakraborti (2010) describe the normalization of the U-statistic, which enables its approximation by a normal distribution in large-sample settings. This transformation facilitates hypothesis testing by allowing researchers to use standard normal critical values, thereby broadening the applicability of the Mann-Whitney U test across various disciplines (*Gibbons & Chakraborti, 2010, p. 261*).

Let $X_1, X_2, X_3, X_4 \dots\dots\dots, X_m$ and $Y_1, Y_2, Y_3, Y_4 \dots\dots\dots, Y_n$ be independent samples from two continuous distribution functions, $F_X(x)$ and $F_Y(x)$, respectively.

Hypothesis: H_0: $F_X(x)=F_Y(x)$ V/s H_1: $F_X(x)\neq F_Y(x)$

Let,

$$T(X_i\ ; Y_j) = \begin{cases} 1, & if\ X_i < Y_j \\ 0, & if\ X_i \geq Y_j \end{cases} \qquad Where\ \begin{matrix} i=1,2,3,\dots\dots m \\ j=1,2,3,\dots\dots n \end{matrix}$$

$T(X_i ; Y_j)$) is an unbiased estimator of $g(F_X(x), F_Y(x)) = PF,G(X < Y)$ and the two-sample U statistic for g is given by $U_1(X;Y) = (m,n)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} T(X_i\ ; Y_j)$ For notational convenience, let us write

$$U = \sum_{i=1}^{m} \sum_{j=1}^{n} T(X_i\ ; Y_j)$$

Then U is the number of values of $X_1, X_2, X_3, X_4 \dots\dots\dots, X_m$ that are smaller than each of $Y_1, Y_2, Y_3, Y_4 \dots\dots\dots, Y_n$ .The statistic U is called the Mann-Whitney statistic.

If m and n are large, we can use the asymptotic normality of U.
under H_0,

$$E(u|H_0) = \frac{mn}{2} \qquad V(u|H_0) = \frac{mn(N+1)}{12}$$

Then large sample test statistic is

$$Z = \frac{U-\left(\frac{mn}{2}\right)}{\sqrt{\frac{mn(m+n+1)}{12}}}$$

Reject H_0 if $Z > Z_\alpha$ Otherwise accept H_0

whose distribution is approximately standard normal. This approximation has been found reasonably accurate for equal sample sizes as small as 6.

Procedure for performing Mann-Whitney U Test:

1. Take observations $X_1, X_2, X_3, X_4 \dots\dots\dots, X_m$ and $Y_1, Y_2, Y_3, Y_4 \dots\dots\dots, Y_n$
2. After taking values compute test statistic $U = \sum_{i=1}^{m} \sum_{j=1}^{n} T(X_i\ ; Y_j)$
3. If the sample size is small then compare test statistics with Mann Whitney table
4. If the sample size is large then compute $Z = \frac{U-\left(\frac{mn}{2}\right)}{\sqrt{\frac{mn(m+n+1)}{12}}}$
5. Compare the value of Z in a normal table and interpret the decision.

3 Proposed Sequential Mann-Whitney U Test

Development of a Sequential Mann-Whitney U test is essential in cases where data keeps on arriving and decisions are required to be made in real-time. Sequential testing, as opposed to conventional fixed-sample testing, permits early termination when sufficient statistical evidence is obtained, and hence it is best utilized in dynamic and high-stakes settings. Clinical trials of drugs are a case in point, where new medicines are being tested against control drugs. A fixed-sample Mann-Whitney U test would demand all the patient data to be gathered before concluding, and this can be time-consuming and might even hold up approval for effective drugs. A sequential approach, however, can permit early termination if an important effect is detected, reducing patient exposure to failed drugs and accelerating approval. Another major application is bank fraud detection, where banks constantly examine patterns of transactions to identify fraudulent patterns. A Mann-Whitney U test can identify systematic differences in transaction behaviours between honest users and suspected fraudsters, but a sequential version enables real-time detection of fraud. If the patterns of fraud are detected early, immediate intervention can be taken to prevent financial losses. In much the same way, in medical diagnosis, automated cancer screening machines analyse biopsy samples to assess whether they are benign or malignant. Rather than having to wait for an entire dataset, a sequential Mann-Whitney U test can continuously analyse diagnostic signals so that early intervention is made in cancer cases and unnecessary surgeries in benign diseases are not performed. These instances exemplify the importance of modifying the Mann-Whitney U test to a sequential framework for enhanced efficiency, reduced costs, better decision-making in various fields, and increased real-life data testing.

The development of the Sequential Mann-Whitney U Test is divided into three parts; that are assumptions, formation of test statistics, and formation of boundaries for decision

a) Assumption

1. The two groups being compared must be independent.
2. Observations within each group should also be independent.
3. The sample size from the two groups should be the same.
4. The observation should be sequentially generated from process.

b) Formation of test statistics

Let $X_1, X_2, X_3, X_4 \ldots\ldots\ldots, X_n$ and $Y_1, Y_2, Y_3, Y_4 \ldots\ldots\ldots, Y_n$ be independent samples from two continuous distribution functions generating in sequentially.

Hypothesis: $H_0: F_X(x)=F_Y(x)$ V/s $H_1: F_X(x)\neq F_Y(x)$

$$T(X_i\ ; Y_j) = \begin{cases} 1, & if\ X_i < Y_j \\ 0, & if\ X_i \geq Y_j \end{cases} \qquad Where\ \ i=1,2,3,\ldots..n \quad j=1,2,3,\ldots..n$$

Then

$$U = \sum_{i=1}^{n} \sum_{j=1}^{n} T(X_i\ ; Y_j)$$

For sequential process the observations are coming sequentially in pair i.e.

(x_1, y_1) , $(x_2, y_2), \ldots \ldots \ldots \ldots \ldots . . (x_n, y_n)$

We computed U statistics sequentially,

For (x_1, y_1)

$$U_1 = T_{11}$$

For (x_2, y_2)

$$U_2 = T_{11} + T_{12} + T_{21} + T_{22}$$

i.e. $U_2 = U_1 + T_{12} + T_{21} + T_{22}$

For (x_3, y_3)

$$U_3 = T_{11} + T_{12} + T_{13} + T_{21} + T_{22} + T_{23} + T_{31} + T_{32} + T_{33}$$

i.e. $U_3 = U_2 + T_{13} + T_{23} + T_{31} + T_{32} + T_{33}$

.

.

.

For (x_j, y_j)

$$U_j = U_{j-1} + \sum_{i=1}^{j} T_{ji} + \sum_{i=1}^{j-1} T_{ij}$$ ….. Sequential form of test statistics (1)

If j is large, we can use the asymptotic normality of U.

under Ho,

$$E(u|H_0) = \frac{j^2}{2} \qquad V(u|H_0) = \frac{j^2(2j+1)}{12}$$

The large sample test statistic is

$$Z_j = \frac{U_j - \frac{j^2}{2}}{\sqrt{\frac{j^2(2j+1)}{12}}}$$

For j-1; $$z_{j-1} = \frac{U_{j-1} - \frac{(j-1)^2}{2}}{\sqrt{\frac{(j-1)^2(2j-1)}{12}}}$$

Now we solve Z_j such that the Z_j can be computed in recursive using Z_{j-1}.

$$Z_j = \frac{U_j - \frac{j^2}{2}}{\sqrt{\frac{j^2(2j+1)}{12}}}$$

$$Z_j = \frac{U_{j-1} + \sum_{i=1}^{j} T_{ji} + \sum_{i=1}^{j-1} T_{ij} - \frac{j^2}{2}}{\sqrt{\frac{j^2(2j+1)}{12}}} \qquad \ldots\ldots\ldots.\ From\ equation(1)$$

$$Z_j = \frac{\sqrt{\frac{(j-1)^2(2j-1)}{12}} \times \frac{U_{j-1} - \frac{(j-1)^2}{2}}{\sqrt{\frac{(j-1)^2(2j-1)}{12}}} + \sum_{i=1}^{j} T_{ji} + \sum_{i=1}^{j-1} T_{ij} - \frac{j^2}{2} + \frac{(j-1)^2}{2}}{\sqrt{\frac{j^2(2j+1)}{12}}}$$

$$Z_j = \frac{z_{j-1}\sqrt{\frac{(j-1)^2(2j-1)}{12}} + \sum_{i=1}^{j} T_{ji} + \sum_{i=1}^{j-1} T_{ij} - \frac{j^2}{2} + \frac{(j-1)^2}{2}}{\sqrt{\frac{j^2(2j+1)}{12}}}$$

$$Z_j = \frac{z_{j-1}\sqrt{\frac{(j-1)^2(2j-1)^2}{12}} + \sum_{i=1}^{j} T_{ji} + \sum_{i=1}^{j-1} T_{ij} - j + \frac{1}{2}}{\sqrt{\frac{j^2(2j+1)^2}{12}}} \qquad ; j = 2,3,4, \ldots\ldots equation(2)$$

For j=1; $$Z_1 = \begin{cases} 1, & x_1 < y_1 \\ -1, & otherwise \end{cases}$$ … …. Equation (3)

The above Z_j is test statistics for Sequential Mann Whitney U Statistics.

Procedure for performing Sequential Mann-Whitney U Test:

1. Check all the assumptions of the Sequential Mann-Whitney U Test
2. Determine $\sum_{i=1}^{j} T_{ji}$ and $\sum_{i=1}^{j-1} T_{ij}$ where $\sum_{i=1}^{j} T_{ji}$ is summation of $X_j < Y_i$ and $\sum_{i=1}^{j-1} T_{ij}$ is summation of $X_i < Y_j$
3. Compute the test statistic Z_j as defined in Section 3.
4. If $|Z_j| \le a$, stop sampling and accept H_0, if $|Z_j| \ge b$, stop sampling and reject H0, and if $a < |Z_j| < b$, take the next sequential observation and repeat the procedure from step 1.

Illustration of above Sequential Mann Whitney U Test procedure of is given below.
Suppose a company wants to determine whether two different service centers provide the same level of customer satisfaction. Customers from each center are surveyed and asked to rate their satisfaction on a scale of 1 to 100. The given continue generating data X represents satisfaction level of service center A and Y represents service center B.

X	34.47	15.6	21.69	24.65	28.24	41.41	17.99	30.57	33.7	11.86	34.3	16.82	12.6
Y	46.5	32.5	40.8	48.87	27.39	58.78	51.01	57.58	55.8	43.92	56.87	23.54	27.84

From above observation table we form next table for just clarification of computation

X	34.47	15.6	21.69	24.65	28.24	41.41	17.99	30.57	33.7	11.86	34.3	16.82	12.6
Y	46.5	32.5	40.8	48.87	27.39	58.78	51.01	57.58	55.8	43.92	56.87	23.54	27.84
$\sum_{i=1}^{j} T_{ji}$	1	2	3	4	4	3	7	7	-	-	-	-	-
$\sum_{i=1}^{j-1} T_{ij}$	0	0	2	3	3	5	6	7	-	-	-	-	-

Step 1: Check all the assumptions of the Sequential Mann-Whitney U Test

Here X and Y values are independent. The values within X and Y are also independent and continuous. A sample size of both the X and Y group is same.

Step 2: Find the boundaries for making decision.

For testing we use specified ASN=10 and required level of significance is 5%. Hence, we put ASN=10 and Alpha=0.05 in equation (4) and equation (5) gives boundaries for testing. After putting these values, we got a=0.477819 and b=2.42164

Step 3: Compute Z_j sequentially and make decision using boundaries.

In this sequential generating data, we take $(X_1, Y_1) = (34.47, 46.5)$ here $X_1 < Y_1$ then Z_1 is always 1.00 according to Z_1 from equation (3) hence $|Z_1|=1.00$ which lies between a and b values hence the decision is continue sampling. Take next observation (X_2, Y_2) calculate $\sum_{i=1}^{2} T_{2i} = 2$ and $\sum_{i=1}^{2-1} T_{i2} = 0$ using these values compute Z_2. For computing Z_2 use formula of Z_j using Z_1 value. The calculated $|Z_2|=0.7745$ so decision is continue sampling.

Step 4: Compute Step 3 repeatedly until the decision of acceptance or rejectance is occur.

Here by sequentially computing at 8th observation $|Z_8|$ =2.6255 > b and decision is reject H_0 and hence process terminate.

In such a way we can perform the Sequential Mann-Whitney U test.

c) Formation of boundaries for decision of sequential test statistics

The exploratory development of the sequential Mann-Whitney U test in the first instance of using very large simulations for boundary values (a and b). The a values were kept ranging between 0.20-0.95 (20 values), while the b values ranged from 1.40 to 3.20 (20 values). Every one of these "a values" was paired with all the different "b values", resulting in a total of 400 combinations of boundaries. To compute the robustness of this boundary, 100,000 simulations were run for each pairing using random values generated from the standard normal distribution Normal (0,1). Once the random numbers were generated, the test statistic Z_j was calculated one by one and was compared to the corresponding border values. The ASN (Average Sample Number) is the number of observations needed to reach a decision of acceptance or rejection. The Type I error rate (α) was approximated by finding the proportion of simulations in which the null hypothesis was rejected. This method allowed for a comprehensive examination of boundary settings, optimizing the balance between sampling efficiency and statistical precision. We explored a simulated data set with ASN values between 2 and 35 and Alpha values between 0.0009 and 0.25 to explore the relationship between ASN, Alpha, and boundary values (a and b, respectively). We employed this data set to build two regression models for *a* and *b,* with ASN and Alpha as predictor variables. Applying numerous transformations enhanced model fit, yielding extremely high results R^2 of 98% for a and 99% for b.

$$a \approx 1/(0.7938 + 0.0330 * ASN - 6.8485 * Alpha + 2.6224ASN * Alpha \quad \dots.Equation(4)$$

$$b \approx 1.0355 + 0.1297 * ln(ASN) - 0.3629\, ln(Alpha) \quad \dots.Equation(5)$$

These models provide accurate estimation of boundary values for a specified ASN and Alpha, leading to an organized method for enhancing the sequential Mann-Whitney U test. With these models, we establish data-driven strategy to dynamically set boundaries, ensuring effective decision-making along sequential testing. In actual practice to achieve a desired ASN (2 to 30) and Alpha (0.0009 to 0.25), use Equations (3) and (4) to find boundary values a and b, respectively. The equations' precision was tested by checking the calculated ASN and Alpha with the given values, which proved close in accordance. The test suggests that the given models can estimate boundary values with high precision in practical conditions. Below is a comparison between the specified and actual ASN and Alpha values as measures of reliability in achieving goal test performance.

Table 1. Values of the stopping bound a and b of the Sequential Mann-Whitney U Test determined using equations 1 and 2 along with the specified and actual values of ASN_0 and α

a	b	ASN (specified)	ASN (Actual)	Alpha (specified)	Alpha (Actual)
0.978868	2.915949	05.00	04.46	0.01	0.01
0.786076	2.331728	05.00	04.96	0.05	0.04
0.630783	2.080118	05.00	05.34	0.10	0.10
0.526725	1.932935	05.00	04.83	0.15	0.18
0.758842	3.005861	10.00	08.68	0.01	0.01
0.477819	2.421640	10.00	11.01	0.05	0.05
0.326621	2.170030	10.00	09.65	0.10	0.10
0.248111	2.022848	10.00	10.40	0.15	0.13
0.619577	3.058456	15.00	13.49	0.01	0.01
0.343225	2.474235	15.00	15.01	0.05	0.05
0.220363	2.222625	15.00	15.08	0.10	0.10
0.162275	2.075443	15.00	16.87	0.15	0.14
0.523502	3.095773	20.00	16.98	0.01	0.01
0.267792	2.511552	20.00	20.63	0.05	0.05
0.166271	2.259942	20.00	22.25	0.10	0.11
0.120564	2.112760	20.00	21.84	0.15	0.14
0.453223	3.124718	25.00	20.59	0.01	0.01
0.219542	2.540497	25.00	24.00	0.05	0.05
0.133501	2.288887	25.00	26.72	0.10	0.11
0.095912	2.141705	25.00	28.24	0.15	0.15

4 Performance evaluation

A simulation study was conducted to compare the Sequential Mann-Whitney U test with the fixed-sample Mann-Whitney U test in terms of key performance indicators like Average Sample Number (ASN), Power, and Power-to-ASN Ratio (PAR). The hypothesis setup was as follows: $H_0 : \theta_1 = \theta_2$ V/s $H_1 : \theta_1 = \theta_2 + \delta\sigma$, where δ reflects a shift in location under H_1. The simulations assessed how the sequential test performs at various sample sizes and significance levels while remaining consistent with the fixed-sample strategy. The Average Sample Number (ASN), which is the estimated number of observations required to make a decision, is a key measure in this assessment. Lower ASN values indicate greater efficiency since fewer samples are needed before stopping the test. Power $(\beta_\delta)$is the probability of correctly rejecting H_0 when H_1 is true. This ensures the test can successfully detect real differences. To measure power versus sample efficiency quantitatively, The concept of PAR has been adopted from the work of Deore and Mahadik (2025), who proposed a efficiency measure in their study on sequential signed-rank tests (*Deore & Mahadik, 2025*)

$$PAR_\delta = \frac{\beta_\delta}{ASN_\delta} \times 100$$

A greater PAR value means that the test possesses high statistical power with fewer samples, making it a useful metric for evaluating overall success of the sequential test. To assess the performance, ASN_0 values of 10, 20, and 30 were selected to cover small and large sample sizes, and significance levels at 0.01 and 0.05. Regression models were employed to find optimized decision boundaries (a, b) for every combination of ASN and α using the past simulations. To maintain a consistent level of significance with the sequential test, the fixed-sample Mann-Whitney U test was applied with the same ASN values as under H_0. Critical values were obtained from the normal table. This ensured that the ASN and α were the same for both tests under H_0 conditions so that they could be compared directly. 100,000 simulations for each scenario, across four other probability distributions, were conducted to examine the performance of tests on varying data structures. Normal, Laplace, lognormal, and shifting exponential distributions were normalized with one variance to be comparable. The following probability density functions and parameter values under H_0 were given:

Table 2. Probability density functions and parameter values under H_0 for the Normal, Laplace, Lognormal, and Shifted Exponential distributions

Distribution	Probability Density Function		Parameter values under H_0	
Normal	$\frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-1}{2}\left(\frac{x-\theta}{\sigma}\right)^2}$	$, \ -\infty < x < \infty$	$\theta = 2\,,$	$\sigma = 1$
Laplace	$\frac{1}{2\lambda} e^{\frac{-1}{\lambda}\lvert x-\theta\rvert}$	$, \ -\infty < x < \infty$	$\theta = 2\,,$	$\sigma = 1$
Lognormal	$\frac{1}{x\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{\ln(x)-\theta}{\sigma}\right)^2}$	$, \ -\infty < x < \infty$	$\theta = 2\,,$	$\sigma = 1$
Shifted Exponential	$\frac{1}{\beta} e^{\frac{(x-\mu)}{\beta}}$	$, x \geq \mu \ and \ \beta > 0$	$\beta = 1\,,$	$\mu = 2$

These distributions were chosen to represent both symmetric (normal, Laplace) and non-symmetric (lognormal, shifted exponential) cases, allowing us to examine how skewness and symmetry impact the test's performance. The performance comparisons were conducted for six different combinations of (α, ASN_0), specifically (0.01,10), (0.05,10), (0.01,20), (0.05,20), (0.01,30), and (0.05,30). These combinations were chosen to evaluate the test's performance across varying significance levels and sample sizes. The corresponding H_1 conditions and design parameters for each case are detailed in Table 3.

Table 3. H_1 and the design parameters of the sequential and fixed tests.

Test	H_1	Design Parameters		
		ASN_0≈10	ASN_0≈20	ASN_0≈30
			α =0.01	
Sequential	$\theta_1 \neq \theta_2$	a=0.7108 b=2.9771	a=0.4565 b=3.0609	a=0.3117 b=3.1115
Fixed	$\theta_1 \neq \theta_2$	N=10 $Z_{\alpha N}$ =2.492	N=10 $Z_{\alpha N}$ =2.491	N=10 $Z_{\alpha N}$ = 2.492
			α =0.05	
Sequential	$\theta_1 \neq \theta_2$	a=0.5384 b=2.4520	a=0.2677 b=2.5115	a=0.1925 b=2.6002
Fixed	$\theta_1 \neq \theta_2$	N=10 $Z_{\alpha N}$ = 1.964	N=10 $Z_{\alpha N}$ = 1.962	N=10 $Z_{\alpha N}$ = 1.964

Table 4,5,6,7 represents simulation outputs of the distributions normal, laplace, lognormal and shifted exponential. Figures 1 to 8 show the ASN_δ, β_δ and PAR_δ graphs with delta shifts for all given distributions. ASN values were considered to define significance thresholds ($\alpha = 0.01, 0.05$) and target ASN_0 values (10, 20, 30). The sequential test produced significant reduction in ASN for all conditions compared to the fixed-sample test. The benefit in efficiency increased for larger shift values, illustrating that the sequential method enables decision-making at higher speed without affecting precision. Across the Normal, Laplace, Lognormal, and Shifted Exponential distributions, the sequential test always required fewer observations to make a conclude. Column "ASN_δ" illustrates extensive ASN_δ comparisons, illustrating how the sequential approach effectively minimizes sample requirements while maintaining statistical reliability.

Power (β) was evaluated with the alternative hypothesis for various shift values (δ). The power levels of the sequential test were similar to or greater than the fixed-sample test, particularly for $\delta > 0.5$. Even for small shifts, the sequential test had consistent power over a variety of sample sizes, showing its robustness in hypothesis testing. Column "β_δ" gives power comparisons for different shift situations. The results show that the sequential test can maintain statistical power while significantly reducing the required sample size. The Power-to-ASN Ratio (PAR) was calculated to determine the trade-off between accuracy and efficiency. PAR is a primary measure to gauge how efficiently the test can achieve statistical significance using minimal sample utilization. The sequential test consistently reported higher PAR_δ values compared to the fixed-sample test, proving its higher efficiency.

The PAR_δ improvement was seen most prominently for all distributions, as the sequential method gained an advantage in handling skewed and symmetric data structures. The PAR_δ comparisons are seen as column "PAR_δ", reinforcing the efficiency of the sequential test in maximizing statistical power while minimizing sample needs. The robustness of the sequential test was checked with the Normal, Laplace, Lognormal, and Shifted Exponential distributions. The sequential test exhibits significant ASN reduction at the expense of maintaining power for non-symmetric distributions (Lognormal, Shifted Exponential), and thus it is the better solution for skewed data. All distribution figures illustrate the PAR_δ trade-off graphically, showing the sequential test's performance under various distributions and confirming its versatility in varying data environments. Simulation results indicate that the Sequential Mann-Whitney U Test surpasses the fixed-sample test by significantly minimizing ASN_δ while maintaining or increasing power across a range of situations. Utilizing PAR as a comparative metric further solidifies the test's superiority in trading statistical power with sample efficiency. These results established the Sequential Mann-Whitney U Test as a successful and adoptable procedure.

5 Conclusion

The Sequential Mann-Whitney U Test is a breakthrough in nonparametric statistical testing, providing a more efficient alternative to the conventional fixed-sample procedure. This research proved that the sequential approach successfully minimizes the necessary Average Sample Number while maintaining tight control of Type I and Type II error rates. By enabling continuous data assessment and a dynamic stopping rule, the test enhances efficiency in practical applications where early decision is essential.

Development of a sequential test statistic was a key element of this research. The Mann-Whitney U statistic was adapted to address a dynamic dataset instead of a fixed sample size. The decision-making process was guided by upper and lower decision boundaries (a, b) that were optimized using aggressive simulation and regression modelling. Such limits guarantee that the test has balanced sample efficiency against statistical accuracy to become flexible in most study environments. The boundary estimating method becomes increasingly flexible through provision for automatic adaptation based on selected ASN and levels of significance. Comprehensive simulations were used to validate the sequential test's performance at various significance levels, ASN targets, and effect sizes. The results show that the test performs well with a variety of distributional structures, including Normal, Laplace, Lognormal, and Shifted Exponential distributions. Notably, the sequential test retains power comparable to or greater than the fixed-sample technique, particularly for bigger effect sizes, highlighting its utility in practical applications.

The sequential test often generated larger PAR values, illustrating its ability to retain statistical power while lowering sample sizes. This measure provides evidence of the test's superiority over traditional methods. In addition, the new regression-based boundary estimation enhances the usability of the method by providing a structured framework for determining decision thresholds. The results have far-reaching consequences in fields where hypothesis testing is essential. Clinical research sees its sequential approach lessen patient recruiting hurdles without losing statistical power. In industrial quality assurance, it can detect process deviations more quickly and save huge sums of money. Financial risk assessment, offers instant checking of market trends, leading to quicker decision-making and risk control. While its advantages are many, the difficulties remain. The computational challenge of real-time updating of test statistics and decision boundaries deserves further examination. Further research might address generalizing this to multivariate settings or incorporating it with machine learning methods to achieve greater flexibility. In summary, the Sequential Mann-Whitney U Test closes the gap between classical nonparametric methods and today's data-oriented decision-making. This work paves the way for future breakthroughs in sequential hypothesis testing by reducing sample usage while ensuring statistical validity. Exploring this technique into higher-dimensional problems, adaptive testing paradigms, and practical applications will make it more relevant and useful in statistical practice.

6 References

Mann, H. B., & Whitney, D. R. (1947). On a test of whether one of two random variables is stochastically larger than the other. *The annals of mathematical statistics*, 50-60.

Rohatgi, V. K., & Saleh, A. K. M. E. (2001). *An introduction to probability and statistics* (2nd ed., p. 629). Wiley.

Gibbons, J. D., & Chakraborti, S. (2010). *Nonparametric statistical inference* (5th ed., p. 261). CRC Press.

Bandyopadhyay, U., & Biswas, A. (1999). Sequential-type nonparametric test using Mann-Whitney statistics. *Journal of Applied Statistics*, *26*(3), 301-308.

Nowak, C. P., Mütze, T., & Konietschke, F. (2022). Group sequential methods for the Mann-Whitney parameter. *Statistical Methods in Medical Research*, *31*(10), 2004-2020.

MacFarland, T. W., Yates, J. M., MacFarland, T. W., & Yates, J. M. (2016). Nonparametric statistics for the biological sciences. *Introduction to nonparametric statistics for the biological sciences using R*, 1-50.

Deore, Ravindra E., Mahadik, S. B., (2025). A New Sequential Signed-Rank Test for One-Sample Location Problem.

Table 4. The β_δ, ASN_δ, and PAR_δ of the sequential and fixed Mann Whitney U test for normal distribution.

	$ASN_0 \approx 10$ α=0.01					
	Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.01	9.59	0.11	0.01	10.00	0.12
0.25	0.07	13.27	0.55	0.02	10.00	0.23
0.50	0.20	12.07	1.65	0.06	10.00	0.64
0.75	0.35	11.09	3.16	0.15	10.00	1.52
1.00	0.50	10.32	4.82	0.30	10.00	2.97
1.25	0.63	9.66	6.50	0.48	10.00	4.84
1.50	0.73	9.08	8.09	0.67	10.00	6.73
1.75	0.82	8.60	9.54	0.83	10.00	8.26
2.00	0.88	8.19	10.78	0.92	10.00	9.23
2.25	0.93	7.86	11.79	0.97	10.00	9.72
2.50	0.96	7.60	12.59	0.99	10.00	9.91
2.75	0.98	7.40	13.19	1.00	10.00	9.98
3.00	0.99	7.25	13.61	1.00	10.00	10.00
3.25	0.99	7.15	13.88	1.00	10.00	10.00
3.50	1.00	7.08	14.06	1.00	10.00	10.00

	$ASN_0 \approx 10$ α=0.05					
	Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.05	9.71	0.50	0.05	10.00	0.52
0.25	0.12	10.52	1.17	0.08	10.00	0.84
0.50	0.27	9.76	2.77	0.18	10.00	1.84
0.75	0.43	8.84	4.84	0.35	10.00	3.49
1.00	0.58	8.09	7.15	0.55	10.00	5.48
1.25	0.70	7.44	9.36	0.74	10.00	7.36
1.50	0.79	6.86	11.52	0.87	10.00	8.74
1.75	0.86	6.43	13.41	0.95	10.00	9.50
2.00	0.91	6.08	15.03	0.98	10.00	9.85
2.25	0.95	5.79	16.34	1.00	10.00	9.96
2.50	0.97	5.58	17.37	1.00	10.00	9.99
2.75	0.98	5.40	18.18	1.00	10.00	10.00
3.00	0.99	5.27	18.77	1.00	10.00	10.00
3.25	0.99	5.18	19.20	1.00	10.00	10.00
3.50	1.00	5.12	19.49	1.00	10.00	10.00

	$ASN_0 \approx 20$ α=0.01					
	Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.01	20.14	0.06	0.01	20.00	0.06
0.25	0.12	25.58	0.47	0.04	20.00	0.18
0.50	0.29	19.00	1.50	0.15	20.00	0.73
0.75	0.44	15.24	2.92	0.38	20.00	1.88
1.00	0.59	13.15	4.50	0.66	20.00	3.32
1.25	0.70	11.54	6.10	0.88	20.00	4.39
1.50	0.80	10.46	7.63	0.97	20.00	4.85
1.75	0.87	9.61	9.02	1.00	20.00	4.98
2.00	0.91	8.93	10.24	1.00	20.00	5.00
2.25	0.95	8.42	11.23	1.00	20.00	5.00
2.50	0.97	8.02	12.07	1.00	20.00	5.00
2.75	0.98	7.72	12.71	1.00	20.00	5.00
3.00	0.99	7.50	13.21	1.00	20.00	5.00
3.25	0.99	7.34	13.56	1.00	20.00	5.00
3.50	1.00	7.22	13.81	1.00	20.00	5.00

	$ASN_0 \approx 20$ α=0.05					
	Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.05	20.73	0.25	0.05	20.00	0.24
0.25	0.17	18.91	0.92	0.12	20.00	0.59
0.50	0.34	13.56	2.49	0.34	20.00	1.68
0.75	0.49	10.95	4.51	0.63	20.00	3.17
1.00	0.62	9.22	6.76	0.87	20.00	4.33
1.25	0.73	8.06	9.08	0.97	20.00	4.85
1.50	0.81	7.25	11.21	1.00	20.00	4.98
1.75	0.87	6.63	13.15	1.00	20.00	5.00
2.00	0.92	6.18	14.83	1.00	20.00	5.00
2.25	0.95	5.86	16.17	1.00	20.00	5.00
2.50	0.97	5.61	17.28	1.00	20.00	5.00
2.75	0.98	5.41	18.11	1.00	20.00	5.00
3.00	0.99	5.28	18.73	1.00	20.00	5.00
3.25	0.99	5.19	19.16	1.00	20.00	5.00
3.50	1.00	5.12	19.47	1.00	20.00	5.00

	ASN_0≈30 α=0.01							ASN_0≈30 α=0.05					
	Sequential			Fixed				Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ	δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.01	29.36	0.03	0.01	29.00	0.04	0.00	0.05	31.86	0.16	0.05	32.00	0.15
0.25	0.15	33.92	0.43	0.05	29.00	0.18	0.25	0.21	27.42	0.77	0.16	32.00	0.49
0.50	0.32	21.98	1.44	0.23	29.00	0.79	0.50	0.40	18.09	2.22	0.48	32.00	1.50
0.75	0.47	16.84	2.81	0.56	29.00	1.92	0.75	0.56	13.63	4.11	0.82	32.00	2.56
1.00	0.61	13.91	4.36	0.85	29.00	2.92	1.00	0.69	10.88	6.30	0.97	32.00	3.03
1.25	0.72	11.97	5.97	0.97	29.00	3.35	1.25	0.78	9.17	8.48	1.00	32.00	3.12
1.50	0.80	10.68	7.52	1.00	29.00	3.44	1.50	0.84	7.97	10.59	1.00	32.00	3.12
1.75	0.87	9.71	8.92	1.00	29.00	3.45	1.75	0.89	7.11	12.57	1.00	32.00	3.13
2.00	0.91	9.00	10.15	1.00	29.00	3.45	2.00	0.93	6.51	14.26	1.00	32.00	3.13
2.25	0.95	8.46	11.18	1.00	29.00	3.45	2.25	0.96	6.08	15.70	1.00	32.00	3.13
2.50	0.97	8.04	12.04	1.00	29.00	3.45	2.50	0.97	5.76	16.88	1.00	32.00	3.13
2.75	0.98	7.74	12.69	1.00	29.00	3.45	2.75	0.98	5.53	17.79	1.00	32.00	3.13
3.00	0.99	7.51	13.18	1.00	29.00	3.45	3.00	0.99	5.36	18.48	1.00	32.00	3.13
3.25	0.99	7.33	13.56	1.00	29.00	3.45	3.25	0.99	5.24	18.98	1.00	32.00	3.13
3.50	1.00	7.22	13.81	1.00	29.00	3.45	3.50	1.00	5.15	19.37	1.00	32.00	3.13

Table 5. The β_δ, ASN_δ, and PAR_δ of the sequential and fixed Mann Whitney U test for Laplace distribution.

	ASN_0≈10 α=0.01							ASN_0≈10 α=0.05					
	Sequential			Fixed				Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ	δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.01	9.61	0.11	0.01	10.00	0.12	0.00	0.05	9.64	0.51	0.05	10.00	0.53
0.25	0.06	13.30	0.47	0.02	10.00	0.21	0.25	0.11	10.56	1.02	0.08	10.00	0.78
0.50	0.16	11.73	1.35	0.05	10.00	0.53	0.50	0.22	9.54	2.33	0.15	10.00	1.55
0.75	0.28	10.98	2.52	0.11	10.00	1.12	0.75	0.35	8.74	4.03	0.27	10.00	2.70
1.00	0.39	10.22	3.81	0.20	10.00	2.03	1.00	0.47	8.06	5.79	0.42	10.00	4.19
1.25	0.50	9.65	5.13	0.32	10.00	3.23	1.25	0.57	7.55	7.59	0.57	10.00	5.67
1.50	0.59	9.21	6.38	0.45	10.00	4.53	1.50	0.66	7.12	9.25	0.70	10.00	7.01
1.75	0.66	8.79	7.54	0.58	10.00	5.82	1.75	0.73	6.74	10.80	0.81	10.00	8.08
2.00	0.73	8.53	8.58	0.70	10.00	6.98	2.00	0.79	6.48	12.15	0.88	10.00	8.84
2.25	0.78	8.24	9.51	0.79	10.00	7.95	2.25	0.83	6.22	13.37	0.93	10.00	9.32
2.50	0.83	8.05	10.30	0.86	10.00	8.62	2.50	0.87	6.02	14.43	0.96	10.00	9.63
2.75	0.86	7.85	11.00	0.91	10.00	9.12	2.75	0.90	5.85	15.35	0.98	10.00	9.82
3.00	0.89	7.71	11.58	0.95	10.00	9.46	3.00	0.92	5.70	16.14	0.99	10.00	9.90
3.25	0.91	7.56	12.10	0.97	10.00	9.67	3.25	0.94	5.58	16.78	1.00	10.00	10.00
3.50	0.94	7.49	12.50	0.98	10.00	9.81	3.50	0.95	5.48	17.33	1.00	10.00	10.00
3.75	0.96	7.31	13.12	0.99	10.00	9.94	3.75	0.97	5.34	18.18	1.00	10.00	10.00
4.00	0.97	7.25	13.37	1.00	10.00	9.96	4.00	0.98	5.27	18.52	1.00	10.00	10.00
4.25	0.98	7.21	13.54	1.00	10.00	9.98	4.25	0.98	5.23	18.78	1.00	10.00	10.00
4.50	0.98	7.16	13.70	1.00	10.00	10.00	4.50	0.99	5.19	19.00	1.00	10.00	10.00
4.75	0.99	7.13	13.82	1.00	10.00	10.00	4.75	0.99	5.16	19.17	1.00	10.00	10.00
5.00	0.99	7.10	13.92	1.00	10.00	10.00	5.00	0.99	5.13	19.34	1.00	10.00	10.00
5.25	1.00	7.08	14.07	1.00	10.00	10.00	5.25	1.00	5.10	19.53	1.00	10.00	10.00

	ASN_0≈20 α=0.01					
	Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.01	19.82	0.05	0.01	20.00	0.06
0.25	0.10	27.13	0.37	0.03	20.00	0.16
0.50	0.24	19.44	1.22	0.11	20.00	0.54
0.75	0.37	15.86	2.32	0.26	20.00	1.32
1.00	0.49	13.74	3.54	0.48	20.00	2.41
1.25	0.59	12.32	4.78	0.69	20.00	3.45
1.50	0.67	11.24	5.98	0.84	20.00	4.22
1.75	0.74	10.43	7.09	0.93	20.00	4.66
2.00	0.79	9.75	8.11	0.98	20.00	4.88
2.25	0.84	9.28	9.00	0.99	20.00	4.96
2.50	0.87	8.91	9.78	1.00	20.00	4.99
2.75	0.90	8.57	10.49	1.00	20.00	5.00
3.00	0.92	8.30	11.08	1.00	20.00	5.00
3.25	0.94	8.08	11.60	1.00	20.00	5.00
3.50	0.95	7.91	12.04	1.00	20.00	5.00
3.75	0.97	7.62	12.73	1.00	20.00	5.00
4.00	0.98	7.51	13.00	1.00	20.00	5.00
4.50	0.99	7.35	13.41	1.00	20.00	5.00
5.00	0.99	7.24	13.69	1.00	20.00	5.00
5.25	1.00	7.20	13.83	1.00	20.00	5.00

	ASN_0≈20 α=0.05					
	Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.05	20.64	0.26	0.05	20.00	0.25
0.25	0.15	19.32	0.79	0.11	20.00	0.53
0.50	0.29	14.14	2.06	0.26	20.00	1.32
0.75	0.42	11.36	3.71	0.50	20.00	2.51
1.00	0.53	9.71	5.44	0.73	20.00	3.64
1.25	0.62	8.57	7.23	0.88	20.00	4.40
1.50	0.69	7.80	8.90	0.96	20.00	4.78
1.75	0.75	7.18	10.49	0.99	20.00	4.94
2.00	0.80	6.76	11.89	1.00	20.00	4.98
2.25	0.84	6.42	13.11	1.00	20.00	5.00
2.50	0.88	6.16	14.21	1.00	20.00	5.00
2.75	0.90	5.94	15.17	1.00	20.00	5.00
3.00	0.92	5.78	15.95	1.00	20.00	5.00
3.25	0.94	5.65	16.63	1.00	20.00	5.00
3.50	0.95	5.53	17.23	1.00	20.00	5.00
3.75	0.97	5.35	18.12	1.00	20.00	5.00
4.00	0.98	5.29	18.45	1.00	20.00	5.00
4.50	0.99	5.20	18.96	1.00	20.00	5.00
5.00	0.99	5.13	19.30	1.00	20.00	5.00
5.25	0.99	5.11	19.44	1.00	20.00	5.00

	ASN_0≈30 α=0.01					
	Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.01	28.99	0.04	0.01	29.00	0.04
0.25	0.12	35.76	0.35	0.04	29.00	0.14
0.50	0.27	23.37	1.14	0.17	29.00	0.57
0.75	0.40	17.88	2.22	0.41	29.00	1.40
1.00	0.51	14.93	3.41	0.67	29.00	2.32
1.25	0.60	12.93	4.64	0.87	29.00	2.98
1.50	0.68	11.63	5.85	0.96	29.00	3.30
1.75	0.74	10.71	6.93	0.99	29.00	3.41
2.00	0.80	10.01	7.96	1.00	29.00	3.44
2.25	0.84	9.42	8.89	1.00	29.00	3.45
2.50	0.87	8.98	9.70	1.00	29.00	3.45
2.75	0.90	8.63	10.40	1.00	29.00	3.45
3.00	0.92	8.38	11.01	1.00	29.00	3.45
3.25	0.94	8.12	11.55	1.00	29.00	3.45
3.50	0.95	7.94	11.97	1.00	29.00	3.45
3.75	0.97	7.64	12.70	1.00	29.00	3.45
4.00	0.98	7.53	12.98	1.00	29.00	3.45
4.50	0.99	7.35	13.40	1.00	29.00	3.45
5.00	0.99	7.24	13.68	1.00	29.00	3.45
5.25	0.99	7.21	13.79	1.00	29.00	3.45

	ASN_0≈30 α=0.05					
	Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.05	31.82	0.16	0.05	32.00	0.15
0.25	0.19	29.01	0.64	0.14	32.00	0.42
0.50	0.34	19.02	1.81	0.38	32.00	1.19
0.75	0.48	14.44	3.34	0.69	32.00	2.15
1.00	0.59	11.81	5.00	0.89	32.00	2.79
1.25	0.68	10.10	6.71	0.97	32.00	3.04
1.50	0.74	8.93	8.31	1.00	32.00	3.11
1.75	0.79	8.04	9.87	1.00	32.00	3.12
2.00	0.84	7.42	11.28	1.00	32.00	3.12
2.25	0.87	6.91	12.55	1.00	32.00	3.12
2.50	0.89	6.51	13.72	1.00	32.00	3.13
2.75	0.91	6.23	14.68	1.00	32.00	3.13
3.00	0.93	6.00	15.49	1.00	32.00	3.13
3.25	0.95	5.83	16.23	1.00	32.00	3.13
3.50	0.96	5.67	16.89	1.00	32.00	3.13
3.75	0.97	5.45	17.83	1.00	32.00	3.13
4.00	0.98	5.36	18.22	1.00	32.00	3.13
4.50	0.99	5.25	18.80	1.00	32.00	3.13
5.00	0.99	5.17	19.20	1.00	32.00	3.13
5.25	1.00	5.14	19.38	1.00	32.00	3.13

Table 6. The β_δ, ASN_δ, and PAR_δ of the sequential and fixed Mann Whitney U test for lognormal distribution.

	ASN_0≈10 α=0.01							ASN_0≈10 α=0.05					
	Sequential			Fixed				Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ	δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.01	9.59	0.11	0.01	10.00	0.12	0.00	0.05	9.59	0.51	0.05	10.00	0.52
0.25	0.07	13.31	0.55	0.02	10.00	0.23	0.25	0.12	10.51	1.16	0.08	10.00	0.85
0.50	0.20	12.12	1.65	0.06	10.00	0.64	0.50	0.27	9.58	2.78	0.18	10.00	1.84
0.75	0.35	11.13	3.13	0.15	10.00	1.52	0.75	0.43	8.82	4.85	0.35	10.00	3.46
1.00	0.49	10.26	4.81	0.30	10.00	2.97	1.00	0.57	8.04	7.12	0.55	10.00	5.48
1.25	0.63	9.65	6.49	0.48	10.00	4.84	1.25	0.69	7.42	9.36	0.73	10.00	7.35
1.50	0.73	9.06	8.10	0.68	10.00	6.76	1.50	0.79	6.88	11.54	0.87	10.00	8.73
1.75	0.82	8.60	9.54	0.83	10.00	8.25	1.75	0.86	6.43	13.41	0.95	10.00	9.50
2.00	0.88	8.20	10.78	0.92	10.00	9.24	2.00	0.91	6.07	15.03	0.98	10.00	9.84
2.25	0.93	7.86	11.80	0.97	10.00	9.72	2.25	0.95	5.78	16.36	1.00	10.00	9.96
2.50	0.96	7.60	12.59	0.99	10.00	9.92	2.50	0.97	5.57	17.38	1.00	10.00	9.99
2.75	0.98	7.40	13.18	1.00	10.00	9.98	2.75	0.98	5.39	18.20	1.00	10.00	10.00
3.00	0.99	7.26	13.61	1.00	10.00	10.00	3.00	0.99	5.27	18.77	1.00	10.00	10.00
3.25	0.99	7.15	13.88	1.00	10.00	10.00	3.25	0.99	5.18	19.20	1.00	10.00	10.00
3.50	1.00	7.09	14.06	1.00	10.00	10.00	3.50	1.00	5.12	19.48	1.00	10.00	10.00

	ASN_0≈20 α=0.01							ASN_0≈20 α=0.05					
	Sequential			Fixed				Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ	δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.01	19.95	0.05	0.01	20.00	0.05	0.00	0.05	20.58	0.25	0.05	20.00	0.24
0.25	0.12	26.26	0.46	0.04	20.00	0.18	0.25	0.17	18.84	0.92	0.12	20.00	0.60
0.50	0.29	18.78	1.52	0.15	20.00	0.74	0.50	0.34	13.58	2.49	0.33	20.00	1.66
0.75	0.45	15.31	2.91	0.38	20.00	1.88	0.75	0.49	10.91	4.53	0.63	20.00	3.17
1.00	0.59	13.03	4.51	0.67	20.00	3.33	1.00	0.62	9.23	6.77	0.87	20.00	4.34
1.25	0.70	11.54	6.11	0.88	20.00	4.40	1.25	0.73	8.07	9.05	0.97	20.00	4.85
1.50	0.80	10.44	7.65	0.97	20.00	4.86	1.50	0.81	7.22	11.21	1.00	20.00	4.98
1.75	0.87	9.57	9.05	1.00	20.00	4.98	1.75	0.87	6.63	13.16	1.00	20.00	5.00
2.00	0.91	8.91	10.23	1.00	20.00	5.00	2.00	0.92	6.19	14.80	1.00	20.00	5.00
2.25	0.95	8.41	11.26	1.00	20.00	5.00	2.25	0.95	5.85	16.19	1.00	20.00	5.00
2.50	0.97	8.02	12.06	1.00	20.00	5.00	2.50	0.97	5.60	17.28	1.00	20.00	5.00
2.75	0.98	7.73	12.70	1.00	20.00	5.00	2.75	0.98	5.41	18.11	1.00	20.00	5.00
3.00	0.99	7.49	13.21	1.00	20.00	5.00	3.00	0.99	5.28	18.73	1.00	20.00	5.00
3.25	0.99	7.33	13.56	1.00	20.00	5.00	3.25	0.99	5.19	19.18	1.00	20.00	5.00
3.50	1.00	7.22	13.81	1.00	20.00	5.00	3.50	1.00	5.12	19.48	1.00	20.00	5.00

	ASN_0≈30 α=0.01							ASN_0≈30 α=0.05					
	Sequential			Fixed				Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ	δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.01	28.66	0.04	0.01	29.00	0.04	0.00	0.05	31.99	0.16	0.05	32.00	0.15
0.25	0.15	34.19	0.43	0.05	29.00	0.19	0.25	0.21	27.25	0.76	0.16	32.00	0.49
0.50	0.32	22.09	1.43	0.24	29.00	0.84	0.50	0.40	18.12	2.21	0.48	32.00	1.51
0.75	0.47	16.94	2.79	0.58	29.00	2.00	0.75	0.56	13.60	4.13	0.82	32.00	2.57
1.00	0.61	13.87	4.37	0.87	29.00	2.99	1.00	0.68	10.92	6.27	0.97	32.00	3.03
1.25	0.71	12.01	5.95	0.98	29.00	3.37	1.25	0.78	9.17	8.47	1.00	32.00	3.12
1.50	0.80	10.66	7.52	1.00	29.00	3.44	1.50	0.84	7.95	10.60	1.00	32.00	3.12
1.75	0.87	9.72	8.92	1.00	29.00	3.45	1.75	0.89	7.09	12.56	1.00	32.00	3.13
2.00	0.91	9.01	10.15	1.00	29.00	3.45	2.00	0.93	6.51	14.28	1.00	32.00	3.13
2.25	0.95	8.46	11.19	1.00	29.00	3.45	2.25	0.95	6.08	15.70	1.00	32.00	3.13
2.50	0.97	8.04	12.03	1.00	29.00	3.45	2.50	0.97	5.76	16.88	1.00	32.00	3.13
2.75	0.98	7.74	12.68	1.00	29.00	3.45	2.75	0.98	5.53	17.79	1.00	32.00	3.13
3.00	0.99	7.50	13.19	1.00	29.00	3.45	3.00	0.99	5.36	18.49	1.00	32.00	3.13
3.25	0.99	7.34	13.55	1.00	29.00	3.45	3.25	0.99	5.24	18.99	1.00	32.00	3.13
3.50	1.00	7.22	13.81	1.00	29.00	3.45	3.50	1.00	5.15	19.36	1.00	32.00	3.13

Table 7. The β_δ, ASN_δ, and PAR_δ of the sequential and fixed Mann Whitney U test for Shifted exponential distribution.

	ASN_0≈10 α=0.01							ASN_0≈10 α=0.05					
	Sequential			Fixed				Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ	δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.01	9.64	0.11	0.01	10.00	0.11	0.00	0.05	9.60	0.51	0.05	10.00	0.53
0.25	0.15	12.95	1.14	0.04	10.00	0.42	0.25	0.21	10.25	2.02	0.13	10.00	1.34
0.50	0.33	11.08	3.00	0.15	10.00	1.47	0.50	0.41	8.72	4.73	0.34	10.00	3.37
0.75	0.49	9.91	4.99	0.32	10.00	3.16	0.75	0.57	7.75	7.40	0.56	10.00	5.64
1.00	0.62	9.11	6.75	0.50	10.00	5.04	1.00	0.68	6.99	9.78	0.74	10.00	7.42
1.25	0.71	8.57	8.26	0.67	10.00	6.66	1.25	0.77	6.49	11.81	0.86	10.00	8.60
1.50	0.78	8.15	9.54	0.79	10.00	7.90	1.50	0.83	6.11	13.53	0.93	10.00	9.28
1.75	0.83	7.85	10.56	0.87	10.00	8.71	1.75	0.87	5.84	14.83	0.96	10.00	9.64
2.00	0.87	7.67	11.33	0.92	10.00	9.23	2.00	0.90	5.65	15.92	0.98	10.00	9.82
2.25	0.90	7.51	11.97	0.95	10.00	9.54	2.25	0.92	5.50	16.81	0.99	10.00	9.91
2.50	0.92	7.39	12.49	0.97	10.00	9.74	2.50	0.94	5.38	17.48	1.00	10.00	9.96
2.75	0.94	7.30	12.88	0.99	10.00	9.85	2.75	0.95	5.29	18.03	1.00	10.00	9.98
3.00	0.95	7.23	13.19	0.99	10.00	9.92	3.00	0.97	5.23	18.46	1.00	10.00	9.99
3.25	0.96	7.17	13.43	0.99	10.00	9.95	3.25	0.97	5.18	18.79	1.00	10.00	10.00
3.50	0.97	7.13	13.62	1.00	10.00	9.97	3.50	0.98	5.14	19.05	1.00	10.00	10.00
3.75	0.98	7.09	13.88	1.00	10.00	9.98	3.75	0.99	5.09	19.42	1.00	10.00	10.00
4.00	0.99	7.07	13.97	1.00	10.00	9.99	4.00	0.99	5.06	19.55	1.00	10.00	10.00
4.25	0.99	7.05	14.04	1.00	10.00	9.99	4.25	0.99	5.05	19.65	1.00	10.00	10.00
4.50	0.99	7.03	14.14	1.00	10.00	10.00	4.50	0.99	5.04	19.72	1.00	10.00	10.00
4.75	1.00	7.02	14.17	1.00	10.00	10.00	4.75	1.00	5.03	19.79	1.00	10.00	10.00
5.00	1.00	7.02	14.20	1.00	10.00	10.00	5.00	1.00	5.02	19.83	1.00	10.00	10.00

	ASN_0≈20 α=0.01					
	Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.01	19.49	0.06	0.01	20.00	0.05
0.25	0.22	21.26	1.02	0.09	20.00	0.44
0.50	0.43	15.22	2.82	0.36	20.00	1.79
0.75	0.58	12.51	4.65	0.68	20.00	3.39
1.00	0.69	10.89	6.37	0.88	20.00	4.39
1.25	0.77	9.86	7.84	0.96	20.00	4.81
1.50	0.83	9.16	9.06	0.99	20.00	4.95
1.75	0.87	8.63	10.11	1.00	20.00	4.99
2.00	0.90	8.24	10.94	1.00	20.00	5.00
2.25	0.92	7.95	11.62	1.00	20.00	5.00
2.50	0.94	7.74	12.18	1.00	20.00	5.00
2.75	0.95	7.56	12.62	1.00	20.00	5.00
3.00	0.97	7.45	12.97	1.00	20.00	5.00
3.25	0.97	7.34	13.26	1.00	20.00	5.00
3.50	0.98	7.27	13.47	1.00	20.00	5.00
3.75	0.99	7.17	13.78	1.00	20.00	5.00
4.00	0.99	7.13	13.90	1.00	20.00	5.00
4.25	0.99	7.10	13.99	1.00	20.00	5.00
4.50	0.99	7.08	14.05	1.00	20.00	5.00
4.75	1.00	7.06	14.11	1.00	20.00	5.00
5.00	1.00	7.05	14.14	1.00	20.00	5.00

	ASN_0≈20 α=0.05					
	Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.05	20.55	0.26	0.05	20.00	0.24
0.25	0.27	15.38	1.79	0.22	20.00	1.10
0.50	0.48	10.85	4.38	0.59	20.00	2.95
0.75	0.62	8.74	7.05	0.85	20.00	4.25
1.00	0.72	7.57	9.48	0.96	20.00	4.80
1.25	0.78	6.81	11.52	0.99	20.00	4.95
1.50	0.84	6.33	13.22	1.00	20.00	4.99
1.75	0.88	5.99	14.63	1.00	20.00	5.00
2.00	0.90	5.75	15.73	1.00	20.00	5.00
2.25	0.93	5.57	16.64	1.00	20.00	5.00
2.50	0.94	5.43	17.37	1.00	20.00	5.00
2.75	0.96	5.33	17.93	1.00	20.00	5.00
3.00	0.97	5.26	18.38	1.00	20.00	5.00
3.25	0.97	5.20	18.72	1.00	20.00	5.00
3.50	0.98	5.16	19.00	1.00	20.00	5.00
3.75	0.99	5.09	19.41	1.00	20.00	5.00
4.00	0.99	5.08	19.52	1.00	20.00	5.00
4.25	0.99	5.06	19.62	1.00	20.00	5.00
4.50	0.99	5.04	19.71	1.00	20.00	5.00
4.75	1.00	5.03	19.78	1.00	20.00	5.00
5.00	1.00	5.03	19.83	1.00	20.00	5.00

	ASN_0≈30 α=0.01					
	Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.01	28.85	0.04	0.01	29.00	0.04
0.25	0.25	25.83	0.97	0.13	29.00	0.46
0.50	0.46	17.02	2.69	0.53	29.00	1.81
0.75	0.60	13.30	4.52	0.85	29.00	2.94
1.00	0.70	11.28	6.23	0.97	29.00	3.35
1.25	0.78	10.09	7.70	1.00	29.00	3.43
1.50	0.83	9.28	8.97	1.00	29.00	3.45
1.75	0.87	8.69	10.03	1.00	29.00	3.45
2.00	0.90	8.33	10.85	1.00	29.00	3.45
2.25	0.93	8.02	11.54	1.00	29.00	3.45
2.50	0.94	7.77	12.13	1.00	29.00	3.45
2.75	0.96	7.60	12.57	1.00	29.00	3.45
3.00	0.97	7.47	12.94	1.00	29.00	3.45
3.25	0.97	7.36	13.22	1.00	29.00	3.45
3.50	0.98	7.28	13.45	1.00	29.00	3.45
3.75	0.99	7.17	13.78	1.00	29.00	3.45
4.00	0.99	7.13	13.89	1.00	29.00	3.45
4.25	0.99	7.10	13.97	1.00	29.00	3.45
4.50	0.99	7.08	14.04	1.00	29.00	3.45
4.75	1.00	7.06	14.10	1.00	29.00	3.45
5.00	1.00	7.05	14.14	1.00	29.00	3.45

	ASN_0≈30 α=0.05					
	Sequential			Fixed		
δ	β_δ	ASN_δ	PAR_δ	β_δ	ASN_δ	PAR_δ
0.00	0.05	32.41	0.16	0.05	32.00	0.15
0.25	0.32	20.88	1.55	0.33	32.00	1.03
0.50	0.53	13.36	4.00	0.79	32.00	2.47
0.75	0.67	10.20	6.55	0.97	32.00	3.02
1.00	0.75	8.45	8.89	1.00	32.00	3.11
1.25	0.81	7.43	10.92	1.00	32.00	3.12
1.50	0.85	6.73	12.68	1.00	32.00	3.12
1.75	0.88	6.25	14.13	1.00	32.00	3.12
2.00	0.91	5.95	15.30	1.00	32.00	3.13
2.25	0.93	5.71	16.29	1.00	32.00	3.13
2.50	0.95	5.55	17.07	1.00	32.00	3.13
2.75	0.96	5.41	17.71	1.00	32.00	3.13
3.00	0.97	5.32	18.18	1.00	32.00	3.13
3.25	0.97	5.25	18.57	1.00	32.00	3.13
3.50	0.98	5.19	18.88	1.00	32.00	3.13
3.75	0.99	5.11	19.33	1.00	32.00	3.13
4.00	0.99	5.09	19.47	1.00	32.00	3.13
4.25	0.99	5.07	19.58	1.00	32.00	3.13
4.50	0.99	5.05	19.68	1.00	32.00	3.13
4.75	1.00	5.04	19.75	1.00	32.00	3.13
5.00	1.00	5.03	19.81	1.00	32.00	3.13

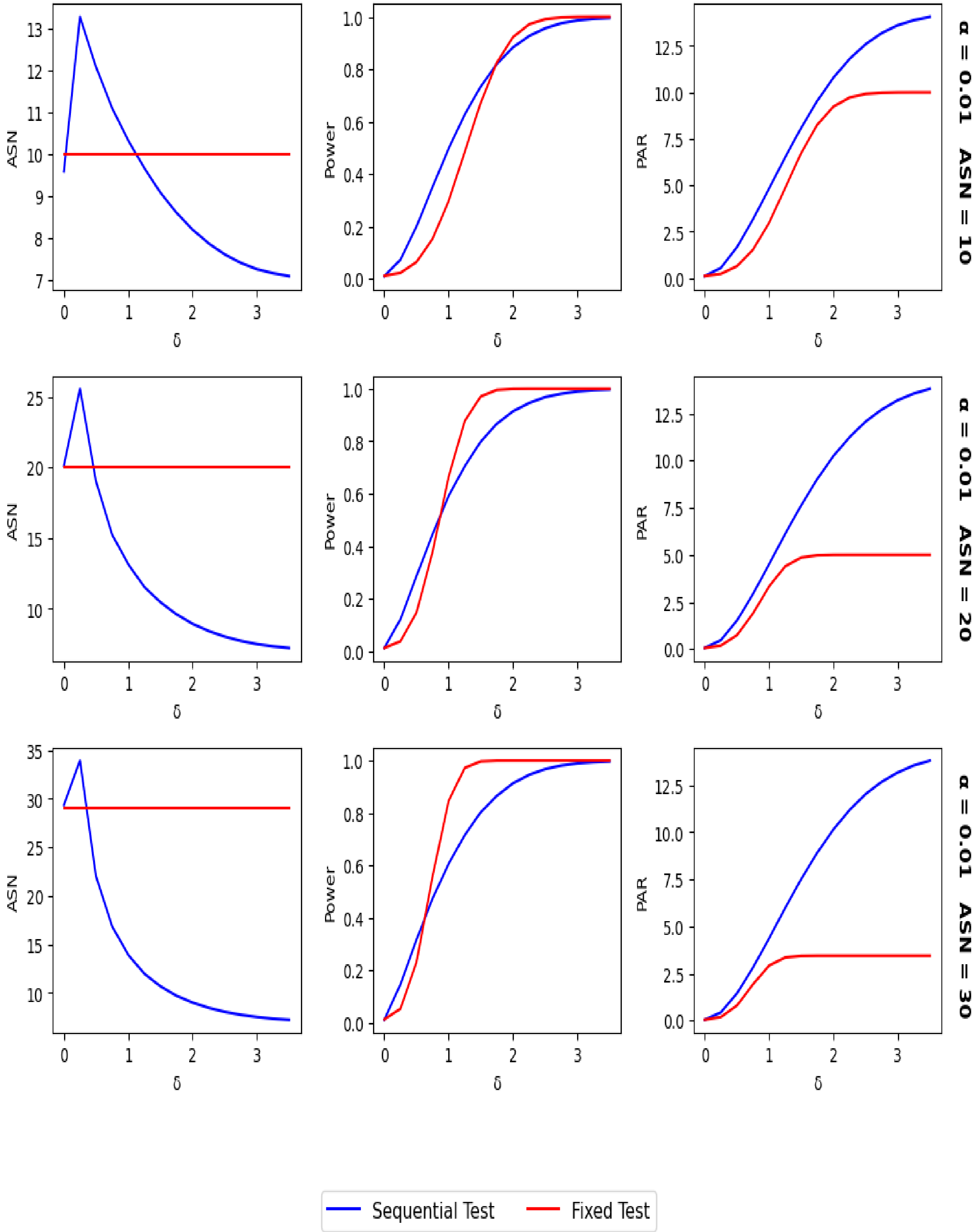

Fig 1. The ASN_δ, β_δ, and PAR_δ of Sequential and Fixed Mann-Whitney U Test for normal distribution at alpha 0.01

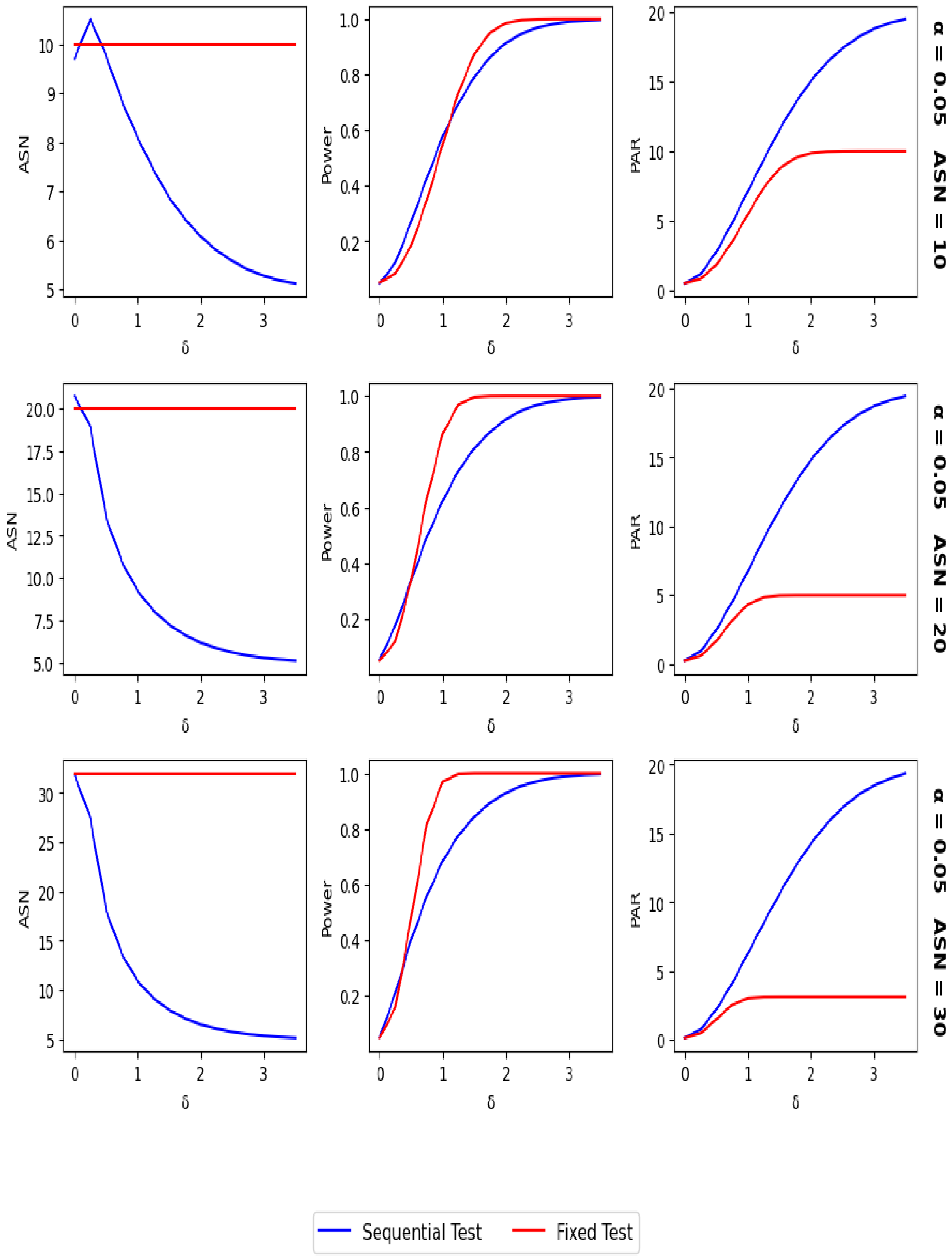

Fig 2. The ASN_δ, β_δ, and PAR_δ of Sequential and Fixed Mann-Whitney U Test for normal distribution at alpha 0.05

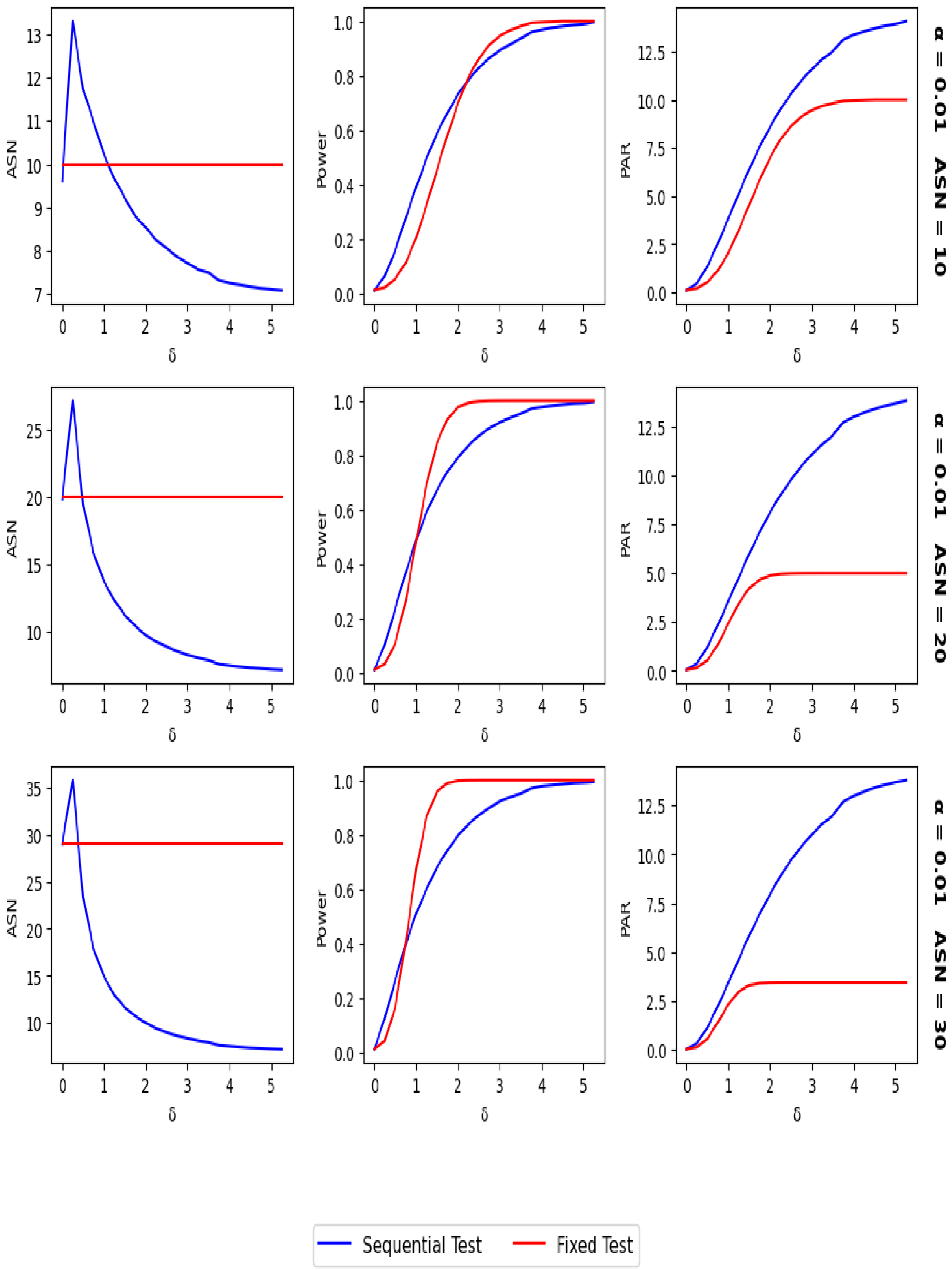

Fig 3. The ASN_{δ}, β_{δ}, and PAR_{δ} of Sequential and Fixed Mann-Whitney U Test for laplace distribution at alpha 0.01

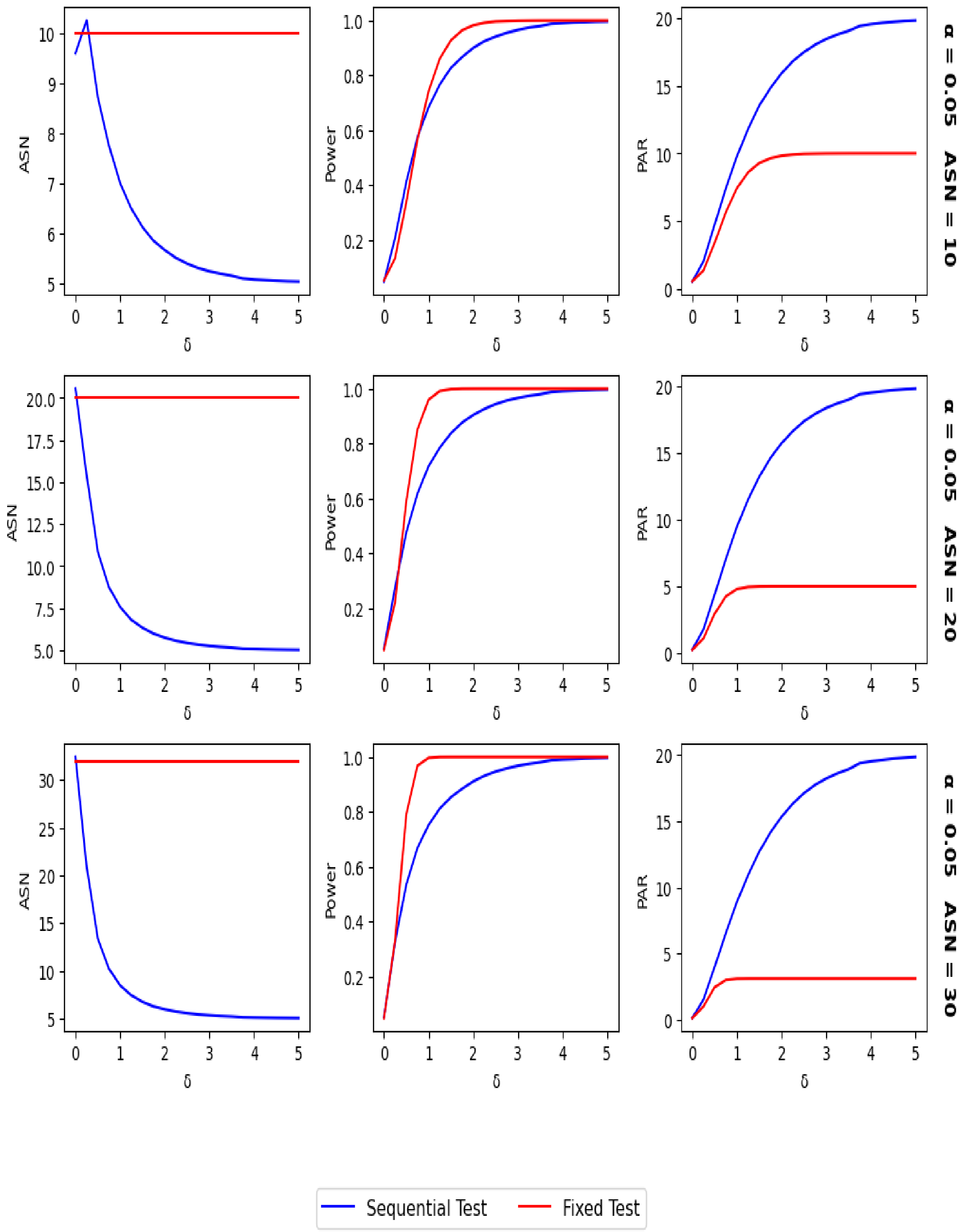

Fig 4. The ASN_δ, β_δ, and PAR_δ of Sequential and Fixed Mann-Whitney U Test for laplace distribution at alpha 0.05

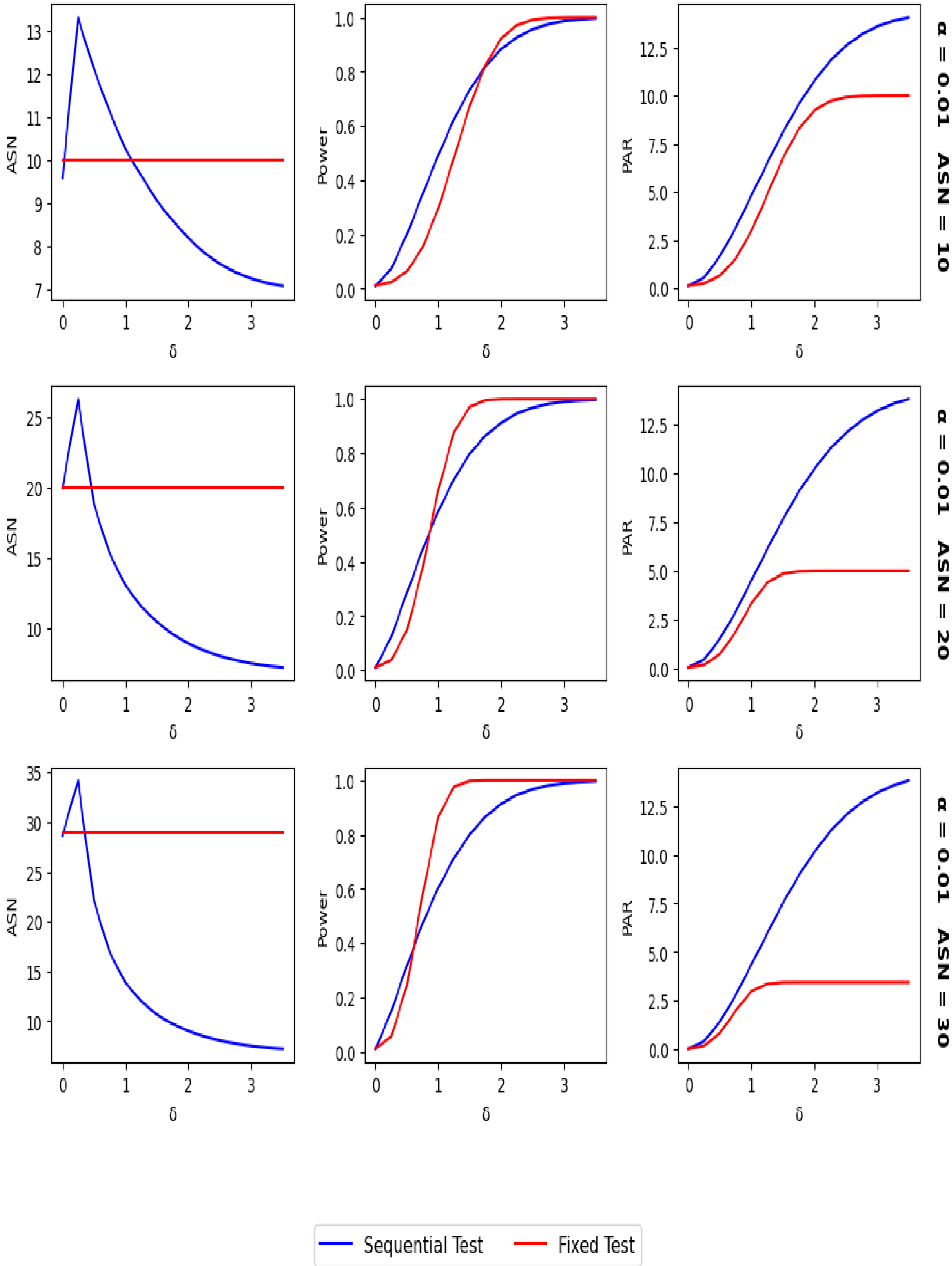

Fig 5. The ASN_δ, β_δ, and PAR_δ of Sequential and Fixed Mann-Whitney U Test for lognormal distribution at alpha 0.01

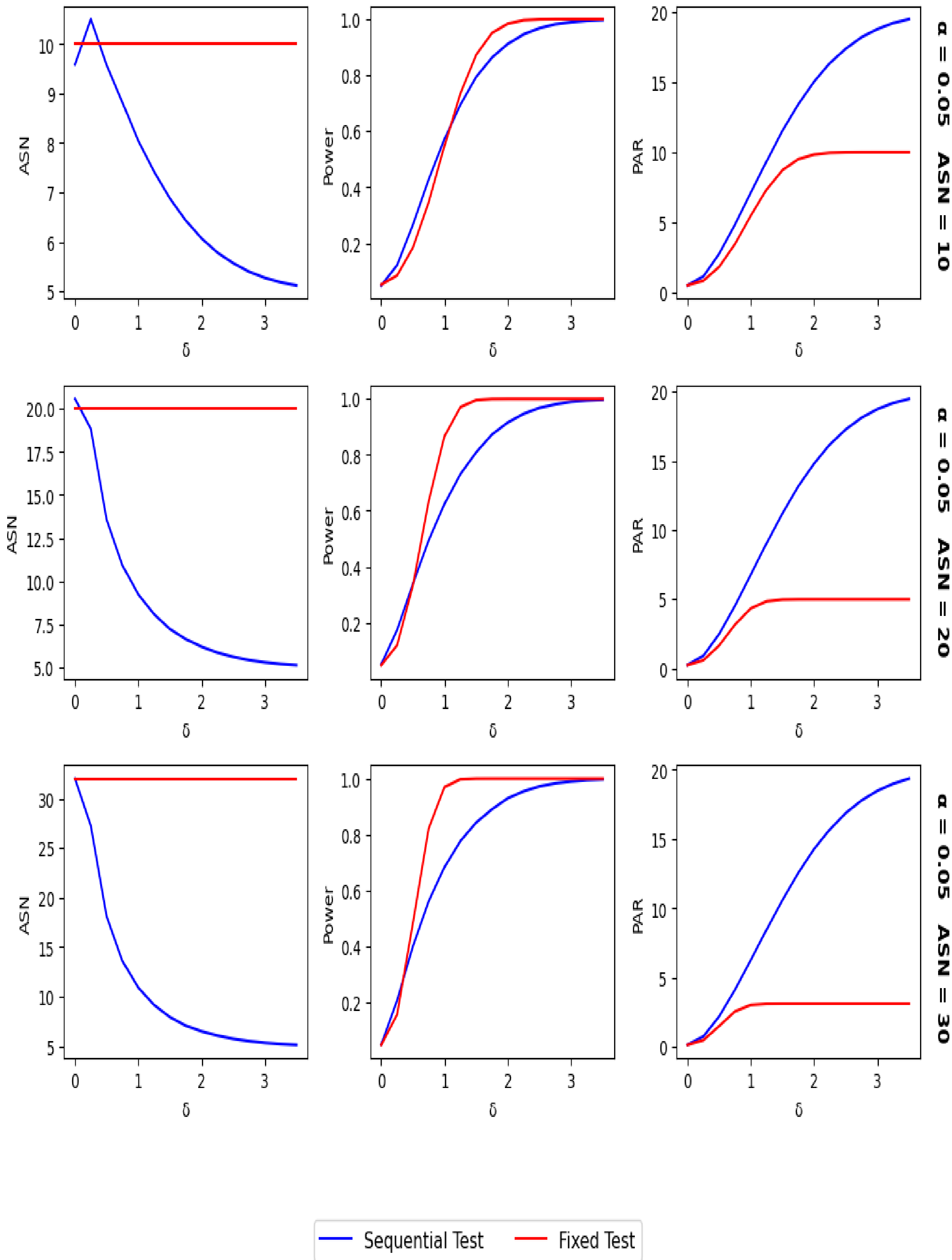

Fig 6. The ASN_δ, β_δ, and PAR_δ of Sequential and Fixed Mann-Whitney U Test for lognormal distribution at alpha 0.05

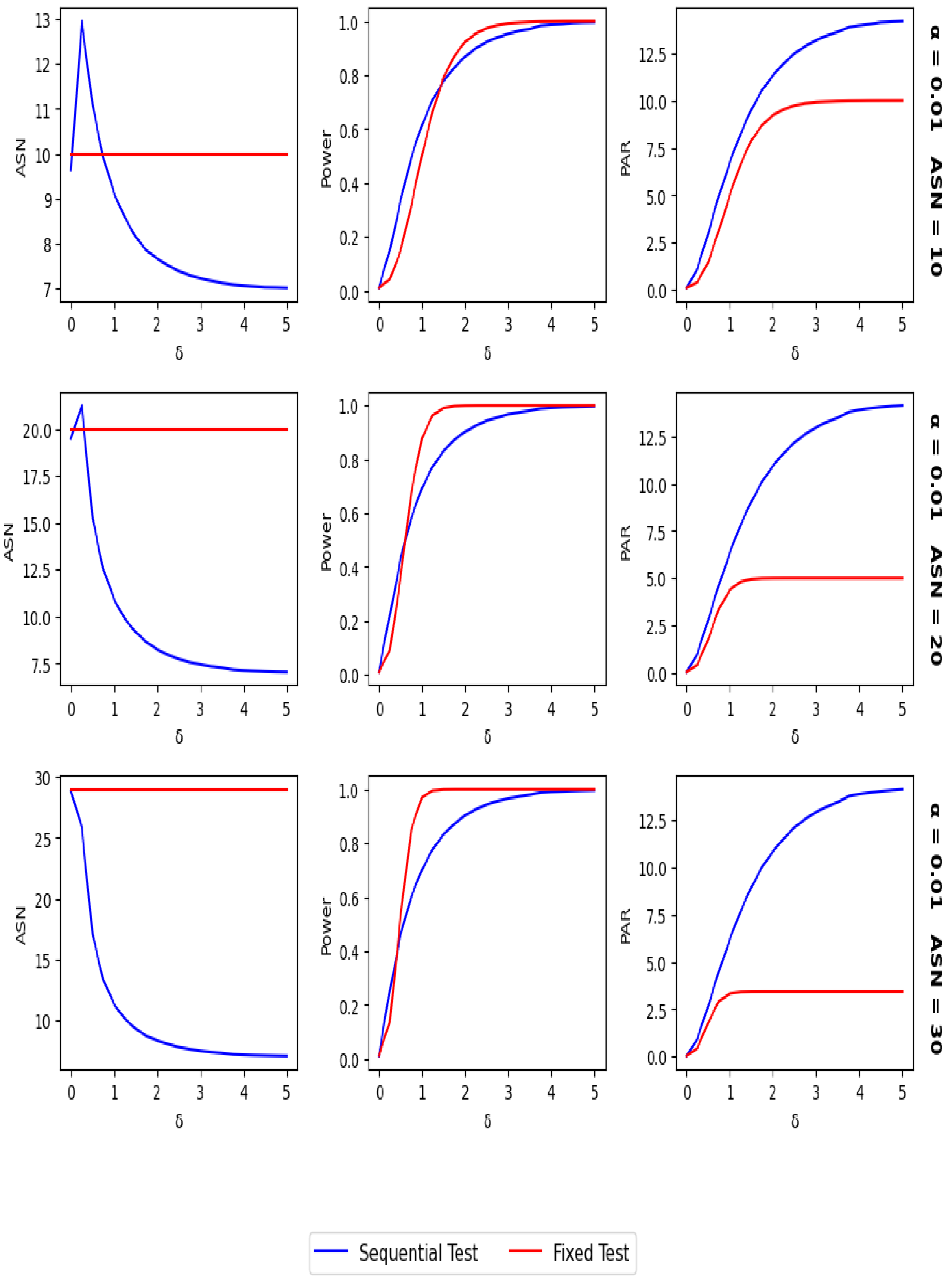

Fig 7. The ASN_δ, β_δ, and PAR_δ of Sequential and Fixed Mann-Whitney U Test for shifted exponential distribution at alpha 0.01

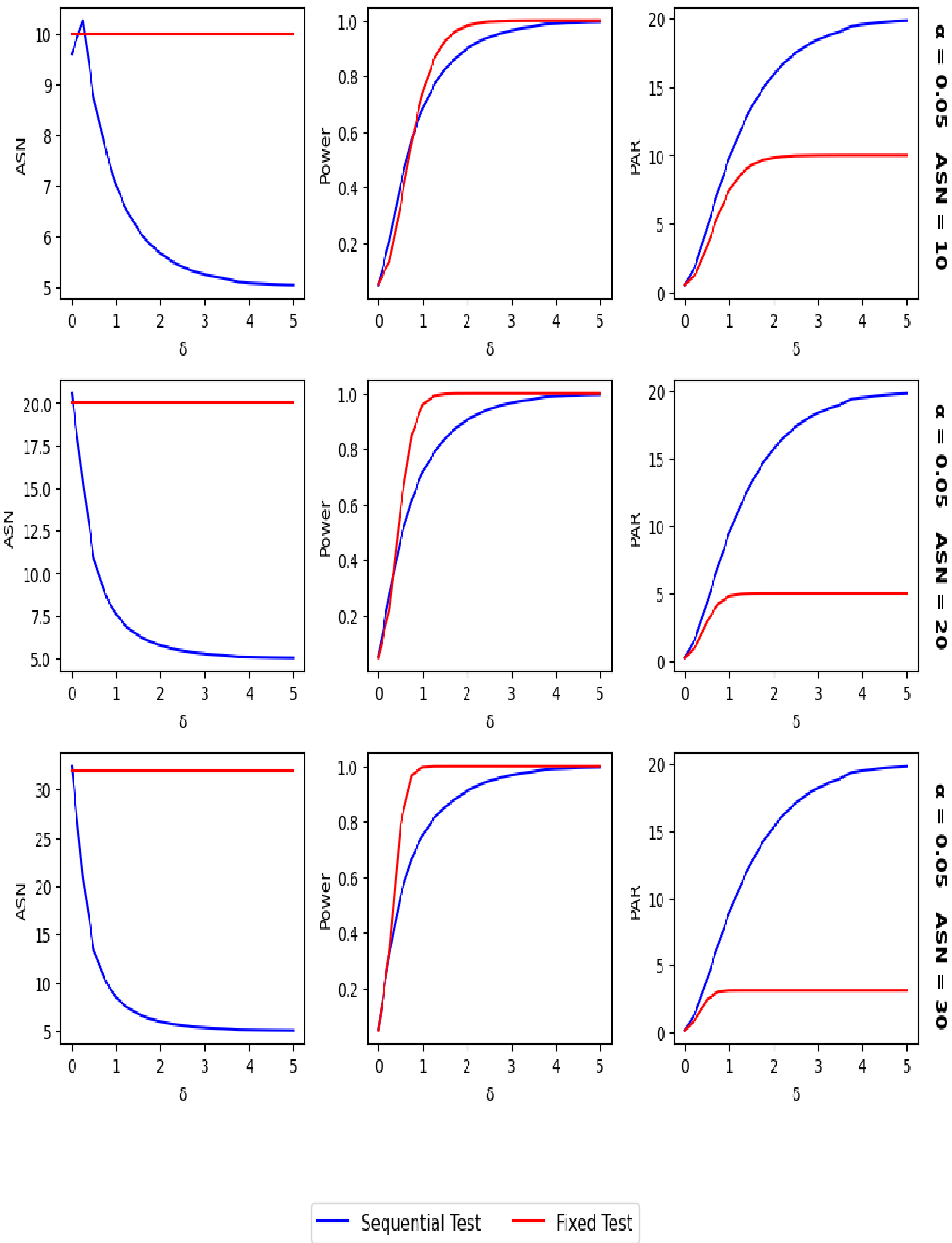

Fig 8. The ASN_δ, β_δ, and PAR_δ of Sequential and Fixed Mann-Whitney U Test for shifted exponential distribution at alpha 0.05

Appendix: Simulated data for the determination of relationships between (a, b) and (ASN_0, α)

a	*b*	ASN_0	α	*a*	*b*	ASN_0	α	*a*	*b*	ASN_0	α
0.20000	2.015789	13.31	0.16	0.436842	2.884211	18.92	0.02	0.673684	2.963158	09.94	0.01
0.20000	2.094737	15.89	0.13	0.436842	2.963158	19.75	0.01	0.673684	3.042105	10.83	0.01
0.20000	2.173684	17.47	0.12	0.436842	3.042105	20.58	0.01	0.713158	1.778947	03.54	0.17
0.20000	2.252632	20.06	0.11	0.476316	1.778947	04.72	0.20	0.713158	1.857895	03.61	0.16
0.20000	2.331579	22.55	0.09	0.476316	1.857895	04.96	0.19	0.713158	1.936842	03.68	0.16
0.20000	2.410526	25.89	0.07	0.476316	1.936842	05.22	0.18	0.713158	2.015789	04.26	0.11
0.20000	2.489474	28.44	0.06	0.476316	2.015789	06.30	0.13	0.713158	2.094737	04.56	0.09
0.20000	2.568421	30.52	0.05	0.476316	2.094737	06.99	0.11	0.713158	2.173684	04.74	0.08
0.23947	1.857895	07.20	0.20	0.476316	2.173684	07.61	0.10	0.713158	2.252632	05.07	0.07
0.23947	1.936842	07.94	0.19	0.476316	2.252632	08.55	0.09	0.713158	2.331579	05.48	0.05
0.23947	2.015789	09.87	0.14	0.476316	2.331579	09.74	0.07	0.713158	2.410526	06.01	0.04
0.23947	2.094737	11.57	0.12	0.476316	2.410526	10.83	0.05	0.713158	2.489474	06.39	0.04
0.23947	2.173684	12.82	0.11	0.476316	2.489474	11.91	0.05	0.713158	2.568421	06.88	0.03
0.23947	2.252632	14.81	0.09	0.476316	2.568421	12.73	0.04	0.713158	2.647368	07.48	0.02
0.23947	2.331579	17.34	0.08	0.476316	2.647368	14.11	0.03	0.713158	2.726316	07.80	0.02
0.23947	2.410526	19.20	0.06	0.476316	2.726316	15.24	0.03	0.713158	2.805263	08.31	0.02
0.23947	2.489474	21.38	0.06	0.476316	2.805263	16.30	0.02	0.713158	2.884211	09.08	0.01
0.23947	2.568421	23.26	0.05	0.476316	2.884211	17.17	0.02	0.713158	2.963158	09.33	0.01
0.23947	2.647368	25.42	0.04	0.476316	2.963158	18.37	0.01	0.752632	1.857895	03.51	0.16
0.23947	2.726316	27.62	0.03	0.476316	3.042105	18.83	0.01	0.752632	1.936842	03.58	0.15
0.23947	2.805263	29.61	0.02	0.515790	1.778947	04.58	0.19	0.752632	2.015789	04.08	0.10
0.27895	1.857895	06.89	0.21	0.515790	1.857895	04.77	0.19	0.752632	2.094737	04.37	0.09
0.27895	1.936842	07.46	0.19	0.515790	1.936842	05.04	0.18	0.752632	2.173684	04.53	0.08
0.27895	2.015789	09.47	0.14	0.515790	2.015789	05.97	0.13	0.752632	2.252632	04.79	0.07
0.27895	2.094737	10.99	0.12	0.515790	2.094737	06.60	0.11	0.752632	2.331579	05.22	0.05
0.27895	2.173684	12.41	0.11	0.515790	2.173684	07.18	0.10	0.752632	2.410526	05.63	0.04
0.27895	2.252632	13.67	0.09	0.515790	2.252632	07.97	0.08	0.752632	2.489474	05.98	0.04
0.27895	2.331579	16.09	0.08	0.515790	2.331579	08.87	0.07	0.752632	2.568421	06.42	0.03
0.27895	2.410526	18.08	0.06	0.515790	2.410526	09.90	0.05	0.752632	2.647368	06.85	0.02
0.27895	2.489474	19.43	0.06	0.515790	2.489474	10.64	0.05	0.752632	2.726316	07.25	0.02
0.27895	2.568421	21.91	0.05	0.515790	2.568421	11.74	0.04	0.752632	2.805263	07.73	0.02
0.27895	2.647368	24.16	0.04	0.515790	2.647368	13.05	0.03	0.752632	2.884211	08.16	0.01
0.27895	2.726316	25.94	0.03	0.515790	2.726316	13.65	0.03	0.752632	2.963158	08.28	0.01
0.27895	2.805263	27.58	0.02	0.515790	2.805263	15.32	0.02	0.792105	2.252632	03.60	0.05
0.27895	2.884211	28.86	0.02	0.515790	2.884211	15.56	0.02	0.792105	2.331579	03.96	0.04
0.27895	2.963158	29.82	0.02	0.515790	2.963158	16.86	0.01	0.792105	2.410526	04.22	0.03
0.31842	1.778947	05.51	0.20	0.515790	3.042105	17.31	0.01	0.792105	2.489474	04.31	0.03
0.31842	1.857895	05.76	0.19	0.555263	1.778947	04.38	0.19	0.792105	2.568421	04.66	0.03
0.31842	1.936842	06.32	0.19	0.555263	1.857895	04.55	0.18	0.792105	2.647368	04.92	0.02
0.31842	2.015789	07.90	0.13	0.555263	1.936842	04.76	0.18	0.792105	2.726316	05.31	0.02
0.31842	2.094737	08.86	0.11	0.555263	2.015789	05.61	0.12	0.792105	2.805263	05.59	0.01
0.31842	2.173684	09.94	0.11	0.555263	2.094737	06.17	0.10	0.792105	2.884211	05.91	0.01
0.31842	2.252632	11.54	0.09	0.555263	2.173684	06.59	0.09	0.831579	2.331579	03.79	0.04
0.31842	2.331579	13.35	0.07	0.555263	2.252632	07.30	0.08	0.831579	2.410526	04.00	0.03
0.31842	2.410526	14.55	0.06	0.555263	2.331579	08.35	0.06	0.831579	2.489474	04.20	0.03

a	*b*	ASN_0	*α*	*a*	*b*	ASN_0	*α*	*a*	*b*	ASN_0	*α*
0.31842	2.489474	16.28	0.05	0.555263	2.410526	08.96	0.05	0.831579	2.568421	04.41	0.02
0.31842	2.568421	18.27	0.04	0.555263	2.489474	09.62	0.05	0.831579	2.647368	04.73	0.02
0.31842	2.647368	19.52	0.03	0.555263	2.568421	10.43	0.04	0.831579	2.726316	04.97	0.01
0.31842	2.726316	21.75	0.03	0.555263	2.647368	11.71	0.03	0.831579	2.805263	05.06	0.01
0.31842	2.805263	23.07	0.02	0.555263	2.726316	12.50	0.03	0.831579	2.884211	05.46	0.01
0.31842	2.884211	24.09	0.02	0.555263	2.805263	13.43	0.02	0.871053	2.331579	03.58	0.04
0.31842	2.963158	25.21	0.02	0.555263	2.884211	14.37	0.02	0.871053	2.410526	03.81	0.03
0.31842	3.042105	26.30	0.01	0.555263	2.963158	14.83	0.01	0.871053	2.489474	03.91	0.03
0.31842	3.121053	27.91	0.01	0.555263	3.042105	15.54	0.01	0.871053	2.568421	04.14	0.02
0.35789	1.778947	05.18	0.20	0.594737	1.778947	04.11	0.18	0.871053	2.647368	04.50	0.02
0.35789	1.857895	05.56	0.19	0.594737	1.857895	04.22	0.18	0.871053	2.726316	04.74	0.01
0.35789	1.936842	05.96	0.19	0.594737	1.936842	04.38	0.17	0.871053	2.805263	04.81	0.01
0.35789	2.015789	07.30	0.13	0.594737	2.015789	05.11	0.12	0.910526	2.331579	03.58	0.04
0.35789	2.094737	08.41	0.11	0.594737	2.094737	05.60	0.10	0.910526	2.410526	03.71	0.03
0.35789	2.173684	09.16	0.10	0.594737	2.173684	05.92	0.09	0.910526	2.489474	03.84	0.03
0.35789	2.252632	10.30	0.09	0.594737	2.252632	06.58	0.08	0.910526	2.568421	04.02	0.02
0.35789	2.331579	12.21	0.07	0.594737	2.331579	07.26	0.06	0.910526	2.647368	04.18	0.02
0.35789	2.410526	13.87	0.06	0.594737	2.410526	08.09	0.05	0.910526	2.726316	04.47	0.01
0.35789	2.489474	15.12	0.05	0.594737	2.489474	08.56	0.04	0.910526	2.805263	04.72	0.01
0.35789	2.568421	15.95	0.04	0.594737	2.568421	09.25	0.04	0.950000	2.410526	03.52	0.03
0.35789	2.647368	18.18	0.03	0.594737	2.647368	10.27	0.03	0.950000	2.489474	03.70	0.03
0.35789	2.726316	19.40	0.03	0.594737	2.726316	11.11	0.02	0.950000	2.568421	03.84	0.02
0.35789	2.805263	20.49	0.02	0.594737	2.805263	11.89	0.02	0.950000	2.647368	04.02	0.02
0.35789	2.884211	22.04	0.02	0.594737	2.884211	12.04	0.02	0.950000	2.726316	04.25	0.01
0.35789	2.963158	23.26	0.02	0.594737	2.963158	13.23	0.01	0.950000	2.805263	04.34	0.01
0.35789	3.042105	24.03	0.01	0.594737	3.042105	13.66	0.01	0.116576	2.120265	21.56	0.14
0.35789	3.121053	24.83	0.01	0.634211	1.778947	04.06	0.18	0.098716	2.119507	26.88	0.15
0.39737	1.778947	05.07	0.20	0.634211	1.857895	04.16	0.18	0.084401	2.089625	27.18	0.16
0.39737	1.857895	05.30	0.19	0.634211	1.936842	04.29	0.17	0.073255	2.088289	29.75	0.16
0.39737	1.936842	05.62	0.18	0.634211	2.015789	04.97	0.12	0.089073	1.973221	21.26	0.20
0.39737	2.015789	06.87	0.13	0.634211	2.094737	05.42	0.10	0.076018	1.979622	22.47	0.19
0.39737	2.094737	07.89	0.11	0.634211	2.173684	05.74	0.09	0.057795	1.964383	26.35	0.21
0.39737	2.173684	08.51	0.10	0.634211	2.252632	06.27	0.08	0.076166	2.008581	24.30	0.18
0.39737	2.252632	10.00	0.09	0.634211	2.331579	06.79	0.06	0.098493	2.089761	25.74	0.16
0.39737	2.331579	11.11	0.07	0.634211	2.410526	07.48	0.05	0.215000	2.075000	14.42	0.14
0.39737	2.410526	12.43	0.06	0.634211	2.489474	08.05	0.04	0.215000	2.450000	25.98	0.07
0.39737	2.489474	13.54	0.05	0.634211	2.568421	08.81	0.04	0.061579	2.021053	30.32	0.18
0.39737	2.568421	14.80	0.04	0.634211	2.647368	09.57	0.03	0.113158	2.021053	18.92	0.16
0.39737	2.647368	16.84	0.03	0.634211	2.726316	10.15	0.02	0.164737	2.021053	15.10	0.15
0.39737	2.726316	17.71	0.03	0.634211	2.805263	11.05	0.02	0.164737	2.152632	18.58	0.13
0.39737	2.805263	18.99	0.02	0.634211	2.884211	11.56	0.02	0.110000	2.000000	18.31	0.17
0.39737	2.884211	20.12	0.02	0.634211	2.963158	12.04	0.01	0.072000	2.050000	27.91	0.17
0.39737	2.963158	21.79	0.01	0.634211	3.042105	12.75	0.01	0.078000	1.990000	23.23	0.19
0.39737	3.042105	22.20	0.01	0.673684	1.778947	03.61	0.17	0.190000	2.200000	18.74	0.12
0.43684	1.778947	04.88	0.20	0.673684	1.857895	03.67	0.16	0.090000	2.130000	28.58	0.15
0.43684	1.857895	05.08	0.19	0.673684	1.936842	03.75	0.16	0.095000	2.123000	27.27	0.15
0.43684	1.936842	05.38	0.18	0.673684	2.015789	04.30	0.11	0.150000	2.133000	20.01	0.13

a	b	ASN_0	α	a	b	ASN_0	α	a	b	ASN_0	α
0.43684	2.015789	06.60	0.13	0.673684	2.094737	4.63	0.09	0.166361	2.381463	27.04	0.08
0.43684	2.094737	07.49	0.11	0.673684	2.173684	4.89	0.08	0.114000	2.110000	22.03	0.14
0.43684	2.173684	08.05	0.10	0.673684	2.252632	5.31	0.07	0.200000	2.400000	24.98	0.08
0.43684	2.252632	09.16	0.09	0.673684	2.331579	5.78	0.05	0.098720	2.109621	26.21	0.15
0.43684	2.331579	10.24	0.07	0.673684	2.410526	6.28	0.04	0.130000	2.209621	24.18	0.12
0.43684	2.410526	11.61	0.06	0.673684	2.489474	6.71	0.04	0.089073	1.973221	20.46	0.20
0.43684	2.489474	12.69	0.05	0.673684	2.568421	7.43	0.03	0.089073	2.052169	25.42	0.17
0.43684	2.568421	13.95	0.04	0.673684	2.647368	7.92	0.03	0.110000	2.219821	25.78	0.12
0.43684	2.647368	15.22	0.03	0.673684	2.726316	8.58	0.02	0.110000	2.225821	26.64	0.12
0.43684	2.726316	15.98	0.03	0.673684	2.805263	8.98	0.02	0.210474	2.000789	12.89	0.16
0.43684	2.805263	17.55	0.02	0.673684	2.884211	9.51	0.01	0.140000	2.235821	23.68	0.12