A Project Report

On

Reflection & Refraction of wave in a non-ideal medium

BY

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Certificate

This is to certify that the project report entitled “**Reflection & Refraction of wave in a non-ideal medium”** submitted by Mr. Mohit Agrawal (ID No.2019B4AA0918H), Mr. Raghav Gupta (ID No. 2019B4A30927H) and Mr. Shubhan Mital (ID No.2019B4A30900H) in partial fulfillment of the requirements of the course MATH F376, Design Project Course, embodies the work done by them under my supervision and guidance.

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ABSTRACT

The paper we are studying and trying to implement presents a highly efficient implicit finite difference scheme for modeling acoustic wave propagation. The scheme is based on the second-order accurate backward difference formula and is designed to handle large time steps, which significantly reduces the computational cost of the simulation. The authors also present a stable and efficient implementation of the scheme using the alternating direction implicit (ADI) method. The effectiveness of the proposed scheme is demonstrated through numerical experiments on a range of test cases, including homogeneous and heterogeneous media, and the results show that the proposed scheme is highly accurate and computationally efficient, making it a promising tool for modeling acoustic wave propagation in real-world applications.

Introduction

Acoustic wave propagation is a fundamental phenomenon that occurs in various scientific and engineering applications, such as medical imaging, non-destructive testing, and seismic exploration. Accurate numerical simulation of acoustic wave propagation is essential for these applications. The finite difference method is a widely used numerical method for simulating acoustic wave propagation. In this paper, Malkoti et al. propose a highly efficient implicit finite difference scheme for simulating acoustic wave propagation.

Methodology

The proposed numerical scheme is based on the implicit finite difference method. The wave equation is a second-order partial differential equation that describes the propagation of acoustic waves. The authors use a second-order accurate central difference scheme for the spatial discretization of the wave equation. The spatial derivative is approximated by the difference between the values of the function at two nearby points, divided by the distance between the points. The central difference scheme is more accurate than the forward or backward difference scheme because it uses the values of the function at both points.

To reduce the computational cost, the authors employ a technique called preconditioning. Preconditioning is a technique in which the original system of linear equations is transformed into a new system that is easier to solve, but which has the same solution. The authors use the Incomplete LU (ILU) factorization as the preconditioner. The ILU factorization is a sparse matrix factorization that approximates the original matrix by a product of lower and upper triangular matrices. It is a widely used preconditioner for solving sparse linear systems.

The authors also use a technique called the alternating direction implicit (ADI) method to further reduce the computational cost. The ADI method splits the original system of linear equations into smaller sub-problems that can be solved independently in a more efficient manner. The ADI method is based on the observation that the matrix resulting from the spatial discretization of the wave equation is tridiagonal, meaning that it has nonzero entries only on the main diagonal and the two adjacent diagonals. The ADI method splits the system of linear equations into two sub-problems, one involving the main diagonal and the upper adjacent diagonal, and the other involving the main diagonal and the lower adjacent diagonal. Each sub-problem can be solved using a simpler matrix factorization method, such as the Thomas algorithm.

Discussion

The derivative operator is a mathematical operation that finds the rate at which a function changes over an infinitesimal interval. In other words, it tells us how much a function changes as its input changes slightly.The derivative of a function f(x) with respect to x is denoted by f'(x) or dy/dx. It is defined as the limit of the difference quotient as the interval between two points in the function approaches zero:

f'(x) = lim(h->0) [f(x+h) - f(x)] / h

The **derivative operator** is important in calculus and many areas of mathematics, physics, engineering, and economics. It is used to find maximum and minimum points, rates of change, slopes of curves, and much more.

The **spectral characteristics** of a derivative operator refer to its eigenvalues, eigenvectors, and other spectral properties. In particular, the spectral properties of a derivative operator depend on the type of derivative being considered, as well as the domain and boundary conditions of the problem. For example, the Fourier transform can be used to analyze the spectral characteristics of the derivative operator. The Fourier transform of the derivative operator transforms it into a multiplication operator, which makes it easier to analyze its spectral properties. The eigenvalues of the derivative operator are then given by the Fourier transform of the coefficients of the differential equation being considered. For the first-order derivative operator, the spectral characteristics include the fact that its eigenvalues are purely imaginary and its eigenvectors are complex exponential functions. The eigenvalues and eigenvectors of the second-order derivative operator depend on the boundary conditions of the problem.

The spectral characteristics of the derivative operator play an important role in the stability and convergence properties of numerical methods for solving differential equations. In particular, the spectral properties of the operator can affect the accuracy and stability of numerical methods, and must be carefully considered in the design and implementation of such methods.

The proposed scheme has several advantages over existing numerical methods for acoustic wave propagation. It is unconditionally stable and converges to the exact solution in the limit of the grid spacing tending to zero. It is also highly efficient and requires less computational resources than existing methods. The scheme is easy to implement and can be applied to a wide range of problems in seismology and geophysics.

The **implicit difference** scheme is a numerical method used to solve partial differential equations, particularly those with nonlinearities or complex boundary conditions. Unlike explicit difference schemes, which involve updating the solution at each time step based on the values at the previous time step, implicit difference schemes involve solving a system of equations at each time step to obtain the updated solution.The implicit difference scheme involves approximating the solution of a partial differential equation using finite differences, but instead of using the values at the current time step, it uses a combination of values at the current and previous time steps. This leads to a system of nonlinear equations, which must be solved at each time step.The implicit difference scheme can also be used to solve the acoustic wave equation, which describes the propagation of sound waves through a medium. The acoustic wave equation is a partial differential equation that can be written as:

$$\partial^2p/\partial t^2 = c^2\nabla^2p$$

where $p(x,t)$ is the pressure, c is the speed of sound, and ∇^2 is the Laplace operator. To solve this equation numerically, we can use the implicit difference scheme, which involves approximating the Laplace operator using values at the current and previous time steps. This leads to a system of linear equations, which must be solved at each time step.The implicit difference scheme for the acoustic wave equation can be written as:

$$p(x, t+\Delta t) - 2p(x, t) + p(x, t-\Delta t) = c^2 \Delta t^2 (\partial^2p/\partial x^2)$$

where Δt is the time step size and $(\partial^2p/\partial x^2)$ is approximated using values at the current and previous time steps. This equation can be rearranged to give an equation for the pressure at the next time step in terms of the pressure at the current and previous time steps. The resulting system of linear equations can be solved using techniques such as the Thomas algorithm. The implicit difference scheme is particularly useful for simulating acoustic wave propagation in complex media, where the boundary conditions may be difficult to express explicitly. It can also be used to simulate wave propagation in anisotropic media, where the speed of sound may vary with direction. However, the implicit difference scheme can be computationally expensive, especially for large-scale simulations, and its stability and accuracy depend on the choice of time step size and spatial grid spacing.

The **central difference scheme** is a numerical method used to approximate the derivative of a function at a point, by using the function values at neighboring points. In particular, the central difference scheme is a finite difference method that uses the values of the function at points symmetrically located around the point of interest. The central difference scheme is commonly used in numerical methods for solving differential equations, where it is used to discretize the derivative terms in the equations. The accuracy and stability of these numerical methods depend on the choice of the step size h, and must be carefully chosen to ensure accurate and stable solutions.The central difference scheme can also be used to solve the acoustic wave equation, which describes the propagation of sound waves through a medium. The acoustic wave equation is a partial differential equation that can be written as:

$$\partial^2p/\partial t^2 = c^2\nabla^2p$$

where p(x,t) is the pressure, c is the speed of sound, and ∇^2 is the Laplace operator. To solve this equation numerically using the central difference scheme, we can approximate the Laplace operator using values at neighboring points in space. For example, the second derivative with respect to x can be approximated using the central difference scheme as:

$$(\partial^2p/\partial x^2) \approx [p(x+\Delta x) - 2p(x) + p(x-\Delta x)] / \Delta x^2$$

where Δx is the spatial step size.

Substituting this approximation into the acoustic wave equation yields:

$$\partial^2p/\partial t^2 = c^2[p(x+\Delta x, t) - 2p(x, t) + p(x-\Delta x, t)] / \Delta x^2$$

This equation can be rearranged to give an equation for the pressure at the next time step in terms of the pressure at the current and previous time steps. The central difference scheme is second-order accurate, meaning that the error in the approximation is proportional to Δx^2. It is a commonly used method for simulating wave propagation in acoustic and seismic applications, where high accuracy is required. However, it can suffer from numerical instability if the time step size is too large or if the spatial grid spacing is not properly chosen.

Work done

1) We read and analyzed the research paper and derived some equations.

Eq (4) from Section 2.2

$$f_i'' + \alpha\left(f_{i+1}'' + f_{i-1}''\right) + \beta\left(f_{i+2}'' + f_{i-2}''\right)$$

$$= a\,\frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + b\,\frac{f_{i+2} - 2f_i + f_{i-2}}{(2h)^2} + c\,\frac{f_{i+3} - 2f_i + f_{i-3}}{(3h)^2}$$

applying fourier on both sides

LHS

$$f_{i+1}'' = f_i'' + f_i''' h + f_i^4 \frac{h^2}{2!} + f_i^5 \frac{h^3}{3!} + \dots$$

$$f_{i-1}'' = f_i'' - f_i''' h + f_i^4 \frac{h^2}{2!} - f_i^5 \frac{h^3}{3!} + \dots$$

$$f_{i+2}'' = f_i'' + 2f_i''' h + 4f_i^4 \frac{h^2}{2!} + 8f_i^5 \frac{h^3}{3!} + \dots$$

$$f_{i-2}'' = f_i'' - 2f_i''' h + 4f_i^4 \frac{h^2}{2!} - 8f_i^4 \frac{h^3}{3!} + \dots$$

Putting the above equations in LHS of eqn (4), we get

$$f_i'' + \alpha\left(2f_i'' + 2f_i^4 \frac{h^2}{2!} + 2f_i^6 \frac{h^4}{4!}\right) + \beta\left(2f_i'' + 8f_i^4 \frac{h^2}{2!} + 32 f_i^6 \frac{h^4}{4!}\right) \quad \text{— (a)}$$

RHS

$$f_{i+1} = f_i + f_i' h + f_i'' \frac{h^2}{2!} + f_i^3 \frac{h^3}{3!} + f_i^4 \frac{h^4}{4!} + \dots$$

$$f_{i-1} = f_i - f_i' h + f_i'' \frac{h^2}{2!} - f_i^3 \frac{h^3}{3!} + f_i^4 \frac{h^4}{4!} + \dots$$

$$f_{i+2} = f_i + 2f_i' h + 2f_i'' h^2 + 4f_i^3 \frac{h^3}{3} + 8f_i^4 \frac{h^4}{3} + \dots$$

$$f_{i-2} = f_i - 2f_i' h + 2f_i'' h^2 - 4f_i^3 \frac{h^3}{3} + 8f_i^4 \frac{h^4}{3} + \dots$$

$$f_{i+3} = f_i + 3f_i' h + 9f_i'' \frac{h^2}{2} + 27 f_i''' \frac{h^3}{6} + 81 f_i^{4} \frac{h^4}{24} + \dots$$

$$f_{i-3} = f_i - 3f_i' h + 9f_i'' \frac{h^2}{2} - 27 f_i''' \frac{h^3}{6} + \dots$$

Putting these equations in RHS of eqn (4), we get

$$\left(\frac{a}{h^2}\right)\left(2f_i'' \frac{h^2}{2!} + 2f_i^{4} \frac{h^4}{24} + \dots\right) + \left(\frac{b}{(2h)^2}\right)\left(4f_i'' h^2 + 4f_i^{4} \frac{h^4}{3!} + \dots\right)$$
$$+ \left(\frac{c}{(3h)^2}\right)\left(18 f_i'' \frac{h^2}{2} + 162 f_i^{4} \frac{h^4}{24} + \dots\right) \quad \text{— (b)}$$

Now comparing the eqn (a) & (b) we get

h^0, f'' : $(1 + 2\alpha + 2\beta) = 2 \frac{1}{2!} (a+b+c)$

h^2, f^4 : $(2) \frac{1}{2!} (\alpha + 2^2 \beta) = 2 \frac{1}{4!} (a + 2^2 b + 3^2 c)$

$\vdots$

↳ eqn (5)

2) Derivation of equation (15):

To derive the wavenumber for the implicit finite difference scheme, we start by discretizing the Laplacian operator using the second-order central finite difference approximation:

∇^2(u) ≈ (u(i-1,j,k) - 2u(i,j,k) + u(i+1,j,k))/h^2 + (u(i,j-1,k) - 2u(i,j,k) + u(i,j+1,k))/h^2 + (u(i,j,k-1) - 2u(i,j,k) + u(i,j,k+1))/h^2 where h is the grid spacing in all three directions.

Substituting this approximation into the wave equation, we get:

$$\partial^2 u/\partial t^2 = c^2(x) [(u(i-1,j,k) - 2u(i,j,k) + u(i+1,j,k))/h^2 + (u(i,j-1,k) - 2u(i,j,k) + u(i,j+1,k))/h^2 + (u(i,j,k-1) - 2u(i,j,k) + u(i,j,k+1))/h^2]$$

Next, we replace the continuous variables with their discrete counterparts, using the notation $u(i,j,k,n)$ to denote the value of u at grid point (i,j,k) and time step n.

$$u(i,j,k,n+1) - 2u(i,j,k,n) + u(i,j,k,n-1) = c^2(x) [(u(i-1,j,k,n) - 2u(i,j,k,n) + u(i+1,j,k,n))/h^2 + (u(i,j-1,k,n) - 2u(i,j,k,n) + u(i,j+1,k,n))/h^2 + (u(i,j,k-1,n) - 2u(i,j,k,n) + u(i,j,k+1,n))/h^2]$$

We can simplify this equation by multiplying both sides by h^2 and rearranging the terms:

$$u(i,j,k,n+1) - 2u(i,j,k,n) + u(i,j,k,n-1) = c^2(x) [u(i-1,j,k,n) + u(i+1,j,k,n) + u(i,j-1,k,n) + u(i,j+1,k,n) + u(i,j,k-1,n) + u(i,j,k+1,n) - 6u(i,j,k,n)]$$

Now, we assume that the solution can be expressed as a plane wave of the form: $u(i,j,k,n) = A \cos(kx\ i + ky\ j + kz\ k - \omega\ n\ \Delta t)$

where A is the amplitude, kx, ky, and kz are the components of the wave vector, ω is the angular frequency, and Δt is the time step size.

Using this solution, we think we can derive the wavenumber.

Work to be done

We will test the proposed numerical scheme for acoustic wave propagation in different scenarios. The numerical experiments will be conducted using a computer program developed in MATLAB and python. We will compare the results obtained using the proposed scheme with those obtained using the explicit finite difference scheme and the analytical solution and will try to find out which scheme is more efficient.

Conclusion

The paper presents a new and highly efficient approach for simulating acoustic wave propagation. The scheme has the potential to improve the accuracy and reduce the computational cost of acoustic simulations in various applications, such as seismic exploration, medical imaging, and non-destructive testing. A highly efficient implicit finite difference scheme for acoustic wave propagation is presented where the authors first derive a new formulation of the acoustic wave equation that is more suitable for finite difference methods. After that, a formulation to develop an implicit finite difference scheme that is highly efficient and accurate is applied.The scheme is shown to have several advantages over existing methods, including faster convergence, improved accuracy, and reduced computational cost.This scheme can be easily adapted to handle anisotropic media and can be extended to solve more complex wave equations (to be studied in more depth in later stages)

There are certain other aspects of the paper that are yet to be explored and will be done in later stages of time.